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The Causal Set Approach to Quantum Gravity

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Abstract

Causal set approach is an attempt to formulate a theory of quantum gravity. This article considers some fundamental issues of causal set theory, including its kinematics, dynamics and some applications (phenomenology), for example: fluctuations in the cosmological constant, entropy of black holes, etc.

Acknowledgments

Firstly, I would like to thank my dissertation supervisor Prof. Fay Dowker who provides me this interesting topic. Without her help, I cannot complete this paper.

Then I would like to thank my course tutor Dr. Tim Evans who helps me with the issues arose during the past academic year.

Contents

1 Introduction

1.1 Classical gravity

The problem of quantum gravity has become more and more interesting and challenging since the construction of both quantum mechanics and general relativity. In the narrow sense, it is just a problem of finding a method to incorporate these two theories. To solve this problem, many attempts and hypotheses have been proposed. And every approach which wants to answer this question needs to answer these two fundamental questions first: “What is quantum mechanics?” and “What is the underlying structure of spacetime?” The latter one on which I would like to focus in this article seems to be more important when we want to find a perfect theory of quantum gravity.

To answer the latter question, we need to go back to the classical theory which we are trying to quantize.

The classical general relativity has a threefold structure like many other physics theories, which are

“the kinematical part”, which tells us “What are we dealing with”,

“the dynamical part”, which tells us “How the physical substance behaves”,

“the phenomenological part”, which tells us “How this physical substance shows itself to us in an accessible way”.

As we have known in general relativity, “the kinematical part” contains a 4 dimensional differentiable manifold *M,* along with a Lorentzian metric g_{ab} on it, and of course a structure which is the so-called causal order relation. “The dynamical part” of general relativity is then manifestly the Einstein equation $G_{ab} = 8\pi T_{ab}$, where $G_{ab} = R_{ab} - g_{ab}R/2$ is the Einstein tensor. When the gravity can be ignored, the background spacetime would go back to the Minkowski spacetime as the metric go back to the η_{ab} (flat metric). Finally, the first two parts will manifest themselves as the phenomena of time, gravity, causality, etc.

1.2 Motivations for quantization

In seeking to reach a theory of quantum gravity, one might ask why we have to quantize the gravitational field. Here, I will present two main reasons.

The first motivation is the idea of unification. There are 4 fundamental forces in nature and three of them have been perfectly described by quantum theory. The theory which unifies them is the well-known Standard Model, including electroweak theory and quantum chromodynamics. Physicists believe that there exists a theory which can reconcile the quantum mechanics and general relativity. The second motivation comes from cosmology and black holes. Einstein's theory of relativity has proved to be successful in many physical areas except some extreme situations, such as the fundamental understanding of the early universe and the final stage of black hole evolution. The quantum fluctuations in these situations are inconsistent with smooth background (differential manifold) in general relativity.

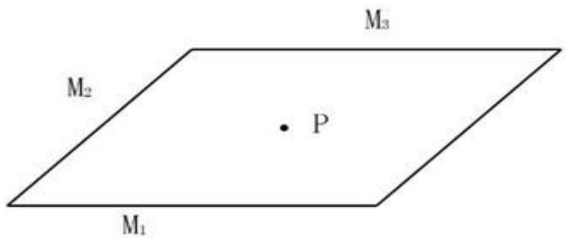
Accordingly, we need to find an approach to incorporate the quantum mechanics and general relativity.

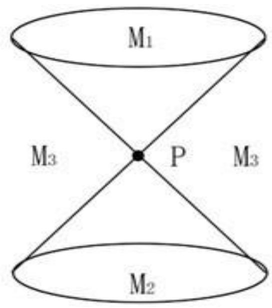
1.3 The causal set theory

In order to solve some well-known problems arises in looking for a theory of quantum gravity, many attempts have been made. Among them, the causal set program is such an attempt.

The causal set program is a theory that embodies the concepts of discreteness and causality. The view of causality is more important than space and time in history, and in general relativity, it becomes more important. Since the causal relation distinguishes the relativistic space and the Newtonian space. (Figure 1)

As to the concept of discreteness, it also has a long history along with the opposite notion of continuum. This concept is the core of quantum mechanics, since the word "quantum" comes from the discreteness of particle. In the recent times, more and more scientists tend to believe that the underlying structure, to which the continuous spacetime is only an approximation, is discrete.





Points in M_1 are to the past of P.
Points in M_3 are to the future of P.
M_2 is the absolute simultaneous surface of P.

Newtonian space

Points in M_3 are spacelike to P.
Points in M_1 are to the future of P.
Points in M_2 are to the past of P.

relativistic space

Figure 1

As these two essential concepts in the causal set theory, spacetime is then replaced by some discrete “elements” which are associated by means of “causal relations”. This is the so-called “partial order”. Of course, the useful notions in continuous spacetime (metric, topology, differential structure, etc.) are abandoned, according to the former statement, they are just approximate concepts to some more fundamental ones. The elements of causal sets have no inner structure. And all one can do is to count the number of elements.

In this paper, I will introduce several aspects of causal sets. In section 2, the origins and the definition of a causal set are given, along with the correspondence relation between discrete structure and continuous structure. Also some relevant topics are presented. This is the kinematics of causal set theory, if we want a perfect theory, the dynamics is necessary as we discussed earlier. So in section 3, a new dynamics for causal set theory is given. In section 4, I will set out the phenomenological part of causal sets. Some achievements and challenges are summarized in this section. In addition, I will discuss the “sum-over-history” approach used in causal set theory and give the reasons why we prefer it to the canonical approach (vector-state approach).

2 Causal Set Kinematics

2.1 Origins of causal set program

In order to demonstrate the origins of causal set idea, I want to quote R. D. Sorkin, who wrote in [2]:

"The tradition of seeing the causal order of spacetime as its most fundamental structure is almost as old as the idea of spacetime itself (in its Relativistic form). …, Robb presented a set of axioms for Minkowski space analogous to Euclid's axioms for plane geometry. In so doing, he effectively demonstrate that, up to an overall conformal factor, the geometry of 4-dim flat spacetime (which I'll denote by M4, taken always with a definite time orientation) can be recovered from nothing more than the underlying point set and the order relation $\prec$ among points (where $x \prec y \Leftrightarrow$ the vector x to y is timelike or lightlike and future-pointing). … In a certain sense, however, these results appear to say more than they really do. Informally, they seem to tell us that M4 can be reconstructed from the relation $\prec$, but in actually carrying out the reconstruction (see below), one needs to know what one is trying to recover is a flat spacetime not just a conformal flat one. Clearly, there's nothing in the relation $\prec$ per se which can tell us that. … But it shows up still more clearly with the curved spacetime of GR, where the natural generalization of the flat space theorems is that a Lorentzian geometry M can be recovered from its causal order only up to a local conformal factor.

Notice that when one says that a Lorentzian manifold *M* is recovered, one is talking about all the mathematical structure that go into the definition of a spacetime geometry: its topology, its differential structure and its metric. … The upshot of all these reconstruction theorems is that, in the continuum, something is lacking if we possess only the causal order, namely the conformal factor or equivalently the "volume element" $\sqrt{-g}d^4x$. … The causal order alone, however, is – in the continuum – incapable of furnishing such a measure."

From the statement above, one realizes that the continuum is not the fundamental structure, there must be something instead. A lot of clues argue that the candidate which can replace the continuum is discreteness. Lacking of a proper cut-off leads to the problem – the infinities of general relativity and quantum field theory. Although the renormalization method ameliorates this problem in QFT, it is still difficult to work with the gravitational field. The technical problems of the definition of a path-integral on a continuous space have never been solved. There are also some problems with the history space of Lorentzian manifold. And a discrete history space solves this problem.

Another argument comes from the black hole's entropy. In [3], J. Henson demonstrated that, with no short cut-off, the "entanglement entropy" of quantum fields seems to be infinite. If one adds this entropy to the black hole's entropy, then a short-distance cut-off of Planck scale must be introduced. This, along with some other analyses, leads to the result that discreteness is unavoidable under Planck scale.

Except causal set program, some other theories of quantum gravity, such as loop quantum gravity, also possess a discrete structure. As J. Henson referred in [3], there are some interesting results come from "analogue model". These models are such ones where objects similar to black holes can be mocked up in condensed state matter system. These analogies argue that there are some more fundamental theories of which the Einstein equation is only an equation of state.

Finally, introducing discreteness can be of great use. Since many problems, which cannot be solved in continuous spacetime, might be explained clearly in the discrete version. At this stage, one finds that the causal set idea, which combines the concepts of both causality and discreteness, is necessary to create a basic structure for a theory of quantum gravity.

