Five dimensional SUSY gauge theories in the context of M theory

Author:
Mohammed Hakeem

Supervisor:
Prof. Amihay Hanany

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Contents

1 Introduction

The intertwining relationship between brane dynamics and non-Abelian gauge theories has long been an intuitive approach to understanding supersymmetric gauge theories in several dimensions. A substantial amount of work has been done in the past decade in realizing supersymmetric field theories, in [9] three dimensional theories were formulated using *D3* branes stretched between *NS5* branes. Four dimensional theories were discussed in [10,11,12], the most relevant to this review being $N = 4$ [7], where brane configurations are realized in terms of M theory. In this dissertation we study brane configurations leading to $N = 1$ supersymmetric theories in five dimensions with gauge group $SU(N_c)$ and N_f hyper multiplets, the M theory realization is then described along with its relationship with $N = 4$ theory in five dimensions.

In section 2 we review the basic properties of five dimensional , $N = 1$ supersymmertic gauge theories with results established mainly by [1] and [2]. The general prepotential for an $N = 1$ supersymmertic gauge theory in 5d with N_c vector multiplets and N_f hypermultiplets is given[], along with the form it takes when the gauge group is chosen to be $SU(N_c)$ and the N_f hypermultiplets transform in the fundamental representation of $SU(N_c)$. The conditions on N_f, N_c and c_{cl} are shown along with the BPS spectrum of the theory.
In section 3 we introduce the concept of (p, q) webs in Type IIB string theory [3], starting with how a vertex can be formed from a *D5* brane ending on *NS5* by bending it into a $(1, 1)$ five brane and a general equation is demonstrated for the bending of an *NS5* when multiple *D5* ending on it . We then move on to (p, q) webs, describing how they can be built to represent $SU(N_c)$ gauge

theories. $SU(2)$ is used as an example to demonstrate how field theoretical results can be obtained from brane configurations. The concept of grid diagrams which are dual to the webs is also explained and how it is used to characterize different theories.

Section 4 starts with a brief description of how Type IIB string theory can be reduced from M theory compactified on a 2-torus [13].In this context (p, q) branes are M theory five branes wrapped around (p, q) cycles on the torus and brane configurations correspond to an *M5* on $\mathbb{R}^{1,3} \times \Sigma$,where Σ is a 2 dimensional surface described by a holomorphic curve [5]. We describe how curves for a simple vertex and the $SU(2)$ web are obtained. Grid diagrams are then used to write down curves for pure $SU(N_c)$ theories. At the end of the section we follow [6] in deriving a general curve for $SU(N_c)$ theories with N_f flavors, it is then put into a form where the 4d limit can be taken, and the resulting curve is in analogy with curves given in [8].

2 Field Theory content

We start by giving a brief discerption of the field theory, with results mainly established in [1],[2]. We consider $N = 1$ supersymmetry in five dimensions, which has eight supercharges. The shortest representations are the vector multiplet containing a vector field, a real scalar and fermions, plus a hyper multiplet containing four scalars and fermions.

The Lorentz group is broken to,

$$SO(1,9) \rightarrow SO(1,4) \times SO(3) \simeq SO(1,4) \times SU(2)_R \tag{2.0.1}$$

with $SU(2)_R$ acting as the global R-symmetry group.

Consider a configuration consisting a vector multiplet transforming in the adjoint representation of the gauge group G, and N_f hypermultiplets transforming in a representation R_f. The Coulomb branch is parameterized by the Cartan sub algebra of the scalars in the vector multiplet. Low energy dynamics are governed by the prepotential $\mathcal{F}$ which is restricted to be at most cubic[1], and is given buy [2],

$$\mathcal{F}(\phi) = \frac{1}{2g_0^2}\phi^i\phi_i + \frac{c_{cl}}{2}d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}(\sum_\alpha |\alpha_i\phi^i|^3 - \sum_f \sum_{w\in R_f} |w_i\phi^i + m_f| \tag{2.0.2}$$

with $\phi = \phi^i T_i$,T_i are the generators of G in the representation R_f, and $d_{ijk} = tr(T_i\{T_j, T_k\})$. The first sum is over the roots of G while the second is over the weights of R_f. The bare coupling is g_0 and the effective coupling which is the metric on the Coloumb branch is given by,

$$(\frac{1}{g_{eff}^2})_{ij} = \frac{\partial^2\mathcal{F}(\phi)}{\partial\phi_i\partial\phi} \tag{2.0.3}$$

c_{cl} is a quantized parameter related to the 5 dimensional Chern-Simons term which characterizes the theory. If we specialize to $G = SU(N_c)$ with N_f hypermultiplet in the fundamental representation of G, the moduli space is given by $\phi = \mathrm{diag}(a_1, a_2 \dots a_{N_c})$ with $\sum_i a_i = 0$, modulo the Weyl group which allows us to set $a_1 \geq a_2 \geq a_3 \dots \geq a_N$. In this case the prepotential simplifies to[3],

$$\mathcal{F} = \frac{1}{2g_0^2} \sum_i^{N_c} a_i^2 + \frac{1}{12}(2 \sum_{i>j}^{N_c} |a_i - a_j|^3 + 2c_{cl} \sum_i^{N_c} a_i^3 - N_f \sum_f^{f} \sum_i^{N_c} |a_i + m_f|^3) \quad (2.0.4)$$

