# ENTANGLEMENT ON SPIN NETWORKS
IN
# LOOP QUANTUM GRAVITY


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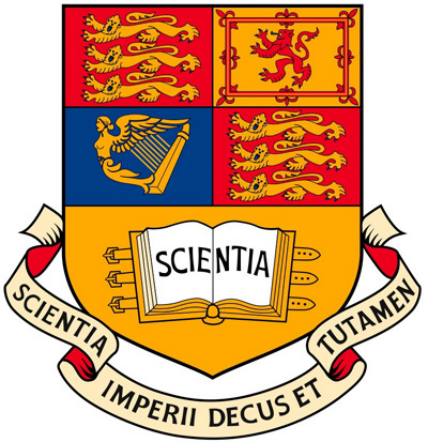



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# Contents

# Introduction

At the beginning of the $20^{\text{th}}$ century, General Relativity has deeply modified the foundation of physics established since Newton. General Relativity was a huge conceptual shift which has achieved great success since then and can now be considered as established knowledge, one of the fundamental properties it taught us being the relationality of spacetime localization. Almost a century after this breakthrough, it can be found behind widespread technologies such as the Global Positioning System (GPS) [33]. A few years later, the quantum theory had a similar impact on the conceptual foundation of classical physics.

Within the framework of modern scientific knowledge, Quantum Mechanics plays a fundamental role in the understanding and the description of natural phenomena. Indeed, as soon as we are dealing with a phenomenon which takes place at very small scales, classical physics is not relevant anymore and we need Quantum Mechanics to explain it. Furthermore, even if we consider macroscopic objects such as a pendulum, we should start by studying the microscopic entities which constitute the pendulum in order to fully capture the essence of its behavior, even if we can already study it, with a certain amount of success, using classical physics. In this sense, Quantum Mechanics provides the fundamental blocks of our understanding of the universe. Consequently, it seems natural to look for a quantum theory of General Relativity in order to fully apprehend the mechanisms which drive what we experiment everyday.

Although these two theories are now considered to be correct thanks to numerous empirical successes, they do not seem to be complete yet. Indeed, both theories carry well-known inconsistencies. For instance, the description of Black Holes involves spacetime singularities where the curvature represented by the Ricci scalar becomes infinite [42]. Near these singularities the laws of classical General Relativity break down. In addition, it is is also well known that with Quantum Field Theory, we must deal with the UV divergences [31]. These different issues are the testimony of the incompleteness of the theories. Relativists believe these theories are the starting point of a bigger conceptual revolution which, at the end, will bring them together.

However, and we will spend some time trying to explain it, it is a tough task to unify General Relativity and Quantum Field Theory (QFT) as they are mathematically and conceptually incompatible. Quantum Gravity would provide the framework necessary to support both theories and resolve the different issues. Furthermore, this unified theory would reduce into one theory or the other in some specific physical regimes. We then would be able to describe gravitational interaction at every regime as from the one of Planck scale. Beyond the resolution of Planck length $l_{\mathcal{P}} \equiv \sqrt{\hbar G/c^3}$, the objects are "hidden" behind a gravitational horizon and cannot be studied. The corresponding energy to the Planck scale is the Planck energy $E_{\mathcal{P}} \equiv \sqrt{\hbar c^5/G}$ from which our current physical laws do not apply anymore.

The quantum theory and General Relativity have both radically changed our vision of the world provided by prerelativistic classical physics. As we have just mentioned it, these upheavals are based on assumptions which appear to contradict each other. Indeed, Quantum Mechanics is formulated on a fixed background whereas the main feature of General Relativity is the background independence. The background independence comes from the fact that the geometry of spacetime fully encodes the degrees of freedom of the gravitational field. Furthermore, in the Schrodinger picture of Quantum Mechanics, we make use of a global time variable in order to compute the expectation values; such a variable does not exist in the general covariant theory of General Relativity. Moreover, we know from the quantum theory that any dynamical field must be quantized. Therefore, combining the fact that the spacetime of GR is dynamical and any dynamical entity of QM is decomposed into quanta, we expect a quantization of spacetime.

These contradictions mentioned above explain why we do not have a successful theory of quantum gravity yet. Indeed, we now manage to describe matter interactions with Quantum Field Theory and it may seem surprising that the usual procedure does not apply easily in the case of General Relativity. The key reason why it is hard to combine the principles of both theories, is that QFT only takes into account Special Relativity. Indeed, QFT is built on a structure provided by the Minkowski background metric. Such a fundamental structure is notably necessary for the axiomatic notions of Locality and Causality. However, and it cannot be stressed enough, the main feature of GR is the background independence making these two notions meaningless. One idea was to consider a splitting of the metric into a Minkowski background metric and a perturbation piece [30]. However, this strategy still violates the independence requirement as well as another feature of GR which is the diffeomorphism invariance. These reasons are sufficient to say that we should look for a background independent non-perturbative theory of Quantum Gravity.

The point of view described above is the one of Loop Quantum Gravity which does not propose a QFT on a background spacetime but on a differential manifold. Therefore, both matter and geometry become dynamical entities and they interact with each other. Obviously, the path followed by Loop Quantum Gravity is not the only one studied, the most famous one being String Theory which tries to describe everything by unifying all the interactions within a single global theory.

The conceptual path to the theory that we have described above induces that Loop Quantum Gravity is an attempt to quantize directly General Relativity and therefore, it is sometimes called Canonical Quantum General Relativity. In this context, we consider a hypersurface representing space which recreates spacetime by evolving with respect to time. Then, we define quantum states of the hypersurface, called spin networks, on which the geometry is quantized. Spin networks which are basically graphs labeled by representations of the gauge group SU(2), can be represented by discretized hypersurfaces constituted of quanta of space. As LQG is background independent, there is no assumed background metric and therefore spin networks, which are the 3D states of geometry, must describe the full metric of space. This means that all the classical geometrical notions such as distance must be reconstructed from the spin networks themselves. More precisely, the lack of background induces the absence of a notion of position without which it is more difficult to define a notion of distance. Indeed, there is not absolute position of a given region of a spin network, but it can be localized with respect to other parts of the graph. Therefore, it is necessary to understand the notions of close and far as relations between parts of a spin network. Consequently, we expect to extract from spin networks information which enable to tell if two regions are more or less distant. The access to such a notion would be an important step in the derivation of an embedding for a spin network which is not, a priori assumed by definition.

In Classical General Relativity, one conventional way to define the spatial distance between two spacetime events is to consider a spacelike curve between them and compute the proper distance of the path [39]. However, such a procedure would require General Relativity tools that we are not able to recover from the quantum framework yet. The strategy followed in this dissertation is to use the fact that spin networks provide relations between different regions of space. How they are related could then tell us how close they are. One natural way to proceed is to seek for the distance as a function of the correlations between the regions. In other words, what we would like to do is use the entanglement as a measure of distance on spin networks.

In the first chapter, we briefly introduce the connection formalism and the fundamental features of LQG. We spend some time describing the Hilbert space of the theory and more precisely the basis states which are formed from spin networks. The mathematical framework of the spin networks is described and we insist on the geometrical interpretation of these labeled graphs through the derivation of the spectrum of the area operator. We then stress that, although this study provides a nice picture of the quantized space, we must be careful when using it.

In the second chapter, we precise how we expect the entanglement to provide a notion of distance and recall some useful definitions of Quantum Information. A first computation of entanglement on spin networks is done in order to become familiar with the notions. We then introduce some further notions, such as coarse-graining, and use them to derive our procedure. All the tools introduced previously are used to calculate the entanglement between two regions in the case of the totally mixed state. This allows to mark the limits to the model and show the conceptual issues attached to the quest of a notion of distance. In the last section of the chapter, we discuss the semi-classical limit of spin networks which we will need to take into account at some point to recover classical notions, such as distance, from the quantum framework of spin networks.

# Chapter 1

# Review of Loop Quantum Gravity

The purpose of this chapter is neither to give a thorough description of the formalism of Loop Quantum Gravity nor the entire conceptual path to the theory. A lot of key aspects of the canonical formalism of Quantum General Relativity will be omitted, however, the aim is to cover and introduce the notions needed to understand the second chapter of this dissertation whose subject is the entanglement on spin networks. There are a lot of review articles which are a nice start to get familiar with the important features of the theory, see for instance [32] [18] [41] [30], as well as the undergraduate-level book by Gambini and Pullin [17]. For detailed information about the path to the theory and some applications, please refer to Rovelli's first book [34]. The book by Thiemann [40] proposes a very detailed description of the mathematical formalism behind the theory.

The review presented here is inspired by the different references quoted above. First we will recall some elements of General Relativity, then we will briefly describe the ADM formalism in order to introduce the Ashtekar variables. Finally we will describe the quantization process and explain in detailed the construction of the Hilbert space of the theory.

We make use of the following index notation:
$\mu$, $\nu$ , ... : tensorial spacetime indices
$a$, $b$, ... : tensorial space indices
$i$, $j$, ... : Lie algebra internal indices

## 1.1 Elements of general relativity

### 1.1.1 Hamiltonian formalism

Before describing the quantization of General Relativity, we will recall some notions about the classical theory. More precisely, we will briefly develop the formalism of canonical General Relativity and stress where the connection formalism comes from. The Hamiltonian formalism of General Relativity [43] is the starting point of the canonical formalism of the theory. The first step is the introduction of the so-called Einstein-Hilbert action from which the theory is derived:

$$S_{EH} = \frac{1}{\kappa}\int d^D x R\sqrt{-g} \tag{1.1}$$

where $R$ is the scalar curvature, $g = det(g_{\mu\nu})$ is the determinant of the metric and $\kappa$ a constant. In the case of 4D gravity, we have $\kappa_{4D} = \frac{16\pi G}{c^3} = \frac{16\pi l_P^2}{\hbar}$. We can show that Einstein's equations derive indeed from this action. Varying the action with respect to the metric we get

$$\delta S_{EH} = \frac{1}{\kappa}\int d^D x \delta(R_{\mu\nu}g^{\mu\nu})\sqrt{-g} + R\delta(\sqrt{-g}) \tag{1.2}$$

$$= \frac{1}{\kappa}\int d^D x \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} \tag{1.3}$$

which gives us the vacuum Einstein's equations. A similar procedure can obviously be done with matter and cosmological terms.

### 1.1.2 3+1 decomposition

With the Einstein-Hilbert action we have the starting point of the Hamiltonian formalism. However, in classical physics, such a formalism is given in terms of a set of canonical variables which creates a problem. Let us consider for instance a mechanical system described by a set of variables $q_i$ and their time derivative $\dot{q}_i$. The corresponding Lagrangian is $L(q_i, \dot{q}_i)$ and the associated action is:

$$S = \int dt L(q_i, \dot{q}_i) \tag{1.4}$$

$$= \int dt (p_i\dot{q}_i - H(p,q)) \tag{1.5}$$

where we have performed the Legendre transformation by defining the canonical momentum $\underline{p}(t) \equiv \frac{\partial L}{\partial \dot{\underline{q}}} = \frac{\delta S}{\delta \dot{\underline{q}}}$. Moreover the Hamiltonian is defined such that $H(p,q) \equiv p_i\dot{q}_i - L$. The equations of the theory are then given by:

$$\dot{\underline{q}} = \{\underline{q}, H\} \quad ; \quad \underline{p} = \{\underline{p}, H\} \tag{1.6}$$

where $\{f, g\}$ is the Poisson bracket of $f$ and $g$ such that $\{f, g\} = \frac{\partial f}{\partial \underline{q}} \frac{\partial g}{\partial \underline{p}} - \frac{\partial f}{\partial \underline{p}} \frac{\partial g}{\partial \underline{q}}$.

In the Hamiltonian formalism, the system is not described by the set of canonical variables $q_i$ and $\dot{q}_i$ but by $q$ and its momenta $p$ at a given instant time $t$. So, we have a natural 3+1 decomposition, between space and time, which emerges if we want to work according to the Hamiltonian formalism. But, such a separation between space and time is not really appropriate as far as General Relativity is concerned because we should treat both space and time the same way. However, such a notion of time is needed by Quantum Mechanics, as presently formulated, in order to compute expectation values. Therefore, the idea is to assume that the manifold $M$, which represents the spacetime we consider, has the topology $M = \mathbb{R} \times \sigma$ where $\mathbb{R}$ represents the time parameter and $\sigma$ the 3D manifold of space.

### 1.1.3 ADM variables

It comes from the 3+1 decomposition that the 3D manifold $\sigma$ foliates into hypersurfaces $\Sigma_t$. In order to understand the meaning of this foliation, let us write the variables introduced by Arnowitt, Deser and Misner in 1960 [3]:

$$q_{ab} := g_{ab} \tag{1.7}$$

$$N_a := g_{a0} \tag{1.8}$$

$$N := \sqrt{-g_{00}} \tag{1.9}$$

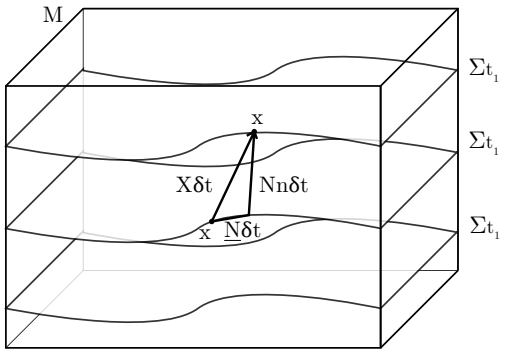


Figure 1.1: Graphical interpretation of the foliation of the manifold $\sigma$ into hypersurfaces $\Sigma_t$.

By doing so, we replace the 10 components of the metric $g_{ab}$ by the 6 components of the induced 3D metric $q_{ab}$ and the 3 components associated with $N$ and $N_a$. The most interesting thing about these variables is that the Lagrangian will not depend on the time derivatives of

$N$ and $N_a$. This makes easier the canonical analysis of the action. Moreover, we can interpret geometrically these so-called ADM variables.

$N$ and $\vec{N}$, respectively called the Lapse and the Shift, allow to describe the evolution with respect to $t$, in a way adapted to the metric, as a normal and a tangential evolution. The metric $q_{ab}$ is the 3D metric induced on the hypersurfaces t = constant $\Sigma_t$. Now, if $n^\mu$ is the unit normal vector to $\Sigma_t$, then the point defined by $x'^\mu = x^\mu + Nn^\mu dt$ is on the hypersurface $\Sigma_{t+dt}$. A similar interpretation can be derived as far as $N_a$ is concerned.

Moreover we define the extrinsic curvature on the hypersurfaces as

$$k_{ab} = \frac{1}{2N}(\dot{q}_{ab} - D_{(a}N_{b)}) \tag{1.10}$$

where $D_a$ is the covariant derivative. In this configuration, we can express the action in terms of the extrinsic curvature $k_{ab}$ and the induced 3D metric $q_{ab}(x)$. In other words, the pair of canonical variables on the phase space $\Gamma$ is $(k_{ab}, q_{ab}(x))$.

### 1.1.4 Connection formalism

#### Triads

In order to derive the connection formalism, we need to reformulate the theory by introducing three vector fields $e^i_a$ called triads, such that the expression of the induced 3D metric is given by:

$$q_{ab}(x) = e^i_a(x)e^j_b(x)\delta_{ij} \tag{1.11}$$

Physically, we can think about the triads as a frame of reference for an inertial observer. We similarly define the co-triads such that:

$$q_{ab}e^a_i e^b_j = e_{ai}e^a_j = \delta_{ij} \tag{1.12}$$

Furthermore, we define the inverse densitized triads of density weight +1 as:

$$E^a_i \equiv \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c \tag{1.13}$$

The densitized triads, also called electric fields, fully encode the geometry of the 3D manifold $\sigma$. They are therefore sufficient to express all geometrical quantities in space. Using the expression $e^i_b = e^a_j q_{ab}\delta^{ij}$, we get the induced metric as the square of the inverse of the densitized triad:

$$E^a_i(x)E^b_i(x) = q(x)q^{ab}(x) \tag{1.14}$$

where $q(x) = det(q_{ab}(x))$. We have defined $3 \times 3 = 9$ densitized triads which may seem redundant compared to the previous 6 components of the 3D induced metric $q_{ab}$. The three

extra components corresponds to the SO(3) symmetry which illustrates the freedom of choice of the local triads. Besides, we define the corresponding triad version of the extrinsic curvature such that $K_a^i e_b^j \equiv k_{ab}$ i.e.

$$K_a^i = \frac{k_{ab} E^{bi}}{\sqrt{q}} \tag{1.15}$$

**Ashtekar's variables**

The new pair of canonical variables on the phase space is $(E_i^a(x), K_a^i(x))$. The next step is to define the new connection $A_a^i$ such that [4]:

$$A_a^i = w_a^i + \beta K_a^i \tag{1.16}$$

where $w_a^i \equiv w_{aj}^i \tau^j$ with $\tau^j$ the generators of $SU(2)$ in the adjoint representation and $w_{aj}^i$ the spin connection. The new internal indices introduced by the triads require indeed a covariant derivative which is associated with the spin connection $w_{aj}^i$ such that:

$$D_a \phi^i = \partial_a \phi^i + w_{aj}^i \phi^j \tag{1.17}$$

$$D_a \phi_i = \partial_a \phi_i + w_{ai}^j \phi_j \tag{1.18}$$

Furthermore, in case we take the derivative of a mixed-indices object, we have the following formula:

$$D_a v_b^i = \partial_a v_b^i + w_{aj}^i v_b^j - \Gamma_{ab}^c v_c^j \tag{1.19}$$

where $\Gamma_{ab}^c$ is the Christoffel symbol. Let us note that in order to define the spin connection $w$, we require the torsion freeness and metricity conditions of the Levi-Civita connections i.e.

$$\Gamma_{ab}^c = \Gamma_{(ab)}^c \quad ; \quad \nabla_a g_{bc} = 0 \tag{1.20}$$

The connection defined in equation 1.16 is the so-called Ashtekar's connection where $\beta$ is an arbitrary parameter called the Immirzi parameter [1]. This new connection with the corresponding electric field $E_i^a = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e_b^j e_c^k$ form a canonically conjugate pair. Indeed, we have the following Poisson brackets:

$$\{A_a^i(x), A_b^j(y)\} = 0 = \{E_i^a(x), E_j^b(y)\} \tag{1.21}$$

$$\{A_a^i(x), E_j^b(y)\} = \beta G \delta_a^b \delta_j^i \delta^{(3)}(x, y) \tag{1.22}$$

[1] The value of this parameter needs to be fixed at the value of $\frac{ln(2)}{\pi\sqrt{3}}$ when we want the formula of Black Holes entropy derived from the LQG theory to match the one derived by Bekenstein and Hawking [5]. The key feature of this constant is whether it is real or complex [19]. This choice has huge consequences as far as the construction of the quantum constraints is concerned, here we will only consider the case of a real connection A.

