Generalized Geometry and the NS-NS Sector of Type II Supergravity

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Abstract

The reformulation of the NS-NS sector of type II supergravity in terms of generalized geometry is discussed. We show that the generalized tangent bundle $TM \oplus T^*M$ admits a natural action of $O(d,d)$, the T-duality group. Generalized differential structures are seen to encode the diffeomorphism and gauge symmetries of the supergravity action and to be invariant under a larger group of symmetries including both B-field transformations and diffeomorphisms. Promoting the structure group to $O(d,d) \times \mathbb{R}^+$ provided the right framework to incorporate the dilaton and to accommodate a generalized metric that unified the NS-NS fields. Generalized analogues of curvature tensors are discussed and used to construct a unique Ricci scalar for the supergravity action.

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Contents

1 Introduction

Generalized geometry originally arose through work by Hitchin on invariant functionals of differential forms for special geometry in low dimensions [1]. The core premise of generalized geometry is to promote the tangent bundle TM of a manifold M to the *generalized tangent bundle* $TM \oplus T^*M$ [2]. This extended bundle provides a way of unifying complex and symplectic geometries as extremal cases of a more general structure, the understanding of which was further developed by Hitchin and his students Gualtieri and Cavalcanti [2]. The key idea is to consider operations on this generalized structure instead of just on the usual tangent bundle [3].

The mathematical framework of generalized geometry has subsequently proven extremely powerful for elucidating important symmetries of string theory when applied to its low energy limit, supergravity. As its name suggests, supergravity is a theory encorporating the principles of supersymmetry and Einstein gravity [4] and was in fact discovered independently of string theory in 1976 [5]. Now recognized as the leading candidate for a theory of quantum gravity, string theory was developed in the late 1960s, originally in an effort to describe the strong interaction [4]. It replaces the idea of particles as point-like objects with 1-dimensional extended objects called strings. Different vibrational modes of a string are identified with various types of elementary particle. One particular mode of vibration gives rise to a spin-2 particle identified as the graviton; gravity thus naturally arises in string theory and it is not put in by hand [6].

At first, the field content of string theory was entirely bosonic with a string living in 26-dimensions. The introduction of fermions in the early 1970s gave birth to supersymmetry and thereby superstring theory [7]. Superstring theory predicts that strings propagate in 10-dimensions. However, this is 6 more dimensions than we are used to experiencing in nature. One way of circumventing the superfluous dimensions is by curling them up small enough for them to escape detection by low energy experiments. A popular type of compactified space is known as a Calabi-Yau manifold [8].

The mid 1980s marked the first superstring revolution and out of this period came five different 10-dimensional superstring theories: type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic [6]. Dualities drove major developments in the second superstring revolution which began in 1995. The five superstring theories could be united as part of a unique fundamental theory with an 11-dimensional non-perturbative solution known as M-theory [4].

A duality of particular significance in our generalized geometry discussion is *T-duality* whose discovery actually preceded the second revolution [4]. T-duality can be conceptually understood by considering two cylindrical spacetimes for two string theories X and Y with compactified dimension S^1 of radius R_X and R_Y respectively. If the two radii are such that $R_X \propto 1/R_Y$ then these two theories are related by T-duality [9]. T-duality results from strings being extended objects which can thus wind around the compactified dimension [9]. Two such string theories which are related by T-duality are type IIA and type IIB [10].

In this dissertation we focus on the Neveu-Schwarz (NS-NS) sector of the (9+1)-dimensional type IIA and type IIB supergravity theories which correspond to the full type IIA and type IIB superstring theories. The field content of the NS-NS sector

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi\}, \tag{1.1}$$

consists of the graviton $g_{\mu\nu}$ ($\mu = 1, ..., 10$) which is a 2nd-rank symmetric traceless tensor, the 2-form potential $B_{\mu\nu}$, and the scalar field ϕ known as the *dilaton*. The bosonic action for the NS-NS sector is given by [11]

$$\mathcal{S}_B = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2)\}. \tag{1.2}$$

where $\mathcal{R}$ is the Ricci scalar, $g = det(g_{\mu\nu})$ is the determinant of the metric tensor, $H = dB$ is the 3-form flux, $(\partial\phi)^2 = \partial_\mu\phi\partial^\mu\phi$ and $\kappa = 8\pi Gc^{-4}$ is a constant. The corresponding field equations for $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ are [11]

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + 2\nabla_\mu\nabla_\nu\phi &= 0, \\ \nabla^\mu(e^{-2\phi}H_{\mu\nu\rho}) &= 0, \\ \nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}R - \frac{1}{48}H^2 &= 0, \end{aligned} \tag{1.3}$$

respectively, where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor and ∇ is the Levi-Civita connection.

The supergravity action (1.2) exhibits two important symmetries; that of diffeomorphism invariance and gauge transformations [12]. Infinitesimally, diffeomorphisms are parameterized by a vector $v \in TM$ and gauge transformations by a one-form $\lambda \in T^*M$. The infinitesimal variations of the fields are given by

$$\delta_v g = \mathcal{L}_v g \qquad \delta_v\phi = \mathcal{L}_v\phi \qquad \delta_v B = \mathcal{L}_v B + d\lambda \tag{1.4}$$

where under gauge transformations we have $B \to B' = B + d\lambda$, $\mathcal{L}_v$ is the usual Lie derivative along a vector field v and $d : \wedge^p T^*M \to \wedge^{p+1}T^*M$ is the exterior derivative. Generalized geometry provides a natural structure for unifying these symmetries by combining the parameters into a single object belonging to the generalized tangent bundle $TM \oplus T^*M$. In this way the extended bundle accommodates a larger symmetry group including both diffeomorphisms and gauge transformations

The symmetry group of greatest interest to us in the following discussion is that of the structure group $O(d, d)$ for the generalized tangent bundle. This is the same as the T-duality group [13]. At scales relevant for supergravity, we are considering particles as point-like objects and so we would not expect to find remnants of a string theory related to a string's extension. We find that not only does generalized geometry provide a framework for a reformulation of type II supergravity theories, it is also adapted to incorporate manifest symmetries of string theory, proving an appropriate tool for gaining deeper insights into such high energy theories.

Generalized geometry is not the first instance of a rewriting of supergravity. Other formulations including the work of Siegel [14, 15] and more recently Hull and Zwiebach considered doubled $2d$-dimensional spaces [16]. Exceptional generalized geometry is an extension of Hitchin's work introduced by Hull [17] and further developed by Pacheco and Waldram [18], for incorporating symmetries of M-theory. It involves a more complicated generalized bundle which admits a natural $E_{d(d)} \times \mathbb{R}^+$ structure relevant to the U-duality group of string theory.

Our approach initially focuses on the original generalized tangent bundle $TM \oplus T^*M$, the space on which our bosonic fields g and B live. We proceed in building objects living on this space by analogy with constructs familiar from our experience of conventional differential geometry. In the first section we see that the natural metric of $TM \oplus T^*M$ gives rise to the $O(d,d)$ structure group. We investigate how this group patches elements of the bundle and its action on generalized differential structures which encode diffeomorphism and gauge symmetries. In the following section we build a generalized analogue of the Riemannian metric which unifies the g and B fields into a single object. The latter portion of this dissertation is dedicated to the extended bundle $(TM \otimes T^*M) \times \mathbb{R}^+$; a space also able to accommodate the dilaton ϕ. Generalized frames enable us to build objects on this space such as a generalized metric which captures the full NS-NS sector. In section 5, we introduce the notion of generalized curvature with analogues of connections, torsion, the Riemann tensor, Ricci tensor and Ricci scalar. Finally we show that we can rewrite the above supergravity action (1.2) using our generalized analogues and reproduce the correct field equations.

2 $O(d,d)$ generalized geometry

In this section we introduce the space, known as the generalized tangent bundle, on which we build the generalized analogues of structures familiar from conventional differential geometry. We will see that a metric naturally emerges, equipping the bundle with an $O(d,d)$ structure group whose action on fibres we investigate. Generalized differential structures are then introduced and their symmetries are explored by deconstructing a general element of $O(d,d)$.

2.1 Linear structure

2.1.1 The natural metric

The generalized tangent bundle is given as the direct sum of the tangent bundle $TM \xrightarrow{\pi} M$ and cotangent bundle $T^*M \xrightarrow{\pi} M$ over a smooth manifold M [2]

$$E = TM \oplus T^*M. \tag{2.1}$$

An element belonging to a fibre of E is a generalized vector $V = v + \lambda$ where $v \in TM$ and $\lambda \in T^*M$. We denote the components of the vector by $V^M = v^\mu + \lambda_\mu$, where M=1,...,2d is the generalized index and μ=1,...,d is our usual spacetime index, with d the dimension of the manifold.

The bundle E is endowed with a natural inner product acting on generalized vectors [10]

$$\langle V, W\rangle = \langle v + \lambda, w + \sigma\rangle = \frac{1}{2}(\sigma(v) + \lambda(w)), \tag{2.2}$$

where $v, w \in TM$ and $\lambda, \sigma \in T^*M$. Correspondingly there is a natural metric, given explicitly in matrix form, by

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{2.3}$$

After diagonalization, this metric becomes

$$\eta' = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{2.4}$$

where it is now clear that it has an indefinite signature (d,d). This metric thus naturally defines the action of the Lie group $O(d,d)$ which leaves it, and consequently the inner product, invariant. The appearance of the group $O(d,d)$ will become ubiquitous throughout our discussion as it provides the generalization of the $GL(d)$ transformations, which act on the fibres of TM, familiar from conventional differential geometry.