Having been given all these reasons for discreteness and causality of spacetime, we are now at the stage to provide a precise definition for a causal set.

2.2 Definition of causal sets

Mathematically, a causal set (or causet for short) is a local finite partially ordered set (or a poset for short), meaning that a set of points C endowed with the relation $\prec$, which possess the following properties:

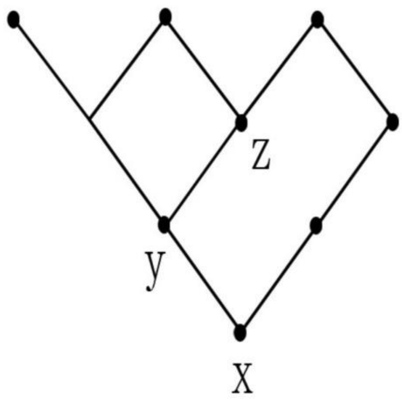
(1) Transitivity: $(\forall x, y, z \in C)\,(x \prec y \prec z \Rightarrow x \prec z)$

(2) Irreflexivity: $(\forall x \in C)\,(x \prec x)$

(3) Local finiteness: $(\forall x, z \in C)\,(\mathrm{card}\{y \in C \mid x \prec y \prec z\} < \infty)$

Where card C is the cardinality of C. The notation "$\prec$" has been explained earlier, for $x \prec y$, it denotes that x is to the past of y, and if two points cannot be connected by $\prec$, it means that those two points are spacelike to each other. The first two properties imply that there is no cycle $x_0 \prec x_1 \prec x_2 \ldots \prec x_n = x_0$. And the third property means discreteness.

Such a structure can be reflected in many different ways. First one is the tree level diagram (Figure 2). This is also known as the so-called Hasse diagram in which the elements of C are represented by dots and the relations between them are represented by lines. In order to simplify the whole graph, one does not need to draw the relations which are shown by the transitivity property. In [2], R. D. Sorkin proposed a second view to describe this structure. He treated a causal set as a matrix M (the "causal matrix") with the rows and columns labeled by the elements of C and with the matrix element M_{jk} being 1 if $j \prec k$ and 0 otherwise.



This is a tree level diagram (Hasse diagram) of a causal set. In this diagram, the dots stand for elements, and the lines stand for causal relations. Because of transitivity, the line between x and z has been ignored. And the elements at the bottom of the lines are to the past of the elements at the top of the lines.

Figure 2

The multiplicity of description for causal set makes it natural to use different ways to discuss the structure induced by the order relation $\prec$. For example, the relation $x \prec y$ can be considered as that x precedes y, that x is to the past of y (or y is to the future of x), that

x is an ancestor of y, or that y is a descendant of x. In particular, if x is an immediate ancestor of y (means that there is no z such that $x \prec y \prec z$), then x is a parent of y (or y is a child of x). For instance, in Figure 2, x and y are ancestors of z, x is a parent of y, y is a parent of z, but x is not a parent of z.

There are of course other interpretations for the relation $\prec$ except the one above. In [2], R. D. Sorkin suggested two. One is that a causal set of finite cardinality is equivalent to a T_0 topological space. This allows us to discuss causal set using the topological methods. The other one is that a causal set can be treated as a function by identifying C with the function that associates to each point $x \in C$ the set past(x) of all its ancestors.

As the basis of quantum gravity, a causal set is meant to be the deep structure of spacetime. Another way of saying this is that it is nothing but a macroscopic approximation to the continuous spacetime which cannot exist on small scales (Planck scale).

Now, for having a first sight of some future parts, one needs to focus on the volume element $\sqrt{-g}d^4x$. It has been proved that one can obtain all the information (differential structure, metric, topology, etc.) of a manifold, given only the volume element and the causal order on the points. However, the volume element cannot be recovered from a Lorentzian manifold. Things change in a discrete structure, one can obtain the volume elements just by counting the number of causal set elements. The slogan can be coined "Geometry = Order + Number", where order denotes to the causal order relations between elements and number denotes the amount of elements.

As discussed above, the definition and its interpretation of a causal set are given, in the meantime, these statements provide a starting point for the "kinematical part" of causal set theory.

2.3 The size of an element

Since the causal set elements have no size as what we usually define, such a question like "How big is a causal set element?" is actually meaningless. What we really want to know is the conversion factor v_0 given by $N = V/v_0$, where N is the number of elements of a causal set and V is the volume of the corresponding region. We hope that $N = V$, it is obviously that this can be satisfied only when we measure the length in units such that $v_0 \equiv 1$. One expects $v_0 \backsim (G\hbar)^2$ (where I have set $c \equiv 1$) using dimensional analysis. But we can do more than this. Consider the entropy of a black hole, $S = A/4G\hbar = 2\pi A/\kappa$, where $\kappa = 8\pi G$. From this formula, one can realize that one bit of entropy belongs to each horizon "plaquette" of size $\kappa\hbar$, and the finite size of these "plaquettes" reflects the underlying spacetime discreteness. The same conclusion comes from the discussion on the so-called entanglement entropy, which indicates that there exists an effective "ultraviolet cut-off" at around $l = \sqrt{\kappa\hbar}$. Another way which concerns the gravitational action-integral $\frac{1}{2\kappa}\int RdV$ gives the same result, $l = \sqrt{\kappa\hbar}$. [2]

Next consider the expression of l, one finds that when $\hbar \to 0$, $l \to 0$, this significant consequence implies that as the quantum limit returns to the classical limit, the discreteness will be back to the continuum, which agrees with our former conclusions that the continuous spacetime is just an approximation to the discrete one.

Now let us proceed to discuss the correspondence principle between discrete structure and the effective continuum description of spacetime.

2.4 An analogy: discrete matter

In many physical researches, it is helpful to introduce some analogous situations with which we are familiar when the original systems are hard to deal with. In general relativity, the fundamental structure of spacetime is a 4 dimensional Lorentzian manifold. However, in the theory of quantum gravity, it is no longer true, this kind of continuum description is only an approximation to a more basic structure. Before exploring the correspondence principle between these two structures, let us first look at a discrete matter model to which they are analogous.

Consider there is a box which occupied by some material substance. The background space is flat and the continuum description is now the mass density of this substance which is of course not the fundamental structure. And we want to know "what is the discrete object" in this model. In order to find this object, it is better for us to recall the critical step of the foundation of quantum mechanics (quantization). Analogously, one just needs to discretize this "mass density". Then, we should prove that this kind of discrete object is the fundamental structure, and we also need to prove that the continuum object is just an approximation to the discrete one and to provide a way of recovering it.

We are now at the stage to discretize the mass density. Initially, I give two assumptions: one is that this material substance is made of identical atoms. The other is the mass density can only be influenced by the number densities of these atoms. We then discretize the continuum object by distributing the atoms such that in a large enough space the number of atoms is proportional to the mass density integrated over that region. In the case of such an analogy, we do not need to give a precise definition of the atom mass since what we care about is the density (like we do not care the size of a causal set element, but its number), provided the mass is small enough so that the space between the atoms is much smaller than the scales on which the mass density varies.

There are many ways to discretize such a material substance. And one might ask “How to choose a better approach?” This depends on which properties of the continuum description we want to preserve and how effective the dynamics is.

Suppose that there is a continuum theory of material substance which possesses the Euclidean invariance property. And there are then many ways to discretize this model which will break the symmetry. In [4], F. Dowker suggested one. Firstly, one can divide the box into cubes small enough so that the density is approximately constant in each. And then, in every cube, put an atom at each vertex of a Cartesian lattice where the spacing lattice is chosen to be inversely proportional to the mass density in the cube. An atom state produced in this way is not invariant under Euclidean transformation (translation and rotation) because there are preferred directions. A “mass density” which is recovered from this discrete state is of course not invariant under Euclidean transformation, either. Here, I want quote F. Dowker, who wrote in [4]:

“If a fully fledged fundamental discrete theory is based on such lattice-like atomic states, this will show up in deviations from exact Euclidean invariance in the continuum approximation to this full underlying theory.”

In order to respect Euclidean invariance, there must be no preferred direction. To achieve this, one can distribute the atoms randomly, in other words, atomic positions should be chosen according to the Poisson distribution such that the expected number of atoms in a given region is equal to the mass in that region (in atomic mass units).

