The Bethe/Gauge Correspondence and Geometric Representation Theory

by

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Abstract

This paper reviews the Bethe/gauge correspondence and its relation with geometric representation theory. The Bethe/gauge correspondence, first introduced by Nekrasov and Shatashvili, connects an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory in two dimensions with an integrable system solvable by the Bethe ansatz. Both sides of the correspondence are discussed and then the Bethe/gauge correspondence is stated. This is followed by a discussion of geometric representations which can be used for the interpretation of the Bethe/gauge correspondence.

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Contents

1 Introduction

The Bethe/gauge correspondence was first formulated by Nekrasov and Shatashvili in [1], [2]. They relate the two branches of physics (or mathematics), integrable systems and gauge theories.

'Bethe' in the Bethe/gauge correspondence means the intergrable systems solvable by the Bethe ansatz technique. Here, the name 'Bethe ansatz' comes from the Bethe's educated guess for the wave function to solve the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ model in 1931 [5]. Today his method is called the coordinate Bethe ansatz. Through the coordinate Bethe ansatz, however, it is hard to see the quantum integrable structure of the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ model. To elucidate the quantum integrability of the model, we need the quantum inverse scattering methods and the algebraic Bethe ansatz developed by the Leningrad School led by Faddeev in the 1980s. The quantum inverse scattering method starts with the quantisation of the Lax formulation, which comes from a classical integrable system, and reproduces the results of the Bethe ansatz. In the process, the algebraic Bethe ansatz is formulated and used to solve the quantum integrable system.

On the other hand, 'gauge' in the Bethe/gauge correspondence refers to the vacua of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories in two dimensions. The peculiar notation $\mathcal{N} = (2, 2)$ means that there are four real supercharges, two with positive chirality and two with positive chirality. One of reason why we pay attention to $\mathcal{N} = (2, 2)$ supersymmetric models in two dimensions is their relation to string theory and conformal theory. In particular, two-dimensional conformal field theories with $\mathcal{N} = (2, 2)$ supersymmetry are considered as candidate vacua for perturbative string theory. In the Bethe/gauge correspondence, the vacua of $\mathcal{N} = (2, 2)$ supersymmetric theories are concerned. Since supersymmetry restrics a theory and makes it well-behaved through many desirable properies such as non-renormalisation theorems and protection of certain parmeters from quantum corrections, these properties are very helpful to analyse the vacua of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories in two dimensions.

Thus the Bethe/gauge correspondence contains those cores of two theories. However the Bethe/gauge correspondence is just an observation. The underlying reason for existence of such correspondence is elusive. In [4], Orlando and Reffert argued that geometrical representation theory is the mathematical foundation of the Bethe/gauge correspondence. They start with the spectrum of $\mathrm{XXX}_{\frac{1}{2}}$ spin chain and then, via geometric representation, relate it with the ground state geometry of the corresponding gauge theory by the Bethe/gauge correspondence.

This paper's aim is to understand the Bethe/gauge correspondence and its relation with geometric representation. Consquently, this paper is composed of four main chapters, which treat quantum integrability, $\mathcal{N} = (2, 2)$ supersymmetrc theory, Bethe/gauge correspondence and geometric representation theory.

We start with a discussion of integrable systems in chapter 2. The concept of the classical integrability of finite and continuous systems is explained. In particular, Lax pairs are introduced and investigate their role of integrability. Their quantum analogue is introduced and the quantum inverse scattering method is discussed. After all the preliminaries are discussed, we solve the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ spin chain by the algebraic Bethe ansatz. In doing so, we obtain the Bethe ansatz equations which play an important role in the Bethe/gauge correspondence. Also we discuss generalised spin chains by introducing local spins, inhomogeneities, and a twist parameter. At last, we discuss a unexpected feature, the Yang-Yang function which is a counterpart of the effective twisted superpotential.

In chapter 3, we discuss the $\mathcal{N} = (2,2)$ supersymmetric gauge theory in two dimensions or the other side of the Bethe/gauge correspondence. First we start with the basics of the gauge theory such as superspaces, superfields, and Lagrangians. Afterwards, we study twisted masses and twisted superpotentials which play important roles in the Bethe/gauge correspondence. Some contents in the theory have no counterpart in four-dimensional theories, so we discuss how they are understood by dimensional reduction from the $\mathcal{N} = 1$ supersymmetric theory in four dimensions. After discussing the basic ingredients of the theory, we analyse the vacuum structure; in particular we look into the Coulomb branch. Thus we consider the effective low energy theory after integrating out the massive fields and calculate the effective twisted superpotential. Finally we then arrive at the vacuum equations.

After explaining the both side of the correspondence, we compare the results of two theories in chapter 4. In particular, we relate the Bethe ansatz equations and Yang-Yang function with the vacuum equations and effective twisted superpotential respectivley. Finally, with some modifications, we arrive at the Bethe/gauge correspondence.

At last, in chapter 5, we explain the geometric representation theory of $\mathfrak{sl}_n$, following the Ginzburg's construction in particular. We only sketch the general outline of his construction; a detailed explanation can be found in [6]. As a preliminary, we discuss the Borel-Moore homology which gives the vector space structure to varieties. Also a list of a few desirable properties are illustrated. After that, we move on to the convolution structure which allows us to have an algebra and its module. All these are preliminary and we finally apply them to the flag variety to construct the $\mathfrak{sl}_n$-module. Equipped with the mathematical tool, we give the physical interpretation of geometric representation theory for the Bethe/gauge correspondence.

2 Quantum Integrable Lattice Model

2.1 Integrable Systems

2.1.1 Integrability

In a classical theory, a Hamiltonian system with $2n$-dimensional phase space and Hamiltonian H is called *Liouville integrable* if it has n independent functions $F_1, \cdots, F_n$ which are conserved ($\{H, F_i\} = 0$) and involutive ($\{F_i, F_j\} = 0$). Then, by Liouville's theorem, the solution of the equations of motion of a Liouville integrable system is obtained by quadrature, which means the solution is obtained through only aglebraic manipulations and some integrals. Given a Hamiltonian of a system, however, there is no general procedure to determine whether the system is integrable or not; that is, a general method to find the n independent conserved quantities in involution is not known. In spite of this difficulty, many techniques to classify and construct the possible integrable systems have been developed. One of them is the *Lax pair* formulation, first introduced by Peter Lax [15]. This involves a Lax pair which implies the existence of conserved quantities and provides properties that serve as a basis for inverse scattering methods. Also, its quantised version provides a starting point for quantum inverse scattering methods hence the following discusses some of its features before moving onto quantum inverse scattering methods.

Let $\mathcal{M}$ be the space of $n \times n$ matrices whose entries are functions on a phase space. If $L, M \in \mathcal{M}$ satisfy

$$\frac{d}{dt}L = [M, L] = ML - LM \tag{2.1}$$

then they are called a *Lax pair*. Then from the above equation, we can see that there are n convserved quantities:

$$\frac{d}{dt}\mathrm{Tr}(L^m) = m\mathrm{Tr}(L^{m-1}\dot{L}) = m\mathrm{Tr}(L^{m-1}[M, L]) = \mathrm{Tr}([M, L^m]) = 0 \qquad \forall m \in \mathbb{N} \tag{2.2}$$

The existence of conserved quantities, however, does not guarantee the Liouville integrability; we have to check that they are in involution. For the involution property, we need to introduce a new notion, the *classical r-matrix*. The involution between the conservsed quantities is equivalent to the existence of a matrix $r \in \mathcal{M} \otimes \mathcal{M}$ such that

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{2.3}$$

Here the subscripts of the matrices indicate the space they act on. For instance, $L_1 := L \otimes \mathbf{1}$. In the case the r-matrix is constant, (2.3) has a solution if and only if the r-matrix satisfies

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0. \tag{2.4}$$

If $r_{12} = -r_{21}$, then (2.4) is called the *classical Yang-Baxter equation.*

Next we consider a classical field theory. The natural question that occurs then is the method of which integrability can be defined in this case. Unfortunately, there does not exist a clear notion of integrability of classical systems of fields. However, since such systems have infinite dimensional phase spaces and the integrability requires infinitely many independent conserved quantities in involution, it is generally accepted that such quantities exist as a naive definition. As in the case of finite systems, the idea of Lax pairs is used to find infinitely many conserved quantities. This time one might think one has to use infinite matrices. However, progress can be made in another way. Rather than using infinite dimensional matrices for Lax pairs, we introduce *spectral parameters*, λ, for Lax pairs. Then (2.1) is modified to

$$\frac{d}{dt}L(\lambda) = [M(\lambda), L(\lambda)]. \tag{2.5}$$

The Lax equation (2.5) gives rise to the *zero curvature condition*:

$$\partial_t V(x,t|\lambda) - \partial_x U(x,t|\lambda) + [U(x,t|\lambda), V(x,t|\lambda)] = 0 \qquad \forall\lambda \tag{2.6}$$

where U and V are matrices called potentials which depend on the spacetime variables and parameters of the theory. Then (2.6) is the compatibility condition, $\Psi_{tx} = \Psi_{xt}$, for the associated linear system:

$$(\partial_x + V)\Psi = 0, \qquad (\partial_t + U)\Psi = 0 \tag{2.7}$$

Thus we can reformulate an original non-linear equation as the compatibility conditions for the associated linear equations. This leads to the *classical inverse scattering method*, which is a powerful tool for solving non-linear partial differential equations. However since this topic is irrelevant to this paper, we do not discuss further. Also (2.6) and (2.7) have geometrical meaning. Further details can be found in [16].

From now on, we impose periodic boundary conditions in the interval $[0, L]$ on fields. Along with the continuous case, one dimensional lattice system with each lattice site labeled by $m \in \{1, \cdots, L\}$ will also be discussed. This will, in the discrete case, impose (2.7) to be modified to

$$\Psi(m+1,t|\lambda) = L(m,t|\lambda)\Psi(m,t|\lambda), \qquad (\partial_t + U(m,t|\lambda))\Psi(m,t|\lambda) = 0. \tag{2.8}$$

where $L(m,t|\lambda)$ is known as the *Lax operator*.

To see how the infintely many conserved quantities arise, we introduce the transtion matrix, monodromy matrix and transfer matrix. The *transition matrix* $T(x,y|\lambda)$ (from y to x) is defined as a special solution of the left equation of (2.7):

$$(\partial_x + V(x|\lambda))T(x,y|\lambda) = 0, \qquad T(y,y|\lambda) = I, \qquad x \geq y \tag{2.9}$$

For lattice systems, the transition matrix $T(n,m|\lambda)$ is defined by

$$T(n,m|\lambda) := L(n|\lambda)\cdots L(m|\lambda), \qquad n \geq m. \tag{2.10}$$

In the above equations, t is fixed, and is omitted. Setting $x = L$, $y = 0$ and $n = L$, $m = 1$ in (2.9) and (2.10) resepectively, we define the *monodromy matrix* $T(\lambda)$, which is the transition matrix for the whole interval or lattice:

$$T(\lambda) := T(L, 0|\lambda); \qquad T(\lambda) := T(L, 1|\lambda). \tag{2.11}$$

Finally we arrive at the *transfer matrix*:

$$\tau(\lambda) := \mathrm{Tr}\, T(\lambda) \tag{2.12}$$

The transfer matrix $\tau(\lambda)$ plays the role of a generating function of conserved charges and the Hamiltonian of a given model is usually expressed in terms of $\tau(\lambda)$. Thus we can think of the information of conserved quantities as being encoded in $\tau(\lambda)$. Similarly to the finite system, for the involution of conserved quantities, we introduce the r-matrix. If the Poisson bracket of $L(\lambda)$ can be expressed in the form

$$\big\{L(k|\lambda) \overset{\otimes}{,} L(l|\mu)\big\} = \delta_{kl}\big[L(k|\lambda) \otimes L(l|\mu), r(\lambda, \mu)\big] \tag{2.13}$$

then the Poisson bracket of $T(\lambda)$ is given by

$$\big\{T(n, m|\lambda) \overset{\otimes}{,} T(n, m|\mu)\big\} = \big[T(n, m|\lambda) \otimes T(n, m|\mu), r(\lambda, \mu)\big] \tag{2.14}$$

It is worth noting here that although we only consider lattice systems, identical results exist for the continuous cases. The $\{A \overset{\otimes}{,} B\}$ in (2.13) and (2.14) represents a $n^2 \times n^2$ matrix whose entries are $\{A_{i_1 i_2}, B_{j_1 j_2}\}$.
Taking the trace of (2.14) over the $n^2 \times n^2$ matrix space, we can obtain the following relation:

$$\{\tau(\lambda), \tau(\mu)\} = 0, \qquad \forall \lambda, \mu. \tag{2.15}$$

Since the Hamilonian and conserved quantities are generated by $\tau(\lambda)$, (2.15) guarantees the integrability of the lattice systems discussed above.

Now let us move on to quantum integrability. Like the classical field theory case, there is no universally accepted definition of quantum integrability. One might think that one can define the quantum analogue of Liouville integrability by promoting functions to operators and replacing Poisson brackets to commutators. However, it turns out this is not useful since this method makes all finite lattice system to be quantum integrable. Thus as in the classical case, we accept the existence of many conserved quantities in involution as a naive definition.