In order to have non trivial fixed points for $N_c > 2$, N_c, N_f, and c_{cl} have to satisfy the following constraints [2],

$$\frac{1}{2} N_f + c_{cl} \in \mathbb{Z} \quad (2.0.5)$$

$$N_f + 2|c_{cl}| \leq 2N_c \quad (2.0.6)$$

A special case is when $N_c = 2$ where we have two pure gauge theories labeled by their $\mathbb{Z}$ theta angle, $d_{ijk} = 0$ and the number of flavors allowed is extended to $N_f < 7$. Along with electrically particles, Instanton are present In the BPS spectrum which is charged under the $U(1)_I$ global symmetries with instanton number I. The central charge is given by,

$$Z = (n_e + I)\phi + m_I \quad (2.0.7)$$

masses of BPS states are equal to their central charge, strings are magnetic monopoles with tensions,

$$T_m = (n_m)_i \frac{\partial \mathcal{F}}{\partial \phi_i} \quad (2.0.8)$$

3 Type IIB setup

We consider Five brane configurations of Type IIB in the context of $SL(2,\mathbb{Z})$, with *D5* branes being a (1,0) multiplet , charged under the RR sector 2-form a nd *NS5* branes being a (0,1) multiplet, charged under the NS sector 2-form. The *D5* branes have world volumes[4],

Brane	World volume	Tension
D5	$(x^0,x^1,x^2,x^3,x^4,x^6)$	$1/g_s l_s^6$
NS5	$(x^0,x^1,x^2,x^3,x^4,x^5)$	$1/g_s^2 l_s^6$

g_s being the Type IIB string coupling, and l_s the string length scale. Dimensions (x^0,x^1,x^2,x^3,x^4) are shared my both *NS5* and *D5*, interesting geometry lies in the (x^6,x^5) plane, where the branes appear as straight lines. The last three dimensions (x^7,x^8,x^8) represent the $SO(3)$ R-symmetry and play no role in the Coulomb branch and correspond to deformations in the Higgs branch.

(p,q) branes are formed as bound states of *D5* and *NS5* branes, their tension being[4],

$$T_{p,q} = |p + \tau q|\, T_{D5} \tag{3.0.9}$$

3.1 Vertices

A *D5* brane ending on an *NS5* brane looks like a point charge in one dimension which causes the *NS5* brane to bend in to the location,

$$x^6 = \frac{g_s}{2}(|x^5| + x^5) \tag{3.1.1}$$

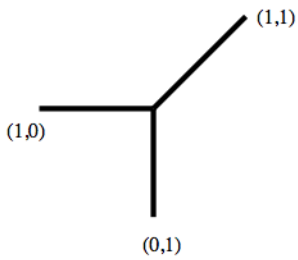
We compose Vertices as our building blocks for the webs to come, by charge conservation they satisfy,

$$\sum_i p_i = \sum_i q_i = 0 \tag{3.1.2}$$

Following [], a quarter or the supersymmetries are preserved if the branes are constrained to obey a the slope condition,

$$\Delta x_6 + i\Delta x_5 || p + \tau q \tag{3.1.3}$$

are horizontal while *NS5* branes are vertical, the simplest vertex (figure) can be viewed as a *D5* ending on an *NS5* brane,thus forming a (1,1) brane.



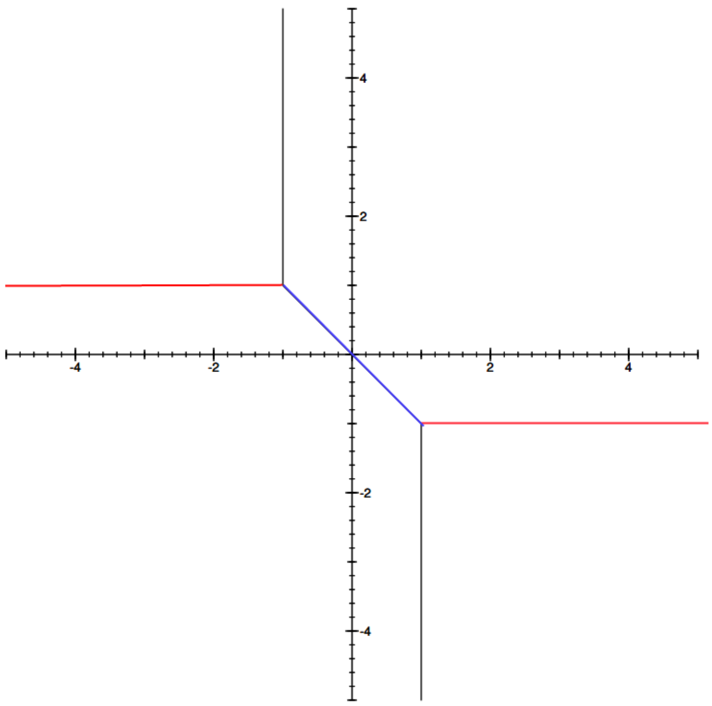
Figure[], vertex extended in the (x^6, x^5) plane.

3.2 Webs

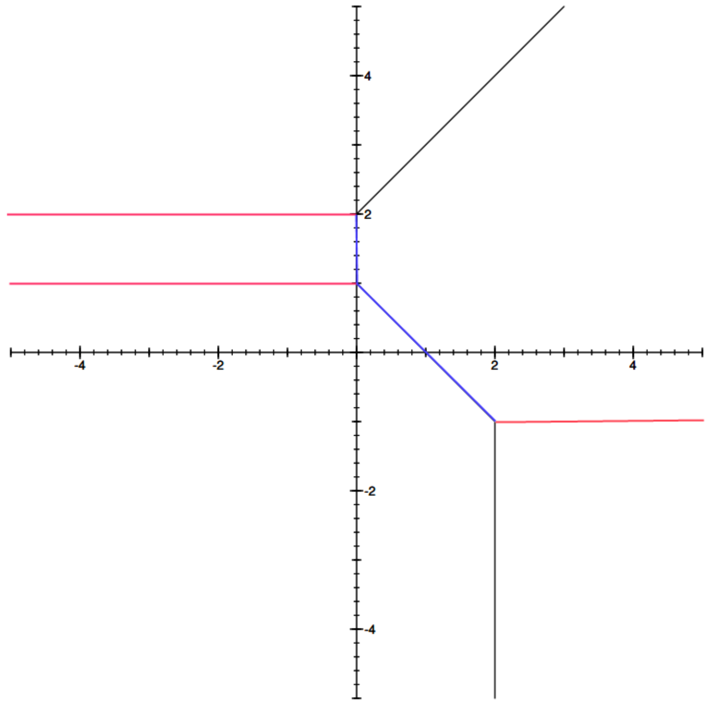
To generalize (3.1.1) we consider an infinite *NS5* brane stretched in x^5 with n_L *D5* branes ending on the left with positions (in x^5) $a_i, i = 1, 2 \ldots n_L$ and n_R branes ending on the right at $b_i, i = 1, 2 \ldots n_R$, the five brane formed is bent according to [4],

