# Causal Set Phenomenology in Friedmann-Robertson-Walker Spacetime

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**Abstract**

The causal set approach to quantum gravity states that spacetime is not a continuous Lorenztian manifold but a causal set composed of discrete spacetime elements together with their causal relations. It is argued that using only causality and discreteness, one can recover a more fundamental theory of gravitation with interesting qualities. Though the theory is incomplete, some phenomenological models have been developed, one of which is discussed here. Since discrete spacetime is locally finite, there are a finite number of paths a free massive particle can take. It will be forced to deviate from its classical geodesic path through a process known as swerving. This model was been developed for Minkowski spacetime, and in this paper we attempt to generalise the model to Friedmann-Robertson-Walker spacetime which is used in cosmology.

# Contents

# Chapter 1

# Introduction

Roughly speaking, most things in our universe can be described by Newtonian physics, which by now is very well understood. There are two regimes where Newtonian physics fails and we must introduce a new framework. On microscopic scales the universe is best described by quantum mechanics (QM). Similarly, in large systems with very large mass, we must use the framework of general relativity (GR). These two frameworks have extensively been confirmed by experiment and are well established as being the best methods of describing these two regimes.

Both theories describe a reality which is radically different to the one we experience in everyday life. In QM a particle does not have a definite position but rather a probability distribution over positions. GR replaces the notion of separate space and time dimensions with the single unified entity of spacetime. However, since both of these theories simplify to classical, non-relativistic Newtonian physics in the appropriate limits, we must accept that the classical world we experience is just an approximation of these underlying theories.

Problems arise when we try to describe the region where these two theories overlap, i.e. systems involving large masses which are microscopic, i.e. around the Planck scale. The problem is that they are not consistent, and we must formulate a new framework, known as quantum gravity. Such a theory should be formulated so that it is approximated in the appropriate limits by QM and GR. In analogue to these theories, one should expect that a theory of quantum gravity may present a similarly radical new way of thinking about the universe. In this paper, we focus on the causal sets approach to quantum gravity, which is a theory of discrete spacetime.

## 1.1 General relativity

Before discussing any potential theory of quantum gravity, it is a good idea to present the main ideas of classical gravity which is best described by GR. Here I will present some key concepts which will become relevant later on. For a complete treatment, see [1][2].

Space and time, previously thought of as separate entities, are unified in GR into the single spacetime. Spacetime is curved, and it is this curvature which we experience as the force of gravity. In this picture, a massive body distorts the surrounding spacetime, and the trajectories of other massive bodies in this region will be altered so that, to an external observer, there will appear to be a force acting between the two bodies. The common analogy is that of a bowling ball (the mass) placed on taut horizontal elastic sheet (spacetime). The bowling ball will cause a large depression in the sheet, and marbles placed near to the bowling ball will roll towards it, as though they were attracted to it. The problem with this analogy is that the curvature of spacetime happens in every dimension, i.e. there is temporal as well as spatial curvature.

We are used to spacetime as the background for other physics, but it itself is the main dynamical entity in GR. It is modelled as a differentiable (and thus continuous) manifold $\mathcal{M}$ with a Lorentzian metric $g$ of signature $(-+++)$. The metric is a 2nd rank symmetric tensor field which describes the geometry of the spacetime, and is therefore the object we focus on. It is described by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor and $R$ is the Ricci scalar, both of which are functions of $g$ and its derivatives, $T_{\mu\nu}$ is the energy momentum tensor for the spacetime (which is conserved $\nabla_\mu T^{\mu\nu}$), $\Lambda$ is the cosmological constant, and $G$ is Newton's gravitational constant. If we move the cosmological constant term to the right hand side, where it can be interpreted as a non-zero vacuum energy density, this equation relates the spacetime curvature with the local energy and momentum. Note that throughout this paper, we will use units where the speed of light $c$ is equal to 1.

A spacetime point $x$, which is a position in space at a specific time, is called an *event*, and spacetime itself is the set of all events. The trajectory of an object through spacetime is known as its *world line*. Indeed it is impossible to stay at a fixed point in spacetime due the continual progression of time. No object can travel faster through space than at the speed of light, and as a result, only certain trajectories are permitted. For one event to influence another, a signal must be transmitted between them. Since nothing can travel faster than light, an event can only affect a future event if the time difference between them is sufficient for a light beam to travel between the two points. This is illustrated in figure 1.1 by the light cone structure.

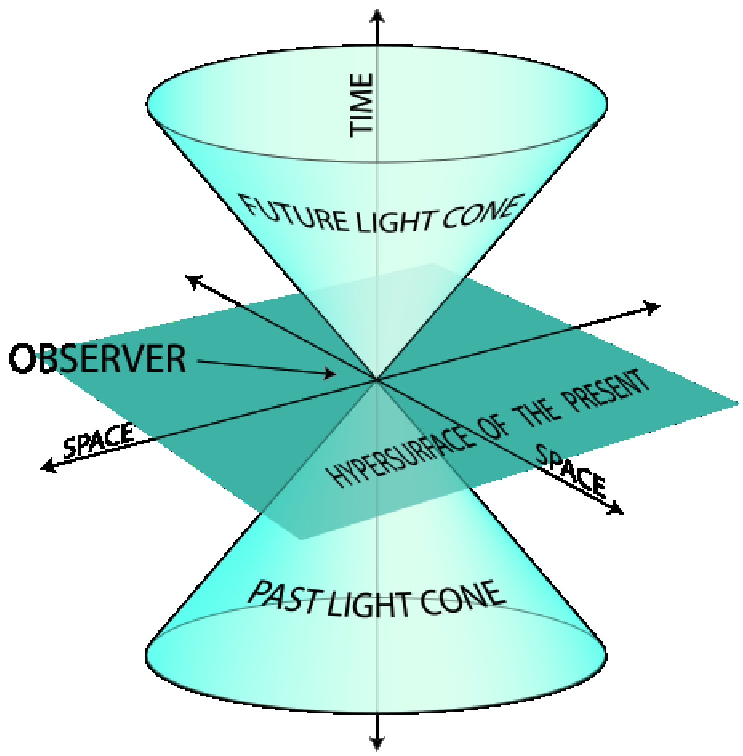


Figure 1.1: [3] Light cones which reflect the causal structure of spacetime. An event at the origin can only be influenced by events in the past light cone and in turn can only influence events in the future light cone. This light cone structure exists for every point in the spacetime.

If an event $y$ lies on or inside the future light cone of $x$, then $y$ is in the *causal future* of $x$: $y \in \mathcal{J}^+(x)$. Similarly, if $y$ lies on or inside the past light cone of $x$, then $y$ is in the *causal past* of $x$: $y \in \mathcal{J}^-(x)$. If $y$ lies on or inside the light cones of $x$ or vice versa, the points are *timelike* or *null* related respectively, otherwise they are *spacelike* related. The causal relations of all the points in the spacetime (i.e. which points are in each others causal past/future) are collectively referred to as the *causal structure*. In Lorentzian spacetimes, only timelike or null trajectories are permitted.

Usually in GR, one starts with the metric $g_{\mu\nu}$ as a solution to equation (1.1), and from the light cones of the metric the casual structure of the spacetime is determined. However, this seems somewhat counterintuitive. Causality is a notion with which we are very familiar from experience: forget to set your alarm clock and you will miss the bus. The causal structure of spacetime seems then to be more physical concept than its metric tensor. Indeed, it is possible to obtain almost all of the metric from the causal structure. Two metrics $g_1$ and $g_2$ which are related by a conformal factor $g_1 = f^2 g_2$, where $f$ is some non-zero function everywhere on the manifold, describe spacetimes which have the same causal structure, and the converse is also true. It has been shown [4][5] that the causal structure determines the metric up to an unknown conformal factor. Since in $3+1$ spacetime dimensions the metric is a 2nd rank $4 \times 4$ symmetrical tensor field, it consists of ten spacetime functions, once of which is the conformal factor, therefore we say that causal structure is nine tenths of the metric.