2.1.2 Quantum Inverse Scattering Method

Now our task is to quantize the structure of the inverse scattering method. The first thing to do is to promote $L(m|\lambda)$, $T(\lambda)$ and $\tau(\lambda)$ to operators. From the previous section, we know that the

Hamiltonian, H, can be expressed in terms of $\tau(\lambda)$. Thus $\tau(\lambda)$ ia a quantum operator acting on a physical Hilbert space $\mathcal{H}$. However, the space on which $T(\lambda)$ acts is larger than $\mathcal{H}$ because $\tau(\lambda)$ is obtained by taking the trace of $T(\lambda)$ over the $n \times n$ matrix space. So, for $T(\lambda)$, we need an n-dimensional space V_a, known as the *auxiliary space*, where $n \times n$ matrices act such that $T(\lambda)$ is an operator acting on $\mathcal{H} \otimes V_a$. In addition, since $T(\lambda)$ is defined as a product of $L(m|\lambda)$'s over the whole lattice, $L(m|\lambda)$ acts on $\mathcal{H}_l \otimes V_a$ where $\mathcal{H}_l$ is a local Hibert space on the l-th site. Summarising, we have:

$$L(m|\lambda) : \mathcal{H}_l \otimes V_a \to \mathcal{H}_l \otimes V_a, \qquad T(\lambda) : \mathcal{H} \otimes V_a \to \mathcal{H} \otimes V_a, \qquad \tau(\lambda) : \mathcal{H} \to \mathcal{H}.$$

The next thing in line is the expression of the quantum version of (2.14) in terms of the *quantum R-matrix*. As is usual in quantisation procedures, Poisson brackets are replaced by commutators. Then if the following relations:

$$\text{(i)} \quad [L_{ij}(p|\lambda), L_{kl}(q|\mu)] = 0 \qquad \text{when} \quad p \neq q \tag{2.16}$$

$$\text{(ii)} \quad R(\lambda, \mu)(L(k|\lambda) \otimes L(k|\mu)) = (L(k|\mu) \otimes L(k|\lambda))R(\lambda, \mu) \tag{2.17}$$

are valid, then the commutation relations between matrix elements of the transtion matirix, $T(n, m|\lambda)$, are given by:

$$R(\lambda, \mu)(T(n, m|\lambda) \otimes T(n, m|\mu)) = (T(n, m|\mu) \otimes T(n, m|\lambda))R(\lambda, \mu). \tag{2.18}$$

For $n = L$ and $m = 1$ in the above equation, there exists similar relations for the monodromy matrix which are rewritten as

$$R(\lambda, \mu)(T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda))R(\lambda, \mu). \tag{2.19}$$

This relation is called the *fundamental commutation relation*. The fundamental commutation relation classifies all quantum integrable systems. Thus constructing a representation of (2.19) is equivalent to constructing a quantum integrable model.

Taking the trace on both sides of (2.19), we obtain the quantum analogue of (2.15), which shows the commutativity of the transfer matrices:

$$[\tau(\lambda), \tau(\mu)] = 0, \qquad \forall \lambda, \mu. \tag{2.20}$$

As in the classical case, an important consequence of (2.20) is the existence of many involutive quantum operators.

Notice that not any $n^2 \times n^2$ matrix can serve as an R-matrix for some integrable system. In fact, the fundmental commutation relation (2.19) imposes a condition on the R-matrix. This leads to the *quantum Yang-Baxter equation*. To derive the quantum Yang-Baxter equation, we consider two different ways to relate $T(\lambda) \otimes T(\mu) \otimes T(\nu)$ and $T(\nu) \otimes T(\mu) \otimes T(\lambda)$ by using (2.19):

$$\begin{aligned} T(\lambda) \otimes T(\mu) \otimes T(\nu) =& (R^{-1}(\lambda,\mu) \otimes I)(I \otimes R^{-1}(\lambda,\nu))(R^{-1}(\mu,\nu) \otimes I) \times (T(\nu) \otimes T(\mu) \otimes T(\lambda)) \\ & \times (R(\mu,\nu) \otimes I)(I \otimes R(\lambda,\nu))(R(\lambda,\mu) \otimes I) \qquad \cdots\cdots \quad \text{(a)} \\ =& (I \otimes R^{-1}(\mu,\nu))(R^{-1}(\lambda,\nu) \otimes I)(I \otimes R^{-1}(\lambda,\mu)) \times (T(\nu) \otimes T(\mu) \otimes T(\lambda)) \\ & \times (I \otimes R(\lambda,\mu))(R(\lambda,\nu) \otimes I)(I \otimes R(\mu,\nu)) \qquad \cdots\cdots \quad \text{(b)} \end{aligned} \tag{2.21}$$

The sufficient condition for validity between (a) and (b) is given by the famous quantum Yang-Baxter eqaution:

$$(I \otimes R(\lambda,\mu))(R(\lambda,\nu) \otimes I)(I \otimes R(\mu,\nu)) = (R(\mu,\nu) \otimes I)(I \otimes R(\lambda,\nu))(R(\lambda,\mu) \otimes I) \tag{2.22}$$

The quantum Yang-Baxter equation can be expressed neatly if we let the subsripts denote the vector spaces on which the corresponding operators act nontrivially.

$$R_{23}(\lambda,\mu)R_{12}(\lambda,\nu)R_{23}(\mu,\nu) = R_{12}(\mu,\nu)R_{23}(\lambda,\nu)R_{12}(\lambda,\mu) \tag{2.23}$$

2.1.3 Algebraic Bethe Ansatz

Now we apply the formalism dicussed so far to a specific example, the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ model whose Hamiltonian is given by

$$H = \sum_{l=1}^{L} \vec{S}_l \cdot \vec{S}_{l+1}. \tag{2.24}$$

$\vec{S}_l (= \frac{1}{2}\vec{\sigma}_l)$ is a spin operator acting on the Hilbert space $\mathcal{H}_l (= \mathbb{C}^2)$ at l-th site.

We take the auxiliary space $V_a = \mathbb{C}^2$ and define the Lax operator acting on $\mathcal{H}_l \otimes V_a$ by

$$L_{l,a}(\lambda) := \lambda I_{l,a} + i\vec{S}_l \cdot \vec{\sigma_a} = \begin{pmatrix} \lambda I_l + iS_l^z & iS_l^- \\ iS_l^+ & \lambda I_l - iS_l^z \end{pmatrix} \tag{2.25}$$

(The notation has been slightly changed: $L(l|\lambda) \to L_{l,a}(\lambda)$)

We can have an alternative expression for the Lax operator by using the permutation operator P in $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by $P(v \otimes w) = w \otimes v$. In terms of $P_{l,a}$, $L_{l,a}(\lambda)$ can be rewritten as

$$L_{l,a}(\lambda) = \Big(\lambda - \frac{i}{2}\Big) I_{l,a} + iP_{l,a} \tag{2.26}$$

From the definition of the monodromy matrix, $T_a(\lambda)$,

$$T_a(\lambda) = L_{L,a}(\lambda) \cdots L_{1,a}(\lambda). \tag{2.27}$$

Also the R-matrix for the $XXX_{\frac{1}{2}}$ model is given by:

$$R_{a,b}(\lambda,\mu) = R_{a,b}(\lambda - \mu) := (\lambda - \mu) I_{a,b} + iP_{a,b} \tag{2.28}$$

From these monodromy matrix and R-matrix, we can show that the fundamental commutation relation (2.19):

$$R_{a,b}(\lambda-\mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{a,b}(\lambda-\mu) \tag{2.29}$$

Let us find out how conserved charges arise from the transfer matrix. First note that the Lax operator has the following properties.

$$L_{n,a}(i/2) = iP_{n,a} \qquad \text{and} \qquad \frac{d}{d\lambda}L_{n,a}(\lambda) = I_{n,a} \quad \text{for any } \lambda \tag{2.30}$$

Then the monodromy matrix at $i/2$ is given by

$$T_a(i/2) = i^L P_{N,a} \cdots P_{1,a} \tag{2.31}$$

Using this, define the shift operator U.

$$U := i^{-L}\mathrm{Tr}_a T(i/2) = P_{1,2}P_{2,3}\cdots P_{L-1,L} \tag{2.32}$$

Indeed, we can easily check that $U^{-1}X_lU = X_{l-1}$ for any local operator X_l on $\mathcal{H}_l$. Since momentum operator P is the generator of infinitesimal shift or shift along one site on the lattice, we have the relation $e^{iP} = U$. Thus our first conserved charge is

$$F_0 = \log\tau(\lambda)|_{\lambda=i/2} \propto P \tag{2.33}$$

The next conserved charge is given by

$$F_1 = \frac{d}{d\lambda}\log\tau(\lambda)|_{\lambda=i/2} = \frac{1}{i}\sum_l P_{l,l+1} \tag{2.34}$$

We can write the Hamiltonian (2.24) in terms of F_1.

$$H = -\frac{1}{2}\sum_l^L (P_{l,l+1} - \frac{1}{2}) \tag{2.35}$$

As mentioned before the Hamiltonian is indeed generated by the transfer matrix. Another consequence of (2.35) is that the Hamitonian and transfer matrix commute.

$$[H(\lambda), \tau(\lambda)] = 0 \tag{2.36}$$

which implies that $H(\lambda)$ and $\tau(\lambda)$ can be simultaneously diagonalised.

Now our next aim is to diagonalise the transfer matrix $\tau(\lambda)$. Frist rewrite the $T_a(\lambda)$ as the matrix form:

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.37}$$

Here $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ are operators on the Hilbert space $\mathcal{H}$. Then the transfer matrix $\tau(\lambda)$ is given by:

$$\tau(\lambda) = \mathrm{Tr}_a T_a(\lambda) = A(\lambda) + C(\lambda) \tag{2.38}$$

Also choosing a basis for $V_a \otimes V_b$, the R-matirx can be writen as:

$$R_{a,b}(\lambda-\mu) = \begin{pmatrix} \lambda-\mu+i & 0 & 0 & 0 \\ 0 & \lambda-\mu & i & 0 \\ 0 & i & \lambda-\mu & 0 \\ 0 & 0 & 0 & \lambda-\mu+i \end{pmatrix} \tag{2.39}$$

From the fundmental commutation relation (2.29), the matrix elements of $T_a(\lambda)$ satisfy:

$$\begin{aligned} [A(\lambda), B(\mu)] &= 0 \\ A(\lambda)B(\mu) &= \frac{\lambda-\mu-i}{\lambda-\mu}B(\mu)A(\lambda) + \frac{i}{\lambda-\mu}B(\lambda)A(\mu) \\ D(\lambda)B(\mu) &= \frac{\lambda-\mu+i}{\lambda-\mu}B(\mu)D(\lambda) - \frac{i}{\lambda-\mu}B(\lambda)D(\mu) \end{aligned} \tag{2.40}$$

Our strategy is to use $B(\lambda)$ and $C(\lambda)$ as creation operator and annihilation operator respectively. For this we need a reference state Ω in $\mathcal{H}$ so that $C(\lambda)\Omega = 0$. Ω can constructed from the spin-up state at l-th site, $|\uparrow\rangle_l$. Then from (2.25), we can see that Lax operator acts on $|\uparrow\rangle_l$ by an upper triangular matrix.

$$L_{l,a}(\lambda)|\uparrow\rangle_l = \begin{pmatrix} \lambda+\frac{1}{2}i & * \\ 0 & \lambda-\frac{1}{2}i \end{pmatrix} |\uparrow\rangle_l \tag{2.41}$$

Here $*$ denotes the irrelevant term.

Let $\Omega \in \mathcal{H}$ be the state with spin-up at all site.

$$\Omega := \bigotimes_{l=1}^{L} |\uparrow\rangle_l \tag{2.42}$$

This is called the *pseudovacuum.*

Then the monodromy matrix $T_a(\lambda)$ acts on Ω by upper triangular matrix.

$$T_a(\lambda)\Omega = \begin{pmatrix} (\lambda+\frac{1}{2}i)^L & * \\ 0 & (\lambda-\frac{1}{2}i)^L \end{pmatrix} \Omega \tag{2.43}$$

Thus we have the state Ω annihilated by $C(\lambda)$.

$$C(\lambda)\Omega = 0 \tag{2.44}$$

Since $B(\lambda)$ plays the role of a creation operator, we assume that other eigenvectors of τ have the form:

$$\Phi(\{\lambda\}) = B(\lambda_1)\cdots B(\lambda_N)\Omega \tag{2.45}$$

$\Phi(\{\lambda\})$ is called the *Bethe vector.* Using the relations (2.40), we arrive at the result of the *algebraic Bethe ansatz* which states that the Bethe vector $\Phi(\{\lambda\})$ is an eigenstate of $\tau(\lambda)$ if the *Bethe ansatz equations* are satisfied (See [10] for the detailed calculation):

$$\left(\frac{\lambda_n+\frac{1}{2}i}{\lambda_n-\frac{1}{2}i}\right)^L = \prod_{m\neq n}^{N} \frac{\lambda_n-\lambda_m+i}{\lambda_n-\lambda_m-i}, \qquad 1 \leq n \leq N \tag{2.46}$$

2.2 Some More Generalisations in Spin Chain

2.2.1 Twisted Boundary Condition

Until now we impose the periodic boundary condition, $\vec{S}_{L+l} = \vec{S}_l$. However we can generalise a spin chain by imposing a different boundary condition by following the method explained in [9]. Let $K \in SU(2)$ be an operator acting on the auxiliary space V_a. In [9], it is shown that if $[K_a \otimes K_b, R_{a,b}(\lambda)] = 0$, we can change the boundary conditions on a spin chain without spoiling its integrability. Let us use this statement and change our boundary condtion. First take our K as

$$K = \begin{pmatrix} e^{\frac{i}{2}\vartheta} & 0 \\ 0 & e^{-\frac{i}{2}\vartheta} \end{pmatrix} \tag{2.47}$$

It is easy to check that our K satisfies $[K_a \otimes K_b, R_{a,b}(\lambda)] = 0$. Then our new monodromy matrix is given by

$$T_a(\lambda) = K L_{L,a}(\lambda) \cdots L_{1,a}(\lambda) = \begin{pmatrix} e^{\frac{i}{2}\vartheta} A(\lambda) & e^{\frac{i}{2}\vartheta} B(\lambda) \\ e^{-\frac{i}{2}\vartheta} C(\lambda) & e^{-\frac{i}{2}\vartheta} D(\lambda) \end{pmatrix} \tag{2.48}$$

Taking the trace, we obtain the transfer matrix:

$$\tau(\lambda) = e^{\frac{i}{2}\vartheta} A(\lambda) + e^{-\frac{i}{2}\vartheta} D(\lambda) \tag{2.49}$$

It results in the change of the eigenvalue compared to (2.43)

$$\left(\lambda + \frac{i}{2}\right)^L + \left(\lambda - \frac{i}{2}\right)^L \quad \longmapsto \quad e^{\frac{i}{2}\vartheta}\left(\lambda + \frac{i}{2}\right)^L + e^{-\frac{i}{2}\vartheta}\left(\lambda - \frac{i}{2}\right)^L \tag{2.50}$$

This leads to a modification of the Bethe ansatz equations.

$$\left(\frac{\lambda_n + \frac{1}{2}i}{\lambda_n - \frac{1}{2}i}\right)^L = e^{i\vartheta} \prod_{m \neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i}, \qquad 1 \leq n \leq N \tag{2.51}$$

Since it is equivalent to impose the following boundary condition:

$$\vec{S}_{L+1} = e^{\frac{i}{2}\vartheta\sigma^z} \vec{S}_1 e^{-\frac{i}{2}\vartheta\sigma^z}\ , \quad \vartheta \in S^1 \tag{2.52}$$

we call this the *twisted boundary condition.*

2.2.2 Local Spin and Inhomogeneity

In the previous sections, we consider the Heisenberg $XXX_{\frac{1}{2}}$ model. This is the simplest model. However, we need more generalisations for the Bethe/gauge correspondence. We can include local spin s_l and inhomogeneities ν_l at each site. For these we need to understand the *Yang-Baxter algebra*, the underlying algebraic stuructre of the spin chain. The Yang-Baxter algebra is a bialgebra whose generators are entries of a monodromy matrix $t_{ij}(\lambda)$ satisfying the fundamental commutation relation (2.19). In this framework the R-matrix plays a role of structure constant.