$$x^6 = \frac{g_s}{2}(\sum_{i=1}^{n_L} |y - a_i| - \sum_{i=1}^{n_R} |y - b_i| + (n_L - n_R) \tag{3.2.1}$$

some simple examples are shown below,



$n_L = 1, n_R = 1$



$n_L = 2, n_R = 1$

3.3 Gauge theories

To construct 5 dimensional gauge theories, in this case $U(N_c)$, we stretch N_c parallel *D5* branes (in x^6)between two *NS5* branes. Say we sperate the *NS5* branes by L_6 in x^6 and position the *D5* branes in x^5 at $a_i, i = 1, 2 \ldots N_{c},$, we now have $N_c - 1$ internal legs between each *D5* brane there bending condition can be found by utilizing (3.0.1).

we let the left and right lower external legs to be (p_L, q_L) and (p_R, q_R) respectively, by the condition (3.1.2) the left and right internal legs between the i'th and $i+1$'th $D5$ branes have charges,

$$(p_L - i, q_L) \tag{3.3.1}$$

$$(p_R + i, q_R) \tag{3.3.2}$$

and the upper external legs have charges,

$$(p_L - N_c, q_L) \tag{3.3.3}$$

$$(p_R + N_c, q_R) \tag{3.3.4}$$

To study gauge theory we need to decouple gravity and massive string modes by considering the limits,

$$L_{max}, g_s, l_s \to 0 \tag{3.3.5}$$

if the gauge coupling is at distance scale L such that,

$$L \gg l_s \gg L_{max} \tag{3.3.6}$$

gravity decouples and massive modes can be integrated out,at low energy this yields a $U(N_c)$ gauge theory in 1+4 dimensions, with a coulomb branch parameterized by $a_i, i = 1, 2 \ldots N_c$ and coupling,

$$\frac{1}{g_0^2} = \frac{L_6}{g_s l_s^2} \tag{3.3.7}$$

As above, quantum effects bend the branes which restricts the locations of the branes,

$$\sum_{i=1}^{N_c} a_i = 0$$

hence one of the classical positions is fixed and a $U(1)$ factor is frozen, breaking the gauge group to $SU(N_c)$.

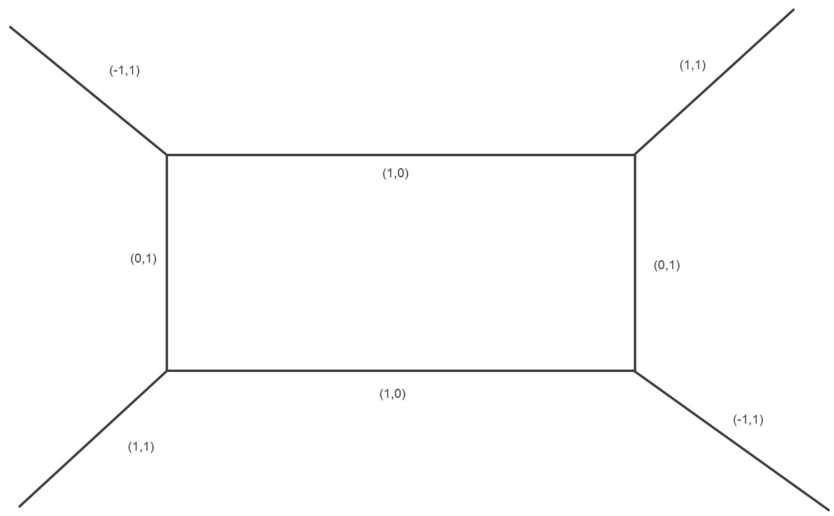


figure [3], pure $SU(2)$ web.

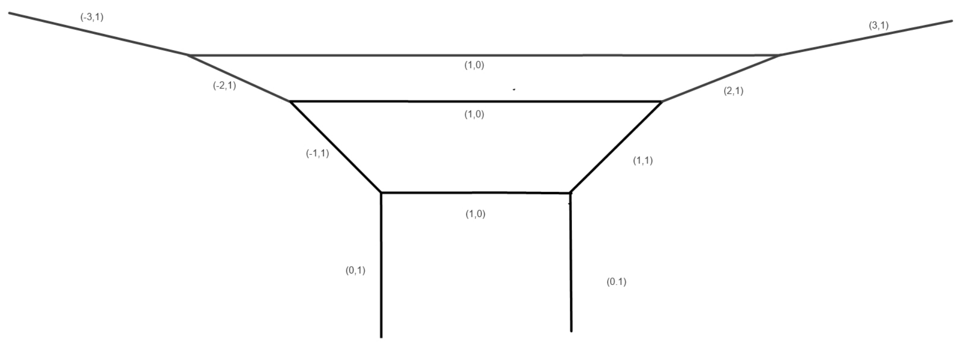


figure [4], pure $SU(3)$ web.

One can add matter hypermultiplets by having N_L semi infinite *D5* branes to the left of the *NS5* branes, and N_R to the right with,

$$N_L + N_R = N_f \tag{3.3.8}$$

their locations $m_i, i = 1 \ldots f$ in x^5 parameterize the quark masses, the charges of the uppermost left and right five branes are now changed to $(p_L - N_c + N_L, q_L)$ and $(p_R + N_c - N_R, q_R)$ respectively.