Clearly the remaining conformal factor is a measure of scale: fixing the spacetime volume of a region R, given by $\int_R \sqrt{-\det(g)}\,\mathrm{d}^4x$, is enough to determine the value of $f$. Since this volume information is required to complete the metric, it is not possible to describe GR solely as a theory of causal structure. When formulating any quantum theory, it is necessary to decide which aspects of the classical theory can be altered or discarded, and which must be kept. Since causality is such a fundamental concept, we will choose not abandon this approach centred on causal structure, but instead introduce a new concept.

## 1.2 Discrete spacetime

One possible resolution is that we model spacetime not as a continuous differentiable manifold, but a discrete one. In his inaugural lecture [6], Riemann stated:

> The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from

> outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.

Although Riemann was concerned with space and not spacetime, the idea is the same: the part of the metric which must be externally imposed on a continuous manifold is contained within a discrete manifold, where the volume of a region may be obtained by counting the number of elements in the region. This is impossible for a continuous manifold which has uncountably many elements.

In seeking a theory of quantum gravity grounded in causal structure, if we consider discrete spacetimes, the volume information required to complete the metric can be found within the spacetime: the volume of a region is equal to the number of elements in said region, and can be retrieved simply by counting! This determines the conformal factor, the remaining tenth of the metric. This idea is commonly expressed in the phrase attributed to Rafael Sorkin [7]:

$$\text{Order} + \text{Number} = \text{Geometry} \tag{1.2}$$

where order refers to the causal order, and number to the number of elements, i.e. the volume.

The idea of discrete spacetime might sound bizarre, but as we will see, it produces interesting results. Indeed, why should we suppose that spacetime continues to be continuous on ever smaller scales? From the the time when GR became widely accepted, it has been the general consensus that spacetime is the background in which physics itself takes place, and hence most theories are 'set', so to speak, in continuous spacetime. There are several issues in modern physics which might be traced to the continuum in which it is based. For example, in quantum field theory (QFT) calculations are made using path integrals, in which one integrates over all possible outcomes of a given process. Since in the continuum there are an infinite number of outcomes to most processes, QFT is plagued with infinities. It is therefore necessary use the technique of renormalisation to avoid these infinities. An in depth discussion of QFT and renormalisation can be found in [9]. Renormalisation is widely accepted because it provides extremely accurate results, but there is a general dissatisfaction with the technique. Pioneer of Richard Feynman said [10]

> The shell game that we play ... is technically called 'renormalization'. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate.

Hopefully this comment by one of its creators indicates the general feeling towards renormalisation. One method, cutoff renormalisation, involves cutting off an integral and ignoring contributions to it from above some finite energy. This can also be interpreted as introducing a minimum scale. In other words, the infinities are avoided by ignoring what happens on scales below the cutoff scale. This is consistent with the idea of spacetime becoming discrete at this minimum scale.

There is also the problem of curvature singularities in GR. In many instances, for example close to the centre of a black hole or immediately after the big bang, at around the Planck scale, the spacetime curvature becomes infinite. It is in these regions that we say that GR 'breaks down'. Since black holes are *the* quintessential quantum gravitational object (extremely massive and extremely small) any quantum gravity theory had better be able to fill in the gaps left by GR. Indeed, when attempting to directly count the degrees of freedom of the horizon of a black hole, one comes to the conclusion that the entropy of the black hole $S_{BH} = \infty$ [8]. However Bekenstein and Hawking showed [2] that that the entropy is finite

$$S_{BH} = \frac{A}{4{l_P}^2} \tag{1.3}$$

where $A$ is the area of the horizon, and $l_P = \sqrt{G\hbar}$ is the Planck length. This finiteness is one of the biggest motivations for a theory of discrete spacetime.

So, at this point we introduce the causal set: a discrete alternative to the Lorentzian manifold of GR. A causal set is a set of spacetime events with an order relation between each element and all other elements, i.e. the causal order. From only discreteness and causality, causal set theory [11][12][13] aims to reproduce all the macroscopic features, the metric, topology, differentiable structure and tensor fields of the continuum manifold as an approximation to the underlying discrete structure.

## 1.3 Kinematics

Mathematically speaking, a causal set is a type of partially ordered set or poset. It is a set $C$ together with a binary order relation $\preceq$ called precedes: If $x \preceq y$ then $x$ is in the causal past of $y$. Using the common genealogical convention, $x$ is an *ancestor* of $y$ and $y$ is a *descendant* of $x$. This paper will use the convention that $\preceq$ is reflexive, i.e. $x \preceq x$. This allows the use of the notation $\prec$ when discussing unambiguously distinct elements, i.e. if $x \prec y$ then $y \neq x$ is implied.

The order relation $\preceq$ satisfies the following three axioms, which hold $\forall x, y, z \in \mathcal{C}$:

1. **Transitivity**: If $x$ is in the causal past of $y$, and $y$ is in the causal past of $z$, the trivially $x$ is the the casual past of $z$

$$\text{If } x \preceq y \text{ and } y \preceq z \text{ then } x \preceq z \tag{1.4}$$

2. **Acyclicity**: No closed timelike curves

$$\text{If } x \preceq y \text{ and } y \preceq x \text{ then } x = y \tag{1.5}$$

3. **Local finiteness**:

$$A(x, y) \equiv \{z \mid x \preceq z \preceq y\} \text{ is finite} \tag{1.6}$$

where $A(x, y)$ is called an Alexandroff set.

Axioms (1.4) and (1.5) state that $C$ is a partially ordered set, and (1.6) defines the discreteness. It is important to reiterate that this is all there is to causal sets. A set of discrete spacetime elements and their causal relations. Any other structures related to the continuum spacetime manifold, which is treated as a macroscopic approximation, come from the underlying causal set. We say that spacetime *is* a causal set.

## 1.4 Causal sets from the continuum

Since spacetime macroscopically appears to be a Lorentzian manifold $(\mathcal{M}, g)$, given $\mathcal{M}$, how can one find a causal set $\mathcal{C}$ to which $\mathcal{M}$ is an approximation. An *embedding* of $\mathcal{C}$ into $\mathcal{M}$ is a map which assigns to every element of $\mathcal{C}$ a point in $\mathcal{M}$, in a way that preserves the causal order. Specifically, the ordering of $\mathcal{C}$ is the causal order of $\mathcal{M}$: $e_i \preceq e_j$ when $p_i \in \mathcal{J}^-(p_j)$. The procedure used is referred to as sprinkling. Points are sprinkled randomly into $\mathcal{M}$ via a Poisson process. The probability of sprinkling $n$ elements into a volume $V$ is

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \tag{1.7}$$

where $\rho$ is the density of the sprinkling. This density is expected to be if the order of the Planck density i.e. unit density in fundamental units. Through this procedure, $\rho V$ elements are sprinkled on average into volume $V$, in keeping with the idea that the number of elements of $\mathcal{C}$ is proportional to the spacetime volume.

The fluctuations are of the order $\sqrt{\rho V}$, and become negligible as $V$ becomes large. There is of course the possibility of voids, regions empty of points, but it can be shown [14] that the probability at least one nuclear sized void exists in the observable universe is less than $10^{252} \times e^{-72}$. This number is small enough that the prefactor has little effect on its value.

Another property that causal set theory retains from classical gravity is Lorentz invariance, and as a result the sprinkling is necessarily random since spacetime volume is an invariant. If the points were distributed in a regular lattice in one frame, they would become highly irregular in a boosted frame. Since the number of points in a region is

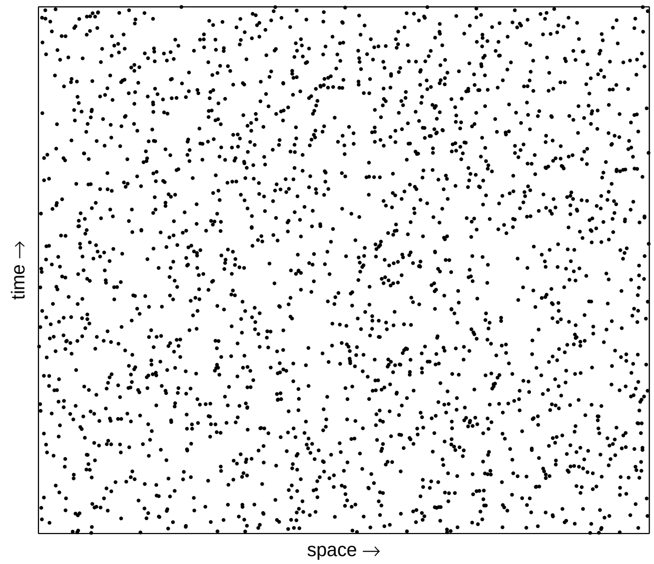


Figure 1.2: Example of a sprinkling into a portion of $1+1$D Minkowski spacetime.

proportional to the volume of the region this would violate Lorentz invariance. The only way to circumvent this problem is by using a random lattice.