Thus constructing a representation of the Yang-Baxter algebra is equivalent to constructing a quantum integrable model. For $\mathfrak{sl}_n$ spin chain, the corresponding Yang-Baxter algebra is $\mathcal{Y}(\mathfrak{sl}_n)[[\lambda]]$ and $t_{ij}(\lambda)$ are given by

$$t_{ij}(\lambda) = \sum_{m=0}^{\infty} t_{ij}^{(m)} \lambda^{-m} \tag{2.53}$$

where $\mathcal{Y}(\mathfrak{sl}_n)$ is Yangian of $\mathfrak{sl}_n$ and $t_{ij}^{(m)}$ are generators of $\mathcal{Y}(\mathfrak{sl}_n)$.

Recall that the crucial point of the algebraic Bethe ansatz is to know the pseudovacuum Ω. Mathematically it corresponds to a highest weight vector of representation of $\mathcal{Y}(\mathfrak{sl}_n)[[\lambda]]$ defined by

$$t_{ii}(\lambda)\Omega = M_i(\lambda)\Omega \quad \text{and} \quad t_{ij}(\lambda)\Omega = 0 \ \text{ if } i > j \tag{2.54}$$

for some scalars $M_i(\lambda) \in \mathbb{C}[[\lambda^{-1}]]$. Here $M(\lambda) = (M_1(\lambda), \cdots, M_n(\lambda))$ is called the highest weight. We can construct a highest weight representation by noticing that there is the evaluation morphism $ev_\nu : \mathcal{Y}(sl_n) \to \mathcal{U}(sl_n)$ $\forall \nu \in \mathbb{C}$ [11, 12]. The parameter, ν, is called *inhomogeneity*. Thus using the $\mathfrak{sl}_n$ representation theory, we have the *evaluation representation*:

$$\rho_\nu^m = \pi_m \circ ev_\nu : \mathcal{Y}(\mathfrak{sl}_n) \xrightarrow{ev_\nu} \mathfrak{sl}_n \xrightarrow{\pi_m} V_m \tag{2.55}$$

where (π_m, V_m) is a representation of $\mathfrak{sl}_n$ with dominant weight $m = (m_1, \cdots, m_n) \in \mathbb{N}^n$. The weight of this evaluation representation is given by $M(\lambda) = (M_1(\lambda), \cdots, M_n(\lambda))$ with $M_j(\lambda) = \lambda - \nu - m_j$. Then, with the evaluation representation, we can give the local spin at each site by using coproduct Δ:

$$\left(\bigotimes_{l=1}^{L} \rho_{\nu_l}^{m^{\langle l \rangle}} \right) \circ \Delta^{(L)} \left(T(\lambda) \right) = \rho_{\nu_1}^{m^{\langle 1 \rangle}} \left(T(\lambda) \right) \otimes \cdots \otimes \rho_{\nu_L}^{m^{\langle L \rangle}} \left(T(\lambda) \right) \tag{2.56}$$

where $m^{\langle l \rangle} = (m_1^{\langle l \rangle}, \cdots, m_n^{\langle l \rangle})$, $l = 1, \cdots, L$, is the dominant weight of the representation of $\mathfrak{sl}_n$ at l-th site. This provides a representation of $\mathcal{Y}(\mathfrak{sl}_n)[[\lambda]]$ with the weight:

$$M_j(\lambda) = \prod_{l=1}^{L} M_j^{\langle l \rangle}(\lambda) = \prod_{l=1}^{L} \left(\lambda^{\langle l \rangle} - \nu^{\langle l \rangle} - m_j^{\langle l \rangle} \right) \qquad j = 1, \cdots, n \tag{2.57}$$

Now apply the above prodecure to $\mathfrak{sl}_2$ spin chain. In this case the monodromy matrix have the following form:

$$T(\lambda) = \begin{pmatrix} t_{11}(\lambda) & t_{12}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) \end{pmatrix} \tag{2.58}$$

Then pseudovacuum Ω satisfies

$$t_{11}(\lambda)\Omega = M_1(\lambda)\Omega\,, \qquad t_{22}(\lambda)\Omega = M_2(\lambda)\Omega\,, \qquad t_{21}(\lambda)\Omega = 0. \tag{2.59}$$

This looks very similar to (2.37), but this time the representation information is encoded in M_1 and M_2. Here $t_{12}(\lambda)$ is a creation operator and by applying the Bethe ansatz we obtain the Bethe equations.

$$\frac{M_1(\lambda_n)}{M_2(\lambda_n)} = \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i}\,, \qquad 1 \leq n \leq N \tag{2.60}$$

Plugging (2.57) into the above equation, we obtain the Bethe equations including the local spins:

$$\prod_{l=1}^{L} \frac{\lambda_n - \nu_l + is_l}{\lambda_n - \nu_l - is_l} = e^{i\vartheta} \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i}\,, \qquad 1 \leq n \leq N \tag{2.61}$$

If we impose the twisted boundary condition, then the most general Bethe ansatz equations is given by

$$\boxed{\prod_{l=1}^{L} \frac{\lambda_n - \nu_l + is_l}{\lambda_n - \nu_l - is_l} = e^{i\vartheta} \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i}\,, \qquad 1 \leq n \leq N.} \tag{2.62}$$

2.3 Yang-Yang Function

For the Bethe ansatz to be valid, the Bethe ansatz equations must have solutions. To prove the existence of solutions, consider the logarithm form of (2.46):

$$L\log\left(\frac{\lambda_n + \frac{1}{2}i}{\lambda_n - \frac{1}{2}i}\right) - \sum_{m\neq n}^{N} \log\left(\frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m + i}\right) = 2\pi i m_n\ , \quad 1 \leq n \leq N \tag{2.63}$$

$m_n \in \mathbb{Z}$ denotes the different choices of branch cuts for the complex logarithms. These are the equivalent equations to the original Bethe ansatz equations (2.46) and thus we have to prove that the logrithmic Bethe ansatz equations (2.63) have solutions. The proof of the existence of solutions of (2.63) is based on a vaiational principle with the action given by

$$\begin{aligned} Y_{\vec{m}}(\lambda) = {} & \frac{L}{2\pi} \sum_{n=1}^{N} \Big[(\lambda_n + \frac{1}{2}i)\log(\lambda_n + \frac{1}{2}i) - (\lambda_n - \frac{1}{2}i)\log(\lambda_n - \frac{1}{2}i) \Big] \\ & - \frac{1}{4\pi} \sum_{n,m=1}^{N} \Big[(\lambda_n - \lambda_m + i)\log(\lambda_n - \lambda_m + i) - (\lambda_n - \lambda_m - i)\log(\lambda_n - \lambda_m - i) \Big] \\ & - i \sum_{n=1}^{N} \lambda_n m_n \end{aligned} \tag{2.64}$$

(We do not discuss the proof for the existence of solutions further. For the complete proof see [13].) This is called the *Yang-Yang function*, first introduced by Yang and Yang in [14]. The logarithmic Bethe ansatz equations (2.63) arise as the extremum conditions for the Yang-Yang function (2.64):

$$\frac{\partial Y_{\vec{m}}(\lambda)}{\partial \lambda_n} = 0 \tag{2.65}$$

If we shift $Y_{\vec{m}}(\lambda)$ by $i\sum \lambda_n m_n$ and define $Y(\lambda) := Y_{\vec{m}}(\lambda) + i\sum \lambda_n m_n$, then the above equation read off as:

$$\frac{\partial Y(\lambda)}{\partial \lambda_n} = i m_n \qquad \text{or} \qquad \exp\left(2\pi \frac{\partial Y(\lambda)}{\partial \lambda_n}\right) = 1 \tag{2.66}$$

Thus we see that the Yang-Yang function serves as a potential of the Bethe ansatz equation.

For the most general Bethe ansatz equations (2.62), the corresponding Yang-Yang function is given by

$$\boxed{\begin{aligned} Y(\lambda) = & \frac{1}{2\pi}\sum_{l=1}^{L}\sum_{n=1}^{N}\Big[(\lambda_n-\nu_l+is_l)\log(\lambda_n-\nu_l+is_l)-(\lambda_n-\nu_l-is_l)\log(\lambda_n-\nu_l-is_l)\Big] \\ & -\frac{1}{2\pi}\sum_{n<m}^{N}\Big[(\lambda_n-\lambda_m+i)\log(\lambda_n-\lambda_m+i)-(\lambda_n-\lambda_m-i)\log(\lambda_n-\lambda_m-i)\Big] \\ & -\frac{i\vartheta}{2\pi}\sum_{n=1}^{N}\lambda_n \end{aligned}} \tag{2.67}$$

Indeed, by plugging (2.67) into (2.66), we can obtain the logarithm of the Bethe ansatz equation (2.62).

3 $\mathcal{N} = (2,2)$ SUSY Gauge Theory in 2D

3.1 Superspace and Superfields

3.1.1 Superspace

Let us consider a field theory in flat Minkowski space $\mathbb{R}^{1,1}$ with the metric $\eta_{\mu\nu} = \text{diag}(-1,1)$ and coordinates (x^0, x^1). Then the associated Poincaré algebra is $\mathfrak{iso}(1,1)$, and its only non-trivial commutation relations are:

$$[M, H] = H, \qquad [M, P] = P \tag{3.1}$$

where $M = iM_{01}$, $H = P^0$, and $P = P^1$.

For $\mathcal{N} = 2$ supersymmetry, we consider a superspace $\mathbb{R}^{1,1|4}$ with four fermionic coordinates (θ^+, θ^-, $\bar{\theta}^+$, $\bar{\theta}^-$) which are related by Hermitian conjuation $(\theta^{\pm})^\dagger = (\bar{\theta}^{\pm})$. Let $Q^{\pm}$, $\bar{Q}^{\pm}$ denote the corresponding supercharges such that they satisfy the supersymmetry algebra:

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = 2(H \mp P), \qquad \{Q_-, \bar{Q}_+\} = 0, \qquad \{Q_+, \bar{Q}_-\} = 0 \tag{3.2}$$

$$[M, Q_{\pm}] = \mp Q_{\pm}, \qquad [M, \bar{Q}_{\pm}] = \mp \bar{Q}_{\pm} \tag{3.3}$$

We can also view the above result as a result of dimensional reduction from the $\mathcal{N} = 1$ supersymmetric theory in four dimensions. From a four-dimensional theory with coordinates $\{X^0, X^1, X^2, X^3\}$, we obatain a two-dimensional theory by considering fields independent of X^1 and X^2. Thus the two-dimensional coordinates are given by $(x^0, x^1) = (X^0, X^3)$. For the fermionic coordinates, we rearrange a Majorana spinor, θ^a, in four dimensions into $(\theta^-, \theta^+) = (\theta^1, \theta^2)$ and $(\bar{\theta}^-, \bar{\theta}^+) = (\theta^3, \theta^4)$. Then the supersymmetry algebra in four dimensions reduces to (3.2) and (3.2).

Defining derivatives $\partial_{\pm} = \partial_0 \pm \partial_1$ whose corresponding coordinates are defined as $x^{\pm} := \frac{1}{2}(t \pm x)$, the supercharges are realised as differential operators acting on superfields:

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm}, \qquad \bar{Q}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm}. \tag{3.4}$$

We also define another set of differential operators:

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm}, \qquad \bar{D}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm} \tag{3.5}$$

It is easy to check that their only non-trivial anti-commutation relations are:

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = -2i\partial_{\pm}, \qquad \{D_{\pm}, \bar{D}_{\pm}\} = 2i\partial_{\pm} \tag{3.6}$$

3.1.2 Chiral and Twisted Chiral Superfields

A *chiral superfield*, Φ, and *antichiral superfield*, $\bar{\Phi}$, are defined by $\bar{D}_{\pm}\Phi = 0$ and $D_{\pm}\bar{\Phi} = 0$, respectively. The most general form of a chiral superfield Φ can be obtained by introducing $y^{\pm} := x^{\pm} - i\theta^{\pm}\bar{\theta}^{\pm}$. In these coordinates, the covariant derivatives read

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - 2i\bar{\theta}^{\pm}\frac{\partial}{\partial y^{\pm}}, \qquad \bar{D}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}}.$$

Thus the most general Φ is independent of $\bar{\theta}^{\pm}$ and can be written as

$$\Phi(y^{\pm},\theta^{\pm}) = \phi(y^{\pm}) + \sqrt{2}\theta^{\alpha}\psi_{\alpha}(y^{\pm}) + 2\theta^{-}\theta^{+}F(y^{\pm}) \tag{3.7}$$

in which ϕ and F depict complex scalar fields, and ψ_α shows a Dirac spinor.

To express supersymmetric Largragians, we introduce fermionic volume elements given by

$$d^2\theta := \frac{1}{2}d\theta^- d\theta^+ \qquad d^2\bar{\theta} := (d^2\theta)^\dagger = -\frac{1}{2}d\bar{\theta}^- d\bar{\theta}^+ \qquad d^4\theta := d^2\theta d^2\bar{\theta}. \tag{3.8}$$

Moreover, we include L flavours of matter fields through Φ^l. Then the kinetic term is given by

$$\mathcal{L}_{\text{kin}} = \int d^4\theta \Phi_l^\dagger \Phi^l \tag{3.9}$$

with a superpotential of the form:

$$\mathcal{L}_W = \int d\theta^2 W(\Phi^1,\cdots,\Phi^L)\Big|_{\bar{\theta}^{\pm}=0} + \text{h.c.} \tag{3.10}$$

where W is a holomorphic function.

