Primordial non-Gaussianities in Hořava-Lifshitz Gravity

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Abstract

We address the issue of the origins of the primordial fluctuations is the Cosmic Microwave Background spectrum, the "seeds" of the Universe. We review the history of the discovery and the origins of the CMB and explain the questions surrounding it. We move on to consider famous cosmological "puzzles", which served as an inspiration for the theory of Inflation, reviewed subsequently. We discuss the mechanism and predictions of the theory and move on to discuss some less famous but equally successful alternatives. We next move on to consider the theory of Hořava - Lifshitz gravity together with its predictions, issues and different versions. We discuss importance of three-point correlation functions in the investigation of primordial fluctuations and attempt calculation of such for the theory in question. We conclude with directions for future work.

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1 Introduction - Standard Model of Cosmology

We begin the discussion by laying out the “Standard Model of Cosmology”, a framework within which the rest of the work will be embedded [1]. We work under the axioms of the Cosmological Principle - a homogeneous, isotropic, adiabatically expanding spacetime governed by the Robertson - Friedman - Walker metric of the form

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right) , \tag{1}$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$, $a(t)$ is the scale factor representing the relative expansion of the Universe, r is the comoving distance, which does not change in time due to the expansion of space itself and the parameter K dictates the following three possible geometrical scenarios:

$$K = \begin{cases} -1 & \text{negative curvature, open Universe,} \\ \ \ 0 & \text{zero curvature, flat Universe,} \\ +1 & \text{positive curvature, closed Universe.} \end{cases} \tag{2}$$

The evolution of the scale factor is governed by the Einstein equations:

$$\frac{1}{a(t)}\frac{d^2a(t)}{dt^2} = -\frac{4\pi}{3}G(\rho + 3p) , \tag{3}$$

$$H^2 + \frac{K}{a^2(t)} = \frac{8\pi}{3}G\rho , \tag{4}$$

where $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble “constant”, p is the pressure and ρ the energy density of the Universe. After some simple manipulation, we obtain the fundamental Friedman equation, governing the expansion of the Universe [2]:

$$\dot{a}^2(t) + K = \frac{8\pi G\rho a^2(t)}{3} . \tag{5}$$

A bit of further derivation leaves us with a piece of additional information - a conservation law of the form

$$\dot{\rho} = -\frac{3\dot{a}(t)}{a}(\rho + p) . \tag{6}$$

This can in turn be solved to give an equation of state of the form

$$p = w\rho , \tag{7}$$

with w a time-independent function. In the present case, we obtain:

$$\rho \propto a(t)^{-3(1+w)} \,. \tag{8}$$

We also define the concept of a "cosmological horizon". Theory of Special Relativity limits the possible speed of particle propagation to a speed of light. Since we assume the "Big Bang" origin of the Universe, and hence its finite age, the distance which we can see is limited by the maximum distance photon could have travelled throughout the lifetime of the Universe. This is the so-called "Particle Horizon", given by

$$d_p = a(t_0) \int_{t_i}^{t_0} \frac{dt}{a(t)} \tag{9}$$

where t_0 is a time at a particular point in the expansion and t_i is the initial singularity time.

The purpose of this work is a full investigation into the origins of the "seeds" of our observed Universe - primordial anisotropies and non-Gaussianities in the spectrum of the Cosmic Microwave Background. Understanding of the origins of this structure is of fundamental importance - these small variations in density of the very early Universe have grown to become what is now known as galaxies, stars, planets and people. We are hence aiming to answer the ambitious question about our own origins.

The work here is organised as follows. In section 2 we discuss the discovery, origins and spectrum of the Cosmic Microwave Background and the meaning of all these features. In section 3 we review the famous three "cosmological puzzles" which partially inspired the development of the theory of Inflation, descibed next. We briefly discuss the recent observational evidence in support of Inflation and in section 4 we ponder over alternative scenarios. In section 5 we introduce the theory of Hořava-Lifshitz gravity, a popular proposal developed in 2009. Finally in section 6 we present original work completed for the purpose of this project and attempt a direct application of Hořava-Lifshitz gravity to the problem of near scale-invariance of the Cosmic Microwave Background spectrum. We conclude with directions for further work and discussion in section 7.

2 The Cosmic Microwave Background

The 1965 discovery of a nearly isotropic Cosmic Microwave Background radiation by Arno Penzias and Robert Wilson can be considered to be one of the greatest breakthroughts in modern Cosmology, which has provided us with a wealth of previously unrealised data and as many answers as new questions. In this section we briefly review the origins of the background radiation, consider some of the implications of its existence and discuss the anisotropies observed in its spectrum.

The Universe is expanding - hence, in accordance with classical thermodynamics of an expanding fluid, we expect for it to have been denser and hence hotter in the past. Following the expansion backwards in time, we reach the point where the temperature was too high for electrons to be bound into atoms - we call this the recombination epoch. If we travel even further back in time we reach a point of temperature at which rapid collisions of photons with free electrons would have kept radiation in thermal equilibrium with matter. The black - body spectrum accurately describes the number density of photons in thermal equilibrium with matter at hight temperature T and at frequency between ν and $\nu + d\nu$ [1]

$$n_T(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(h\nu/k_B T) - 1}\,, \tag{10}$$

where h is the Planck's constant and k_B is the Bolzmann's constant. As time went by, the matter became cooler and less dense, but it can be shown that the spectrum remained unchanged. For, suppose there exist a certain time, t_L say, at which radiation went from being in thermal equilibrium with matter to a free expansion. A photon of frequency ν at a later time t would have a frequency of $\nu a(t)/a(t_L)$ at the time t_L. The number density of photons between frequency ν and $\nu + d\nu$ at time t would be

$$n(\nu,t)d\nu = \left(\frac{a(t_L)}{a(t)}\right)^3 n_T(t_L)\left(\nu\frac{a(t)}{a(t_L)}\right) d\left(\nu\frac{a(t)}{a(t_L)}\right)\,, \tag{11}$$

where the cubic factor arises from the dilution of photons due to the cosmic expansion. Using this expression in (10) we see that the factors of $a(t)/a(t_L)$ cancell out everywhere except for the exponential, such that

$$n(\nu,t)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(h\nu/k_B T(t)) - 1} = n_{T(t)}(\nu)d\nu\,. \tag{12}$$

The photon density retains the black - body spectrum, with the redshifted temperature $T(t) = T(t_L)a(t_L)/a(t)$. The fact that the Universe should be filled with black-body radiation had been first realised in 1940s by George Gamov [3] and collaborators [4] and the temperature of the radiation had been calculated to be around 3 K. Famous 1965 discovery and subsequent measurements have confirmed this prediction, and Cosmic Microwave Background is currently the most precisely measured black body spectrum observed in nature [5].

At the time of initial discovery, Cosmic Microwave Background was thought to have been perfectly isotropic. Indeed, this is the main reason lying behind its detection - Penzias and Wilson recognised the radiation as “background” as it was independent of the direction of observation. Of course, the spectrum of the Cosmic Microwave Background does present small variations in direction and those deviations from isotropy provide some of the most important pieces of information about the evolution of the Universe. We divide the anisotropies into two categories, depending of their origins. Hence we talk about the “late-time” or “secondary” anisotropies, arising as a result of Earth's motion, scattering of photons by intergalactic electrons, known as the Sunyaev-Zel'dovich effect and “primordial” anisotropies, which have their origins in the early Universe - at and before the surface of last scattering. These primordial density fluctuations are considered to be the “seeds” of structure observed in today's Universe. A quest for obtaining a satisfactory explanation for these early structures has motivated much of the modern and most groundbreaking research in Cosmology. Any scenario aiming to achieve such an explanation must however face an important problem: the spectrum of the Microwave Background is nearly homogeneous. Moreover, any inhomogeneities must be predicted to be nearly scale-invariant, meaning that the amplitude of primordial fluctuations is approximately constant. If that was not enough to deter any attempts, observational evidence provides us with an extremely well-defined amplitude measurement - accurate to one part in 100,000. Any wannabe structure-formation scenario must essentially provide all those details with a painstaking accuracy.

3 Inflation

One of the most acclaimed and popular proposals aiming to explain the early structure formation mentioned in the previous section is the Theory of Inflation. In this section we review motivations behind the original proposal of 1981, introduce the theory itself and explain how it solves some of the most famous cosmological "puzzles". Finally, we discuss the recent experimental evidence for the proposal.