Besides, within this framework the constraints of General Relativity which arise from the Hamiltonian formalism take the form

$$\epsilon_{ijk} F^i_{ab} E^{aj} E^{bk} = 0 \tag{1.23}$$

$$G^i \equiv D_a E^{ai} = 0 \tag{1.24}$$

$$F^i_{ab} E^{bi} = 0 \tag{1.25}$$

where $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b$ is the curvature. The simplicity of the Poisson brackets with the new variables is the key feature of the formalism. Indeed, if the Poisson brackets didn't take this nice form, it would be too complicated to quantize and derive the canonical commutation relations. Furthermore, we note that with this new pair of canonical variables, the structure of the phase spaces of General Relativity and $SU(2)$ Yang-Mills Theory is the same.

## 1.2 Quantization of the theory

### 1.2.1 Outlook of the construction of the Hilbert space

We have described in the previous section the rough construction of the new pair of canonical variables introduced by Ashtekar. As we saw, this configuration leads to very simple Poisson brackets which makes a lot easier the quantization and more precisely, the promotion of the brackets to commutators. Furthermore, such a construction is very helpful in order to compute the quantum constraints of General Relativity.

In order to realize the full quantum implementation we need to construct the quantum constraints associated with the classical ones (Gauss's constraint, the diffeomorphism constraint and the Hamiltonian constraint), promote the Poisson brackets to commutators and the canonical variables to quantum operators.

In the previous setting of the metric formalism, we would have defined states as wave functions of the 3D induced metric $q_{ab}$. The natural consequence of the connection formalism introduced above is to define quantum states as functional of the connection. These quantum states live in a Hilbert space with a scalar product allowing the computation of expectation values. Furthermore, in order to get physical results, we need to impose the quantum constraints on this Hilbert space. Therefore, if we write $\mathcal{H}$ the kinematical unconstrained Hilbert space and $\mathcal{H}_{phys}$ the Hilbert space of physical quantum states, we will follow the procedure:

$$\mathcal{H} \overset{SU(2)}{\longrightarrow} \mathcal{H}_0 \overset{Diff}{\longrightarrow} \mathcal{H}_{diff} \overset{\widehat{\mathcal{H}}}{\longrightarrow} \mathcal{H}_{phys} \tag{1.26}$$

where $\mathcal{H}_0$ and $\mathcal{H}_{diff}$ are the Hilbert spaces respectively after implementation of the gauge constraint and the diffeomorphism constraint. This means that we will start from a Hilbert space of unconstrained states and then impose one by one the constraints. In fact we will only treat the first two steps and focus on the resulting Hilbert space. Its basis will be the object of interest of the second chapter.

### 1.2.2 Holonomies

Following the procedure described above, the first thing we need to achieve is the construction of the unconstrained Hilbert space. However, in order to implement the constraints afterwards, we require this Hilbert space to carry a unitary representation of $SU(2)$ and a unitary representation of $Diff(\sigma)$. We will first deal with the $SU(2)$ representation.

One of the main difficulties about the promotion of Poisson brackets to commutators is that the brackets of the canonical variables show singularities. In such a case, these are called distributional Poisson brackets. In order to carry out the calculations, one can smear the fields

$A_a^j$ et $E_j^a$ by integrating them against chosen test functions. This allows to get rid off the delta functions. But the smearing of the canonical variables is not obvious. At first we could be tempted to try the following smearing:

$$E(f) \equiv \int_\sigma d^3x f_a^j E_j^a \tag{1.27}$$

$$F(A) \equiv \int_\sigma d^3x F_j^a A_a^j \tag{1.28}$$

where $f_a^j$ and $F_j^a$ are the smearing fields. This choice gives us the following non-distributional Poisson brackets:

$$\{F(A), F'(A)\} = 0 = \{E(f), E(f')\} \tag{1.29}$$

$$\{E(f), F(A)\} = \beta G \int_\sigma d^3x F_j^a f_a^j \tag{1.30}$$

such that $F = F' + g^j G_j$ and $f = f' + g^j G_j$, where the prime denotes the part of the constraint which is not proportional to the Gauss constraint $G_j$. The problem with this choice of test functions is that the the smeared variables don't have a nice behavior under gauge transformations. A lot of ideas have been tried in order to solve this issue and the only solution which has been found is the use of Wilson loops. These functions are defined as traces of the holonomies of the connection along a curve $\gamma$ satisfying $\gamma(0) = \gamma(1)$ i.e. a closed curve. We will discuss later the smearing in more details but we are describing now how Wilson loops are constructed.

The holonomy of the connection A along a curve $\gamma$, denoted $hol[A, \gamma] \in SU(2)$, is the unique solution to the equation

$$\begin{cases} \frac{d}{ds} hol[A, \gamma_s] = hol[A, \gamma_s] A(\gamma(s)) \\ \\ hol[A, \gamma](0) = \mathbb{1}_{SU(2)} \\ \\ hol[A, \gamma_s] \equiv hol[A, \gamma_1] \end{cases} \tag{1.31}$$

where $\gamma_s(t) \equiv \gamma(st)$ with $s \in [0, 1]$ and $A(\gamma(s))$ is the 1-form defined by

$$A(\gamma(s)) \equiv \frac{A_a^j(\gamma(s))\tau_j}{2\dot{\gamma}^a(s)} \tag{1.32}$$

An explicit solution is given using the path ordering $\mathcal{P}$:

$$hol[A, \gamma] = \mathcal{P}exp(\int_\gamma A) = \mathcal{P}exp \int_0^1 ds A(\gamma(s)) \tag{1.33}$$

$$= \mathbb{1}_{SU(2)} + \sum_{n=1}^{\infty} \int_0^1 dt_1 \int_{t_1}^1 dt_2 \ ... \ \int_{t_{n-1}}^1 dt_n A(\gamma(t_1)) ... A(\gamma(t_n)) \tag{1.34}$$

Physically, a holonomy measures how data that we parallel transport along a closed curve fail to be preserved [28]. Furthermore, if we think about holonomies as functionals of the path $\gamma$, we see that they contain exactly the same information as the connection.

### 1.2.3 Structure of the kinematical Hilbert space

As one of the motivation for the use of holonomies is the gauge invariance, we should check the behavior of these objects under such a transformation. Using the fact that under a gauge transformation $g$, we have the following transformation for the connection

$$A \to A_g = g^{-1}dg + g^{-1}Ag \ , \tag{1.35}$$

the holonomy transforms locally as follows:

$$hol[A,\gamma] \to hol[A_g,\gamma] = g(\gamma(0))hol[A,\gamma]g(\gamma(1))^{-1} \tag{1.36}$$

More precisely, this means that a gauge transformation localized in the bulk of the curve $\gamma$ won't change the holonomy but the holonomy will change homogeneously under a gauge transformation on the end points of $\gamma$. Now we can introduce the Wilson loops defined as traces of holonomies:

$$\mathcal{W}[A,\gamma] \equiv -Tr(hol[A,\gamma]) \tag{1.37}$$

The key idea behind the construction of Loop Quantum Gravity was to choose the states $\Psi_\gamma = -Tr(hol[A,\gamma])$ as the basis states of the Hilbert space. However, constructing the Hilbert space that way, we end up with a over-complete basis. The solution was found by Rovelli and Smolin in [37] with the use of so-called "spin networks" which will be defined in the next section.

Let us first consider a graph $\Gamma_E$ which is a finite and ordered collection of smooth oriented curves $\gamma_i$, $i = 1, ..., E$. These curves, which will be called "edges" from now on, are embedded in the $3D$ manifold $\sigma$. Ideally we would like to work without this embedding, which would lead to the notion of abstract spin networks, but we won't treat this aspect in our definition of the Hilbert space. We associate with each $\gamma_i$ a holonomy $hol[A,\gamma_i]$, so that $hol[A,\gamma_i]$ is the holonomy with respect to the connection $A$ along the edge $\gamma_i$. Then we consider a function $f_E$ on the $E$ holonomies

$$f_E(hol_1, ..., hol_E) \tag{1.38}$$

where $hol_i \equiv hol[A,\gamma_i]$. As presented above, a holonomy $hol[A,\gamma_i]$ is an element of the Lie group $SU(2)$. Thus, what we do is that we assign an element $g_i \in SU(2)$ to each edge and therefore we assign an element of $SU(2)^E$ to the graph $\Gamma_E$ so that $f_E : SU(2)^E \to \mathbb{C}$. Consequently, the couples $(\Gamma_E, f_E)$ define the functionals of the connection $A$:

$$Cyl_{\Gamma_E,f_E}(A) \equiv f_E(hol_1, ..., hol_E) \tag{1.39}$$

These functionals are called cylindrical functions and they allow to build a scalar product.

### 1.2.4 Inner product

#### Definition of the scalar product

The construction of the inner product requires a very interesting property of cylindrical functions which is the possibility to rewrite any cylindrical function $Cyl_{\Gamma_n,f_n}(A)$ as another one defined on a graph $\Gamma'_n$, such that $\Gamma_n$ is a subgraph of $\Gamma'_n$. This means that two couples $(\Gamma, f)$ and $(\Gamma', f')$ can define the same cylindrical function such that $Cyl_{\Gamma',f'}(A) = Cyl_{\Gamma,f}(A)$. In other words, if we consider two cylindrical functions such that $Cyl_{\Gamma',f'} = Cyl_{\Gamma'',f''}$, we can rewrite them so that they are defined on the same graph $\Gamma$ where we would have $\Gamma = \Gamma' \cup \Gamma''$. Therefore, we can define an inner product between two functionals constructed on the same graph:

$$\langle \Psi_{\Gamma,f} | \Psi_{\Gamma,g} \rangle = \int \prod_i dh_i \overline{f(hol_1, ..., hol_E)} g(hol_1, ..., hol_E) \tag{1.40}$$

where $dh$ is the Haar measure of SU(2)[2]. Furthermore, this inner product defines the norm $\|\Psi_{\Gamma,f}\| = \langle \Psi_{\Gamma,f} | \Psi_{\Gamma,f} \rangle^{1/2}$. Thus, for a given graph $\Gamma$ with $E$ edges, we can define, with the Haar measure, the unconstrained kinematical Hilbert space $\mathcal{H}_\Gamma = L^2(SU(2)^E)$ as the completion in this norm of the space of the functionals. In order to derive the kinematical Hilbert space of LQG, we just need to sum over all possible graphs $\Gamma$:

$$\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\Gamma \tag{1.42}$$

#### Invariance of the scalar product

We have mentioned that we require the Hilbert space $\mathcal{H}$ to carry unitary representations of $SU(2)$ and $Diff(\sigma)$, which means that the scalar product must be invariant under these transformations. We have already seen that despite the behaviour of the connection under a $SU(2)$ gauge transformation, the holonomy transforms homogeneously:

$$hol[A, \gamma] \to hol[A_g, \gamma] = g(\gamma(0)) hol[A, \gamma] g(\gamma(1))^{-1} \tag{1.43}$$

[2]The Haar measure of SU(2) satisfies the following properties:

$$\int_{SU(2)} dh = \mathbb{1},\ dh = d(\alpha h) = d(h\alpha) = dh^{-1}\ \ \forall \alpha \in SU(2) \tag{1.41}$$

Therefore the cylindrical functions transform the following way:

$$\begin{aligned}\Psi_{\Gamma,f}(A) =& f(hol_1, ..., hol_E) && (1.44)\\ &\to f_g(hol_1, ..., hol_E) && (1.45)\\ &= f\left(g(\gamma_1(0))\, hol_1 g(\gamma_1(1))^{-1}, ..., g(\gamma_E(0))\, hol_E g(\gamma_E(1))^{-1}\right) && (1.46)\\ &= \Psi_{\Gamma,f}(A_g) && (1.47)\\ &= \Psi_{\Gamma,f_g}(A) && (1.48)\end{aligned}$$

Using the fact that the Haar measure is $SU(2)^E$ invariant, this shows that the inner product is invariant under $SU(2)$ gauge transformations. Under a diffeomorphism defined by the smooth, continuous and invertible map $\phi : \sigma \to \sigma$, the connection transforms as a 1-form i.e.

$$A \to \phi^{-1} A \tag{1.49}$$

and so the transformation of the holonomy is

$$hol[A, \gamma] \to hol[\phi^{-1} A, \gamma] = hol[A, \phi.\gamma] \tag{1.50}$$

where $(\phi.\gamma)(s) \equiv (\phi(\gamma(s)))$. Besides, according to the expression of the inner product defined in equation 1.40, the integrand depends on the holonomy but not explicitly on the graph. Therefore a diffeomorphism leaves the inner product invariant.

Now that we have described the rough structure of the unconstrained Hilbert space, it is time to construct its basis and by doing so, introduce the concept of spin network.

### 1.2.5 Construction of the basis

The first step in constructing the basis is to define the notion of colored graph. Recall that we have previously considered a graph $\Gamma_E$ with $E$ edges $\gamma_i$, $i = 1, ..., E$ embedded in the 3D manifold $\sigma$. We now denote $e_i$ the edges, we label each edge with a half-integer spin quantum number $j_e \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}$ and we color each node with an intertwiner $\mathcal{I}_n$.

The coloring by a spin number is actually the assignment of an irreducible representation of $SU(2)$, and we note $\mathcal{H}_j$ the corresponding Hilbert space. The intertwiner at a given node is the intertwiner between the representations associated with the edges joined at the node i.e. the invariant map between the representations spaces $\mathcal{H}_{j_e}$:

$$\mathcal{I}_n : \bigotimes_{e\ ingoing} \mathcal{H}_{j_e} \to \bigotimes_{e\ outgoing} \mathcal{H}_{j_e} \tag{1.51}$$

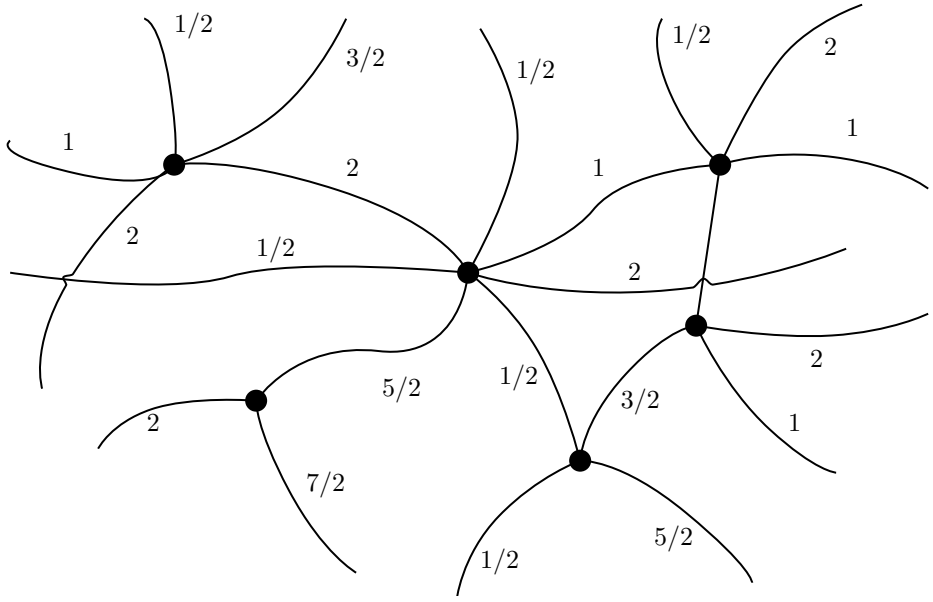


Figure 1.2: An example of complicated spin network.

More precisely, if we consider a p-valent node with the links joined at this node colored by the spin numbers $j_1, ..., j_p$, we associate the Hilbert space $\mathcal{H}_n$ with the node such that:

$$\mathcal{H}_n = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes ... \otimes \mathcal{H}_{j_p} \tag{1.52}$$

Then, the choice of an element within the invariant subspace of this space gives us an intertwiner. Therefore, if we write the decomposition of $\mathcal{H}_n$ into irreducible representations the following way

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes ... \otimes \mathcal{H}_{j_p} = \bigoplus_k (\widetilde{\mathcal{H}}_k)^{m_k} \tag{1.53}$$

the intertwiner we choose for the node $n$ is an element of the subspace $\widetilde{\mathcal{H}}_0^{m_0}$. However, such a procedure is only possible if the graph satisfies some conditions which are necessary in order to have an invariant subspace. When we consider the tensor product of three representation spaces corresponding to the spin numbers $j_1$, $j_2$ and $j_3$, there exists an intertwiner space if the Clebsch-Gordan conditions are satisfied:

$$\begin{cases} j_1 + j_2 + j_3 = N \in \mathbb{Z} \\ |j_1 - j_2| \leq j_3 \leq (j_1 + j_2) \end{cases} \tag{1.54}$$

One interpretation of these conditions is that one of the three representations must appear when we compute the tensor product of the two others. For instance, when we compute the

decomposition of the tensor product $V^{3/2} \otimes V^{3/2}$:

$$V^{3/2} \otimes V^{3/2} = V^3 \oplus V^2 \oplus V^1 \oplus V^0 \tag{1.55}$$

$V^2$ appears in the decomposition, hence the existence of the intertwiner subspace to the tensor product space:

$$V^{3/2} \otimes V^{3/2} \otimes V^2 = V^5 \oplus 2V^4 \oplus 3V^3 \oplus 4V^2 \oplus 3V^1 \oplus V^0 \tag{1.56}$$

We notice that, in this case, the intertwiner subspace has dimension 1. The Clebsch-Gordan conditions can also be interpreted graphically. In order to do so, we need to display each $j_i$-representation by $2j_i$ lines and then connect every line of one representation with another line of another representation (see figure 1.3 ).

