Theoretical Studies of Magnetic Monopole

from Maxwell to non-perturbative lattice formulation

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Declaration

I here with certify that all material in this thesis which is not my own work has been properly acknowledged.

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I hope whoever is reading this thesis would gain as much pleasure as I was in writing it.

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Abstract

The researches on magnetic monopole, both theoretically and experimentally, have inspired many generations for it had the potential to provide new physics beyond the standard model. Despite the lack of either experimental or observational evidence of their existence, ideas and techniques that were originally invented for the purpose of studying magnetic charges have already played important roles in theoretical high-energy physics. In this thesis, the historical development of monopole researches would be reviewed and the discrete space-time lattice modification of the monopole theory would be discussed in more details.

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1 Introduction

The theory of electromagnetism (EM) was one of the most profound and fruitful theories in modern physics. From a theoretical respective, the ideas used to construct the theory had laid down the foundations of some much more fundamental theories. Among them, there are the theory of quantum electrodynamics, the electroweak theory and what eventually leads to the complication of the Standard Model in particle physics. Alternative thinking of electric induction was also the initiative that inspired Albert Einstein to build the theories of special and general of relativity [1]. There are also countless experimental apparatus built on the basis of electromagnetism, in fact, one can rarely think of a piece of equipment in the arsenal of the experimentalists that does not, to some extent, involve the uses of electromagnetism. This thesis, however, would focus on one specific property of the EM theory, which is the arguably plausible existence of a magnetic charge, magnetic monopole. We would show, as the chapter progresses, despite being nothing but a simple concept when the idea was initially proposed, its later development showed such a great potential that would not only refresh our thinking about the existing theories but also provide route to completely new theories of a much wider physical picture. The layout of the thesis follows the historical timeline as we will start from the classical theory of Maxwell's in 1870s and end up discussing the non-perturbative lattice C-boundary condition approach that has been developed over the last decade.

2 Classical/Semi classical theories of magnetic monopoles

"A bar magnet has two poles, and they cannot be separated into two independent monopoles." This is what everyday experience tells people how one should describe the basic nature of classical magnetism.

The concept of magnetic monopole was long regarded only as the hypothetical particle that was merely introduced to Maxwell's theory of electromagnetism for computational convenience. Back then, it was believed that the existence of an independent magnetic charge had no physical reality [2]. Despite being only a concept, because it would restore the electric-magnetic dual symmetry, it had proven its value in practical calculations, for example in Ref. [3]. Besides the opinions of the majority of scientific society at the time, in 1894, Pierre Curie made a suggestion that it might be possible for the real independent magnetic charge to exist in nature [4]. Almost all the later developments on the subject would be under the concept of quantum mechanics, yet I believe that a classical description would still be necessary, not only as a conceptual introduction but also to provide a less-abstract picture for discussion and to set the classical limit.

2.1 Maxwell equation

The classical theory of electromagnetism was regarded as one of the earliest of unified theories. It was James Clerk Maxwell who combined the works of earlier researchers in, at the time was been regarded as, separate fields of electricity and magnetism to present a unified theory of electromagnetism [5]. Probably inspired by Faraday's law of magnetic induction [6], the changing magnetic field produces an electric field, Maxwell realised that a changing electric field should also generate a magnetic field. He carried this ideas further and wrote down a set of four equations that we now recognise as Maxwell equations [5]:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{j} & \nabla \times \mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} \end{aligned} \tag{2.1}$$

Although not manifestly obvious in this representation, the set of equations possess the Lorentz symmetry: invariance under spatial rotation (group

SO(3), or O(3) if parity included) [7]. By the virtue of this property of symmetry, despite the fact that these equations were constructed prior to the days of Einstein's theory of relativity and the theory of quantum mechanics, it stood alongside very few that had survived the scientific revolutions of the 20th century.

However, there is one thing that makes the theory less "satisfactory" to many great minds of the time [8][9]. The lack of magnetic charge and magnetic current spoils the other type of symmetry that makes the theory mathematically imperfect[3]. The original Maxwell equations had only electric sources, the electric charge density, ρ, and the electric current density, $\mathbf{j}$. This enables people to write the set of four equations into two subsets of two equations each, the sourced ones and the source-less ones. However, if we were able to introduce the magnetic sources (magnetic charge and magnetic current) into the theory, the electric-magnetic dual symmetry would be restored and four equations become, essentially, one! To mathematicians (and many others, author included), the beauty of one completely unified equation is irresistible and because the Maxwell equation does not forbid the existence of such an object, the original version of the equations can be easily modified.

2.2 Dual Maxwell equation

After introducing the magnetic charge and the corresponding current, the modified Maxwell equations read:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho_e & \nabla \cdot \mathbf{B} &= 4\pi\rho_m \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{j}_e & -\nabla \times \mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \frac{4\pi}{c}\mathbf{j}_m \end{aligned} \tag{2.2}$$

where ρ_e and $\mathbf{j}_e$ are the original electric charge density and electric current density respectively. In addition, the magnetic charge density ρ_m and magnetic current density $\mathbf{j}_m$ are brought into the equations. It is easy to see how the dual-symmetry is restored. In other words, it is trivial to notice that the dual Maxwell equations are invariant under the exchange of electric and magnetic components.

Mathematically, the electric-magnetic exchange can be written in a complex form:

[3] This was arguably the very first initiative of introducing the monopole into the theory. Unfortunately, no references or any prove were found to support this clasim.

$$\begin{aligned} \nabla \cdot (\mathbf{E} + i\mathbf{B}) &= 4\pi(\rho_e + i\rho_m) \\ \nabla \times (\mathbf{E} + i\mathbf{B}) &= i\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E} + i\mathbf{B}) + i\frac{4\pi}{c}(\mathbf{j}_e + i\mathbf{j}_m) \end{aligned} \tag{2.3}$$

For convenience, we take natural units where, $c = \hbar = 1$, and ignore the numerical factor of 4π. The electric-magnetic exchange symmetry, or duality transformation, has the form of a rotation of the complex phase θ.

$$\begin{aligned} &\mathbf{E} + i\mathbf{B} \longrightarrow e^{i\theta}(\mathbf{E} + i\mathbf{B}) \\ &\rho_e + i\rho_m \longrightarrow e^{i\theta}(\rho_e + i\rho_m) \\ &\mathbf{j}_e + i\mathbf{j}_m \longrightarrow e^{i\theta}(\mathbf{j}_e + i\mathbf{j}_m) \end{aligned} \tag{2.4}$$

For further uses in later sections, I shall note that there is a symmetry group corresponds to the transformation called U(1).

2.3 Semi-classical quantization

The charge quantization condition could be derived from the semi-classical treatment in quantisation of the angular momentum. Consider a static system consists of an electric charge (e) and a magnetic charge (g) separated by a distance R. The rule of quantization of angular momentum reads:

$$\mathbf{J} \cdot \mathbf{R} = eg/c = n\hbar/2, n = 0, \pm 1, \pm 2, \ldots$$

or

$$eg = m\hbar c, \tag{2.5}$$

where m is a half integer. And we shall see later this is the essential requirement that Dirac proposed for electromagnetic charges to inevitably exist in quantum theory.

In later section of this review, we would encounter somewhat more mathematically sophisticated expressions of the same theory from different aspects of understanding. Nevertheless, the laws of electromagnetism remain at the heart of all the attempts to explain the magnetic monopole, which is governed by the Maxwell equations. These are the equations of motion all the hypothetical construction of monopole theory must obey.

3 Monopoles in quantum mechanics

3.1 Dirac monopoles

Despite the theory of monopole had been investigated by many prior to the birth of quantum mechanics, the subject was not treated as importantly since the classical approach does not provide any insight or explanation of its reason of existence. The attitude of the scientific society did not change until the paper published by the British physicist Paul Dirac in 1931[10]. The theory proposed in this paper was originally supposed to give a theoretical value of the fundamental electric charge, or as in the original paper the fine structure constant. But instead, Dirac worked out a connection between the unit electric charge and a unit magnetic monopole, hence proposed the existence of a particular symmetry between electricity and magnetism, analogues to the dual symmetry we have seen in the classical modification of the Maxwell theory. He argued the necessity of such a symmetry that without it the important experimental ratio $\hbar c/e^2$, fine structure constant, would remain theoretically completely undetermined.

Dirac's argument was based on the fact of redundant information of a physical system should not have any physical meaning and hence should not be observable despite the conditions under which any experiments are to perform. For example, let us consider the simple case of a single particle whose equation of motion is represented by a wave function ψ of the form $\psi = Ce^{i\theta}$, where C and θ denote the amplitude and phase respectively. In this example, the redundant degree of freedom is the complex phase, which does not have physical meanings, either it should have. In order to avoid the ambiguity in the practical applications of the theory, we shall propose the condition that over a integer number of rotation around a loop, $2\pi n$, the wave function remains the same.

Therefore, we can write the wave functions that describe the electromagnetic fields as,

$$\psi = Ce^{i\theta} = Ce^{i2\pi n\theta}, \quad n \in \mathbb{Z} \tag{3.1}$$

Let us then consider the case where the wave function of electromagnetic field vanishes in three-dimensional space. Since the phase space of a general wave function is complex it would require two conditions to vanish. So in general it will vanish along a line, called nodal line[10].

In any given finite volume of space, we could have three different scenarios.

- The trivial case is where the nodal lines do not cross the closed surface of space.
- The second case is that all the nodal lines enter the volume would necessarily leave. In another words, all the nodal lines would cross the closed surface at least twice, and the magnetic flux would always be zero as the incoming lines always have outgoing counter-flux.

For our purpose of discussion, the interesting case is

- if the nodal line ends inside a three-dimensional closed surface.

We would then see a non-zero magnetic flux crossing it. We can write down the required change restriction in phase condition as:

$$2\pi n + \frac{e}{\hbar c} \cdot \int (\mathbf{H} \cdot d\mathbf{S}) = 0, \tag{3.2}$$

The first term represents the vanishing conditions we applied earlier, where n is the number of end point of nodal lines inside the space. We can solve for the simplest non-trivial case when $n = 1$, this means we have only a single pole inside the surface and we integrate over the volume to get

$$4\pi\mu = 2\pi\hbar c/e. \tag{3.3}$$

We can regard μ as the field strength of a single pole (end of a nodal line) inside the surface. Effectively, $\mu \equiv g$ is the fundamental unit of a Dirac magnetic charge. It is easy to spot that we have obtained the same quantization condition as what we have seen in the semi-classical approach. Therefore, Dirac argued that the quantization of the equations of motion of charges in the electromagnetism is possible if the multiples of unit charges satisfying the Dirac quantization condition:

$$e_0 g_0 = \frac{1}{2}\hbar c, \tag{3.4}$$

Note that the theory does not fix the value of either charge but only the product, so it would be rather surprising if the monopole did not exist [10]. In nature, although the magnetic charges are yet to be discovered, the electric charges do obey such a condition.

3.2 Basic properties of an isolated monopole

Let us then consider what type of field such a monopole would cause. Analogous to the properties that of an electric charge, we would expect the magnetic monopole generates a field of the form,

$$\mathbf{B}(\mathbf{r}) = \frac{g}{4\pi r^3}\mathbf{r} \tag{3.5}$$

In the model, we have assumed the present of a free magnetic charge at the origin of he space. This configuration must satisfy the dual Maxwell equations (or more precisely the equivalent the field equation of), as well as the requirement of charge quantization. In order to define a Hamiltonian, which is the essential description in quantum field theories, one must introduce a vector potential A and the magnetic field would be defined as the curl of that potential, $\mathbf{B} = \nabla \times \mathbf{A}$. However, any continuous potential would not fit the requirement of the dynamics, since

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{3.6}$$

More generally, we can consider the field strength tensor of monopole–free electromagnetism

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.7}$$

The dual tensor is

$$F^{\dagger}_{\nu\mu} = \frac{1}{2}\epsilon_{\alpha\beta\nu\mu}F^{\alpha\beta} \tag{3.8}$$

The dual Maxwell equation in the field notation can be written as

$$\partial_\nu F_{\nu\mu} = -4\pi j_\mu, \quad \partial_\nu F^{\dagger}_{\nu\mu} = -4\pi k_\mu \tag{3.9}$$

where j_μ and k_μ are the electric and magnetic currents respectively.