$(\mathcal{M}, g)$ is said to approximate $\mathcal{C}$ provided that, with a relatively high probability, $\mathcal{C}$ could have come from sprinkling $(\mathcal{M}, g)$. Note that the sprinkling is not unique. There are an infinite number of possible sprinklings which would approximate as $\mathcal{M}$. The casual set theory hauptvermutung, or fundamental conjecture, states that if two manifolds are good approximations of $\mathcal{C}$ then they should be similar. What is meant by similar is debated, making it difficult to prove this conjecture, although it has been proved for $\rho \longrightarrow \infty$ [15], and there has been more recent progress [16].

Although a causal set $C$ may be obtained from a manifold $(\mathcal{M}, g)$ through this procedure, it is important to note that not every causal set can be embedded in a manifold. It may be required to focus on causal sets which are 'manifold-like', but again this is a hard quality to identify. There are techniques for determining the dimension of a causal set, but this only applies to causal sets which conform to the notion of having a dimensionality. There is the technique of *coarse-graining* [17] which removes some elements to produce a manifold-like causal set from one which is non manifold-like.

Before moving onto the focus of this paper, phenomenology, there is one other consideration to make. It is often stated in literature on causal sets, as above, that 'spacetime *is* a causal set', but of course this is an oversimplification. It is very useful to think of a manifold as an approximation to a single causal set, since it becomes easy to understand the key feature of the theory: discreteness. However since we want a quantum theory of gravity, it will most likely have more features common to quantum theories. For example the statement might be that spacetime *is* a superposition of or path integral over all causal sets which can be embedded into the approximating spacetime manifold. There might be some contributions from causal sets which are not even manifold-like. Which causal sets contribute, and how they are put together into what *is* the spacetime are questions which do not yet have answers, and which we will ignore for now.

# Chapter 2

# Swerves in Minkowski spacetime

To any physical theory, there are three aspects: kinematics, dynamics, and phenomenology. The kinematics of causal set are well defined and were discussed in section 1.3. The dynamics of causal set theory have not yet been fully established, although the classical sequential growth dynamics model [18] is quite promising, and until more experimental evidence comes to light, progress is slow. It is possible however to move on to phenomenology to try and develop some models to see what observable effects a discrete spacetime might produce. In this paper the focus is on the phenomena of swerves [14].

## 2.1 Swerves

In the continuum, massive particles follow timelike geodesics. In a discrete spacetime however there is no notion of a straight line. Since discrete spacetime is locally finite, there are only a finite number of paths a free massive particle can take, forcing it to deviate from its classical geodesic path.

Consider a massive particle moving freely in Minkowski space $\mathbb{M}^4$, or rather, propagating in a causal set $\mathcal{C}$ created by sprinkling into $\mathbb{M}^4$. This propagation can be thought of as the particle moving in discrete steps along a sequence of elements $e_i \in \mathcal{C}$ called a *chain* (The notation $e_n$ will be used interchangeably to refer to the element of $\mathcal{C}$ and the corresponding point sprinkled in $\mathbb{M}^4$). Classically, the position and momentum of a particle are sufficient to determine its subsequent motion. Analogously, is it possible to determine the trajectory of particle situated at $e_n$ with momentum $p_n$? A massive particle will follow a timelike geodesic, so it will move to some element $e_{n+1}$ such that $e_n \preceq e_{n+1}$. The assumption is made that only a certain proper time $\tau_f$ into the past of a particle is relevant to its future path. Although this introduces an element of nonlocality, it is necessary in order

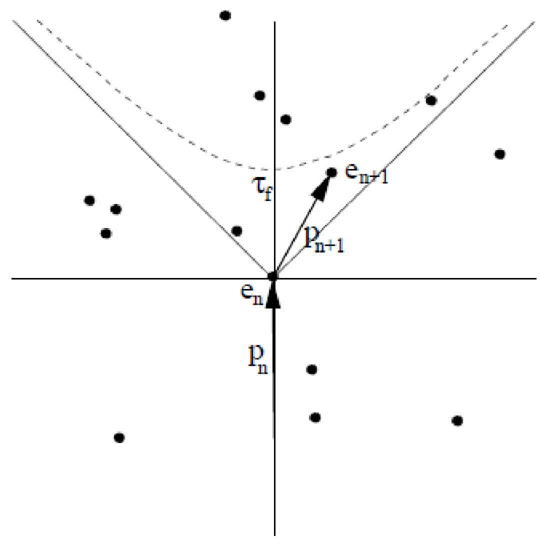


Figure 2.1: A section of 1+1D Minkowski space in the rest frame of the particle, where time is the vertical axis and space the horizontal. The particle arrived at $e_n$ with momentum $p_n$ along the time axis (due to the choice of frame). The dotted line is the hyperbola of points a proper time $\tau_f$ from $e_n$. The new momentum $p_{n+1}$ at $e_{n+1}$ is proportional to the vector from $e_n$ to $e_{n+1}$, and $e_{n+1}$ is chosen to minimise $|p_n - p_{n+1}|$. The size of $\tau_f$ is exaggerated for clarity.

to proceed. $\tau_f$ will be called the forgetting time. Thus, $e_{n+1}$ must be chosen from the elements $e_i$ within $\tau_f$ of the causal future of $e_n$, as illustrated in figure 2.1.

The procedure for constructing trajectories is as follows: starting with $e_n$ and $p_n$, choose a point $e_{n+1}$ from the region less than $\tau_f$ into the causal future of $e_n$ (note that spacetime volume of this region is infinite, so the existence of such a point is guaranteed by the Poisson distribution (1.7) and the local finiteness condition (1.6)). The new momentum $p_n + 1$ is proportional to the vector from $e_n$ to $e_{n+1}$. $e_{n+1}$ is chosen so that $|p_n - p_{n+1}|$ is minimised, for example in figure 2.1 $e_{n+1}$ is chosen so that $p_{n+1}$ is as close to vertical as possible. Repeat this process starting with $e_{n+1}$ and $p_{n+1}$. Continuing in this manner will inductively generate the trajectory of the particle.

In this model, the trajectory remains close to a timelike geodesic (i.e. straight line) for scales of $\tau_f$ and smaller, but will deviate from this path on scales $\gg \tau_f$, due to the fluctuations in momentum, which could potentially grow very large. The dynamics are approximately Markovian, i.e. the future state depends only upon the present state, not on the sequence of events that preceded it. This is true provided that $\tau_f$ is sufficiently small. It must also be larger than the scale of discreteness of course. There are a large range of values between the planck scale and 'sufficiently small', but there is a tradeoff. If $\tau_f$ is too small, the swerves will be very violent, but the larger $\tau_f$ is, the more nonlocal the swerves become.

There are a few problems with this model, one of which being that it is not really intrinsic to the causal set. The process is one which takes place in the $\mathbb{M}^4$, not in $\mathcal{C}$. One can attempt to build models which are more intrinsic to $\mathcal{C}$, which will be discussed in the following section. There are other problems, the particle is treated as point-like and the motion is deterministic and not quantum. However, as is usually the case in phenomenology, the point is to produce a simple model which will accent one or two aspects of the theory to illuminate what physical phenomena might be attributed to those aspects.