Finally, one would like to know whether the distribution of atoms is the underlying structure, and how to recover continuum theory of this material substance. To answer these two questions, we can use the discretization inversely: a continuum mass density is a good approximation to an atomic state if that atomic state could be a discretization of the mass density [4]. An important step cannot be neglected: one must check that if there are two continua which are approximate to the same atomic state, they must be close to each other.

After discussing this analogous situation, we have a brief understanding about the correspondence principle between discreteness and the effective continuum description. Now let us go on to see the causal set program itself.

2.5 The continuum approximation

In modern quantum mechanics (or quantum field theory), there must be a classical theory (or a classical field) when we talk about its corresponding quantum theory. In other words, this classical theory is an approximate theory to the quantum one. In the case of quantum gravity, we also need to find a program to which the general relativity is only a classical approximation. In this paper, the causal set theory is such a program, and now I will give the correspondence principle between them. In the matter analogy, the mass density is treated as the continuum (classical) object, and the atoms are its discrete (quantum) description. Armed with this model, we can then explore the similar relation between causal sets and the effective continuum description of spacetime.

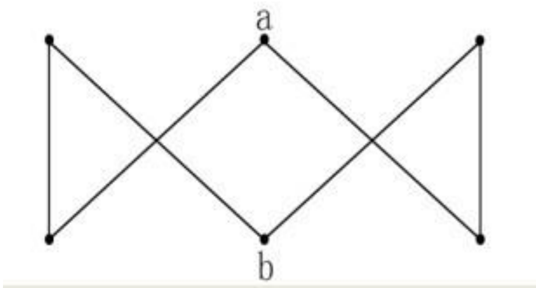
In the case of quantum gravity, the elements of causal sets play the role of atoms, and in general relativity, the volume plays the role of mass density. This volume is an integral of the volume elements over the region *R*, where the volume element is $\sqrt{-g}d^4x$, i.e. the volume is $\int_R \sqrt{-g}d^4x$. Now, we can discretize the continuum spacetime follow the same steps. Let us distribute the elements (atoms) in such a way that the number of them in a large enough region is the volume (mass density) of that region in fundamental volume units[1].

According to all these discussions, the correspondence principles can be stated as: the causal order of a causal set corresponds to the causal order of continuous spacetime, and the nu-

[1]The fundamental volume unit is proportional to l_0^4 (i.e. $v_0 \backsim l_0^4$), where l_0 is the length of an element (Actually, this length is of Planck scale).

mber of elements in a causal set corresponds to the volume of the corresponding region. This is just a rough definition, in order to obtain a more explicit interpretation for this approximation.

I firstly give the definition of embedding. A causal set C whose elements are points in a Lorentzian manifold (M, g_{ab}), and whose order is the one induced on those points by the causal order of that spacetime, is said to be an embedding of C into (M, g_{ab})[2]. [3] Not all causal sets can be embedded into all manifolds. In [3], J. Henson provided an example, the causal set in Figure 3 cannot be embedded into 1+1 dimensional Minkowski space, but it can be embedded into 2+1 dimensional Minkowski space. This means that there might be many different manifolds into which the same causal set can be embedded. However, not all these manifolds can be approximate to this causal set. For example, two manifolds must be similar to each other, if both of them are approximate to the same causal set (this will be discussed later).



This is a Hasse diagram of the “crown” causal set. This causal set cannot be embedded into 1+1 dimensional Minkowski space: if the above Hasse diagram is imagined as embedded into a 2 dimensional Minkowski spacetime diagram, the points at which elements a and b are embedded are not correctly related. In no such embedding can the embedded elements have the causal relations of the crown causal set induced on them by the causal order of 1+1 dimensional Minkowski space. The causal set can however be embedded into 2+1 dimensional Minkowski space, where it resemble a 3 pointed crown, hence its name. [3]

Figure 3

[2] It should be an embedding of the isomorphism class of the causal set. Ignoring the differences between isomorphism classes and particular instances of causal sets has few effects in this article.

In order to explain the discretization process specifically, one needs to introduce the concept of sprinkling which is a random selection of points from a spacetime according to Poisson process. As we expect, such a process is exactly Lorentz invariance in Minkowski space. However, the elements cannot be the fundamental description of spacetime under the sprinkling process only. Since in the case of quantum gravity, the elements themselves are spacetime, and if we take away the background spacetime, these elements would be nothing but a matter without any structure. There must be some extra conditions on them.

Up to now, we keep on talking about the element and its continuum approximation – volume, one would ask "What about the causal order of spacetime?" The answer to this question can solve our difficulty arises above. By counting the number of the elements we can only get the volume information of a Lorentzian manifold. If we want to obtain all the other information, we need the causal structure. For each point of the spacetime, the causal past (future) of point p can be defined as the set of curves. These curves should be timelike or lightlike. Then the causal order of the spacetime is the collection of these past and future sets. Let us therefore endow the sprinkled elements with the causal order, i.e. if two elements x and y are sprinkled respectively at points p and q in the continuous spacetime, and they satisfy such condition: if $x \prec y$, then we can say that p is to the past of q.

Having been given all these conclusions above, we can provide a precise definition for the correspondence principle: A Lorentzian manifold (M, g_{ab}) is said to be an approximation to a causal set C if C can come from sprinkling (M, g_{ab}) with relatively high probability[3]. In [3], J. Henson gave an expression for this probability: the probability for sprinkling n elements into region of volume V is

[3]The practical meaning of "relatively high probability" is decided on a case-by-case basis. It is usually assumed that the random variable (function of the sprinkling) in question will not be wildly far from its mean in a causal set. Beyond this, standard techniques involving χ^2 tests exist to test the distribution of sprinkled points for Poisson statistics.

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!}, \tag{1}$$

where ρ is a fundamental density which is of order of Planck scale, and the probability only depends on the volume of the region.

In the discrete matter analogy, we have obtained the result that if two continua are approximate to the same atomic state, then they are "close" to each other. As we have followed the same steps for the quantum gravity, we can gain the similar consequence which is the central conjecture of causal set theory – the so-called "Hauptvermutung". So far, this conjecture cannot be proved in a general sense. The main reason is that it is difficult to define precisely the distance on the space of a Lorentzian manifold.

2.6 Coarse graining

From the steps above we can obtain that the causal sets to quantum gravity kinematics has been taken under the assumption that the continuous spacetime is an emergent phenomenon of discrete structure which is the fundamental entity. Note that if a causal set *C* could have come from sprinkling a Lorentzian manifold (M, g_{ab}), then *C* is said to be faithful embeddable in *M.* For such a causal set *C*, if one changes some causal order relations in *C* to form a new causal set C_1, then this new causal set might be no longer embeddable into any manifold. Of course, no one wants such small influence to be physical significant. So we need to introduce a new method – coarse graining to eliminate this influence. The notion of coarse graining procedure can be interpreted as: causal set C_1 can be obtained from causal set *C* by removing some points with fixed probability in *C.* For example, if we want to perform a $\frac{1}{q}$: 1 coarse graining of a causal set, we just need to throw away every element with fixed probability $p = 1 - q$. This seems to be analogous to sprinkling. In this case, a continuum spacetime can be considered as an approximation to a coarse grained set of many discrete elements. More specifically, a manifold *M* could correspond to a set of elements which are causal sets with no continuum approximation, but which have a common coarse graining causal set to which *M* is an approximation. The view of course graining is useful, and in order to address it, we need the dynamics in next section.

2.7 Reconstruct the continuum

Now we already have the full knowledge of the correspondence principle between discreteness and continuous spacetime. What we want to know next is how one uses it to reconstruct the continuum entity (i.e. the Lorentzian manifold). There are some important points we need to pay attention: one can only use the sprinkling to discretize a continuum spacetime. The causal sets are the fundamental structure of spacetime.

The notion of faithful embedding gives us the criterion for a manifold to approximate to a causal set. Then we can proceed to recover a manifold under the help of these arguments.

The causal sets possess a structure that is rich enough to be extended to provide a lot of information for manifolds which need recovering. The information contains differential structure, topology, metric, etc. And one of the most important information is dimension. Physicists have proposed several valid approaches to estimate it, in [2], R. D. Sorkin listed three of them:

"*Myrheim-Meyer dimension.* Let $N = |I|$ be the number of elements in I and let R be the number of relations in I (i.e. pairs x, y such that $x \prec y$). Let $f(d) = \frac{3}{2}\binom{3d/2}{d}^{-1}$. Then $f^{-1}(R/\binom{N}{2})$ is a good estimator of d when $N \gg (27/16)^d$.

REMARK The *Myrheim-Meyer* estimator is coarse-graining invariant on average (as is the next).