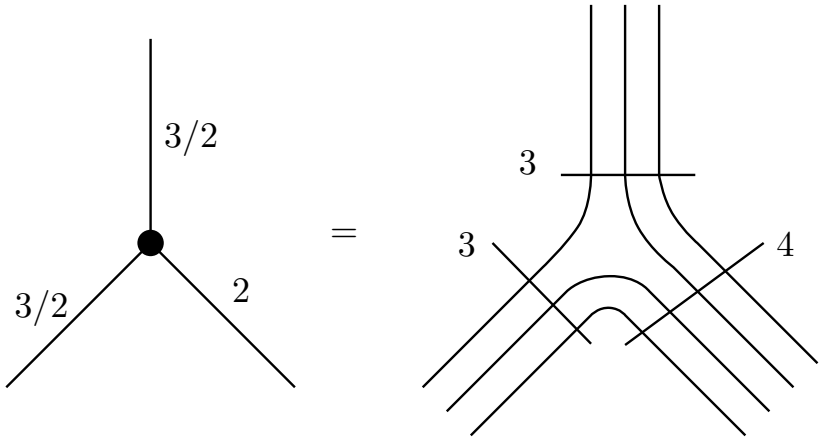


Figure 1.3: Graphical interpretation of an intertwiner.

But every representation is the product of fundamental ones. Therefore, when such a graphical construction is possible, it means that we can contract every fundamental representation with itself, and so we have an intertwiner. Consequently, the Clebsh-Gordan conditions are nothing but the necessary conditions to make such a construction possible.

We define the spin network embedded in $\sigma$ associated with the graph $\Gamma$ as the triplet $(\Gamma, \vec{j_e}, \vec{\mathcal{I}_n})$, where we color each edge of the graph by a spin number $j_e$ and each node by an intertwiner $\mathcal{I}_n$. We can then define a spin network state as the cylindrical function over the spin network $S$ associated with the graph $\Gamma$:

$$\Psi_S(A) = \Psi_{\Gamma, f_S}(A) \equiv f_S(hol[A, \gamma_1], ..., hol[A, \gamma_n]) \tag{1.57}$$

which can be rewritten the following way:

$$\Psi_S(A) = \left(\bigotimes_i R^{j_i}\left(hol[A,\gamma_i]\right)\right).\left(\bigotimes_n \mathcal{I}_n\right) \tag{1.58}$$

where $R^{j_i}\left(hol[A,\gamma_i]\right)$ is the $SU(2)$ spin-$j_i$ irreducible representation of the $SU(2)$ element $hol[A,\gamma_i]$. One can now enunciate one of the fundamental properties of Loop Quantum Gravity: spin network states form an orthonormal basis of the Hilbert space. Indeed, using Peter-Weyl theorem, which states that the Wigner matrices form an orthogonal basis of $L^2(SU(2))$, and the definition of the scalar product, we have:

$$\langle \Psi_S | \Psi_{S'} \rangle \equiv \delta_{SS'} \tag{1.59}$$

$$\equiv \delta_{\Gamma\Gamma'}\delta_{\vec{j}\,\vec{j}'}\delta_{\vec{\mathcal{I}}\vec{\mathcal{I}}'} \tag{1.60}$$

We will also make use of an abstract basis state denoted $|S\rangle \equiv |\Gamma, \vec{j_e}, \vec{\mathcal{I}_n}\rangle$ such that:

$$\Psi_S(A) = \langle A|S\rangle \tag{1.61}$$

Furthermore, the gauge invariance of the states is immediate looking at what we did above in 1.36 as well as the definition of the intertwiners. Therefore, we have just constructed an orthonormal basis for the kinematical Hilbert space associated with a given graph $\Gamma$ and implement the $SU(2)$ constraint by choosing an intertwiner at each node. Consequently, if we note $E$ the number of edges and $V$ the number of vertices, we have defined the Hilbert space $\mathcal{H}^0_\Gamma \equiv L^2(SU(2)^E/SU(2)^V)$ of wave functions. We then construct the kinematical Hilbert space $\mathcal{H}_0$ of the theory by summing over all the possible graphs. The next step is to implement the diffeomorphism constraint on the space $\mathcal{H}_0$.

We have already mentioned the fact that the unconstrained Hilbert space $\mathcal{H}$ carries a unitary representation of $Diff(\sigma)$ i.e.

$$\Psi(A) \to \Psi(\phi^{-1}A) \tag{1.62}$$

Let $U$ be a unitary representation of $Diff(\sigma)$. The action of this representation on a spin network state $|S\rangle$ is:

$$U(\phi)|S\rangle = |\phi.S\rangle \tag{1.63}$$

Therefore, what one need to find are the states that are invariant under diffeomorphism transformations i.e. such that $U\Psi = \Psi$. However, such states are not in the Hilbert space $\mathcal{H}_0$ of gauge invariant states but they are in $\mathcal{H}^*$, which is the dual of the space spanned by the spin network states. Hence the invariant Hilbert space $\mathcal{H}_{diff}$ we are looking for is actually the

diffeomorphism invariant part of $\mathcal{H}^*$. But, if we work on an extension of the Hilbert space, the inner product must be extended as well. We won't give the details of the derivation but the procedure leads to the definition of so-called spin-knots denoted $|s\rangle$. A spin-knot $|s\rangle$ is an equivalence class of $|S\rangle$ under the action of $Diff(\sigma)$. These entities are defined in $\mathcal{H}^*$ and so we take the adjoint to get an element in $\mathcal{H}_{diff}$

### 1.2.6 Aside on the meaning of diffeomorphism invariance

The two features of General Relativity that we would like to emphasize are the background independence and the diffeomorphism invariance. Einstein theory taught us that the spacetime metric and the gravitational field carry the same information. Therefore, as soon as we want to develop a quantum theory of the gravitational field, it is the same as deriving a quantum theory of the metric. This means that we cannot build the theory over a background metric. One consequence is that there is no notion of absolute position and thus, any notion of position must be understood with respect to other parts of space. Furthermore the quantum states of geometry must describe themselves the structure of spacetime.

Indeed, in Loop Quantum Gravity the geometry of space is built from spin networks and so, they carry all the information needed to reconstruct space. Thus, every object is spacetime-localized with respect to its neighbours. This leads to the concept of diffeomorphism invariance. If we displace every object of the space the same way and at the same time, we get a state which is equivalent to the previous one. Consequently, in General Relativity a state does not correspond to a solution of Einstein's equations but to a set of solutions equivalent under active diffeomorphism, i.e. an equivalence class.

It is important to distinguish passive diffeomorphisms which link the same object in different coordinate systems and the active diffeomorphisms which relate an object to another one in the same coordinate system. The invariance under passive diffeomorphisms refers only to the invariance under change of coordinates. This is not an intrinsic property of the theory but only a consequence of the formulation. Every theory can be formulated in a way satisfying the passive diffeomorphism invariance. This is not the case of the invariance under active diffeomorphisms which requires a background background-independent framework.

### 1.2.7 Operators on spin networks

With the introduction of spin networks, we have constructed an orthonormal basis for the Hilbert space and we have shown how the gauge invariance is implemented. What we would like to do now is construct operators and require them to be gauge invariant. The gauge invariance is a necessary condition in order to derive meaningful results from these operators. The operators

which satisfy this condition are called "Dirac's observables" and they are characterized by vanishing Poisson brackets with the constraints.

Recall that the starting point of the theory built so far was the choice of the pair of canonical variables which are the connection $A_a^i$ and its conjugate momentum $E_i^a$. We define two field operators associated with each one of them as follows:

$$\widehat{A}_a^i(\tau)\Psi_S(A) = A_a^i(\tau)\Psi_S(A) \tag{1.64}$$

$$\frac{1}{8\pi G}\widehat{E}_i^a(\tau)\Psi_S(A) = -i\hbar\frac{\delta}{\delta A_i^a(\tau)}\Psi_S(A) \tag{1.65}$$

However, these operators are not well-defined on the unconstrained Hilbert space and thus, we need to realize a smearing with some appropriate test functions. When we have introduced the Wilson loops, we mentioned that this choice was motivated by the need to find a suitable smearing of the variables which satisfies the gauge invariance requirement. Therefore, one can first define an operator based on the connection $A$. Indeed, the holonomy itself defines an operator which is the multiplicative operator. However, in order to promote this operator to a Dirac observable, we need to take the trace of the holonomy along a closed curve and so the definition of the observable is given by the so-called Wilson loops :

$$\widehat{W}[\gamma] = -Tr(\widehat{hol}[A,\gamma]) \tag{1.66}$$

For instance, the action on the spin network state $\Psi_S(A)$ gives:

$$\widehat{W}[\gamma]\Psi_S(A) = -Tr(hol[A,\gamma])\Psi_S(A) \tag{1.67}$$

It is more difficult to construct the second operator which appears to be the derivation operator.

We have stressed before that the connection $A$ is in particular a one-form so it can be easily smeared in one-dimension and give a well defined operator. The second operator which is defined as the derivative of the functional will be distributional and therefore, it is also required to be smeared suitably. Looking at the expression of Poisson brackets and knowing that the smearing of $A$ is done in one dimension, we deduce that the smearing of $E_i^a$ can be done in 2 ( $=$ dim $\sigma - 1$) dimensions. First, let us define the 2-form $(*E^j)_{ab} \equiv E_{ab}^j = E_j^c\epsilon_{abc}$ of density weight 0 and consider the so-called electric fluxes:

$$\widehat{E}_j(\Sigma) = \int_\Sigma *E_j \tag{1.68}$$

where $\Sigma$ is a two-dimensional open surface. The transformation of such a quantity is not nice under gauge transformations yet. Nevertheless, $E_j(\Sigma)$ is a fundamental component which will be useful in order to compute a more complicated operator which will be gauge invariant.

### 1.2.8 Area operator

We have already shown in 1.36 that the gauge invariance condition is satisfied as far as the Wilson loops are concerned. We have mentioned above that the electric fluxes introduced in 1.68 can be understood as elementary components of another operator which will be gauge invariant. The solution is given by the following operator [**?**]:

$$\widehat{Ar}(\Sigma) = \lim_{N\to\infty} \sum_n \sqrt{\widehat{E}^2(\Sigma_n)} \tag{1.69}$$

with $\widehat{E}^2(\Sigma_n) \equiv \sum_i \widehat{E}_i(\Sigma_n)\widehat{E}_i(\Sigma_n)$ where we have decomposed the surface $\Sigma$ into $N$ small pieces $\Sigma_n$ such that $\lim_{N\to\infty} \Sigma_n = 0$ and for each N, $\cup_n \Sigma_n = \Sigma$. Such an operator is gauge invariant and self-adjoint [34]. Now we can evaluate this operator on a spin network, assuming that there will be no more than one edge going through a surface $\Sigma_n$:

$$\widehat{Ar}(\Sigma)|S\rangle = \hbar \sum_{p\in(\Sigma\cup\Gamma)} \sqrt{j_p(j_p+1)}|S\rangle \tag{1.70}$$

$\sqrt{j_p(j_p+1)}$ is the value of the Casimir operator for the corresponding representation and so, it represents the contribution of each edge labeled by $j_p$ crossing a surface $\Sigma_n$. This means that spin networks are eigenstates of this operator. Let us stress the fact that the equation 1.70 does not correspond to the full spectrum of the operator. Indeed, without mentioning it, we have excluded the possibility for the surface $\Sigma$ to intersect $S$ at nodes.

The diagonalization of the operator $\widehat{Ar}(\Sigma)$ on spin network states carries a very important physical meaning. Let us recall that if we consider a 2-dimensional surface $\Sigma$ embedded in the 3-dimensional space $\sigma$, then the area of $\Sigma$ is given by:

$$Area(\Sigma) = \int_\Sigma d^2s|E| \tag{1.71}$$

Moreover, in the classical case, we can derive the classical version $\widehat{Ar}_{Cl}$ of the operator $\widehat{Ar}(\Sigma)$ in terms of the densitized triads. We denote $n_a(\vec{s}) = \epsilon_{abc}\frac{\partial x^b(\vec{s})}{\partial s_1}\frac{\partial x^c(\vec{s})}{\partial s_2}$ a normal vector to $\Sigma$ where $\vec{s} = (s_1, s_2)$ are the local coordinates. The expression of the metric on $\Sigma$ induced by $q_{ab}$ is then given by:

$$q_{ab}^{(\Sigma)} = q_{ab} - n^{-2}n_a n_b \tag{1.72}$$

This allows to write the following expression for the operator $\widehat{Ar}_{Cl}(\Sigma)$:

$$\widehat{Ar}_{Cl}(\Sigma) = \int_\Sigma ds_1 ds_2 \sqrt{q^{(\Sigma)}} \tag{1.73}$$

Let us now recall that the densitized triads and the 3D metric are related thanks to the formula $E_i^a E_i^b = qq^{ab}$. Let us contract this last expression with the normal vectors $n_a$ et $n_b$:

$$qq^{ab}n_a n_b = E_i^a E_i^b n_a n_b \tag{1.74}$$

But the term $q^{ab}n_a n_b$ is equal to the ratio of the determinant of the metrics $\frac{q^{(\Sigma)}}{q}$ [30]. This leads to an explicit expression of $q^{(\Sigma)}$ in terms of the triads:

$$q^{(\Sigma)} = E_i^a E_i^b n_a n_b \tag{1.75}$$

Hence the following expression for the classical operator:

$$\widehat{Ar}_C l(\Sigma) = \int_\Sigma d^2 s \sqrt{n_a E_i^a n_b E_j^b \delta^{ij}} \tag{1.76}$$

Comparing the last expression with 1.71, it is immediate to see that $Ar(\Sigma)$ is the area of the surface $\Sigma$. Therefore, the measurement of the surface of any physical area will give a value within the spectrum of the operator. In other words, any physical surface can be decomposed into elementary quanta of area. Reintroducing the physical units, the spectrum of the operator $\widehat{Ar}(\Sigma)$, that we can from now on call "area operator", is given by:

$$Ar(\Sigma) = 8\pi\beta\hbar G \sum_i \sqrt{j_i(j_i+1)} \tag{1.77}$$

This also means that there is a minimal value of area which corresponds to the smallest eigenvalue of the operator. A similar procedure can be done with the volume operator and a length operator for which similar results apply (see for instance [7] [36] [26]). We will not derive these operators but we will emphasize two facts. First, the definition of the length operator is more complicated than the one of the two other operators and its physical interpretation far less intuitive. Secondly, while the area operator acts on edges of a spin network, the action of the volume operator can be reduced to the action on the vertices, and more precisely on the intertwiners.

### 1.2.9 Physical interpretation of spin networks

We have just described the fact that spin networks diagonalize both the area and the volume operators. More precisely, we have mentioned that only nodes contribute to the spectrum of the volume operator. This induces that the volume of a given region of a spin network is actually the sum of n terms where each term is associated with one node. In other words, we associate a volume to each node of the region and the overall volume is equal to the sum of the volume of each node. By doing so, we attach a "chunk" of space to each vertex of the spin network.

Therefore, space is represented by a collection of chunks of space with quantized volume. Two chunks of space whose corresponding nodes are linked by an edge, are adjacent. In that case, the edge which connects the two nodes cuts the elementary surface which separates the

two chunks. But we have shown that the area of the surfaces is determined by the coloring of the edges. This means that two adjacent chunks are separated by a surface with a quantized area (see figure 1.4 for heuristic explanation).

We will now stress the relation between this picture and the diffeomorphism invariance. The chunks of space that we have just described are not objects in a space but encode the space itself. Furthermore, the only notion of distance there is, is the one given by the position of the chunks with respect to each other. Thus, the spin knot states $|s\rangle$ which are the basis elements of $\mathcal{H}_{diff}$ are the quantum states of space defining themselves the physical space.

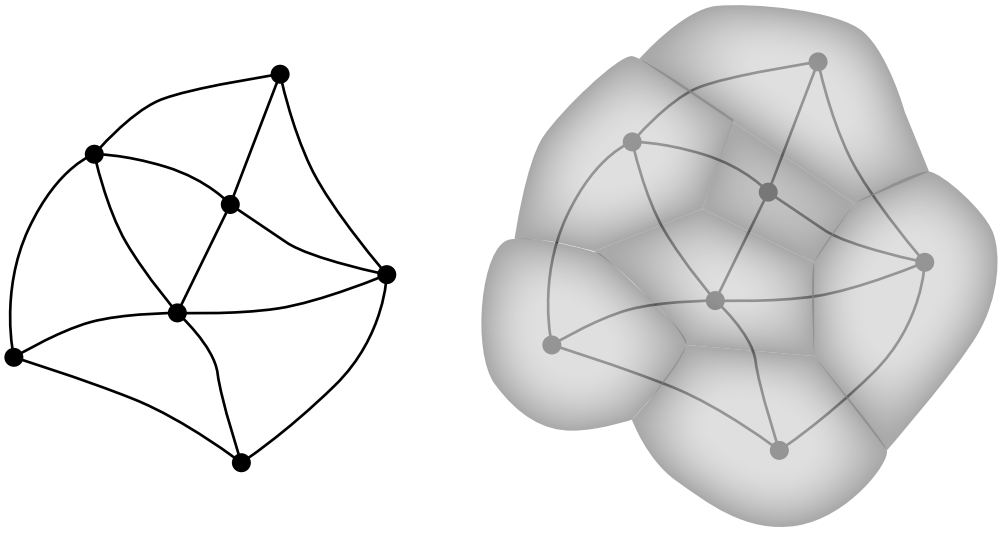

Figure 1.4: Picture of the space as a collection of adjacent chunks of space.