The kinetic Lagrangian is invariant under the global flavour symmetry group $U(L)$. However, unlike the kinetic term, the superpotential term usually breaks the flavour symmetry. For instance, consider the complex mass term:

$$\mathcal{L}_m = \int d^2\theta m_l^k \Phi_k^\dagger \Phi^l + \text{h.c.} \tag{3.11}$$

If m_l^k are all different, the complex mass term breaks the flavour group, $U(L)$, down to its maximal torus, $U(1)^L$.

In addition to chiral superfields, there is a special kind of superfields in two dimensions. It is the *twisted (anti) chiral superfields* Σ ($\bar{\Sigma}$) defined by:

$$D_-\Sigma = \bar{D}_+\Sigma = 0, \qquad D_+\bar{\Sigma} = \bar{D}_-\bar{\Sigma} = 0. \tag{3.12}$$

They can also be understood by dimensional reduction. Recall that in four dimensions the super-covariant derivatives' anticommutators are given by $\{D_a, D_b\} = -2i(\gamma^\mu)_{ab}\partial_\mu$. After dimensional reduction, we only consider σ^0 and σ^3, and thus obtain $\{D_\pm, \bar{D}_\mp\} = 0$ whose counterpart, in four dimensions, is not trivial. The triviality in two dimensions leads us to define the twisted chiral superfields (3.12).

For the general form of twisted (anti)chiral superfields, we employ a neat trick which limits us to carry out the same procedure as in chiral superfields to progress. Let us begin by introducing 'twisted' fermionic coordinates [17]: $(\vartheta^-,\vartheta^+) := (\theta^-,\bar{\theta}^+)$ and $(\bar{\vartheta}^-,\bar{\vartheta}^+) := (\bar{\theta}^-,\theta^+)$, and defining the 'twisted' version of covariant derivatives: $\tilde{D}_\pm = \frac{\partial}{\partial\vartheta^\pm} - i\bar{\vartheta}^\pm\partial_\pm$ and $\bar{\tilde{D}}_\pm = -\frac{\partial}{\partial\bar{\vartheta}^\pm} + i\vartheta^\pm\partial_\pm$. Then the twisted chiral superfield condition (3.12) reads $\tilde{D}_\pm\Sigma = 0$, which is similiar to the antichiral superfield condition. Thus repeating the procedure with $\tilde{y}^\pm := x^\pm + i\vartheta^\pm\bar{\vartheta}^\pm$, one obtains

$$\Sigma(\tilde{y}^{\pm},\bar{\vartheta}^{\pm}) = \sigma(\tilde{y}^{\pm}) + \sqrt{2}\bar{\vartheta}^{\alpha}\bar{\chi}_{\alpha}(\tilde{y}^{\pm}) + 2\bar{\vartheta}^{-}\bar{\vartheta}^{+}E(\tilde{y}^{\pm}) \tag{3.13}$$

where σ is a complex scalar, $(\tilde{\chi}_-, \tilde{\chi}_+) = (\chi_-, \bar{\chi}_+)$ is a Dirac spinor, and E is a complex auxiliary field.

3.1.3 Vector Superfields

Consider the U(1) gauge transformation of a chiral superfield, Φ, given by $\Phi \mapsto e^{iq\Lambda}\Phi$ in which Λ is also a chiral superfield. Then the kinetic term (3.9) is not invariant under this transformation. To compensate for this, we introduce a *vector superfield*, V, defined by the reality condition $V = V^\dagger$. If we impose the supersymmetric gauge transformation, $V \mapsto V - (\Lambda + \Lambda^\dagger)$, on V, then the modified Lagrangian,

$$\mathcal{L}_{\text{kin}} = \int d^4\theta \Phi^\dagger e^{2qV}\Phi, \tag{3.14}$$

becomes invariant under the U(1) gauge transformation. In the Wess-Zumino gauge, the component expansion of the vector superfield is given by:

$$\begin{aligned} V = \theta^-\bar{\theta}^-(A_0 - A_1) + \theta^+\bar{\theta}^+(A_0 + A_1) - \sqrt{2}\theta^-\bar{\theta}^+\sigma - \sqrt{2}\theta^+\bar{\theta}^-\sigma^\dagger \\ + 2i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) - 2i\bar{\theta}^-\bar{\theta}^+(\theta^-\lambda_- + \theta^+\lambda_+) - 2\theta^-\theta^+\bar{\theta}^-\bar{\theta}^+ D. \end{aligned} \tag{3.15}$$

Compared to the four-dimensionsional case, V has an additional complex scalar field σ. This is another consequence of dimensional reduction due to the guage field $A_\mu^{(4)}$ yielding a two-dimensional field $(A_0, A_1) = (A_0^{(4)}, A_3^{(4)})$ and a complex scalar field $\sigma = \frac{1}{\sqrt{2}}(A_1^{(4)} - iA_2^{(4)})$.

Note that we bring over the same symbol, σ, from the case of twisted chiral superfields (3.13) for the lowest component. The reason for this becomes apparent when we consider the *superfield strength* defined by

$$\Sigma = \frac{1}{\sqrt{2}}\bar{D}_+ D_- V, \tag{3.16}$$

which turns out also be a twisted chiral superfield as it satisfies (3.12). The same argument is valid for the replicated use of Σ. In terms of $\tilde{y}^\pm$, Σ can be expanded as

$$\Sigma = \sigma(\tilde{y}^\pm) - \sqrt{2}i\bar{\theta}^-\lambda_-(\tilde{y}^\pm) + \sqrt{2}i\theta^+\bar{\lambda}_+(\tilde{y}^\pm) + \sqrt{2}\theta^+\bar{\theta}^-\left(D(\tilde{y}^\pm) - iF_{01}(\tilde{y}^\pm)\right) \tag{3.17}$$

Here $F_{01} = \partial_0 A_1 - \partial_1 A_0 = *F$ is the electric field and completely determines F in two dimensions. We also have the kinetic term of the vector superfield, V, which is written in terms of the superfield strength as

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4e^2}\int d^4\theta \Sigma^\dagger \Sigma \tag{3.18}$$

Now consider the nonabelian gauge theory with $G = U(N)$. In this case, V transforms as $e^V \mapsto e^{-\Lambda^\dagger} e^V e^{-\Lambda}$ where the chiral superfield Λ is $\mathfrak{g}$-valued. We can emulate the same procedure as in the abelian case if we introduce the gauge covariant derivatives:

$$\mathcal{D}_\pm := e^{-V} D_\pm e^V \qquad \bar{\mathcal{D}}_\pm := e^V \bar{D}_\pm e^{-V}. \tag{3.19}$$

Then the super field strength Σ is defined by:

$$\Sigma := \frac{1}{2}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\} \tag{3.20}$$

with the twisted chiral condition $\bar{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$ also satisfied by the Bianchi identity as in the abelian case. To obtain the gauge kinetic term for the non-abelian gauge we take a trace such that

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4e^2}\int d^4\theta \mathrm{Tr}(\Sigma^\dagger \Sigma). \tag{3.21}$$

3.2 Some More Special Features in 2D

3.2.1 Twisted Masses

Similarly to four dimensions, we can give masses to chiral superfields in two dimensions as well by introducing a quadratic term in the superpotential (3.11). In two dimensions, as pointed out by Hanany and Hori [18], there is another kind of mass called *twisted mass*, $\tilde{m}$, whose analogue in four dimensions is non-existent.

We introduce the twisted mass term in a menifestly supersymmetric way as in [17]. Consider an abelian gauge theory with L chiral matters Φ^l. In the absence of the superpotential, this theory possesses a global flavour symmetry group $H^{\max} = U(L)/U(1)$. The idea is to weakly gauge $H^{\max}$, or to introduce $\mathfrak{h}^{\max}$-valued vector superfield, $\hat{V}$, and freeze it to its vev. Then every derivative of the component fields is zero, and $\hat{A}_\mu$ and fermions are zero by Lorentz invariance. To preserve supersymmetry, the supersymmetry transformation of $\hat{V}$ requires $[\hat{\sigma}, \hat{\sigma}^\dagger] = 0$. (See [20] for the supersymmetry transformation of non-abelian vector superfield.) This implies $\hat{\sigma}$ is diagonalisable. Thus only $U(1)^L/U(1)$ instead of $H^{\max}$ can be gauged. We introduce the corresponding abelian vector superfields $\tilde{V}^l$ whose only non-trivial components are $\tilde{\sigma}^l =: \tilde{m}^l$. Then the kinetic terms of matter fields are modified to

$$\begin{aligned} &\mathcal{L}_{\text{kin},\tilde{m}} = \int d^4\theta \Phi_k^\dagger (e^{2\tilde{V}})^k_l \Phi^l \\ &\text{where} \quad \tilde{V} = -\tilde{m}\theta^-\bar{\theta}^+ - \tilde{m}^\dagger\theta^+\bar{\theta}^-\,, \qquad \tilde{m} = \begin{pmatrix} \tilde{m}^1 & & 0 \\ & \ddots & \\ 0 & & \tilde{m}^L \end{pmatrix} \end{aligned} \tag{3.22}$$

through which we observe how the chiral superfields acquire the masses.

Notice that the twisted masses also break the global symmetry. This implies that the superpotential and the twisted mass are only compatible with special choices of W and $\tilde{m}$. We will see how this works when we dicuss the Bethe/gauge correspondence. Also note that the above procedure shows why there are no twisted masses in four dimensions. In four dimensions, vector superfields have no complex scalar field components, σ. Thus if we apply the above procedure to the flavour vector superfield $\hat{V}$ in four dimensions, then it should vanish.

3.2.2 Twisted Superpotential

By inspection of the supersymmetry transformation of the superfield strength (3.17), we can construct two supersymmetric and gauge-invariant couplings in terms of D and F_{01}. The first is known as the *Fayet-Iliopoulos term*:

$$\mathcal{L}_r = -rD \tag{3.23}$$

while the other is known as the *ϑ-term*:

$$\mathcal{L}_\vartheta = \frac{\vartheta}{2\pi} F_{01} \tag{3.24}$$

We can express the above two terms in a combined form due to Witten [21]. First note that

$$\int d^2\vartheta \Sigma\Big|_{\vartheta^\pm=0} = \frac{1}{\sqrt{2}}(D - iF_{01}), \qquad \int d^2\bar{\vartheta}\bar{\Sigma}\Big|_{\bar{\vartheta}^\pm=0} = \frac{1}{\sqrt{2}}(D + iF_{01}) \tag{3.25}$$

(Here $d^2\vartheta := -\frac{1}{2}d\theta^+ d\bar{\theta}^-$ and $d^2\bar{\vartheta} := (d^2\vartheta)^\dagger = -\frac{1}{2}d\theta^- d\bar{\theta}^+$)
From these equations, we can write

$$\mathcal{L}_{r,\vartheta} = -rD + \frac{\vartheta}{2\pi}F_{01} = \frac{i\tau}{\sqrt{2}}\int d^2\vartheta\Sigma\Big|_{\vartheta^\pm=0} - \frac{i\bar{\tau}}{\sqrt{2}}\int d^2\bar{\vartheta}\Sigma^\dagger\Big|_{\bar{\vartheta}^\pm=0} \tag{3.26}$$

where $\tau := ir + \frac{\vartheta}{2\pi}$.
This can be viewed as a special case of the *twisted superpotental*, $\tilde{W}(\Sigma)$, which is a holomorphic function of Σ. Its Lagrangian, expressed through

$$\mathcal{L}_{\tilde{W}} = \int d^2\vartheta \tilde{W}(\Sigma)\Big|_{\vartheta^\pm=0} + \int d^2\bar{\vartheta}\bar{\tilde{W}}(\Sigma^\dagger)\Big|_{\bar{\vartheta}^\pm=0} = \frac{1}{\sqrt{2}}\frac{\partial\tilde{W}}{\partial\sigma}(D - iF_{01}) + \frac{\partial^2\tilde{W}}{\partial\sigma^2}\lambda_-\bar{\lambda}_+ + \text{h.c.}, \tag{3.27}$$

gives a clear picture that once reduced becomes (3.26) when $\tilde{W}(\Sigma) = \frac{i\tau}{\sqrt{2}}\Sigma$.
From now on we recale superfield strength $\Sigma \to \sqrt{2}\Sigma$ (while leaving V unchanged) to make the linear twisted superpotential have the form $\tilde{W}(\Sigma) = i\tau\Sigma$. Then the factor of the gauge kinetic terms (3.18) (3.21) changes to $\frac{1}{2e^2}$.

The ϑ-term has a special physical meaning due to Coleman [22]. In four dimensional spacetime it is a well-known fact that the constant background electric field cannot exist in a large region of space. If such a situation was possible, an electron-positron pair would be produced and attracted to the opposite boundaries hence screening the electric field. This process would continue until the energy of the field becomes totally converted to pair production. Therefore the field would disappear. This phenomenon is the famous *Schwinger pair production.* However, this effect changes in two dimensional spacetime. In two dimensions, Maxwell's equations imply that the electric field is constant throughout space. If there is a point charge of charge e at x_0, $d*F = e\delta(x - x_0)$ tells us that the electric field jumps by e when crossing the point charge. Thus when an electron-positron pair is produced in a background electric field, E, space is divided into three different region shown by Figure 1.