Midpoint scaling dimension. Let I = interval (a, b) and let $m \in I$ be the (or a) "midpoint" defined to maximize N' = min{|interval(a, m)|, |interval(m, b)|}. Then $\log_2(N/N')$ estimates d.

A third dimension estimator. Let K be the total number of chains[4] in I. Then $\ln N/\ln\ln K$ estimates d. However, the logarithms mean that good accuracy sets in only for exponentially large N."

Here, I is an interval in a causal set C. And all these three estimators assign a dimension to it. For some interval A in Minkowski space M (d dimension), $I \approx A$.

There are of course some other estimators except the three given above. All these estimators can help us to reconstruct a manifold. For example, given the function $f^{-1}(R/\binom{N}{2})$ in the first estimator and a causal set, then if a manifold approximates to this causal set, its dimension is $f^{-1}(R/\binom{N}{2})$.

[4]Here chain (antichain) means linear order where any two elements are (not) related to each other. In particular, any linearly ordered subset of an order is a chain.

On the contrary, if the results which come from such two dimension estimators are not the same, then we can say that the causal set *C* has no continuum approximation. Apart from the methods for recovering dimension, there are some ways to retrieve the other geometries, such as topological structure. One such example which is given by J. Henson in [3] is an estimator of timelike distance. And the expression is:

$$Vol\big(I(x,y)\big) = \frac{\pi}{24} d(x,y)^4, \tag{2}$$

where x and y are two timelike points, $d(x,y)$ is the distance between them, *Vol*(*R*) is the volume of region *R*, and $I(x,y)$ is the interval between x and y. As we have known that the volume of the region *R* is approximate to the number of the sprinkled elements in the corresponding region. Hence we get the volume *Vol*, and then we can calculate the distance via equation (2). There are also some methods of measuring timelike distance in curved spacetime, I will not list them here.

Instead of saying a causal set *C* is faithfully embeddable, one can introduce a new notion of "manifoldlikeness" to describe the similar situation. Such a concept is better than the faithful embedding because it makes sense even though there are some "small fluctuations" which are actually of no physical significance. And the measure of timelike distance and some other geometric structures can be used to find an embedding for *C*. This idea can be applied to any causal set which is embeddable into manifolds. These attempts provide a measure of manifoldlikeness for *C*.

In [3], J. Henson presented two features of causal set structure after getting the discrete/continuum correspondence principle. The first one is that there is no barrier to sprinkling into manifolds with spatial topology change, provided it is degeneracy of the metric at a set of isolated points that enables topology change, and not the existence of closed timelike curves – and in this discrete theory the set of histories can be characterized. Secondly, the structure can represent manifolds of any dimension – no dimension is introduced at the kinematics level, as it is in Regge-type triangulations.

2.8 Lorentz invariance

There are many quantum gravity programs which depend on the hypothesis that the underlying structure of spacetime is discrete. But not all these (actually most of them) ideas respect the local Lorentz invariance (boost and rotation). This is a big problem when we want the continuum structure which is an approximation to the discrete one to preserve this symmetry. The causal set theory, however, is successful in preserving local Lorentz invariance and perhaps this is the most apparent property that distinguishes it from other theories which also possess a discrete structure. This is an important criterion for judging whether an approach is a good one to quantum gravity.

In the classical theory (continuum description), where the concept of local Lorentz invariance makes sense, we mean boosts and rotations when we talk about Lorentz invariance. For example, in Minkowski space, physical quantities satisfy global Lorentz invariance as we have seen in special relativity (it changes to local Lorentz invariance when the background spacetime is curved). Hence, we need to preserve Lorentz invariance in the case of the discrete spacetime which corresponds to a Minkowski space. A lattice-like structure, for example, will violate global Lorentz invariance in Minkowski space (similar as the lattice system in the matter analogy).

In the matter analogy, if one wants to obtain a system which preserves Euclidean invariance, a random distribution of atoms (Poisson distribution) works it out. Correspondingly, in the case of discrete structure, the only one that possesses such a process is the causal set theory. Now I will discuss how this works itself out precisely.

Before discussing the causal set, let us recall the atoms/mass density model. If we divide the whole space into small cubes, and put the atoms at each vertex of the lattice. Obviously, this atomic state would break Euclidean invariance since there are preferred directions. In order to avoid this problem, we should discretize the continuum object randomly according to Poisson process. Such a random process can help us to build an atomic state which respects Euclidean invariance.

Things are similar in the case of a locally Lorentz invariant causal set. Analogously, we may consider a grid of coordinate system for a continuous spacetime, and place each element at the vertex of the grid. This way of discretizing a spacetime would obviously violate local Lorentz invariance (the preferred directions break the rotational invariance). So, this kind of discrete model cannot be considered as the fundamental structure of spacetime. What we need is a

Poisson process, and in the case of causal set theory, it is the sprinkling process. As we have already known that the probability *P*(*n*) given in equation (1) only depends on the volume of the region *R*. And in [3], J. Henson argued that:

"In Minkowski spacetime, to establish Lorentz invariance of Poisson process rigorously we need only note the theorems proving the existence and uniqueness of the process with the distribution (1) for all measurable subsets of R^d and its invariance under all volume preserving linear maps, which of course includes Lorentz transformations."

In a curved spacetime, there is no global Lorentz invariance, but the local one instead, since the curved spacetime looks locally like a Minkowski spacetime, namely physical quantity will be described in local Lorentz systems in general relativity.

We can now say that respecting local Lorentz invariance is the main property that apart the causal set theory from possible discrete structure theories. And the range of choosing a good approach to quantum gravity is extremely restricted if local Lorentz invariance has to be preserved.

2.9 Discussion on other approaches

One can utilize sprinkling to incorporate Lorentz symmetry in causal set program. What about the other approaches to quantum gravity? In fact, they cannot do the same thing. Nevertheless, each of them provides a different opinion on whether we have to incorporate local Lorentz invariance. A brief introduction of how these approaches work is given here.

The first example is the spin-foam approaches whose underlying structure is also discrete (this can be seen as collections of discrete pieces of data [3]). Unlike the causal set theory, there is no explicit correspondence principle between Lorentzian manifolds and the discreteness of spin-foam approach so far.

Different opinions on whether local Lorentz invariance is preserved in spin-foam models are under investigation. On one hand, some people believe that spin-foams with Planck scale discreteness are not invariant under local Lorentz transformations. For instance, near one particular frame, a continuum region can be approximated, however, in other frames this region will be badly approximated. On the other hand, some people believe that there is rotational invariance in spin foam models. But arguments which prove this statement is wrong have been presented: if spin-foam histories are used, then the macroscopic properties of the universe will not be presented in some systems, it is apparently that these models break Lorentz invariance. Both of the two views need to be supported by predictions or observations to prove them true in the future.

Another two programs in [2] are listed here:

"In loop quantum gravity program, the spectra of certain operations (e.g. the areas of 2D surface) are claimed to be discrete, although as yet the physical Hilbert space and operators have not been identified. Nevertheless, some arguments have been provided as to how the problems of spacetime singularities and black hole entropy might be solved in LQG. But without the physical observables, this type of discreteness could circumvent the arguments mentioned in the introduction or in the previous paragraphs, or even whether it would exist in a completed form of loop gravity, is not clear as yet.

In dynamical triangulations, discreteness is used to solve the problems, of defining the path integral and coming to grips with technical issues in a manageable way, notably the Wick rotation. However, in this approach the discreteness is not considered fundamental and a continuum limit is so sought. As the "causal dynamical triangulations" program is in the happy situation of possessing a working model, it would be of great interest for the debate on discreteness to see what becomes of black hole entropy (or more general forms of horizon entropy) as the cut-off is removed."

3 Causal Set Dynamics

Having discussed the kinematical part of the causal set theory, we are now going to explore the dynamical part. Initially, the most important question is how to construct a dynamics for causal set program. In order to obtain a consistent quantum theory of gravity via causal set program, we have abandoned a lot of geometric properties (metric, topology, differential structure, etc.) of a Lorentzian manifold which is only considered as an effective description of spacetime. And what we have now are only the elements and the causal order relations between them. So the only remaining structure is the number of elements in causal sets. According to these statements, what we need to construct now is a "manifoldlike" causal set. To see this more clearly, I will introduce a 3-level model of causal set at first. [4] Consider a causal set with N elements, where N is a large enough number, with such a structure: there are 3 levels (hence its name) in this causal set, the first level is possessed by about $N/4$ number of elements with no ancestors, the second level is possessed by about $N/2$ number of elements with ancestors in level 1, and the remaining elements are in the third level with ancestors in level 2. Such a causal set has no continuum description (the time interval is too short (Plank scale) to form a manifold), i.e. it is not a "manifoldlike" causal set.