3.1 Motivations - Cosmological Puzzles

Grand Unified Theories of particle physics predict existence of a simple symmetry group unifying all particle interactions, most famously that of $SU(5)$ after Georgi and Glashow, which, at a certain energy scale of order $M \approx 10^{16}$ GeV, is spontaneously broken to the well-known symmetry group of the Standard Model, namely $SU(3) \times SU(2) \times U(1)$. If proved to be correct, such scenario would allow for the existence of scalar fields carrying non-zero magnetic charge - magnetic monopoles. The argument is as follows [1]: the scalar fields before the symmetry breaking would be causally disconnected, and hence uncorrelated at distance larger than the horizon distance, which is the furthest away the light could travel since the initial singularity. In the standard Big Bang Cosmology the horizon distance for early times was of the order of

$$t \approx \frac{1}{\sqrt{G\,(k_B T)^4}}. \tag{13}$$

The nuber density of monopoles produced at the time where the temperature dropped down to M/k_B is of the order of

$$t^{-3} \approx \left(GM^4\right)^{3/2}\,, \tag{14}$$

which is smaller than the photon density of M^3 by a factor of order $\left(GM^2\right)^{3/2}$. At the Grand Unification Scale of 10^{16} GeV and with the Newton's constant $G \approx (10^{19}\,\mathrm{GeV})^{-2}$ this factor is of order 10^{-9}. With at least 10^9 Cosmic Microwave Background photons per nucleon today, this would mean the observable density of at least one magnetic monopole per nucleon today. As we cannot observe any at all, something must be wrong with our reasoning. Some argue that this argument points towards the fact that the model of Cosmology used here is not entirely correct. Others question the “reality” of this problem, as it is based on the assumption of existence of the Grand Unification, which itself has not yet been proven to exist beyond any doubt. As such, we deem this puzzle to be of least severity, however still, significantly, exposing incompleteness in our understanding of the early Universe.

The second puzzle is the so-called “flatness problem”. Observational evidence coming from Type Ia supernovae points towards a vanishing spatial curvature parameter Ω_K, where

$$\Omega_K = \frac{K}{a_0^2 H_0^2}\,, \tag{15}$$

which is also a value favoured by the data from the Cosmic Microwave Background. Although oservationally there is still some room for a small, non-vanishing Ω_K, it is considered safe to assume $|\Omega_K| < 1$. According to Hubble equation, Ω_K is just a present value of a dimensionless, time-dependent curvature parameter

$$\Omega_K(t) = \frac{K}{a^2 H^2} = \frac{K}{\dot{a}^2} \tag{16}$$

and we know that in the matter dominated universe, from the time the temperature dropped to about 10^4 K until present $a(t)$ has been increasing like $t^{2/3}$, so Ω_K must also have been increasing as $t^{2/3}$ or as T^{-1}. Thus if $|\Omega_K| < 1$ as observed, then at 10^4 K the curvature parameter could not have been greater than 10^{-4}. At earlier times, for example in the radiation era, $a(t)$ has been increasing as $t^{1/2}$, leading to the conclusion that the curvature parameter was at most about 10^{-16} at the temperature of 10^{10} K and even smaller at earlier times. The smallness of the curvature or the flatness of the Universe is not in itself a problem or iconsistency, but nothing in the current model accounts for such a small value of the curvature and the physical reasons behind it seem unclear.

The final, most ”serious” puzzle is the so-called Horizon Problem. The observations of the Comsic Microwave Background show its high degree of homogeneity and isotropy. The horizon size in matter- or radiation-dominated Universe is of order t, which, as the scale factor $a(t)$ has been increasing as $t^{2/3}$ since the time of last scattering, at the time of last scattering was of order of

$$d_H \approx \frac{1}{H_0\,(1+z_L)^{3/2}}\,. \tag{17}$$

This means that the horizon at the time of last scattering at present subtends an angle of order

$$\frac{d_H}{d_A} \approx \frac{1}{\sqrt{1+z_L}}\,, \tag{18}$$

where

$$d_A \approx \frac{1}{H_0\,(1+z_L)} \tag{19}$$

is the anglular diameter distance to the surface of last scattering. With the current value of the cosmological redshift of $z_L \approx 1100$ this gives the angle of 1.6^o. Hence in a matter or radiation dominated Universe no physical influence could have smoothed out initial inhomogeneities and brought points at a redshift z_L that are separated by more than a few degrees to the same temperature.This is in contradiction with our initial statement about the isotropy of the Cosmic Microwave Background.

3.2 The Theory of Inflation

Proposed by Alan Guth in 1981 [6], the theory of Inflation provides a modification to the "standard model" thus solving the aforementioned cosmological puzzles. The proposal claims that in order to answer to the horizon and flatness problems, prior to radiation- or matter-dominated periods, the early Universe has been dominated by slowly varying vacuum energy and the scale factor $a(t)$ must have undergone a rapid, exponential expansion by a factor $e^{\mathcal{N}}$. The particle physics details of the process have not yet been entirely specified, but the hypothethical scalar field responsible for the inflation has been dubbed as "inflaton". The original model, as proposed by Guth, based on the idea of a delayed first-order phase transition in which the scalar field was trapped in a local minimum of a potential and then leaked through the potential barrier in order to slowly roll down towards the true minumum of the potential. The energy of empty space would have remained constant while the scalar field was trapped, which corresponded to a constant rate of expansion and $a(t)$ growing exponentially [1]. After leaking through the potential barrier the roll of the scalar field towards the global minimum corresponds to the present Universe. This concept, however, has soon encounteded some problems. The transition from two phases could not have occured everywhere simultaneously but instead came in small "bubbles" of the true vacuum, which have expanded into the background of the false vacuum where the scalar field has been trapped. The latent heat released in this phase transition would have been trapped in the bubble walls, leaving the interior of the bubble empty and as a result leading to a high degree of inhomogeneity and anisotropy. Guth's suggestion that the bubbles could have merged has also been rejected on the basis of the argument that the background false vacuum space must have continued to inflate so that the bubble walls were moving away from each other too fast to have ever joint.

The initial idea has soon become replaced by the so-called "new inflation" as devised by Linde, Albrecht and Steinhardt in 1982. In this modification, the phase transition occurs forming bubbles again, but the potential is modified such that for low temperature the potential barrier is very small and hence the scalar field in the interior of the bubble starts near the zero point. The field rolls down slowly in the potential as the Universe undergoes expansion. The field energy is eventually converted into ordinary particles filling the bubble and our observable Universe is supposed to fill in one small part of such bubble.

We can now look into how the theory of Inflation resolves each of the three cosmological puzzles. As for the monopoles problem, the period of exponential expansion occuring before the time the monopoles were produced would have greatly extended the horizon and the expansion occuring after the monopole production, but before photon creation in reheating, would have reduced the monopole to photon ratio. The observational limit on numer of magnetic monopoles per photon is currently of order of 10^{-39} which provides a good indication for the number of inflation e-foldings to be $\mathcal{N} = 23$. As for the flatness problem, during the expansion $H = \dot{a}/a$ would have been roughly constant, hence $\Omega_K(t) = |K| / a^2 H^2$ would have been decreasing roughly like a^{-2}. If $\Omega_K(t)$ was of order of unity at the beginning of inflation, then $\Omega_K(t_I)$ at the end of inflation would have been of order $e^{-2\mathcal{N}}$ and hence today we would have

$$\Omega_K(t_0) = \Omega_K = \frac{|K|}{a_0^2 H_0^2} = e^{-2\mathcal{N}} \left(\frac{a_I H_I}{a_0 H_0} \right)^2 . \tag{20}$$

Hence the flatness problem is solved completely if the inflation has the lower bound of

$$e^{\mathcal{N}} > \frac{a_I H_I}{a_0 H_0} \tag{21}$$

and calculations involving the observed density of the Universe locate that value at $\mathcal{N} > 62$. Finally, the horizon problem finds its solution in the theory of Inflation as well. During the inflationary period the observable part of the Universe would have occupied a tiny space and there would have been enough time for everything in this space to be homogenised. To quantify what it means for Inflation again, recall that the size of the horizon at the time t_L of last scattering is

$$d_H(t_L) = a(t_L) \int_{t_*}^{t_L} \frac{dt}{a(t)} \tag{22}$$

with t_* being the time of the beginning of Inflation. We assume that during inflation $a(t)$ increased exponentially at a rate H_I so that

$$a(t) = a(t_*) \exp\left(H_I(t - t_*)\right) = a(t_I) \exp\left(-H_I(t_I - t)\right) \tag{23}$$

where t_I is the time at the end of Inflation. Using $N = H_I(t_I - t_*)$ and noting that to observe the hight degree of isotropy in the Cosmic Microwave Background we need $d_H(t_L) > d_A(t_L)$ we obtain the same condition again:

$$e^{\mathcal{N}} > \frac{a_I H_I}{a_0 H_0} . \tag{24}$$

3.3 Experimental evidence

The already-famous and slightly controversial announcement of the BICEP2 experiment's results in March of 2014 [7] left the entire community buzzing about confirmation of one of the most important theories in modern physics - Inflation. The experiment, running on the South Pole between 2010 and 2012 aimed to detect signs of inflationary gravitational waves in the B-mode power spectrum. Theory of Inflation predicts production of primordial background of stochastic gravitational waves of a characterisctic spectral shape which are believed to have imprinted a tracable polarization pattern on the Cosmic Microwave Background. This polarization was predicted to contain a "curl" or a "B-mode" at angular scales that are not accessible via primordial density perturbations. The amplitude of the signal depends directly on the energy scale of inflation, hence its detection provides a unique confirmation both for the theory and the associated energy scale. The group behind BICEP2 has announced detection of inflationary B-modes to accuracty of about 5σ, which is ranked within "discovery" range. Some doubt has been cast upon the result almost immediately, however. Amongst other publications, Flauger, Hill and Spergel have questioned the validity of the result, sugesting the signal might have been a consequence of combined Galactic foreground noise and lensed E-mode signals [8]. The current consensus is that more data is needed - that includes both further BICEP experiments as well as additional information gathered by Planck Satelite, amongst others. As such, Inflation has escaped the "discovery" certainity range and yet remains an open problem.