## 2.2 Intrinsic models of swerves

Before discussing specific models, some terms related to causal set theory must be defined:

- A *link* is an irreducible relation, i.e. for two distinct elements $x$ and $y$, $x \prec y$ and $\nexists\, z$ such that $x \prec z \prec y$.
- A *chain* is a totally ordered subset of $\mathcal{C}$. An $n$ *chain* is a chain with $n$ elements and its *length* is $n-1$, the number of links.
- A *longest chain* between two elements $x$ and $y$ is a chain whose length is maximal amongst chains between those two elements. There may be multiple longest chains for any $x$ and $y$. The length of a longest chain is denoted $d(x,y)$. If there is a link between $x$ and $y$, then $d(x,y)=1$.
- A *path* is a chain consisting entirely of links, i.e. a set of elements $x \prec y \prec z \prec ...$ such that $d(x,y)=1$, $d(y,z)=1$, ...

Notice that the length of longest chain automatically satisfies the triangle inequality of Minkowski space. For $x \prec y \prec z$:

$$d(x,z) \geq d(x,y) + d(y,z) \tag{2.1}$$

since the longest chain between $x$ and $z$ only cannot be shorter than the longest chain via $y$. In classical gravity, a geodesic between two points is the maximal curve between those two points. The closest approximation to a geodesic in a causal set is therefore a longest chain [19].

We will now discuss two models of swerves [20] which do not rely on notions from the manifold, and are more intrinsic to the causal set.

### 2.2.1 Model 1

Since we are now working strictly within the causal set, the notion of the forgetting time $\tau_f$ must be replaced the forgetting number $n_f$, an integer $\gg 1$. The trajectory of the particle is a chain $...e_{n-2} \prec e_{n-1} \prec e_n \prec e_{n+1}...$ which is constructed according to the following rules [20]:

Starting with a particle trajectory $...e_{n-1}$, $e_n$ the next element $e_{n+1}$ is chosen which satisfies

- $d(e_{n-1}, e_n) = n_f$
- $d(e_{n-1}, e_{n+1}) = 2n_f$.

A unique $e_{n+1}$ is not guaranteed by this model, however there will almost surely be a finite number, one of which can be chosen at random. This is a Markov process of order 2.

For large distances in sprinklings into Minkowski spacetime, it has been shown[19] that $d(x, y) \approx \alpha T$, where $T$ is the proper time between $x$ and $y$, and $\alpha$ is a constant which depends on the dimension. In the same paper it is shown that, for large $n_f$, $e_{n+1}$ should lie close to the hyperboloid of points a proper distance $n_f/\alpha$ from $e_n$ and the hyperboloid of points proper distance $2n_f/\alpha$ from $e_{n-1}$. In this case the trajectory should remain approximately straight, despite swerving slightly.

One can think of the trajectory either as consisting only of the elements $...e_{n-1}, e_n, e_{n-1}...$, or as the whole chain, i.e. $...e_{n-1}, e_n, e_{n-1}...$ *and* the $n_f - 1$ elements in the longest chain between each $e_i$ and $e_{i+1}$. Without the second condition, each segment would be geodesic, but not the whole chain. Since second condition guarantees that the chain between $e_{n-1}$ and $e_{n+1}$ is also a longest chain of length $2n_f$, it is approximately geodesic over segment between $e_{n-1}$ and $e_{n+1}$. This spreads the geodesic property along the trajectory, although strictly speaking any longer segment, e.g. between $e_{n-1}$ and $e_{n+2}$, is not necessarily a longest chain. Possible variations of this model involve varying the forgetting number at each step, this is explored later.

### 2.2.2 Model 2

In this model [20], the trajectory is constructed as a path, i.e. a chain consisting only of links. Starting with $...e_{n-n_f}, e_{n-n_f+1}, ..., e_{n-1}$, the point $e_n$ is chosen according to the following rules:

- $d(e_{n-1}, e_n) = 1$
- $d(e_{n-n_f}, e_n) + ... + d(e_{n-2}, e_n) + d(e_{n-1}, e_n)$ is minimised.

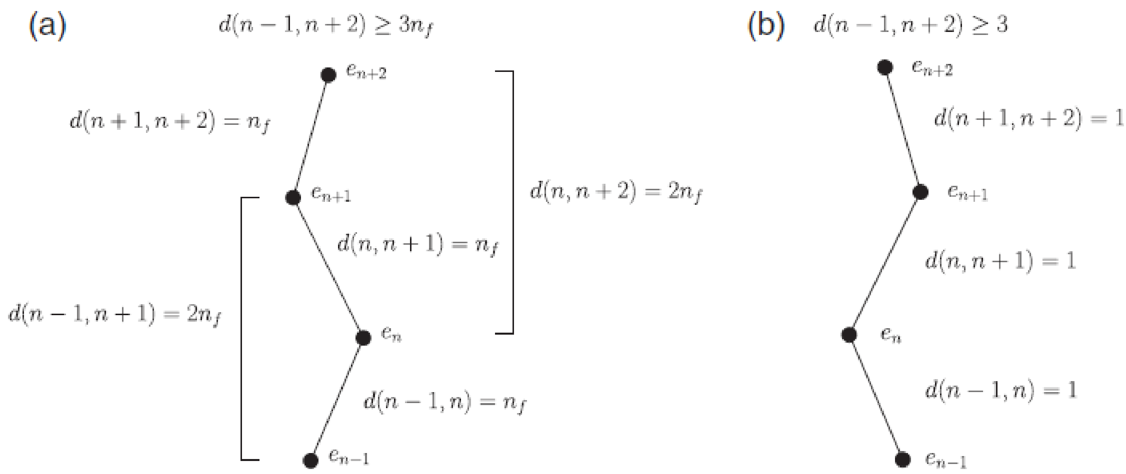


Figure 2.2: Trajectories constructed according to (a)model 1 and (b)model 2

As in model 1, a unique $e_n$ is not guaranteed, and one is chosen at random in such cases. In the event that the past trajectory has fewer than $n_f$ elements, the minimisation is done over the available elements.

Since the trajectory is a path, there must be a chain of length $n_f$ between $e_{n-n_f}$ and $e_n$, thus $d(e_{n-n_f}, e_n) \geq n_f$. The sum in (2.2.2) must be over all lengths in order to distribute the geodesic property along the path. If we had required $d(e_{n-n_f}, e_n)$ be minimised, the path would have been geodesic between $e_{n-n_f}$ and $e_n$, then between $e_n$ and $e_{n+n_f}$ and so on, but not between say, $e_{n-1}$ and $e_{n+1}$ for example.

## 2.3 Diffusion equation

Since the swerving process occurs on such small scales, it can be described macroscopically by a diffusion equation. The diffusion takes place in the space $\mathbb{H}^3 \times (M)^4$, where $\mathbb{H}^3$ is the mass shell and $(M)^4$ is Minkowski spacetime, because of the diffusion in momentum and spacetime respectively, and the diffusion evolves in the proper time.

We have a scalar probability distribution $\rho \equiv \rho(p^\nu, x^\mu; \tau)$ on $\mathbb{H}^3 \times (M)^4$, a function of momentum $p^\nu$, spacetime position $x^\mu$ and proper time $\tau$. Note that although the full 4-momentum is written as an argument, it only has 3 independent components, the 3-momentum, since we are on the mass shell. In [21] a method of describing stochastic evolution on a maniold of states is set up, and using this together with requirement of

Lorentz invariance, we arrive at the following equation:

$$\frac{\partial \rho}{\partial \tau} = k\nabla_p^2 \rho - \frac{1}{m} p^\mu \frac{\partial}{\partial x^\mu} \rho \tag{2.2}$$

where $\nabla_p^2$ is the Laplacian on $\mathbb{H}^3$, $m$ is the mass of the particle, and $k$ is a constant, which will depend on the parameters of the swerving process, namely the forgetting time $\tau_f$. A detailed derivation of this equation is given in [20].

This equation is the unique Poincaré invariant, Markovian, relativistically causal diffusion law in vacuum with a positive definite probability density $\rho > 0$. When the $m$ is constant, there is a single free parameter, $k$. As a result of these uniqueness properties, this equation can be used for any Lorentz invariant, Markovian, causal microscopic model. So, even if the models described above are incorrect in detail, the macroscopic description of swerves is robust as a phenomenological model of the effects of a Lorentz invariant 'uncertainty' in spacetime.