The natural metric can be used to raise and lower generalized indices. By contracting with η_{MN}, a generalized vector can be converted to a 'generalized one-form',

$\eta_{MN} W^N = W_M$. The metric actually provides an isomorphism between E and E^*, the 'dual generalized tangent bundle', since

$$\eta : E = TM \oplus T^*M \to E^* = (TM \oplus T^*M)^* = T^*M \oplus TM \cong E. \tag{2.5}$$

We can therefore equally think of W_M as generalized vectors and write a generalized tensor as

$$A^{M_1 \dots M_r}_{N_1 \dots N_q} \cong A^{M_1 \dots M_{r+q}} \in E^{\otimes r+q}. \tag{2.6}$$

2.1.2 Patching

We would like to understand how the generalized vectors transform when going from one coordinate patch to another, that is, how the fibres of E are sewn together. The vector component $v \in TM$ of a generalized vector $V = v + \lambda$ is globally defined across coordinate patches U_i and U_j [13]

$$v_{(i)} = v_{(j)}, \tag{2.7}$$

but to determine the patching of the one-form λ, we have to consider how the B-field transforms over coordinate patches. Note that here we have suppressed the usual $GL(d)$ coordinate transformations. By including the action of $M^{\mu}{}_{\nu} = (\partial x'^{\mu} \backslash \partial x^{\nu}) \in GL(d)$ we would have $v'^{\mu} = M^{\mu}{}_{\nu} v^{\nu}$, with coordinate maps x'^{μ} and x^{μ} on charts U_i and U_j respectively.

Unlike the field strength H, the B-field is not defined globally, but only up to a patching with a one-form Λ that encodes its topology. On the overlap of two coordinate patches $U_i \cap U_j$ we have [11]

$$B_{(i)} = B_{(j)} - d\Lambda_{(ij)}, \tag{2.8}$$

where $d : \wedge^p T^*M \to \wedge^{p+1} T^*M$ is the exterior derivative. The patching of the two-form $B_{\mu\nu}$ is akin to the patching of the one-form electromagnetic potential A_μ over a $U(1)$ bundle [13]. In the case of A_μ there is instead a zero-form $\theta(x) \in \mathbb{R}$ which parameterizes members $e^{i\theta(x)}$ of the $U(1)$ structure group.

Under infinitesimal variations of the B-field on patch U_i we have

$$\delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} + d\lambda_{(i)}. \tag{2.9}$$

According to the patching condition of (2.8), $\mathcal{L}_v B_{(i)}$ on patch U_j will be given by

$$\mathcal{L}_v (B_{(j)} - d\Lambda_{(ij)}) = \mathcal{L}_v B_{(j)} - \mathcal{L}_v d\Lambda_{(ij)}, \tag{2.10}$$

and thus we require that

$$d\lambda_{(i)} = d\lambda_{(j)} + \mathcal{L}_v d\Lambda_{(ij)}, \tag{2.11}$$

on $U_i \cap U_j$ to ensure that the variation is consistently defined across the manifold. Using the nil-potency of d and the identity for the Lie derivative $\mathcal{L}_v w = (di_v + i_v d)w$, we can write equation (2.11) as

$$d\lambda_{(i)} = d\lambda_{(j)} + di_v d\Lambda_{(ij)}, \tag{2.12}$$

where $i_x : \wedge^p T^*M \to \wedge^{p-1}T^*M$ is the interior product and is a contraction with the first argument i.e. for $\omega \in \wedge^2 T^*M$, $i_x\omega = x^\mu \omega_{\mu\nu}$. By integration we obtain the transformation for the one-form components of the generalized vectors across coordinate patches [11]

$$\lambda_{(i)} = \lambda_{(j)} + i_v d\Lambda_{(ij)}. \tag{2.13}$$

We have utilized the fact that the existence of an integration constant is inconsequential since λ plus a constant term defines the same gauge transformation.

We have found that the entire generalized vector $V = v + \lambda$ on $U_i \cap U_j$ is given by:

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + (\lambda_{(j)} + i_{v_{(j)}} d\Lambda_{(ij)}). \tag{2.14}$$

We can summarize the transformation in matrix form as

$$\begin{pmatrix} v_{(i)} \\ \lambda_{(i)} \end{pmatrix} = \begin{pmatrix} M_{(ij)} & 0 \\ C_{(ij)}M_{(ij)} & M^{-T}_{(ij)} \end{pmatrix} \begin{pmatrix} v_{(j)} \\ \lambda_{(j)} \end{pmatrix}, \tag{2.15}$$

where $C_{(ij)} = d\Lambda_{(ij)}$ with $d\Lambda : v \mapsto i_v d\Lambda$ and we have explicitly included the conventional action of $M \in GL(d)$, where $(M^{-T}) = (M^{-1})^T$.

At first sight one might expect that doubling the dimension of a d-dimensional space by the addition of another d-dimensional space would simply promote the relevant structure group $GL(d)$ to $GL(2d)$. However, we have seen that because of the nature of the construction of E, precisely the combination of tangent and cotangent bundles, the structure group we obtain in this instance is in fact the larger group of $O(d,d)$. We have so far seen $O(d,d)$ acting on the fibres of E but we do not yet have the most general form of an element $O \in O(d,d)$. This is discussed in section (2.3.1).

2.2 Differential structure

2.2.1 The Dorfman derivative

The Dorfman derivative provides a generalization of the Lie derivative. Unlike the latter, which just encodes diffeomorphisms, the Dorfman derivative combines both diffeomorphisms and gauge transformations, as can be seen from the appearance of the gauge parameter λ in its definition [11]:

$$L_V W = L_V(w + \sigma) = \mathcal{L}_v w + \mathcal{L}_v \sigma - i_w d\lambda, \tag{2.16}$$

where $\mathcal{L}_x y$ is the usual Lie derivative.

Let us recall the action of the Lie derivative on an generic tensor field T along a vector field v [19]

$$\begin{aligned} \mathcal{L}_v T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_q} &= v^\sigma \partial_\sigma T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_q} \\ &- (\partial_\lambda v^{\mu_1}) T^{\lambda \mu_2 \ldots \mu_r}_{\nu_1 \ldots \nu_q} - (\partial_\lambda v^{\mu_2}) T^{\mu_1 \lambda \ldots \mu_r}_{\nu_1 \ldots \nu_q} - \ldots \\ &+ (\partial_{\nu_1} v^\lambda) T^{\mu_1 \ldots \mu_r}_{\lambda \nu_2 \ldots \nu_q} + (\partial_{\nu_2} v^\lambda) T^{\mu_1 \ldots \mu_r}_{\nu_1 \lambda \ldots \nu_q} + \ldots \quad . \end{aligned} \tag{2.17}$$

The first term is interpreted as the translation of the tensor in the v direction whereas the remaining terms correspond to the action of matrices ${(\partial v)_\nu}^\lambda$ in the adjoint representation of $GL(d)$ [12] as illustrated in appendix A.

We can more easily make a comparison between the Lie and Dorfman derivatives by writing the latter in an explicitly $O(d,d)$ covariant form with the help of a generalized partial derivative, given by [11]

$$\partial_M = \begin{pmatrix} \partial\mu \\ 0 \end{pmatrix} . \tag{2.18}$$

With this, equation (2.16) becomes:

$$L_V W^M = V^N \partial_N W^M + (\partial^M V^N - \partial^N V^M) W_N, \tag{2.19}$$

(see appendix B for details). The Dorfman derivative now takes the same form as the Lie derivative but with the adjoint matrices belonging instead to the algebra of $\mathfrak{o}(d,d)$ [11]. Again we see that in the generalized case, there is a natural $O(d,d)$ action replacing that of the usual $GL(d)$.

Finally, we note that it is straightforward to extend the action of the Dorfman derivative to a generic tensor T as follows

$$\begin{aligned} L_V T^{M_1 \ldots M_r} &= V^N \partial_N T^{M_1 \ldots M_r} \\ &+ (\partial^{M_1} V^N - \partial^N V^{M_1}) {T_N}^{M_2 \ldots M_r} + \ldots \\ &+ (\partial^{M_r} V^N - \partial^N V^{M_r}) {T^{M_1 \ldots M_{r-1}}}_N . \end{aligned} \tag{2.20}$$

2.2.2 The Courant bracket

The generalized substitute for the Lie bracket is provided by the Courant bracket, the natural bracket on our generalized space. For two generalized vectors $V = v + \lambda$ and $W = w + \sigma$, it is defined as [1]:

$$[\![V, W]\!] = [v, w] + \mathcal{L}_v \sigma - \mathcal{L}_w \lambda - \frac{1}{2} d(i_v \sigma - i_w \lambda), \tag{2.21}$$

where $[v, w]$ is the usual Lie bracket. Again we can see it is a structure encapsulating diffeomorphisms parameterized by v and gauge transformations parameterized by λ. It is in fact the antisymmetrization of the Dorfman derivative [3], shown as follows

$$\begin{aligned}
[\![V,W]\!] &= [v,w]+\mathcal{L}_v\sigma-\mathcal{L}_w\lambda-\frac{1}{2}d(i_v\sigma-i_w\lambda)\\
&= \frac{1}{2}([v,w]+[v,w]+\mathcal{L}_v\sigma-\mathcal{L}_w\lambda+\mathcal{L}_v\sigma-di_v\sigma-\mathcal{L}_w\lambda+di_w\lambda)\\
&= \frac{1}{2}([v,w]-[w,v]+\mathcal{L}_v\sigma-\mathcal{L}_w\lambda+i_vd\sigma-i_wd\lambda) \qquad (2.22)\\
&= \frac{1}{2}(\mathcal{L}_vw-\mathcal{L}_wV+\mathcal{L}_v\sigma-\mathcal{L}_w\lambda+i_vd\sigma-i_wd\lambda)\\
&= \frac{1}{2}((\mathcal{L}_vw+\mathcal{L}_v\sigma-i_wd\lambda)-(\mathcal{L}_wV+\mathcal{L}_w\lambda-i_vd\sigma))\\
&= \frac{1}{2}(L_VW-L_WV).
\end{aligned}$$

We can also write it in an $O(d,d)$ covariant form

$$[\![V,W]\!]^M=V^N\partial_NW^M-W^N\partial_NV^M-\frac{1}{2}(V_N\partial^MW^N-W_N\partial^MV^N), \qquad (2.23)$$

(see appendix B for more details).