4 Alternatives

However promising it may be, until proven correct beyond all doubt, the theory of Inflation leaves space for theoretical alternatives. There exist a few competing scenarios which aim to explain the early structure formation without the need for inflationary expansion and possibly solve some of the other outstanding problems along the way. In this section, we review several of these alternative theories and discuss their consequences and predictions.

4.1 Variable Speed of Light

Dubbed as controversial and heretic, Variable Speed of Light theories base on abandoning the special-relativistic postulate of constancy of speed of light c in vacuum [9]. There exist several different ways of obtaining this effect. First, the so-called hard breaking of Lorentz symmetry claims existence of a preferred frame in physics and speed of light which is variable in time, usually in the very early Universe. Both the matter content of the Universe and the laws of physics evolve in time, while the dynamical postulate claims that the Einstein's field equations remain valid. The second type are the Bimetric VSL theories, postulating existence of two independent metrics - one for gravity and one for matter, such that the speed of graviton is different from the speed of massless matter particles. Other proposals include color-dependent speed of light or space-time varying c aiming to preserve Lorentz invariance. In any case, those theories not only successfully explain some of the cosmological puzzles, make contact with quantum gravity, but are also able to reproduce the required Harrison-Zeldovich spectrum of Gaussian fluctuations.

Let's consider the mechanism for production of the primordial fluctuation spectrum from a scalar-tensor bimetric gravity theory, as outlined in [9] and [10]. We consider a scalar field ϕ minimally coupled to the gravitational field described by the space-time metric $g_{\mu\nu}$ via the matter metric

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + B\partial_\mu\phi\partial_\nu\phi \,, \tag{25}$$

such that the total action reads $S = S_g + S_\phi + \hat{S}_M$. S_g is the usual gravitational Einstein-Hilbert action while the scalar action reads

$$S_\phi = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \,. \tag{26}$$

In a frame where a fixed speed of light is chosen, one observes dynamically-determined speed of gravitational disturbances v_g. In a scenario where $v_g \approx 0$, the fluctuations in ϕ have been calculated and a spectral index of $n_s \approx 0.98$ has been found. The tensor fluctuations have been calculated with $n_t = -0.027$ and the tensor to scalar ratio can be shown to satisfy $r \geq 0.014$. The value of these parameters can be an important distinguisher between the bimetric theory and the theory of Inflation.

4.2 Variable Speed of Sound

As discussed in [11], a theory with a varying speed of sound can act as a proxy for a varying speed of light theory. A theory containing two metrics - one for massless particles and one for gravity, distinguishes the speeds of the two. In such setup, a frame in which gravity is unaffected can be defined, forcing "light" to travel much faster in such frame. A new mechanism for producing scale-invariant fluctuations has been proposed basing on the varying speed of sound hypothesis. We consider here unmodified Einstein gravity with $w > -1/3$ and assume that the speed of sound is density-dependent and diverging with conformal time like

$$c_s \propto \eta^{-\alpha} \,. \tag{27}$$

The density fluctuations are described by a modified harmonic oscillator equations. In therms of the curvature perturbations ζ, the equation for the related variable $v = a\zeta$ is that of Einstein gravity:

$$v'' + \left[c_s^2k^2 - \frac{z''}{z}\right] = 0 \,, \tag{28}$$

where $z \propto a/c_s$. This equation can be exactly solved after employment of Bessel functions, but a WKB approximation helps to establish initial conditions of the problem and it gives

$$v \sim \frac{e^{ik\int c_s d\eta}}{\sqrt{c_s k}} \sim \frac{e^{-\beta c_s k\eta}}{\sqrt{c_s k}} \,, \tag{29}$$

with $\beta = 1/(\alpha - 1)$ and up to a phase. Equation (28) can be transformed into a Bessel equation, yielding a result

$$v = \sqrt{\beta\eta}\left(AJ_\nu(\beta c_s k\eta) + BJ_{-nu}(\beta c_s k\eta)\right) \tag{30}$$

with A and B k-independent numbers of order 1 and

$$\nu = \beta\left(\alpha - \frac{3(1-w)}{2(1+3w)}\right). \tag{31}$$

Now it is easy to derive that the scale-invariance requirement of

$$k^3\zeta^2 = \text{const.} \tag{32}$$

translates to $\nu = 3/2$, i.e.

$$\alpha = \alpha_0 = 6\frac{1+w}{1+3w}. \tag{33}$$

If we wish to rephrase this requirement, we discover that it leads to $c_s \propto \rho$ for all w. We can also derive the size of the amplitude of fluctuations for near scale-invariant spectrum. Considering a curvature fluctuation ζ which is time independent outside of the horizon, one obtains [12]

$$k^3\zeta^2 \sim \frac{(5+3w)^2}{1+w}\frac{\rho}{M_{Pl}^4 c_s}. \tag{34}$$

If we rephrase this requirement in terms of characteristic density rho_* which triggers the divergence of c_s, we discover that

$$k^3\zeta^2 \sim \frac{(5+3w)^2}{1+w}\frac{\rho_*}{M_{Pl}^4} \sim 10^{-10}. \tag{35}$$

4.3 Modified On-Shellness

As reviewed in [13], using a class of theories with higher order spatial derivatives it may be possible to obtain deviation from strict scale invariance of primodial density fluctuations. Under assumption that the gravitational equations are those of Einstein as laid out in equations (3) and (4), the most important equation governing cosmological perturbations is:

$$v'' + \left[c^2k^2 - \frac{a''}{a}\right]v = 0. \tag{36}$$

The curvature perturbation is then given by $\zeta = -v/a$ while the modes are labelled by a comoving k, the conserved charge associated with translational invariance. The physical wave-number of the mode, however, is given by $p = k/a$. Since we are considered a general class of higher-derivative theories, the dispersion relation associated to the theory has neccesarily been modified as well, and is of form

$$E^2 = p^2\left(1 + (\lambda p)^{2\gamma}\right). \tag{37}$$

The speed c can be found from the above as

$$c = \frac{E}{p} \propto \left(\frac{\lambda k}{a}\right)^{\gamma}, \tag{38}$$

and hence, considering a fixed comoving mode we see that the frequancy dependent speed of light translates into a time-dependent speed of light.

The dispersion relation of the type (37) has long been known to be able to produce a scale-invariant spectrum of primordial fluctuations without need for Inflation [14]. Perhaps more intestring is the perspective of embarking on a concrete higher-derivative theory streaming from different motivations completely and taking a closer look at how it tests in the context of primordial non-Gaussianities. This is what we move on to next.

5 Hořava-Lifshitz gravity

This section introduces the theory of Hořava-Lifshitz gravity and it is based on authors original publication of [15] and references [16] and [17]. One must distinguish between canonical approach to the theory, which involves a classical split of space-time into space and time components, as discussed in this section, and covariant approach, aiming to preserve four-dimensional covariance of General Relativity, which we do not discuss within the scope of the present work.

5.1 Motivations

General Relativity is an extremely successful and celebrated framework which together with the quantum theories of electrodynamics, chromodynamics and flavordynamics successfully explains a great majority of phenomena observed in nature. However, there exist several motivations for unifying the two, yet incompatible, ways of describing interactions into one complete framework, which would provide a consistent description of all known processes. The first, probably most obvious motivation is the striving of scientific community to obtain the unification of existing theories into so-called "Theory of Everything". Since all the remaining interactions have already been unified by the way of the Standard Model of particle physics, it is not unreasonable to expect gravity to follow suit. All the attempts to construct a coherent "semiclassical" theory, where gravity is ruled by classical theory and all the other fundamental forces are viewed in the quantum framework have failed so far. The second motivation may seem more solid, since it derives from experimental results and observations. Despite its acclaim, General Relativity fails to explain gravitational behaviour near singularities, the Big Bang being one of them. In the small-scale, high-density limit of the Big Bang, the frameworks of General Relativity and Quantum Mechanics need to be simultaneously applicable. Quantum effects remain most relevant in such an environment due to the small scale involved while gravitational effects become significant as a result of high mass density. It this context, the theory is shown to be incomplete.

In order to achieve unification between the two theories, one would require for gravity to be quantized (accommodated within the mathematical and conceptual frameworks of quantum theory). It is here, that the struggle begins; General Relativity seems to resist all know quantization techniques, yielding non-informative divergences and infinities as a result of the process. Indeed, if we consider the general relativistic action of Einstein and Hilbert

$$S_{EH} = \frac{2}{\kappa^2} \int d^4x \sqrt{g} R \ , \tag{39}$$

and analyse the mass dimensions of its constituents we note that

$$\kappa^2 = 8\pi G = (M_{Pl})^{-2} \ . \tag{40}$$

is not only a mass-dimensionful quantity, but has negative mass dimension. The standard argument involving the computation of the superficial degree of divergence tells us that the theory is non-renormalisable and hence computations will yield infinite results.