In order to express the motion of the particle we denote the world line of any given particle in terms of its proper time τ, and its four-coordinates $z_\mu = z_\mu(\tau)$. Thus, according to Lorentz's equation, the motion of the charged particles obeys

$$m_e(\frac{d^2 z_\mu}{d\tau^2}) = e\frac{dz^\nu}{d\tau}F(\mathbf{z})_{\nu\mu} \tag{3.10}$$

and

$$m_g(\frac{d^2 z_\mu}{d\tau^2}) = g\frac{dz^\nu}{d\tau}F^\dagger(\mathbf{z})_{\nu\mu} \tag{3.11}$$

While in the Eq. (3.6), the reason of vanishing flux is purely mathematical, Eq.(3.10) and Eq. (3.11) fail because the field strength are to be taken at any point $\mathbf{z}$ in space-time and are there infinitely great and singular [11]. It turns out that the presence of monopole forces the conventional equations of motion to fail at somewhere on the surface.

The way around this problem was to introduce a hypothetical object, along which we could place the singularies. Suppose there is a large surface containing many monopoles, and we can divide it into a network of smaller closed surfaces surrounding one pole each. We restrict Eq(3.11) to fail at one point on every closed surface, so that it will fail on a line of such points forming a string, the *Dirac String*. As a consequence of this configuration, every pole must be attached to such a string. These strings do not corresponding to any physical observables and the choice of variables to describe these strings must be arbitrary and do not influence any physical phenomena. Mathematically, we can write the new expression of the field strength including the strings represented by singularities

$$F^\dagger_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + 4\pi \sum (G^\dagger_{\mu\nu})_g, \tag{3.12}$$

where each $G^\dagger_{\mu\nu}$ is a field quantity which vanishes everywhere except on the sheets traced out by the strings. One natural choice of $G^\dagger_{\mu\nu}$ is to define a singular vector potential [12]. Consider:

$$\mathbf{A}(\mathbf{r}) = \frac{g}{4\pi r}\frac{\mathbf{r}\times\hat{\mathbf{n}})}{r - \mathbf{r}\cdot\hat{\mathbf{n}}} \tag{3.13}$$

where $\mathbf{r}$ is the position vector and $\hat{\mathbf{n}}$ is a unit vector. Under this particular choice of singularity, the resulting magnetic field will be identical to a Coulomb-like field as in Eq.(3.5), apart from in the direction of the vector n and this is a line of singularities.

We can easily visualize this field by considering a semi-infinite long solenoid with one end at the origin. It carries electric current of the magnitude $\mathbf{j}_e = g/2\pi\mathbf{R}$, where $\mathbf{R}$ is the radius of the solenoid cross-section. Analogous to the example of the complex phase in wave function if we propose that the solenoid is infinitesimally thin and then, in the classical limits, it would be necessarily unobservable and therefore unphysical. As far as the physics is concerns, it will appear to be an isolated monopole at the origin. Argued along this line, Dirac concluded [11], quantum electrodynamics allows the existence of point-like magnetic monopoles attached to the end of an unphysical string provided the Dirac quantization condition is satisfied and the string does not cross the charged particles.

The effective quasi particle with magnetic charges analogous to the Dirac monopole has been observed recently in spin ice, a frustrated magnetic condensed matter system [13][14]. The formation of these monopoles occurs when the dipole moment of the electronic degrees of freedom fractionalizes, which is essentially a phase transition in the spin ice at high-dimensional fractionalization. One thing worth mention is that the Dirac strings attached to the monopoles in the spin ice system are actually observables and therefore the particles are not quantized. It is the "string soup" characteristic of the system makes the strings energetically unimportant and practically impossible to locate to any specific monopole in the system. Nevertheless, these effective monopoles have very similar properties to actual Dirac monopoles and can provide new ways to study the physics about them.

The theory of Dirac monopole was undeniably important as it pioneers the way that people would think about the otherwise purely theoretical object. However, the theory had its drawbacks. Among which the most inconvenient is the fact that it attempted to introduce a new unphysical object, namely Dirac string, in order to clarify the existence of the originally hypothetical monopole. And also because of the fact that the monopole does not occur automatically in QED, and it turns out to be rather difficult to add such modification in it [15].

4 t' Hooft-Polyakov monopole

Back in 1970s, people had moved on from the formulation of QED towards the construction of a theory that unifies the electromagnetic, weak and strong interactions. Along its development, a new theory of magnetic monopoles proposed independently by both Dutch scientist G. 't Hooft and Russian (USSR) physicist A.M. Polyakov caught peoples' attention once again. In their papers, they demonstrated that in the broken phase of the unified gauge field in which the $U(1)_{em}$ is embedded the field equation has a non-trivial solution, and the solution turns out to be a new type of particle carrying non-zero magnetic charge, 't Hooft-Polyakov monopole. [16][17]

The development of the unified theory laid down the foundations of the Standard Model in particle and high-energy physics. Back then, physicists started to adapt the ideas of symmetry and gauge theory to formulate the unified theory [18][19]. In the proceeding chapters we would start by reviewing some of the basics in gauge theory. Then, we would reconstruct the models that 't Hooft and Polyakov used to get this new monopole solution. After that, we would show that this solution is nothing specific to our particular choices of model but a general feature of any unified theory leaves a unbroken $U(1)_{em}$ to describe electromagnetism.

4.1 Gauge theory

In quantum field theory, the equation of motion is described by Lagrangians. Because of the redundant degrees of freedom in the Lagrangian, one can perform transformations between different choices of gauges within that gauge group, or symmetry group, without alter any physics about the original theory. In these cases, we say that Lagrangians remain invariant under symmetry transformation groups. So the challenge for theorists became to find the "correct" choice of gauge under which the desirable physical quantity is most manifestly accessible. In the case of quantum electrodynamics, the field equation is described by an abelian gauge theory with the symmetry group $U(1)_{em}$. Similar to what we have done in previous chapters, we can see that by replacing the scale factor with a complex phase and the corresponding scale transformation will simply become a phase transformation in complex plane. [20]

Let us consider the lagrangian of the quantum electrodynamics, it has a relatively simple form:

$$\mathscr{L}[g(\phi)] = \mathscr{L}[\phi] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\phi}(i\gamma^{\mu}D_{\mu} - m)\psi \tag{4.1}$$

where $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ is the conventional covariant derivative for the coupling e, and $F_{\mu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the strength tensor.

The invariant property of the Lagrangian is easy to spot if we perform the gauge transformation on the wave function,

$$\psi(x) \to \exp(i\alpha(x))\psi(x); \tag{4.2}$$

and to the gauge field,

$$A_{\mu}(x) \to A_{\mu} - \frac{1}{e}\partial_{\mu}\alpha(x). \tag{4.3}$$

These transformations are said to be local because there are space-time dependences in the equations. In the case where the gauge field A_{μ} decouples, we observe that the Lagrangian can be invariant only if the phase α is constant in the whole space-time. Another way of saying this is that the symmetry has become global.

Later, in 1954, Chen Ning Yang and Robert Mills attempted to construct a non-abelian gauge theory to generalize the gauge invariance of the electromagnetism, known as the Yang-Mills theory [21][22]. Their ideas later leads to the construction of the SU(2), describes the weak interaction, and SU(3), describes quantum chromo dynamics for strong interactions. The unified theory of all three types of natural interactions were developed later, using a much more complicated group:

$$\text{Standard Model} \equiv SU(3)_{\text{color}} \times [SU(2) \times U(1)]_{\text{electroweak}}. \tag{4.4}$$

This is the Standard Model of the particle physics.

4.2 Spontaneous symmetry breaking

Let us move on to refresh our minds with some of the basic ideas of spontaneous symmetry breaking. For the simplest case in 2D space, in which we consider two different types of potentials:

$$V_1 = |\psi|^2, \text{ and } V_2 = -m^2|\psi|^2 + \lambda|\psi|^4, \tag{4.5}$$

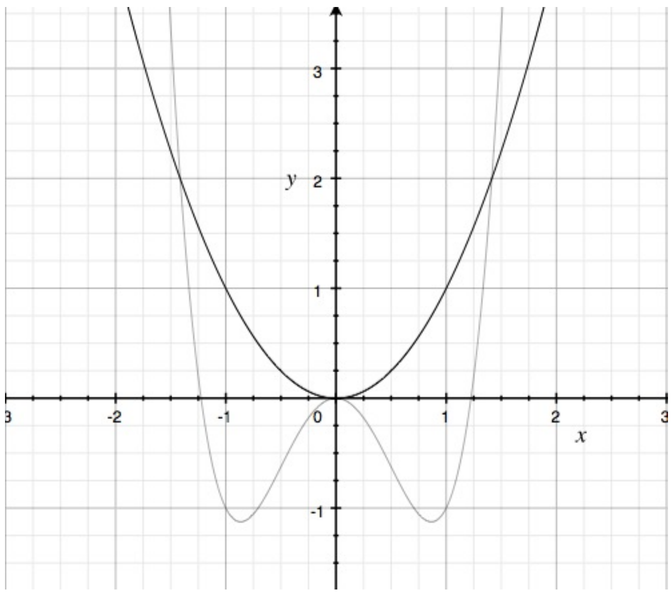

Figure 1: spontaneous symmetry breaking: V1, potential in black has a minimum at the origin; V2, the grey potential has two minima, reflect symmetric about the y-axis.

Although both of the potentials are symmetric in the unbroken phase, simply by looking at the shape of the potential well, we can see that in the case of V2, we have two degenerated minima at $|\psi| = \pm(-m^2/2\lambda)^{1/2}$. These two minima in the potential means that the system would have two vacuum states related to each other by a simple reflection transformation. As far as the symmetry's concern, these two vacua are completely identical, the choice of which the actual particle would fall into is totally random. However, as soon as the system fells into one of the vacuum state, the reflection symmetry of the original Lagrangian would be spontaneously broken. In some symmetry groups more complicated than U(1), the symmetry can also be partially broken. It is possible for the vacuum to remain invariant under a subgroup of the original symmetry group. We can take one of the simple case of SU(2), and consider the scenario in which the group is broken by an scalar field $\Phi = \sum_i \phi^i \sigma^i$, where σ^i is the Pauli matrices. We can choose $\phi = \phi^3\sigma^3$ without loss of generality (wlog) since all the vacuum states are identical which makes the choice arbitrary. In this case, the field transforms as $\phi \to g^\dagger \phi g$, where g is the SU(2) matrix $\exp(i\alpha\sigma^3)$. Then, even after one particular choice of vacua breaks the full SU(2) symmetry, the scalar field is still invariant under U(1) [7].

4.3 t'Hooft-Polyakov monopoles

We would now follow the reasoning by 't Hooft in [16]. Consider a magnetic flux ϕ entering at one spot on a 2-sphere. For a contour around the spot, we have a magnetic potential field A, where $\oint(A \cdot dx) = \phi$. We can rewrite the field in terms of local gauge field, we have $A = \nabla\Lambda$. Due to the redundancy of the theory, the gauge field Λ is multivalued. We further require all fields to be single valued, so ϕ must be an integer multiplied by 2π, a complete gauge rotation along the contour. Therefore, we can write $\phi = m2\pi$, where m is the winding number. In the abelian gauge, another spot is necessarily required for the flux lines to come out. However, in a non-Abelian theory with compact cover, a 4π (2π) rotation may be shifted towards a constant without singularity, thus we could obtain a vacuum all around the sphere. Follow this argument, the magnetic monopole with twice (or once in some cases) the fundamental charges would be allowed in non-abelien theories, provided the electromagnetic $U(1)_{em}$ is a subgroup of such a gauge group with compact covering group. This leads to the consideration for a non-trivial solution in the non-abelian Higgs-Kibble system.

Conventionally, the gauge is chosen in which the Higgs field is a vector in a fixed direction in space-time. 't Hooft promoted a different condition of the gauge that the Higgs field is chosen so that it is $\Omega(\theta, \phi)$ times the vector of our choices, where $\Omega(\theta, \phi)$ is the gauge transformation that brings vacuum to a non-zero vector potential outside the kernel that is at the origin of the three-dimension space. This gauge would cause a different boundary condition at the infinity that corresponds a solution of a stable monopole occurring at the origin. For future references, the analogy of this boundary condition would later play an important role in the lattice theory of monopole formation.