We have found that the generalized differential structures capture a larger group of symmetries than their conventional analogues, including both diffeomorphisms and gauge transformations. This is good news, as after all we want a framework that captures both of these symmetries of type II supergravity. Additionally we acknowledge that unlike the Lie bracket, the Courant bracket does not satisfy the Jacobi identity signifying that it is not a bracket belonging to any Lie algebra [1].

2.3 Symmetries

To expose the symmetry properties of objects residing in E, we need to find the form of a general element $O \in O(d,d)$ and observe how it acts. We have already seen in section (2.1.2) an $O(d,d)$ rotation in action on fibres. In this section we proceed to construct the most general element $O \in O(d,d)$ by exploring the algebra of a subgroup $SO(d,d) \subset O(d,d)$.

2.3.1 An element of $O(d,d)$

We can identify the connected subgroup $SO(d,d) \subset O(d,d)$ which, in addition to the inner product, preserves the orientation of $TM \oplus T^*M$. Orientable spaces are desirable in general as they admit a top form that vanishes nowhere, enabling the integration of differential forms. Furthermore the incorporation of spinor fields into our discussion would automatically fix the orientation of the generalized space meaning that $SO(d,d)$ is the appropriate subgroup to consider. The Lie algebra of $SO(d,d)$ is defined as [3]:

$$\mathfrak{so}(E) = \{R \mid \langle RV, W\rangle + \langle V, RW\rangle = 0 \quad \forall\ V, W \in E\}, \tag{2.24}$$

i.e. the set of automorphisms which are antisymmetric with respect to the inner product. We note that $\mathfrak{so}(E)$ decomposes as [1]

$$\mathfrak{so}(E) = End(TM) \oplus \wedge^2 T^*M \oplus \wedge^2 TM, \tag{2.25}$$

and hence a general element $R \in \mathfrak{so}(E)$ can be represented as [2]

$$R = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}, \tag{2.26}$$

where $A \in End(TM)$, $\beta \in \wedge^2 TM$ and $B \in \wedge^2 T^*M$ is our B-field, and hence $A : TM \to TM$, $\beta : T^*M \to TM$ and $B : TM \to T^*M$.

We can view B as a smooth map by dint of the interior product

$$\begin{aligned} B:\ \ TM &\to T^*M \\ v &\mapsto B(v) \equiv i_v B = v^\mu B_{\mu\nu}. \end{aligned} \tag{2.27}$$

Considering the action of the B-field only, such that we have

$$R_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \tag{2.28}$$

we can exponentiate to obtain a member of $SO(d,d)$, call it e^B, allowing us to see how B acts upon fibres of E as follows

$$e^B \equiv e^{R_B} \approx \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix}, \tag{2.29}$$

then

$$e^B V = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = v + \lambda + B(v) = v + \lambda + i_v B. \tag{2.30}$$

This is known as the B transform or the B-field transformation [3].

The endomorphism A defines another subgroup of transformations which we exponentiate to find

$$R_A = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \quad \implies \quad e^A \equiv e^{R_A} = \begin{pmatrix} e^A & 0 \\ 0 & e^{(A^{-T})} \end{pmatrix}. \tag{2.31}$$

Noting that $e^{(A^{-T})} = ((e^A)^T)^{-1}$, we rewrite e^A and $e^{(A^{-T})}$ as general matrices M and M^{-T} respectively. Then, performing this transformation on a generalized vector

$$e^A(v + \lambda) = Mv + M^{-T}\lambda, \tag{2.32}$$

we recognise that $M \in GL(d)$. The familiar $GL(d)$ transformations on TM are thus embedded as a subgroup of $O(d, d)$.

If we now substitute B for the patching two-form quantity C (as seen in section (2.1.2)) and combine the transformations $e^{B=C}$ and e^A, we obtain the subgroup that patches elements of E (c.f. equation (2.15))

$$e^{B=C}e^A = \begin{pmatrix} \mathbb{1} & 0 \\ C & \mathbb{1} \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & M^{-T} \end{pmatrix} = \begin{pmatrix} M & 0 \\ CM & M^{-T} \end{pmatrix}. \tag{2.33}$$

These matrices belong to the semi-direct product subgroup $GL(d) \ltimes \mathbb{R}^{d(d-1)/2}$ [17]. Therefore the structure group $O(d, d)$ acting on the fibres reduces to that of $GL(d) \ltimes \mathbb{R}^{d(d-1)/2}$.

Finally, isolating the contribution of the bi-vector β and exponentiating, we have

$$e^\beta \equiv e^{R_\beta} = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}. \tag{2.34}$$

This acts on a generalized vector by sending $v + \lambda \mapsto v + \lambda + \beta(\lambda) = v + \lambda + i_\lambda\beta$. The action attributed to β is not as significant in our discussion of generalized spaces as the other transformations and so to this effect its existence expresses a breaking of symmetry amongst the subgroups of $O(d, d)$ [2].

2.3.2 Symmetries of the Courant bracket

The Lie bracket on vector fields is preserved under the family of diffeomorphisms $Diff(M)$. The Courant bracket also possesses diffeomorphism invariance but enjoys an additional symmetry under the action of the B-field.

We can represent the diffeomorphism invariance of the Lie bracket on the tangent bundle $\pi : TM \to M$ diagramatically [3]:

$$\begin{array}{ccc} TM & \xrightarrow{F} & TM \\ \pi\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{f} & M \end{array} \tag{2.35}$$

where (f, F) are a pair of diffeomorphisms, $f : M \to M$ and $F : TM \to TM$. For F to preserve the Lie bracket

$$F([v,w]) = [F(v), F(w)] \quad \forall\, v, w \in TM, \tag{2.36}$$

it must be equal to the pushforward of f, defined as $f_* : T_pM \to T_{f(p)}M$ [3]. In a similar vein we define another pair of diffeomorphisms (g, G) such that $g : M \to M$ and $G : TM \oplus T^*M \to TM \oplus T^*M$. In the generalized case by requiring that $G = f_* \oplus f^*$, we have diffeomorphism invariance of the Courant bracket [3]

$$G([\![V, W]\!]) = [\![G(V), G(W)]\!] \quad \forall\, V, W \in E. \tag{2.37}$$

Let us now show that the Courant bracket is invariant under the action of the B-field transformations e^B. This symmetry only exists iff B is a closed two-form, $dB = 0$. The proof is as follows [2]

$$\begin{aligned} [\![e^B(V), e^B(W)]\!] &= [\![v + \lambda + i_vB, w + \sigma + i_wB]\!] \\ &= [v + \lambda, w + \sigma] + \mathcal{L}_v i_w B - \mathcal{L}_w i_v B - \frac{1}{2}d(i_v i_w B - i_w i_v B) \\ &= [v + \lambda, w + \sigma] + \mathcal{L}_v i_w B - i_w d(i_v B) \\ &= [v + \lambda, w + \sigma] + i_{[v,w]}B + i_w \mathcal{L}_v B - i_w d(i_v B) \\ &= [v + \lambda, w + \sigma] + i_{[v,w]}B + i_w i_v dB \\ &= e^B([\![v + \lambda, w + \sigma]\!]) + i_w i_v dB \end{aligned} \tag{2.38}$$

where in going from the second line to the third we have made use of the identity $\mathcal{L}_v w = (di_v + i_v d)w$ to obtain $di_v i_w B = \mathcal{L}_v i_w B - i_v d(i_w B)$.

We have already seen that the natural inner product is left unchanged by $Diff(M)$. The inner product is also invariant under the action of the B-field

$$\begin{aligned} \langle e^B(V)\,,\, e^B(W)\rangle &= \langle v + \lambda + i_vB\,,\, w + \sigma + i_wB\rangle \\ &= \frac{1}{2}(i_v(\sigma + i_wB) + i_w(\lambda + i_vB)) \\ &= \frac{1}{2}(i_v\sigma + i_w\lambda + i_v i_w B + i_w i_v B) \\ &= \langle V, W\rangle \end{aligned} \tag{2.39}$$

by the asymmetry of the two-form B, $i_w i_v B = -i_v i_w B$.

Together the group of diffeomorphisms and B-field transformations acts as the semi-direct product group $Diff(M) \ltimes \Omega^2_{closed}$, preserving the inner product and Courant bracket [2]. This displays a fundamental feature of generalized geometry compared to conventional differential geometry:- key structures of the generalized space are invariant under a larger group of symmetries to which the B-field transformations contribute as well as diffeomorphisms. In fact it can be shown that $Diff(M) \ltimes \Omega^2_{closed}$ is the *only* group of symmetries preserving the Courant bracket [3].

3 The generalized metric

We would like to find a generalization of the Riemannian metric g. We motivate the generalized metric's construction by recognising that the introduction of g to a manifold reduces the structure group of the tangent bundle from $GL(d)$ to the subgroup $O(d)$. The corresponding subgroup in the generalized case is $O(d) \times O(d)$. By investigating the consequences of the splitting $O(d,d) \to O(d) \times O(d)$ we find an explicit matrix representation for the generalized metric G.

Under the decomposition $O(d,d) \to O(d) \times O(d)$, E splits into two d-dimensional orthogonal subbundles [13]

$$E = C_+ \oplus C_-, \tag{3.1}$$

such that the natural metric also splits into two, a positive definite metric on C_+ and a negative definite metric on C_-, so that the action of $O(d)$ preserves the inner product on each subspace separately. This allows us to define a positive definite metric G on $E = TM \oplus T^*M$ by [3]

$$G = \langle \ \ , \ \ \rangle|_{C_+} - \langle \ \ , \ \ \rangle|_{C_-} \quad > 0. \tag{3.2}$$

We can view G as an automorphism of E that squares to the identity, $G^2 = \mathbb{1}$ or as a matrix invariant under $O(d) \times O(d)$.