The only two possible steps from this discovery is either giving up the idea of quantum gravity altogether or modifying the original theory in order to allow for quantization. Since either way a change of an approach is necessary, considered in this work is a proposed theory, known as Hořava-Lifshitz gravity, stated by Prof. Petr Hořava in 2009.
The central topic of this report is the theory of Hořava-Lifshitz gravity and the work herein has been organised as follows. In section 2 a brief introduction to the concept of renormalization is presented, followed by the explanation of why General Relativity is not renormalizable. Hořava's theory is then discussed, together with all its assumptions and consequences and its power-counting renormalizability is shown. The section concludes with a worked example of an application of the theory, used in obtaining rotationally invariant black hole solutions. In section 3 the covariant extension of Hořava-Lifshitz gravity is presented and its applications to spherically symmetric, static solutions are discussed. In section 4 the application of covariant Hořava-Lifshitz gravity to modelling galaxy rotation curves is presented, the results of which lead to paper [33]. Section 5 contains the discussion of results obtained and section 6 contains conclusions of the work, the summary of obtained results and proposed areas for further study.

5.2 The action

We begin with the famous Einstein-Hilbert action in the units of $c = \hbar = 1$

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x R \sqrt{-g} \tag{41}$$

with G_N Newton's gravitational constant, R Ricci curvature scalar and $g = det g_{ij}$ the determinant of the metric tensor and by noting that it contains only the lowest order terms in curvature expansion [16]. The natural first step towards extending the theory would be to add higher order terms to the expression and investigate properties and renormalizability of such an action. As summarised by S. Weinberg in reference [17], there exist three known approaches towards constructing a renormalizable, higher derivative theory of gravitation. The first approach includes adding matter fields and imposing further symmetries on the theory, in the hope of obtaining cancellation of divergences. The second includes renormalization through rearrangement of the perturbation series and comes under the name of "resummation". The final approach discusses existence of composite gravitons.

The discussion below focuses on the second approach, that of resummation. By adding terms of higher order in curvature to Einstein-Hilbert action of equation (41), one modifies the Lagrangian density to the form of [17]

$$\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g}R - g_1\sqrt{g}R^2 - g_2\sqrt{g}R_{\mu\nu}R^{\mu\nu} \ , \tag{42}$$

where g_1, g_2 are coupling constants and where the metric in question is treated under a decomposition into a flat "background" Minkowski metric plus fluctuations quanta

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}h_{\mu\nu} \ . \tag{43}$$

Instead of treating the last two terms of equation (42) as perturbations and expanding in powers of g_1 and g_2, upon which the standard divergences would appear, one sums up the contributions to the quadratic curvature terms which in turn are quadratic in the gravitational field $h_{\mu\nu}$.

The modified graviton propagator dressed in quantum loop corrections can now be expressed as a resummation of terms corresponding to one - loop Feynman diagrams. If each graviton propagator corresponds to a term of $1/k^2$ and each loop to a term of $\alpha g_1 G k^4$ (where α is some dimensionless function of the ratio g_2/g_1), one can construct a full expression for graviton propagation,

$$\frac{1}{k^2} + \frac{1}{k^2}\alpha g_1 G k^4 \frac{1}{k^2} + \frac{1}{k^2}\alpha g_1 G k^4 \frac{1}{k^2}\alpha g_1 G k^4 \frac{1}{k^2} + ... = \frac{1}{k^2 - \alpha g_1 G k^4} \ , \tag{44}$$

where the summation is obtained by using the geometric series. The cost of this modification, however, cannot be overlooked. The modified graviton propagator can be decomposed as

$$\frac{1}{k^2 - \alpha g_1 G k^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/\alpha g_1 G} \ . \tag{45}$$

The expression is dominated by the $1/k^4$ term in high energy regimes but at the same time clearly exhibits an additional pole at $k^2 = 1/\alpha g_1 G$. This corresponds to a new massive degree of freedom which is forbidden by the theory due to the requirement for a graviton to be massless, which in turns results from the scale of gravitational interaction being infinite. Those so called ghost excitations cause violation of unitarity of the theory and hence such a straightforward extension of General Relativity must be abandoned in favour of a more sophisticated approach, as discussed next.

As observed above, a naive addition of higher derivatives to the action evokes ghost propagation. It ought to be noted however, that it is solely due to inclusion of higher order derivatives in time coordinate; these constitute additional kinetic terms, and hence are the truly problematic elements. In the classical theory of General Relativity, space and time are diffeomorphic (transition maps between coordinate patches on space-time manifold are differentiable isomorphisms), it is impossible to include additional spatial derivatives without including the temporal ones. Hořava's proposition includes breaking of four-dimensional diffeomorphism of General Relativity, where time and space coordinates mix, in favour of a model exhibiting anisotropy between space and time. The way of inducing space and time anisotropy is through defining an ultraviolet fixed "Lifshitz" point of renormalization group transformations which is invariant under the scaling $\overrightarrow{x} \to b\overrightarrow{x}$ and $t \to b^z t$, where z is the critical exponent. [18] In this new choice of scaling, the mass dimensions of time and space change to

$$[\text{time}]{=}-z \qquad \text{and} \qquad [\text{space}]{=}-1\ .$$

With such a choice, it is possible to add higher order spatial derivatives to the action, hence making it renormalizable, without adding higher order time derivatives and evoking ghost degrees of freedom.

The natural metric to impose on such a system will be the one which separates time as a preferred direction. The metric complying with those conditions comes under the name of Arnowitt-Deser-Misner (ADM) decomposition of space-time [19],

$$ds^2 = -c^2N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\ , \tag{46}$$

where N is the lapse function, N^i is the shift function and g_{ij} is the three-dimensional spatial metric of signature (+++). In this decomposition and with $z = 1$, Einstein-Hilbert action of general relativity takes the form of

$$S_{EH} = \frac{2}{\kappa^2}\int dt d^3x\sqrt{g^{(3)}}N\left(K_{ij}K^{ij} - K^2 + R^{(3)}\right)\ , \tag{47}$$

where the extrinsic curvature,

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\ , \tag{48}$$

$g^{(3)}$ is the determinant of the spatial metric, dot represents time derivative and $R^{(3)}$ is the curvature scalar in three dimensions representing potential for graviton.

The action proposed by Hořava in his paper of [15] is build basing on symmetry principle and taking care of keeping the dimensions correct. Keeping the dynamical critical exponent z arbitrary, the most general action takes the form of [16]

$$S_{HL} = \frac{2}{\kappa^2}\int dt d^3x\sqrt{g^{(3)}}N(K_{ij}K^{ij} - \lambda K^2 - V(g_{ij}, N))\ , \tag{49}$$

where V is the potential containing higher order spatial derivatives and can be expressed in terms of the Riemann tensor and its derivatives.

In the expression above, λ is a new free coupling of the theory, which needs to be equal to 1 if General Relativity is to be exactly recovered in the infrared limit. Various studies on the possible constraints on the value of this parameter both from the mathematical point of view [20] and the experimental perspective [21], [22] have been done up to now, although there is no symmetry in the theory which would impose this value to be exactly 1. Throughout the majority of this work, the value of $\lambda = 1$ is assumed, unless explicitly kept unspecified in the formulas.

Focusing for a moment on the specific case of $z = 3$, one notes the mass dimension $[V] = 6$ and the fact that the mass dimension of the Riemann tensor $[R] = 2$. As a result, one needs to introduce all possible terms of mass-dimension 6 into the potential. Keeping in mind that in three dimensions the Weyl tensor vanishes automatically and the Riemann tensor can be decomposed into the Ricci tensor, Ricci scalar and the metric, allows one to arrive at the following list of possible summands in the potential:

$$\left\{(\mathrm{Ricci})^3, [\nabla(\mathrm{Ricci})]^2, (\mathrm{Ricci})\,\nabla^2(\mathrm{Ricci}), \nabla^4(\mathrm{Ricci})\right\} .$$

Clearly, there exist a number of possible terms that include the above and hence could be added on to the potential, generating different versions of the theory. In what follows, the discussion in reference [16] is followed and two possible versions of the theory are discussed.