Mathematically, the choice of the non-abelian gauge group is arbitrary as long as the conditions stated above are satisfied. However, what we are really looking for is a theory that would work in the real physical world. We now know that in the electro-weak theory, the $U(1)_{em}$ group is embedded in the $(SU(2) \times U(1))_{\text{electroweak}}$ group. Unfortunately, this group does not have compact covering group and therefore would not necessarily yield monopole solutions. However, it was realized [23] in a Grand Unification Theory (GUT), of which the Standard Model group is embedded in, some of the candidates do indeed have compact covering, eg. SU(5), hence they must always yield monopole solutions. In a higher order of unified theory, theory of everything that includes gravity or "M-theory", a similar argument follows that they would always allow monopole solution analogous to the 't Hooft-Polyakov monopole. [24]

4.4 Non-trivial monopole solution

Consider the Higgs isovector fields $\phi_a(x), a = 1, 2, 3$. The lagrangian is

$$\mathscr{L} = \frac{1}{2}\pi_a^2 + \frac{1}{2}(\nabla\phi_a)^2 - \frac{1}{2}\mu^2\phi_a^2 + \frac{1}{4}\lambda(\phi_a^2)^2, \tag{4.6}$$

where π is the conjugated momentum of the Higgs fields. This was a very common lagrangian which had been studied for some time but what really special is that we can apply a special boundary condition, $\phi^2(\pm\inf) = \mu^2/\lambda$, so that the model would allow the vacuum to be perturbed within a finite volume. Using Eular-Lagrangian equation to find the equation for the extremal, we get

$$\nabla^2\phi_a + \mu^2\phi_a - \lambda\sum_b \phi_b^2\phi_a = 0. \tag{4.7}$$

Solve for ϕ_a, and we find the solution of the form

$$\phi_a = x_a u(r) r^{-1}. \tag{4.8}$$

The function $u(r)$ is subject to the equation

$$u'' + \frac{2}{r}u' + (\mu^2 - \frac{2}{r^2})u - \lambda u^3 = 0. \tag{4.9}$$

In turn, the boundary condition, as we have stated earlier in this chapter, becomes $u(\inf) = \mu/\sqrt{\lambda}$. This is the well-known "hedgehog" configuration. Note that we have not yet couple to any external fields and the energy of the solitary hedgehog solution diverges linearly at large distances, as we would expect. We can solve the divergence problem by coupling it to the Yang-Mills field. This would change the partial derivatives to the covariant derivative of the form,

$$D_\mu\phi_a = \partial_\mu\phi_a + g\epsilon_{abc}A_b^\mu\phi_c. \tag{4.10}$$

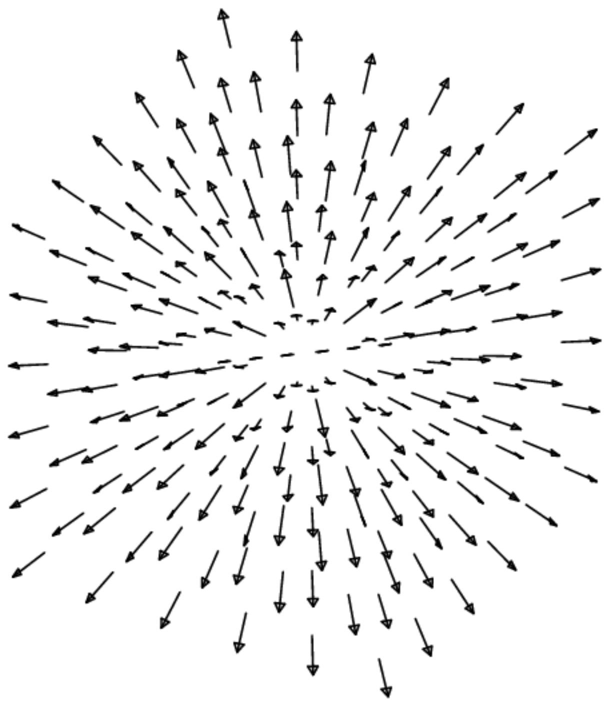

Figure 2: "hedgehog" configuration corresponding to a monopole. [25]

Field solutions become

$$\phi_a(x) = x_a u(r) r^{-1}$$
$$A^a_\mu(x) = \epsilon_{\mu ab} x_b (a(r) - \frac{1}{gr^2}), \tag{4.11}$$

where functions u and a are subject to the differential equations:

$$u'' + \frac{2}{r}u' + (\mu^2 - 2g^2 a(r)^2)u - \lambda u^3 = 0$$
$$a'' + \frac{4}{r}a' - \frac{3}{r^2}a - g^2 r^2 a^3 - g^2 u^2 a = 0. \tag{4.12}$$

Because of the gauge symmetry of the Yang-Mills fields, the inhomogeneity of the distribution of the direction of the ϕ_a becomes unrealizable and makes no contribution to the energy. The energy density falls as $\rho \sim 1/2g^2r^4$, which means it turns out to be finite [25]. It also means there is a long-range magnetic Coulomb-type force between monopoles. The model with normal boundary conditions has been well studied [26][27][28], and the particle spectrum consists of one massless vecton and two massive vectons. If the massless one is photon, the mass of the electrically charged W-bosons are determined by the coupling and the value of the vacuum, $m_W = gv$, and the scalar Higgs field gained mass during spontaneous symmetry breaking, $m_H = \sqrt{2\lambda}v$, then the hedgehog configuration obeys Eq[4.11] is a massive monopole solution. However, the field solutions can only be solved numerically and we will re-visit their forms in later chapters. Nevertheless, in the relatively simple Georgi-Glashow model [29], based on SO(3), we can estimate the monopole mass as $137 m_W$ and m_W is no heavier than $53 GeV/c^2$ in this model.

4.5 More about quantization

The virtue of Dirac's monopole is that he showed that the quantum mechanics does not preclude the existence of it. Moreover, if the monopole really exists, it will imply the reason the electric charge quantization. At the time when Dirac proposed his theory, the later was merely a conclusion or rather a consequence of the former idea, and it made people wonder if the two ideas are really connected within somewhat more fundamental laws of nature. It was the old days when the quantization could not be explained otherwise. We now have alternative ways of understanding the electric charge quantization in the language of group theory [30]. For the electromagnetism gauge group U(1)em, we can write the phase factors as, $\Omega(x) = e^{i\omega(x)}$. We may then interpret this expression as a unitary representation of the gauge group $U(1)_{em}$ of real numbers, hence $\omega(x) \to e^{i\omega(x)q}, \omega(x) \in \mathbb{R}$. Consider the mapping,

$$\Omega = e^{i\omega} \to D(\Omega) = e^{i\omega q_r} \tag{4.13}$$

If we require $D(\Omega)$ also to be a representation, then q_r must be an integer other wise $D(\Omega)$ becomes not single valued as a function of Ω. The field in the $U(1)_{em}$ theory transforms with the representation q_r takes the form

$$\psi_r'(x) = D^r(x)\psi_r(x), \quad D^r(x) = \Omega(x)^{q_r} \tag{4.14}$$

We would, of course, require the action to be $U(1)_{em}$ gauge invariant. Therefore, we must adapt the corresponding covariant derivative $D_\mu^r = \partial_\mu -$

$iq_r e A_\mu(x)$. It then follows that the charge eq_r are a multiple of a fundamental charge unit e.

Further more, we can generalize the above derivation and prove that the charges will necessarily be quantized if the gauge group corresponding to the field equation is compact, that is, if $U(1)_{em}$ is compact[31]. The real beauty this new approach is that the $U(1)_{em}$ is automatically compact in a unified gauge theory in which $U(1)_{em}$ is embedded in a non-abelian semi-simple group. To aid the understanding, we can make comparison in the simpler case of quantization of angular momentum. The angular momentum is quantized because its operator obeys nontrivial commutation relations with other operators in the theory. The eigenvalue of J_z is required by such algebra to be integer multiples of $1/2\hbar$, where we can consider $1/2\hbar$ to be, in analogy, the "fundamental charge" of angular momentum. The electric charge operator obeys the similar commutation relations provided a non-abelian semi-simple compact group and the quantization condition follows as a result. One thing worth mention is that the conclusion is valid even in the phase of spontaneous symmetry broken.

These two rather distinct approaches are in fact not independent, according to what ‘t Hooft and Polyakov have shown in the construction of monopoles. What they managed to achieve was to have changed our viewpoint from the consistency of the monopole existence demonstrated by Dirac to the necessity of their existence in a grand unified theory.

5 Search for monopoles

5.1 Cosmological defects

We have shown, up to this point in this review, the natural laws of physics seem to allow the existence of magnetic monopole as stable particles. The argument follows that they would have been produced in the Big Bang or shortly afterwards when the unified theory was broken in the very early stage of the universe [32][33]. Tom Kibble proposed the mechanism of how these monopoles may be produced in 1976 [34], and the basic idea is known as Kibble mechanism.

For the continuum of the literature, we would once again express our argument using the SU(2) Georgi-Glashow model. Although this model is much less complicated than any of the candidates of the Hot Big Bang model [35], we can still see the outline of the mechanism and defect formation.

In the event of the Big Bang or very shortly after, the universe is very hot, provided the temperature is high enough, we could assume that the GUT symmetry would be unbroken at that time and the scalar Higgs field was zero. As soon as the universe started to cool down and expand, the phase transition occurred and the universe went into a gauge symmetry broken phase. This whole process took place when the universe is very young, only 10-35 second old. During the phase transition, the Higgs field became non-zero and for the system to be in a minimum energy state the direction of the vector would have to be the same everywhere. However, even in the stage as early as that, we still need to obey the law of relativity, that restricts the information travel speed no faster than the speed of light. If we consider two spatially separated points, the choices of the directions of Higgs field of these two points has to be completely independent to each other. It follows that, due the symmetry, the choices of directions are random and totally uncorrelated given that the separation is large enough and the minimum energy requirement still holds at short distances. Therefore, it follows that after the transition, we can roughly picture the structure this stage of the universe consists of domains of size ξ within which the Higgs field is uniform. This correlation length ξ cannot be longer than the particle horizon, which is roughly the age of the universe at that time. When two of these domains meet, the continuum of the Higgs field requires the overlapping be smooth. However, in the cases of more than two domains, the field cannot continuously interpolate between all domains without vanishing in the middle. In the vanishing region, we would expect the formation of topological defects. In our SO(3) model, the spherical symmetry broken would result in the formation of the localized cube-like defects. Related to the theory reviewed in the previous sections, this defect is the magnetic monopole (or anti-monopole) that carries the non-zero magnetic charge (anti-charge). Similar type of the monopole is expected in the GUT symmetry broken phase [36]. However, because the probabilities of both monopole and anti-monopole production are equally likely, they can initially meet and annihilate each other. After a while, this initial process stops and the number density decreases only die to the expansion of the universe. We can estimate the initial number density of the monopole or anti-monopole is roughly the same as the number density of the domains. That is equivalent of saying for each domain we expect the production of at least one monopole or anti-monopole [32]. And an estimate of the number density at the present time after considering the annihilation and universe expansion was calculated by Preskill [37], and he showed that it should be comparable to the number density of the nucleons. Recall the mass of the proposed GUT monopole, $\sim 10^{16} GeV$, which is many orders of magnitude higher than the nucleons. It is easy to conclude that this prediction simply cannot be valid. This question is known as the monopole problem.

The monopole problem, along side with horizon problem [38] and flatness problem [39], had troubled cosmologists for many decades and it was not until 1980 Alan Guth proposed an alternative form of the mechanics known as the idea of inflation that could potentially solve all of these questions [40]. Inflation is, by its own right, a very interesting and vast idea that is obvious beyond the scope of this thesis and therefore I should refer the viewers to Ref [40] for detailed review. The basic concept of inflation is that shortly after the Big Bang, the universe expanded at an accelerated rate. If the inflation occurred after the GUT symmetry breaking, it could potentially dilute the monopole density and restrict it at a level that is acceptable to the observation and there is in a large number of theoretical models supporting this theory.

Another interesting point to mention is that there are also models in which the monopoles lighter than the GUT monopoles, known as intermediate mass monopole, were to be formed at the end of the inflation or shortly afterwards [25]. If any of these models turned out to be the true picture, it would be even more important to study the precise mechanism of the monopole formation and the time evolution after their production.

5.2 Experiments

After the publication of ‘t Hooft and Polyakov's paper, there is no surprise that experimentalists had tried so hard to find the real particles that either have been existed in nature or produced in the high energy particle experiments.