There is an isomorphism between TM and C_+ [2]. This means that we can introduce a positive definite map $M : TM \to T^*M$ whose graph, the set of pairs (x, Mx), defines the subundle C_+:

$$C_+ = \{x + Mx \ \ \forall \ \ x \in TM\}. \tag{3.3}$$

The mapping $M : TM \to T^*M$ implies $M \in T^*M \otimes T^*M$ and hence we can decompose M into its symmetric and antisymmetric parts; $M = g + B$, with $g \in sym^2 T^*M$, the familiar Riemannian metric, and $B \in \wedge^2 T^*M$ our two-form B-field. Hence a general element of C_+ is written $X_+ = x + (B + g)x$. Respecting the orthogonality condition between C_+ and C_-, the other subbundle is then given by

$$C_- = \{x + (B - g)x \ \ \forall \ \ x \in TM\}, \tag{3.4}$$

with general element $X_- = x + (B - g)x$.

We can demonstrate that g is indeed the Riemannian metric by taking the inner product of two vectors $X_+, Y_+ \in C_+$

$$\begin{aligned}\langle X_+, Y_+\rangle &= \langle x + g(x) + B(x)\,,\, y + g(y) + B(y)\rangle \\ &= \langle x + g(x) + i_x B\,,\, y + g(y) + i_y B\rangle \\ &= \frac{1}{2}(i_x g(y) + i_y g(x) + i_x i_y B + i_y i_x B) \\ &= g(x,y) > 0\end{aligned} \tag{3.5}$$

again resulting from the asymmetry of B, $i_y i_x B = -i_x i_y B$.

We want to find the form that our generalized metric G takes. By using the definition (3.2), we can uncover how G acts on the individual components of $X_\pm$ to build a matrix representation for the metric. Expressing a generalized vector $X \in C_+ \oplus C_-$ as $X = X_+ + X_-$, we have

$$\begin{aligned}G : E = C_+ \oplus C_- &\rightarrow E^* \cong E \\ X = X_+ + X_- &\mapsto G(X) = \langle X_+ + X_-,\ \ \rangle|_{C_+} - \langle X_+ + X_-,\ \ \rangle|_{C_-} \\ &\cong X_+ - X_-.\end{aligned} \tag{3.6}$$

We start with the case $B = 0$, such that we are now dealing with the sets

$$C_{g\pm} \equiv \{x \pm g \ \ \forall \ \ x \in TM\}, \tag{3.7}$$

and denote vectors in $C_{g\pm}$ by $X_{g\pm} = x \pm g$. Noting that $2x = X_+ + X-$, it follows that [20]

$$2G(x) = X_+ - X_- = 2g(x) \ \text{ and } \ G^2 = \mathbb{1} \implies 2G(g(x)) = 2x. \tag{3.8}$$

The simplest form that G can take is:

$$G_g = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{3.9}$$

To incorporate the B-field transformations, we note that

$$e^B X_{g\pm} = e^B(x \pm g(x)) = (x \pm g(x) + B(x)) \tag{3.10}$$

and utilise the observation that $C_\pm$ ($C_{g\pm}$) are the ± 1 eigenspaces of G (G_g) [3]

$$G(X_\pm) = \pm X_\pm \quad \text{and} \quad G_g(X_{g\pm}) = \pm(X_{g\pm}). \tag{3.11}$$

We then obtain an expression for G containing both g and B by computing that

$$\begin{aligned} e^B X_{g\pm} &= e^B G_g X_{g\pm} \\ &= e^B G_g e^{-B} e^B X_{g\pm} \\ &= e^B G_g e^{-B} X_{\pm} \\ \implies & e^B G_g e^{-B} = G, \end{aligned} \tag{3.12}$$

and finally we have our generalized metric

$$G = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}. \tag{3.13}$$

We can check this expression for G by acting on a general element $X_+ = x + \xi \in C_+$:

$$G = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} x \\ \xi \end{pmatrix}. \tag{3.14}$$

From this we obtain

$$\begin{aligned} -g^{-1}B(x) + g^{-1}(\xi) = x \\ \implies \xi = g(x) + B(x), \end{aligned} \tag{3.15}$$

as required. We have a similar result for the subbundle C_-.

We have found that the generalized metric is an object which unifies the g and B fields and that its introduction corresponds to the structure group reduction $O(d,d) \to O(d) \times O(d)$. In conventional differential geometry, we identify the Riemannian metric g with the coset space

$$GL(d)/O(d), \tag{3.16}$$

whereas in the generalized case, G parametrizes the coset

$$O(d,d)/(O(d) \times O(d)). \tag{3.17}$$

Another starting point for the generalized metric's construction is to introduce generalized frames which encode the metric g and B-field and from these construct the form of G. This is the method outlined in section (4.2.2).

4 Extension to $O(d,d) \times \mathbb{R}^+$

So far, we have seen the unification of the supergravity fields g and B within the generalized metric but we are still to incorporate the last remaining degree of freedom afforded by the dilaton ϕ into our discussion. In this section we provide the necessary framework for its inclusion by extending the generalized tangent bundle and introducing generalized conformal split frames. We finish by describing how generalized structures introduced in previous sections, namely the Dorfman derivative and generalized metric, can be easily extended to objects living in a *weighted bundle* $\tilde{E}$. We begin with a review of frames from conventional geometry.

4.1 Frames

4.1.1 Generalized frames

On a coordinate patch U_i of TM, we can introduce a set of linearly independent vector fields $\{\hat{e}_a\}$ that do not rely upon any underlying coordinate system. The basis set $\{\hat{e}_a\}$ defines a local frame over U_i and we refer to $a = 1, ..., d$ as frame indices [21]. We cannot necessarily determine these frames globally just as we may not be able to cover a manifold with a single coordinate chart.

The frame bundle is the bundle associated with these basis vectors, defined as [21]

$$F \equiv \{(p, \{\hat{e}_a\}) \quad \forall \ p \in M\}. \tag{4.1}$$

This is manifestly a principal bundle as the elements of the fibres are themselves members of $GL(d)$. We can choose our basis vectors to satisfy [19]

$$g(\hat{e}_a, \hat{e}_b) = \eta_{ab}, \tag{4.2}$$

such that $\{\hat{e}_a\}$ now comprises an orthonormal basis and g is a general metric. The form of η_{ab} is chosen according to the signature of the manifold we are dealing with. For a Lorentzian spacetime, η_{ab} would be the Minkowski metric. In our case, g is the Riemannian metric and η_{ab} is that of Euclidean space. Notice how the introduction of a metric g reduces the structure group from $GL(d)$ to $O(d)$ and thus defines a G-structure. A G-structure is a principal subbundle $P \subset F$ with fibre G [21]. In this case, $G = O(d)$ and the subbundle P is given by

$$P \equiv \{(p, \{\hat{e}_a\}) \quad \forall \ p \in M \ | \ g(\hat{e}_a, \hat{e}_b) = \eta_{ab}\}. \tag{4.3}$$

We can expand a vector v in terms of this orthonormal basis as $v = v^a \hat{e}_a$. The components v^a transform on each fibre of P according to

$$v^a \mapsto v'^a = M^a{}_b v^b, \tag{4.4}$$

and the orthonormal bases transform as

$$\hat{e}_a \mapsto \hat{e}'_a = \hat{e}_b (M^{-1})^b{}_a, \tag{4.5}$$

where $M \in O(d)$ [21]. Lastly we note that there exists a set $\{e^a\}$ for the cotangent bundle

T^*M which we used to define the cotangent frame bundle.

The construction of generalized frames runs in parallel to the above discussion. We introduce a basis $\{\hat{E}_A\}$ for the generalized tangent bundle E with orthonormality condition

$$\eta(\hat{E}_A, \hat{E}_B) = \eta_{AB}, \tag{4.6}$$

where $\eta = \frac{1}{2}\left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$ is the natural metric, $\eta_{AB} = \frac{1}{2}\left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right)$ and $A = 1, ..., 2d$ we refer to as generalized frame indices. These frames form the bundle with structure group $O(d,d)$

$$F = \{(p, \{\hat{E}_A\}),\ \forall\ p \in M\ \mid\ \eta(\hat{E}_A, \hat{E}_B) = \eta_{AB}\}, \tag{4.7}$$

and transform according to

$$\hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B (O^{-1})^B{}_A, \tag{4.8}$$

with $O \in O(d,d)$.

4.1.2 Weighted frames

To incorporate the dilaton into our generalized geometry picture we need to add an extra degree of freedom to our space. We do this by extending the generalized bundle E to a *weighted* bundle, given by [11]

$$\tilde{E}_{(p)} \equiv (detT^*M)^p \otimes E. \tag{4.9}$$

As a consequence, the structure group is promoted from $O(d,d)$ to $O(d,d) \times \mathbb{R}^+$. We refer to the elements of the fibres of $\tilde{E}_{(p)}$ as weighted vectors, denoted $\tilde{V}$. They are of the form

$$\tilde{V}^M = \begin{pmatrix} \sqrt{-g}v^\mu \\ \sqrt{-g}\lambda_\mu \end{pmatrix} \equiv \begin{pmatrix} \tilde{v}^\mu \\ \tilde{\lambda}_\mu \end{pmatrix}, \tag{4.10}$$

where $\sqrt{-g} \in det(T^*M)$ such that $\tilde{v}^\mu$ is a tensor density of weight 1 and so in this example $\tilde{V}^M \in \tilde{E}_{(1)} \equiv \tilde{E}$. The components, $\tilde{v}$ and $\tilde{\lambda}$, transform under $O(d,d) \times \mathbb{R}^+$ as tensor densities according to

$$\tilde{v}^\mu \mapsto \tilde{v}'^\mu = det\left(\frac{\partial x}{\partial x'}\right)\frac{\partial x'^\mu}{\partial x^\mu}\tilde{v}^\mu, \qquad \tilde{\lambda}_\mu \mapsto \tilde{\lambda}'_\mu = det\left(\frac{\partial x}{\partial x'}\right)\frac{\partial x^\mu}{\partial x'^\mu}\tilde{\lambda}_\mu. \tag{4.11}$$

Employing the isomorphism provided by the natural metric η as discussed in section (2.1.1), generalized tensors of weight p will be denoted

$$\tilde{T}^{M_1...M_r} \in\ \tilde{E}^{\otimes r}_{(p)} \equiv (detT^*M)^p \otimes E \otimes ... \otimes E, \tag{4.12}$$

so now when grouping tensors together we consider their weight p as well as their rank.