Now may be the time to stress that we have encountered two kinds of discreteness, through the process of quantization, which are conceptually very different. The first one is related to the use of spin networks and so the discretization of space. Nevertheless, this is not a quantum discreteness as the one of the energy of a single harmonic oscillator for instance. This is rather the analogue of the plane wave expansion method where we can focus on a single node and treat it analytically. The second discreteness related to the spectral properties of the volume and the area operators corresponds to the discretization of the geometry which occurs at quantum scale.

Finally we would like to underline that even if we make an extensive use of the picture presented in 1.4, we must keep in mind that we don't actually deal with chunks of space (as

it could be understood in a classical way) but rather with specific modes of an interaction involving the area and the volume of the studied entity.

### 1.2.10 Chunks of space as polyhedra

The geometrical interpretation of the algebraic data that we have just described is a first step in the attempt to describe the 3D geometry from the degrees of freedom displayed by the spin network. It has been known for a few years that, as far as four-valent nodes are concerned, the intertwiner state can be understood as the state of a quantum tetrahedron. Such a result is quite intuitive as the dual picture of a 4-valent node is a chunk of space with four neighbours. A neighbour is represented by a link which carries the area of the interface between the two neighbours. Therefore, we consider an entity with four faces which naturally leads to the concept of a quantum tetrahedron. This picture will be used in the context of the toy model in order to compute the volume associated with an intertwiner.

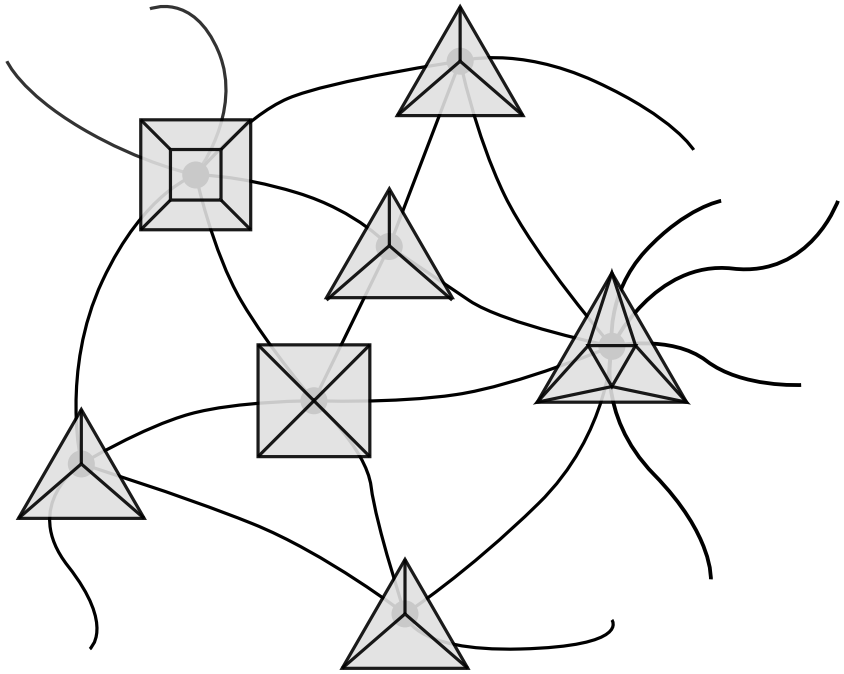

Figure 1.5: Heuristic picture of the space where each n-valent intertwiner is represented by a polyhedron with $n$ faces using the Schlegel diagrams.

In the article [8], they show that we can generalize this correspondence to the case of n-valent nodes. Indeed, at the quantum level, it is possible to identify the state of an intertwiner with

the one of a quantum polyhedron. Such a procedure leads to the representation of space given in figure 1.5 where we have used the Schlegel diagrams in order to represent the tetrahedra. The Schlegel diagrams are the result of a plane projection of convex polyhedra. This is a very useful representation since it allows capturing the symmetries of a 3D polyhedron in the plane. Although this result is, in particular, relevant for the study of semi-classical states, note however that the derivation of the volume of the quantum polyhedra does not generally give the same values as the eigenvalues of the volume operator derived from the classical formula.

## 1.3 Explicit calculations on spin networks

In this section, we introduce a toy model of spin network that will be used in the next chapter. In order to apply some of the features introduced previously, we will carry out some explicit calculations on this simple spin network.

The toy model that we consider is a checkerboard spin network with four-valent vertices. Besides, each edge is labeled by $j = \frac{1}{2}$, and so is associated with the fundamental representation of $SU(2)$ (see figure 1.6).

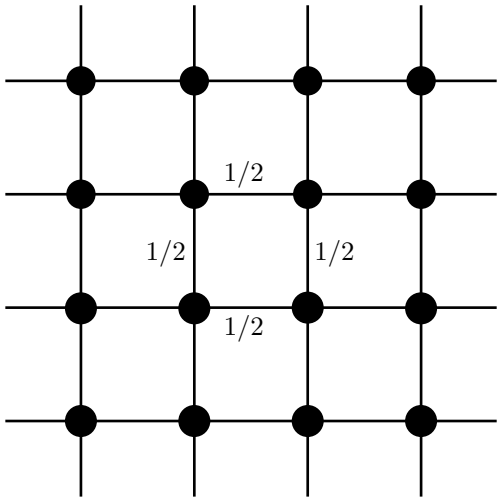


Figure 1.6: The checkerboard spin network used as toy model.

The first thing we would like to compute is an intertwiner associated with a node. The Hilbert space at each vertex is given by the tensor product of the Hilbert spaces associated with the edges i.e. $(V^{1/2})^{\otimes 4}$, where $V^{j_i}$ is the vector space of the spin-$j_i$ representation. Moreover, we know that such a product can be decomposed into a direct sum of irreducible representations of $SU(2)$. In our situation, such a decomposition is given by the formula:

$$(V^{1/2})^{\otimes 2n} = \bigoplus_{k=0}^{n} (C_{2n}^{n+k} - C_{2n}^{n+k+1}) V^k \tag{1.78}$$

where $C_{2n}^{n+k} = \binom{2n}{n+k}$ are the binomial coefficients. Explicitly, as far as our set-up is concerned:

$$(V^{1/2})^{\otimes 4} = 2V^0 \oplus 3V^1 \oplus V^2 \tag{1.79}$$

As the decomposition displays it, there is a degeneracy factor 2 for $V^0$, which means that the invariant subspace, i.e. the intertwiner space, has dimension 2. In other words, we can find

two linearly independent tensors which span the space. As mentioned earlier, in the case of a 3-valent node, the edges joined at the vertex must verify the Clebsch-Gordan conditions. Furthermore, if we consider the tensor product of Hilbert spaces $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$, the invariant subspace will be formed by invariant tensors with $2(j_1 + j_2 + j_3)$ indices. As a matter of fact, such a tensor is unique up to normalization and an expression for the normalized tensor is given by the famous Wigner $3j$-symbols:

$$Qv^{\alpha_1\alpha_2\alpha_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \tag{1.80}$$

where the normalization factor is determined such that:

$$\overline{Qv^{\alpha_1\alpha_2\alpha_3}} Qv_{\alpha_1\alpha_2\alpha_3} = 1 \tag{1.81}$$

Furthermore, every intertwiner associated with a node of valence four or more can be expressed in terms of the trivalent ones. For instance, in the case of a 4-valent node, we get an intertwiner by contracting two 3-valent ones:

$$v_i^{\alpha_1\alpha_2\alpha_3\alpha_4} = i^{\alpha_1\alpha_2\alpha} i_\alpha^{\alpha_3\alpha_4} \tag{1.82}$$

where the spin $i$ representation would be the one associated with a virtual link (see figure 1.7):

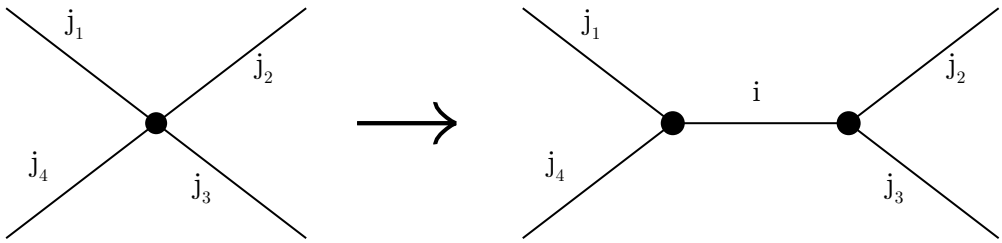


Figure 1.7: Decomposition of a four-valent node into two three-valent nodes.

Using the Wigner symbols, we can explicitly write the spin network state associated with the theta spin network displayed figure 1.8:

$$\Psi_S(A) = R^{1/2}(hol_1)_a^b R^1(hol_2)_i^j R^{1/2}(hol_3)_c^d (\mathcal{I}_1)^{aic} (\mathcal{I}_2)_{bjd} \tag{1.83}$$

$$= (hol_1)_a^b R^1(hol_2)_i^j (hol_3)_c^d \begin{pmatrix} 1/2 & 1 & 1/2 \\ a & i & c \end{pmatrix} \begin{pmatrix} 1/2 & 1 & 1/2 \\ b & j & d \end{pmatrix} \tag{1.84}$$

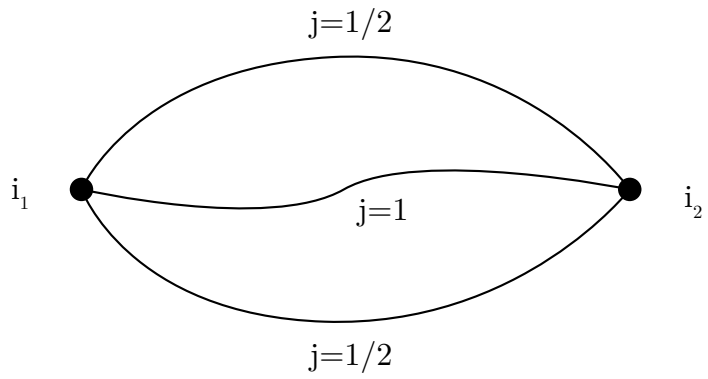


Figure 1.8: The "theta" spin network.

In our set-up, the decomposition of the intertwiner is not necessary as we have chosen very simple vertices. Indeed, we just need to find two independent invariant tensors with four indices. Looking at the construction of $\mathcal{H}^{1/2}$, we guess that we can form these tensors using $\epsilon^{ab}$ and $\sigma^a_{ib}$. The matrices $\sigma^a_{ib}$ are given by the Pauli matrices:

$$\sigma^a_{ib} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{1.85}$$

and $\epsilon^{ab}$ is the invariant tensor such that $\sigma^{ab}_i = \sigma^a_{ic}\epsilon^{cb}$ i.e.

$$\sigma^{ab}_i = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \tag{1.86}$$

Two invariant 4-index tensors are therefore given by

$$v^{abcd} = \epsilon^{ab}\epsilon^{cd} \tag{1.87}$$

$$w^{abcd} = \sigma^{ab}_i \sigma^{cd}_i \tag{1.88}$$

and so, $v^{abcd}$ and $w^{abcd}$ span the intertwiner space of the 4-valent node denoted $Int((V^{1/2})^{\otimes 4})$. In order to color the node, we just need to choose and fix an element of this space. For the sake of simplicity, let us choose the intertwiner given by $v + w$.

Following the procedure given in [38], we can now compute the quanta of volume associated with this intertwiner which matches the eigenvalue of the volume operator. In the special case of a 4-valent node, we can compute the volume corresponding to the intertwiner as the volume of a tetrahedron. In order to characterize the geometry of the tetrahedron, we can make use of

numerous geometrical quantities such that the length of its edges, its volume or the area of its faces. A nice set of independent quantities is given by the four vectors $\vec{\Lambda}_p$, $p = 1, ..., 4$ normal to each surface and whose norm is equal to the area of the corresponding face. One of the key properties of these quantities is that they satisfy the closure relation given by:

$$\vec{C} = \sum_{p=1}^{4} \vec{\Lambda}_p = \vec{0} \tag{1.89}$$

Furthermore, the volume $V$ satisfies the following formula:

$$V^2 = \frac{2}{9}\epsilon_{ijk}\Lambda_1^i\Lambda_2^j\Lambda_3^k = \frac{2}{9}det\Lambda \tag{1.90}$$

where $\epsilon^{ijk}$ is the totally antisymmetric tensor. In the case where the tetrahedron is small compared to the scale of curvature, and so that we can consider the metric locally flat, we can derive a formula in terms of the triads:

$$\Lambda_p^i = \frac{1}{2}\epsilon_{jk}^i \int_p e^j \wedge e^k \tag{1.91}$$

Following a usual quantization procedure, we can then promote the vectors $\vec{\Lambda}_p$ to operators. This leads to the commutation relation:

$$[\Lambda_p^i, \Lambda_q^j] = i\delta_{pq} l_0^2 \epsilon_k^{ij} \Lambda_p^k \tag{1.92}$$

where $l_0$ is the elementary length such that $l_0^2 = 8\pi G l_{Planck}^2$. In our situation, the operators $\Lambda_p^i$ acts on the fundamental representation of $SU(2)$ because the faces are colored by $j = \frac{1}{2}$. Consequently, they must be proportional to the Pauli matrices, the coefficient of proportionality being given by the commutation relation:

$$L_p^i = l_0^2 \tau^i \tag{1.93}$$

where $\tau^i = \frac{\sigma^i}{2}$. Thus, the action of the operator $V^2$ on a four-index tensor $z^{abcd}$ is given by the following formula:

$$(V^2 z)^{abcd} = \frac{2}{9}(\frac{l_0^2}{2})^3 \epsilon^{ijk} \sigma_{ia'}^a \sigma_{jb'}^b \sigma_{kc'}^c z^{a'b'c'd} \tag{1.94}$$

We can now apply this result to the invariant tensors found earlier. Using the fact that $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ and $\epsilon\sigma_i\epsilon = \sigma_i*$, we get :

$$V^2 v = -\frac{i l_0^6}{18} w \quad ; \quad V^2 w = \frac{i l_0^6}{6} v \tag{1.95}$$

i.e.

$$V^2 = -\frac{i l_0^3}{16} \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \tag{1.96}$$

By diagonalizing the matrix, we get the eigenvalues: $V^2 = \pm\frac{l_0^2}{6\sqrt{3}}$. Finally, using the definition of $l_0$, the quantum of volume associated with the intertwiner is:

$$\frac{1}{\sqrt{6\sqrt{3}}}\left(\frac{8\pi\hbar\beta G}{c^3}\right)^{3/2} \tag{1.97}$$

# Chapter 2

# Entanglement on spin networks

The brief review of Loop Quantum Gravity presented in Chapter 1 has introduced the main concepts of the theory. Besides, we have described in details the construction of the basis of the Hilbert space and the physical interpretation of the spin networks. In this chapter, we will use these notions and our toy model in order to discuss how we could use the entanglement as a measure of distance, within the background independent framework. In order to do so, we will follow the procedure suggested in [25]. First, we will explain carefully the difficulties of such a construction and how we can use the results of Quantum Information and Condensed Matter Physics. Then, we will present the notions of correlations and coarse-graining. Finally we will use all the tools introduced to present the calculation of the entanglement between two separated regions of a spin network, and discuss how we could define from these results a notion of distance.

## 2.1 Outlook

We have stressed in several occurrences that one of the fundamental feature of General Relativity is the background independence which is closely related to the diffeomorphism invariance of the theory. Background independence means that the theory is not built on a given background and therefore there is no background metric which defines the geometric notions. Loop Quantum Gravity which proposes a canonical quantization of the theory implements the diffeomorphism constraint with the use of the spin knots as basis states. Therefore, these states are not are not constructed over a spacetime but define the spacetime themselves and, so they must reconstruct the geometry at the quantum level.

One issue which arises when we consider a background-independent framework is the question of localization. Indeed, in the absence of reference frame there is no notion of position, the only information available being the position of a region of a spin network with respect to another one. Furthermore, without notion of position, it is difficult to derive a sense of the distance between two regions of the space.

We have described in the previous chapter that the spectrum properties of the volume and the area operators allow to derive a heuristic picture of space as a collection of adjacent chunks. However, this does not provide a full and clear description of the geometrical properties of spin networks. In this chapter, we will try to understand more precisely this problem and discuss a possible reconstruction of a notion of distance in such a background independent formalism. Another related issue, which will not be treated here, would be to make sense of the notion of distance when we consider a superposition of spin networks.

But spin networks are not a tool specific to Loop Quantum Gravity. Indeed, spin networks are very similar to the spin systems which are extensively used in both Quantum Information and Condensed Matter Physics. The key difference is that spin networks do not rely on any background whereas spin systems are embedded in a background equipped with a metric. Although there are some conceptual differences, it is very interesting to look at these systems as they display some properties that we would like to try to replicate with spin networks. Indeed, it is now well-known that there is a decrease of the correlations as the graph distance between two qubits [1] grows (see [16] for instance). Therefore, we would like to investigate a way of reconstructing a notion of distance through the study of the correlations on a spin network.

[1] A qubit is a object that transforms as a tensor in the fundamental representation of the Lie Algebra su(2).

The project to derive a notion of distance thanks to the entanglement is also motivated by Quantum Field Theory. Indeed, when we compute the correlations of a field at two different points of spacetime x and y, the two-point correlation function $\langle\phi(x)\phi(y)\rangle$ decreases with respect to the distance $d(x, y)$ between the two points as $1/d^2$. At first we could expect a similar relation as far as the correlations on our checkerboard spin network are concerned.