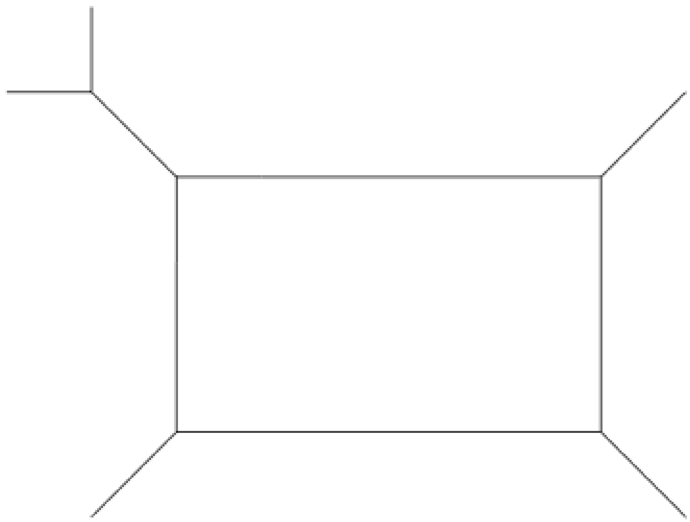


figure [5], $SU(2)$ with $N_f = 1$

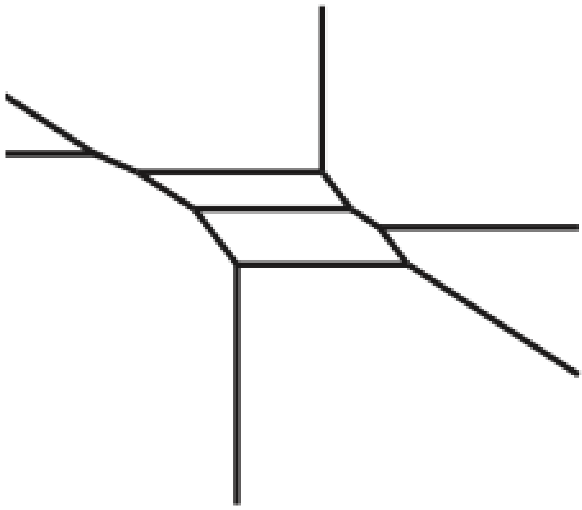


figure [6], $SU(3)$ with $N_f = 2$

An intuitive example that can demonstrate the properties of (p, q) webs is that of pure $SU(2)$ gauge theory (figure). We can immediately read off two field theory parameter, one being the mass of the W boson due to a fundamental string stretched between the two *D5* branes,

$$m_W = \Delta x^5 T_s \tag{3.3.9}$$

with T_s being the string tension and Δx^5 the distance between the two *D5* branes, which we normalize so that $m_W = \phi$, the scalar in our vector multiplet. We also have an instanton due to a *D1* string stretched between the two *NS5* branes, it has mass ,

$$m_I = \Delta x^6 \, |\tau| \, T_s \tag{3.3.10}$$

If we deform the web to a point where the *D5* branes ar coincident ($\phi = 0$) we see that

$$m_I = m_0 = \frac{1}{g_0^2} \tag{3.3.11}$$

At this point (the origin of the moduli space) the W bosons massless and the gauge group is enhanced to $SU(2)$., however at a generic point of the moduli space ($\phi \neq 0$) the W bosons gain mass,

$$m_W = \phi \tag{3.3.12}$$

and,

$$m_I = \frac{1}{g_0^2} + \phi \tag{3.3.13}$$

the global guage group is now just $U(1)$. The effective coupling is,

$$(\frac{1}{g^2})_{eff} = \frac{1}{2}(2m_0 + 4\phi) = m_0 + 2\phi \tag{3.3.14}$$

A monopole corresponding to a *D3* brane wrapping the face of the web its tension s given by the area of the face.

$$T_m = \text{area of face} = \phi(\phi + m_0) \tag{3.3.15}$$

We know from field theory results that $T_m = \frac{\partial \mathcal{F}}{\partial \phi}$ and as,

$$\frac{\partial T_m}{\partial \phi} = (\frac{1}{g^2})_{eff} \tag{3.3.16}$$

the prepotential takes the form,

$$\mathcal{F}(\phi) = \frac{1}{2}m_0\phi^2 + \frac{1}{3}\phi^3 \tag{3.3.17}$$

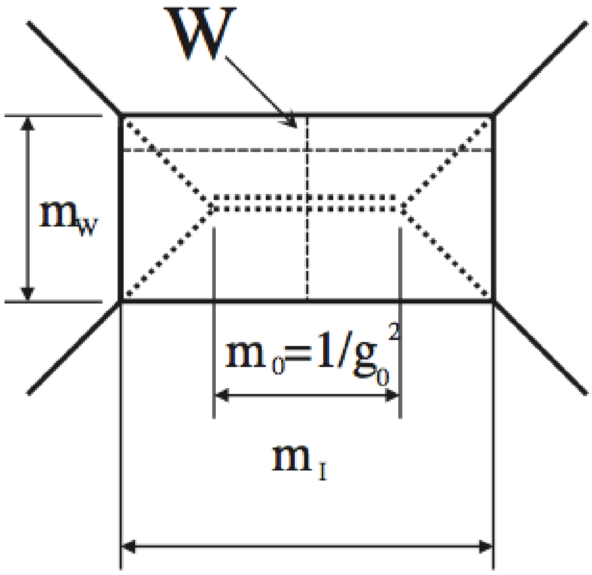
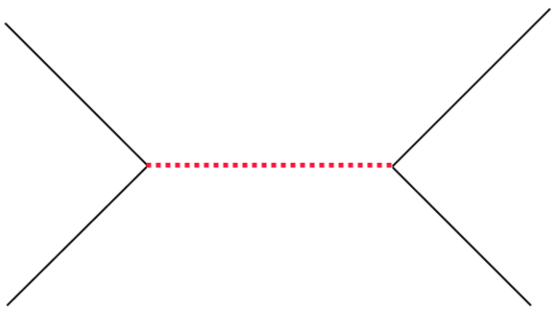


figure [7], particle states in $SU(2)$.