This kind of 3-level model is an obstruction for causal set theory to arrive at a theory of quantum gravity, so we need a new dynamics to solve this problem.

When one is trying to look for this dynamics, there is a question arising – What is quantum mechanics? In general, there are two interpretations for quantum mechanics now, one is the state-vector approach, and the other is the "sum-over-histories" approach. But neither of them can provide a perfect interpretation for quantum mechanics (perhaps that is why we cannot find an effective theory of quantum gravity).

In the case of causal set theory, I would like to use the "sum-over-histories" approach in constructing the new dynamics since the state-vector approach is not suited (I will give the reasons later).

There is more than one way to build a dynamics for quantum causal set theory. Here, I will introduce two of them [2]:

"On one hand, one could try to mimic the formulation of other theories by seeking a causet-invariant analogous to the scalar curvature action, and then attempting to build from it some discrete version of a gravitational "sum-over-histories". On the other hand, one could try to identify certain general principles or rules powerful enough to lead, more or less uniquely, to a family of dynamical laws sufficiently constrained that one could then pick out those members of the family that reproduce the Einstein equation in an appropriate limit or approximation. (By way of analogy, one could imagine arriving at general relativity either by seeking a spin-2 analog of Poisson's equation or by seeking the most general field equations compatible with general covariance and locality.)"

In this paper, I will mainly use the second method to construct a new dynamics.

3.1 The CSG models

In order to construct a dynamics for quantum causal set theory, we need not only the kinematics of causal sets, but also a dynamical framework which is the so-called "quantum measure theory". With the help of the causal sets kinematics and quantum measure theory, one can extremely narrow the range of the physical principles which can be used to build the dynamics (these principles are general covariance and locality in the case of the causal set).

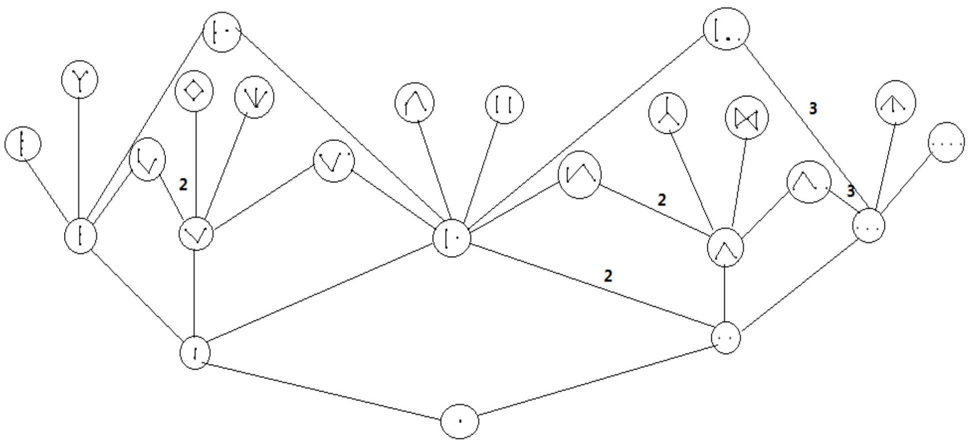
Initially, let us see what happens in classical causal set theory, where I will use the "transition probability" instead of the "quantum probability" (quantum measure). Having been given this "transition probability", along with kinematics of causal sets, one can endow the causal sets with a particular growth process. Then the causal sets can be considered as some "evolution in time"[5]. And this growth process should be a random process rather than a deterministic one, i.e. such a process is analogous to a classical stochastic process. Introducing the so-called "classical sequential growth" models – CSG models, which are a set of stochastic processes on the space of finite past[6] causal sets.

In these models, a causal set is growing randomly under certain rules. Briefly speaking, a gr-

[5]More precisely, this should be seen as the growth of the casual sets is time.

[6]Past finite means that each element has finite ancestors.

owth process of a partially ordered set is to impose a certain distribution on elements – add new elements on the old ones with particular probabilities, where this probability is the one we referred earlier (transitivity probability). In particular, the procedure is: given a single element as the starting point, then add a new element on it, where this new element can be to the future of or spacelike to the first element. There are several possibilities for adding the new element, and certain probabilities which has mentioned above are assigned to these possibilities. These "transition probabilities" are constrained by the physical rules which we have selected for the dynamics. Such a stochastic process is shown in Figure 4. And this growth process can be viewed in another way: it is a transition from one partial causal set to a new one under certain "transition probabilities". In this process, there will be manifestly infinite causal sets created (keep adding new elements)



This is a Hasse diagram of the growth process of partially ordered set which is of finite causal sets. All these causal sets are connected by lines which are just lines (not the same as lines in a causal set which represent the causal order). Two different causal sets may give birth to a same one, that is how the word "partial" comes from. One can add new elements to the elements of causal sets at the bottom of each line to create new causal sets at the top. Repeat doing so, we will get an infinite-element partially ordered set. (Note that the numbers shown on the lines mean that there is more than one way (shown by the numbers) to generate a new causal set.) [3]

Figure 4

According to the growth process stated above, the possible types of dynamics are more than one. In order to gain a useful dynamics, some further restrictions should be imposed on it to narrow its possibilities. There are two conditions ([2], [4]) which are discrete general covariance and Bell causality. The former one is a discrete analogy to the general covariance in general relativity. Firstly, we can label the causal set elements (first born to be labeled 0, the second is 1, etc.). And this labelling is just a kind of "pure number", as what we have done in general relativity (the equations are independent of the choice of coordinate frames). The second condition is presented to ban the superluminal influence, i.e. it means that any birth taking place in one region cannot be influenced by other ones in a region which is spacelike to the former one. Taking these two conditions, a new class of models which are called the "Rideout-Sorkin" models come out, along with the "transition probability" given below:

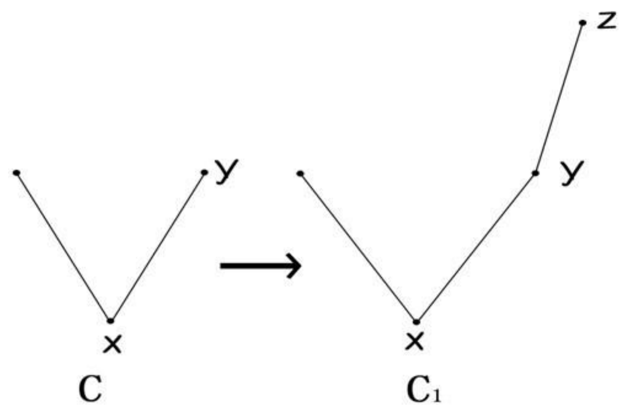
$$\frac{\lambda(\varpi,m)}{\lambda(n,0)}, \tag{3}$$

this is a transition from $C \rightarrow C_1$, where ϖ is the number of the ancestors of new element, *m* is the number of the parents of new element , *n* is the number of elements in the causal set *C* as illustrated in Figure 5. The function λ is expressed by:

$$\lambda(\varpi, m) = \sum_{k=m}^{\varpi} \binom{\varpi-m}{k-m} t_k, \tag{4}$$

where t_k is a sequence of non-negative numbers. The dynamical rules can be determined by these "coupling constants". In such models, a newly born element chooses elements of cardinality *k* to be its ancestors with relative probability t_k. [2]

The CSG models with their properties given above are very useful for causal set dynamics. In the meantime, some results in these models have been applied to quantum theories. For example [3], the large causal set (i.e. the type that is most likely to be found from a uniform probability distribution on causal sets with a large number of elements) is not a manifoldlike one,



This is the diagram of a transition from C to C_1, where in C_1 we add a new element z onto the original causal set C. In this diagram, element z has two ancestors (x and y), and a parent (y), i.e. in this case, n is 3, m is 1, and ϖ is 2.

Figure 5

but has a “flat” shape instead. This kind of causal set is the “Kleitman-Rothschild” causal set. To find a dynamics which can solve this problem, the classical sequential models show their advantages. These growth models can easily overcome the “large number problem” by lowering the probabilities of appearance of these causal sets.