The important point is that we will not be able to display such a relation between the correlations and the distance in our framework because we do not have any idea, a priori, of what it means for two regions on a spin network to be close or far. The idea is rather to compute the correlations and then deduce the distance by speculating the formula relating both. Nevertheless, in order to check the consistency of the results, we could compare the distance resulting of our procedure with an approximation of the distance provided by the geometric picture. Indeed, we can compute what we will call a geometrical distance as a ratio of a volume and a surface provided by the spectra of the geometrical operators. However, once again we would like to shed light on the danger of thinking about the chunks of space as classical entities and therefore, it will be necessary not to attach too much importance to this distance. Finally, looking at the results on spin systems, we expect the correlations to vanish very quickly with the distance. Thus, the measure of distance provided by our procedure would be of interest only for close enough regions [2].

[2]By close we mean that the minimum number of vertices between the two regions is small compared to the total number of vertices.

## 2.2 Entanglement and correlations

The entanglement is a fundamental feature of every quantum system which can be decomposed into several systems. It has been discovered in the early days of Quantum Mechanics and became rapidly an object of great interest [15]. Later, a new field saw the light using the entanglement as a fundamental tool, namely Quantum Information [29]. Despite the great success achieved by the studies on the entanglement, a lot of aspects remain obscure. Besides, numerous tools are available in order to evaluate the correlations, depending on the physics we intend to display. We shall now briefly introduce some of these tools.

A first measure is given by the quantum mutual information. Let us consider two quantum systems $A$ and $B$ and let us note $\mathcal{H}_A$ and $\mathcal{H}_B$ the corresponding Hilbert spaces. The composite system is described by the mixed state $\rho_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ and the two systems $A$ and $B$ are described by the reduced density matrices $\rho_A$ and $\rho_B$ such that:

$$\rho_A = Tr_B(\rho_{AB}) \tag{2.1}$$

$$\rho_B = Tr_A(\rho_{AB}) \tag{2.2}$$

We can then define the Von-Neumann entropy, or entanglement entropy, of a system X as:

$$S(X) = -Tr(\rho_X log \rho_X) \tag{2.3}$$

Denoting $\{\lambda_i\}$ the eigenvalues of the density matrix $\rho_X$, we can rewrite the expression 2.3 the following way:

$$S(X) = -\sum_i \lambda_i log \lambda_i \tag{2.4}$$

The quantum mutual information of the system $A + B$ is defined according to the formula:

$$\mathcal{I}(A:B) \equiv S(A) + S(B) - S(A,B) \tag{2.5}$$

$$= S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \tag{2.6}$$

From now on, we will drop the indices for the mixed state and write $\rho_{AB} = \rho$. The quantum mutual information physically represents the total amount of correlations between two systems, and therefore, it takes into account both the classical and the quantum correlations.

In the case where we are only interested by the quantum correlations, we will try to compute the entanglement. However, the entanglement is often impossible to compute explicitly as soon as we consider complicated states. As far as pure states are concerned, such a computation is reasonably straightforward. Indeed, in this case, the entanglement is directly given by the Von

Neumann entropy. If we denote $|\psi\rangle$ the pure state, the density matrix is $\rho = |\psi\rangle\langle\psi|$ and the entanglement is given by:

$$\mathcal{E}(\rho) = S(\rho) = -Tr(\rho log \rho) \tag{2.7}$$

If now we consider an arbitrary mixed state, it is more difficult to find a computable formula. Indeed, most of the definitions of entanglement involve extremizations, which are often tough to evaluate. Nevertheless, since we consider systems of qubits, a suitable definition is the one of the entanglement of formation. Let us write the mixed state the following way:

$$\rho = \sum_i \omega_i |\psi_i\rangle\langle\psi_i| \tag{2.8}$$

which is by definition a mixture of pure states $\rho = |\psi\rangle\langle\psi|$. But, we have already mentioned the fact that for a pure state, the entanglement is equal to the Von-Neumann entropy:

$$\mathcal{E}(A:B, |\psi\rangle\langle\psi|) = S(\rho_A) = S(\rho_B) \tag{2.9}$$

$$= -Tr(\rho_A log \rho_A) = -Tr(\rho_B log \rho_B) \tag{2.10}$$

We then define the entanglement of formation of the mixed state $\rho$ as the average of the entanglement of the pure states appearing in the decomposition minimized over the whole range of possible decompositions [44]:

$$\mathcal{E}(A:B, \rho) = \min_{\{\psi_i\}} \sum_i \omega_i \mathcal{E}(A:B, |\psi_i\rangle\langle\psi_i|) \tag{2.11}$$

Let us finally note that if we are interested in the purely classical correlations $\mathcal{C}(\mathcal{A}:\mathcal{B})$, we can get the amount by taking the difference of the definitions above:

$$\mathcal{C}(A:B) \equiv \mathcal{I}(A:B) - \mathcal{E}(A:B) \tag{2.12}$$

## 2.3 Preliminary computation of entanglement

Before focusing on the entanglement between two regions of a spin network, we will briefly discuss the calculation of the entanglement entropy for an arbitrary region of space. In order to do so, we will closely follow the procedure described by W. Donnelly in [13].

Let us consider a spin network, with $N$ nodes, associated with a graph $\Gamma$ that we split into two regions. More precisely, we choose a region $\Omega$ on the spin network and therefore, the other region is defined as the conjugate $\overline{\Omega}$ of the first one. Thus, the Hilbert space $\mathcal{H}_\Gamma$ associated with the whole graph is such that:

$$\mathcal{H}_\Gamma = \mathcal{H}_\Omega \otimes \mathcal{H}_{\overline{\Omega}} \tag{2.13}$$

By doing so, we realize a Schmidt decomposition of the states in $\mathcal{H}_\Gamma$. Therefore, if we consider a state $|\psi\rangle \in \mathcal{H}_\Gamma$, there exists a set of states $\{|\psi_i^\Omega\rangle, |\psi_i^{\overline{\Omega}}\rangle\}$ and some real numbers $\{s_p\}$ such that:

$$|\psi\rangle = \sum_p \sqrt{s_p} |\psi_i^\Omega\rangle \otimes |\psi_i^{\overline{\Omega}}\rangle \tag{2.14}$$

Then, we can define the mixed state in $\mathcal{H}_\Omega$ associated with the region $\Omega$ as the reduced density matrix:

$$\rho(\Omega) = Tr_{\mathcal{H}_{\overline{\Omega}}}(|\psi\rangle\langle\psi|) \tag{2.15}$$

$$= \sum_p s_p |\psi_p^\Omega\rangle\langle\psi_p^\Omega| \tag{2.16}$$

And similarly, the density matrix associated with the region $\overline{\Omega}$ is given by:

$$\rho(\overline{\Omega}) = \sum_p s_p |\psi_p^{\overline{\Omega}}\rangle\langle\psi_p^{\overline{\Omega}}| \tag{2.17}$$

Finally, as presented earlier, the entanglement entropy is simply as follows:

$$S(\rho_\Omega) = S(\rho_{\overline{\Omega}}) = -\sum_p s_p log s_p \tag{2.18}$$

Consequently, the Schmidt decomposition, and more precisely the Schmidt coefficients, give direct access to the entanglement entropy of the region $\Omega$. We denote $\partial\Omega$ the boundary of the region $\Omega$. The boundary is the region composed of the edges such that one endpoint is inside the region and the other one outside the region. For now we assume that the boundary of $\Omega$ intersects $P$ edges, and we introduce a new node on each edge in the boundary $\partial\Omega$. Actually, we split each edge $e \in \partial\Omega$ into two smaller edges such that one is in $\Omega$ and the other one in $\overline{\Omega}$. The new nodes are therefore associated with a trivial intertwiner which is the unit matrix of

the representation corresponding to the spin labeling the edge. If we denote $N_\Omega$ the number of nodes in the region $\Omega$, we have the following repartition of intertwiners:

$$\underbrace{\underbrace{\mathcal{I}_1 + ... + \mathcal{I}_P}_{\in\partial\Omega} + \underbrace{\mathcal{I}_{P+1} + ... + \mathcal{I}_{P+N_\Omega}}_{\in\Omega} + \underbrace{\mathcal{I}_{P+N_\Omega+1} + ... + \mathcal{I}_N}_{\in\overline{\Omega}}}_{\in\Gamma} \tag{2.19}$$

Furthermore, the expression of the trivial intertwiner $\mathcal{I}_p$ is given by:

$$\mathcal{I}_p = \frac{1}{\sqrt{2j_p+1}} \mathbb{1}_{V^{j_p}} \tag{2.20}$$

$$= \frac{1}{\sqrt{2j_p+1}} \sum_{a_p=1}^{2j_p+1} |e_{a_p}\rangle\langle e_{a_p}| \tag{2.21}$$

where $\{|e_{a_p}\rangle\}$ is an orthonormal basis of the $2j_p+1$-dimensional representation space $V^{j_p}$. By applying this decomposition to every new intertwiner, we get an expression for the spin network state

$$|S\rangle = \Big( \prod_{p=1}^{P} \frac{1}{\sqrt{2j_p+1}} \Big) \sum_{\vec{a}} |S_\Omega, \vec{a}\rangle \otimes |S_{\overline{\Omega}}, \vec{a}\rangle \tag{2.22}$$

where we sum over the n-tuples $(a_1, ..., a_p)$. The state $|S_\Omega, \vec{a}\rangle$ is associated with the extended graph of the region $\Omega$ where we have added an intertwiner on the boundary. Then, using the fact that spin network states form an orthogonal basis, we see that the expression 2.22 corresponds to a Schmidt decomposition. Following what we did previously in 2.15, it is straightforward to see that the reduced density matrix is given by:

$$\rho(\Omega) = \frac{1}{N} \sum_{\vec{a}} |S_\Omega, \vec{a}\rangle\langle S_\Omega, \vec{a}| \tag{2.23}$$

and the corresponding entanglement entropy is:

$$S(\Omega) = \sum_{p=1}^{P} log(2j_p + 1) \tag{2.24}$$

The most important thing to stress is that the entanglement entropy only depends on the spin numbers attached to the edges on the boundary of the region $\Omega$. Furthermore, we have implicitly chosen the boundary region to intersect only edges and therefore to be composed only of these edges. If now we allow the boundary to contain nodes, it appears that the entanglement entropy actually depends on the intertwiners on the boundary, those which were already there and those created by the procedure described above. Such a feature was not obvious. Indeed, as the group elements of SU(2) are attached to the edges, the degrees of freedom are attached to them. Thus, we could expect the entropy to depend on the edges and not the nodes.

Keeping in mind the results established with this first calculation, we can now proceed with our study.

## 2.4 Derivation of a notion of distance

### 2.4.1 Outlook of the procedure

Because we are interested in the computation of the distance between two regions $A$ and $B$ on a spin network, we want to investigate the entanglement induced by the region outside $A$ and $B$. Indeed, we expect this measure to give us access to the distance with respect to the exterior metric which would be the physical value we are looking for [25]. Let us first give some notations that we will use later. We call $C$ the region exterior to $A$ and $B$ such that we have the following relation between the boundaries:

$$\partial C = \partial A \cup \partial B \tag{2.25}$$

Furthermore, if we write $\Gamma$ the whole graph, we denote $\Gamma_X$ the connected graph with all the vertices inside the region X (see figure 2.1).

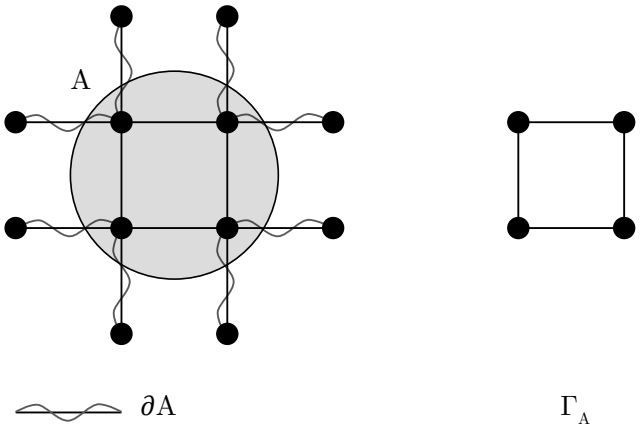


Figure 2.1: An edge is in the boundary $\partial A$ of the region $A$ if one endpoint is inside the region and the other one outside. The region $\Gamma_X$ corresponds to the graph of the whole region less the boundary.

The key idea that we will use later is that the boundary of a region $X$ can be thought as the result of the coarse-graining of $X$. This point of view will be important in order to compute the entanglement since we will consider a state on the boundary of the region $C$.

### 2.4.2 Coarse-graining of a spin network

The principle of the coarse-graining is to change the "resolution" of a diagram, namely a spin network in the context of Loop Quantum Gravity. By change of resolution we mean reducing the number of elements of a fine graph in order to get a rougher graph, while keeping the most important information contained in the original graph. In other words, the aim is to capture the essential information of a graph and express it with fewer variables. Let us illustrate this concept with a bounded connected region $A$ on a spin network $S$ associated with the graph $\Gamma$ [25] [2].

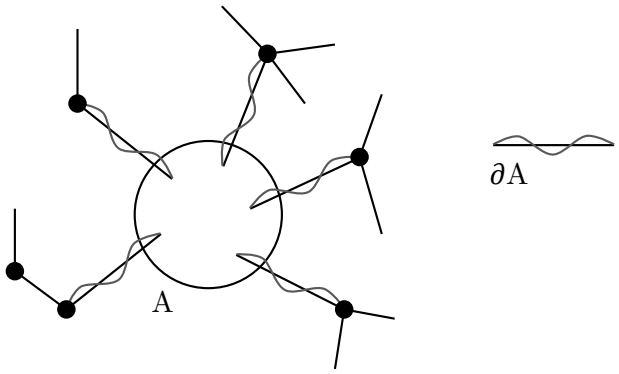


Figure 2.2: The boundary state of a region is independent of the structure of the region and therefore we can think of the interior of the region as a big vertex.

The boundary space is independent of the structure of the graph inside the region i.e. independent of $\Gamma_A$. This is why we can think of the boundary $\partial A$ as the coarse-grained space of $A$ where $\Gamma_A$ would be reduced to a single vertex. We will now explain how the reduction of a whole graph to a simple vertex is possible.

The coarse-graining of a graph requires two steps which are the coarse-graining of the nodes and the coarse-graining of the links. In order to deal with the whole graph we just have to repeat and combine the two steps.

#### Coarse-graining of nodes

Let us consider an open graph with mono-valent vertices at the exterior. Earlier we have used the Peter-Weyl theorem in order to justify that spin networks form a basis of the kinematical Hilbert space. We write the Wigner matrices $R^j_{mm'}(U_e)$ with $U_e = hol[A, \gamma_e] \in SU(2)$.

These matrices form a basis of $L^2(SU(2))$ and we write $|j,m,n\rangle$ the vectors of that basis such that:

$$R^j_{mn}(U_e) = \langle U|j,m,n\rangle \tag{2.26}$$

Therefore, if we consider a subgraph $\Gamma$, a basis of the corresponding Hilbert space $\mathcal{H}_\Gamma$ is given by the vectors $|j_e, m_e, n_e\rangle$ where $e = 1, ..., E$, with $E$ the number of edges and $m_e, n_e$ are the usual quantum magnetic numbers of the spin-j representation such that $m_e, n_e = -j, ..., +j$. Furthermore, we associate the quantum numbers $m_e$ and $n_e$ with one of the two nodes which constitute the endpoints of the edge $e$, such that they transform according to the gauge transformation defined at this vertex. Thus, if we label by $v$ the vertices, instead of the notation $|j_e, m_e, n_e\rangle$, the basis vectors are denoted $|j_e, m_{ve}\rangle$, where the edges joined at the vertex $v$ are labeled by $e$. By doing so, we associate all the quantum numbers related to the same node, and so, for every state in $\mathcal{H}_\Gamma$ we have:

$$|\psi\rangle = \sum_{j_e, m_{ve}} c_{j_e, m_{ve}} |j_e, m_{ve}\rangle \tag{2.27}$$

We can now rewrite this state by doing the distinction between the edges on the boundary ($e \in \partial A$) and the ones inside the region ($e \in \Gamma_A$):

$$|\psi\rangle = \sum_{\substack{j_b, m_b \\ b \in \partial A}} \sum_{\substack{j_e, m_{ve} \\ e \in \Gamma_A}} c_{j_b, m_b, j_e, m_{ve}} |j_b, m_b, j_e, m_{ve}\rangle \tag{2.28}$$

The coarse-graining of the region $A$ will put together every vertex in $\Gamma_A$ so that the interior is reduced to a single vertex while the edges in the boundary space remain unchanged. Furthermore, in the case of spin networks, we have a condition of gauge invariance at each vertex. Therefore, before the coarse-graining, the Hilbert space of the region $A$ is given by the tensor product of the intertwiner spaces associated with each vertex i.e.

$$\mathcal{H}_A \equiv \bigotimes_{v \in A} Int(\bigotimes_{e \in v} V^{j_e}) \equiv \bigotimes_{v \in A} Int_v \tag{2.29}$$

After the coarse-graining, we get a state which lives in the boundary space given by $\bigotimes_{e \in \partial A} V^{j_e}$ that we require to be gauge invariant:

$$\mathcal{H}_{\partial A} = Int(\bigotimes_{e \in \partial A} V^{j_e}) \tag{2.30}$$

Thus, in the case of spin networks, the coarse-graining is the map:

$$\Theta : \bigotimes_{v \in A} Int_v \longrightarrow Int(\bigotimes_{e \in \partial A} V^{j_e}) \tag{2.31}$$

Explicitly, in the formula derived in 2.28, we only consider after the coarse-graining the quantum numbers associated with the boundary space. Thus, the basis vectors of the Hilbert space $\mathcal{H}_{\widetilde{\Gamma}}$, where $\widetilde{\Gamma}$ is the graph $\Gamma$ after the coarse-graining, are:

$$|j_b, m_b\rangle \tag{2.32}$$

Therefore, noting that we must have $j_b \neq j_e$ and $m_b \neq m_{ve}$, one see that the quantum numbers $(j_e, m_{ve})$ are the numbers needed to remove the degeneracy of the vectors $|j_b, m_b\rangle$. If we consider a state $|\psi\rangle \in \mathcal{H}_{\widetilde{\Gamma}}$ ,then we get the density matrix by tracing over these indices:

$$\rho = Tr_{j_e, m_{ve}} |\psi\rangle\langle\psi| \tag{2.33}$$

Now, in order to take into account the invariance under gauge transformation which arises when we deal with spin networks, we need to contract the Wigner representation matrices with intertwiners. Therefore, denoting $\{\mathcal{I}_v\}$ the basis of intertwiners, the coefficients $c$ take the following form:

$$c_{j_b, m_b, j_e, m_{ve}} = \sum_{\{\mathcal{I}_v\}} c_{j_b, m_b, j_e, \mathcal{I}_v} \mathcal{I}_v^{m_{ve}} \tag{2.34}$$

We can then apply a procedure similar to the one above in order to derive the corresponding matrix density.