The frames relevant for $\tilde{E}$ are given by a *conformal* basis $\{\hat{E_A}\}$, satisfying [11]

$$\eta(\hat{E}_A, \hat{E}_B) = \Phi^2 \eta_{AB}, \tag{4.13}$$

where $\Phi \in det(T^*M)$ is a conformal factor that is frame dependent whose value we leave undefined for now. The frame bundle for $\tilde{E}$ comprised of these bases is called the *generalized structure bundle* and is given by

$$\tilde{F} \equiv \{(p, \{\hat{E}_A\}) \ \forall \ p \in M \ | \ \eta(\hat{E}_A, \hat{E}_B) = \Phi^2 \eta_{AB}\}. \tag{4.14}$$

It is a principal bundle with structure group $O(d, d) \times \mathbb{R}^+$.

4.1.3 Split frames

It is necessary to introduce a particular type of conformal frames called *split frames.* They provide the required structure to correctly define the patching of objects on $\tilde{E}$. We define a split frame $\{\hat{E}_A\}$ for $\tilde{E}$ by

$$\hat{E}_A = \begin{pmatrix} \hat{E}_a \\ E^a \end{pmatrix} = \begin{pmatrix} (\det\, e)(\hat{e}_a + i_{\hat{e}_a} B) \\ (\det\, e) e^a \end{pmatrix}, \tag{4.15}$$

where the conformal factor Φ now takes the value $(\det\, e) \in det(T^*M)$ and B is our familiar two-form B-field. The basis $\{\hat{E}_A\}$ is clearly conformal, as

$$\eta(\hat{E}_A, \hat{E}_A) = (\det\, e)^2 \eta_{AB}. \tag{4.16}$$

These split frames transform according to

$$\hat{E}_A \mapsto \hat{E'}_A = \hat{E}_B (M^{-1})^B{}_A, \tag{4.17}$$

where M belongs to $GL(d) \ltimes \mathbb{R}^{d(d-1)/2}$. The structure group reduces from $O(d, d)$ to $GL(d) \ltimes \mathbb{R}^{d(d-1)/2}$ and thus these frames define a G-structure as a principal subbundle of $\tilde{F}$. This subgroup is the same as we found in section (2.3.1) for the patching of elements of E.

Lastly we note that expressing a generalized vector's components in frame indices as $\tilde{V}^A = \tilde{v}^a + \tilde{\lambda}_a$, we expand a generalized vector with respect to the generalized basis in the usual way

$$\tilde{V} = \tilde{V}^A \hat{E}_A = \tilde{v}^a \hat{E}_a + \tilde{\lambda}_a E^a, \tag{4.18}$$

where by construction we have $\tilde{v} = \tilde{v}^a (\det\, e) \hat{e}_a \in (det T^*M) \otimes TM$ an $\tilde{\lambda} = \tilde{\lambda}_a (\det\, e) e^a \in (det T^*M) \otimes T^*M$.

4.2 $O(d,d) \times \mathbb{R}^+$ structures

4.2.1 Extended Dorfman derivative

The Dorfman derivative now acting on a general weighted vector $\tilde{W} = \tilde{w} + \tilde{\sigma} \in \tilde{E}_{(p)}$ takes the same form as seen in section (2.2.1)

$$L_V \tilde{W} = \mathcal{L}_v \tilde{w} + \mathcal{L}_v \tilde{\sigma} - i_{\tilde{w}} d\lambda. \tag{4.19}$$

The distinction between the weighted and unweighted cases becomes apparent when we look at the action of the Lie derivative on the individual components of $\tilde{W}$ as now we are dealing with tensor densities $\tilde{w}$ and $\tilde{\sigma}$. We find [22]:

$$\begin{aligned} \mathcal{L}_v \tilde{w}^\mu &= v^\nu \partial_\nu \tilde{w}^\mu - \tilde{w}^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu)\tilde{w}^\mu, \\ \mathcal{L}_v \tilde{\sigma}_\mu &= v^\nu \partial_\nu \tilde{\sigma}_\mu + (\partial_\mu v^\nu)\tilde{\sigma}_\nu + p(\partial_\nu v^\nu)\tilde{\sigma}_\mu, \end{aligned} \tag{4.20}$$

where p is the weight. By using the generalized partial derivative defined in equation (2.18), we can write the Dorfman derivative now in an $O(d,d) \times \mathbb{R}^+$ covariant form

$$L_V W^M = V^N \partial_N W^M + (\partial^M V^N - \partial^N V^M) W_N + p(\partial_N V^N) W^M. \tag{4.21}$$

4.2.2 Extended generalized metric

The subsequent discussion gives an overview of the methodology behind incorporating the generalized metric into the weighted bundle $\tilde{E}_{(p)}$. We follow that of reference [11] which may be consulted for further details.

In section (3) we considered the splitting of the structure group $O(d,d)$ of E to the subgroup $O(d) \times O(d)$. Now, we are interested in the subgroup of $O(p,q) \times O(q,p) \subset O(d,d) \times \mathbb{R}^+$, where $p + q = d$. This subgroup again defines a splitting of E into two d-dimensional subbundles $E = C_+ \oplus C_-$, with the natural metric splitting into a metric of signature of (p,q) on C_+ and (q,p) on C_-.

We set up a split conformal frame $\{\hat{E}_A\}$ appropriate for the subbundles defined by

$$\hat{E}_A = \begin{pmatrix} \hat{E}_a^+ \\ \hat{E}_{\bar{a}}^- \end{pmatrix} = \begin{pmatrix} e^{-2\phi}\sqrt{-g}(\hat{e}_a^+ + e_a^+ + i_{\hat{e}_a^+} B) \\ e^{-2\phi}\sqrt{-g}(\hat{e}_{\bar{a}}^- - e_{\bar{a}}^- + i_{\hat{e}_{\bar{a}}^-} B) \end{pmatrix}, \tag{4.22}$$

where $\{\hat{E}_a^+\}$ and $\{\hat{E}_{\bar{a}}^-\}$ are orthonormal bases for C_+ and C_- respectively with $a, \bar{a} = 1...d$. As these frames are conformal we have a conformal factor appearing in the orthogonality condition

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB}, \tag{4.23}$$

where in this case we have $\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}$. Using these newly introduced frames we can construct an $O(p,q) \times O(q,p)$ invariant metric, given by

$$G = \Phi^{-2}(\eta^{ab}\hat{E}_a^+ \otimes \hat{E}_b^+ + \eta^{\bar{a}\bar{b}}\hat{E}_{\bar{a}}^- \otimes \hat{E}_{\bar{b}}^-). \tag{4.24}$$

In addition to defining two subbundles, the group splitting also gives a globally defined conformal factor $\Phi \in det(T^*M)$. Recall that in general, given a tensor density of weight p,

$$\tilde{T}^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_q} = (\sqrt{-g})^p T^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_q}, \tag{4.25}$$

we can convert easily between tensor densities and tensors by multiplying by $\sqrt{-g}$ to the appropriate power. In this case, it is the conformal factor Φ that provides a mapping between weighted and unweighted objects, that is, an isomorphism between E and $\tilde{E}_{(p)}$. We choose the conformal factor to be

$$\Phi = e^{-2\phi}\sqrt{-g}. \tag{4.26}$$

Therefore in terms of the $O(d,d)$ metric, the new metric is a weighted object given by

$$\tilde{G} = \frac{1}{(e^{-2\phi}\sqrt{-g})^2}G. \tag{4.27}$$

This is the appropriate generalized structure to capture all the bosonic degrees of freedom $\{g, B, \phi\}$. It can be viewed as parameterizing the coset space:

$$(O(d,d) \times \mathbb{R}^+)/(O(p,q) \times O(q,p)). \tag{4.28}$$

5 Generalized curvature

We are now in a position to build generalized objects associated with curvature that are compliant with the $O(d,d) \times \mathbb{R}^+$ structure. In this section we introduce a generalized connection and torsion tensor with the use of conformal split frames. These provide the necessary ingredients to build a generalization of the Riemann curvature tensor which we present in section (5.3). The key results stated are those found in [11].

5.1 Generalized connections

A generic connection quantifies the change of a tensor field along an integral curve of a vector field. A conventional connection ∇_μ acting on a vector v^a is of the form [19]

$$\nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b \tag{5.1}$$

where $\mu = 1, ..., d$ are spacetime indices, $a, b = 1, ..., d$ are frame indices and $\omega_\mu{}^a{}_b$ is the spin connection. The spin connection is not a tensor itself but its transformation properties ensure that $\nabla_\mu v^a$ does transform as a tensorial quantity.