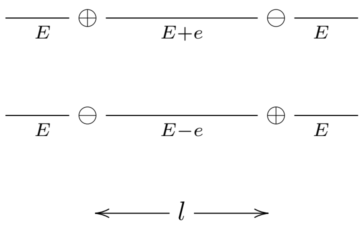

Figure 1: Two oposite charges in a background electric field E

The energy difference between these configurations and the vacuum is given by:

$$\Delta\mathcal{E} = \frac{1}{2}\int dx[(F_{01})^2 - E^2] = \frac{l}{2}[(E \pm e)^2 - E^2] \tag{3.28}$$

For the pair prodution, $\Delta\mathcal{E} > 0$ is required. This implies that it is energetically unfavourable to produce a pair in the vacuum if $|E| \leq \frac{e}{2}$. It is only when $|E| > \frac{e}{2}$, will pairs be produced until $|E| \leq \frac{e}{2}$. This means that the shift of E by ne ($n \in \mathbb{Z}$) describes the same physics. In terms of $\vartheta = \frac{2\pi E}{e}$ introduced by Coleman, the physics of the system is the same with a period of 2π. ϑ is also called the vacuum angle since it labels vacua of the massive Schwinger model [23]. The Lagrangian for F_{01} is given by

$$\Big(\mathcal{L}_{gauge} + \mathcal{L}_{\vartheta}\Big)\Big|_{F_{01}} = \frac{1}{2e^2}F_{01}^2 + \frac{\vartheta}{2\pi}F_{01}. \tag{3.29}$$

Then we see that the ϑ-term is a source term of the background electric field, $\frac{\vartheta}{2\pi}$. This constant electric field contributes to the vacuum energy. The energy density stored in this field is $\frac{e^2}{2}(\frac{\vartheta}{2\pi})^2$, but pair production confines the value of ϑ by $|\vartheta| \leq \pi$. Thus the vacuum energy density contributed by the ϑ-term is

$$\min_{n\in\mathbb{Z}}\left\{\frac{e^2}{2}\Big(\frac{\vartheta}{2\pi} - n\Big)^2\right\} \tag{3.30}$$

This, as we will see later, affects the vacuum equation by causing a shift in the twisted superpotential.

In $U(N)$ gauge theroy, the FI-term and ϑ-term change to:

$$\mathcal{L}_r = -r\mathrm{Tr}D \qquad \mathcal{L}_{\vartheta} = \frac{\vartheta}{2\pi}\mathrm{Tr}F_{01} \tag{3.31}$$

Again we can combine and express the FI-term and ϑ-term in a twisted superpotential.

$$\mathcal{L}_{\tilde{W}} = i\tau\int d^2\vartheta\mathrm{Tr}\Sigma\Big|_{\vartheta^{\pm}=0} + \text{h.c.} \tag{3.32}$$

Even in the non-abelian case, ϑ is still periodic. Since the first Chern class, $c_1 = \frac{1}{2\pi}\mathrm{Tr}\int F$, should be an integer (with appropriate boundary condions), $e^{i\int\mathcal{L}_\theta d^2x}$ is invariant under $\vartheta \mapsto \vartheta + 2\pi$, which implies the physics is periodic with respect to ϑ.

3.3 Vacuum Structure

3.3.1 Classical Analysis

Consider a $U(1)$ gauge theory with L_{f} fundamental matters Q^l and $L_{\bar{\mathrm{f}}}$ anti-fundamental matters $\bar{Q}^k$. We turn off the superpotential, so we can give masses to the chiral fields via general twisted masses $\tilde{m}_{\mathrm{f}}^l$ and $\tilde{m}_{\bar{\mathrm{f}}}^k$. Also we include the FI term and ϑ-term via the linear twisted potential, $\tilde{W}(\Sigma) = i\tau\Sigma$. Then the Lagrangian of the theory is

$$\mathcal{L} = \mathcal{L}_{kin,\tilde{m},\mathrm{f}} + \mathcal{L}_{kin,\tilde{m},\bar{\mathrm{f}}} + \mathcal{L}_{gauge} + \mathcal{L}_{r,\vartheta}. \tag{3.33}$$

which after component expansion, allows us to check from the F-term equation that $F^l = \bar{F}^k = 0$, where F^l and $\bar{F}^k$ are the auxiliary field components of Q^l and $\bar{Q}^k$ respectively. Also, from the D-term equation, one can confirm that

$$D = -e^2\left(\sum |\phi^l|^2 - \sum |\bar{\phi}^k|^2 - r\right) \tag{3.34}$$

for D equal to the auxiliary field component of V.

The above demonstrates that the scalar potential is then given by

$$U = \frac{1}{2}e^2\left(\sum_{l=1}^{L_{\mathrm{f}}} |\phi^l|^2 - \sum_{k=1}^{L_{\bar{\mathrm{f}}}} |\bar{\phi}^k|^2 - r\right)^2 + \sum_{l=1}^{L_{\mathrm{f}}} |\sigma - \tilde{m}_{\mathrm{f}}^l|^2 |\phi^l|^2 + \sum_{k=1}^{L_{\bar{\mathrm{f}}}} |\sigma + \tilde{m}_{\bar{\mathrm{f}}}^k|^2 |\bar{\phi}^k|^2. \tag{3.35}$$

To analyse the supersymmetric vacua, we have to find the zeros of U. Let us proceed first by considering the case where σ is fixed. The corresponding class of solutions is called the *Higgs branch* since the $U(1)$ gauge invariance is sponataneously broken. If the twisted masses are zero and $L_{\bar{\mathrm{f}}} = 0$, a solution exists only if $r > 0$ and the resulting vaccum mainfold is $\mathbb{CP}^{L_{\mathrm{f}}-1}$. Similary, if the twisted masses are zero and $L_{\mathrm{f}} = 0$, the solution exists only if $r < 0$ and the resulting vaccum mainfold is $\mathbb{CP}^{L_{\bar{\mathrm{f}}}-1}$. A more general case of the Higgs branch is discussed in [17] and [24]. The class which preserves the $U(1)$ gauge invariance is called the *Coulomb branch*. In this case ϕ^l and $\bar{\phi}^k$ vanish. We can achieve this by imposing ϕ^l and $\bar{\phi}^k$ to be massive and thus frozen at low energies. For the vanishing twisted masses, this requires a large, slowly varying σ with the solution only existing for $r = 0$.

Now consider a $U(N)$ gauge theory with the same Lagrangian (3.33). After integrating out the auxiliary fields, the scalar potential is given by

$$U = \frac{1}{2}e^2\mathrm{Tr}\left(\sum_{l=1}^{L_{\mathrm{f}}} |\phi^l|^2 - \sum_{k=1}^{L_{\bar{\mathrm{f}}}} |\bar{\phi}^k|^2 - r\right)^2 + \frac{1}{2e^2}\mathrm{Tr}[\sigma, \sigma^\dagger]^2 + \cdots . \tag{3.36}$$

In this equation for $U = 0$ requires $\mathrm{Tr}[\sigma, \sigma^\dagger]^2 = 0$ which forces σ to be diagonalisable. This will be a crucial point when we extend the analysis of the abelian case to non-abelian cases in the next section. In this paper, we will not discuss the vacuum structure further for its structure becomes more complicated with additional branches including mixed branches. For details, see [18].

3.3.2 Quantum Effects

In this section, we discuss the quantum Coulomb branch of the theory. Thus we consider the low energy effective theory for slowly varing σ after integrating out all the massive fields. The corresponding effective action is given by

$$e^{iS_{\text{eff}}[\Sigma]} = \int \prod_j \mathcal{D}\Phi^j e^{iS[\Sigma,\Phi^j]}. \tag{3.37}$$

To find out S_{eff}, we need a description of how the terms in the Lagrangian are affected by quantum corrections and renormalisations. Indeed, supersymmetry elucidates this problem and restricts the way in which they influence each other by the *decoupling theorem* and the *non-renormalisation theorem*.

The decoupling theorem states that when te F-terms and the twisted F-terms are deformed they cannot mix. Moreover D-terms cannot enter into F-terms or twisted F-terms although the reverse is possible; in other words the effective D-term may have the F-term or twisted F-term couplings. The idea behind proving the decoupling theorem involves promoting parameters to superfields [19]. Consider a general supersymmetric Lagrangian:

$$\mathcal{L} = \int d^4\theta K(\Phi^i, \bar{\Phi}^i, \Sigma^j, \bar{\Sigma}^j, \gamma^a) + \Big(\int d^2\theta W(\Phi^i, \lambda^b) + \text{h.c.}\Big) + \Big(\int d^2\vartheta \tilde{W}(\Sigma^j, \xi^c) + \text{h.c.}\Big) \tag{3.38}$$

in which γ^a, λ^b and ξ^c are parameters. We want to check whether γ^a and λ^b can enter into $\tilde{W}_{\text{eff}}$ and whether γ^a and ξ^c into W_{eff} at the low energy limit. To see this, we promote γ^a and λ^b to chiral superfields, Γ^a and Λ^b, respectively, and ξ^c to a twisted chiral superfield, Ξ^c. We then enlarge the theory with a Lagrangian, $\mathcal{L}_\epsilon$, which contains the kinetic terms of Γ^a, Λ^b, Ξ^c:

$$\frac{1}{\epsilon}\int d^4\theta\Big(\sum_a |\Gamma^a|^2 + \sum_b |\Lambda^b|^2 - \sum_c |\Xi^c|\Big) \tag{3.39}$$

By supersymmetry, Γ^a and Λ^b cannot enter into $\tilde{W}_{\text{eff}}$ while Ξ^c cannot enter into W_{eff}. This is valid for any ϵ. In the limit $\epsilon \to 0$, the kinetic terms (3.39) of the promoted fields become very large. This means any variation of the promoted fields results in a very large action and they are required to be frozen at constant values. Thus in this limit the original effective action is recovered and we see that not only are there no mixing of parameters between the F-term and the twisted F-term, but also there are no parameters of the D-term which enter into the twisted F-term. For the decoupling of the D-term and F-term, we promote γ^a to twisted chiral superfields, Γ^a (including the minus sign in the first term in (3.39)) and the apply the same argument. In particular, the decoupling of F-terms and twisted F-terms implies that we may assume the absence of the superpotential when we calculate the effective twisted superpotential, $\tilde{W}_{\text{eff}}$.

The non-renormalisation theorem states that the F-terms and twisted F-terms do not change when the D-term is deformed. The idea behind proving this is to demote the fields to parameters by introducing an additional D-term:

$$\Delta_\epsilon S = \frac{1}{\epsilon}\int d^4\theta\Big(\sum_i |\Phi^i|^2 - \sum_j |\Sigma^j|^2\Big). \tag{3.40}$$

In the limit $\epsilon \to 0$, D-terms are very large and the fields become parameters. Thus all the quantum fluctuations are suppressed and there is no renormalisation of the F-term and twisted F-term. However, we know, by the decoupling theorem, that the demoted parameters cannot enter into the F-terms and twisted F-terms. Thus for any ϵ the F-terms and twsited F-terms do not get renormalised, and this proves the non-renormalisation theorem. There also are other approaches to proving the theorem based on symmetry, holomorphy and supergraphs [25, 26]. The non-renomalisation theorem, in particular, implies that integrating out high frequency modes of Σ does not affect the effective twisted superpotential $\tilde{W}_{\text{eff}}$.

After we calculate the effective Lagrangian with the help of the decoupling theorem and the non-renormalisation theorem, the next in the agenda is to specify the supersymmetric ground state in terms of σ. We start with the most general effective supersymmetric Lagrangian containing terms with at most two derivatives and four fermions:

$$\mathcal{L}_{\text{eff}} = \int d^4\tilde{K}_{\text{eff}}(\Sigma, \Sigma^\dagger) + \Big(\int d^2\vartheta \tilde{W}_{\text{eff}}\Big|_{\vartheta^\pm=0} + \text{h.c.}\Big) \tag{3.41}$$

Then the D-term equation reads off as:

$$\frac{\partial \mathcal{L}_{\text{eff}}}{\partial D^\dagger} = \frac{\partial \tilde{K}_{\text{eff}}}{\partial\sigma\partial\sigma^\dagger}D + \frac{\partial \bar{\tilde{W}}_{\text{eff}}}{\partial\sigma^\dagger} + \text{fermions} = 0. \tag{3.42}$$

Assuming the Kähler metric is non-degenerate, we can obtain the effective scalar potential:

$$U_{\text{eff}}(\sigma) = -\left(\frac{\partial \tilde{K}_{\text{eff}}}{\partial\sigma\partial\sigma^\dagger}\right)^{-1}\frac{\partial \tilde{W}_{\text{eff}}}{\partial\sigma}\frac{\partial \bar{\tilde{W}}_{\text{eff}}}{\partial\sigma^\dagger} = -\left(\frac{\partial \tilde{K}_{\text{eff}}}{\partial\sigma\partial\sigma^\dagger}\right)^{-1}\left|\frac{\partial \tilde{W}_{\text{eff}}}{\partial\sigma}\right|^2 \tag{3.43}$$

For the vacua of the theory, we have to find the zeros of $U_{\text{eff}}(\sigma) = 0$ or

$$\frac{\partial \tilde{W}_{\text{eff}}}{\partial\sigma} = 0. \tag{3.44}$$

At this point, there is another key element which must be accounted for, that is a modification due to the ϑ-term. Recall that the ϑ-term sources the constant electric field and the vacuum energy density stored in this field is given by (3.30). We have to choose a $n \in \mathbb{Z}$ which minimises (3.30) which causes the shift of ϑ. Let n^* be the integer which minimises (3.30). Then there is a shift $\vartheta \to \vartheta - 2\pi n^*$ which in turn results in the shift of the twisted potential $\tilde{W}(\Sigma) \to \tilde{W}(\Sigma) - in^*\Sigma$. These shifts lead to the modification of the vacuum equation as

$$\frac{\partial \tilde{W}_{\text{eff}}}{\partial\sigma} = in^*, \qquad \text{or} \qquad \exp\Big(2\pi\frac{\partial \tilde{W}_{\text{eff}}}{\partial\sigma}\Big) = 1. \tag{3.45}$$