figure[8], length of dotted red line corresponds to $1/g_0^2$

3.4 Deformations

Deformations of the web play an important role in gauge theory. A **local deformation** is one that does not change the locations of the external legs and hence can be realized by giving VEVs to the scalar fields in the vector multiplet , and hence parameterized by ϕ. The number of local deformations is given by the number of internal faces [3].

$$n_G = \sharp(\text{local deformations}) = \text{rank}(\text{local gauge group}) = \sharp(\text{internal face}) \tag{3.4.1}$$

and by Euler's formula,

$$n_G = E - V + 1 \tag{3.4.2}$$

where E is the number of internal edges and V is the number of verticies. The metric on the Coulomb branch (effective coupling) is given by the sum of the masses of the edges moved by the deformation, weighted by there displacements squared,

$$(\frac{1}{g^2})_{eff} = 2 \sum_{edges} m_i \delta_i^2 \tag{3.4.3}$$

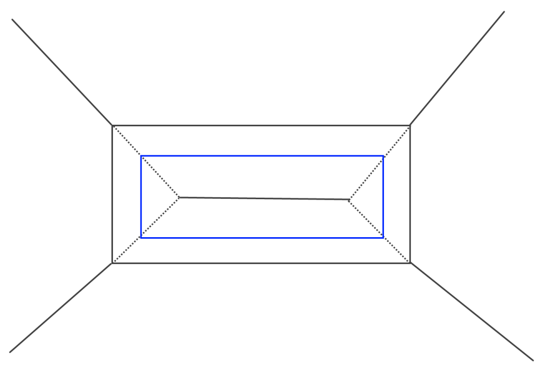


figure [9], a local deformation

Global deformations are ones which do change the asymptotic location of the external legs and there is one deformation associated with each leg. We are only interested in deformations which give new webs so we discard translations in the (x^6, x^5) plane and an additional deformation constrained by the web. We are left with,

$$n_G = n_X - 3 \tag{3.4.4}$$

with n_G being the number of global deformations and n_X the number of external legs.The bare coupling $\frac{1}{g_0^2}$ can obtained geometrically by deforming the web locally so that the $D5$ are coincident (ie. $\phi = 0$) , $\frac{1}{g_0^2}$ is then given by () for the $SU(2)$ case [3],

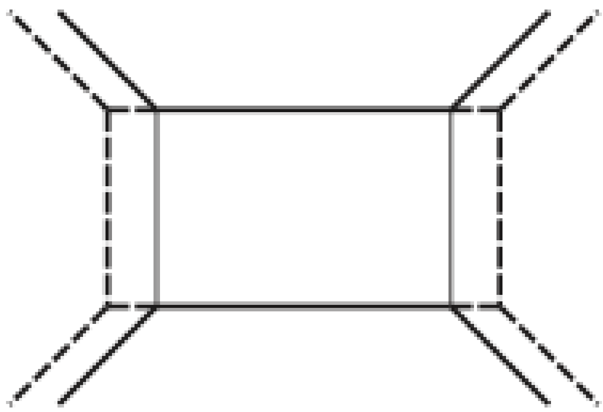


figure [10], a global deformation

3.5 Grid diagrams

Grids diagrams are an important tool in the classification of five dimensional theories, a thorough description on how to construct grids diagrams is given in [3], here I give a brief description of the concepts and results involved.

A grid diagram is a convex set of points, lines and polygons that lie in a 2-dimensional integer lattice and are labeled by $(a, b) \in \mathbb{Z}^2$. Grid diagrams are dual to (p,q) webs in the following way,

- A (p,q) edge in the web is mapped to a vector $(-q, p)$ (line).
- vertices in the web are mapped to polygons.
- Faces are mapped points on the diagram.

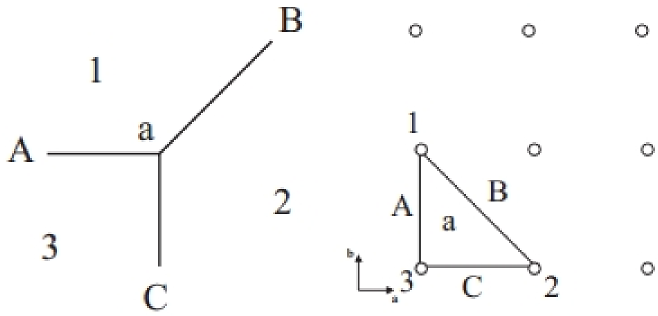
By charge conservation at each vertex we have,

$$\sum_{\text{edges}\in\text{vertex}} (p, q) = \sum_{\text{lines}\in\text{polygon}} [-q, p] \tag{3.5.1}$$

Using a grid diagram one can calculate the dimension of the Coloumb branch by,

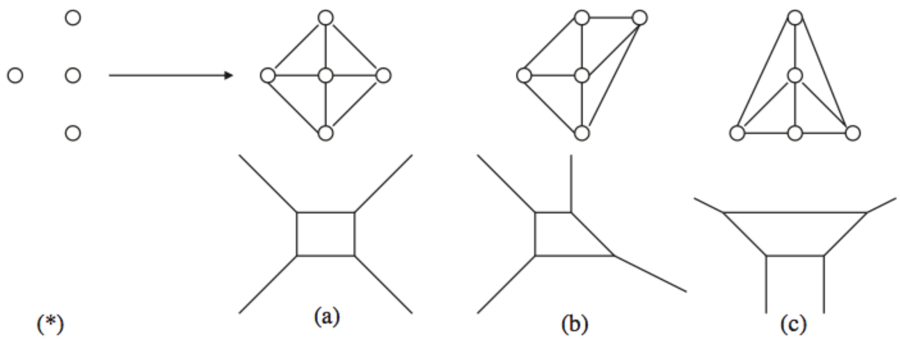
$$\dim(\text{Coulomb branch}) = \sharp(\text{internal points}) \tag{3.5.2}$$

The residual $SL(2, \mathbb{Z})$ symmetry of the web is carried through by the grid diagram description.



figure[11], a single vertex in the (x^6, x^5) plane, along with its dual grid diagram in (a, b).