**Coarse-graining of links**

The second step of the coarse-graining of a graph consist in the coarse-graining of the links. The idea is to put together several links, which share the same starting point and the same endpoint, in order to get a bigger single link. We will denote these links as "parallel links". If we consider a single link, it is characterized by the number $j$ labeling it. Besides, we associate with this spin number, two magnetic quantum numbers which respectively label a basis in $V^j$ and $V^{j^*}$. Keeping in mind that, as far as the Lie Algebra $su(2)$ is concerned, a representation and its conjugate are equivalent, the Hilbert space associated with a link is given by:

$$\bigoplus_j V^j \otimes V^j \tag{2.35}$$

where the sum is over the irreducible representations. If now we consider several parallel links labeled by the index $p$, the corresponding Hilbert space is:

$$\bigotimes_p \Big( \bigoplus_{j_p} (V^{j_p} \otimes V^{j_p}) \Big) = \bigoplus_{j_p} \Big( (\bigotimes_p V^{j_p}) \otimes (\bigotimes_p V^{j_p}) \Big) \tag{2.36}$$

Once again, we can decompose the spaces $\bigotimes_p V^{j_p}$ into irreducible representations that we label by the spin number $\widetilde{j}$. We emphasize that the states labeled by $\widetilde{j}$, provided by this procedure, are different than the previous ones labeled by $j$. We can use these new states to compute the density matrix corresponding to the coarse-grained link.

**Coarse-graining of a graph**

The coarse-graining of a graph is nothing but the combination of the two previous steps. Figure 2.3 shows a graphical illustration of the procedure in the case of the checkerboard spin network.

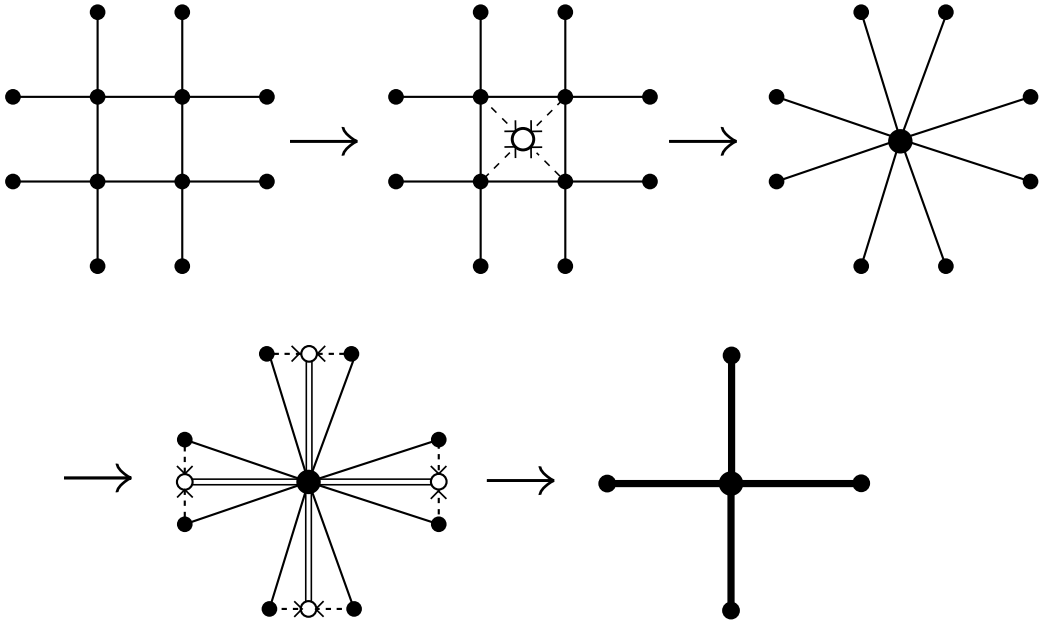

Figure 2.3: The coarse-graining of a graph is the result of the merging of the vertices into a single vertex and then parallel links are combined together in order to form bigger links.

Even if we do not derive the explicit formula for the matrix density, we expect the coarse-graining state to be a mixed state. Therefore, the entropy related to this state will be bound by $S_{min}$ and $S_{max}$, where $S_{min} = 0$ is the case of a pure state and $S_{max} = log(d)$ with $d = dim(\mathcal{H}_{\partial A})$ is obtained with the totally mixed state:

$$\rho = \frac{\mathbb{1}_{\partial A}}{dim(\mathcal{H}_{\partial A})} \tag{2.37}$$

Furthermore, we have already noticed that the coarse-graining requires to choose some data that we want to preserve through the process. Thus, we should not get the totally mixed state since this corresponds to the situation where we erase all the information about $\Gamma_A$, i.e. we don't preserve any information about the interior of the graph of the region $A$. We will see later that the choice of the mixed state, and so, the choice of the data that we wish to preserve, is fundamental in order to derive a notion of distance.

### 2.4.3 Derivation of the mixed state

Let us now explicitly derive the mixed state, as suggested in [25] [24]. We recall that our purpose is to compute the correlations between two regions $A$ and $B$ induced by the exterior region $C$, such that $\partial C = \partial A \cup \partial B$, because we want the distance between the regions to be with respect to the metric in $C$. We call $\Gamma_{(X)}$ the graph obtained from $\Gamma$ after the coarse-graining of the region $X$ i.e. after we have have replaced the interior of the region $X$ by a single vertex. In our configuration, we have:

$$\Gamma_{(C)} = A \cup B \cup \partial C = A \cup B \cup \partial A \cup \partial B \tag{2.38}$$

Therefore, we define the mixed state $\rho_{AB}$ of the two regions $A$ and $B$ as the density matrix obtained after the coarse-graining of the region $C$. The resulting mixed state correlates the region $A$ and $B$. If we reshape the spin network according to the figure 2.4, we confirm that $\partial C = \partial A \cup \partial B$, but more important, we understand that looking only at the boundary of $C$, it is not possible to know how big is the graph distance between the two regions.

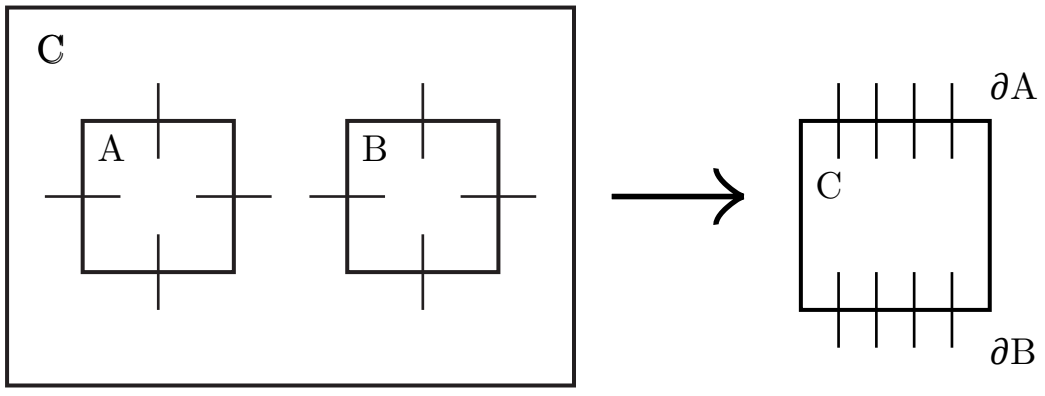


Figure 2.4: We reshape the spin network in order to get a situation similar to figure 2.2.

The distance is encoded in the region $C$ and this is precisely that information we would like to preserve through the coarse-graining. Moreover, we have just mentioned that the coarse-graining of the region will lead to the definition of the mixed state $\rho_{AB}$. Consequently, the strategy is to use the calculation of the entanglement associated with this state in order to define a notion of distance.

The space of the region $C$ is the one given by the tensor product of the intertwiner spaces associated with each node outside $A$ and $B$. According to what we have done above, the coarse-graining of the region $C$ will lead to the boundary state which lives in the intertwiner space on $\partial C$ i.e.

$$\mathcal{H}_{\partial C} = Int(\bigotimes_{e\in\partial C} V^{j_e}) \tag{2.39}$$

The first thing we would like to emphasize is that the boundary state does not live in the space $\mathcal{H}_{\partial A}\otimes\mathcal{H}_{\partial B}$. We would have obtained a state in $\mathcal{H}_{\partial A}\otimes\mathcal{H}_{\partial B}$ if we had considered boundary states separately in the two regions. Indeed, if we fix an intertwiner in each boundary region space $\mathcal{H}_{\partial A}$ and $\mathcal{H}_{\partial B}$, and then take an overall intertwiner, it lives in $\mathcal{H}_{\partial A}\otimes\mathcal{H}_{\partial B}$. This tensor product space is actually included in the one we are interested in:

$$\mathcal{H}_{\partial A}\otimes\mathcal{H}_{\partial B}\subset\mathcal{H}_{\partial C} \tag{2.40}$$

Let us give a little more information about the difference between these two spaces. We focus on the very simple situation where the regions $A$ and $B$ consist of one single vertex. The count of the dimensions of the spaces is the following:

$$\begin{aligned}
\dim(\mathcal{H}_{\partial A}\otimes\mathcal{H}_{\partial B}) &= \dim(\mathcal{H}_{\partial A})\times\dim(\mathcal{H}_{\partial B}) && (2.41)\\
&= \dim\Big(Int\Big((V^{1/2})^{\otimes 4}\Big)\Big)\times\dim\Big(Int\Big((V^{1/2})^{\otimes 4}\Big)\Big) && (2.42)\\
&= \dim\Big(Int\Big(2V^0\oplus 3V^1\oplus V^2\Big)\Big)^2 && (2.43)\\
&= 2^2 = 4 && (2.44)
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
\dim(\mathcal{H}_{\partial C}) &= \dim\Big(Int\Big((V^{1/2})^{\otimes 8}\Big)\Big) && (2.45)\\
&= \dim\Big(Int\Big(14V^0\oplus 28V^1\oplus 20V^2\oplus 7V^3\oplus V^4\Big)\Big) && (2.46)\\
&= 14 && (2.47)
\end{aligned}$$

This non negligible difference of degrees of freedom comes from the difference of gauge freedom. When we consider two different vertices, we have the freedom to choose the intertwiners

and so impose under which transformation we want the invariance to be. When considering the $SU(2)$ invariant subspace $Int\Big((V^{1/2})^{\otimes 8}\Big)$, we implicitly make the choice to fix the same transformation at both vertices. If we use the full local freedom so that we allow the possibility of having invariance under different gauge transformations, we get the smaller space $Int\Big((V^{1/2})^{\otimes 4}\Big) \otimes Int\Big((V^{1/2})^{\otimes 4}\Big)$. To sum up, we see that if we allow more gauge freedom, we reduce the degrees of freedom. It is illustrated by the count of the dimensions of the invariant spaces. Therefore, it is legitimate to wonder why we are more interested in $\mathcal{H}_{\partial C}$ than $\mathcal{H}_{\partial A} \otimes \mathcal{H}_{\partial B}$. Let us recall one more time that the purpose of this study is to derive the distance between the regions $A$ and $B$ with respect to the metric on $C$, and therefore, we want to compute the correlations induced by the state in $\bigotimes_{v \in C} Int_v$.

We will now look at the result of the coarse-graining. Graphically, the coarse-graining reduces the region $C$ to a virtual edge linking the two boundary vertices of $\partial A$ and $\partial B$. More precisely, the resulting space decomposes into a sum of Hilbert space labeled by $j$, where $j$ is the spin associated with the virtual edge created like so. This virtual edge is here to ensure the $SU(2)$ invariance at the vertices. Furthermore, the values of $j$ possible are such that we can find an intertwiner subspace to $V^j \otimes \bigotimes_{e \in \partial X} V^{j_e}$. Consequently, in the simple case where $\mathcal{H}_{\partial A} = \mathcal{H}_{\partial B} = Int\Big((V^{1/2})^{\otimes 4}\Big)$, the possible values of $j$ are 0, 1 and 2.

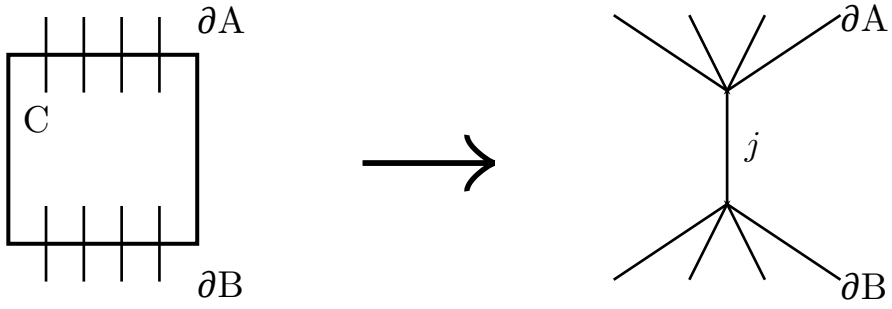


Figure 2.5: The coarse-graining of the region $C$ makes appear a virtual link labeled by $j$ which is necessary in order to maintain the gauge invariance at each vertex. The resulting state on the boundary of $C$ is then decomposed on a basis labeled by $j$.

In the case where $j = 0$, the vertices associated with $\partial A$ and $\partial B$ are not connected by the virtual edge. The corresponding space is actually $\mathcal{H}_{\partial A} \otimes \mathcal{H}_{\partial B}$. This is another way to see that $\mathcal{H}_{\partial A} \otimes \mathcal{H}_{\partial B}$ is a subspace of $\mathcal{H}_{\partial C}$. With the virtual edge added to the vertices, the new

intertwiner spaces are given by:

$$Int(V^j \otimes \bigotimes_{e\in\partial X} V^{j_e}) \tag{2.48}$$

where $X = A, B$. Hence,

$$Int(\bigotimes_{e\in\partial C} V^{j_e}) = \bigoplus_j \Big( Int\Big(V^j \otimes \bigotimes_{e\in\partial A} V^{j_e}\Big) \otimes Int\Big(V^j \otimes \bigotimes_{e\in\partial B} V^{j_e}\Big)\Big) \tag{2.49}$$

From this formula we can explicitly check that for $j = 0$, we get $\mathcal{H}_{\partial A} \otimes \mathcal{H}_{\partial B}$. Furthermore, we can already anticipate the fact that the entanglement will deeply depends on the value of the spin $j$.

With the procedure described above, we have seen that in order to get a mixed state of the regions $A$ and $B$, we can realize a coarse-graining of the region $C$. This leads to a state lying in the boundary space $\mathcal{H}_{\partial C}$ which is nothing but the intertwiner subspace of the tensor product of the representation spaces associated with the edges lying in the boundary i.e. $\mathcal{H}_{\partial C} = Int(\bigotimes_{e\in\partial C} V^{j_e})$. Let us now go back to our toy model and derive explicitly this intertwiner subspace. Before investigating the special case where both $A$ and $B$ are made of a single vertex, let us keep it general. We write the Hilbert spaces associated with the regions the following way:

$$\mathcal{H}_A = \bigotimes_{e\in A} V^{j_e} \quad ; \quad \mathcal{H}_B = \bigotimes_{e\in B} V^{j_e} \tag{2.50}$$

But any representation $R(g)$ of SU(2) can be decomposed into irreducible representations such that:

$$R(g) = \bigoplus_k \mathcal{R}(g) \otimes \mathbb{1}_{d_k} \tag{2.51}$$

where $d_k$ is the dimension of the corresponding representation space. Therefore we can decompose the Hilbert spaces the following way:

$$\mathcal{H}_A = \bigotimes_{e\in A} V^{j_e} = \bigoplus_k (V^k \otimes \sigma_k^A) \tag{2.52}$$

where $\sigma_k^A$ is the degeneracy space associated with the spin-$k$ representation. The dimension $d_k^A \equiv dim(\sigma_k^A)$ is the multiplicity of the spin-$k$ subspace of $\mathcal{H}_A$. In the case where the region $A$ is composed of $2k$ qubits, we have the following relation:

$$d_j^{(2k)} = \binom{2k}{k+j} \frac{2j+1}{k+j+1} \tag{2.53}$$

Obviously the previous propositions hold for the region $B$ as well. The decomposition 2.52 gives us direct access to the intertwiner subspace of a region. Now, according to the coarse-graining

procedure, the state we are interested in is a singlet state in $\mathcal{H}_A \otimes \mathcal{H}_B$ i.e. an intertwiner in $\bigotimes_{e\in A,B} V^{j_e}$. Using the decomposition 2.52, we get:

$$\mathcal{H}_A \otimes \mathcal{H}_B = \Big(\bigotimes_{e\in A} V^{j_e}\Big) \otimes \Big(\bigotimes_{e\in B} V^{j_e}\Big) \tag{2.54}$$

$$= \Big(\bigoplus_{j_A} \Big(V^{j_A} \otimes \sigma^A_{j_A}\Big)\Big) \otimes \Big(\bigoplus_{j_B} \Big(V^{j_B} \otimes \sigma^B_{j_B}\Big)\Big) \tag{2.55}$$

$$= \bigoplus_{j_A}\bigoplus_{j_B} \Big(V^{j_A} \otimes V^{j_B}\Big)\Big(\sigma^A_{j_A} \otimes \sigma^B_{j_B}\Big) \tag{2.56}$$

We can construct the space $Int(\mathcal{H}_A \otimes \mathcal{H}_B)$ as the space of spin-0 states which is obtained when $j_A = j_B = j$:

$$\mathcal{H}_0 \equiv Int(\mathcal{H}_A \otimes \mathcal{H}_B) \tag{2.57}$$

$$= Int(\bigotimes_{e\in A,B} V^{j_e}) = \bigoplus_j V^0_{(j)} \otimes (\sigma^A_j \otimes \sigma^B_j) \tag{2.58}$$

where $V^0_{(j)}$ is the singlet state in $V^j \otimes V^j$. This space $\mathcal{H}_0$ is the intertwiner space on $\partial C$ and the label $j$ corresponds to the spin of the virtual edge between the boundary vertices. The dimension of the intertwiner subspace is given by the formula:

$$\Delta = dim(\mathcal{H}_0) = \sum_j d^A_j d^B_j \tag{2.59}$$

In the case where $\mathcal{H}_{A,B} = (V^{1/2})^{\otimes 4} = 2V^0 \oplus 3V^1 \oplus V^2$, we have $d^A_0 = 2$, $d^A_1 = 3$ and $d^A_2 = 1$ and therefore $\Delta = 14$. This is what we found above when we computed directly $dim\Big(Int\Big((V^{1/2})^{\otimes 8}\Big)\Big)$.