5.3 Detailed balance

One of the proposed symmetries V could exhibit comes under the name of "detailed balance" and originates from condensed matter systems. It requires for V to be derivable from a superpotential W

$$\begin{aligned} V &= E^{ij}\mathcal{G}_{ijkl}E^{kl} \\ E^{ij} &= \frac{1}{\sqrt{g}}\frac{\delta W}{\delta g_{ij}} \\ \mathcal{G}_{ijkl} &= \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk}\right) - \frac{\lambda}{1-3\lambda}g_{ij}g_{kl} \end{aligned} \tag{50}$$

where W is a Euclidean action in D spatial dimensions. Such a requirement provides a way of obtaining a $(D+1)$-dimensional action from a D-dimensional one, and essentially imposes the potential to be of the form [16]:

$$V = \frac{4}{w^4}C_{ij}C^{ij} - \frac{4\mu}{w^2}\epsilon^{ijk}R_{ij}\nabla_j R^l{}_k + \mu^2 R_{ij}R^{ij} - \frac{\mu^2}{(1-3\lambda)}\left(\frac{1-4\lambda}{4}R^2 + \Lambda_W R - 3\Lambda_W^2\right) , \tag{51}$$

with the Cotton tensor C^{ij} playing the role of the Weyl tensor in three dimensions and defined as

$$C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}} \nabla^k \left(R^j{}_l - \frac{1}{4} R \delta^j{}_l \right) \ , \tag{52}$$

and w, μ and Λ_W couplings of suitable dimensions.

This restriction significantly reduces the number of terms one needs to include in the action and greatly simplifies quantization. The potential (**??**) however, is not parity invariant and furthermore the last term of expression, which acts as a cosmological constant exhibits a wrong sign in comparison with observations. The assumption of detailed balance is arbitrary and strictly simplifying. It does not possess a physical motivation, and hence it is often abandoned it in favour of the "projectability condition", described next.

5.4 Projectability condition

Another condition one can impose onto the theory in question is that of "projectability", which essentially means assuming that the lapse is a function of time only, $N = N(t)$. Under this assumption, one can rescale the time variable in such a way that $dtN(t) = dt'$. As remarked in [23], the projectability condition can always be enforced onto standard General Relativity locally, as a gauge choice. In the action under consideration, the condition greatly reduces the number of independent terms in the potential V. Since N is a function of time only there are no invariant terms to be build out of it, which would contain spatial derivatives only. One can thus conclude that V should contain all the curvature invariants one can construct out of g_{ij}, which for previously considered case of $z = 3$ means including up to six spatial derivatives. Since we are in process of searching for an effective theory in three spatial dimensions, we can simplify further by noting that the Cotton tensor vanishes identically, the Riemann tensor can be expressed in terms of the Ricci tensor and by using Bianchi identities, commutator identities and integration by parts [23].
As a result, for $z = 3$, the derivative expansion of V becomes:

$$\begin{aligned} -V &= \alpha \quad R^{(3)} + \beta_1 (R^{(3)})^2 + \beta_2 R^{(3)}_{ij} R^{(3)ij} + \gamma_1 (R^{(3)})^3 + \gamma_2 R^{(3)} R^{(3)}_{ij} R^{(3)ij} \\ &+ \quad \gamma_3 R^{(3)i}{}_j R^{(3)j}{}_k R^{(3)k}{}_i + \gamma_4 R^{(3)} \nabla^2 R^{(3)} + \gamma_5 \nabla_i R^{(3)}_{jk} \nabla^i R^{(3)jk} \ , \end{aligned} \tag{53}$$

where $\alpha, \beta_1, ..., \gamma_5$ are couplings to be fixed by experiment and of mass dimensions

$$\begin{aligned} [\alpha] &= 4 \\ [\beta_1] &= [\beta_2] = 2 \\ [\gamma_1] &= [\gamma_2] = [\gamma_3] = [\gamma_4] = [\gamma_5] = 0 \end{aligned} \tag{54}$$

If the infrared regime is defined to be the region where $\gamma_1 \left(R^{(3)}\right)^3 \ll \beta_1 \left(R^{(3)}\right)^2 \ll \alpha R^{(3)}$, one can immediately see that Hořava-Lifshitz action can be approximated as Einstein-Hilbert action plus higher order terms. Starting from action in equation (49) and allowing free couplings to be small and the bare cosmological constant to coincide with its General

Relativistic value of $\lambda = 1$, one needs to rescale time to $t \to \frac{\tau}{M^2}$, such that $dt \to d\tau \frac{1}{M^2}$. The mass dimensions $[g_{ij}] = [N] = 0$, so these quantities stay invariant under the time reparametrization. The mass dimension of extrinsic curvature $[K_{ij}] = z$, which is an inverse dimension to the dimension of time. This implies that each term of K_{ij} should be rescaled by M^2 and consequently both $K_{ij}K^{ij}$ and K^2 are rescaled by M^4, so that finally

$$\begin{aligned} S_{HL} &= \int \frac{1}{M^2} d\tau d^3x \sqrt{g^{(3)}} N M^4 \left[K_{ij}K^{ij} - \lambda K^2 - \frac{V}{M^2} \right] \\ &= M^2 \int d\tau d^3x \sqrt{g^{(3)}} N \left[K_{ij}K^{ij} - \lambda K^2 - \frac{V}{M^2} \right] \\ &= S_{EH} + \int \mathcal{O}\left(\frac{R^{(3)}}{M^2} \right) d\tau d^3x \ , \end{aligned} \tag{55}$$

where one identifies $M^2 = [4\pi G]^{-1} = 2M_{Pl}^2$ and indeed in the infrared limit $R^{(3)} \ll M^2$ and $S_{HL} \simeq S_{EH}$.

5.5 Consequences

The first, desired consequence of employing space-time anisotropic model is the change of mass dimensionality of the coupling constant of the theory. Consider the mass dimensions of the Hořava-Lifshitz action in (41), where $[S_{HL}] = 0$ as usual and $[\kappa^2]$ is to be determined. One has

$$[dt] = -z \qquad [d^3x] = -3 \qquad [g_{ij}] = [N] = 0 \qquad [K_{ij}] = [K^{ij}] = z \ ,$$

so that one arrives at the equation for mass dimensions of coupling constant

$$\begin{aligned} 0 &= \left[\kappa^2\right] + (-z) + (-3) + 2z \\ \left[\kappa^2\right] &= -z + 3 \ . \end{aligned} \tag{56}$$

In the specific case of $z = 3$, which is acceptable from the point of view of the theory, κ^2 is indeed mass dimensionless and the theory exhibits power-counting renormalizability, as desired.

Another consequence of employing a Lifshitz-type model in the theory is the immediate violation of Lorentz invariance, which comes as a cost of keeping the quantum field theory finite. This can be shown as follows. Consider a scalar field action for a field φ with $z = 2$:

$$S = \int dt d^3y \left[\frac{1}{2} \left(\dot{\varphi}(y)\right)^2 - \frac{1}{2} \overrightarrow{\partial} \varphi(y) \left(M^2 - \Delta\right) \cdot \overrightarrow{\partial} \varphi(y) - U\left(\varphi(y)\right) \right] \tag{57}$$

Taking the field derivative of the action

$$
\begin{aligned}
\frac{\delta S}{\delta\varphi(x)} &= \int dt d^3y \Big\{\dot{\varphi}(y)\partial_0\delta^{(4)}(x-y) - M^2\overrightarrow{\partial}\varphi(y)\cdot\overrightarrow{\partial}\delta^{(4)}(x-y) + \\
&\quad +\overrightarrow{\partial}\delta^{(4)}(x-y)\Delta\cdot\overrightarrow{\partial}\varphi(y) - U'(\varphi(y))\delta^{(4)}(x-y)\Big\} \\
&= -\ddot{\varphi} + M^2\Delta\varphi - \Delta^2\varphi - U'(\varphi)\ . \qquad (58)
\end{aligned}
$$

If one now assumes that the field in question is free, i.e. $U(\varphi(y)) = \frac{m^2}{2}(\varphi(y))^2$ and considers a plane wave solution $\varphi = \varphi_0 \exp\left(i\left(\omega t - \overrightarrow{p}\cdot\overrightarrow{r}\right)\right)$ of equation (58), one arrives at

$$
\begin{aligned}
0 &= \left[\omega^2 - M^2\overrightarrow{p}^2 - \left(\overrightarrow{p}^2\right)^2 - m^4\right]\varphi\ , \qquad (59) \\
\omega^2 &= M^2\overrightarrow{p}^2 + \left(\overrightarrow{p}^2\right)^2 + m^4\ . \qquad (60)
\end{aligned}
$$

In order for all terms to be of mass dimension one, one needs to rescale $\omega = M\tilde{\omega}$ such that

$$
\begin{aligned}
M^2\tilde{\omega}^2 &= M^2\overrightarrow{p}^2 + \left(\overrightarrow{p}^2\right)^2 + m^4\ , \\
\tilde{\omega}^2 &= \overrightarrow{p}^2 + \mu^4 + \frac{\left(\overrightarrow{p}\right)^4}{M^2}\ , \qquad (61)
\end{aligned}
$$

where $\mu = m^2/M$. One identifies the first two terms with a relativistic, Lorentzian dispersion relation while the non-relativistic, third term is a consequence of the Lifshitz model. One can now recognise that in the low energy (infrared) region where $|\overrightarrow{p}| \ll M$, the non-relativistic term is of negligible order. This region stays in agreement with Lorentz invariance and the theory is compatible with observation. In the higher energy, ultraviolet limit, however, this ratio cannot be ignored; in this "Lifshitz" region dispersion relation is non-relativistic and Lorentz invariance is lost. In order for the theory to agree with the observed Lorentz covariance at low energy scales, one can fix the value of the theory regulator M as very large, for instance approaching the order of the Planck mass in equation (**??**). Since no experiments have ever reached that region, the third term of (61) is considered to be negligible and the theory stays in the Lorentz regime, at least as far as the experimental procedure goes. Theoretically, the inconsistency remains in place; however there exists no experimental evidence suggesting the theory should necessary exhibit Lorentz symmetry in the ultraviolet regimes.