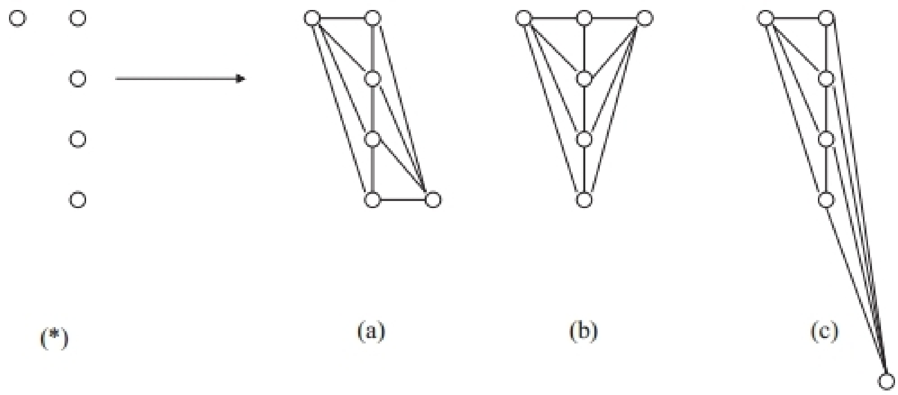
Grid diagrams provide an important insight into the special case of $G = SU(2)$. For an $SU(2)$ we need two parallel D branes which correspond to a column of three points in the grid diagram. The Coulomb branch in one dimensional so we need one internal point and four external points. The left point is fixed by $SL(2, \mathbb{Z})$ and we are left with 3 possible choices for the point on the left, possible webs are shown below,



figure[12],possible webs for $SU(2)$.

(a) and (b) correspond to theories with theta angles $\Theta = 0$ (mod 2) and $\Theta = 1$ (mod 2) respectively. The web (c) contains paralell external legs which leads to 6d particles which are charged under global symmetry, if such particles decouple from the low-energy theory (c) will give to a new theory with the same Coulomb branch, but conservatively we assume such theories don't exist.

Pure $SU(N_c)$ gauge theories can be classified using the concept of grid diagrams, from section (3.3) we need N_c horizontal $D5$ branes, which corresponds to a column of $N_c + 1$ points on the grid. Two additional points are required on both sides of the column, one of which can be fixed by the residual $SL(2, \mathbb{Z})$ symmetry. There are now $2N_c+1$ locations one the grid where the point can be placed without breaking convexity, these locations correspond to the possible values of c_{cl} for $N_f = 0$. Charge conjugation ($c_{cl} \to -c_{cl}$) can be viewed as a $\mathbb{Z}_2$ symmetry constructed by an $SL(2, \mathbb{Z})$ combined with a rotation.



figure[13], possible configurations of $SU(3)$, (*) point on the left is fixed by $SL(2, \mathbb{Z})$, (a) $c_{cl} = 0$, (b) $c_{cl} = N_c$, (c) $c_{cl} = -N_c$

4 Curves

4.1 M theory

To interpret the brane configurations in M theory, we first consider Type IIA string theory in 9+1 dimensions as M theory compactified on a circle in the x^{10} direction[13]. By comparing brane tensions in Type IIA and M theory, the parameters of the two theories can be mapped to each other by,

$$R_{10} = g_s l_s \tag{4.1.1}$$

$$\frac{R_{10}}{l_p^3} = \frac{1}{l_s^2} \tag{4.1.2}$$

where R_{10} is the radius of the circle on which x^{10} is compactified, $l_p$being the plank length. In the strong coupling limit $g_s \to \infty$, $R_{10} \to \infty$ and the theory is described by 10+1 dimensional super gravity. The limit $g_s, R_{10} \to 0$ takes us to 9+1 dimensional Type IIA string theory.

We further compactify Type IIA on a direction x_4 on circle with radius R_A by T duality this is equivalent to Type IIB string theory compactified on a radius,

$$R_B = \frac{l_s^2}{R_A} \tag{4.1.3}$$

and string coupling,

$$g_s' = \frac{g_s l_s}{R_A} \tag{4.1.4}$$

This is equivalent to m theory compactified on a 2-torus with sizes R_{10}, R_A, By (4.1.3) taking the limit $R_{10}, R_A \to 0$ implies,

$$R_B = \frac{l_p^3}{R_A R_{10}} \to \infty \tag{4.1.5}$$

hence Type IIB is recovered from M theory and the wrapping modes of the *M2* brane become light and an extra dimension x^i is created.

In the M theory picture *D5* and *NS5* branes are interpreted as *M5* wrapped around a $(1, 0)$ and $(0, 1)$ cycles on the torus respectively, (p, q) branes correspond to (p, q) cycles.

4.2 Complex coordinates

$SL(2, \mathbb{Z})$ acts the complex structure of the 2-torus, hence on the winding numbers (p, q) related to the x^4 and x^{10} directions. The fact that the orientation of the Type IIB five branes in the (x^6, x^5) plane means that [5],

$$\Delta x^6 : \Delta x^5 = \Delta x^4 : \Delta x^{10} \tag{4.2.1}$$

This enables us to define complex coordinates,

$$w = x^6 + ix^{10} \tag{4.2.2}$$

$$v = x^4 + ix^5 \tag{4.2.3}$$

as x^{10} and x^6 are compact it is convenient to define,

$$s = exp(w/R_{10}) \tag{4.2.4}$$

$$t = exp(-iv/R_A) \tag{4.2.5}$$

Thus in the M theory picture, our brane configuration is described by an *M5* brane on $\mathbb{R}^{1,3} \times \Sigma$, where Σ is a 2-dimensional surface embedded in $\mathrm{T}^2 \times \mathbb{R}^2$

which is parameterized by s and v. The curve can be written in the form of a holomorphic function $F(s,t)$ such that,