Unlike most of the hypothetical particles in particle physics, once confirming its existence, it is actually quite easy to detect magnetic monopole. These particles are very stable and can only be destroyed by monopole-antimonopole annihilation, so it would not decay in laboratory timescale. It is also believed that at the core of monopoles, the GUT symmetry is restored [41], and it would catalyze the decay of otherwise considered stable nucleons [42][43]. It also carries strong magnetic charge, which means if one could imagine a charged particle passing through a superconducting ring, the changing magnetic field would induce a current in the ring and the current can be measured to determine the magnetic charge very accurately. Ionization loss of a monopole can also be detected when it passes through matter. Although due to the unknown detailed ionization process for monopoles, the results may be hard to spot or distinguish from other ionization processes.

The experimental aspects of finding this extraordinary new spice of particles are an interesting and active field of researches. Numerous experiments and techniques had been developed over the years and I would not include further details of them in this review. Viewers are advised to refer to [44] [45] [46] for more details.

6 Theoretical studies

In short, the current status of experimental monopole research provides no solid evidence of its existence to announce the discovery [4]. We shall then focus on the theoretical studies and computational simulations of it.

6.1 Monopole as soliton [5]

The hedgehog configuration of the ‘t Hooft-Polyakov monopoles cannot be turned continuously into the uniform vacuum state, so we say that it is topological stable. This configuration is an example of a topological defect or soliton [33]. This type of monopole configuration is nothing but the simplest non-trivial sector of the static soliton solution in 3+1 dimensions constructed out of spin-1 fields, the gauge fields.

We shall now show briefly how the monopole emerges purely as gauge fields soliton solution. A natural choice would be the gauge to describe a free electromagnetic system, the Abelian group $U(1)_{em}$. Unfortunately, it does not yield solitary solution. In an abelian group, the solution exists as a packet and will necessarily dissipate. The choice is then restricted to non-Abelian groups and according to “what else can it be theorem” [7], the next simplest candidate will be the non-Abelian SU(2) gauge group of which the Abelien $U(1)_{em}$ is a subgroup embedded in it. This triplet of gauge fields is mentioned in the previous section as the Yang-Mills fields. However, is has also been shown that the set of pure Yang-Mills fields also fail to yield any static soliton solutions [51][52][53], although Yang and Wu [54] did show that the model allows singular solutions. One possible way to overcome this problem is to enlarge the SU(2) further by coupling it to a triplet of scalar fields developed by Georgi and Glashow in 1972 [29]. This new model consists of scalar fields $\phi^a(x,t)$, the Higgs fields, and vector fields $A^a_\mu(x,t)$ in 3+1 dimensions. The space index a will transform according to local SU(2) and for a given value, ϕ^a is a scalar and A^a_μ is a vector under Lorentz transformation. The trivial solution would be the vacuum where no stable particles exist and we are looking for the solutions that satisfy two conditions:

1. Static,

[4] there are, however, some claims of the monopole-like event happening on various occasions. For details, see [47][48]

[5] The calculations and approaches in this chapter are largely inspired by [30] [49] [50], viewers are advised to refer to those for more details.

2. $A_0^a(x) = 0 \;\; \forall a, x$.

The second condition was used to restrict us to pure-magnetic charge for mathematical simplicity. Although it also has been shown [55] that the same lagrangian also yields dyon solution, particle carries both electric and magnetic charges and can be thought as the excitation of the ground state monopole solution. We also require the solution to have finite-energy, because we expect monopole to be a physically real particle, which cannot be arbitrarily energetic.

The approach would be divided into three steps: Firstly, we would find the vacuum solutions hence identify the set of allowed boundary conditions for which the finite-energy requirement would be satisfied. We would then make a homotopy classification of these boundary conditions. And finally, amongst these possible configurations, we search for a finite-energy solution.

To start with, we write down the Hamiltonian, $\mathscr{H} = \int d^3x(-\mathscr{L}(4.1))$, and we set it to reach a minimum. The trivial solution would be when the Yang-Mills fields vanish and the covariant derivatives got reduced to normal partial derivative that also vanish. Because of the gauge invariance of the original lagrangian, we would obtain a family of degenerate vacuum solutions $\mathscr{H} = 0$. For each of these solutions, $\phi \equiv \{\phi^a\}$ must have a fixed magnitude, but can point in any directions in internal space. Recall the local SU(2) gauge symmetry contains in it a global rotational symmetry of the scalar fields ϕ. It means all the solutions within the family correspond to $\mathscr{H} = 0$ are related to each other by this symmetry. Let us then move on to non-zero but finite energy, $\mathscr{H} \neq 0$. This condition can be achieved by setting the boundary conditions: the fields approach some $\mathscr{H} = 0$ configuration at spatial infinity sufficiently fast. In terms of Hamiltonian, we can see that this condition requires the covariant derivatives to vanish as $r \to \infty$. Note that due to the coupling fields, the partial derivative itself does not need to vanish. It turns out that some components of A_μ^a can even fall as slowly as $1/r$, and it will still be consistent with the finiteness of the energy. The general condition would become that the allowed values of ϕ^a at the boundary lie on a spherical surface in internal space. The radius of the sphere would be the fixed magnitude of ϕ in the vacuum solution, determined by the specific parameters of the lagrangian. Bear in mind that we are considering a 3+1 space-time, therefore we can have another physical boundary of the entire space which is also a 2-sphere. Hence, the set of allowed boundary conditions are the set of all non-singular mappings of the physical spherical surface to the internal space 2-sphere, $f : S_2^{\text{phy}} \to S_2^{\text{int}}$. Suching mappings fall into a denumberable infinity of homotopy class, which forms a group. Divide the group into *sectors* each parameterized by a topological parameter Q in such way that field configurations from one sector in the group cannot be continuously deformed into another sector.

The $Q = 0$ sector would be the trivial vacuum solution. $Q = 1$ field configuration is the one that would have the scalar fields pointing radially outward with its internal directions parallel to the coordinate vector, the hedgehog configuration in Figure 4.4. Therefore, we have wed that the finite-energy configurations of this model arose entirely from the boundary conditions consideration of the fields, analogous to what ‘t Hooft and Polyakov did in 1974.

The topological parameter, however, has more meaning rather than a label. We would now show that the monopole charge is proportional to Q and hence prove that field configurations in the $Q = 1$ sector indeed corresponds to a monopole solution.

This can be the most easily shown using ‘t Hooft's definition of a gauge-invariant field strength tensor:

$$F_{\mu\nu} \equiv \phi^a G^a_{\mu\nu} - \frac{1}{g}\epsilon^{abc}\phi^a D_\mu \phi^b D_\nu \phi^c. \tag{6.1}$$

where $G^a_{\mu\nu}$ is the original field tensor in the lagrangian. This effective U(1) field strength tensor has a dual with non-zero divergence hence we can apply the dual Maxwell equation and obtain:

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{1}{2g}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu \phi^a \partial^\rho \phi^b \partial^\sigma \phi^c. \tag{6.2}$$

On the other hand, one can then define a topological current [56]:

$$k_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu \phi^a \partial^\rho \phi^b \partial^\sigma \phi^c. \tag{6.3}$$

and the dual Maxwell could be re-written as:

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{4\pi}{g}k_\mu. \tag{6.4}$$

Analogous to the electric current, we denote the magnetic current as $(1/g)k_\mu$, and the magnetic field satisfies the dual magnetic Gaussian equation:

$$\nabla \cdot \mathbf{B} = 4\pi \frac{k_0}{g}.$$

Hence, the magnetic charge can be obtained by integration:

$$g_m = \int \frac{k_0}{g} d^3x = \frac{Q}{g}. \tag{6.5}$$

Note that if one imposes the condition of Yang-Mills coupling, $e = g\hbar$, equation above then becomes Schwinger quantization condition, which has double the value of the fundamental charges restricted by Dirac's condition. In the simplest non-trivial case where $Q = 1$, we have reproduced the 't Hooft-Polyakov monopole. The curious reader might be interested in the sectors of which the Q number is greater than 1, and we would come back to it in later chapters for reasons would become clearer then.

6.2 Magnetic charge in continuum

For the rest of this thesis, in order to make comparison between different approaches and due to the special treatment we would use for various types of simulations, we would introduce a new expression of the SU(2) Georgi-Glashow model. Although the physical content is no different than the expressions we have used before (4.1), this new expression is more compatible to lattice formulation, which we would discuss as the main frame work for the non-purterbative study of the subject.

Consider the SU(2) Georgi-Glashow lagrangian of the form:

$$\mathscr{L} = -\frac{1}{4}\text{Tr }\ F_{\mu\nu}F^{\mu\nu} + \text{Tr }\ [D_\mu, \phi][D^\mu, \phi] - m^2\text{Tr }\ \phi^2 - \lambda(\text{Tr }\ \phi^2)^2, \tag{6.6}$$

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative and the field strength tensor can be written as $F_{\mu\nu} = [D_\mu, D_\nu]/ig$. Both ϕ and A_μ are Hermitian and traceless 2x2 matrix, which makes it possible to expand them in terms of group generators. The common choice of generators for the gauge group SU(2) is the Pauli matrices [???],

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.7}$$

This choice of generators, $T^a \propto \sigma^a$, corresponds to SU(2) adjoint representation in which the fields can be expressed in terms of components, as $\Phi = \phi^a \sigma^a$ and $A_\mu = A_\mu^a \sigma^a$. The monopole solutions with an extended scalar field occurs if we choose the vacuum expectation value Tr $\Phi^2 = -\frac{m^2}{2\lambda} = \nu^2$, and the SU(2) symmetry breaks into $U(1)_{em}$.

The effective U(1) field strength we have seen 6.1 could be written as:

$$\mathscr{F}_{\mu\nu} = \text{Tr } \hat{\Phi} F_{\mu\nu} - \frac{i}{2g} \text{Tr } \hat{\Phi}[D_\mu, \hat{\Phi}][D_\nu, \hat{\Phi}], \tag{6.8}$$

where $\hat{\Phi} = \Phi/\sqrt{2\text{Tr } \Phi^2}$ represents the direction of the symmetry breaking. We have chosen this specific form because it makes it easy for us to see the outcomes of gauge fixing. If we fix the unitary gauge, in which $\Phi \propto \sigma_3$, it gets reduced to the Abelian form as in the monopole-free Maxwell equations where $\mathscr{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu^3 - \partial_\nu \tilde{A}_\mu^3$ and $\mathscr{B} = \nabla \times \tilde{A}^3$. In the gauge fixing scenario, fields Φ becomes diagonal after the gauge transformation R(x)

$$\tilde{\Phi}(x) \equiv R(x)^\dagger \Phi(x) R(x) = \frac{\sqrt{2\text{Tr } \Phi^2}\sigma^3}{2}. \tag{6.9}$$

And the gauge field takes the form

$$\tilde{A}_\mu = R^\dagger A_\mu R - \frac{i}{g} R^\dagger \partial_\mu R. \tag{6.10}$$

For a general unitary gauge $\Phi \propto \sigma_a$, where $a = 1, 2, 3$. We can alternatively write in terms of the diagonal elements of the gauge field after the transformation:

$$\mathscr{F}_{\mu\nu}^a = \partial_\mu \tilde{A}_\nu^{aa} - \partial_\nu \tilde{A}_\mu^{aa}. \tag{6.11}$$

Note that this is not the conventional field strength tensor, but they are related by

$$\mathscr{F}_{\mu\nu} = \mathscr{F}_{\mu\nu}^1 - \mathscr{F}_{\mu\nu}^2, \tag{6.12}$$

and the anti-symmetry of the tensor was preserved because of the tracelessness of the transformed gauge field,

$$\mathscr{F}^1_{\mu\nu} = -\mathscr{F}^2_{\mu\nu}, \tag{6.13}$$

The conserved magnetic current is defined via the dual Maxwell equation (**??**) as,

$$j^a_\mu = \partial^\nu \mathscr{F}^{\dagger a}_{\mu\nu}, \tag{6.14}$$

which shares the anti-symmetric properties of the tensor and hence indicating that there is only one monopole species. This current is the Noether current corresponds to the symmetry and one can obtain the Noether charge by integrating the 0^{th} component of the current over a finite volume, which is the magnetic charge of the monopole.

$$Q = \int_V d^3x j_0 = \pm\frac{2\pi}{g}. \tag{6.15}$$

Note that these current and charge are the Noether's definition associated to the symmetry, and the there are structurally very different from those mentioned before in the topological argument. However, it has been proven that they are related and correspond to the same physical quantities[75].