For our generalized space, we want to promote this to a connection of a generalized weighted vector $\tilde{V}^A$

$$D_M \tilde{V}^A = \partial_M \tilde{V}^A + \tilde{\Omega}_M{}^A{}_B \tilde{V}^B, \tag{5.2}$$

where $\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Sigma_M \delta^A{}_B$ are the connection coefficients. Here $\Omega_M{}^A{}_B$ is the $O(d,d)$ part of the connection and $\Sigma_M \delta^A{}_B$ is the part corresponding to $\mathbb{R}^+$, and D is simply a differential operator upon which we place no other restraints other than it be linear and first-order [11]. We can extend the action of D to a generalized tensor $\tilde{T} \in \tilde{E}^{\otimes r}_{(p)}$

$$\begin{aligned} D_M \tilde{T}^{A_1 \dots A_r} &= \partial_M \tilde{T}^{A_1 \dots A_r} \\ &+ \Omega^{A_1}_{M\ \ B} T^{B A_2 \dots A_r} + \dots + \Omega^{A_r}_{M\ \ B} \tilde{T}^{A_1 \dots A_{r-1} B} \\ &- p \Lambda_M \tilde{T}^{A_1 \dots A_r}. \end{aligned} \tag{5.3}$$

Considering for a moment the $O(d,d)$ structures of the bundle only, the connection reduces to $D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B$ and we can use equation (4.6) to express the flat metric in a non-coordinate basis

$$\eta_{AB} = \eta_{MN} \hat{E}_A{}^M \hat{E}_B{}^N. \tag{5.4}$$

It is now straightforward to see that metric compatibility of the $O(d,d)$ part of the connection implies that it is antisymmetric in its frame indices

$$\begin{aligned} \nabla_M \eta_{AB} &= \partial_M \eta_{AB} - \Omega_M{}^C{}_A \eta_{CB} - \Omega_M{}^C{}_B \eta_{AC} \\ &= -\Omega_{MBA} - \Omega_{MAB} \\ &= 0 \\ \implies \Omega_{MBA} &= -\Omega_{MAB}. \end{aligned} \tag{5.5}$$

With clearly defined weighted vector and one-form components in equation (4.18), we can define the usual action of a conventional connection on them i.e. $\nabla_\mu \tilde{v}^a$ and $\nabla_\mu \tilde{\lambda}_a$. We then define the generalized connection in terms of the conventional connection ∇, using the split frame, as

$$(D^{\nabla}_M \tilde{V}^A) \hat{E}_A = \begin{pmatrix} (\nabla_\mu \tilde{v}^a) \hat{E}_a + (\nabla_\mu \tilde{\lambda}_a) E^a \\ 0 \end{pmatrix}. \tag{5.6}$$

Note the absence of the one-form component in this expression. This gives the correct form of the generalized connection required for patching.

In conventional geometry, once we have a metric g there is a unique torsion-free, metric compatible connection [19]. Now that we have a well-defined generalized connection and generalized metric, it is natural to ask if there is a generalized analogue of the Levi-Civita connection. This is considered in section (5.4).

5.2 Generalized torsion tensor

The torsion tensor T is a multilinear map defined by [19]

$$\begin{aligned} T : TM \times TM &\rightarrow TM \\ v, w &\mapsto T(v,w) \equiv \nabla_v w - \nabla_w v - [v,w], \end{aligned} \tag{5.7}$$

where $\nabla_x = x^\mu \nabla_\mu$ is a generic conventional connection along the vector field x and $[v,w]$ is the usual Lie bracket. Geometrically the torsion expresses the failure of the closure of an infinitesimal loop traversed by vectors and their parallel transports [21]. In components we find that the torsion is given by

$$\begin{aligned} T^\lambda{}_{\mu\nu} v^\mu w^\nu &= v^\mu \nabla_\mu w^\lambda - w^\nu \nabla_\nu v^\lambda - [v,w]^\lambda \\ &= v^\mu \partial_\mu w^\lambda + v^\mu \omega_\mu{}^\lambda{}_\nu w^\nu - w^\nu \partial_\nu v^\lambda \\ &- w^\nu \omega_\nu{}^\lambda{}_\mu w^\mu - v^\mu \partial_\mu w^\lambda + w^\nu \partial_\nu v^\lambda \\ &= v^\mu w^\nu (\omega_\mu{}^\lambda{}_\nu - \omega_\nu{}^\lambda{}_\mu) \\ &\Longrightarrow T^\lambda{}_{\mu\nu} = \omega_\mu{}^\lambda{}_\nu - \omega_\nu{}^\lambda{}_\mu \end{aligned} \tag{5.8}$$

and hence the torsion is antisymmetric in its lower two indices $T^\lambda{}_{\rho\sigma} = -T^\lambda{}_{\sigma\rho} \in TM \otimes \wedge^2 T^* M$.

We can rewrite T in terms of Lie derivatives as

$$T^\lambda{}_{\mu\nu} v^\mu w^\nu = [v,w]^\lambda_\nabla - [v,w]^\lambda = (\mathcal{L}^\nabla_v w - \mathcal{L}_v w)^\lambda, \tag{5.9}$$

where the appendage ∇ on the Lie bracket and derivative instructs us to replace partial derivatives by covariant ones. Furthermore for a vector x we can write

$$v^\mu T^\lambda{}_{\mu\nu} x^\nu \equiv (T_{(v)})^\lambda{}_\nu x^\nu = (\mathcal{L}^\nabla_v - \mathcal{L}_v)^\lambda{}_\nu x^\nu, \tag{5.10}$$

where $(T_{(v)})^\lambda{}_\nu$ can be viewed as a matrix living in the adjoint representation of $GL(d)$. The above expression in fact holds for any tensor.

With this adjoint action in mind, we define the generalized torsion in parallel to the above

$$V^P T^M_{PN} X^N \equiv (T_{(V)}).X = L^D_V X - L_V X, \tag{5.11}$$

where L^D_V is the Dorfman derivative with D, the generalized connection, in place of ∂. This equation also holds for a generic generalized tensor. The above expression leads us to think of the torsion as a map $T : E \rightarrow \mathrm{adj}(\tilde{F}) \approx \wedge^2 E \oplus \mathbb{R}$ where $\mathrm{adj}(\tilde{F})$ is the bundle corresponding to the adjoint representation of the structure group for $\tilde{F}$.

We can count how many independent components the torsion has by splitting it into symmetric S and antisymmetric A parts as follows

$$T^M{}_{PN} = A^M{}_{PN} - S_P\delta^M{}_N, \tag{5.12}$$

From the definition of the generalized torsion, we have

$$A_{MNP} = -\tilde{\Omega}_{[MNP]} = -3\Omega_{[MNP]} \quad \text{and} \quad S_M = -\tilde{\Omega}^Q_{QM} = \Lambda_M - \Omega^Q_{QM}. \tag{5.13}$$

Hence we find that T has $\binom{2d}{3} + 2d$ components. Therefore $T \in \wedge^3 E \oplus E$ and does not belong to $(E \otimes \wedge^2 E) \oplus E$ as may be expected from (5.11).

For completeness, we state without proof an explicit expression for the torsion components in frame indices

$$T_{ABC} = -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D\eta_{AC} - \Phi^{-2}\langle \hat{E}_A, L_{\Phi^{-1}\hat{E}_B}\hat{E}_C\rangle. \tag{5.14}$$

Here we have given a basis $\{\Phi^{-1}\hat{E}_A\}$ for E in terms of the conformal basis of $\tilde{E}$ by multiplying with the appropriately weighted factor as described in section (4.2.2).

5.3 Generalized Riemann curvature tensor

In conventional geometry, curvature is quantified by the Riemann tensor $\mathcal{R} \in \wedge^2 T^*M \otimes TM \otimes T^*M$. It is derived from the connection and is defined by [21]

$$\begin{aligned} \mathcal{R} : TM \times TM \times TM &\to TM \\ v, w, u &\mapsto R(v,w,u) \equiv \nabla_v\nabla_w u - \nabla_w\nabla_v u - \nabla_{[v,w]}u, \end{aligned} \tag{5.15}$$

where, in components, $\nabla_{[v,w]} = [v,w]^\nu\nabla_\nu = (v^\mu\partial_\mu w^\nu - w^\mu\partial_\mu v^\nu)\nabla_\nu$.

It is tempting to interpret $\mathcal{R}$ as a differential operator from the form of the expression above; however we can show that it does indeed behave like a tensor, that is, as a multilinear map, by

$$\begin{aligned} \mathcal{R}(v, gw, hu) &= \nabla_{fv}\nabla_{gw}hu - \nabla_{gw}\nabla_{fv}hu - \nabla_{[fv,gw]}hu \\ &= fg\nabla_v(\nabla_w(h)u + h\nabla_w u) \\ &- gf\nabla_w(\nabla_v(h)u + h\nabla_v u) - fg\nabla_{[v,w]}(hu) \\ &= fg\nabla_v(w[h]u + h\nabla_w u) - gf\nabla_w(v[h]u + h\nabla_v u) \\ &- fg[v,w][h]u - fgh\nabla_{[v,w]}u \\ &= fgh(\nabla_v\nabla_w u - \nabla_w\nabla_v u - \nabla_{[v,w]}u) \\ &= fghR(v,w,u), \end{aligned} \tag{5.16}$$

where f, g, h are scalar functions and notationally $\nabla_x f = x^\mu \partial_\mu f = x[f]$.

Geometrically the Riemann tensor measures the difference between parallel transporting a tensor one way and then the other way around a closed loop. We can see this by letting v and w be the coordinate basis vector fields. The commutator $[v, w]$ then vanishes and we have

$$\mathcal{R}(v, w, u) = \nabla_v \nabla_w u - \nabla_w \nabla_v u = [\nabla_v, \nabla_w]u. \tag{5.17}$$

In flat space, parallel transport is unambiguous - it does not depend on the path taken between two points. It is the path dependence of parallel transport which embodies the notion of intrinsic curvature when we deal with non-Euclidean spaces. The relation between the Riemann tensor and the torsion tensor T given explicitly in coordinates is encapsulated in the following expression

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} v^\sigma = [\nabla_\mu, \nabla_\nu] v^\rho + T^\lambda{}_{\mu\nu} \nabla_\lambda v^\rho. \tag{5.18}$$

It would seem reasonable to construct a generalized Riemann tensor to have the same form as that of equation (5.15) but with the conventional connection and bracket replaced with their generalized analogues. We have

$$\begin{aligned} R : E \times E \times E &\rightarrow E \\ V, W, U &\mapsto R(V, W, U) = [D_V, D_W]U - D_{[\![V,W]\!]}U. \end{aligned} \tag{5.19}$$

We can check whether this object is a tensor, as we did for the non-generalized version

$$\begin{aligned} R(fV, gW, hU) &= [D_{fV}, D_{gW}]hU - D_{[\![fV,gW]\!]}hU \\ &= fhg([D_V, D_W]U - D_{[\![V,W]\!]}U) - \frac{1}{2} h \langle V, W \rangle D_{(fdg - gdf)} U. \end{aligned} \tag{5.20}$$

We see that it is nonlinear in V and W and hence not acting as a tensor [23]. However, all is not lost, as we can rescue the notion of generalized curvature if we restrict V and W to lie in orthogonal subspaces of E such that $\langle V, W \rangle = 0$, that so this object becomes tensorial [11]. Despite this victory, this object is still not unique (see section (6.1) for further discussion).