Next step is to identify and describe the questions which the CSG models can answer. This might be a big difficulty for any other theories which also possess a random process and the principle of general covariance. In the quantum mechanics (under the state-vector interpretation), these questions can be stated as “What are the observables” and “What are the meanings of them”. To answer them is involved in solving the problem of time. It is probably that this problem exists in any theory which possesses a random process and general covariance, including the CSG models. Nevertheless, physicists who are working on these models have made some progress by providing a physical interpretation for “observables” of some generic CSG models. Some more general models are under exploring and several of them will be used in the “quantum sequential model” (the QSG model, which will be discussed later).

Another application of the CSG models comes from the aspect of cosmology (the so-called “cosmic renormalization” [3]). Consideration on these models gives rise to a “bouncing” cosmos: the universe initially expands from a starting point which has a “minimal size”, and then keeps on expanding until arrives at a “maximal size” point, finally, collapses down to its original size. This is just one cycle in a set of infinite cycles of expansion and contraction. Each dynamics between a starting point and a final point is a CSG model with a different “renormalized” set of free parameters, where these parameters always converge to a small set after a plenty of cycles. In particular, the total number of elements (between a big bang and crunch) of the cycle becomes large no matter what the number is in the first few cycles.

In the previous paragraph, the notion of “quantum sequential growth” model appears. Its name may mislead us to understand it as a quantum limit of the “classical sequential growth” models as we usually do in general sense. Unlike most of the corresponding relation between quantum theories and their classical approximations, these “classical sequential growth” models cannot be treated as a classical limit of some quantum theory, but a classical stochastic process which is analogous to the Bronian motion and its relationship with the quantum particle. In order to manifest this situation, we need to use the quantum measure instead of the probability measure in the CSG models to create a quantum model. Thus, what we have finally is the quantum version of growth process where, in this case, the classical process is just an intermediate step to an ultimate dynamics.

The CSG models are simple and clear as an approach to causal set dynamics. However, there still remain some challenges in seeking a full quantum theory. For instance, the relativistic causality principle is hard to be generalized to a quantum case. But many other ways of providing a growth model are under consideration, and there are also many different principles which can be used to constrain the dynamics we need. Further discussion is necessary on the way to construct an effective quantum dynamics.

3.2 Actions and amplitudes

Now I will introduce another approach to dynamics of causal set theory. As we can see in the subtitle, the cores of this approach are actions and amplitudes. One main barrier to this method is that we have no expression for the amplitude $\exp(iS)$. At first, we want to find an action for the causal set. Since we already have the Einstein action in general relativity, an immediate thought of finding such an action is to obtain a function of causal set from which the Einstein action (an effective continuum description) can be recovered as we have done in the causal set kinematics section (such as the recovery of dimension, distance, etc.). In general relativity, the Einstein action is an integral of local quantity on a Lorentzian manifold. As a result of this, an action for a causal set should also be some kind of local action, which means that what we need to do next is to recover the locality. Physicists who are concentrating on this task have made some progress (this will be discussed in next section).

In order to gain an action for causal sets, one needs to identify which kind of causal set is validity. Here, I suggest the causal sets that possess a fixed number of elements. Then what we should do is to sum over all these causal sets, where this approach is also known as the "unimodular" sum over histories. The particular process is: "slowly vary" the action near the classical solutions, and "quickly vary" it elsewhere. Here, "elsewhere" means that not only the causal sets that are not manifoldlike ones, but also those causal sets whose continuum approximations are not Lorentzian manifolds that can be accepted in general relativity. A useful method is to limit the history space to the causal sets which are faithfully embeddable into manifolds. This approach can be used to produce an effective model: 2 dimensional quantum gravity. Since the action of this model is trivial for manifolds with fixed topology on them (in this case the only degree of freedom of a spatial slice is the length, which can be easily recovered from the causal set because it is approximated by the number of elements in an inextendable antichain). In the meantime, the only degrees of freedom of a Lorentzian manifold can be given by the volume. And we may consider the variation of the conformal factor as that of the density of sprinkling for causal sets, then the history space can be seen as the set of causal sets which are embeddable into a fixed Lorentzian manifold. Now some results about causal sets embeddable into 2 dimensional Minkowski space have already come out. A large causal set of this type can be faithfully embeddable into a causal interval of Minkowski. The existence of techniques which prove these statements argues that analytical results are possible in the 2 dimensional dynamics. So some of the requirements for the causal set to be used as the regulator for 2 dimensional quantum gravity have been achieved. [3]

3.3 Recover the locality

As mentioned in last section, the recovery of locality is essential for the continuum of dynamics of causal set theory (also essential for the phenomenological part which will be considered later). In order to see how to reconstruct the continuum locality clearly, let us first look at the problem in causal set to its solution on a light-cone lattice.

Suppose there is a lattice which has a 2 dimensional Minkowski space as a continuum approximation. In this case we will use the light-cone coordinates *u, v*, where the metric is $ds^2 = dudv$. Let us introduce a function ϕ to this coordinate system, where ϕ can be considered as a map:

$$\phi: (u, v) \rightarrow \mathbb{R}.$$

The vertices of the lattice lie on the points (*na*, *ma*), where *a* is the spacing of the lattice and *n*, *m* are integers. Note that there is a crucial concept which can help us reconstruct the locality – "nearest neighbours", where these neighbours can be identified near any points. So, the following equations can be true:

$$\frac{\partial\phi\,(u,v)}{\partial u} \approx \frac{\phi\,(u,v)-\phi\,(u-a,v)}{a}, \tag{5}$$

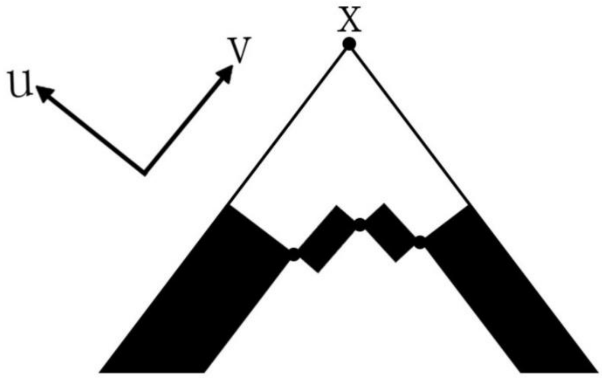
$$\frac{\partial\phi\,(u,v)}{\partial v} \approx \frac{\phi\,(u,v)-\phi\,(u,v-a)}{a}, \tag{6}$$

then one can find that the d'Alambertian at point (*u*, *v*) is:

$$\Box\phi(u,v) = \frac{\partial^2\phi}{\partial u\partial v} \approx \frac{\phi\,(u,v)-\phi\,(u,v-a)-\phi\,(u-a,v)+\phi\,(u-a,v-a)}{a^2}. \tag{7}$$

All these expressions come from the fact that the "nearest neighbours" (here, they are (*u-a*, *v*), (*u*, *v-a*) (*u-a*, *v-a*)) to the certain point (*u*, *v*) can be identified. Obviously, however, the Lorentz invariance has been violated in this model. [3] Apart from this, another problem is how to identify the "nearest neighbours" in a causal set discretization of the same 2 dimensional Minkowski space.

To answer this question, let us consider an element x with coordinate (u_x, v_x) in the light-cone coordinate frame. Then from the discussion on the timelike distance between two elements in previous paragraphs, one can say that the “nearest neighbours” to the past of element x are the ones which have the fewest elements causally between them and x. Define a set of elements in a causal set C with a particular element y, where element z satisfy: $\{z \in C: y \prec z \prec x\} = I(y, x)$. And we hope that $|I(y, x)| = 0$, element y is said to be a nearest element to the past of x. However, there is a problem in such a sprinkling of 2 dimensional Minkowski space: there are infinite many elements (nearest neighbours to the past of x) with probability 1. To explain this problem precisely, it is helpful to introduce this model: a certain element x with its coordinate (u_x, v_x), and several nearest elements linked to the past of x. Choose a maximal u coordinate u_{max} and a minimal v coordinate v_{min} from the coordinates of all its neighbours, form such a region: $u_{max} < u < u_x$, $v < v_{min}$. Then the elements in this region are obviously spacelike to the original neighbours, they are also to the past of x. Note that the area of this region is infinite, so it is clearly that in this region there must be at least one element sprinkled from Minkowski space with probability 1. Repeat this procedure we can obtain a set of infinite number of such element. This situation is illustrated in Figure 6. Actually, the same conclusions can be generalized



This is a Hasse diagram with 4 elements. 3 of them are in the light-cone and to the past of x, these 3 elements are spacelike to each other. To make sure that the new sprinkled elements to be the nearest elements of x, we need that these new elements to be sprinkled into the shaded region. It can be clearly seen that two of the shaded regions are of infinite area, so there would be some elements with probability 1 sprinkled there. [3]

Figure 6

to higher dimensional space.