$$F(s,t) = 0 \tag{4.2.6}$$

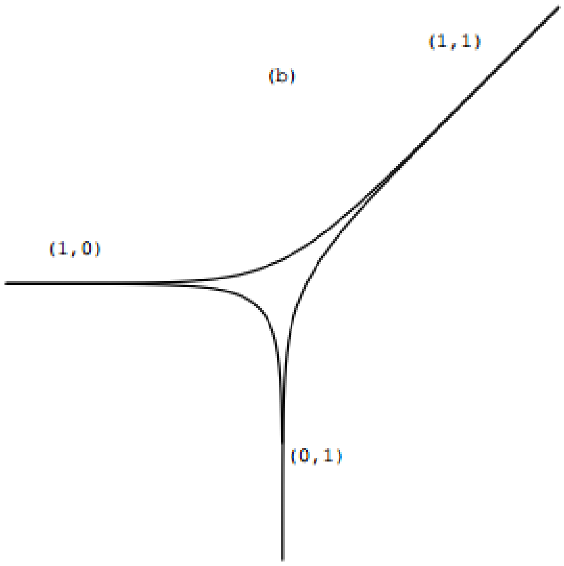
clearly $s = \mathit{constant}$ and $t = \mathit{constant}$ correspond to *NS5* and *D5* branes respectively. A simple example would be the curve described by a single vertex, the curve can be computed by considering the asymptotics of s and t [5] ,

- $s \to 0$, $x^6 \to -\infty$ so $F(s,t) = w - 1$
- $t \to 0$, $x^5 \to -\infty$ so $F(s,t) = t - 1$
- $s, t \to \infty$, $s \sim t$

hence we have [5],

$$F(s,t) = t + s - 1 \tag{4.2.7}$$

(4.2.7) represents a smoothed out vertex in the (x^6, x^5) plane figure [13] , in the 5 dimensional limit $(R_B \to \infty)$ the vertex and their singularities are retrieved.



figure[13], a simple vertex as a smoothed out M theory curve.

Similarly the curve for $SU(2)$ is given by,

$$F(s,t) = As + t + ABst + Ast^2 + ts^2 \tag{4.2.8}$$

with

$$A \sim 2\exp(L_4/2g_0^2) \quad B \sim 2\exp(m_W L_4/2) \tag{4.2.9}$$

4.3 Curves from grids

As was demonstrated in [3], Curves representing a brane configuration can be obtained simply from the grid diagram. Take a (p, q) brane described by the equation,

$$m + (-qx^6 + px^5)T_s = 0 \tag{4.3.1}$$

with m being the transverse position of the brane. Changing variables to (s, t) we have,

$$As^{-q}t^p = 1 \tag{4.3.2}$$

where $|A| = exp(mR_A^4)$, this can be rearranged to give,

$$A_1 s^{a_1} t^{b_1} + A_2 s^{a_2} t^{b_2} = 0 \tag{4.3.3}$$

where,

$$(a_1, b_1) - (a_2, b_2) = (-q, p), |A_1/A_2| = exp(mR_A^4) \tag{4.3.4}$$

So every point (a, b) in the grid diagram can be associated with a monomial $As^a t^b$, for a general grid we have,

$$F(s,t) = \sum_{i \in \text{points on grid}} A_i s^{a_i} t^{b_i}$$

(4.3.5)

coefficients are obtained by exhausting (4.2.11) to a loop of points in the diagram , consider a loop of n points with coefficients $A_1 \ldots A_n$. We assign A_1 arbitrarily and determine A_2 by,

$$|A_1/A_2| = \exp(m_{1,2}R_A) \tag{4.3.6}$$

with m_{12} being the transverse position of the brane we cross to get from A_1 to A_2, similarly,

$$|A_i/A_{i+1}| = \exp(m_{i,i+1}R_A^4) \tag{4.3.7}$$

Tackling the pure $SU(N_c)$ grid with the same approach, we have a sum of N_c+1 monomials corresponding to the column of points,

$$(\sum_{i=0}^{N_c} u_i t^i)s \tag{4.3.8}$$

the vertical position of the point to the right of the column $(2, c_{cl})$ characterizes the theory and translate in the curve as the monomial

$$As^2 t^{c_{cl}} \tag{4.3.9}$$

using the residual $SL(2,\mathbb{Z})$ symmetry the point to the left of the column can be placed at $(0, c_{cl)}$, putting these results together we have the curve,

$$As^2 t^{c_{cl}} + (\sum_{i=0}^{N_c} u_i t^i)s + Bt^{c_{cl}} = 0 \tag{4.3.10}$$

for constants A, u_i, C, after some rearrangement and re scaling of s we have,

$$s^2 + P(t)s + 1 = 0 \tag{4.3.11}$$

where

$$P(t) = t^{c_{cl}}(\sum_{i=0}^{N_c} u_i t^i) \tag{4.3.12}$$

4.4 General $SU(N_c)$

Now we move on to a more general construction given in [6],which will allow us to give restrictions on the possible gauge theories. We study curves that describe $SU(N)$ configurations with N_f flavors, following []we focus on configurations with two *NS5* branes, hence the polynomial will be as most quadratic in t. For convenience we redefine $s = exp(-w)$, the general form of the curve should be

$$A(t)s^2 + B(t)s + C(t) = 0 \tag{4.4.1}$$

for values of t where $C(t) = 0$, a solution of the curve is $s = 0$ ($x^6 \rightarrow \infty$), hence the zeros of C correspond to semi infinite *D5* branes to the right, similarly the zero of A correspond to semi infinite branes to the right and zeros of B correspond the "color" *D5* branes.