It is shown in [14] that there is an upper bound placed on $k$. This is due to the fact that the momentum diffusion described in equation (2.2), supposing the particles were protons, would cause hydrogen gas to heat up spontaneously. If we assume that a heating rate of a millionth of a degree or more would have already been detected, then we require that $k \leq 10^-56kg^2m^2s^{-3}$. It was hoped that high energy cosmic rays [22] might be shown to be an effect of swerves, but this was shown to be inconsistent with the upper bound on $k$ set in the laboratory. However, this may be resolved by developing a more sophisticated model. It was assumed that $k$ depends only on $\tau_f$, but it is possible that it could depend on other local quantities, for example temperature and pressure. A model might be developed where the particles are treated quantum mechanically as finite wave packets. In this paper we will look at developing a new model of swerves grounded in expanding spacetime.

# Chapter 3

# Swerves in Friedmann-Robertson-Walker spacetime

All the models of swerves in the previous chapter are based in Minkowski space, but little work has been done to describe the phenomenon in more exotic spacetimes, which may be more physically relevant. The effect of swerves is thought to be small, but it may become more apparent on larger scales. For this reason, it is important that we try and develop a description of swerves in a cosmological setting, as particles travelling on cosmic scales will have much more time to swerve. The standard model of cosmology is Friedmann-Robertson-Walker spacetime (FRW). In the following section some important results will be introduced, but a more complete and detailed introduction to cosmology and FRW spacetime is given in [23].

## 3.1 FRW spacetime

FRW spacetime is a way of modelling an expanding universe and, by tracing the expansion back in time, is the model used to describe the big bang. The FRW metric is a homogeneous and isotropic solution to the Einstein equations (1.1) for which the spatial component is time-dependent. In the most general form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{3.1}$$

where a(t) is called the scale factor, and describes the expansion. $k$ is a constant which describes the curvature of the 3-dimensional space: $k > 0$ corresponds to positive curvature (elliptical), $k < 0$ corresponds to negative curvature (hyperbolic), and $k = 0$ describes flat (Euclidean) space. These coordinates are referred to as comoving coordinates, since they move along with the expansion of the universe, i.e. the Hubble flow.

In this model the energy momentum tensor is homogeneous and isotropic. In analogy with fluid dynamics, where a fluid composed of molecules is treated as continuous, the assumption is made that on large scales the universe is approximately a perfect fluid, wherein the molecules are galaxies. In the cosmological rest frame, we have:

$$T_{tt} = \rho(t) \text{ and } T_{ij} = P(t) g_{ij} \tag{3.2}$$

where $\rho$ is the energy density, $P$ is the pressure, and $g_{ij}$ is the spatial part of the metric (including the scale factor $a^2(t)$).

Using (3.1) and (3.2), the solutions to the Einstein equations (1.1) are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{3.3}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi G P - \frac{k}{a^2} + \Lambda. \tag{3.4}$$

These are known as the Friedmann equations, and they can be rearranged to the form:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \tag{3.5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + P) + \frac{\Lambda}{3}. \tag{3.6}$$

The first equation is simply a statement of conservation of the energy-momentum, i.e. $\nabla_\mu T^{\mu\nu} = 0$. The second equation describes the acceleration of the expansion of the universe. From this it is clear that a positive cosmological constant increases the rate of expansion, and the energy density and pressure, which are positive definite, decrease the rate of expansion. The latter is a result of gravitational attraction of the matter in the universe slowing down the expansion.

The FRW model is far from perfect, since the matter content of the universe is not really a continuous fluid, it is granularly distributed with multitudes of empty space in between. Indeed, if the universe was truly homogeneous and isotropic, there would be no galaxies or planets, which were only able to form due to inhomogeneities in early universe. However the model has been extremely useful in cosmology, and the initial inhomogeneities immediately following the big bang would not have been extreme, so there are many theories based on taking perturbations of FRW [24].

For simplicity in the following discussion, we will focus on flat FRW, where $k = 0$, with a zero cosmological constant, and use the Cartesian form of the metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{3.7}$$

Note that although we call this 'flat', it is only really spatially flat. The spacetime manifold itself is indeed curved.

## 3.2 Momentum evolution in FRW spacetime

In order to discuss swerves in FRW spacetime, we need to know about the timelike geodesics of a massive particle in that spacetime, specifically how the momentum of such particles evolves. Starting with the action for a massive particle:

$$S = \int \mathcal{L}\mathrm{d}\tau = \frac{1}{2m}\int \left( g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - m^2 \right) \mathrm{d}\tau \tag{3.8}$$

where $\mathcal{L}$ is the Lagrangian and $\tau$ is proper time along the geodesic. In FRW, this becomes

$$S = \frac{1}{2m}\int \left( -\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + a^2(t)\left( \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \right) - m^2 \right) \mathrm{d}\tau. \tag{3.9}$$

Since the spacetime is homogeneous and isotropic, we can choose coordinates where the motion of the particle is along the $x$ axis. Hence $\mathrm{d}y/\mathrm{d}\tau = \mathrm{d}z/\mathrm{d}\tau = 0$. In this case, the action becomes

$$S = \frac{1}{2m}\int \left( -\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + a^2(t)\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - m^2 \right) \mathrm{d}\tau. \tag{3.10}$$

The geodesic equations are simply the Euler-Lagrange equations of this action:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial(\partial x^\mu/\partial\tau)}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0. \tag{3.11}$$

Since there is no explicit dependence on $x$, we choose to calculate the corresponding Euler-Lagrange equation, which gives

$$a^2(t)\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\epsilon}{m} \tag{3.12}$$

where $\frac{\epsilon}{m}$ is a convenient constant term. Since the 4-momentum is defined as $p^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, we have

$$p(t) = \frac{\epsilon}{a^2(t)} \tag{3.13}$$

where $p = m\frac{\mathrm{d}x}{\mathrm{d}\tau}$ is the magnitude of the 3-momentum $|\mathbf{p}|$ since $\mathrm{d}y/\mathrm{d}\tau = \mathrm{d}z/\mathrm{d}\tau = 0$. When the space is expanding, i.e. the scale factor and hence distances are increasing, so momentum will decrease. This is simply cosmological redshift.

Although it is possible to calculate the Euler-Lagrange equation corresponding to $t$, it is easier to find the equation for $t$ using the fact that a timelike geodesic satisfies $\mathrm{d}s^2 = -\mathrm{d}\tau^2$. Expanding this gives

$$1 = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - a^2(t)\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \tag{3.14}$$

and hence the equation for $t$ is

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1 + a^2(t)\frac{p^2}{m^2}} \tag{3.15}$$

where we have taken the positive square root so that the time coordinate $t$ increases in the same direction as $\tau$. In the case where the particle is at rest $p = 0$, $t$ is equal to the proper time.

Understanding how the 3-momentum changes in the classical case is crucial to developing a model of swerves. In Minkowski space, the 3-momentum for a free particle is a constant, and the background spacetime is the same as the system evolves. Neither of these things are true in FRW. Since we are interested in measurement, we need to determine how the momentum evolves in the *local inertial frames* of the galaxies it is passing through, not the comoving cosmological frame as we just calculated. A local inertial frame is a system of coordinates in which the metric is approximately Minkowskian in a certain region.

Suppose the particle passes a galaxy at time $t_1$. Under the coordinate transformation

$$x' = a(t_1)x \tag{3.16}$$

$$t' = t \tag{3.17}$$

the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2 \tag{3.18}$$

$$= -\mathrm{d}t'^2 + \frac{a^2(t')}{a^2(t_1)}\mathrm{d}x'^2. \tag{3.19}$$

In these coordinates, at time $t_1$, the metric is simply the Minkowski metric. The momentum is

$$p'(t) = m\frac{\mathrm{d}x'}{\mathrm{d}\tau} \tag{3.20}$$

$$= a(t_1)m\frac{\mathrm{d}x}{\mathrm{d}\tau} \tag{3.21}$$

$$= a(t_1)p(t) \tag{3.22}$$

$$= \frac{a(t_1)\epsilon}{a^2(t)} \tag{3.23}$$

Suppose that at some later time $t_2$, the particle passes another galaxy. Its momentum in the local inertial coordinates of this galaxy, with metric

$$\mathrm{d}s^2 = -\mathrm{d}t''^2 + \frac{a^2(t'')}{a^2(t_2)}\mathrm{d}x''^2 \tag{3.24}$$

is given by

$$p''(t) = \frac{a(t_2)\epsilon}{a^2(t)}. \tag{3.25}$$

in a similar way. Notice that

$$a(t_1)p'(t_1) = \epsilon = a(t_2)p''(t_2) \tag{3.26}$$

so we have a relation between the 3-momenta of a particle as measured in successive galaxies:

$$p'' = \frac{a(t_1)}{a(t_2)}p'. \tag{3.27}$$

This relation will become extremely useful in the discrete spacetime model.