Now that all the tools have been prepared we may simply set out to calculate the effective twisted superpotential for the vacuum equation (3.45). Let us first discuss the abelian case with the Lagrangian (3.33) and then extend this to the non-abelian case. Our strategy follows the idea outlined by [21] (See [19] for the path integral calculation). From the general form of the Lagrangian of the twisted superpotential (3.27), we know $\mathrm{Re}(\tilde{W}'_{\text{eff}})$ and $\mathrm{Im}(\tilde{W}'_{\text{eff}})$ are the coefficients of D and F_{01} in the Lagrangian $\mathcal{L}_{\tilde{W}_{\text{eff}}}$. Comparing with the Lagrangian (3.33), we can identify $-r_{\text{eff}} = \mathrm{Re}(\tilde{W}'_{\text{eff}})$ and $\frac{\vartheta_{\text{eff}}}{2\pi} = \mathrm{Im}(\tilde{W}'_{\text{eff}})$. Thus we can obtain the effective twisted superpotential $\tilde{W}_{\text{eff}}$ by calculating r_{eff} and ϑ_{eff}. The starting point is the D-term equation (3.34). In the classical analysis, $|\phi^l| = |\bar{\phi}^k| = 0$ for large σ. However, quantum mechanically, we have to consider the expectation values $\langle|\phi^l|^2\rangle$, $\langle|\bar{\phi}^k|^2\rangle$. Thus from the Lagrangian (3.33), we can see that the one-loop correction to the vev of the D-term equation (3.34) is given by

$$\delta\left\langle\frac{D}{e^2}\right\rangle = \sum_l \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+|\sigma-\tilde{m}^l_{\text{f}}|^2} - \sum_k \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+|\sigma+\tilde{m}^k_{\bar{\text{f}}}|^2} \tag{3.46}$$

This integral diverges. Let us control this divergence by regularising it with a mass scale μ:

$$\delta\left\langle\frac{D}{e^2}\right\rangle = \sum_l \int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{k^2+|\sigma-\tilde{m}^l_{\text{f}}|^2} - \frac{1}{k^2+\mu^2}\right) - \sum_k \int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{k^2+|\sigma+\tilde{m}^k_{\bar{\text{f}}}|^2} - \frac{1}{k^2+\mu^2}\right)$$

This leads to

$$r_{\text{eff}} = r - \frac{1}{2\pi}\sum_l \log\frac{|\sigma-\tilde{m}^l_{\text{f}}|}{\mu} + \frac{1}{2\pi}\sum_k \log\frac{|\sigma+\tilde{m}^k_{\bar{\text{f}}}|}{\mu} \tag{3.47}$$

which then gives the one-loop correction of the effective twisted superpotential:

$$\delta\tilde{W}_{\text{eff}} = \frac{1}{2\pi}\sum_{l=1}^{L_{\text{f}}}(\sigma-\tilde{m}^l_{\text{f}})\left(\log\frac{\sigma-\tilde{m}^l_{\text{f}}}{\mu}-1\right) + \frac{1}{2\pi}\sum_{k=1}^{L_{\bar{\text{f}}}}(-\sigma-\tilde{m}^k_{\bar{\text{f}}})\left(\log\frac{-\sigma-\tilde{m}^k_{\bar{\text{f}}}}{\mu}-1\right). \tag{3.48}$$

Indeed, we can check that this is compatible with (3.47) by differentiating $\delta\tilde{W}_{\text{eff}}$:

$$\mathrm{Re}\left(\frac{\partial\delta\tilde{W}_{\text{eff}}}{\partial\sigma}\right) = \frac{1}{2\pi}\sum_l \log\frac{|\sigma-\tilde{m}^l_{\text{f}}|}{\mu} - \frac{1}{2\pi}\sum_k \log\frac{|\sigma+\tilde{m}^k_{\bar{\text{f}}}|}{\mu} \tag{3.49}$$

We can see that this is the (negative of the) second term in the right hand side of (3.47). The imaginary part is given by

$$\mathrm{Im}\left(\frac{\partial\delta\tilde{W}_{\text{eff}}}{\partial\sigma}\right) = \frac{1}{2\pi}\sum_l\left(\arg(\sigma-\tilde{m}^l_{\text{f}})+2\pi n^l\right) - \frac{1}{2\pi}\sum_k\left(\arg(-\sigma-\tilde{m}^k_{\bar{\text{f}}})+2\pi n^k\right) \tag{3.50}$$

This corresponds to the correction for $\frac{\vartheta}{2\pi}$ and branch cuts are chosen in a way that n^l and n^k minimise the energy density (3.30) (See [19]). Thus we see that (3.48) is the right expression for the one-loop correction. Moreover, there are no higher order corrections. The non-renormalisation theorem prevents further corrections to enter when integrating out the high frequency modes of Σ while for the case of higher loop diagrams, super-renormalisability of this theory tells us that they should vanish in the large σ limit. Thus we have obtained the *exact* effective twisted superpotential given by

$$\tilde{W}_{\text{eff}}(\Sigma) = i\tau(\mu)\Sigma + \frac{1}{2\pi}\sum_{l=1}^{L_{\text{f}}}(\Sigma-\tilde{m}^l_{\text{f}})\left(\log\frac{\Sigma-\tilde{m}^l_{\text{f}}}{\mu}-1\right) + \frac{1}{2\pi}\sum_{k=1}^{L_{\bar{\text{f}}}}(-\Sigma-\tilde{m}^k_{\bar{\text{f}}})\left(\log\frac{-\Sigma-\tilde{m}^k_{\bar{\text{f}}}}{\mu}-1\right) \tag{3.51}$$

This kind of effective superpotentials was first prososed in [27] and was first calculated for the $\mathbb{CP}^{L-1}$ sigma model from a different viewpoint in [28]. Plugging (3.57) into (3.45), the vacuum equation read off as

$$\frac{\prod_{l=1}^{L_{\mathrm{f}}}(\sigma-\tilde{m}_{\mathrm{f}}^{l})}{\prod_{k=1}^{L_{\bar{\mathrm{f}}}}(\sigma-\tilde{m}_{\bar{\mathrm{f}}}^{k})}=\mu^{L_{\mathrm{f}}-L_{\bar{\mathrm{f}}}}(-1)^{L_{\bar{\mathrm{f}}}}e^{2\pi i\tau} \tag{3.52}$$

Now let us move on to non-abelian gauge theory. Recall that for the gound state, σ should be diagonalisable since $\mathrm{Tr}([\sigma,\bar{\sigma}]^2)=0$ from (3.36). Thus we only focus on the maximal torus $U(1)^N$ of $U(N)$. This implies we can think of a $U(N)$ gauge theory as N copies of abelian theories. This leads to N vacuum equations:

$$\exp\left(2\pi\frac{\partial\tilde{W}_{\mathrm{eff}}}{\partial\sigma^n}\right)=1\,,\qquad 1\leq n\leq N \tag{3.53}$$

The contributions to the effective twisted superpotential by each matter are

$$\text{Fundamental}\quad Q^l:\quad \delta\tilde{W}^{l}_{\mathrm{eff,f}}=\frac{1}{2\pi}\sum_{n=1}^{N}(\sigma^n-\tilde{m}_{\mathrm{f}}^{l})\Big(\log\frac{\sigma^n-\tilde{m}_{\mathrm{f}}^{l}}{\mu}-1\Big) \tag{3.54}$$

$$\text{Anti-fundamental}\quad \bar{Q}^k:\quad \delta\tilde{W}^{k}_{\mathrm{eff},\bar{\mathrm{f}}}=\frac{1}{2\pi}\sum_{n=1}^{N}(-\sigma^n-\tilde{m}_{\bar{\mathrm{f}}}^{k})\Big(\log\frac{-\sigma^n-\tilde{m}_{\bar{\mathrm{f}}}^{k}}{\mu}-1\Big) \tag{3.55}$$

$$\text{Adjoint}\quad \Psi:\quad \delta\tilde{W}^{j}_{\mathrm{eff,a}}=\frac{1}{2\pi}\sum_{\substack{m,n=1\\ m\neq n}}^{N}(\sigma^n-\sigma^m-\tilde{m}_{\mathrm{a}}^{j})\Big(\log\frac{\sigma^m-\sigma^n-\tilde{m}_{\mathrm{a}}^{j}}{\mu}-1\Big) \tag{3.56}$$

(For the bi-fundamental matter see [3].
Summing up, the effective twisted superpetential is given by

$$\boxed{\begin{aligned}\tilde{W}_{\mathrm{eff}}(\sigma)=i\tau\sum_{n=1}^{N}\sigma^n&+\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{l=1}^{L_{\mathrm{f}}}(\sigma^n-\tilde{m}_{\mathrm{f}}^{l})\Big(\log\frac{\sigma^n-\tilde{m}_{\mathrm{f}}^{l}}{\mu}-1\Big)\\ &+\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{l=k}^{L_{\bar{\mathrm{f}}}}(-\sigma^n-\tilde{m}_{\bar{\mathrm{f}}}^{k})\Big(\log\frac{-\sigma^n-\tilde{m}_{\bar{\mathrm{f}}}^{k}}{\mu}-1\Big)\\ &+\frac{1}{2\pi}\sum_{\substack{m,n=1\\ m\neq n}}^{N}\sum_{l=j}^{L_{\mathrm{a}}}(\sigma^n-\sigma^m-\tilde{m}_{\mathrm{a}}^{j})\Big(\log\frac{\sigma^m-\sigma^n-\tilde{m}_{\mathrm{a}}^{j}}{\mu}-1\Big)\end{aligned}} \tag{3.57}$$

Plugging the above into (3.53), we obtain the vacuum equation:

$$\boxed{\frac{\prod_{l=1}^{L_{\mathrm{f}}}(\sigma^n-\tilde{m}_{\mathrm{f}}^{l})}{\prod_{k=1}^{L_{\bar{\mathrm{f}}}}(\sigma^n+\tilde{m}_{\bar{\mathrm{f}}}^{k})}=\mu^{L_{\mathrm{f}}-L_{\bar{\mathrm{f}}}}(-1)^{L_{\bar{\mathrm{f}}}-L_{\mathrm{a}}}e^{2\pi i\tau}\prod_{m\neq n}^{N}\prod_{j=1}^{L_{\mathrm{a}}}\frac{\sigma^n-\sigma^m+\tilde{m}_{\mathrm{a}}^{j}}{\sigma^n-\sigma^m-\tilde{m}_{\mathrm{a}}^{j}}\,,\qquad 1\leq n\leq N} \tag{3.58}$$

4 Bethe/Gauge Correspondence

Let us open this chapter by summarising the results of the last two chapters. From the Bethe side, we studied $\mathfrak{sl}_2$ spin chains. After applying the algebraic Bethe ansatz, we obtained N Bethe equations (2.62):

$$\prod_{l=1}^{L}\frac{\lambda_n-\nu_l+is_l}{\lambda_n-\nu_l-is_l}=e^{i\vartheta}\prod_{m\neq n}^{N}\frac{\lambda_n-\lambda_m+i}{\lambda_n-\lambda_m-i}\,,\qquad 1\leq n\leq N. \tag{4.1}$$

Also we discussed the Yang-Yang action (2.67) which played the role of the potential of the Bethe equations:

$$\begin{aligned}Y(\lambda)=&\frac{1}{2\pi}\sum_{l=1}^{L}\sum_{n=1}^{N}\Big[(\lambda_n-\nu_l+is_l)\mathrm{log}(\lambda_n-\nu_l+is_l)-(\lambda_n-\nu_l-is_l)\mathrm{log}(\lambda_n-\nu_l-is_l)\Big]\\&-\frac{1}{2\pi}\sum_{n<m}^{N}\Big[(\lambda_n-\lambda_m+i)\mathrm{log}(\lambda_n-\lambda_m+i)-(\lambda_n-\lambda_m-i)\mathrm{log}(\lambda_n-\lambda_m-i)\Big]\\&-\frac{i\vartheta}{2\pi}\sum_{n=1}^{N}\lambda_n\end{aligned} \tag{4.2}$$

From the gauge side, we discussed the vacuum structure on the Coulomb branch of $\mathcal{N}=(2,2)$ supersymmetric gauge theory. We obtained the exact twisted effective superpotential (3.57) which we mind the reader of:

$$\begin{aligned}\tilde{W}_{\mathrm{eff}}(\sigma)=i\tau\sum_{n=1}^{N}\sigma^n&+\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{l=1}^{L_{\mathrm{f}}}(\sigma^n-\tilde{m}_{\mathrm{f}}^l)\Big(\mathrm{log}\frac{\sigma^n-\tilde{m}_{\mathrm{f}}^l}{\mu}-1\Big)\\&+\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{l=k}^{L_{\bar{\mathrm{f}}}}(-\sigma^n-\tilde{m}_{\bar{\mathrm{f}}}^k)\Big(\mathrm{log}\frac{-\sigma^n-\tilde{m}_{\bar{\mathrm{f}}}^k}{\mu}-1\Big)\\&+\frac{1}{2\pi}\sum_{\substack{m,n=1\\m\neq n}}^{N}\sum_{l=j}^{L_{\mathrm{a}}}(\sigma^n-\sigma^m-\tilde{m}_{\mathrm{a}}^j)\Big(\mathrm{log}\frac{\sigma^m-\sigma^n-\tilde{m}_{\mathrm{a}}^j}{\mu}-1\Big)\end{aligned} \tag{4.3}$$

From the the twisted effective superpotential, we determined the set of N coupled vacuum equations (3.58):

$$\frac{\prod_{l=1}^{L_{\mathrm{f}}}(\sigma^n-\tilde{m}_{\mathrm{f}}^l)}{\prod_{k=1}^{L_{\bar{\mathrm{f}}}}(\sigma^n+\tilde{m}_{\bar{\mathrm{f}}}^k)}=\mu^{L_{\mathrm{f}}-L_{\bar{\mathrm{f}}}}(-1)^{L_{\bar{\mathrm{f}}}-L_{\mathrm{a}}}e^{2\pi i\tau}\prod_{m\neq n}^{N}\prod_{j=1}^{L_{\mathrm{a}}}\frac{\sigma^n-\sigma^m+\tilde{m}_{\mathrm{a}}^j}{\sigma^n-\sigma^m-\tilde{m}_{\mathrm{a}}^j}. \tag{4.4}$$

As can be seen above, both sides have very similar equations. However, for the perfect correspondence, we need some modification. At a glance, the lowest component of the chiral superfield σ^n corresponds to the rapidity λ_n. As of the sign on the RHS of (4.4), we can remove it by shifting τ to relate τ to ϑ. The number of matter fields are restricted to $L=L_{\mathrm{f}}=L_{\bar{\mathrm{f}}}$ and $L_{\mathrm{a}}=1$. Then we see that $\tilde{m}_{\mathrm{f}}^l$, $\tilde{m}_{\mathrm{f}}^l$ and $\tilde{m}_{\bar{\mathrm{f}}}^l$ should be ν_l-is_l, ν_l-is_l and i respectively and the matter contents of the gauge theory are

- L fundamental fields Q^l with twisted masses $\tilde{m}_{\mathrm{f}}^l=\nu_l-is_l$

- L anti-fundamental fields $\bar{Q}^k$ with twisted masses $\tilde{m}^l_{\bar{\mathrm{f}}} = -\nu_l - is_l$

- one adjoint field Ψ with twisted mass $\tilde{m}_{\mathrm{a}} = i$

Note that local spins are half integers ($s_l \in \frac{1}{2}\mathbb{N}$) and that twisted masses are related by local spins, that is twisted masses cannot be freely chosen. They should be compatible with local spins. One way to resolve this is to turn on the superpotential. Recall that we introduced twisted masses by weakly gauging the flavour symmetry and that the existence of the superpotential term breaks the flavour symmetry to some extent. Thus we can restrict the value of twisted masses by choosing the appropriate superpotential. This will not alter our vacuum equation due to the decoupling theorem. Consider the following superpotential:

$$W(Q, \bar{Q}, \Psi) = \sum_{l=1}^{L} \bar{Q}_l \Psi^{2s_l} Q^l. \tag{4.5}$$

Then the residual flavour symmetry is given by

$$\bar{Q}_l \mapsto \bar{Q}_l e^{(-\nu_l - is_l)} \qquad \Psi \mapsto e^i \Psi \qquad Q^l \mapsto e^{(\nu_l - is_l)} Q^l. \tag{4.6}$$

Thus the above superpotential (4.5) is compatible with twisted masses required by the Bethe/gauge correspondence.