5.4 Generalized Levi-Civita connection

We are now ready to discuss a compatible, torsion-free generalized connection. By compatible we mean that the connection only sees the subgroup $O(p, q) \times O(q, p)$ and hence we are interested in defining a connection by its action on elements of the two orthogonal subbundles C_+ and C_-, as introduced in section (4.2.2). We follow closely the constructions and results of [11].

Recall we have the orthonormal bases $\{\hat{E}_a^+\}$ and $\{\hat{E}_{\bar{a}}^-\}$ for C_- and C_+ respectively.

We can therefore represent a generalized vector $\tilde{W} \in \tilde{E}$ as:

$$\tilde{W} = \tilde{W}^A \hat{E}_A = \tilde{w}_+^a \hat{E}_a^+ + \tilde{w}_-^{\bar{a}} \hat{E}_{\bar{a}}^- . \tag{5.21}$$

We wish our generalized connection to act on elements of each subbundle in a way compatible with the Riemannian metric g. This requires replacing the generic conventional connection we had in our previous definition (5.6) with the Levi-Civita connection. The symbol ∇ will denote the Levi-Civita connection from now on. Then in accordance with equation (5.6), we define the generalized analogue of the Levi-Civita connection simply by

$$D_M^\nabla \tilde{W}^a = \begin{pmatrix} \nabla_\mu \tilde{w}_+^a \\ 0 \end{pmatrix}, \qquad D_M^\nabla \tilde{W}^{\bar{a}} = \begin{pmatrix} \nabla_\mu \tilde{w}_-^{\bar{a}} \\ 0 \end{pmatrix}, \tag{5.22}$$

on C_+ and for C_- respectively.

It can also be shown that it is always possible to construct the connection to be torsion free [11]. The four torsion-free, metric compatible components of $D_M W^N$ are given by

$$D_M W^N \quad \begin{cases} D_{\bar{a}} w_+^b = \nabla_{\bar{a}} w_+^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c w_+^c \\ D_a w_-^{\bar{b}} = \nabla_a w_-^{\bar{b}} - \frac{1}{2} H_a{}^b{}_{\bar{c}} w_+^{\bar{c}} \\ D_a w_+^b = \nabla_a w_+^b - \frac{1}{6} H_a{}^b{}_c w_+^c - \frac{2}{9}(\delta_a{}^b \partial_c \phi - \eta_{ac}\partial^b \phi) w_+^c + A^{+b}_{a\ c} w_+^c \\ D_{\bar{a}} w_-^{\bar{b}} = \nabla_{\bar{a}} w_-^{\bar{b}} - \frac{1}{6} H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}} - \frac{2}{9}(\delta_{\bar{a}}{}^{\bar{b}} \partial_{\bar{c}} \phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}} \phi) w_-^{\bar{c}} + A^{+\bar{b}}_{\bar{a}\ \bar{c}} w_-^{\bar{c}} \end{cases} \tag{5.23}$$

where H is the field strength, $A^\pm$ are undetermined tensors and the appearance of the dilaton ϕ is due to the choice of conformal factor $\Phi = e^{-2\phi}\sqrt{-g}$. The first two expressions are unique whereas the last two contain the tensors $A^\pm$ and are thus ambiguous. However, due to the properties of $A^\pm$

$$\begin{aligned} A^{+a}_{a\ b} &= 0, \quad A^+_{abc} = -A^+_{acb}, \quad A^+_{[abc]} = 0, \\ A^{-\bar{a}}_{\bar{a}\ \bar{b}} &= 0, \quad A^-_{\bar{a}\bar{b}\bar{c}} = -A^-_{\bar{a}\bar{c}\bar{b}}, \quad A^-_{[\bar{a}\bar{b}\bar{c}]} = 0, \end{aligned} \tag{5.24}$$

we can act with the differential operators on elements of C_- and C_+ with appropriate index contractions as to eliminate the undetermined tensors in the last two expressions of (5.23). We find

$$\begin{aligned} D_a w_+^a &= \nabla_a w_+^a - 2(\partial_a \phi) w_+^a, \\ D_{\bar{a}} w_-^{\bar{a}} &= \nabla_{\bar{a}} w_-^{\bar{a}} - 2(\partial_{\bar{a}} \phi) w_-^{\bar{a}}, \end{aligned} \tag{5.25}$$

and thus now have a torsion-free, compatible connection with four unique components.

6 Type II NS-NS sector as generalized geometry

We are interested in unique generalized objects from which we can build a supergravity action. In this section we use the unique connections to obtain a unique generalization of the Ricci tensor and Ricci scalar. In the final section we show that our generalized geometry constructs define a generalization of Einstein gravity and produce the correct field equations for the NS-NS sector of type II supergravity.

6.1 Unique curvature

Let us recall from conventional geometry that the Ricci tensor is formed by a contraction of the Riemann tensor

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\rho}{}_{\mu\rho\nu}, \tag{6.1}$$

and that we can define the action of the Ricci tensor acting on a vector v by

$$\mathcal{R}_{\mu\nu} v^{\mu} = [\nabla_{\mu}, \nabla_{\nu}] v^{\mu}. \tag{6.2}$$

We can also take the trace of the Ricci tensor, which gives us the Ricci scalar

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}. \tag{6.3}$$

and since this is a scalar, and thus a coordinate independent quantity, it can give us meaningful information about the curvature of a manifold.

In the generalized case we have already noted that we do not have a uniquely defined Riemann curvature tensor; however we can still form a unique tensorial generalized Ricci. We utilize the unique differential operators we found in the previous section and build a generalized version of Ricci in analogy with equation (6.2). We have [11]

$$\begin{aligned} R^{+}_{a\bar{b}} v^{a}_{+} = {} & [D_a, D_{\bar{b}}] v^{a}_{+} = {} & (D_a D_{\bar{b}} - D_{\bar{b}} D_a) v^{a}_{+}, \\ R^{-}_{\bar{a}b} v^{\bar{a}}_{-} = {} & [D_{\bar{a}}, D_b] v^{\bar{a}}_{-} = {} & (D_{\bar{a}} D_b - D_b D_{\bar{a}}) v^{\bar{a}}_{-}. \end{aligned} \tag{6.4}$$

Upon closer inspection we can see that, due to the index contractions, these two expressions actually pertain to the same object $R^{+}_{a\bar{b}} = -R^{-}_{\bar{b}a} = -R^{-}_{\bar{a}b} \equiv R_{a\bar{b}}$ and thus our generalized Ricci is unique. As in the case of the generalized Riemann tensor, this object does not represent a true tensor unless we restrict the differential operators to lie in orthogonal subspaces. We can write the explicit form of the generalized Ricci tensor as [11]

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \frac{1}{2} e^{2\phi} \nabla^c (e^{-2\phi} H_{cab}), \tag{6.5}$$

where we have chosen the two orthonormal frames to be aligned such that $e^{+}_{a} = e^{-}_{\bar{a}}$.

There is a technicality that is worth noting. Due to the non-tensorial nature of the generalized Riemann tensor we cannot simply perform a contraction like ${R^P}_{MPN} = R_{MN}$ in analogy with (6.1) to obtain the Ricci tensor in the generalized coordinate basis. However, we can still naturally define R_{MN} by

$$R_{MN} = \begin{pmatrix} R_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & R_{\bar{a}\bar{b}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2d}R\delta_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & \frac{1}{2d}R\delta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{6.6}$$

with R the generalized Ricci scalar. Then, writing the generalized metric as

$$G_{MN} = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \delta^{\bar{a}\bar{b}} \end{pmatrix}, \tag{6.7}$$

we see that in the generalized picture there does indeed exist a contraction faithful to the conventional case (6.3)

$$R = G^{MN} R_{MN}. \tag{6.8}$$

The generalized Ricci scalar R is unique and is given explicitly by [11]

$$R = \mathcal{R} + 4\nabla^2\phi - 4(\nabla\phi)^2 - \frac{1}{12}H^2. \tag{6.9}$$

where $\nabla^2\phi = \nabla^\mu\nabla_\mu\phi$ and $(\nabla\phi)^2 = \nabla^\mu\phi\nabla_\mu\phi$. We now have all the ingredients necessary to write down the equations of motion for the NS-NS fields $\{g, B, \phi\}$ of type II supergravity using the generalized geometry formalism.