7 Computational monopole theory

Up to this point of this review, we have discussed the basic ideas of how to construct different types of magnetic monopoles and talked fairly little about their physical properties. Indeed, if we would ever actually discover these interesting particles in experiments, what should we expect to see?

Consider the equations of motion for Dirac monopoles in classical theory. The field quantities $F_{\mu\nu}(z)$ (3.11) are spatially dependent and therefore can be taken to where the particle is situated, infinitely great and singular [11]. One can even argue that these isolated Dirac monopoles may not even have any physical meanings at all. The infinities may be due to the existence of the strings that attached to the monopole yet extending to infinity. However, the problem shall be solved in the construction of the ‘t Hooft-Polyakov monopole which has finite energy. The problem then became really practical.

7.1 Lattice modification

Gauge invariance is formulated in the position space that makes the lattice modification a natural candidate as a regulator for any gauge theories. In lattice field theory, instead of the continuous space-time where the original theories were constructed, we discretize the space-time hence introduces an artificial cut-off to the infinite quantities in these theories. The visually simplest model is a three-dimensional theory, it can be thought as a classical statistic system. We denote the lattice spacing as a and the energy of any monopole constructions inside the lattice would be confined inside the lattice. The energy would be proportional to the inverse lattice spacing, and the monopole would vanish in the continuum limit $a \to 0$.

We use link variable to represent the gauge field in the lattice formulation, the link variables Ui is defined on links between the lattice points. In the compact formulation, the link variables can be constructed out of the continuum vector potential Ai as a complex number with unit norm.

$$U_i = \exp(iaeA_i) \tag{7.1}$$

The action is,

$$S = \beta \sum_{x} \sum_{i<j} P_{ij}(\mathbf{x}), \quad \text{where} \tag{7.2}$$

$$P_{ij}(\mathbf{x}) = U_i(\mathbf{x})U_j(\mathbf{x}+\mathbf{i})U_i^*(\mathbf{x}+\mathbf{j})U_j^*(\mathbf{x})$$

is the plaquette, path-ordered product of four link variables around an elementary closed loop. The magnetic field strength can be given as the complex phase of the plaquette. Substitute Eq(7.1) into Eq(7.2), we obtain

$$P_{ij}(\mathbf{x}) = \exp[iae(A_i(\mathbf{x}) + A_j(\mathbf{x}+\mathbf{i}) - A_i(\mathbf{x}+\mathbf{j}) - A_j(\mathbf{x}))] \qquad (7.3)$$

Consider for each elementary closed loop, we can rewrite the ordered path in terms of magnetic flux. Because, as we have seen in Maxwell's formulation, the flux is related to the curl of the field it is analogous to the path around the elementary loop. Therefore, Eq(7.3) can be written as:

$$P_{ij}(\mathbf{x}) = \exp[iea(a\epsilon_{ijk}B_k)] \qquad (7.4)$$

In three-dimensions, the total charge inside each lattice cube would be equal to the total flux coming out of that cube, which is simply the sum of the flux of six plaquttes on the sides. The main advantage of this treatment is that the finite lattice spacing would provide an ultraviolet cutoff leads to a finite value in the monopole mass. Note that in the continuum limit where the lattice spacing $a \to 0$, the energy vanishes, indicating the monopole inside the cube would disappear.

7.1.1 Perturbative calculation

For most theoretical studies in physics, one can often rely on the methods of perturbative approach. One can often deduce an approximate solution to the problem at some levels argue that this would be the classical approximation of the full solution. Then, we can treat the corrections as small perturbations and can often add to the approximated solution linearly. This approach does not work well in the full 't Hooft-Polyakov monopole solution. The calculations of semiclassical quantum corrections of the monopole are difficult and even the value of the classical quantities are hard to calculate.

To see this, let us consider the classical solution of the field equation (4.11), which can be solved analytically up to the form:

$$\phi^a = \frac{r_a}{gr^2}H(gvr) \qquad (7.5)$$

$$A_i = -\epsilon_{aij}\frac{r_j}{gr^2}[1-K(gvr)] \qquad (7.6)$$

where the values of functions $H(\xi)$ and $K(\xi)$ are determined by the specific Higgs fields that breaks the symmetry and the coupling strength of the coupled fields. These values can only be obtained numerically. Once these values are worked out, it can be integrated in the energy functional to get the total energy of the configuration, which is effectively the mass of the particle. The mass of the topological soliton is considered the most natural quantum observables [49].

The classical monopole mass then has the form

$$M_{cl} = \frac{4\pi m_W}{g^2} f(z) \tag{7.7}$$

where $f(z)$ is a function of $z = m_H/m_W$. Trivial solution is when the condition $f(0) = 1$ is satisfied [58][59]. Physically, it means that in the case of a massless Higgs field, the classical monopole mass gets reduced to

$$M_{cl} = \frac{4\pi m_W}{g^2}, \tag{7.8}$$

proportional to the mass of the exchange bosons corresponding to the model. In practice, the value of the function $f(z)$ has been calculated to high accuracy. And the asymptotic expressions for small and large values of z have been found [60][61][62]. For small-z, we can write it as an expansion as

$$f(z) = 1 + \frac{1}{2}z + \frac{1}{2}z^2(\ln 3\pi z - \frac{13}{12} - \frac{\pi^2}{36}) + O(z^3) \tag{7.9}$$

For large z, they found:

$$f(z) = 1.7866584240(2) - 2.228956(7)z^{-1} + 7.14(1)z^{-2} + O(z^{-3}) \tag{7.10}$$

In the quantum mechanical treatment, one may refer the mass of the soliton as the energy difference between sector $Q = 0$ and $Q = 1$, as we have seen in section 6.1. If the perturbation theory works, we need to, Firstly, find the classical solution $\phi_0(x)$, and assume the quantum correction is small enough to be considered as fluctuations $\delta(t, x)$ around the classical solution. Then, the whole solution can be written as:

$$\phi(t,x) = \phi_0 + \delta(t,x) \tag{7.11}$$

We would now discuss the calculation of the process to leading order [63][64]. Firstly, the higher terms of the lagrangian must be ignored and the correction field is of the form $U(\delta) = \frac{1}{2}V''(\phi_0(x))\delta^2$, which is an harmonic potential. For this approximation, the energy level would be given by the solution of the eigenvalue equation. The one-loop correction is then of the form

$$M_{\text{1-loop}} = M_0 + \frac{1}{2}\sum_{\text{har}}(\omega^1_{\text{har}} - \omega^0_{\text{har}}), \tag{7.12}$$

where ω^1_{har} is the energy levels in $Q = 1$ sector, and ω^0_{har} corresponds to the trivial vacuum. The calculation is difficult but still possible, in theory, and it has been done in the $1+1$ $\lambda\phi^4$ kink model [65]. The calculation for monopole would follow the same procedure but with many extra complications. Technically, for start, the background solution is not known except for the case near BPS limits, even then, the eigenvalue equation does not have analytical solution [66]. It is also hard to maintain the gauge or rotational symmetry due to the ultraviolet divergence. Although fortunately for the monopole, the renormalization problem is not in the worrying list. The one-loop expression would automatically cancel the running of the couplings[67]. Till the date of this review, only the leading logarithmic quantum correction near the BPS limits has been calculated and even in that case, one can spot an interesting result of its logarithmical divergence. The reason for this is the effect named after Coleman and Weinberg [68], which, for short, makes it impossible to reach the BPS limit in any quantum theory. The divergence term $\phi^4 \log\phi$ also restricts the stability condition of the monopole and we are forced to work with the constraint $m_H \gtrsim g m_W$. This means we can only work with extremely weak coupling and the whole quantum correction would become too small to measure at all! Furthermore, we are left with a mass hierarchy and to overcome it one need an even larger lattice.

Up to this point of the discussion, we could see that although, in theory, we can have perturbation theory working to some very restrictedly conditioned cases, the perturbative approach is not suitable for the monopole model. We shall then turn our attention towards the non-perturbative theories.

7.2 Non-perturbative studies

The conventional methods of lattice simulation are to consider the creation and annihilation operators of a topological defect. It was a popular choice because it follows the spirits of lattice Monte Carlo simulation and if we adapt the analogy between phase transition and topological defect, one can use the expectation value as a order (disorder) parameter [70][71][72][73]. However, this type of treatment seems not so fit to the monopole studies. To start with, these formulations are very complicated. If we consider one cube cell in a large lattice space, we need a pair of creation and annihilation operators inside the volume and a path connecting them can be seen as the world line of the particle. One can immediately spot the problem as one can take a 'time slice' of the space at some specific time and it would not have to be magnetically neutral charged. In order to solve the problem, we need, at least, add another pair of operators inside the cube cell and propagate in the opposite direction to cancel the magnetic charge. To make the situation more complicated, our monopole has an infinitely ranged magnetic field created with the monopole particle sitting in the middle and affect through out the whole lattice space. Against our favor, one can even argue that the expectation value is ill-defined as it would always vanish because of the finite lattice size.

After all, although in theory, this approach can provide the test ground for a very large range of observables and the ideas behind the theory are straight forward, its complication is the vital factor to limit its own development. Besides these technical problems, the way that the theory introduces the monopole solution is also not satisfactory. The sketching of the idea can be seen in Figure 3. By putting operators into the cube cell, it would force the monopole to be in the system and the non-zero magnetic charge is the structural feature very artificially applied to the system. Recall the 't Hooft-Polyakov theory in section 4, the monopole should arise as a general result of spontaneous symmetry breaking. In other words, its appearance should be, at its very basic, natural and much less artificial.

For the rest of this thesis, we would consider a relatively new approach to the theoretical study of the monopole [74]. The technique used in this method has a very close analogy to the original construction of the 't Hooft-Polyakov monopole. In the 1974's papers, the major, if not the only, modification required for the monopole solution is the non-trivial boundary condition at infinity. The monopole construction method used in this method requires nothing but a boundary condition which is analogous to the that in the orig-

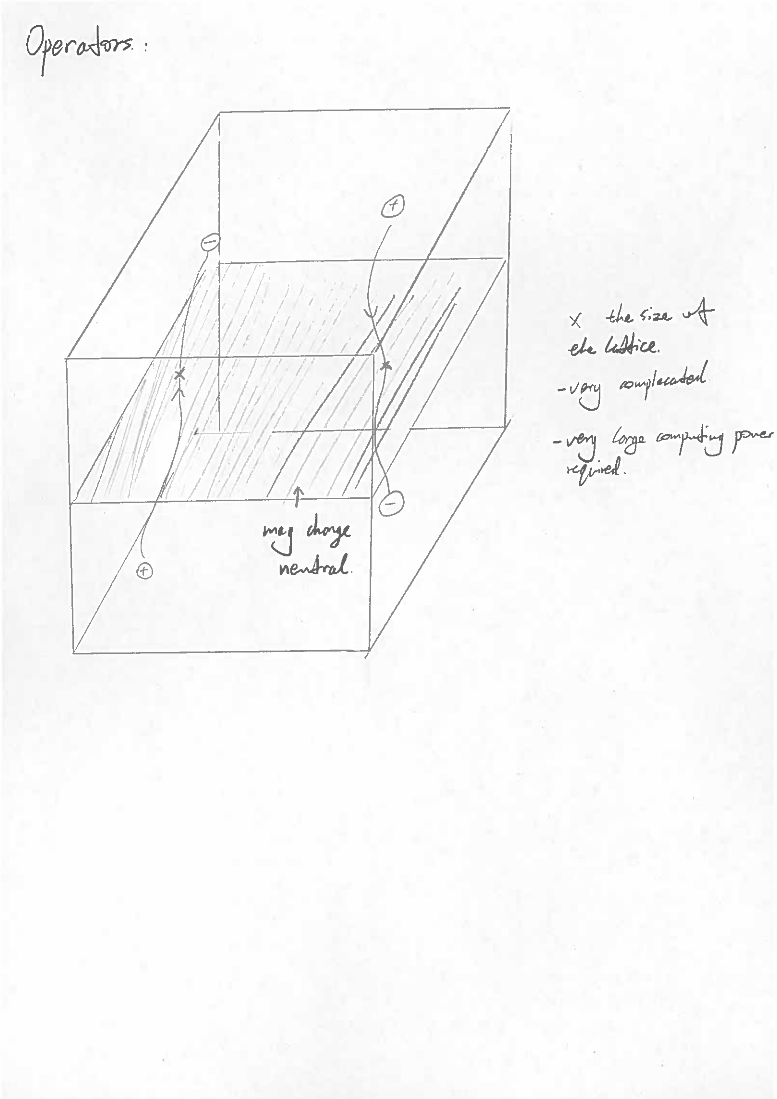

Figure 3: Sketching of proposed operators in one cell cube

inal papers called the twisted C-boundary condition.