The problem of infinite number of “nearest neighbours” gives a reason why it is harder to recover the locality for causal set than that for a lattice model. Another problem is that if we want to recover the locality, the Lorentz invariance will be sacrificed. Nevertheless, it is possible to recover the d’Alambertian in Minkowski space of even dimension, which depends on the fact that the Green’s function of the d’Alambertian in 2 and 4 dimensions can be expressed in terms of causal relations.

3.4 The problem of general covariance

In the Einstein’s general theory of relativity, the principle of general covariance plays an important role. In general sense, general covariance means that physical quantities and mathematical expressions of physical laws are independent of the choice of coordinate frame. In the case of quantum gravity, what we really want to know is how to express this principle and how to recover it.

From the classical theory, we know that physical theorems can be manifest and simple when they are expressed in a certain coordinate system, these labeled systems themselves have no physical meaning. However, things become harder if we throw away these “meaningless” systems. This issue is much more complicated in the case of quantum gravity. Because there is no consensus on the interpretation of quantum mechanics. When one wants to express the dynamics in terms of the quantities in a labeled frame, there is a good way to respect general covariance, which is to form some covariant quantities. But the problem is that it will be difficult for us to find a physical meaning for these covariant quantities. What we need to do then is to find the physical meaning.

Process has been made on this issue by the Rideout-Sorkin dynamics in causal set theory. This model provides a measure on the space of labeled causal sets where the labelling is the order of the birth of elements. This labelling has no physical meaning according to the principle of discrete general covariance. In order to preserve general covariance, we need only the covariant subsets to be physical significant where a covariant subset is one that it contains a labeled causal set and all its other labellings. The dynamics provides each covariant causal set A a number $\mu(A)$ which is the probability that the causal set is an element of A. To determine whether a causal set C is an element of A, one can ask countable logical combinations of questions, "Is the finite causal set b a stem[7] in C?" We cannot do this when there exists a non-isomorphic causal set C_1 which has the same stem as C. Then such a causal set C is called a rogue causal set. Nevertheless, these rogues have measure zero, we can eliminate them from the space directly. [4] This result is not true for all kinds of growth models.

3.5 The problem of time

The knowledge of "time" develops along with the development of human beings in history. People try to interpret it in terms of both philosophy and physics in different epochs. Initially, we treated "time" as an independent entity which was separated from space. So we used to refer the word "now" when we were dealing with physical problems. Things did not change until Einstein proposed his theory of relativity which has been proved to be the most successful theory to describe "time". We then realize that "space and time" (actually spacetime) are a single entity. This theory seems to be contradicted with our general sense of the concept of "now" because spacetime cannot be split into "space and time", hence there is no place for the moment "now", but "here and now".

In causal set theory, the dynamics can be described as a sequential growth process. The elements are given birth one by one which shows how events happen in a causal set background. And the order for such a process is a partial order, not a total one. This means that there is no physical meaning for "now", but for the event (i.e. "here and now", which agrees with our former statement).

Such a view seems to be held in general relativity. For example, an event X occurred before another event Y (i.e. Y is to the future of X), then we can say that there must be infinite events which occurred between X and Y. The same situation makes sense easily in the case of causal set because of its partial order.

Another experience of time is that it will not stop. We have never found a boundary for con-

[7] A stem of a causal set is a finite subset which contains all its ancestors.

tinuum spacetime in general relativity (i.e. both space and time have no boundaries). But the solutions of Einstein equation for a spacetime with boundaries are as much as that for the spacetime without boundaries, and general relativity cannot account for this situation by itself. In a Rideout-Sorkin model, however, it has been proved that time cannot stop. As stated in the previous paragraphs, a sequential model will have an infinite many elements, i.e. each element will have a descendant (this has been proved by J. Henson). [4] If this is true in a quantum case, then it implies that time will never come to an end, and in the meantime, the causal sets will continue to grow in the singularities of big bang or a black hole where the continuum description no longer makes sense.

It is based on our general sense of time when we say that time will not stop. Thus one can conclude that there would be no end for the growth process of causal sets. Conversely, some people believe that the causal set cannot be of infinite many elements, but a large number of elements instead (large here means that there are arbitrarily many stages in growth process, where there are finite elements in each). So far we do not have enough evidence to support this view, it is still a strong candidate in explaining the "boundary problem" of time.

3.6 The quantum case

In order to construct a quantum dynamics, we need to generalize the probability measure to a quantum version – quantum measure.

"A quantum measure of a subset, *A*, of Ω is calculated in the familiar way by summing and then squaring the amplitudes of the fine grained elements of *A*." [4]

A quantum measure is different from a classical probability measure because of the non-additive property of the quantum measure theory. This difference has been shown in the famous double slit experiment.

To generalize the classical case to a quantum case for causal set theory, firstly we construct the quantum measure from the transition amplitudes which replace the transition probability via "sum and square". General covariance and the quantum version of Bell causality can be used to form the transition amplitudes. The problem is that it is difficult to find a quantum Bell causality condition.

Finally a covariant causal quantum measure is still hard to be interpreted. Here I quote F. Dowker, who wrote in [4]:

"The interference between histories and consequent non-additive nature of the quantum measure mean that we are exploring new territory here. … a first attempt at a realist interpretation for quantum measure theory. It relies on the adequacy of predictions such as, "Event *X* is very very like to occur," to cover all the predictions we want to make, which should include all the predictions we make in standard quantum mechanics with its Copenhagen interpretation. This adequacy is explicitly defined by Kent and I have tended to agree with this judgement. However, the quantum measure is the result of taking a conservation approach to Quantum Mechanics (no new dynamics) whilst making histories primary, maintaining fundamental spacetime covariance and taking a completely realist perspective. As such it deserves to be preserved with."

4 Causal Set Phenomenology

After introducing the kinematical and dynamical parts of causal set theory, the last step is to discuss the phenomenology of this program, without which a theory is not a complete one. In order to see this clearly, one would like to know what predictions can be made from causal set theory and how to use causal set idea to explain the phenomena which have not been interpreted accurately. Or in a more phenomenological way, how the substance (discrete spacetime) of causal sets manifests itself in an accessible way to us. One prediction is that no violation of standard, undeformed local Lorentz invariance will be observed, which is, however, a negative prediction. So we need some more useful predictions to support our theory. In order to do so, it is better to see some questions at first. For examples, “What is the origin of the entropy of black holes”, “Is the cosmological constant exactly zero?” etc. Discussion on these issues can be of great help.

4.1 Black hole entropy

When we first learn the concept of the entropy of a black hole, we may consider it as the number of “microstates” inside the horizon. But it is difficult to reconcile this consideration with the fact that the processes inside the horizon should be ignored. This contradiction means that the entropy of a black hole can be seen as a property of the exterior region.

Now let us deal with the black hole’s entropy in causal set theory. In order to identify “microstates” which hold for the entropy of a black hole, one would like to use the causal links[8] crossing the horizon in the neighbour of the hypersurface Σ for which the entropy is sought. These links can serve as the candidates for “microstates”, where the entropy can be calculated just by counting the causal links as what we usually do for the gas in thermodynamics. And the results for counting links should be proportional to the area of the horizon, because the black ho-

[8]Here, link means that the relation between x and y cannot be shown by transitivity, i.e. y is a “nearest neighbour” to x.

le's entropy satisfies that $S \backsim A$.

There is no reason why the coefficient of proportionality cannot be divergent or zero. To answer this question, two different black hole geometries which are referred in [1] are given here.

One is in equilibrium (the 4 dimensional Schwarzschild metric) and the other is far from equilibrium (the horizon that represents the earliest part of a black hole which formed from the collapse of a spherical star). In the case of 4 dimensional Schwarzschild metric, the dimension was reduced to 2 by dimensional reduction. Given a certain definition of the number N of causal links has a volume of $c(\pi 2/6)A$ in the $\hbar \to 0$ limit, where A is the horizon area and c is a constant. In the case of the second geometry, the same result was obtained. In [1], R. D. Sorkin believed that this is not a trivial coincidence, but represented "that something like a number of horizon states has been evaluated for any black hole far from equilibrium." Then we need initially to identify the constant c, which is also unknown. Secondly, we should check that both of the null and spacelike hypersurfaces generate the same results. Accordingly, we need to pay attention to the definition of the "near horizon link", since an improper definition would result in a vanishing or infinite answer.