We will now derive a basis of this space. Then, we will be able to pick a state in this space which will be our mixed state. Let us write $|j, a_j, b_j\rangle$ the components of the basis of the intertwiner space on the boundary of $C$, where $|a_j\rangle$ and $|b_j\rangle$ are the basis vectors of the degeneracy spaces. Explicitly, we have:

$$a_j = 1, ..., d^A_j = dim(\sigma^A_j) \tag{2.60}$$

$$b_j = 1, ..., d^B_j = dim(\sigma^B_j) \tag{2.61}$$

Therefore, the basis states associated with the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are given by $|j, m, a_j\rangle$ and $|j, m, b_j\rangle$, where $m$ is the usual quantum number such that $-j \leq m \leq +j$. Following the formula given in 2.58, we write the basis vectors the following way:

$$|j, a_j, b_j\rangle = |j\rangle_{V^j_A \otimes V^j_B} \otimes |a_j\rangle_{\sigma^A_j} \otimes |b_j\rangle_{\sigma^B_j} \tag{2.62}$$

where $|j\rangle_{V_A^j \otimes V_B^j}$ is the spin-0 state in $V_A^j \otimes V_B^j$. Hence the explicit formula:

$$|j, a_j, b_j\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{m=+j} |j, -m, a_j\rangle_A \otimes |j, m, b_j\rangle_B \tag{2.63}$$

Before pursuing with the procedure, let us apply what we have just done in some particular cases. Recall that when we consider two qubits, we have $(V^{1/2})^{\otimes 2} = V^{1/2} \otimes V^{1/2} = V^1 \oplus V^0$ and a basis vector of the one dimensional intertwiner subspace is given by the EPRB state:

$$|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \equiv \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle) \tag{2.64}$$

using the convention of Quantum Information: $|1\rangle = |j = 1/2, m = 1/2\rangle$ and $|0\rangle = |j = 1/2, m = -1/2\rangle$. We can find this state by applying the formula 2.63 for $j = 1/2$. Similarly, in the case of four qubits where $(V^{1/2})^{\otimes 4} = 2V^0 \oplus 3V^1 \oplus V^2$, we can derive two basis vectors of the invariant space by antisymmetrising each qubit and then symmetrize between the couple of pairs [12]. For instance, one basis vector is given by:

$$\begin{aligned}
&\frac{1}{\sqrt{12}}\Big((|0011\rangle - |0011\rangle) + (|0011\rangle - |1001\rangle) \\
&\qquad +(|0011\rangle - |1010\rangle) + (|1100\rangle - |1100\rangle) \\
&\qquad +(|1100\rangle - |0110\rangle) + (|1100\rangle - |0101\rangle)\Big) \\
&= \frac{1}{2\sqrt{3}}\Big(2|0011\rangle + 2|1100\rangle - |1001\rangle - |1010\rangle - |0110\rangle - |1010\rangle\Big)
\end{aligned}$$

### 2.4.4 Calculation of the entanglement

With the basis given in 2.63, we can pick a mixed state and derive the corresponding entanglement. Therefore, the issue is now to choose an appropriate state in order to get what we are interested in, namely the distance between the regions. Indeed, the choice of a state corresponds to the choice of the data preserved during the coarse-graining. At some point, we will have to make an arbitrary assumption in the calculation and define the distance from there.

Although we have stressed this would not be the resulting state of the coarse-graining, we shall first compute the entanglement in the case of the totally mixed state. The result will give us a first idea of the dependence of the distance on the value of the virtual link $j$. Furthermore, such a result might be useful in order to normalise our definition of distance as the totally mixed state gives the maximum value of the entanglement.

In the case of the totally mixed state, the density matrix is proportional to the identity matrix such that:

$$\rho = \frac{\mathbb{1}_{\mathcal{H}_0}}{\Delta} \tag{2.65}$$

where $\Delta = dim(\mathcal{H}_0) = \sum_j d_j^A d_j^B$. Using the expression of the basis vectors, we explicitly have:

$$\rho = \frac{1}{\Delta} \sum_{j,a_j,b_j} |j, a_j, b_j\rangle\langle j, a_j, b_j| \tag{2.66}$$

$$= \frac{1}{\Delta} \sum_j \sum_{a_j,b_j} |j\rangle\langle j|_{V_A^j \otimes V_B^j} \otimes |a_j\rangle\langle a_j|_{\sigma_j^A} \otimes |b_j\rangle\langle b_j|_{\sigma_j^B} \tag{2.67}$$

We already know that for a pure state, the entanglement is just the Von-Neuman entropy of the reduced density matrix: $S(\rho_A) = S(\rho_B)$. For given $j$ and $a_j$, the reduced density matrix $\rho_{j,a_j}$ obtained by tracing over the space $\mathcal{H}_B$ is:

$$\rho_{j,a_j} = Tr_B |j, a_j, b_j\rangle\langle j, a_j, b_j| \tag{2.68}$$

Because we trace over $\mathcal{H}_B$, there are $d_j^B$ different states $\rho_{j,a_j}$. These matrices are diagonal and only $2j+1$ terms are non vanishing so that:

$$\rho_{j,a_j} = \frac{1}{2j+1} \mathbb{1}_{V^j} \otimes |a_j\rangle\langle a_j|_{\sigma_j^A} \tag{2.69}$$

We get the density matrix $\rho_{j,A}$ by summing over all the possible $a_j$'s, and then $\rho_A$ by summing over all possible $j$'s:

$$\rho_A = \frac{1}{\Delta} \sum_{j,a_j,b_j} d_j^B \Big(\frac{1}{2j+1}\Big) \mathbb{1}_{V^j} \otimes |a_j\rangle\langle a_j|_{\sigma_j^A} \tag{2.70}$$

As we have explained it earlier, we are interested in the entanglement of formation which is the average of the entanglement of the pure states, appearing in the decomposition, minimized over the whole range of possible decompositions. Let us first compute the entanglement of the reduced density matrix $\rho_{j,a_j}$. In this case, the result is given by the Von-Neuman entropy:

$$S(\rho_{j,a_j}) = S\Big(\frac{\mathbb{1}_{V^j}}{2j+1}\Big) \tag{2.71}$$

$$= -(2j+1)\Big(\frac{1}{2j+1} log\Big(\frac{1}{2j+1}\Big)\Big) \tag{2.72}$$

$$= -log\Big(\frac{1}{2j+1}\Big) = log(2j+1) \tag{2.73}$$

Hence, by taking the average [23]:

$$\mathcal{E}(A:B,\rho) = \frac{1}{\Delta} \sum_{j,a_j,b_j} S(\rho_{j,a_j}) \tag{2.74}$$

$$= \frac{1}{\Delta} \sum_j d_j^A d_j^B S(\rho_{j,a_j}) \tag{2.75}$$

$$= \frac{\sum_j d_j^A d_j^B log(2j+1)}{\sum_j d_j^A d_j^B} \tag{2.76}$$

In the case where $\mathcal{H}_{A,B} = (V^{1/2})^{\otimes 4}$, we have:

$$\mathcal{E}(A:B,\rho) = \frac{1}{14}\Big(4log(1) + 9log(3) + 1log(5)\Big) \tag{2.77}$$

$$= \frac{9}{14}log(3) + \frac{1}{14}log(5) \tag{2.78}$$

It is obvious from the formula 2.74 that the entanglement grows with the size of the regions. Indeed, the more qubits the boundary of the regions $A$ and $B$ contains, the more they are correlated. However, two pairs of regions of different size can be at the same graph distance, supposing that the distance is given by the position of the centers of the regions (see figure 2.6).

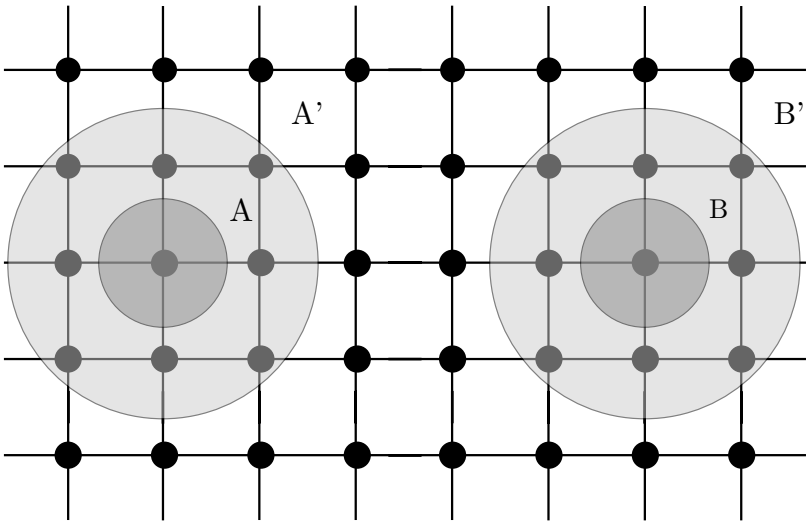


Figure 2.6: The regions $A$ and $B$ are at the same graph distance as the regions $A'$ and $B'$. However the entanglement between bigger regions is more important and therefore it is important to normalize the results in order to get meaningful data.

Therefore, it is necessary to normalise our value of the entanglement. In other words, the definition of distance we will choose needs to give the same value whatever the size of the regions is. We can do that by considering a ratio of our result with a value taking into account the size of the regions.

Let us now consider the general case where the state is an arbitrary combination of the basis vectors $|j, a_j, b_j\rangle$:

$$\rho = \sum_j \sum_{a_j,b_j} \omega^{(j)}_{a_j,b_j} |j\rangle\langle j|_{V_A^j \otimes V_B^j} \otimes |a_j\rangle\langle a_j|_{\sigma_j^A} \otimes |b_j\rangle\langle b_j|_{\sigma_j^B} \tag{2.79}$$

$$\text{with} \sum_{a_j,b_j} \omega^{(j)}_{a_j,b_j} = 1$$

For a state of this type, we can show that the value of the entanglement is [24]:

$$\mathcal{E}(A:B,\rho) = \sum_{j,a_j,b_j} \omega^{(j)}_{a_j,b_j} log(2j+1) \tag{2.80}$$

If we consider the case where we fix the value of $j$ and don't take into account the entanglement between the degeneracy spaces, we find a very easy formula for the entanglement with an explicit dependence on $j$:

$$\mathcal{E}(A:B,\rho) = log(2j+1) \tag{2.81}$$

### 2.4.5 Interpretation of the entanglement

The last case we have described is not very interesting as far as the construction of the distance is concerned, however, it allows to draw a few conclusions. First of all, the entanglement increases as the value of $j$ grows and so, we expect the distance between the regions to decrease. On the contrary, if the value of $j$ is small compared to the spin numbers labeling the edges on the boundary, we will expect the regions to be far away. This induces the idea that two regions very close on the spin network, i.e. with only a few vertices between them, are not necessarily close in the physical space. Consequently, the value of $j$ has a huge impact on the entanglement and therefore, on the physical distance we expect to derive. Once again, this sheds light on the idea that the physical space cannot be intuitively derived from the graphical structure of the spin network.

Given an embedding of a spin network in $\mathbb{R}^3$, there might be an important difference between the distance in the embedding and the actual physical distance. Let us discuss this point of view following the work of Smolin and Markopoulou in [27]. The model considered by the authors is based on a graph with $N$ nodes where there can only be one edge between two nodes. A very simple example of such a graph is represented figure 2.7.

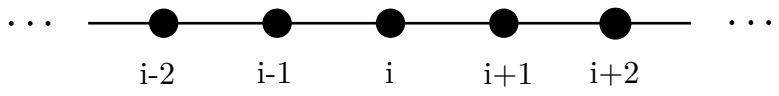


Figure 2.7: Example of a graph where two nodes are connected by one single edge. The only information provided by the graph is the adjacency between the nodes.

Two vertices are said to be adjacent if they are related by one edge. Equally, nodes are said to be nearest neighbours of a given node $P$ if we can reach them, starting from $P$, by traveling

along a single edge. The next nearest neighbours of a given node are those reached by traveling along two edges. Mathematically, we can define an adjacency matrix $\mathcal{A}$. The adjacency matrix $\mathcal{A}$ is a $N \times N$ matrix such that $\mathcal{A}_{ij} = 2$ if the nodes $i$ and $j$ are nearest neighbours, $\mathcal{A}_{ij} = 1$ if the nodes are next nearest neighbours and $\mathcal{A}_{ij} = 0$ otherwise. In the case of the graph given figure 2.7, we have:

$$\mathcal{A} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \tag{2.82}$$

This model is similar to spin networks in the sense that we can think about the nodes of the model as the $N$ elementary building blocks of a universe. However, the only information available is contained in the adjacency matrix. There is no quantum numbers on the edges and no intertwiner at each node. In order to make appear a notion of distance, we will assume there is a non relativistic low energy limit and that we can embed this graph into $\mathbb{R}^3$. Without discussing the procedure allowing such a embedding, the result is that we gain access to coordinates at each node. We denote $x_i^a$ the coordinates of the $i$-th node, where $i = 1, ..., N$ and $a = 1, 2, 3$. This embedding is actually an approximation in which we expect classical physics to emerge. The low energy limit requires in particular a minimum length in the embedding such that:

$$|x_i^a - x_j^a|_{min} = l \tag{2.83}$$

where an appropriate choice for the minimum length $l$ would be the Planck scale. Furthermore, we define the average distance such that:

$$L^2 \equiv \langle (x_i^a - x_j^a)^2 \rangle \tag{2.84}$$

The important feature of this model is that there is no direct relation between the distance in the embedding and the adjacency on the graph. Let us illustrate this point of view with an example. Let us consider a node $x_i$ on the graph. We can define the nearest neighbours of $x_i$ in the embedding as the nodes $x_j$ which are in a sphere with radius $r$ and center $x_i$. There is no requirement that this definition of nearest neighbours in the embedding matches the graphical one.

This model sheds light on several key properties of spin networks. The notion of distance in some given embedding of the graph into $\mathbb{R}^3$ is not necessarily related to the physical one. More precisely, we could have two regions considered far away according to the distance in the embedding, but strongly correlated, and therefore really close in the physical space. This

corroborates the idea that the geometric picture of the space obtained by gluing polyhedra may be highly misleading and that the physical distance may have nothing to do with the network distance. In any case, we should expect any notion of distance derived according to the method described in this dissertation to be very counter-intuitive.

## 2.5 Classical limit

The attempt to define a notion of distance on spin networks falls within the scope of the project to retrieve the features of General Relativity from the theory of Loop Quantum Gravity. More precisely, the overall question is how the classical limit appears, or in other words, whether the theory contains the main feature of General Relativity in a semi-classical limit. We will focus this study on the case of spin networks with only four-valent nodes.

If going from the classical level to the quantum level is quite straightforward because there is well-defined procedure to follow, it is much more difficult to go back to the classical world once the quantum theory is established. Indeed, it requires to properly define the classical limit which is not an obvious task at all. In Quantum Mechanics, the classical limit emerges with the coherent states. Coherent states are a family of wave packets which minimize the spread of both position and classical momenta. In the context of non-relativist particles, wave packets behave like classical particle as far as the dynamic is concerned.

We have mentioned several times that even if a four-valent intertwiner can be well represented by a tetrahedron, this is wrong to think about space as a gluing of tetrahedra. One reason is that the only information available are the volume and the surface of the faces which is not enough to fully encode the shape of a tetrahedron. Indeed, classically we define the shape of a tetrahedron by at least six variables whereas the volume and the surfaces provide only five variables. Nevertheless, in the case of our toy model, because the intertwiner is dual to a regular tetrahedron, we fully describe the shape with the data provided by the spin network. Thus, we could be tempted to think that the picture of a gluing of tetrahedra has better chance to hold.

There is another reason why this geometric picture poses a problem. If we consider an edge which is connected to two nodes; this edge is dual to a surface whose area is determined by the coloring of the edge. However, the shape of the surface depends on the other edges joined at the nodes. More precisely, the shape is independently defined by the algebraic data attached to each node. Consequently, each surface possesses a single area but two different shapes. We will not discuss this problem later on but this is important to keep in mind this subtlety.

Consider a state in the spin network basis. We are unable to associate a 3D classical geometry with this state because of the quantum spread. Indeed, as we have just said, the variables do not fully describe the dual picture of the nodes, namely the tetrahedron. Therefore,

there is an uncertainty regarding the shape around the classical one. We need to find quantum states which approximate a given classical geometry. More precisely, these states will be peaked on the intrinsic and the extrinsic geometry of space.