6.2 Generalized action and field equations

Recall the Einstein-Hilbert action with no matter content is given by [19]

$$\mathcal{S}_H = \frac{1}{2\kappa}\int |\mathrm{vol}_g|\mathcal{R} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\mathcal{R}. \tag{6.10}$$

The corresponding vacuum field equations are given by $\mathcal{R}_{\mu\nu} = 0$. In our discussion we have been solely focusing on the NS-NS sector; we have no matter fields present and so the above equation is the relevant action from which to build a generalized analogue. We write

$$\mathcal{S} = \frac{1}{2\kappa^2}\int |\mathrm{vol}_{\tilde{G}}|R, \tag{6.11}$$

where R is our generalized Ricci scalar and $|\mathrm{vol}_G|$ is the volume form associated with the generalized metric $\tilde{G}$. Our generalized geometry formalism naturally defines a volume form for the action by $|\mathrm{vol}_G| = d^dx\sqrt{-g}(e^{-2\phi})$. Furthermore, throughout our discussions of generalized geometry we have been working in d-dimensions and not the $(9+1)$ dimensions appropriate for our supergravity theory. Taking all this into consideration, we write our action as

$$
\begin{aligned}
\mathcal{S} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}R\} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R}+4\nabla^2\phi-4(\nabla\phi)^2-\frac{1}{12}H^2)\} \\
&= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R}-\frac{1}{12}H^2)+4(e^{-2\phi}\nabla^\mu\nabla_\mu\phi-e^{-2\phi}\nabla^\mu\phi\nabla_\mu\phi)\} \\
&= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R}-\frac{1}{12}H^2) \\
&+ 4(\nabla^\mu(e^{-2\phi}\nabla_\mu\phi)-\nabla^\mu e^{-2\phi}\nabla_\mu\phi-e^{-2\phi}\nabla^\mu\phi\nabla_\mu\phi)\} \\
&= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R}-\frac{1}{12}H^2) \\
&+ 4(\nabla^\mu(e^{-2\phi}\nabla_\mu\phi)+2e^{-2\phi}\nabla^\mu\phi\nabla_\mu\phi-e^{-2\phi}\nabla^\mu\phi\nabla_\mu\phi)\} \\
&= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R}-\frac{1}{12}H^2)+4e^{-2\phi}\nabla^\mu\phi\nabla_\mu\phi\} \\
&= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\{e^{-2\phi}(\mathcal{R}+4(\partial\phi)^2-\frac{1}{12}H^2)=\mathcal{S}_B,
\end{aligned}
\tag{6.12}
$$

dropping the total derivative term. We have thus reproduced the type II supergravity action for the NS-NS sector!

The field equations corresponding to the generalized action are equivalent to $R_{ab}=0$ and $R=0$. The symmetric and antisymmetric components of R_{ab} give the equations of motion for the fields g and B respectively

$$
\begin{aligned}
\mathcal{R}_{ab}-\frac{1}{4}H_{acd}H_b{}^{cd}+2\nabla_a\nabla_b\phi &= 0 \\
\nabla^c(e^{-2\phi}H_{cab}) &= 0.
\end{aligned}
\tag{6.13}
$$

For the dilaton ϕ we have a field equation corresponding to $R=0$, given by

$$
\mathcal{R}+4\nabla^2\phi-4(\partial\phi)^2-\frac{1}{12}H^2=0. \tag{6.14}
$$

These equations precisely match those of (1.3).

7 Discussion

We introduced generalized geometry as an extension of the familiar tangent bundle TM by the cotangent bundle T^*M to the generalized tangent bundle $E=TM\oplus T^*M$. We have seen that the bundle E is endowed with an $O(d,d)$ structure group, which is of relevance for T-duality, and that generalized differential structures on this generalized space naturally encode the infinitesimal symmetry transformations (1.4) evident in supergravity. The B-transformation played a significant role as a subgroup of $O(d,d)$ and, as a symmetry of the Courant bracket, highlighted a novel feature of generalized geometry with no conventional differential geometry counterpart.

By further extending the generalized bundle to $\tilde{E}$ with structure group $O(d,d) \times \mathbb{R}^+$ we showed that we could accommodate all the degrees of freedom in the NS-NS sector. We demonstrated that a generalized metric $\tilde{G}$ unifies the fields $\{g, B, \phi\}$ and reduces the structure group to $O(p,q) \times O(q,p)$. In ten dimensions this corresponds to:

$$\{g, B, \phi\} \in \frac{O(10,10)}{O(9,1) \times O(1,9)} \times \mathbb{R}^+. \tag{7.1}$$

We were able to find a generalization of the torsion tensor and construct a torsion-free, metric compatible generalized connection. Despite the generalized Riemann tensor not being unique, we were still able to use the connection to build a unique generalized Ricci tensor and find a unique Ricci scalar. We established that generalized geometry constructs provide a generalized version of the 10-dimensional type II supergravity action for the NS-NS sector that correctly reproduces the field equations.

A logical extension of this study is to include the remaining fields of type II supergravity, those of the Ramond-Ramond potentials and gravitini and dilatini fermions, into the generalized structure. Futhermore, we do not have to restrict our discussions to type II theories. As mentioned in the Introduction, extending the generalized bundle in such a way as to admit the action of an exceptional group, as described in [12], gives a framework relevant for 11-dimensional supergravity.

Generalized geometry has found great applicability in the study of Calabi-Yau manifolds and the understanding of mirror symmetry [3]. It also promises to illuminate aspects of non-geometric backgrounds, a phenomenon found in string theory [13]. We have seen that generalized geometry is well adapted to supergravity theories and can provide useful insights into string theory symmetries, but in fact has much more to offer than this alone.

A Adjoint action

We can understand the interpretation of the adjoint action in equation (2.17) by considering the infinitesimal expansion of a $GL(d)$ group element M acting on a $(2,1)$-tensor T

$$T^{\mu\rho}{}_{\nu} \mapsto M^{\mu}{}_{\sigma} M^{\rho}{}_{\lambda} T^{\sigma\lambda}{}_{\gamma} (M^{-1})^{\gamma}{}_{\nu}. \tag{A.1}$$

From the Taylor series expansion of M we obtain to leading order $M \approx \mathbb{1} + A$ where $A \in \mathfrak{gl}(d)$. Thus our transformed tensor components become

$$(\mathbb{1}+A)^{\mu}{}_{\sigma}(\mathbb{1}+A)^{\rho}{}_{\lambda} T^{\sigma\lambda}{}_{\gamma}(\mathbb{1}-A)^{\gamma}{}_{\nu} = T^{\mu\rho}{}_{\nu} - T^{\mu\rho}{}_{\gamma} A^{\gamma}{}_{\nu} + A^{\rho}{}_{\lambda} T^{\mu\lambda}{}_{\nu} + A^{\mu}{}_{\sigma} T^{\sigma\rho}{}_{\nu}. \tag{A.2}$$

which is of the same form as that of equation (2.17) with the matrices A belonging to the adjoint representation.

B $O(d,d)$ covariant Dorfman derivative and Courant bracket

We note that given the definition of the generalized covariant derivative (2.18), we also have

$$\partial^{M} = \frac{1}{2}\begin{pmatrix} 0 \\ \partial_{\mu} \end{pmatrix}, \tag{B.1}$$

from raising an index with the inverse of the natural metric $\eta^{MN} = 2\left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$. Likewise, we use the metric to lower the indices of a generalized vector $W^{N} = w^{\mu} + \sigma_{\mu}$ and obtain

$$W_{N} = \frac{1}{2}\begin{pmatrix} \sigma_{\mu} \\ w^{\mu} \end{pmatrix}. \tag{B.2}$$

The equivalence between equations (2.16) and (2.19) can be shown as follows

$$\begin{aligned}
L_{V} W^{M} &= \mathcal{L}_{v} w^{\mu} + \mathcal{L}_{v} \sigma^{\mu} - i_{w} d\lambda \\
&= [v,w]^{\mu} + [v,\sigma]_{\mu} - i_{w}(\partial_{\nu}\lambda_{\mu} - \partial_{\mu}\lambda_{\nu}) \\
&= v^{\nu}\partial_{\nu} w^{\mu} - w^{\nu}\partial_{\nu} v^{\mu} + v^{\nu}\partial_{\nu}\sigma_{\mu} + \partial_{\mu} v^{\nu}\sigma_{\nu} - w^{\nu}(\partial_{\nu}\lambda_{\mu} - \partial_{\mu}\lambda_{\nu}) \\
&= v^{\nu}\partial_{\nu} w^{\mu} + v^{\nu}\partial_{\nu}\sigma_{\mu} + \sigma_{\nu}\partial_{\mu} v^{\nu} + w^{\nu}\partial_{\mu}\lambda_{\nu} - w^{\nu}\partial_{\nu} v^{\mu} - w^{\nu}\partial_{\nu}\lambda_{\mu} \\
&= V^{N}\partial_{N} W^{M} + (\partial^{M} V^{N} - \partial^{N} V^{M}) W_{N}.
\end{aligned} \tag{B.3}$$

For the Courant bracket we have

$$
\begin{aligned}
[\![V,W]\!]^M &= [v,w]^\mu + \mathcal{L}_v\sigma_\mu - \mathcal{L}_w\lambda_\mu - \frac{1}{2}d(i_v\sigma - i_w\lambda)_\mu \\
&= \mathcal{L}_v w^\mu + \frac{1}{2}\mathcal{L}_v\sigma_\mu - \frac{1}{2}\mathcal{L}_w\lambda_\mu + \frac{1}{2}i_v d\sigma_\mu - \frac{1}{2}i_\sigma d\lambda_\mu \\
&= v^\nu\partial_\nu w^\mu - w^\nu\partial_\nu v^\mu + \frac{1}{2}(v^\nu\partial_\nu\sigma_\mu + \sigma_\nu\partial_\mu v^\nu - w^\nu\partial_\nu\lambda_\mu - \lambda_\nu\partial_\mu w^\nu \\
&+ v^\nu(\partial_\nu\sigma_\mu - \partial_\mu\sigma_\nu) - w^\nu(\partial_\nu\lambda_\mu - \partial_\mu\lambda_\nu)) \\
&= v^\nu\partial_\nu w^\mu + v^\nu\partial_\nu\sigma_\mu - w^\nu\partial_\nu v^\mu - w^\nu\partial_\nu\lambda_\mu \\
&- \frac{1}{2}(\lambda_\nu\partial_\mu w^\nu + v^\nu\partial_\mu\sigma_\nu - \sigma_\nu\partial_\mu v^\nu - w^\nu\partial_\mu\lambda_\nu) \\
&= V^N\partial_N W^M - W^N\partial_N V^M - \frac{1}{2}(V_N\partial^M W^N - W_N\partial^M V^N)
\end{aligned}
\tag{B.4}
$$

where we have made use of the identity $\mathcal{L}_x y = (i_x d + d i_x)y$.

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