To normalize the coefficient of t^2 to 1, we move all semi infinite *D5* to the left of the *NS5*, which corresponds to the transformatio+??n $s \rightarrow \frac{C}{s}$ giving,

$$s^2 + B(t)s + C(t) = 0 \tag{4.4.2}$$

with

$$B(t) = c_1 t^n \prod_{k=1}^{N_c} (t - a_k) \tag{4.4.3}$$

and

$$C(t) = c_2 t^m \prod_{k=1}^{N_f} (t - b_k) \tag{4.4.4}$$

m and n are integers that we will impose conditions on, resealing and taking t to be large ($x^5 \rightarrow \infty$) we have the leading order polynomial,

$$s^2 + t^{N_c+n}s + t^{N_f+m} = 0 \tag{4.4.5}$$

with solution,

$$s = \frac{1}{2}(-t^{N_c+n} \pm \sqrt{t^{2(N_c+n)} - 4t^{N_f+m}}) \tag{4.4.6}$$

asymptotically we have the following roots for different n, m, N_c ,and N_f,

- $2(N_c+n) > N_f+m$, $s \sim t^{N_c+n}, t^{-N_c-n+N_f+m}$
- $2(N_c+n) < N_f+m$, $s \sim t^{(N_f+m)/2}$
- $2(N_c+n) = N_f+m$, $s \sim t^{N_c+n}$

The case $2(N_c+n) < N_f+m$ corresponds to *NS5* branes crossing, which leads to new massive states being created. This case should be discarded as according to [6] an $SU(N_c)$ theory with $N_f > 2N_c$ leads to non trivial fixed points , hence we focus on theories with $2(N_c+n) \geq N_f+m$.

Taking the limit $t \to 0$, (4.4.2) reduces to,

$$s^2 + t^n s + t^m = 0 \tag{4.4.7}$$

with solution,

$$s = \frac{1}{2}(-t^n \pm \sqrt{t^{2n} - 4t^m})$$

again asymptotically we have,

- $2n > m$, $s \sim t^{m/2}$
- $2n < m$, $s \sim t^n, t^{m-n}$
- $2n = m$, $s \sim t^n$

Taking $2n > m$ will lead to semi infinite "flavor" *D5* branes crossing, so we only consider $2n \leqslant m$. We still have an $SL(2,\mathbb{Z})$ transformation which acts as $t \to tw^l$ taking $m \to m+2l$ and $n \to n+l$, this allows us to set $n=0$, hence,

$$m \geq 0 \tag{4.4.8}$$

which leaves us with the constraint,

$$2N_c \geq N_f + m$$

rearranging we get

$$m \leq 2N_c - N_f \tag{4.4.9}$$

under parity ($t \to 1/t$),

$$m \to 2N_c - N_f - m$$

hence we are left with $(2N_c - N_f - 1)/2$ different values for m which correspond to the different values of c_{cl} found in[2]. The parity operator translates to charge conjugation $c_{cl} \to -c_{cl}$, hence we identify,

$$c_{cl} = N - m - N_f/2 \tag{4.4.10}$$

4.5 Four dimensions

We now turn to reducing our curve to four dimensions, this requires putting (4.4.2) in a form where we can take $R_A \to 0$. Consider,

$$\sin(\frac{v - a_k}{2R_A}) = \frac{1}{2}(\exp i(\frac{v - a_k}{2R_A}) - \exp -i(\frac{v - a_k}{2R_A})) \tag{4.5.1}$$

substituting $t = \exp(-iv/R_A\ )$ and multiplying through by $-\exp i(\frac{v}{2R_A})$ we have,

$$-\exp i(\frac{v - a_k}{2R_A})\sin(\frac{v - a_k}{2R_A}) = \frac{1}{2}(t^{1/2} - t^{-1/2}\exp(-i(\frac{a_k}{R_A}))) \tag{4.5.2}$$

multiplying by $t^{1/2}$ and letting $\tilde{a}_k = \exp -i(\frac{a_k}{R_A})$ gives,

$$-2t^{1/2}\exp i(\frac{v}{2R_A})\sin(\frac{v - a_k}{2R_A}) = (t - \tilde{a}_k) \tag{4.5.3}$$

letting $c_1 = -\exp i(\frac{v}{2R_A})/R_A$ in (4.4.3) we can write $B(t)$ as,

$$B(t) = 2R_A t^{N/2} \prod_{k=1}^{N_c} \sin(\frac{v - a_k}{2R_A}) \tag{4.5.4}$$

similarly for $C(t)$we have,

$$C(t) = t^{N/2+m} R_A \prod_{k=1}^{N_c} \sin(\frac{v - b_k}{2R_A}) \tag{4.5.5}$$

putting the results together, we can write (4.4.2) as

$$s^2 + 2sR_A t^{N_c/2} \prod_{k=1}^{N_c} \sin(\frac{v - a_k}{2R_A}) + t^{N/2+m} R_A \prod_{k=1}^{Nf} \sin(\frac{v - b_k}{2R_A}) = 0 \tag{4.5.6}$$

$a_k = e^{-i\tilde{a}_k/R_A}$ are know the position of the *D5* branes and $b_k = e^{-i\tilde{b}_k/R_A}$ the positions of the *NS5* . Taking the 4d limit ($R_A \to \infty, R_B \to 0$), this reduces to,

$$s^2 + 2s \prod_{k=1}^{N_c} (v - a_k) + \prod_{k=1}^{Nf} (v - b_k) = 0 \tag{4.5.7}$$

which is analogous to the curves describing $N = 2$ SQCD in 4 dimensions[8].

5 Conclusion

In this dissertation (p, q) webs were introduced, and how they are used to formulate $N = 1$ supersymmetric gauge theories. We focus on the case $G = SU(N_c)$ although concepts could be easily generalized to symplectic and orthogonal groups by the insertion of orientfolds. Grid diagrams are then used to classify different theories, giving the same restrictions that arise in the field theory picture. The M theory description of the brane configurations are then introduced through holomorphic curves which,in the 4d limit, were shown to be analogous to the curves given in[8].

(p, q) webs have provided a very rich venue for studying supersymmetric field theories in five dimensions, it allows us to visualize field theories from a geometrical perspective which paves the way for future work. The interplay between webs, grids and curves is quite enlightening, as it relates three mathematical concepts in the context of field theory.

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