## 3.3 Diffusion equation

Before we try to build a discrete model, it will be helpful to express how the distribution of the comoving 3-momentum changes, like equation (2.2), in FRW spacetime without the effect of swerves. Consider a comoving box of volume $V_1$ in the 3-dimensional hypersurface at time $t_1$ containing $N$ particles with a distribution of momenta $p$ (in the rest frame of the galaxy). In the box the density of particles is some function $\rho(p, t_1)$. At a later time $t_2$, the volume of the box is $V_2$, related to $V_1$ via

$$\frac{V_1}{a^3(t_1)} = \frac{V_2}{a^3(t_2)} \tag{3.28}$$

Assuming that the same number of particles enter and exit the box, $N$ will remain the same, and since the average density $\approx$ number/volume, the new average density is

$$\rho(t_2) = \frac{N}{V_2} = \frac{N}{V_1}\frac{a^3(t_1)}{a^3(t_2)} = \frac{a^3(t_1)}{a^3(t_2)}\rho(t_1). \tag{3.29}$$

Distribution in space dilutes as we know due to expansion, but what about the momenta? A particle with momentum $p$ at $t_1$ will, according to equation (3.27), have momentum

$$p(t_2) = \frac{a(t_1)}{a(t_2)}p(t_1) \tag{3.30}$$

at time $t_2$. Since we have a distribution of momenta, consider the following: The number of particles with momentum in the range $\Delta p$ centred on $p$ at time $t_1$ is $\rho(p, t_1)\Delta p$. Consider these same particles at the later time $t_2$. According to (3.30) the particles have slowed down to momenta $\frac{a(t_1)}{a(t_2)}p$ in the range $\frac{a(t_1)}{a(t_2)}\Delta p$. The number of particles in the box is conserved, gives the relation

$$\rho(p, t_1)\Delta p = \rho\left(\frac{a(t_1)}{a(t_2)}p, t_2\right)\frac{a(t_1)}{a(t_2)}\Delta p. \tag{3.31}$$

If the system evolves for a small amount of time $\delta t \ll 1$, so that $t_1 = t$ and $t_2 = t + \delta t$, then (3.31) becomes

$$\rho(p, t)\Delta p = \rho\left(\frac{a(t)}{a(t+\delta t)}p, t+\delta t\right)\frac{a(t)}{a(t+\delta t)}\Delta p. \tag{3.32}$$

Taylor expanding

$$a(t+\delta t) = a(t) + \delta t\dot{a}(t) + O(\delta t^2) \tag{3.33}$$

and therefore, neglecting terms of second or higher order in $\delta t$, we have

$$\frac{a(t)}{a(t+\delta t)} \approx \frac{1}{1+\delta t\frac{\dot{a}(t)}{a(t)}} \approx 1 - \delta t\frac{\dot{a}(t)}{a(t)}. \tag{3.34}$$

The $\Delta p$s in (3.32) cancel, giving

$$\rho(p, t) = \rho\left(p - \delta t\frac{\dot{a}(t)}{a(t)}p, t+\delta t\right)\left(1 - \delta t\frac{\dot{a}(t)}{a(t)}\right). \tag{3.35}$$

Expanding first the $t$ argument of $\rho$:

$$\rho(p, t) = \rho\left(p - \delta t\frac{\dot{a}(t)}{a(t)}p, t\right)\left(1 - \delta t\frac{\dot{a}(t)}{a(t)}\right) \tag{3.36}$$

$$+ \delta t\frac{\partial\rho}{\partial t}\left(p - \delta t\frac{\dot{a}(t)}{a(t)}p, t\right) \tag{3.37}$$

and then the $p$ argument

$$\rho(p,t) = \rho\left(p,t\right) - \delta t\frac{\dot{a}(t)}{a(t)}\rho\left(p,t\right) - \delta t\frac{\dot{a}(t)}{a(t)}p\frac{\partial\rho\left(p,t\right)}{\partial p} + \delta t\frac{\partial\rho\left(p,t\right)}{\partial t}. \tag{3.38}$$

Rearranging this, and cancelling the $\delta t$s, gives

$$\frac{\partial\rho}{\partial t} = H\rho + Hp\frac{\partial\rho}{\partial p} \tag{3.39}$$

where $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the so called Hubble parameter. This is the equation for the evolution of the momentum distribution of particles in FRW without the effect of swerves.

At this stage, we may put forward a conjecture: Swerves in FRW spacetime may be described by a diffusion equation similar to equation (2.2) for swerves in Minkowski space but including terms to describe the effect of spacetime expansion. By adding the RHS of equation (3.39) to the RHS of (2.2), we have such an equation:

$$\frac{\partial\rho}{\partial\tau} = k\nabla_p^2\rho - \frac{1}{m}p^\mu\frac{\partial}{\partial x^\mu}\rho + H|\mathbf{p}|\frac{\partial\rho}{\partial|\mathbf{p}|} + H\rho \tag{3.40}$$

where we have emphasised that the $p$ in (3.39) is the magnitude of the 3-momentum. This equation is discussed further in [20].

There are two important scales involved in this problem: The forgetting time $\tau_f$ and the Hubble time $1/H$ (the age of the universe). While equation (3.40) is certainly a reasonable conjecture, and is good to fall back on, it is likely that it will fail at certain values of these scales. We expect it will be a good approximation when $1/H \gg \tau_f$, i.e. at late times in the universe, when the spacetime is not highly curved. At scales where $1/H \approx \tau_f$, i.e. at around the Planck time after the big bang however, this equation will fail, since quantum gravitational effects will dominate at these scales. Therefore it is necessary to to try a build model of swerves based in FRW and derive an equation directly from that model.

## 3.4 Modelling swerves in FRW spacetime

Suppose we have a causal set $\mathcal{C}$ obtained by sprinkling into an FRW spacetime manifold $\mathcal{M}$. The particle moves through the causal set in the same manner as section 2.1, along a chain of elements $e_i \in \mathcal{C}$. We want to be able to predict the future trajectory of the particle given some initial position $e_n$ and momentum $p_n$.

What might a sprinkling into FRW look like? As stated earlier, FRW coordinates are known as comoving coordinates since they move with the expansion of the universe. If two galaxies are situated in space at $\mathbf{x}_1$ and $\mathbf{x}_2$ in these coordinates, they will remain at

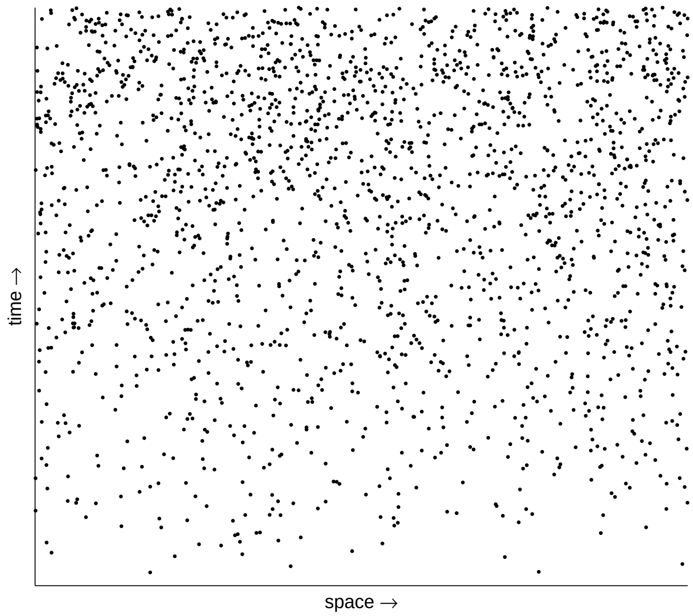


Figure 3.1: Example of sprinkling into a section $1+1$D FRW spacetime. As time increases, the sprinkling appears to become 'more dense'. This is not true however as the sprinkling is always done with uniform density. In these coordinates, spacetime volume is increasing, so the number of sprinkled elements also increases.

those spatial coordinates, however the distance $d$ between them will increase in time. In a sprinkling of elements of $C$ into the spacetime, the number of points sprinkled between the two galaxies will increase in time, since the number of elements is proportional to the spacetime volume, which is increasing in time. This is illustrated in Figure 3.1.