The third, fundamental and immediate consequence of the construction of the theory is the breaking of the four-dimensional diffeomorphism of space-time, one of the most fundamental results of General Relativity. In Hořava-Lifshitz theory of gravity, the time is given a preferred direction, which the theory provides for by the way of utilising the ADM decomposition of space-time and equipping space-time manifold with a preferred structure of a codimension-one foliation $\mathcal{F}$, by slices of constant time. The transition functions of the manifold become the foliation-preserving diffeomorphisms, $\mathrm{Diff}(M, \mathcal{F})$ and the generators of the diffeomorphism transform the fields of the theory as

$$
\begin{aligned}
\delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f \dot{g}_{ij} \ , && (62)\\
\delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + \dot{f} N_i + f \dot{N}_i \ , && (63)\\
\delta N &= \zeta^j \partial_j N + \dot{f} N + f \dot{N} \ . && (64)
\end{aligned}
$$

The result is limiting, as discussed in the paragraph below and one may hope to change it by further modification of the theory, in order to reach experimental and conceptual agreement with well tested General Relativity. Such a modification will be discussed in section 3 on covariant Hořava-Lifshitz gravity.

Final consequence, resulting from breaking of the four-dimensional diffeomorphism of space-time to three-dimensional, foliation preserving diffeomorphism is the emergence of an additional degree of freedom and hence an additional scalar polarization of the graviton. Hypothetical graviton is a massless particle (due to infinite-range interaction) spin 2 (due to the requirement for the metric to be symmetric and gauging). One needs to turn attention to Goldstone's theorem [24], which predicts emergence of nonzero vacuum expectation value of a boson field. In other words, due to symmetry breaking, Goldstone's theorem requires appearance of a new, massless scalar particle in the spectrum of possible excitations. This result presents a big, initially omitted, problem to the new theory of gravity as no such scalar modes are predicted otherwise and no such have been observed to exist. For that reason we will turn to study the so-called covariant extension of Hořava-Lifshitz gravity, discussed in section 3.

6 Cubic action

As explained in section 3, it is well known that some fluctuations in density would be formed in the early universe as a result of a fast change in the speed of propagation. The standard single field inflationary models predict a Gaussian spectrum of these primordial fluctuations. The estimation of the size of the non-Gaussian corrections has been predicted in [25], [26] and [27], while [28] discusses the importance of the prediction of this corrections as a validator for any theory aiming to compete with the Inflationary model. Correlation functions, subject to cosmological consistency relations such as the one relating a particular geometrical limit of the three-point function of density perturbations to the spectrum and tilt of the two-point function [29]

$$
\lim_{k_1 \to 0} \langle \zeta_{\vec{\kappa}_1} \zeta_{\vec{\kappa}_2} \zeta_{\vec{\kappa}_3} \rangle = -(2\pi)^3 \delta^3 (\sum_i \vec{k}_i)(n_s - 1) P_{k_1} P_{k_3} \ , \qquad (65)
$$

provide an important gauge for validity of a given theory and its predictions of the primordial spectrum. Hence they are often a first stop in checking the rationale of the theory as a claimant to the role of a substitute for Inflation. We attempt an estimation of the size of non-Gaussianities deriving from the theory of Hořava - Lifshitz gravity. We hence investigate the previously-performed computations of the quadratic action for the theory and we attempt a calculation of the cubic action and as a result a three-point correlation function of Hořava - Lifshitz gravity, in hope to shed some light onto its predictions versus those of the theory of Inflation.

As mentioned in the previous chapter, the theory of Hořava - Lifshitz gravity in its most general and hence least constrained form involves a potential with number of terms of the order of 10^2. For the reason of limited time available for this project and tediousness of calculation, we shall not attempt to derive a cubic action for this version of the theory. Instead, we focus, in turn on the theory constrained by previously explained projectability and detailed balance conditions.

6.1 Projectability

Following reference [30], we consider Hořava - Lifshitz gravity action with the assumption of projectability, $N = N(t)$:

$$S = \frac{M_{Pl}^2}{2} \int dt d^3x N\sqrt{-g} \left((K_{ij})K^{ij} - \lambda K^2) +^{(3)} R - 2\Lambda + V\right) , \tag{66}$$

and in order to derive the cubic action we consider a metric perturbation of the form

$$N = 1 + \alpha(t) , \qquad N_i = \partial_i y , \qquad g_{ij} = e\delta_{ij} + 2(\delta_{ij} + k^{-2}\partial_i\partial_j)\zeta - 2k^{-2}\partial_i\partial_j\chi . \tag{67}$$

Following the approach developed in [31] we find the momentum constraint - the equation of motion for the shift N^i by varying the action of (66). We use the expression for the extrinsic curvature from equation (48)

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \tag{68}$$

and after variation and integration by parts the momentum constraint reads

$$\nabla_j K_i^j - \lambda \nabla_i K = 0 . \tag{69}$$

The three dimensional diffeomorphism and time reparametrisation invariances allow us to choose the following gauge in order to remove residual degree of freedom and simplify calculation as a result:

$$h_{ij} = a(t)^2 e^{2\zeta}\delta_{ij} , \qquad N_i = \partial_i y , \qquad N(t) = 1 . \tag{70}$$

Hence, to leading order, the momentum constraints solves as

$$\partial^2 y = -a^2 \frac{1}{c_\zeta^2} \dot{\zeta} , \tag{71}$$

where we assume $\lambda \neq 1$ and define the speed of sound

$$c_\zeta^2 = \frac{1-\lambda}{3\lambda - 1} . \tag{72}$$

The three dimensional Ricci scalar can be computed to be

$$ {}^{(3)}R = -2a(t)^{-2}e^{-2\zeta}(\partial_i\partial^i + 2\partial^2\zeta)\,, \tag{73} $$

and hence the quadratic action is found to be

$$ S_2 = \int dtd^3x \left(-a^3\frac{1}{c_\zeta^2}\dot{\zeta}^2 + a(\partial\zeta)^2 \right)\,. \tag{74} $$

We notice that for $c_\zeta^2 > 0$ the action has a wrong sign - the kinetic term is negative and hence the scalar graviton, whose propagation is described by the above action becomes a ghost. The other possibility, avoiding ghost propagation, is that of $c_\zeta^2 < 0$. But this is the unstable case and the time scale of the instability goes like $1/|c_\zeta|M$, where M is the UV scale at which the higher derivative terms become important. In order to avoid the instability within the age of the Universe, we need $|c_\zeta| \sim H_0/M$. Hence we either need $\lambda \approx 1$ or the UV scale M to be very low in order to avoid this instability. In order to describe this problem more accurately, we need to consider higher order, cubic interactions. Following the same steps as above but this time to third order, the cubic action is found to be

$$ S_3 = \int dtd^3x \left\{ a\zeta\partial_i\zeta\partial^i\zeta - \frac{3a^3}{c_\zeta^2}\zeta\dot{\zeta}^2 + \frac{3}{2a}\zeta\left(\partial_i\partial_j y\partial^i\partial^j y - (\partial^2 y)^2\right) - \frac{2}{a}\partial^2 y\partial_i\zeta\partial^i y \right\}\,. \tag{75} $$

Using this action, we can discuss another problem of this version of the theory - that of strong coupling. Restoring the Planck scale and normalising our variable as

$$ \zeta_{norm} = M_{Pl}c_\zeta^{-1/2}\zeta \tag{76} $$

we see that in the $c_\zeta \approx 0$, so $\lambda \approx 1$ case the cubic interactions blow up, resulting in the loss of predictability due to uncontrollable quantum cubic corrections. For this reason, we abandon further analysis of the projectable version of Hořava - Lifshitz theory and brand it as pathological. Instead, we take a closer look at the detailed balance condition.