### 2.5.1 Coherent tetrahedra

A four-valent node is dual to a tetrahedron. Following closely the procedure suggested in [38], let us first study the geometry of the classical shape. One way to fully characterize the classical geometry of a tetrahedron is to use 4 areas and 2 angles between the faces. However, a more efficient way is to use the four normal vectors $\vec{\Lambda}_p$ which satisfy the closure relation:

$$\sum_p \vec{\Lambda}_p = \vec{0} \tag{2.85}$$

When we introduced the triads $e^i_a$, we mentioned they have the meaning of a frame. Let us consider a node $P$ of a tetrahedron, three edges meet at this node. We associate a vector $\vec{e}_a$ with each one of them. The direction of the vector is the one of the edge and its norm is equal to the length of the edge. Thus, the three vectors define the unit of the coordinate length as displayed figure 2.8.

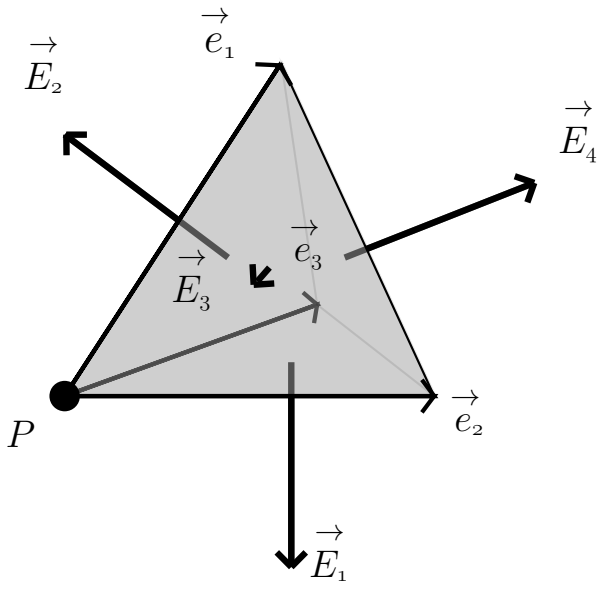


Figure 2.8: The triads are associated with three edges of the tetrahedron meeting at the same point whereas the densitized triads are normal to the faces.

The three triads define completely the tetrahedron. Indeed, we can express six independent parameters which characterise completely the shape:

$$\vec{e}_1 \cdot \vec{e}_1 \ ; \ \vec{e}_2 \cdot \vec{e}_2 \ ; \ \vec{e}_3 \cdot \vec{e}_3 \tag{2.86}$$

$$\vec{e}_1 \cdot \vec{e}_2 \ ; \ \vec{e}_2 \cdot \vec{e}_3 \ ; \ \vec{e}_1 \cdot \vec{e}_3 \tag{2.87}$$

Let us now recall some of the formulas introduced in the first chapter. The three triads define the 3D metric such that

$$q_{ab} = e^i_a e^j_b \delta_{ij} \tag{2.88}$$

and the densitized triads are given by:

$$E^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c \tag{2.89}$$

The vectors $\vec{E}^a$ are normal to the surfaces spanned by $\vec{e}_b$ and $\vec{e}_c$. We define a new index $p = 1, 2, 3, 4$ and denote the four normal vectors as $\vec{E}^p$. In this configuration the closure relation is given by:

$$\sum_{p=1}^{4} \vec{E}^p = \vec{0} \tag{2.90}$$

Furthermore, the dihedral angles are given by $\vec{E}^a \cdot \vec{E}^b$. Let us now reproduce what we did when we computed the volume of the quantum tetrahedron. We quantized the vectors $\vec{E}_p$ by introducing four vectors $\vec{\Lambda}_p$:

$$\vec{E}_p = 8\pi G\hbar\beta\vec{\Lambda}_p \tag{2.91}$$

where the four hermitean operators $\Lambda_p$ satisfy the commutation relation

$$[\Lambda^i_p, \Lambda^j_q] = i\delta_{pq} l_0^2 \epsilon^{ij}_k \Lambda^k_p \tag{2.92}$$

so that they are the generators of the rotation group. If we are interested in the commutation relation of two dihedral angles, we have for instance the following formula:

$$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_2 \cdot \vec{E}_3] = (8\pi G\hbar\beta)^4 [\vec{\Lambda}_1 \cdot \vec{\Lambda}_2, \vec{\Lambda}_2 \cdot \vec{\Lambda}_3] \tag{2.93}$$

$$= (8\pi G\hbar\beta)^4 [\Lambda^i_1 \Lambda^j_2 \delta_{ij}, \Lambda^k_2 \Lambda^l_3 \delta_{kl}] \tag{2.94}$$

$$= (8\pi G\hbar\beta)^4 \delta_{ij}\delta_{kl} \Lambda^i_1 \Lambda^l_3 \underbrace{[\Lambda^j_2, \Lambda^k_2]}_{i\epsilon^{jk}_m \Lambda^m_2} \tag{2.95}$$

$$= i(8\pi G\hbar\beta)^4 \Lambda_{j1}\Lambda_{k3}\Lambda_{m2}\epsilon^{jkm} \tag{2.96}$$

$$= i(8\pi G\hbar\beta)\vec{E}_2 \cdot (\vec{E}_1 \times \vec{E}_3) \tag{2.97}$$

of the form $[\widehat{A}, \widehat{B}] = \widehat{C}$. And therefore we can apply the Heisenberg incertitude relation

$$\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle \widehat{C} \rangle| \tag{2.98}$$

where $\Delta X = \sqrt{\langle \widehat{X}^2 \rangle - \langle \widehat{X} \rangle^2}$ is the standard deviation and represents the incertitude associated with the observable $\widehat{X}$. In our case, we have the following relation:

$$\Delta(\vec{E}_1 \cdot \vec{E}_2)\Delta(\vec{E}_2 \cdot \vec{E}_3) \geq \frac{1}{2}(8\pi G\hbar\beta)|\langle \vec{E}_2 \cdot (\vec{E}_1 \times \vec{E}_3) \rangle| \tag{2.99}$$

We are looking for states which minimise the incertitude and therefore which saturate the relation 2.99. In other words, we want the relative incertitude to vanish i.e.

$$\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|\vec{E}_a||\vec{E}_b|} \ll 1 \tag{2.100}$$

So, we want to derive quantum states which minimise the spread of the dihedral angles around the classical values.

### 2.5.2 SU(2) coherent states

The SU(2) coherent states are those which minimise the uncertainty of the angular momentum. More precisely, if we consider a particle with angular momentum $\vec{J}$, the dispersion of the angular momentum is given by the standard deviation $\Delta \equiv |\langle \vec{J}^2 \rangle - \langle \vec{J} \rangle^2|$ where $\vec{J}$ is the 3-vector formed from the hermitean generators of SU(2) satisfying $[J^i, J^j] = i\epsilon^{ijk}J^k$. The coherent states are the states which minimise $\Delta$. We call $j$ the quantum number associated with the total angular momentum so that the basis states of the Hilbert space $\mathcal{H}^j$ are denoted $|j, m\rangle$.

Let us recall some properties of the angular momentum in Quantum Mechanics. First of all, we have the following spectrum properties:

$$\begin{cases} (\underline{J}^{AB})^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle \\ J_z^{AB}|j, m\rangle = \hbar m|j, m\rangle \end{cases} \tag{2.101}$$

Furthermore, because the operators $J_x$ and $J_y$ do not commute with the operator $J_z$, the basis state $|j, m\rangle$ is an eigenstate of neither $J_x$ nor $J_y$. Therefore, we cannot predict with certitude the result of a measurement of the angular momentum in the directions $Ox$ and $Oy$. Let us rewrite the two operators the following way:

$$J_x = \frac{1}{2}(J_+ + J_-) \text{ and } J_y = \frac{1}{2i}(J_+ - J_-) \tag{2.102}$$

Consequently, the vectors $J_x|j,m\rangle$ and $J_y|j,m\rangle$ are linear combination of $|j,m+1\rangle$ and $|j,m-1\rangle$, and so:

$$\langle j,m|J_x|j,m\rangle = 0 = \langle j,m|J_y|j,m\rangle \tag{2.103}$$

Moreover, we have:

$$\langle j,m|J_x^2|j,m\rangle = \frac{1}{4}\langle j,m|J_+^2 + J_-^2 + J_+J_- + J_-J_+|j,m\rangle \tag{2.104}$$

$$= \frac{1}{2}\langle j,m|\underline{J}^2 - J_z^2|j,m\rangle \tag{2.105}$$

$$= \frac{\hbar^2}{2}(j(j+1) - m^2) \tag{2.106}$$

$$= \langle j,m|J_y^2|j,m\rangle \tag{2.107}$$

Thus, in the state $|j,m\rangle$:

$$\Delta J_x = \Delta J_y = \hbar\sqrt{\frac{1}{2}\Big(j(j+1) - m^2\Big)} \tag{2.108}$$

So, the uncertainty is minimized for $m = j$. In this case, we have $\Delta J_x = \Delta J_y = \sqrt{\frac{j}{2}}$. These vectors $|j,j\rangle$ of maximal weight saturate the incertitude relations $\Delta J_x \Delta J_y \geq \frac{1}{2}|\langle J_z\rangle|$ which becomes:

$$\Delta J_x \Delta J_y = \frac{j}{2} \tag{2.109}$$

These are the coherent states. Starting from these vectors, we can construct an infinite number of coherent states by rotating the state $|j,j\rangle$:

$$|j,\widehat{n}\rangle = G(\widehat{n})|j,j\rangle \tag{2.110}$$

where $G(\widehat{n}) \in SU(2)$ is a group element which rotates the direction $(0,0,1)$ into the direction defined by the unit vector $\widehat{n}$. Explicitly, we can write:

$$G(\widehat{n}) = e^{i\theta sin\phi J_x} e^{-i\theta cos\phi J_y} \tag{2.111}$$

The family of states $|j,\widehat{n}\rangle$ labeled by the parameter $\widehat{n}$ is the set of coherent states. While the uncertainty of the vector $|j,j\rangle$ is minimized in the direction $Oz$, the one of $|j,\widehat{n}\rangle$ is minimized in the direction $\widehat{n}$. To check this statement, we can use the following facts:

$$\langle j,\widehat{n}|J_a|j,\widehat{n}\rangle = jn_a \tag{2.112}$$

$$\langle j,\widehat{n}|J_a^2|j,\widehat{n}\rangle = \frac{j}{2} + j(j-\frac{1}{2})n_a^2 \tag{2.113}$$

This means that the vector $|j,\widehat{n}\rangle$ describes on average a vector on $\mathbb{R}^3$ of direction $\widehat{n}$ and norm $j$. Hence:

$$\Delta^2 \equiv \langle j,\widehat{n}|\vec{J}^2|j,\widehat{n}\rangle - \langle j,\widehat{n}|\vec{J}|j,\widehat{n}\rangle\langle j,\widehat{n}|\vec{J}|j,\widehat{n}\rangle = j \tag{2.114}$$

In addition, we can decompose the coherent states in terms of the $J_z$ eigenstates:

$$|j,\widehat{n}\rangle = \sum_m \phi_m(\widehat{n})|j,m\rangle \tag{2.115}$$

One of the main feature of the states derived above is the vanishing of the relative uncertainty $\frac{\Delta J}{\sqrt{\langle\vec{J}\rangle^2}}$ when $j$ grows large. Furthermore, these states provide a complete basis for the irreducible representations. Therefore, for a given irreducible representation $j$, the states $|j,\widehat{n}\rangle$ resolve the identity such that:

$$\mathbb{1}_j = d_j \int_{S^2} d^2\widehat{n}|j,\widehat{n}\rangle\langle j,\widehat{n}| \tag{2.116}$$

### 2.5.3 Coherent intertwiners

The SU(2) coherent states are the fundamental blocks of the coherent intertwiners $|\mathcal{I}(\underline{j},\underline{\widehat{n}})\rangle$. An important property of the SU(2) coherent states which turns out to be primordial in the construction of the coherent intertwiners is the following. If we consider a tensor product of SU(2) coherent states $\bigotimes_i |j_i,\widehat{n}_i\rangle$, the vanishing of the relative uncertainty is preserved. More generally, we know that, as soon as the closure relation is satisfied, the states admit a semi-classical limit. Thus, if we note $j_i\widehat{n}_i$, with $i = 1,...,E$, the vectors normal to the faces associated with a given node, they satisfy the closure relation:

$$\sum_{i=1}^{E} j_i\widehat{n}_i = 0 \tag{2.117}$$

and we associate with this node the coherent state:

$$\bigotimes_{i=1}^{E} |j_i,\widehat{n}_i\rangle \tag{2.118}$$

The last thing to do in order to get a coherent intertwiner, is to project down to the invariant subspace, which is done by averaging over the groupe SU(2) i.e.

$$|\mathcal{I}(\underline{j},\underline{\widehat{n}})\rangle = \int_{SU(2)} dh \bigotimes_{i=1}^{E} h|j_i,\widehat{n}_i\rangle \tag{2.119}$$

These states form the so-called coherent interwiners or Livine-Speziale intertwiners. The fundamental property is that the group averaging preserves the semi-classical property of these states. Furthermore, in [22], the authors have shown that the norm of the intertwiners is

peaked as far as the states satisfying the quantum closure condition $\sum_{i=1}^{E} j_i \widehat{n}_i$ are concerned. This induces that quantum correlations are strongly dominated by the contribution of the semi-classical states, in the case of the large spin limit. As a matter of fact, the procedure we have just described holds for nodes of arbitrary valence. A case of particular interest is the one of the coherent tetrahedron, whose semi-classical behavior has been described previously. In this situation we have explicitly:

$$|\mathcal{I}(\underline{j}, \underline{\widehat{n}})\rangle = \int_{SU(2)} dh\ h|j_1, \widehat{n}_1\rangle \otimes h|j_2, \widehat{n}_2\rangle \otimes h|j_3, \widehat{n}_3\rangle \otimes h|j_4, \widehat{n}_4\rangle \tag{2.120}$$

This state satisfy the semi-classical property $\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|\vec{E}_a||\vec{E}_b|} \ll 1$ for large spin $j$. This case is very important since it enters in the definition of the vertex amplitude in the spin foam model [22].

### 2.5.4 Coherent states

In order to construct the semi-classical states, a lot of model have been proposed. One of them is the one used in the recent calculations about graviton propagator [35] [21] [11] [9]. The semi-classical states are obtained as a superposition over spins of spin networks where the nodes are given by the Livine-Speziale intertwiners. We denote $c_j$ the coefficients of this superposition and their expression is the one originally proposed by Rovelli in [35]:

$$c_j(j_0, \xi) = exp\Big( - \frac{(j - j_0)^2}{2\sigma_0} \Big) exp\Big( - i\xi j \Big) \tag{2.121}$$

where we choose $\sigma_0 \approx (j_0)^k$ with $0 < k < 2$. The state is associated with a graph $\Gamma$ where each edge $e$ is labeled by a spin $j_0$ and an angle $\xi$, the nodes are labeled by a set of unit vectors $\vec{n}$ where each vector corresponds to an edge emanating from the node. More precisely, the vectors $\vec{n}$ are actually associated with the normal vectors of the faces of the tetrahedra and $j_0$ represents the average area of the corresponding face. Thus, the formula provides states which are peaked on both the angle and the area. In other words, the relative dispersion of both $\xi$ and $j_0$ vanish for the large spin limit.

# Conclusion

In this dissertation, we have discussed the dual picture of spin networks and how a notion of distance could arise within the background-independent formalism. We identify regions of a given spin network with a dual picture of quantum geometry: nodes are dual to chunks of space and edges are dual to surfaces shared by two adjacent chunks. Furthermore, one of the most remarkable results of Loop Quantum Gravity is the quantum discreteness of the area and the volume operators. However, while the introduction of these operators is quite straightforward, it is much more difficult to define a length operator. In order to define such an operator, the strategy followed is to make full use of the dual picture introduced with the two latter geometrical operators. More precisely, the length operator can be expressed as a sum of elementary length operators. The elementary length operator measures the length of the intersecting curve between two surfaces dual two edges emanating from the same node. Even if the length operator defined like so has a discrete spectrum as well as an appropriate semi-classical limit, the definition is very complicated and the physical interpretation not suitable to compute distances between parts of the spin network. The difficulty to define such an operator is an indication of the problems we face in defining a notion of distance. Furthermore, the fact that the definition of the length operator is so little obvious is testament to the fact that we should not get too tied up to the nice picture of the dual space. Indeed, we tried in this dissertation to shed light, through numerous arguments, on the fact that the picture of a space made of a gluing of polyhedra can be highly misleading.

The study of the entanglement carried out in the second chapter brings additional information. First of all, the adjacency of the nodes on a spin network does not mean they are close in the physical space. Indeed, if as expected the entanglement decreases with the distance, it means that two adjacent nodes could be very distant in the physical space if the virtual edge is labeled by a very low spin number $j$. On the contrary, when computing the correlations between two regions, the coarse-graining procedure could lead to a virtual link with a very high spin $j$ which would mean that the two regions are close in the physical space, and this,

even if the regions are not directly related by an edge. In any case, it seems that the coloring of the edges would be more determinant in the construction of the physical space than the adjacency of the nodes and more generally the shape of the graph.

The model proposed in this dissertation is an attempt to use Quantum Information tools in order to measure distance. The problem we encountered is the choice of state, once we have derived the basis of the invariant subspace of the tensor product of the space associated with the two regions. Indeed, it will be required to make an arbitrary assumption and use the results to define the measure of distance. We focused on the case of the totally mixed state in order to get a first taste of the behavior of the entanglement on a spin network. However, we have stressed many times that we do not expect this case to be of any relevance as far as the computation of the distance is concerned. Indeed, because this corresponds to the situation where we lost all the information from the original graph during the coarse-graining process, it is not suitable to quantify how distant the regions are. As a matter of fact, this measure would be appropriate in the case of Black Holes where the extreme density would make the regions as if they were blended into each other.

As suggested by the structure of this dissertation, we would need, in order to go further, to complicate the procedure by taking into account the semi-classical limit and therefore consider a spin network with coherent intertwiners. Furthermore, it would be interesting to make sense of the superposition of spin networks and more precisely to define what it means for two diffeomorphism invariant objects to be far away. This question is a strongly related topic which could deeply enlightens the issue treated here.

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