Naively, one might expect that swerves in FRW can be described in the same manner as in Minkowski spacetime: Choose the next element $e_{n+1}$ and define the new momentum as proportional to the vector from $e_n$ to $e_{n+1}$, choosing $e_{n+1}$ so that $|p_{n+1} - p_n|$. To understand why this will not work in FRW spacetime, we need to recall some concepts from differential geometry [25].

In a differentiable manifold $\mathcal{M}$, a vector at a point $p$ in the manifold exists in the tangent space at that point $T_p(\mathcal{M})$, which is like a copy of Minkowski spacetime at that point. When the manifold itself is Minkowski spacetime, the tangent space is the same at every point, and vectors in Minkowski spacetime can be treated as though they exist in the spacetime. This is why we can say that $p_{n+1}$ is proportional to the vector between $e_n$ and

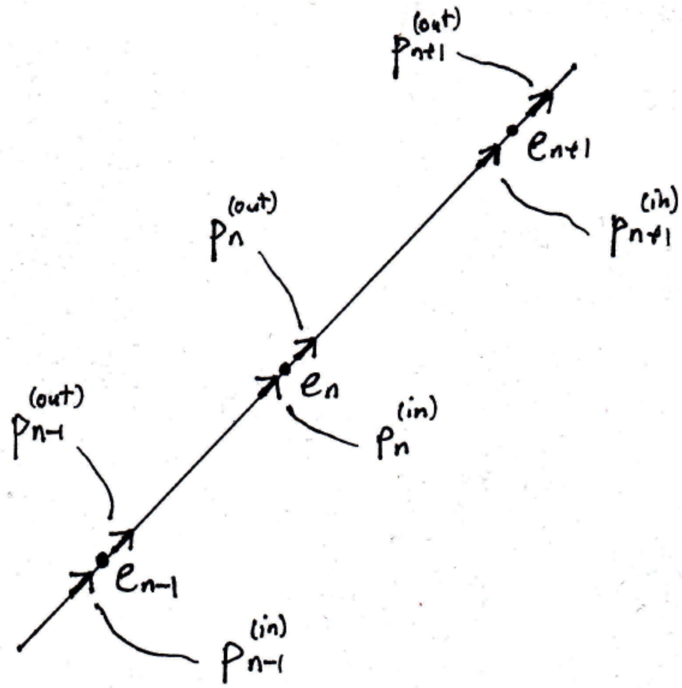


Figure 3.2: The classical trajectory of the particle. In this figure the 'elements' $e_i$ are simply points chosen at intervals along the trajectory of the particle, and do not represent the sprinkling of a causal set. The momenta at each point is in the local inertial coordinates for that point. This diagram illustrates what happens without swerves.

$e_{n+1}$, and we can talk about minimising the difference between $p_{n+1}$ and $p_n$, even though they exist at different points. However, in any spacetime other than Minkowski, i.e. FRW, this is not possible. Vectors at different points live in different tangent spaces, and cannot be compared directly. We will have to develop a model which takes this into account.

In section 3.2, we found a relation (3.27) between the momentum at successive points as measured by observers in the local inertial frame at each point. We will refrain from thinking about swerves for a moment, and just consider a particle travelling from point to point along its classical trajectory, as illustrated in Figure 3.4. Suppose the particle arrives at $e_n = (t_n, \mathbf{x}_n)$ with arrival momentum $p_n^{(\mathrm{in})}$. It will leave $e_n$ with the same momentum, the departure momentum

$$p_n^{(\mathrm{out})} = p_n^{(\mathrm{in})}. \tag{3.41}$$

The particle then travels to the point $e_{n+1}$, where it arrives with momentum $p_{n+1}^{(\mathrm{in})}$. According to (3.27)

$$p_{n+1}^{(\mathrm{in})} = \frac{a_n}{a_{n+1}} p_n^{(\mathrm{out})} \tag{3.42}$$

where $a_i = a(t_i)$. Since we have a way of expressing $p_{n+1}^{(\mathrm{in})}$ in terms of $p_n^{(\mathrm{out})}$, which exists in the same tangent space as $p_n^{(\mathrm{in})}$, the arrival momentum $p^{(\mathrm{in})}$ of the particle can be compared at successive points.

So, in a model with swerves, starting with the initial position $e_n$ and the arrival momentum $p_n^{(in)}$ at $e_n$, the particle will arrive at the next point $e_{n+1}$ with momentum $p_{n+1}^{(in)}$. Given $p_{n+1}^{(in)}$, we can calculate the departure momentum $p_n^{(out)}$ at $e_n$ using (3.42). The crucial change when reintroducing swerves is that this will not be equal to the arrival momentum at the same point:

$$p_n^{(\mathrm{out})} \neq p_n^{(in)}. \tag{3.43}$$

This is illustrated in figure Figure 3.4. In a similar procedure to section 2.1, the point $e_{n+1}$ is chosen from the region a proper time $\tau_f$ into the causal future of $e_n$. It is chosen so that the change in momentum is minimal. $p_n^{(in)}$ and $p_n^{(out)}$ are both expressed in the coordinate system with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{a^2(t)}{a^2(t_n)} \mathrm{d}\mathbf{x}^2 \tag{3.44}$$

At $t_n$, when both momenta are calculated, this is the Minkowski metric $\eta_{\mu\nu}$, and the magnitudes of the 4-momenta are

$$\eta_{\mu\nu} p_n^{(in)\mu} p_n^{(in)\nu} = \eta_{\mu\nu} p_n^{(out)\mu} p_n^{(out)\nu} = -m^2 \tag{3.45}$$

and we have the inequality

$$\eta_{\mu\nu} p_n^{(in)\mu} p_n^{(out)\nu} < -m^2. \tag{3.46}$$

The point $e_{n+1}$ is chosen so that $\eta_{\mu\nu} p_n^{(in)\mu} p_n^{(out)\nu}$ is maximised. This procedure is repeated with $e_{n+1}$ and $p_{n+1}^{(in)}$ and so on, and the trajectory is derived inductively. This is our model of swerves in FRW spacetime. Note that we have chosen to focus on $p^{(in)}$, the arrival momentum, but we could just as easily to use the departure momentum $p^{(out)}$. The former was chosen since the arrival momentum is what would be measured by an observer.

It has proven difficult to proceed any further with this model, and how to extract a diffusion equation remains unclear. We expect it will not be Lorentz invariant like (2.2), since there is clearly a preferred frame: the cosmological rest frame. It is difficult to even develop an intrinsic model in analogy with 2.2, since when working within the confines of the causal set there are very few parameters to manipulate that would seem to distinguish a model of FRW from (2.2.1) and (2.2.2).

One such parameter is the forgetting number $n_f$. Recall that the forgetting time is the amount of proper time into the particles past which is relevant to its future trajectory.