6.2 Detailed balance

To begin with, we review a derivation of the quadratic action as presented in reference [32] and hence begin with the with the action of Hořava - Lifshitz gravity without projectability but with detailed balance condition imposed

$$
\begin{aligned}
S_H &= \frac{M_{\rm pl}^2}{2}\int dtd^3x\sqrt{g}N\Bigg\{K_{ij}K^{ij} - \lambda K^2 + \xi R - 2\Lambda + \eta\, a^i a_i - \frac{1}{M_4^2}R_{ij}R^{ij} + \frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}R^2 \\
&+\frac{2\eta}{\xi M_4^2}\left[\frac{1-4\lambda}{4(1-3\lambda)}Ra^i a_i - R_{ij}a^i a^j\right] - \frac{\eta^2}{4\xi^2 M_4^2}\frac{3-8\lambda}{1-3\lambda}(a^i a_i)^2 + \frac{2}{M_6^2 M_4}\epsilon^{ijk}R_{il}\nabla_j R^l_k \\
&+\frac{2\eta}{\xi M_6^2 M_4}C^{ij}a_i a_j - \frac{1}{M_6^4}C_{ij}C^{ij}\Bigg\}\,, \qquad (77)
\end{aligned}
$$

where we have used the redefinitions of some of the couplings in the detailed balance potential of equation (51)

$$
M_{Pl}^2 = \frac{4}{k}\,, \qquad M_6^2 = \frac{w^2}{2}M_{Pl}^2\,, \qquad M_4^2 = \frac{M_{Pl}^4}{\mu^2}\,, \qquad \xi = \frac{\Lambda_W}{(1-3\lambda)M_4^2}\,, \qquad (78)
$$

and we perturb the metric as

$$
N = 1+\alpha\,, \qquad N_i = \partial_i y\,, \qquad g_{ij} = e^{2\zeta}\delta_{ij}\,. \qquad (79)
$$

To second order, the Ricci scalar and tensor can be calculated as

$$
R_{ij} = -\partial_i\partial_j\zeta - \delta_{ij}\partial^2\zeta + \partial_i\zeta\partial_j\zeta - \delta_{ij}\partial_k\zeta\partial^k\zeta \qquad (80)
$$

$$
R = -e^{-2\zeta}\left(4\partial^2\zeta + 2(\partial\zeta)^2\right) \qquad (81)
$$

For the quadratic action, we only need to calculate K_{ij} and K to first order, as they appear in the action quadratically. Hence we obtain

$$
K^{(1)}_{ij} = \dot\zeta\delta_{ij} - \partial_i\partial_j y\,, \qquad (82)
$$

$$
K^{(1)} = 3\dot\zeta - \partial^2 y\,. \qquad (83)
$$

After a straightforward substitution, we obtain the quadratic action as

$$
\begin{aligned}
S^{(2)} &= \frac{M_{Pl}^2}{2}\int dtd^3x\Big\{3(1-\lambda)\dot\zeta^2 - 2(1-3\lambda)\dot\zeta(\partial^2 y) + (1-\lambda)(\partial^2 y)^2 + 2\xi(\partial\zeta)^2 \\
&- 4\xi\alpha\partial^2\zeta + \eta(\partial_i\alpha)(\partial^i\alpha)\frac{2(1-\lambda)}{1-3\lambda}\frac{1}{M_4^2}(\partial^2\zeta)^2\Big\}\,. \qquad (84)
\end{aligned}
$$

As discussed in the reference [32], we have set $\Lambda = 0$ in action (77) by hand. This is done as to establish whether the dynamics of the scalar mode in this version of the theory exhibits improved behavious compared to the projectability condition. The presence of a large cosmological constant Λ disturbs this discussion as it has both phenomenological and practical consequences, hence we work under the assumption that if a resolution to the large cosmological constant problem was to be found, the scalar mode obey the following dynamics. It is, however, an outstanding issue and must not be overlooked before reaching the final conclusions.We next attempt to integrate out the non-dynamical degrees of freedom y and α using their equations of motion. The variation of the quadratic action with respect to y gives

$$(1-\lambda)\partial^2 y - (1-3\lambda)\partial^2\dot{\zeta} = 0\,, \tag{85}$$

which solves to

$$\partial^2 y = \frac{1-3\lambda}{1-\lambda}\dot{\zeta} = \frac{1}{c_\zeta^2}\dot{\zeta}\,. \tag{86}$$

Variation of the action with respect to α in turn gives

$$\eta\partial^2\alpha + 2\xi\partial^2\zeta = 0 \tag{87}$$

and solves to

$$\alpha = -\frac{2\xi}{\eta}\zeta\,. \tag{88}$$

After a direct substitution, the quadratic action, now expressed in terms of the dynamical field ζ only, reads

$$S^{(2)} = \frac{M_{Pl}^2}{2}\int dtd^3x \left\{2\frac{1}{c_\zeta^2}\dot{\zeta}^2 + 2\xi\left(\frac{2\xi}{\eta}-1\right)\zeta\partial^2\zeta \ - 2c_\zeta^2\frac{1}{M_{Pl}^2}(\partial^2\zeta)^2\right\}\,. \tag{89}$$

We can now substitute a plane - wave solution of the form

$$\zeta = \zeta_0 e^{i(p.x-\omega.t)}\,. \tag{90}$$

in order to obtain a dispersion relation for the scalar, which reads

$$\omega^2 = \xi\left(\frac{2\xi}{\eta}-1\right)c_\zeta^2 p^2 + \frac{1}{M_{Pl}^2}c_\zeta^4 p^4\,. \tag{91}$$

We can immediately see that we run into some problems, as the dispersion relation does not have the expected p^6-terms, as discussed in the previous section. In order to obtain these, we would apparently need to add even higher order terms to the superpotential and hence the action. We proceed nevertheless with the calculation of the cubic action.

Since K_{ij} appears quadratically in the action of (77), for the cubic action we now need to compute it up to second order:

$$K_{ij}^{(2)} = (1-\alpha)\left(\dot{\zeta}\delta_{ij} - \partial_i\partial_j y\right) + 2\dot{\zeta}\zeta\delta_{ij}\,, \tag{92}$$

$$K^{(2)} = (1-\alpha)\left(3\dot{\zeta} - \partial^2 y\right) - 2\zeta\partial^2 y\,. \tag{93}$$

Following the same procedure as for the quadratic action, we next expand to cubic order each of the terms in the action, presented here term by term:

$$K_{ij}K^{ij} = (1-2\alpha)\left(3\dot{\zeta}^2 + (\partial^2 y)^2 - 2\dot{\zeta}\partial^2 y\right) \tag{94}$$

$$-\lambda K^2 = -\lambda\left\{(1-2\alpha)\left(9\dot{\zeta}^2 - 6\dot{\zeta}\partial^2 y + (\partial^2 y)^2\right) - 4\left(3\dot{\zeta} - \partial^2 y\right)\zeta\partial^2 y\right\} \tag{95}$$

$$\xi R = -\xi\left\{2\left(2\partial^2\zeta(\partial\zeta)^2\right) + 8\zeta^2\partial^2\zeta\right\} \tag{96}$$

$$\eta a^i a_i = \eta\,(1-2\alpha)\left(\partial_i\alpha\partial^i\alpha\right) \tag{97}$$

$$-\frac{1}{M_4^2}R_{ij}R^{ij} = -\frac{8}{M_4^2}\partial^2\zeta\partial_k\zeta\partial^k\zeta \tag{98}$$

$$\frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}R^2 = \frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}\left\{16\,(1-4\zeta)\,(\partial^2\zeta)^2 + 16(\partial\zeta)^2\partial^2\zeta\right\} \tag{99}$$

$$\frac{1-4\lambda}{4(1-3\lambda)}Ra^i a_i = -\frac{1-4\lambda}{4(1-3\lambda)} \tag{100}$$

$$-R_{ij}a^i a^j = \partial_i\partial_j\zeta\partial^i\alpha\partial^j\alpha + \partial^2\partial_i\alpha\partial^i\alpha \tag{101}$$

$$\frac{3-8\lambda}{1-3\lambda}(a^i a_i)^2 = 0 \tag{102}$$

$$\epsilon^{ijk}R_{il}\nabla_j R^l_k = 0 \tag{103}$$

$$C^{ij}a_i a_j = \left(\partial_m\partial^2\zeta + \frac{1}{2}\partial_m(\partial\zeta)^2\right)\epsilon^{mij}\partial_i\alpha\partial_j\alpha = 0 \tag{104}$$

$$C_{ij}C^{ij} = 0\,. \tag{105}$$

Finally,

$$\sqrt{g}N = e^{3\zeta}(1+\alpha) \tag{106}$$

such that after substitution into (77) the full cubic action, after some simplification, can be written as:

$$
\begin{aligned}
S_3 \quad &= \frac{M_{\rm pl}^2}{2}\int dtd^3x\Bigg\{6(3\lambda-1)\alpha\dot{\zeta}^2+2(\lambda-1)\alpha(\partial^2 y)^2-4(3\lambda-1)\alpha\dot{\zeta}\partial^2 y \\
&+4\lambda\left(3\dot{\zeta}-\partial^2 y\right)\zeta\partial^2 y-4\xi\partial^2\zeta\left[(\partial\zeta)^2+2\zeta^2\right]-2\eta\alpha\left(\partial_i\alpha\partial^i\alpha\right)-\frac{8}{M_4^2}\partial^2\zeta\partial_k\zeta\partial^k\zeta \\
&+\frac{1-4\lambda}{(1-3\lambda)}\frac{4}{M_4^2}\partial^2\zeta\left(-4\zeta\partial^2\zeta+(\partial\zeta)^2\right)+\frac{2\eta}{\xi M_4^2}\left(\partial_i\partial_j\zeta\partial^i\alpha\partial^j\alpha\right)\Bigg\}. \qquad (107)
\end{aligned}
$$