Here, I give one definition of "near horizon link" which is investigated by D. Don:

"Let H be the horizon of the black hole and Σ, as above, the hypersurface for which we seek the entropy S. The counting is meant to yield the black hole contribution to S, corresponding to the section $H \cap \Sigma$ of the horizon. We count pairs of sprinkled points (x, y) such that

(i) $x \prec \Sigma,\ H$ and $y \succ \Sigma,\ H$.

(ii) $x \prec y$ is a link.

(iii) x is maximal[9] in (past Σ) and y is minimal[9] in (future Σ) $\cap$ (future H)."

4.2 Cosmological constant

One of the most successful predictions in causal set program is that the cosmological constant is not exactly zero. This prediction shows that Λ should fluctuate around a certain value, where

[9]A maximal/minimal element of an order is one without descendants/ancestors.

the fluctuation is of order of $N^{-\frac{1}{2}}$ (N is the number of a causal set in a particular cosmological epoch). Assume that this certain value is zero, and take *N* to be the number of elements of today, then one can find that the value of Λ is not zero which is consistent with recent observations. Now I show how this happens.

In causal set theory, the number of elements *N* in the universe so far can be treated as a kind of "parameter time". This can be seen clearly in the CSG models, since new elements are added one by one in these models. Accordingly, for a quantum case of causal set, we should sum over all causal sets of fixed number of elements *N*. And in continuum approximation, correspondingly, we should integral at fixed volume. Such a procedure gives rise to what is called "unimodular gravity". In the classical theory, the unimodular procedure leads to action principle [2]:

$$\delta\left(\int \left(\frac{1}{2\kappa}R - \Lambda_0\right)dV - \lambda V\right) = 0, \tag{8}$$

where Λ_0 is the "bare " cosmological constant. *V* is the volume, κ = $2\pi G$, and λ is a lagrange multiplier to help fix the value of *V*. Changing the expression of this equation by replacing Λ_0 with Λ, we will obtain a term which is of the form ΛV, meaning that Λ and *V* are conjugate to each other in a quantum case. Using the uncertainty principle we find that:

$$\delta V \delta\Lambda \sim \hbar, \tag{9}$$

from the Poisson fluctuation in *N* of size $\delta N \sim \sqrt{N}$, we find:

$$\delta V \sim \delta N \sim \sqrt{N} \sim \sqrt{V}, \tag{10}$$

finally, substitute (10) into (9) we obtain:

$$\delta\Lambda \sim N^{-\frac{1}{2}}, \tag{11}$$

we have set $\hbar$ = 1 in (11).

This formula proves the assumption which we made at the beginning of this section. It has been proved by observations that the fluctuation is currently in order of magnitude 10^{-120} (in natural units).

4.3 Swerving particles

The third one is about how a particle moves on a causal set. In general relativity, every particle travels along the geodesics if the gravitational field is the only background. One can obtain the future states via the equations of motion, with the initial condition (position and velocity). Then the question will be how to recover the geodesic motion when a particle is moving on a causal set. Apparently, if we want to find the relations between a continuum approximation and a discrete structure, it is better to use a sprinkled causal set. In this case, the best way to model a timelike geodesic motion of a particle is moving on a chain. The velocity cannot be defined at ever point, so, in order to recover it, some part of a chain which is to the past of the particle should be considered. And the causal set should be a local one. There is a strict version for this locality in [3] "the dynamics of the particle to the future of an element is only affected by the past within some maximum proper time of that element". If one wants this statement to be hold, then the velocity cannot be arbitrary precise, which means that the particle possesses a random acceleration – it swerves away from the geodesic. And one can use a Lorentz invariant diffusion in velocity space to model this swerves effectively. Although these conditions are not necessary, this model is still a useful one.

One of the applications of this swerving model is to explain the astronomical problem of high energy cosmic rays (HECRs). Though it is incompatible with experiments, we still hope that a quantum velocity diffusion may solve this problem.

5 “Sum-Over-Histories” Approach

In this article, we use “sum-over-histories” (spacetime) approach in our causal set program. And now the reasons are given here.

Initially, let us recall two different interpretations for quantum mechanics, which are Schrodinger’s state-vector framework and sum-over-histories framework. In the former interpretation, we use wave functions to describe the point-like particles, and the Hilbert space is also a critical notion. The measurements are represented in terms of operators. Unlike the classical theory, what one can do in quantum mechanics is to predict probabilities for measurements (except some particular situations) rather than to obtain exact values for them according to uncertainty principle. In this framework, the most important objects are the state-vectors which we can calculate from Schrodinger equation. And the physical interpretations can be made in terms of observables. As we have known in quantum field theory, the canonical quantization approach is based on this framework.

In the sum-over-histories point of view, things become much more different. Quantum mechanics can be interpreted as “a modified stochastic dynamics characterized by a nonclassical probability-calculus in which alternatives interfere”. [1] Here, this kind of approach is mainly attributed to Richard Feynman. In this framework, the essential object is the spacetime history, and unlike the state-vector framework, the observables are no longer important, but are replaced by something which J. Bell called “beables” (this can be understood as “be-ables”). [1] When we use this approach to quantize a classical field, it naturally leads to a spacetime way of quantization instead of a spatial one (canonical quantization).

Though I prefer the sum-over-histories/spacetime approach to the state-vector/canonical quantization approach, I have to say that there are several advantages for both of them. The state-vector framework can be expressed clearly in a mathematical, but the sum-over-histories cannot. The advantages of the sum-over-histories will be discussed later.

Despite the advantages of mathematical expression, there is a big problem with the state-vector approach, which is known as “the problems of time”, where part of these problem has been discussed in section 3.5.

The first problem in canonical approach comes from the projection postulate. In this postulate, the projection is determined by the order of the observations in time. But it leads to some serious difficulties in certain theories. For example, the causal set theory, where the time itself is a physical quantity to be observed.

Secondly, when we talk about the state-vector framework in quantum mechanics, a Hilbert space which formed by these vectors is necessary. And such state-vectors that come from Schrodinger equations evolve in time, since Schrodinger equations are differential equations, we need to construct a Hilbert space depending on continuous time. In a causal set theory, however, the fundamental structure is discrete spacetime which contradicts with our former statement. And there is of course more than one theory depending on such a discrete structure, so a canonical quantization approach seems inappropriate.

The third problem concerns how we explain the early universe. In the state-vector framework the essential object is the observable, and in the early universe there are no observers (hence no observables). But we want a theory of quantum gravity which can describe the early universe up to its starting point (big bang). In this sense, the state-vector approach is not a good way to quantize gravitational field.

All of these problems will not appear in the sum-over-histories framework. In this framework, "Time itself doesn't need to be recovered, because it is there from the very beginning as an aspect of the spacetime metric. … And the early universe existed just as much as we ourselves do here and now". [1] It also makes sense in a discrete structure since there is no "integral" but "sum" instead in this framework. In comparing these two frameworks, it seems that the state-vector framework is a more comprehensive on measurement than the sum-over-histories framework. But this disadvantage is not a big difficulty for the sum-over-histories which it is for the state-vector approach, because measurement is not a fundamental concept in the sum-over-histories framework. And measurement scheme which state-vector framework possesses only makes sense in the non-relativistic quantum mechanics, it breaks down in quantum field theory. Now we proceed we see how this happens.

In quantum field theory, assume that observables constructed from the operators within a certain region can be measured by operations confined to that region. In doing so would break the causality condition. In order to see this clearly, suppose there are 3 regions A, B and C such that points in *A* and *C* are timelike to points in *B*, and points in *A* are spacelike to points in *C*, where *A* and *C* are to the past and future of *B* respectively. If a man in *A* wants to transmit a signal to another man in *C*, then there must be a third man in *B* who receives the signal from *A* and then transmit it to the man in *C*. Thus, in quantum field theory, one cannot decide which observables can be measured consistently with causality. This is actually another disadvantage of state-vector framework, which will not exist in the sum-over-histories approach.

6 Conclusions

Causal set theory is constructed from the concepts of discreteness and causality. This approach provides a solution for many problems on the way to quantum gravity. We propose new kinematics and dynamics for this theory. And the new causal set dynamics – the CSG model is still a "classical" one, where "classical" here is not the one in general sense. The most critical step next is to quantize it, to be able to do so might be very hard but interesting.

Now physicists are concentrating on the relevant topics of causal set theory and many recent achievements have not been presented in this paper. Though there remain many questions arise in seeking to reach a theory of quantum gravity, the progress which has been made by causal set program makes it an attractive theory.

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