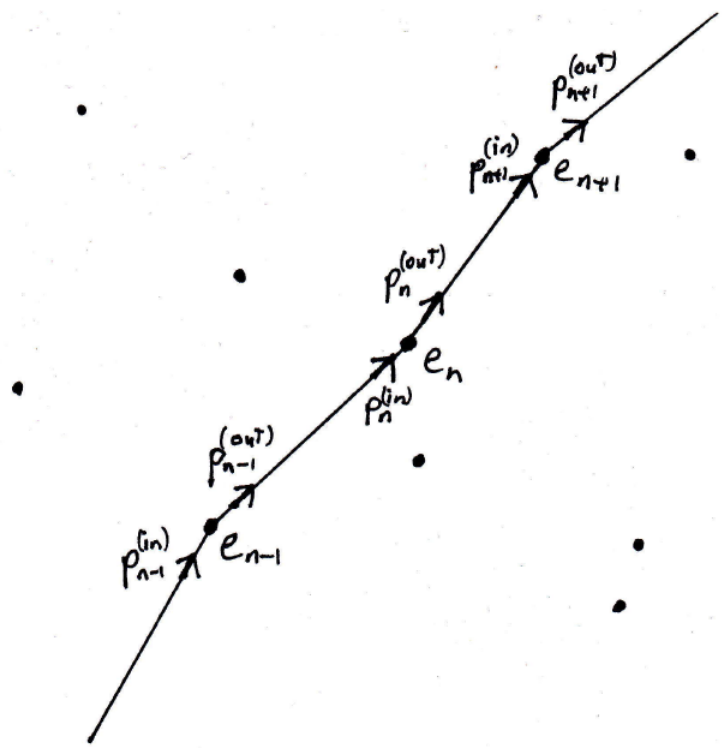


Figure 3.3: The trajectory of the particle in a sprinkling of a causal set into FRW spacetime. The particle leaves each point with a slightly different momentum than it arrived with.

FRW has a definite 'beginning', the big bang, and only as the particle travels along a sequence of elements of $C$ will those elements become physically relevant to it's future trajectory. We might consider a model where the forgetting number is increasing. The length of longest chain between $e_n$ and $e_{n+1}$ will be

$$d(e_n, e_{n+1}) = n_f(t_n) \tag{3.47}$$

where $n_f(t_n)$ is the forgetting number, a monotonically increasing function of time, where it is the forgetting number with which the particle *leaves* $e_n$ (this is a convention). This is complicated since time is now a discrete variable, and $n_f(t_n)$ must be integer valued, but a simple example might be something like

$$n_f(t_n) = m + n \tag{3.48}$$

where the constant $m$ is the forgetting number at some initial element $e_0$, so that $d(e_0, e_1) = m$. For example, suppose $e_0$ is the earliest element sprinkled into FRW spacetime, immediately after the big bang. A particle at this element would have no past trajectory, so $m = 0$.

In analogy with 2.2 the model, similar to (2.2.1), can be expressed as

- $d(e_n, e_{n+1}) = m + n$
- $d(e_{n-1}, e_{n+1}) = 2m + 2n - 1$.

So we require that the longest chain between $e_{n-1}$ and $e_{n+1}$ is equal to the sum of the longest chains between $e_{n-1}$ and $e_n$, and $e_n$ and $e_{n+1}$. The trajectory will be then be approximately geodesic along this segment.

While this model might have interesting properties, it is very flawed. Since the future trajectory depends on the number of steps already taken, the process is no longer Markovian, and is non-local. The reason the notion of forgetting time was introduced in the first place was because of the discreteness. It is impossible to determine the particles instantaneous momentum at a point, as it is in the continuum, so we had to resort to looking into the recent past of the particle. This introduces an element of non-locality as well, but is necessary in order to make progress. The level of non-locality in (3.4) is far too high however.

As stated earlier, in Minkowski spacetime, a geodesic in the continuum can be described by a longest chain in the causal set [19]. We have assumed that this is also true in FRW, but it would be shortsighted to assume that this is true for all spacetimes. It will presumably hold in regions of a curved spacetime where the curvature is low, i.e. close to flat Minkowski spacetime. However in regions where the spacetime is highly curved (and thus where quantum gravitational effects will be more noticeable), for example when $1/H \approx \tau_f$, a new notion of a causal set geodesic will have to be introduced.

# Chapter 4

# Discussion

## 4.1 The validity of FRW spacetime

The motivation for having a model of swerves grounded in FRW, aside from seeing how the swerves model translates to a simple example of a curved spacetime, is because we expect that the effect of swerves will be so small that a long time will be required for the deviation from the classical momentum of a particle to grow large enough to be observable. On these scales, the effect of the expansion of the universe will become relevant and must be taken into account. However, simply building in a model in FRW spacetime may not be sufficient for this purpose.

An important axiom in cosmology is the cosmological principle, which states that on a sufficiently large scale, the universe is homogenous and isotropic [26]. In FRW the matter is assumed to be a perfect, homogeneous and isotropic fluid which pervades the universe, as discussed earlier. In this simple model, the Einstein equations (1.1), 10 coupled, nonlinear, partial differential equations in 4 variables are reduced to 2 ordinary differential equations in one variable: the Friedmann equations (3.3) and (3.4). This assumption makes cosmological models much easier to understand, allowing great amounts of progress to be made, not to mention it is supported observationally by the homogeneity and isotropy of the cosmic microwave background (CMB) [27].

When dealing with an entity as large as a universe, as cosmologists do, and large clusters of galaxies can be thought of as 'small', it is not unreasonable to treat matter as a fluid. However, when we want to describe the motion of an individual particle, this assumption is not ideal. A particle travelling large distances through spacetime will certainly not experience homogeneity and isotropy, as it navigates galaxies, black holes, and huge expanses of empty space. Even at early times, when the universe was an extremely dense soup of particles and the assumption might be more accurate, the particle would not be able to propagate freely as it would interact with the other matter. It may be worthwhile to consider different models of the universe, more suited to describing particle motion, as candidates in which to build our model of swerves on the cosmic scale.

## 4.2 Alternatives to FRW

There are several alternatives to FRW in the field of cosmology which do not assume homogeneity [28]. There is for example the Lemaitre-Tolman-Bondi (LTB) model [29], [30], [31], which describes a spherically symmetric cloud of expanding or contracting dust, usually with a void at the centre, in which Earth is placed. Such models can accurately model our universe under the right conditions. There are also 'swiss cheese' models which consist of multiple regions of LTB or Schwarzschild spacetime embedded in an FRW background [32].

There are many such models, but one type we believe could be particularly useful for describing the propagation of a massive particle are Linquist Wheeler (LW) models [33], [34], [35]. In these models, the matter content of the universe is distributed as discrete islands. Space is covered by a lattice of cells containing equal mass $M$, which is in the centre of the cell. It is assumed that the gravitational influence of the surrounding cells is approximately spherically symmetric, therefore the inside the cells spacetime is Schwarzschild, with metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2m}{r}\right)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{4.1}$$

where $m$ is the central mass. The type of lattice and the number of cells depends on the properties of the space, for example in flat space, the only possible lattice is cubic, with an infinite number of cells. Note that LW models are not exact solutions to the Einstein equations, but rather approximate solutions, although it has been shown that a model with only two cells can be an exact solution of the Einstein equations (1.1) [36].

Dynamically, the cell boundaries move away from or towards their respective centres, which globally translates to either expansion or contraction. The lattice is built in a hyperspherical 3-space, and the radius of the hypersphere in an embedding in Euclidean 4-space acts like a scale factor $a_{LW}$, and satisfies the equation

$$\left(\frac{\dot{a}_{LW}}{a_{LW}}\right)^2 = \frac{2m}{a_{LW}^3 sin^3\psi} - \frac{1}{a_{LW}^2} \tag{4.2}$$

which has the same form as (3.3). $\psi$ is the angle subtended at the centre of the hypersphere between vectors in the Euclidean embedding 4-space connecting the centre of the hypersphere with the centre and spherical boundary of one cell. It is expected that a LW based model of swerves might provide more accurate than its FRW counterpart, since a mostly empty spacetime containing a lattice of localised masses is closer to reality than one filled with continuous matter.

## 4.3 Conclusions

While developing the phenomenology of causal set theory is perhaps not as important as finding the correct dynamics, having information about the potential observable effects of spacetime discreteness will be crucial in the development of the theory, not least because it is quite unique among quantum gravity theories. Of course without said dynamics, any discussions of phenomenology is necessarily heuristic, but it is certainly not without merit. For example, aside from the development of the model of swerves in Minkowski space [14][20] discussed in 2, it has also been possible to predict the cosmological constant $\Lambda$ successfully [17][37].

The work presented here has begun to set up how one might build a model of swerves in an expanding universe. Hopefully in the future it will be possible to fully develop the model for FRW and other spacetimes more complex than Minkowski, and perhaps gain an understanding of how geodesics and the trajectories of particles can be understood in the context of a causal set.

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