However, before we start to look at this nice theory, we need to do some preparation and first of which is the discretization of the continuum magnetic charge.

7.2.1 Lattice discretization

In continuum, we have seen the way to view the monopole configuration as soliton solutions in terms of the topological stability. The transformation into the discrete space-time is everything but trivial. Even in the most fundamentals of the theory, the 4D lattice space-time, on its own, has a completely different topology to that of the continuum theory.

Let us consider how do we normally describe a quantum system. Instead of a field equation or field configuration, a density operator often characterizes the system. Analogous to those of the statistical mechanics and thermodynamics, we can use the ideas of ensemble of configuration and partition function to describe the system. The partition function is of the form:

$$Z = \int D\phi \exp(-S), \tag{7.13}$$

where S is the action. A standard technique is to carry out a Wick rotation, and change the quantum system into the Euclidean space. The physical interpretation of the partition function is that of a path integral. It is essentially integrating over a large number of field configurations, almost none of which are the solutions of field equation. The discrete analogy of the topological parameter in the soliton argument is the winding number and for the whole system, the winding number is well defined [74]. In cases of zero winding number, although there can still be localized object that behave in all ways like a topological defects, the overall number of configurations correspond to both positive and negative, or anti, winding numbers must be equal, and hence neutralize the whole system. If the non-zero value of the winding number occurs, we know that apart from those defect-antidefect pair, we have in the system a true defect.

We can then restrict ourselves to the cases where the total winding number is a constant. This in turn, changes the type of the ensemble to the micro-canonial Z_q, where we q is the total winding number. We can then consider the free energy of the given ensemble characterized by q as,

$$F_q = -\ln Z_q \tag{7.14}$$

The free energy difference between F_1 and F_0 is exactly the classical mass of the defect. The difficult is that in lattice simulations, one can measure neither the partition function nor the free energy. Instead, the only possible measurement is the expectation values correspond to a specific operator and can be written as

$$\langle \hat{\mathscr{O}} \rangle = Z^{-1} \int D\phi \hat{\mathscr{O}} \exp(-S) \tag{7.15}$$

7.2.2 Magnetic charge in lattice

Let us then consider the lattice version of section 6.2. Similarly, we would start by defining the general SU(2) gauge group coupled to a scalar field Φ. What we need is a expression of remaining $U(1)_{em}$ after the SU(2) gauge was partially broken by the scalar field Φ. We would derive the topological charge by looking at the time slice of the 4D lattice. As shown in the equation (7.1), the gauge field in the continuum theory can be expressed as link variables $U_i(\mathbf{x})$, which are SU(2) matrices defined on links $(\mathbf{x}, \mathbf{x}+\mathbf{i})$. The scalar field Φ is defined on the lattice sites.

The symmetry and invariance of the lagrangian must be preserved in the lattice theory, hence the fields are invariant under gauge transformations $\Lambda(\mathbf{x})$, a SU(2)-valued function defined on the lattice.

$$\Phi(\mathbf{x}) \to \Lambda^\dagger(\mathbf{x})\Phi(\mathbf{x})\Lambda(\mathbf{x}), \tag{7.16}$$

$$U(\mathbf{x}) \to \Lambda^\dagger(\mathbf{x})U(\mathbf{x})\Lambda(\mathbf{x}+\mathbf{i}). \tag{7.17}$$

The next step is to make a gauge transformation that diagonalizes Φ, and confirm that it is an Abelian gauge transformation. Recall the trick we have used in the comtinuum case, and we can apply it here since the trick is still valid in the lattice formulation. First, we notice that the gauge Λ can be chosen in such way that it would transform Φ to the z-direction and hence it would have zero measure in the partition function. We can use this property to define the unit vector

$$\hat{\Phi} = \Phi(\Phi^2)^{-1/2} \tag{7.18}$$

Use the definition above, we can therefore have a gauge transformation into the unitary gauge. To see this, denote the gauge transformation that diagonalizes Φ.

$$R(\mathbf{x}) \propto i(\sigma^3 + \hat{\Phi}(\mathbf{x}) \tag{7.19}$$

$$R(\mathbf{x})_\Lambda \propto i(\sigma^3 + \Lambda^\dagger \hat{\Phi}(\mathbf{x})\Lambda) \tag{7.20}$$

We are now ready to define the residual Abelian gauge transformation as,

$$\tilde{\Lambda} = R^\dagger \Lambda R_\Lambda \tag{7.21}$$

We perform the transformation on link variables,

$$\tilde{U_i} = R^\dagger(\mathbf{x}) U_i R(\mathbf{x} + \mathbf{i}) \tag{7.22}$$

Then the transformed link variables is invariant under the transformation induced by $\tilde{\Lambda}$,

$$\tilde{U_i} = \tilde{\Lambda}^\dagger(\mathbf{x}) \tilde{U_i}(\mathbf{x}) \tilde{\Lambda}(\mathbf{x} + \mathbf{i}). \tag{7.23}$$

This new transformation is unitary with its determinant equal to one. It is also diagonal, according to the construction we proposed to Λ. Combine all the restrictions and we can now state that the transformation is of the form

$$\tilde{\Lambda} = \exp(i\lambda\sigma^3) \tag{7.24}$$

The next step is to get an expression for the magnetic charge density in the lattice. For simplicity and to make a clearer connection to the later section when we construct the 't Hooft-Polyakov monopole in the lattice, we would carry out our discussion based on the discrete version of the lagrangian (6.6). The method of discretization is standard and after the Wick rotation to Euclidean space, we can write it as,

$$\begin{aligned} \mathscr{L}_E = \quad & 2\sum_\mu [\text{Tr } \Phi(\mathbf{x})^2 - \text{Tr } \Phi(\mathbf{x}) U_\mu(\mathbf{x}) \Phi(\mathbf{x} + \mu) U_\mu^\dagger(\mathbf{x})] \\ & + \frac{2}{g^2} \sum_{\mu<\nu} [2 - \text{Tr } U_{\mu\nu}(\mathbf{x})] + m^2 \text{Tr } \Phi^2 + \lambda \text{Tr } (\Phi^2)^2 \end{aligned} \tag{7.25}$$

where the plaquette $U_{\mu\nu}$ is defined as

$$U_{\mu\nu}(\mathbf{x}) \equiv U_\mu(\mathbf{x})U_\nu(\mathbf{x}+\boldsymbol{\mu})U_\mu^\dagger(\mathbf{x}+\boldsymbol{\nu})U_\nu^\dagger(\mathbf{x}) \tag{7.26}$$

Compare to the continuum version, we can roughly thought of the link variables U_μ as to $\exp igA_\mu$. And we can make use of the unit vector we have defined before to get a projection operator $\Pi_+ = (1+\hat{\Phi})/2$, hence use the operator to define a projected link variable

$$u_\mu(x) = \Pi_+(x)U_\mu\Pi_+(x+\hat{\mu}) \tag{7.27}$$

Physically, this is the compact Abelian gauge field corresponds to the unbroken U(1) subgroup. Therefore, we have the Abelian field strength tensor of the form

$$\alpha_{\mu\nu} \equiv \frac{2}{g}\arg\mathrm{Tr}\;\; u_\mu(x)u_\nu(x+\hat{\mu})u_\mu^\dagger(x+\nu)u_\nu^\dagger(x) \tag{7.28}$$

and the obtain the lattice version of the magnetic field as $\hat{B}_i = \frac{1}{2}\epsilon_{ijk}\alpha_{jk}$. Take its divergence we then obtain the lattice magnetic density

$$\sum_{i=1}^{3}[\hat{B}_i(x+i) - \hat{B}_i(x)] \in \frac{4\pi}{g}\mathbb{Z} \tag{7.29}$$

One thing worth mention is that its already quantized. By the virtue of our construction, the magnetic field would be well-defined and gauge-invariant, and automatically conserved and fit to our requirement for a real physical entity.

7.2.3 Boundary conditions

We have set the frame-work for which the continuum 't Hooft-Polyakov monopole theory can be simulated in the lattice modification. Both the vector fields and the scalar fields have been well defined in term of the lattice theory. However, the big question still remains, which is how do we trigger the monopole formation?

In the original theory, this was achieved by setting the special boundary conditions at infinity. And we have noted in the previous section that the total magnetic charge inside a finite volume is obtained by a path integration over the boundaries. Therefore, the task left for us is to find the lattice boundary condition analogous to that in the continuum formulation and in such way it can fix the total charge. One natural choices in general lattice theory is to set the periodic boundary conditions. And it is a good place to start our discussion.

The partition function for each topological sector can be written in the form that is similar to the one people would often use in the statistical mechanics. Let us denote the length of the time in the Euclidean space-time by T. We can have the form,

$$Z_Q = \exp(-|N|MT)Z_0 \tag{7.30}$$

where N is the number of the monopole in the system and Z_0 is the partition function of the trivial vacuum sector. The value of M is the quantum mechanical mass of one isolated monopole and we can rearrange Eq. (**??**) to have a expression for a monopole mass in $Q = 1$ sector.

$$M_{Q=1} = -\frac{1}{T}\ln\frac{Z_1}{Z_0}. \tag{7.31}$$

In order to simulate the mass, we need to obtain an expression for the $Q = 1$ sector partition function and Z_0, and the mass would be the difference in the free energy between these two sectors. Unfortunately, the conventional periodic boundary condition does not yield possible solution. It does not even allowed non-zero magnetic charge within the system [76].

$$\Phi(\mathbf{x} + N\mathbf{j}) = \Phi(\mathbf{x}), \qquad U_k(\mathbf{x} + N\mathbf{j}) = U_k(\mathbf{x}). \tag{7.32}$$

The repeated lattice cells consists of one monopole and one anti-monopole would always be at the same number, hence fixed the total charge of the system to zero. It was reported [77], the simulation result would be largely subject to the size of the lattice on certain boundary conditions. It makes the result very restricted and even for the very large lattice size, one still cannot make reasonable comparison with the classical result. However, we can still learn something from it. The periodic boundary condition guarantee that the actual boundary is not physical as it would not have finite-size effect to the physics. This is useful because if anyone wants to determine the monopole mass, they need to measure the free energy. The contribution of the finite-size effect to the free energy can possibly dominate over its actual value. However, we also know that we do not necessarily need the complete periodic condition. We may only require to have the boundary conditions periodic up to the symmetries of the lagrangian. Among which, the no physical boundary constraint would be preserved if one can maintain the translation invariance, it is easy to understand as we want to treat the cells in the lattice identically.

Therefore, we look for candidate that satisfied the requirement. Recall the classical monopole solution of the field equations in continuum (4.11).

$$\Phi(\mathbf{x}) \approx \frac{x_k \sigma_k}{r}, \qquad A_i(\mathbf{x}) \approx \frac{\epsilon_{ijk} x_j \sigma_k}{2r^2} \tag{7.33}$$

Consider the adjacent cell cubes and if we move from one cell to another we reverse the sign of the spatial coordinate, we can then write the set of fields transformations:

$$x_j \to -x_j : \Phi \to -\sigma_j \Phi \sigma_j, \qquad A_i \to \sigma_j A_i \sigma_j, \tag{7.34}$$

which is relatively straight forward to write in the lattice terms. It suggests:

$$\Phi(\mathbf{x} + N\mathbf{j}) = -\sigma_j \Phi(\mathbf{x}) \sigma_j, \tag{7.35}$$

$$U_k(\mathbf{x} + N\mathbf{j}) = \sigma_j U_k(\mathbf{x}) \sigma_j \tag{7.36}$$

The effects of these boundary conditions are easily spotted as in the cases of the projection $\Pi_\pm$. This implies $\Pi_+(\mathbf{x} + N\mathbf{j}) = \sigma_j \Pi_-(\mathbf{x}) \sigma_j$ and the most direct observables are the lattice field strength tensors. They have the boundary conditions of the field strength tensor.

$$\alpha_{ij}(\mathbf{x} + N\mathbf{k}) = -\alpha_{ij}(\mathbf{x}) \tag{7.37}$$

The change in the sign when we apply the boundary condition means the direction of the magnetic flux is reversed at the boundary. Physically, it means the boundary connects two cells each is the charge-conjugated copy of the other. This is, however, not sufficient to suggest non-zero total charge of the whole system. The arguments become the choice of the specific path integral along the lattice boundary for which the details are stated in [74]. It was shown in that paper, for some specific choice of boundary conditions, there will be a flux π through each of the halves of the boundary and from the boundary condition (7.37) we can get a total flux of 2π.