Finally, with some modification, we arrive at the perfect match between both sides. A summary is shown in the table below.

Integrable Model			Gauge Theory
$\mathfrak{sl}_2$ spin chain			vacuum structure on Coulomb branch of $\mathcal{N} = (2,2)$ susy theory in 2D
magnon number	N	$U(N)$	gauge group
Yang-Yang action	$Y(\lambda)$	$\tilde{W}_{\mathrm{eff}}(\sigma)$	effective twisted superpotential
Bethe ansatz equation	$e^{2\pi dY} = 1$	$e^{2\pi d\tilde{W}_{\mathrm{eff}}} = 1$	vacuum equation
length	L	$U(L)$	flavour group
rapidity	λ_n	σ^n	lowest component of the twisted chiral superfield
inhomogeneity, local spin	$\nu_l - is_l$	$\tilde{m}^l_{\mathrm{f}}$	twisted mass of the fundmental field
inhomogeneity, local spin	$-\nu_l - is_l$	$\tilde{m}^l_{\bar{\mathrm{f}}}$	twisted mass of the anti-fundmental field
twist parameter	ϑ	τ	complex gauge coupling

Table 1: Dictionary in the Bethe/gauge correspondence

5 Geometric Representation Theory

We have seen that there is a correspondence between integrable systems and gauge theories. However, this correspondence was merely an observation and the underlying reason for the correspondence was elusive. In [4], Orlando and Reffert argue that the mathematical foundation of the Bethe/gauge correspondence is the geometric representation theory. Our aim of this chapter is to undertand their explanation.

5.1 Borel-Moore Homology

To construct representations of $\mathfrak{sl}_n$ geometrically, we give a vector space structure to varieties. For this we make use of homology, more specifically we will use the *Borel-Moore homology*, first introduced in [29]; as it will become apparent later, it has many desirable properties which allow us to manufacture convolution algebras in the following section. Among several equivalent definitions of the Borel-Moore homology (See [30] for alternative definitions), we give a description analogous to the singular homology. Recall that given a topological space X, the group of n-chains, $C_n(X)$, is the free abelian group with basis given by the set of singular n-simplices in X. Then for any $\sigma = \sum_i n_i\sigma_i \in C_n(X)$, the *support* of σ, defined by $\mathrm{supp}(\sigma) := \bigcup_{i|n_i\neq 0}\sigma(\Delta^n)$, is compact since only finite a number of coefficients are non-zero. In particular, the support of a cycle is compact. The idea of Borel-Moore homology is to allow non-compact cycles. For this, one might replace the direct sum with the direct product:

$$C_n(X) = \bigoplus_{\{\sigma:\Delta^n\to X\}} \mathbb{Z}\sigma \qquad \longrightarrow \qquad C'_n(X) = \prod_{\{\sigma:\Delta^n\to X\}} \mathbb{Z}\sigma \tag{5.1}$$

However, we have to be careful with this replacement since the boundary map might be ill-defined. For instance, take the disc D^2 as our topological space X and consider infinitely many rays, $\sigma_i : \Delta^1 \to X$ from 0 to ∂X. Then for the 1-chain, $\sigma = \sum_i \sigma_i \in C'_n(X)$, the coefficient of 0 in $\partial_1(\sigma_i)$ is given by $\sum_i \partial_1(\sigma_i) = -\infty$. Thus the boundary map is not well-defined. To fix this problem, we impose a condition on $C'_n(X)$ such that $\sigma = \sum_i n_i\sigma_i \in C'_n(X)$ is *locally finite*. That is, for any $x \in X$, there is a neighbourhood $U \subseteq X$ of x such that $\{\sigma_i \,|\, n_i \neq 0 \text{ and } \sigma_i(\Delta^n) \cap U \neq \emptyset\}$ is finite. We are now in a position to define the Borel-Moore homology with $C'_n(X)$.

Definition 5.1. For the chain complex: $\cdots \to C'_{n+1}(X) \xrightarrow{\partial_{n+1}} C'_n(X) \xrightarrow{\partial_n} C'_{n-1}(X) \to \cdots$, the *n-th Borel-Moore homology group* is $H_n(X) := \mathrm{ker}\partial_n/\mathrm{im}\partial_{n+1}$

Note that if X is compact, the BM homology and ordinary homology are equivalent. However, if X is not compact, the ordinary singular homology has no fundamental class. One crucial feature of BM homology is the existence of a fundamental class for any complex algebraic variety. We list some properties of the BM homology which will be needed in the Ginzburg construction. For the proof and explanation, see [6].

(i) Fundamental class: Let Y be an (not necessarily compact) algebraic variety of complex dimension n. Then there is a well-defined fundamental class $[Y] \in H_{2n}(Y)$. If Y has n-dimensional irreducible components, $Y_1, \cdots, Y_k$, then the fundmental classes $[Y_1], \cdots, [Y_k]$ form a basis for the vector space $H_{\text{top}}(X) = H_{2n}(X)$.

(ii) Pushforward: Let $f : X \to Y$ be a proper map. Then there is a pushforward $f_* : H_*(X) \to H_*(Y)$

(iii) Pullback: Let $f : X \to Y$ be a locally trivial fibre bundle with smooth fibre of dimension d. Then there is a pullback $f^* : H_*(Y) \to H_{*+d}(X)$. In particular, for a trivial fibration $f : Y \times F \to Y$ the pullback is given by $c \mapsto c \boxtimes [F]$ where $\boxtimes$ is the Kunneth isomorphism $H_*(Y) \otimes H_*(F) \simeq H_*(Y \times F)$

(iv) Intersection Pairing: Let M be a smooth, oriented manifold of real dimension n and Z, Z' two closed subsets (each of which is a homotopy retract of an open subset). Then we can define a bilinear pairing:

$$\cap : H_i(Z) \times H_j(Z') \to H_{i+j-n}(Z \cap Z'). \tag{5.2}$$

5.2 Convolution Algebra

In this section we construct the convolution algebra and its module with the help of the properties of BM homology. First we discuss the convolution product. Let M_1, M_2, M_3 be connected, oriented, smooth manifolds whose real dimensions are d_1, d_2, d_3. Let $Z_{12} \in M_1 \times M_2$ and $Z_{23} \in M_2 \times M_3$ be closed subsets in the sense of (iv) in the previous section. Also, define projection maps given by $\pi_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$.

$$\begin{tikzcd} & M_1 \times M_2 \times M_3 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ Z_{12} \subseteq M_1 \times M_2 & M_1 \times M_3 & Z_{23} \subseteq M_2 \times M_3 \end{tikzcd}.$$

We also need one more element called the set-theoretic composition defined by

$$Z_{12} \circ Z_{23} := \left\{ (m_1, m_3) \in M_1 \times M_3 \,\middle|\, \exists m_2 \in M_2 \text{ s.t. } (m_1, m_2) \in Z_{12},\, (m_2, m_3) \in Z_{23} \right\}. \tag{5.3}$$

We assume that $\pi_{13} : \pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(Z_{23}) \to M_1 \times M_3$ is proper. Then this implies that its image $Z_{12} \circ Z_{23}$ is a closed subset in $M_1 \times M_3$. Finally we define the convolution product as

$$\begin{aligned} * : H_i(Z_{12}) \times H_j(Z_{23}) \quad &\to \quad H_{i+j-d_2}(Z_{12} \circ Z_{23}) \\ (c_{12}, c_{23}) \quad &\mapsto \quad c_{12} * c_{23} := (\pi_{13})_* \Big(\big(c_{12} \boxtimes [M_3] \big) \cap \big([M_1] \boxtimes c_{23} \big) \Big). \end{aligned} \tag{5.4}$$

Note that in the above definition we have used the BM holomology propeties in the order of fundamental class(i), pullback(iii), intersection pairing(iv), pushforward(ii). One more important fact is that the convolution product is associative. Considering the fourth manifold M_4 and a closed subset $Z_{34} \subseteq M_3 \times M_4$, we can check that the following associativity holds in BM homology.

$$(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34}) \tag{5.5}$$

where $c_{12} \in H_*(Z_{12})$, $c_{23} \in H_*(Z_{23})$, $c_{34} \in H_*(Z_{34})$.

Now we can construct a convolution algebra. First consider a special case $M_1 = M_2 = M_3 = M$. Let N be a variety, $\pi : M \to N$ a proper map and set $Z_{12} = Z_{23} = Z := \{(m_1, m_2) \in M \times M | \pi(m_1) = \pi(m_2)\}$. Then we can easily check that $Z \circ Z = Z$. From this, we have a convolution map: $H_*(Z) \times H_*(Z) \to H_*(Z)$. Thus $(H_*(Z), *)$ is an associative algebra and the unit is the fundamental class of $M_\Delta := \{(m, m)|m \in M\} \subseteq Z$.

We can construct a $H_*(Z)$-module as well. First choose $x \in N$ and set $M^x = \pi^{-1}(x)$. This time, we consider $M_1 = M_2 = M$ and $M_3 = \{\mathrm{pt}\}$. We also alter Z_{12} and Z_{23} so that $Z_{12} = Z$ and $Z_{23} = M^x \times \{\mathrm{pt}\}$. Since $Z \circ M^x = M^x$, we have a convolution map $H_*(Z) \times H_*(M^x) \to H_*(M^x)$. This allows us to obtain a $H_*(Z)$-module, $H_*(M^x)$.

For the Ginzburg's construction, we need one more information: the relationship between the convolution product of varieties and the convolution product for their conormal bundles. Let X_1, X_2, X_3 be complex manifolds and $Y_{12} \subseteq X_1 \times X_2$, $Y_{23} \subseteq X_2 \times X_3$ complex submanifolds. Also let $Y_{13} = Y_{12} \circ Y_{23}$. As usual we denote the projection maps by $\pi_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ and $p_{ij} : T^*(X_1 \times X_2 \times X_3) \to T^*(X_i \times X_j)$. Then for the conormal bundles $Z_{ij} = T^*_{Y_{ij}}(X_i \times X_j)$, we have the following theorem.

Theorem 5.2. *If $\pi_{12}^{-1}(Y_{12}) \cap \pi_{23}^{-1}(Y_{23})$ is transverse and the map $\pi_{13} : \pi_{12}^{-1}(Y_{12}) \cap \pi_{23}^{-1}(Y_{23}) \to Y_{13}$ is a smooth locally trivial oriented fibration with a smooth base Y_{13} and smooth, compact fibre F, then the following holds:*

(i) We have a set-theoretic equality $Z_{12} \circ Z_{23} = Z_{13}$;

(ii) The map $p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13}$ is a smooth locally trivial oriented fibration with fibre F;

(iii) In $H_*(Z_{13})$, we have an equation: $[Z_{12}] * [Z_{23}] = \chi(F)[Z_{13}]$, where $\chi(F)$ is the Euler characteristic of F.

5.3 Ginzburg Construction

Now we apply the convolution structure to flag varieties to construct $\mathfrak{sl}_n$-modules. Let $\mathcal{F}(\mathbb{C}^L)$ denote the set of n-step partial flags in $\mathbb{C}^L$. Observe that connected components of $\mathcal{F}(\mathbb{C}^L)$ are parametrised by $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{N}^n$ where $\sum_i \mu_i = L$. We denote the corresponding connected component by $\mathcal{F}_\mu(\mathbb{C}^L)$.

$$\mathcal{F}_\mu(\mathbb{C}^L) := \Big\{ V_\bullet = (0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^L) \,\big|\, \dim(V_i/V_{i-1}) = \mu_i \Big\} \tag{5.6}$$

$$\mathcal{F}(\mathbb{C}^L) := \bigcup_{\mu \,|\, \mu_1 + \cdots + \mu_n = L} \mathcal{F}_\mu(\mathbb{C}^L). \tag{5.7}$$

Let $\mathcal{N} := \{X \in \mathrm{End}(\mathbb{C}^L) | X^n = 0\}$ and define

$$M := \Big\{ (X, V_\bullet) \in \mathcal{N} \times \mathcal{F}(\mathbb{C}^L) \,\big|\, X(V_i) \subseteq V_{i-1} \text{ for } 1 \leq i \leq n \Big\} \tag{5.8}$$

We can identify M with the cotangent bundle $T^*\mathcal{F}(\mathbb{C}^L)$. The reason why we denote it by M is that $T^*\mathcal{F}(\mathbb{C}^L)$ plays a role of manifold M in the colvolution structure discussed in the previous section. Also $\mathcal{N}$ is a counterpart of N. In this case, the fibres of the projection map $M \to \mathcal{N}$ are called *n-step Springer fibres* and we denote the fibre over $X \in \mathcal{N}$ by $\mathcal{F}(\mathbb{C}^L)^X$. As we did in the last section, we define

$$Z := \Big\{ \Big((X, V_\bullet), (X', V'_\bullet) \Big) \in M \times M \,\Big|\, X = X' \Big\} \subseteq M \times M \tag{5.9}$$

Applying the result in the previous section, we know $H_*(Z)$ is an associative algebra and $H_*(\mathcal{F}(\mathbb{C}^L)^X)$ is a $H_*(Z)$-module for any $X \in \mathcal{N}$. However, we want $\mathfrak{sl}_n$-module and only need a certain subalgebra of $H_*(Z)$. Indeed, this will be the vector space spanned by the fundamental classes of the irreducible components of Z. We denote it by $H_{\mathrm{top}}(Z)$. Then we can state the following theorem.