We proceed with the variation of the action in order to integrate out the non-dynamical fields y and α. Variations with respect to y and α produced no informative results in the course of this investigation, so in order to achieve a rough idea of leading and sub-leading contributions to the action we take the liberty of substituting the previously-derived, first order expressions for the non-dynamical variables, bearing in mind that from now on we are working in approximation. We hence use

$$
\partial^2 y=\frac{1-3\lambda}{1-\lambda}\dot{\zeta}=\frac{1}{c_\zeta^2}\dot{\zeta}\,, \qquad (108)
$$

$$
\alpha=-\frac{2\xi}{\eta}\zeta\,, \qquad (109)
$$

and substitute into the cubic action to obtain

$$
\begin{aligned}
S_3 \quad &= \frac{M_{\rm pl}^2}{2}\int dtd^3x\Bigg\{\frac{12\xi}{\eta}(1-3\lambda)\zeta\dot{\zeta}^2+\frac{4\xi}{\eta}(1-\lambda)\frac{1}{c_\zeta^4}\zeta\dot{\zeta}^2+\frac{8\xi}{\eta}(3\lambda-1)\frac{1}{c_\zeta^2}\zeta\dot{\zeta}^2 \\
&+4\lambda\left(3-\frac{1}{c_\zeta^2}\right)\frac{1}{c_\zeta^2}\zeta\dot{\zeta}^2-4\xi\partial^2\zeta\left[(\partial\zeta)^2+2\zeta^2\right]+\frac{16\xi^3}{\eta^2}\zeta\partial_i\zeta\partial^i\zeta-\frac{8}{M_4^2}\partial^2\zeta\partial_k\zeta\partial^k\zeta \\
&+\frac{1-4\lambda}{(1-3\lambda)}\frac{4}{M_4^2}\partial^2\zeta\left(-4\zeta\partial^2\zeta+(\partial\zeta)^2\right)+\frac{8\xi}{\eta M_4^2}\partial_i\partial_j\zeta\partial^i\zeta\partial^j\zeta\Bigg\}. \qquad (110)
\end{aligned}
$$

Finally, we have obtained an expression for the cubic action in terms of the dynamical field ζ only and can perform some analysis.

6.3 Discussion

We have performed here initial computation necessary for the estimation of deviations from Gaussianity in the primordial spectrum of Cosmic Microwave Background fluctuations, necessary for assertion whether the theory of Hořava-Lifshitz gravity, as a higher derivative theory, stands a chance of explaining those effects and hence serving as an alternative to the theory of Inflation.

Even at this initial stage, however, we have run into some severe problems. Firstly, against our initial expectations, the most constrained version of the theory - that with the projectability condition imposed - produced a pathological dynamics for the scalar graviton mode. Having as a result rejected this option, we have moved on to the detailed balance condition, which in turn appears to be haunted by the issue of a large cosmological constant Λ. Following the advice of the authors of reference [32], we have fixed this problem by hand and worked under a supposition that one day it may be solved. After a computation of the quadratic action we have already spotted that the dispersion relation obtained as a result does not contain the desired p^6 terms, which are responsible for production of non-Gaussian effects, as discussed in [13]. At this stage we felt it appropriate to abandon further derivation of the three-point correlation function and arrived at the conclusion that if the desired effect was to be achieved, one would need to add higher order terms to the detailed balance superpotential W. Adding fourth order terms to W would result in appearance of sixth and eight order terms in the potential V and hopefully p^6 terms in the dispersion relation, unless some cancellation occurs. The possible fourth order terms one could add are R^2, $R_{\mu\nu}R^{\mu\nu}$, $R\nabla^i a_i$, $R^{ij}a_i a_j$, $Ra_i a^i$, $(a_i a^i)^2$, $(\nabla^i a_i)^2$ and $a_i a_j \nabla^i a^j$. Addition of these terms would render the calculation considerably more involved and hence we leave it as an open end for future investigation.

7 Conclusions

A full investigation into the current status of our knowledge of the origins on primordial non-Gaussianities in the Cosmic Microwave Background spectrum has been performed. As a result, we have learned to appreciate the importance of the topic, reviewed the most acclaimed and popular scenario of Inflation as well as a few interesting alternative theories, which manage to arrive at similar predictions, such as Variable Speed of Light / Sound theories. We have then moved on to consider the theory of Hořava - Lifshitz gravity, which has been summarised and attempt to apply it to the topic in question. After failing to achieve tangible results, mostly due to the time constraints of the project, we left the question open for future work. We feel it is crucial that some of the most burning, outstanding issues in modern physics are answered. Equally, we deem it eye-opening to be able to review so many different angles of approaching the same, fundamental, question and using drastically different means arrive at seemingly similar conclusions. We believe it is a good indication of the fact that our current understanding of the Universe, however hight esteem we may have of it, is by no means complete and consistent. This works as an encouragement for future generations to study this fundamental problems.

References

[1] S. Weinberg, *Cosmology,* (2008) Oxford Univ. Press

[2] A. Friedmann, Z. Phys. **21** (1924) 326 [Gen. Rel. Grav. **31** (1999) 2001].

[3] G. Gamow, Phys. Rev. **70** (1946) 572.

[4] R. A. Alpher, H. Bethe and G. Gamow, Phys. Rev. **73** (1948) 803.

[5] M. J. White, astro-ph/9903232.

[6] A. H. Guth, Phys. Rev. D **23** (1981) 347.

[7] P. A. R. Ade *et al.* [BICEP2 Collaboration], Phys. Rev. Lett. **112** (2014) 241101 [arXiv:1403.3985 [astro-ph.CO]].

[8] R. Flauger, J. C. Hill and D. N. Spergel, JCAP **1408** (2014) 039 [arXiv:1405.7351 [astro-ph.CO]].

[9] J. Magueijo, Rept. Prog. Phys. **66** (2003) 2025 [astro-ph/0305457].

[10] M. A. Clayton and J. W. Moffat, Int. J. Mod. Phys. D **12** (2003) 697 [astro-ph/0203164].

[11] J. Magueijo, Phys. Rev. Lett. **100** (2008) 231302 [arXiv:0803.0859 [astro-ph]].

[12] J. Garriga and V. F. Mukhanov, Phys. Lett. B **458** (1999) 219 [hep-th/9904176].

[13] G. Amelino-Camelia, M. Arzano, G. Gubitosi and J. Magueijo, Phys. Rev. D **87** (2013) 12, 123532 [arXiv:1305.3153 [gr-qc]].

[14] J. Magueijo, Class. Quant. Grav. **25** (2008) 202002 [arXiv:0807.1854 [gr-qc]].

[15] P. Horava, Phys. Rev. D **79** (2009) 084008 [arXiv:0901.3775 [hep-th]].

[16] T. P. Sotiriou, J. Phys. Conf. Ser. **283** (2011) 012034 [arXiv:1010.3218 [hep-th]].

[17] S. Weinberg, "Ultraviolet Divergences in Quantum Theories of Gravitation," in: "General Relativity, An Einstein Centrary Survey," Cambridge University Press (1979) eds: S. W. Hawking, W. Israel.

[18] J. Alexandre, Int. J. Mod. Phys. A **26** (2011) 4523 [arXiv:1109.5629 [hep-ph]].

[19] C. W. Misner, K. S. Thorne, J. A. Wheeler, "Gravitation," W. J. Freeman and Company (1973)

[20] A. M. da Silva, Class. Quant. Grav. **28** (2011) 055011 [arXiv:1009.4885 [hep-th]].

[21] S. Dutta and E. N. Saridakis, JCAP **1005** (2010) 013 [arXiv:1002.3373 [hep-th]].

[22] K. Lin, S. Mukohyama and A. Wang, Phys. Rev. D **86** (2012) 104024 [arXiv:1206.1338 [hep-th]].

[23] T. P. Sotiriou, M. Visser and S. Weinfurtner, JHEP **0910** (2009) 033 [arXiv:0905.2798 [hep-th]].

[24] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. **127** (1962) 965.

[25] T. J. Allen, B. Grinstein and M. B. Wise, Phys. Lett. B **197** (1987) 66.

[26] T. Falk, R. Rangarajan and M. Srednicki, Phys. Rev. D **46** (1992) 4232 [astro-ph/9208002].

[27] V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, Nucl. Phys. B **667** (2003) 119 [astro-ph/0209156].

[28] A. Gangui, F. Lucchin, S. Matarrese and S. Mollerach, Astrophys. J. **430** (1994) 447 [astro-ph/9312033].

[29] P. Creminelli and M. Zaldarriaga, JCAP **0410** (2004) 006 [astro-ph/0407059].

[30] K. Koyama and F. Arroja, JHEP **1003** (2010) 061 [arXiv:0910.1998 [hep-th]].

[31] J. M. Maldacena, JHEP **0305** (2003) 013 [astro-ph/0210603].

[32] D. Vernieri and T. P. Sotiriou, Phys. Rev. D **85** (2012) 064003 [arXiv:1112.3385 [hep-th]].

[33] Alexandre, Jean and Kostacinska, Martyna Galaxies, 2-1, (2013), DOI = 10.3390/galaxies2010001 ,http://www.mdpi.com/2075-4434/2/1/1