The result may not be as excited as it may look at the first glance. The flux is only limitedly defined modulo 2π, and the most it can achieve is to force the magnetic charge to be either even or odd.

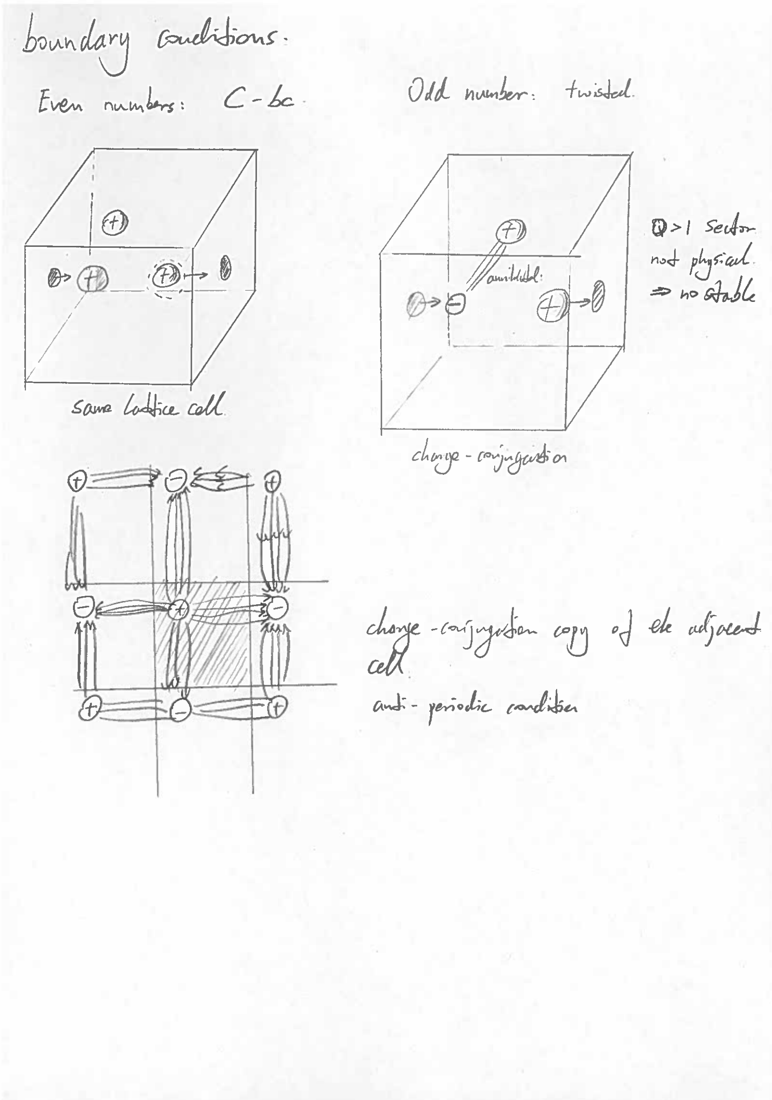

Figure 4: Sketchings to explain the difference between C-boundary and the twisted boundary

It was then another candidate, which is the C-boundary conditions proposed by Kronfeld and Wiese [78]. It states

$$\Phi(\mathbf{x} + N\mathbf{j}) = -\sigma_2 \Phi(\mathbf{x}) \sigma_2, \tag{7.38}$$

$$U_k(\mathbf{x} + N\mathbf{j}) = \sigma_2 U_k(\mathbf{x}) \sigma_2 \tag{7.39}$$

It is easy to spot the similarities between these two conditions. And the former one was known as the "twisted" C-boundary conditions. The twisted condition does not restrict itself to σ_2 although it can be sort locally as the same condition. It turns out the C-condition would only allow even value of the magnetic charge.

Both conditions seem to lack, to some extent, the certainty in the absolute value of the magnetic charge. However, if we consider the partition functions corresponding to each of those conditions we can get the one-monopole solution out of it. The configuration follows that for any non-zero monopole mass and provided the time interval is large enough, the partition functions are suppressed to their minimum possible number of monopoles.

$$Z_C = \sum Z_{2k} = Z_0(1 + \hat{\mathscr{O}}(e^{-2MT})) \tag{7.40}$$

$$Z_{tw} = \sum Z_{2k+1} = Z_1(2 + \hat{\mathscr{O}}(e^{-2MT}) \tag{7.41}$$

In Z_C where only the even number are allowed, the $N = 0$ will dominate the result. In Z_{tw} case, the partition function is dominated by the one-monopole solution!

7.2.4 Measurable quantities

Once we have constructed the lattice in such way, we are guaranteed to get non-zero magnetic charges in the system. However, it is not straightforward to pull the desired information out of the simulation. This is due to both the properties of the lattice Monte Carlo method [79] and the way how we approached the problem.

Prior to our discussion, we need to figure out what can we extract out of the system. Unlike the operator approaches, as we have mentioned in the beginning of this section, which in theory should provide all the information there is about the system. The twisted boundary condition methods rely on the measurement of the configuration ensemble, in particular, the partition function and its corresponding free energy. Up to the date of this review, only the masses of the monopole [63] and information about its form factor [81] have been studied in the deeply broken phase with weak coupling. We shall start with the mass result.

1. **Mass of the monopole:**
   This is naturally proposed as the easiest, yet not in technical terms, and the most straightforward observable of the soliton monopole. Once its value in the quantum theory has been determined non-perturbatively, one can compare its value to that in the classical limit. The quantum equivalent of the classical limit is deep in the broken phase, where the mass parameter m^2 in the lagrangian becomes much smaller than zero.

   We shall only continue our study on the partition functions and read off the expression of the free energy difference in two not continuously connected soliton sectors arisen from our choices of twisted or C-boundary conditions. It is easy to see,

$$-\frac{1}{T}\ln\frac{Z_{tw}}{Z_C} = M - \frac{\ln 2}{T} + \hat{\mathscr{O}}(e^{-2MT}) \tag{7.42}$$

   In the limits $T \to \infty$, we have he above equation equal to monopole mass. However, this is not useful for practical uses as neither partition functions nor free energy are directly measurable in Monte Carlo simulation. Furthermore, because we have applied different boundary conditions on the partition function, we cannot even write their ratio as expectation value. We can, nevertheless, define a new integration variable in the twisted case in such way that it satisfy the C-condition. This is achievable because there are locally the same. After doing that, we can write the action $S \to S + \Delta S$. The expectation value of the ratio then becomes $Z_{tw}/Z_C = \langle \exp(-\Delta S)\rangle_C$ and perfectly defined in the C-boundary condition. In practice, though, it has also little uses because the overlap with the vacuum is very small and extremely high statistics are necessary to even get any meaningful result.

   In comparison, another approach makes much more sense in practice. Let us differentiate the mass with respect to m^2.

$$\frac{\partial M}{\partial m^2} = L^3(\langle \mathrm{Tr}\ \Phi^2\rangle_{tw} - \langle \mathrm{Tr}\ \Phi^2\rangle_C) \tag{7.43}$$

If we can choose a large value of m^2 where we are definitely in the symmetric phase, where the monopole mass vanishes. We can then integrate this equation to any point where the parameter value, m^2, is very defined. Note that, in theory we can choose any point to start the integration, another obvious choice would be the classical mass limit. We do, however, expect a phase transition when the symmetric phase got broken and at which point the derivative would be singular. The actual simulation using this method has been carried out in [63], and the viewer can refer to the paper for more details on the simulation.

2. For any local operator $\hat{\mathscr{O}}$, the form factor is defined as

$$\mathrm{f}(\mathbf{p_2}, \mathbf{p_1}) = \langle \mathbf{p_2} | \hat{\mathscr{O}}(0) | \mathbf{p_1} \rangle \tag{7.44}$$

We can easily relate the form factor to the scattering amplitude between the monopole and the particle corresponds to the creation operator $\hat{\mathscr{O}}$, and hence this can be used to study the interaction of the monopole with other particles. The theory can provide, by the virtue of twisted boundary conditions, useful information about the real physics of the monopole and guide the potentially accessible experiment observations.

Classically, the form factor is given by the Fourier transform of the classical profile of the operator in the monopole configuration. This means that once again we could compare our quantum calculation with classical solution in cases of weak coupling. Similar calculation has been carried out in 1+1 kink model where the precise classical solutions are known [82]. The real difficulties in this approach are the ways to define good operators with correct boundary conditions in the monopole configuration. Studies has been extended to the SU(N) adjoint Higgs [83] cases and one could, in theory, obtain a numerical approximation of the operators by a correct diagonalization. However, it is interesting that although the standard diagonalization can always been performed, the choice of path integral could not be agreed in all cases to guarantee the monopole formation except of the SU(N) cases where N is even. Recall that the important discovery in the monopole history is for the GUT monopoles where the proposed symmetry group was SU(5). So the theory in continuum does allow monopole solution and has nothing to prevent the interaction with other particles. Whether this is due to the limitations of the lattice simulation or there is something deep in the theory remain an open question stand with many others, suggesting the potential further studies on this fascinating subject.

8 Summary

I have reviewed the development of the magnetic monopole theory over years of development and it was really amazing to see how little an idea it was yet to expand and influence so many aspects on modern physics.

It was long before any quantitative arguments about the monopole was proposed people have speculated that a bar magnet could consists of two "poles". Later when they found that it seemed that these poles could not be separated. Maxwell developed his set of equations, in which the magnetic Gaussian showed a vanishing divergence of the magnetic field indicating that the magnetic charge does not exist. It was mathematicians who found that the little "broken" symmetry of the theory could be restored by additional magnetic charges and currents. Probably inspired by the beauty of a restored symmetry and also by the virtue of quantum mechanics, Dirac showed that the new quantum theory at the time allowed the existence of monopole. Furthermore, its existence would explain the quantization of the electric charge. Later on, 't Hooft-Polyakov monopoles were to represented to be the necessary consequences of symmetry breaking. The hedgehog configuration is, in fact, the non-trivial $Q = 1$ sector of the soliton solution. Although the experimental prove of the real particles are yet to be confirmed, theoretical developments and the simulations showed great potential in many ways.

9 Discussion

It would be humble for me to present this review as a concluded even as an introduction to the subject. Along the writing of it, many ideas were considered yet had no sections to fit in the original document. We, therefore, included some of the interesting discussions in this section with the hope that some of those may inspire any of the further studies on the subject.

1. The real starting point of the quantitative study of the monopole is the paper Dirac presented in 1931. In his paper, he wrote down the quantization condition of which the product of the electric and magnetic charges must be quantized and being an integer multiple of some fundamental value (3.4). But interestingly, almost all the systems except that of Dirac's produce another quantization condition, Schwinger's condition, which gives precisely twice the value of the Dirac conditions. Many literatures and relative books have been reviewed yet it seems that even Schwinger himself could not provide a solid explanation. The most convincing idea may be the one suggested by Milton, that provide all the monopole species can be expressed as the Dirac type with a infinitesimal string attached to it. It can then be shown that the infinite string with open ends would satisfy the Schwinger's condition and that of the semi-infinite string would obey Dirac's condition. As illustrated in Fig 5. Then, I would like to suggest that the condition is depend on the actual symmetry of the monopole configuration with respect to the singular nodal lines attached to the configuration. However, other multiples of the Dirac's value has also been suggested in some models, where their arguments were based on the charge differences. And a conclusive explanation is yet to be found.

2. Besides the types of monopoles I have stated in this review, there are other possible species that have either theoretical or experimental importances to some extent, here I would state some examples of interest:

    - **Abelian monopole**
    These monopoles are not the consequences of non-abelian symmetry group spontaneously broken, but merely the concept analogous to the ideas of gluons in QCD to explain the phenomenon of confinement. We imagine to monopoles separated by a distance in a condensed matter system. The analogy of the confinement can be expressed if we can imagine the field lines connecting two charges for a magnetic flux tube. [96] The properties of this type

of object have been studied extensively, and can potentially provide good simulation methods and theoretical explanations to the confinement theory. See Figure 6 for an illustration.