Theorem 5.3. *There is a surjective algebra homomorphism $\mathcal{U}(\mathfrak{sl}_n) \to H_{top}(Z)$*

Since $H_{\mathrm{top}}(Z)$ is a subalgebra of $H_*(Z)$, we need to find the appropriate subsets of $\mathcal{F}(\mathbb{C}^L) \times \mathcal{F}(\mathbb{C}^L)$ to express the above homomorphism explicitly. Let $\alpha_i = (0, \cdots, \underset{i-\mathrm{th}}{1}, \underset{(i+1)-\mathrm{th}}{-1}, \cdots, 0) \in \mathbb{Z}^n$. For a partition μ, if $\mu + \alpha_i$ (respectively $\mu - \alpha_i$) is a partition as well, then we define:

$$Y_{\mu+\alpha_i,\mu} := \left\{ (V_\bullet, V'_\bullet) \in \mathcal{F}_{\mu+\alpha_i}(\mathbb{C}^L) \times \mathcal{F}_\mu(\mathbb{C}^L) \,\middle|\, \begin{array}{l} V_j = V'_j \quad \forall j \in \{1, \cdots, n-1\} \backslash \{i\} \\ V_i \supseteq V'_i \quad \text{with } \dim(V_i) = \dim(V'_i) + 1 \end{array} \right\} \tag{5.10}$$

$$Y_{\mu-\alpha_i,\mu} := \left\{ (V_\bullet, V'_\bullet) \in \mathcal{F}_{\mu-\alpha_i}(\mathbb{C}^L) \times \mathcal{F}_\mu(\mathbb{C}^L) \,\middle|\, \begin{array}{l} V_j = V'_j \quad \forall j \in \{1, \cdots, n-1\} \backslash \{i\} \\ V_i \subseteq V'_i \quad \text{with } \dim(V'_i) = \dim(V_i) + 1 \end{array} \right\} \tag{5.11}$$

for $i = 1, \cdots, n-1$.

For the Chevalley generator $\{h_i, e_i, f_i\}$ of $\mathfrak{sl}_n$, the homomorphism $\mathcal{U}(\mathfrak{sl}_n) \to H_{\mathrm{top}}(Z)$ is given by:

$$e_i \mapsto \sum_\mu \Big[T^*_{Y_{\mu+\alpha_i,\mu}} \big(\mathcal{F}_{\mu+\alpha_i}(\mathbb{C}^L) \times \mathcal{F}_\mu(\mathbb{C}^L) \big) \Big] \tag{5.12}$$

$$f_i \mapsto \sum_\mu \Big[T^*_{Y_{\mu-\alpha_i,\mu}} \big(\mathcal{F}_{\mu-\alpha_i}(\mathbb{C}^L) \times \mathcal{F}_\mu(\mathbb{C}^L) \big) \Big] \tag{5.13}$$

$$h_i \mapsto \sum_{\mu}(\mu_i - \mu_{i+1})\Big[T^*_{\Delta}\big(\mathcal{F}_\mu(\mathbb{C}^L) \times \mathcal{F}_\mu(\mathbb{C}^L)\big)\Big] \tag{5.14}$$

where $\Delta \in \mathcal{F}_\mu(\mathbb{C}^L) \times \mathcal{F}_\mu(\mathbb{C}^L)$ is the diagonal subvariety.
Then the explicit map by each Chevalley generator in the $\mathfrak{sl}_n$-module $H_*(\mathcal{F}(\mathbb{C}^L)^X)$ is given by

$$e_i : H_*(\mathcal{F}_\mu(\mathbb{C}^L)^X) \to H_*(\mathcal{F}_{\mu+\alpha_i}(\mathbb{C}^L)^X) \tag{5.15}$$

$$f_i : H_*(\mathcal{F}_\mu(\mathbb{C}^L)^X) \to H_*(\mathcal{F}_{\mu-\alpha_i}(\mathbb{C}^L)^X) \tag{5.16}$$

$$h_i : H_*(\mathcal{F}_\mu(\mathbb{C}^L)^X) \to H_*(\mathcal{F}_\mu(\mathbb{C}^L)^X) \tag{5.17}$$

The next thing to do is to check that $H_*(\mathcal{F}(\mathbb{C}^L)^X)$ is irreducible or not. Before that, we remind a basic fact from linear algebra. Recall that every $L \times L$ matrix is similar to a Jordan form matrix. Since all eigenvalues of a nilpotent matrix are zero, we can identify it with a Jordan form whose blocks are strictly upper triangular. Let denote the sizes of the blocks by decreasing sequence $\nu = (\nu_1 \geq \cdots \geq \nu_m)$ where $\sum_i \nu_i = L$. Thus there is a bijection between nilpotent matices and ν. In this case we say a nilpotent matrix is of Jordan type ν.

Theorem 5.4. *Let $X \in \mathcal{N}$ be a matrix of Jordan type ν and let λ be the conjugate of ν. (i.e. $\lambda_i = |\{j | \nu_j \geq i\}|$) Then $H_{top}(\mathcal{F}(\mathbb{C}^L)^X)$ is the irreducible $\mathfrak{sl}_n$-module of highest weight λ.*

5.4 Physical Implication

In this section, we focus on a specific example of the Bethe/gauge correspondence. We choose the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ model without inhomogeneity. On the gauge side, by the dictionary 1, the corresponding theory has the superpotential $W = \sum \bar{Q}_l \Psi Q^l$ (4.5) and its low energy limit is the non-linear sigma model(NLSM) on $T^*Gr(N,L)$ (See [2]). Now let us see how geometric representation theory relates the two theories.

First we construct a $\mathfrak{sl}_2$-module. In this case we consider 2-step partial flags in $\mathbb{C}^L$, which is the disjoint union of all the Grassmannians, $Gr(N,L)$, for fixed L:

$$Gr(L) = \bigcup_{N=0}^{L} Gr(N,L) \tag{5.18}$$

Also our $\mathcal{N}$ is given by $\mathcal{N} = \{X \in \mathrm{End}(\mathbb{C}^L) | X^2 = 0\}$. Then $X \in \mathcal{N}$ consists of one-by-one blocks and two-by-two blocks. We choose $X = 0$. Then we have a $\mathfrak{sl}_2$-module $H_*(Gr(L)^0) = H_*(Gr(L))$. Moreover, since our X is a matrix of Jordan type $\nu = \underbrace{(1, \cdots, 1)}_{L}$, by theorem 5.4, $H_{\mathrm{top}}(Gr(L))$ is a simple $\mathfrak{sl}_2$-module with dominant weight $(L,\, 0)$.

Then each generator acts on the space $H_*(T^*Gr(L))$ by

$$e : H_*(Gr(N+1, L)) \to H_*(Gr(N, L)) \tag{5.19}$$

$$f : H_*(Gr(N, L)) \to H_*(Gr(N+1, L)) \tag{5.20}$$

$$h = \bigotimes_{N=0}^{L} (L-2N) Id_{H_*(T^*Gr(N,L))} \tag{5.21}$$

Let V be the fundamental representation of $\mathfrak{sl}_2$ and $V^{\otimes L} = \bigotimes_{N=0}^{L} V_{L-2N}$ be its L times tensor product where V_{L-2N} is $L-2N$ weight space. Then through the above actions, $H_*[T^*Gr(L)] \simeq V_{L-2N}$ and $H_*[T^*Gr(L)] \simeq V^{\otimes L}$. These facts relate both side of the Bethe/gauge correspondence as follows.

The Hilbert space of the $\mathrm{XXX}_{\frac{1}{2}}$ spin chain is $V^{\otimes L}$. Via geometric representation theory, $V^{\otimes L}$ can be identified with $H_*[T^*Gr(L)]$, which is the ground states of the non-linear sigma models on all the $T^*Gr(N, L)$ for $N = 0, 1, \cdots, L$. Thus we idenfity the two spaces which stem from the two theories. Also the spectrum of the N magnon sector is V_{L-2N} and from the geometric representation theory it can be identified with $H_*[T^*Gr(N, L)]$, which is the ground state of the corresponding supersymmetric gauge theory. It reveals that there is a one-to-one correspondence between the solutions to the Bethe ansatz equations and the minima of the twisted superpotential of the NLSM, which is a result of the Bethe/gague correspondence. A summary is shown in the table below.

Physics	Mathematics
spectrum of $XXX_{\frac{1}{2}}$ spin chain	$\mathfrak{sl}_2$ representation $V^{\otimes L} \simeq H_*[T^*Gr(L)]$
ground states of the NLSM on $T^*Gr(N, L)$	cohomology $H^*[T^*Gr(N, L)]$
spectrum for the N magnon sector	weight space $V_{L-2N} \simeq H_*[T^*Gr(L)]$
ground states of $XXX_{\frac{1}{2}}$	hw representation $V(L) \simeq H_{\mathrm{top}}[T^*Gr(L)]$
gauge/Bethe correspondence	geometric representation of $\mathfrak{sl}_2$

Table 2: Bethe/gauge correspondence and geometric representation theory

6 Conclusion

In a summary, the Bethe/gauge correspondence relates quantum integrable models and supersymmetric gauge theories, and its mathematical structure can be understood by the geometric representation theory. We saw that the spectrum of the Bethe solvable spin chain shares very similar features with the vacuum structure on the Coulomb branch of $\mathcal{N} = (2, 2)$ susy gauge theory in two dimensions. In particular, the Bethe ansatz equations have the similar forms with the vacuum equations of the gauge theory and the Yang-Yang function corresponds to the effective twisted superpotential. Moreover we checked that each parameter which determines the theory on one side has its counterpart on the other side. Finally we saw that the mathematical foundation of the Bethe/gauge correspondence is geometric representation theory. Particularly we related the spectrum of $\mathrm{XXX}_{\frac{1}{2}}$ spin chain with the target space geometry of the low energy limit of the corresponding gauge theory via geometric representation of $\mathfrak{sl}_2$.

Every aspect of the Bethe/gauge correspondence has not been covered in this paper. Let us mention some further topics briefly. Throuout the paper, we only considered the isotropic XXX spin chain. However, we can do more. Indeed, the Bethe/gauge correspondence exists even for anisotropic XXZ and XYZ spin chains, which are not covered in this paper. In this case the corresponding gauge theory can be obtained by compactifying spaital dimenstions. Recall that $\mathcal{N} = (2, 2)$ supersymmetric gauge theory in two dimensions can be obtained by dimensional reduction from $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions. Rather than reducing all the two spatial dimensions, consider compactifying one of them to a circle with radius R. Then the corresponding component of the vector field in four dimensions yields Kaluza-Klein modes with masses. Since the masses are inversely proportional to the radius R, the Kaluza-Klein modes are very massive for the small R, and are intergrated out in low energy limit. Thus we obtain an effectively two-dimensional thory with $\mathcal{N} = (2, 2)$ supersymmetry at low energies. The vacuum equations of the resulting theory turn out to correnspond to the Bethe ansatz equations of anistropic XXZ spin chain, which confirms the Bethe/gauge correspondence. For the anistropic XYZ spin chain, we compactify two spatial dimensions to torus, and then apply the same argument as above. Then the vacuum equations of the resulting theory correspond to the Bethe equations for the anistropic XYZ spin chain. For further study, see [1].

Another interesting topic for the further study is matching gauge theories via the Bethe/gauge correspondence. Consider two gauge theories with $G_1 = U(N)$ and $G_2 = U(L - N)$. By the dictionary, N and $L - N$ correspond to magnon sectors in a spin chain. However, since the spin chain has length L, only difference of two magnon sectors is labeling the spin and the one magnon sector can be obtained by simply flipping the spins of the other, which does not affect the physics. Thus their Bethe ansatz equations are same, which means our two starting gauge theories have same vacuum structure by the Bethe/gauge correspondence. See [3] for more information including supergroup symmetry.

In spite of the surprising correspondence of two seemingly unrelated theories, there still remains much room for further developments. First of all, there is a missing point in the Bethe/gauge correspondence. According to the dictionary, a spin chain model correspondends the low energy configurations of a gauge theory. However the counterpart of the gauge theory itself on the Bethe side is mysterious. That is, for the perfect match, we have to find the corresponding object in the Bethe side beyond the vacuum structure of the gauge theory

We also saw that the Bethe/gauge correspondence has a nice interpretation in terms of the geometric represention theory by considering the simplest case, $\mathrm{XXX}_{\frac{1}{2}}$ spin chain with the other parameters turned off. However, there are more parameters in spin chain, including inhomogeneities and twist parameter, and there exist the counterparts in the gauge side. Thus we have to develop a way to understand the these paremeters in the language of geometric representation. Moreover the spin chain can have more general symmetry group or supergroup and even in this case the Bethe/gauge correspondence is valid [3]. To encompass the general cases, it is necessary to develop the geometric representations for all these groups. Nakajima's theory of quiver varieties [7, 8] is closely related to geometric representation theory and may open a new way of the Bethe/gauge correspondence.

One of the key object in the Bethe/Gauge correspondence is the Bethe ansatz equations, which is a consequence of underlying Yang-Baxter algebra. For the Heisenberg XXX model, the Yang-Baxter algebras is given by Yangian and thus constructing a spin chain model is equivalent to constructing a representation of Yangian. Therefore, as we express the representation of $\mathfrak{sl}_2$ in terms of Grassmannians, the Bethe/gauge correspondence may be a good starting point of constructing a geometric representation of Yangian in terms of Grassmannians.

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