- **Dyons**
  Consider the excited states of a ground state monopole in which it interacts with some energetic particles. Provided the coupling is strong enough, we can imagine the process in which the daughters can consist of a particle with electric charge and an effectively excited state of a monopole with electric charges. This object is called Dyon (Figure 7) and it is proposed [] that its formation can actually be simulated using the twisted boundary condition methods in lattice theory. The difficulties are once we have entered the strong coupling regime there is no existing theory to explain the physical interaction.

- **Supersymmetric monopole**
  These are the type of monopole that has been suggested in the supersymmetric theory or (super)string theory. The treatment is somehow much easier than that in the quantum theory since the quantum correction is trivial under supersymmetric conditions.

3. **Strong coupling simulation**
   We have attempted to adapt the theories that have been developed in [81] and tried to extend it to a strongly coupled system. However, because of the shortage of time and the complexity of the lattice Monte Carlo simulation, we failed to get any result of analytical importance. Nevertheless, the suggested behavior could still be useful in further studies. An ideal case sketching is attached in Figure 8.

4. **Mass scales**
   In the theory of 't Hooft – Polyakov monopole, we have two mass scales that can be related to the actual isolated monopole mass. In the classical solution, we have the mass of the interacting boson appearing explicitly in the equation. Yet the information about the Higgs mass is encoded in the function $f(z)$ (**??**). In the weak coupling or deep in the broken phase simulations, the values of both masses have been set to similar values in order to compare the result with previous reports. Since the treatment was non-perturbative, we do not have to restrict ourselves to those values. Simulations with various values of these mass scales could be carried out and by comparing the profile of the monopole and studying its spreading, we would be essentially probing the quantum structure of this objects.

A Figures and sketchings

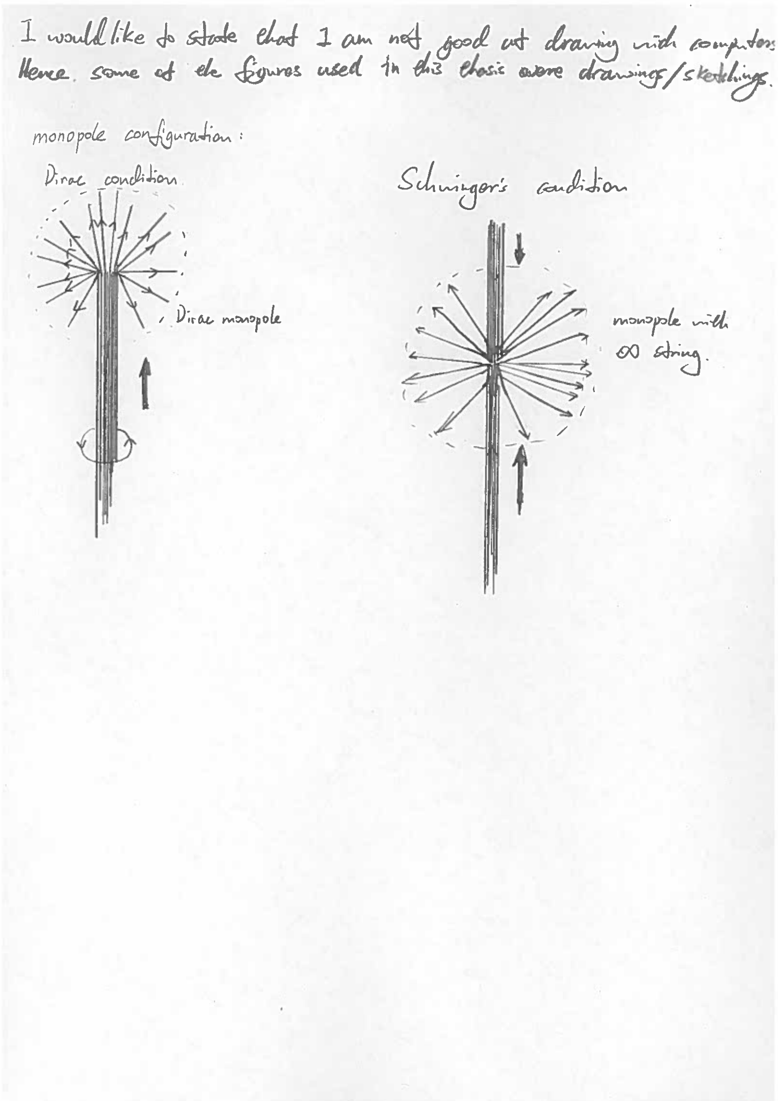

Figure 5: Sketching of different types of strings attach to a monopole

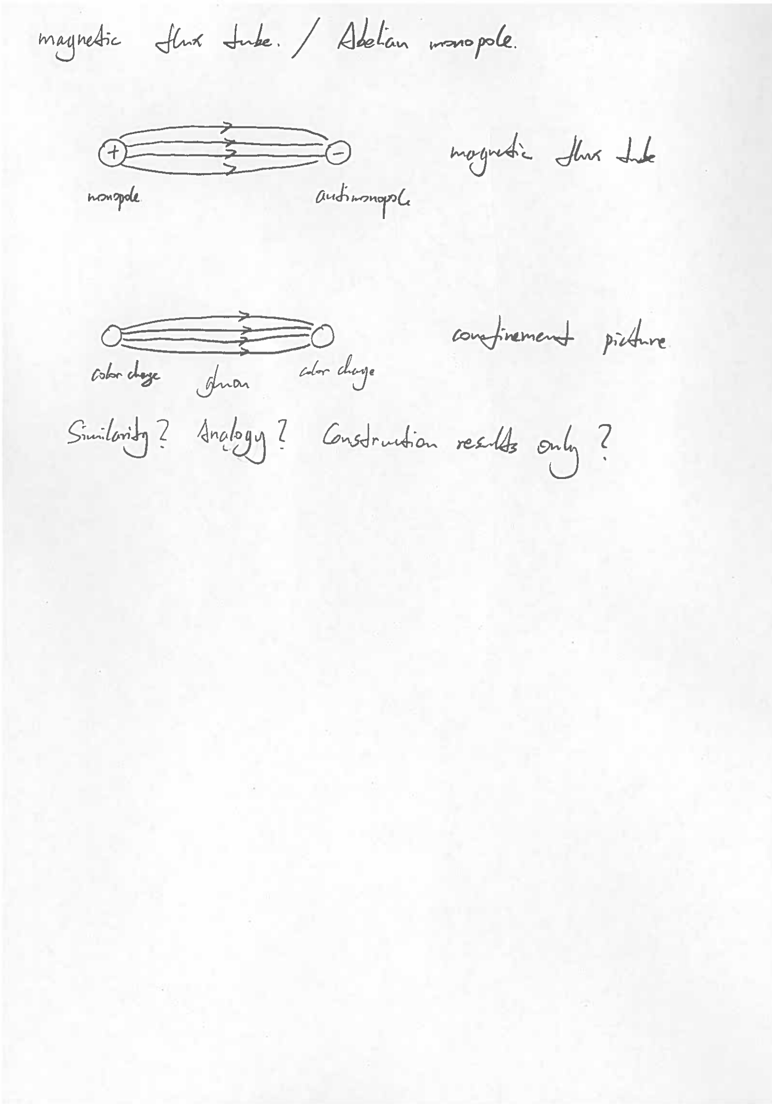

Figure 6: Sketching of flux tube in analogous to the confinement

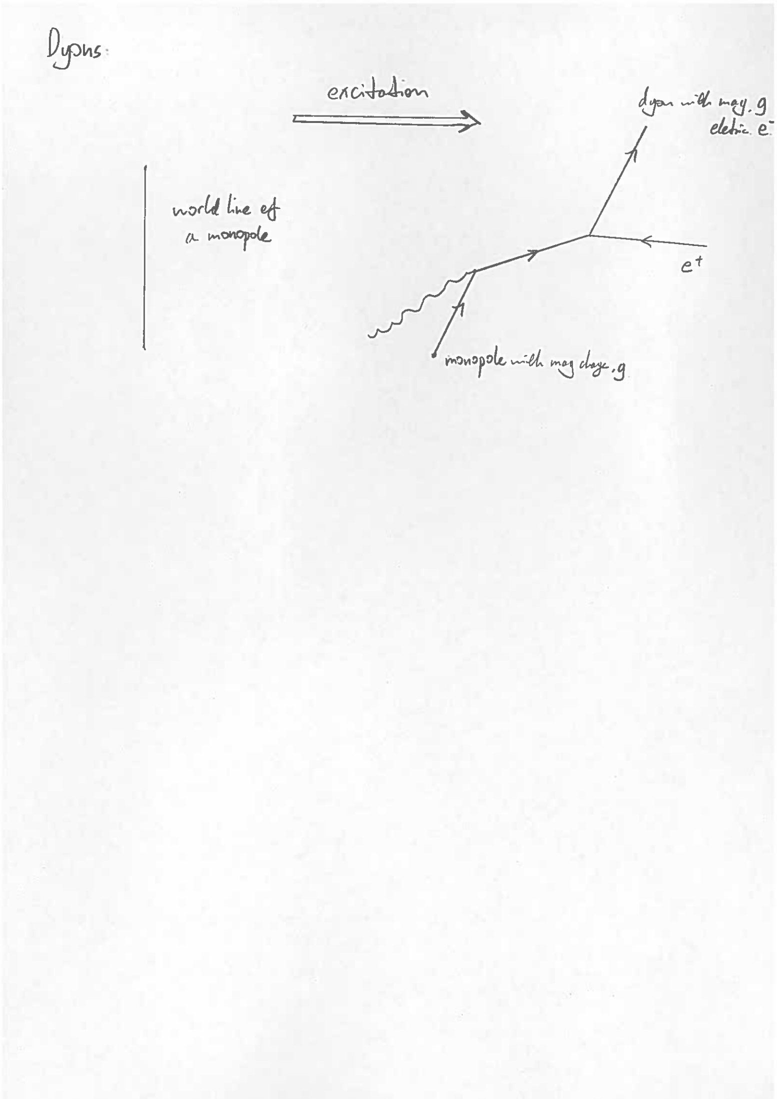

Figure 7: Sketching of the excitation of the ground state monopole to form a dyon and a particle with opposite electric charge

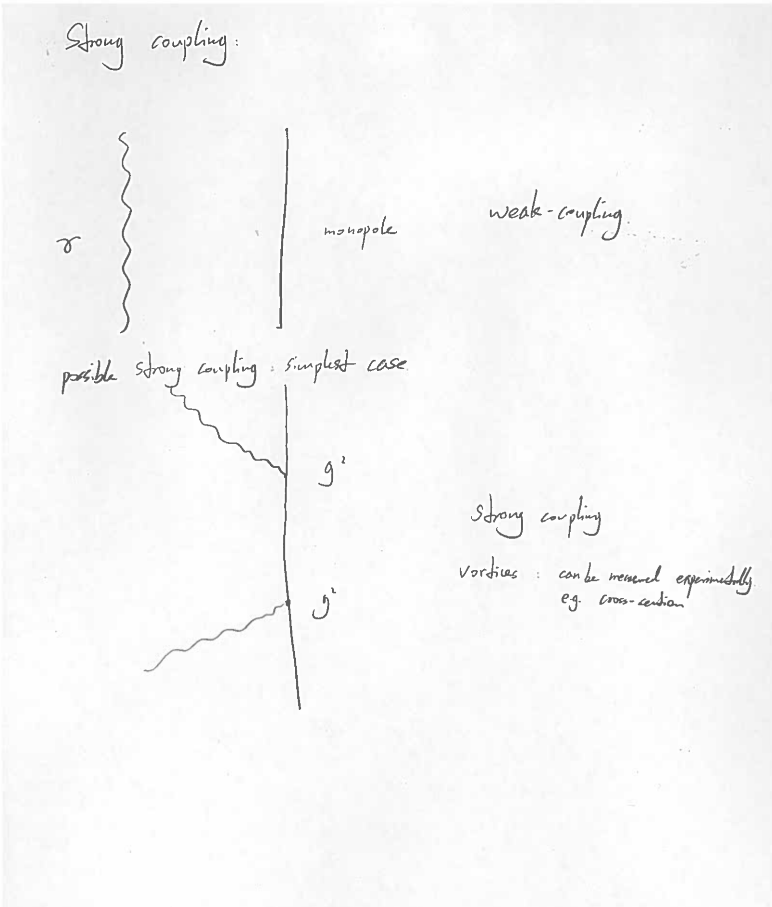

Figure 8: Sketching of the idealized strong coupling interaction

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