Generalized Geometry and Double Field Theory

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September 20, 2013

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Submitted in partial fulfillment of the requirements for the degree of Master of Science of Imperial College London

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Abstract

The goal of this thesis is to review the foundations of double field theory. In order to achieve this, we begin by presenting a self-contained introduction into generalized geometry. More specifically, we study the symmetries of the generalized tangent bundle $TM \oplus T^*M$, the structures that arise related to the natural Courant bracket on $TM \oplus T^*M$, as well as the generalized complex structures, which provide a unified framework for studying complex and symplectic geometry. We end the presentation by introducing the concept of the generalized metric. Next, after a short discussion of some fundamental ideas of string theory, we explore T-duality in a toroidally compactified spacetime. Finally, we introduce double field theory, by first motivating our constructions, and then presenting the way it was founded, from the cubic action to the background independent and the generalized metric formulation. We also discuss its gauge algebra and we end our work with a short account of some open questions on double geometry.

Acknowledgments

First of all, I would like to thank my supervisor, prof. Hull, for suggesting a dissertation on this exciting topic and for the very useful and clarifying discussions we had during the summer. I would also like to thank the lecturers of the QFFF courses, all of whom are pioneer physicists as well as enthusiastic teachers, particularly prof. Hanany and prof. Waldram for some interesting discussions throughout the year.

In addition, I am grateful to all my classmates for the very friendly atmosphere they created; specifically, Amel and Lukas for their companionship, and Davide for some tough chess games I really enjoyed! Also, I would like to thank Maria for being an amazing roommate and for making this exciting but hard year in London a lot more fun!

Finally, special thanks to my family. They have always been extremely supportive and we always have such a great time together! Through our daily communication we have stayed very close, even though we were miles apart. Thank you!

The completion of this thesis was in partial fulfilment of the graduate program which was jointly financed by the Act "Program for awarding SSF scholarships with a process of individualized assessment of the academic year 2012-2013" from funds of the OP "Education and Lifelong Learning" of the European Social Fund (ESF) and the NSRF (2007-2013).

Contents

1 Introduction

String theory was first introduced in the late 60's in order to model strong interactions. Although it was soon replaced by quantum chromodynamics, string theory was found to contain a massless spin-2 particle and it was known that the only such particle possible is the graviton [1]. Since it naturally included general relativity, people started thinking of it as a unified theory of all interactions. However, it also contained a tachyon, i.e. a particle with negative $(\text{mass})^2$ and it was consistent only in 26 spacetime dimensions. The best way to get rid of the tachyon is to introduce fermionic degrees of freedom for the string and make it supersymmetric. Many versions of superstring theories existed until the "first superstring revolution" in 1985, when the requirement of anomaly cancellation left only 5 consistent theories in 10 dimensions: Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. During the next decade they were shown to be related by a web of dualities and so it was proposed that all of them are different limits of a fundamental theory, called M-theory, living in 11 dimensions and whose low energy limit is 11-dimensional supergravity.

In fact, supergravity was developed independently around the seventies, based on the idea of gauging the superPoincaré algebra, in analogy with the gauging of the Poincaré algebra, which produces general relativity. However, with the first superstring revolution, the different supergravity theories, in different dimensions, were realized to be the low energy limits of the corresponding string theories. In this work there is no need to explore supergravity; we will just keep in mind that supersymmetry is a very powerful symmetry between bosons and fermions and so it puts very rigid constraints on the possible field content of a theory. We will be interested only in a small subset of these fields, namely the metric g, the Kalb-Ramond B-field and the dilaton ϕ, which appears as a universal massless sector in all superstring and supergravity theories, and is generally considered to be the background field content of spacetime.

The idea of duality was not new in physics during the second superstring revolution, but it is at least as old as Maxwell theory, where the duality between electric and magnetic charges was studied. Roughly, two theories are said to be dual to each other if both describe exactly the same physics. These two theories may contain different degrees of freedom, in which case there should be an (invertible, of course) transformation relating them. A field theory can even be self-dual in the sense that there may be transformations relating the expansions of the theory around different points in parameter space. In string theory, a web of dualities was found, relating all of the five consistent theories, and that was what motivated the idea that there is a more fundamental theory, M-theory, with its various limits corresponding to the superstring theories.

T-duality was one of the first dualities to be discovered in string theory, one reason being the fact that it is a perturbative weak-weak duality, i.e. it relates expansions around weakly coupled points in the parameter space of the theories, where perturbation theory can be trusted. It identifies the physics around different backgrounds, something that is strange from the field theory point of view. T-duality can be precisely formulated in toroidally compactified backgrounds, i.e. backgrounds with some of the directions forming circles. Even the simplest, bosonic case is particularly interesting: T-duality roughly states that the physics of a string on a circle of radius R is the same as the physics on a circle of radius $1/R$. This stems from the fact that a (closed) string can have both momentum modes as well as winding modes in the compactified directions (we call them "modes" because they will take discrete values due to quantum mechanics and topology respectively-more on this later). An exchange of momentum and winding modes, together with the inversion of the radius leaves the physics invariant. Actually, if more than one dimensions are compactified, the Kalb-Ramond background field comes into play and, apart from affecting the mass spectrum, it also gets changed under T-duality. As we will see, the

T-duality group gets richer as the number of compactified dimensions increases.

T-duality has a very deep meaning, which will not be completely clarified until the formulation of M-theory. However, it is apparent that strings see a different geometry from the usual one: for them, a very small radius of a spacetime direction is equivalent to a very large one.
Additionally, if we explore the superstring theories, we see that the inversion of the radius interchanges Type IIA and Type IIB theories, by interchanging the various solitonic (non-perturbative) objects of the two theories, called branes.
There is also a strong-weak superstring duality, called S-duality, relating strongly coupled regimes to weakly coupled ones. S-duality does not commute with T-duality, so their combination, called U-duality, is nontrivial. U-duality is supposed to contain all the information about the symmetries of string theory.

T-duality is not at all obvious by simply observing the string action, on the contrary, naively, it is very surprising. So, even from the beginning, there has been an interest in making T-duality a manifest symmetry of string theory, see [2] and [3], quickly followed by the first concrete works with doubled coordinates, [4], [5]. In order to understand what this doubling of coordinates means, we should note that, even in first quantised string theory, we can introduce coordinates $\tilde{x}$ dual to x, which physical correspond to coordinates conjugate to the winding modes, in analogy with the fact that the usual coordinates x are conjugate to the momentum modes. However, the idea of making the $\tilde{x}$ dependence explicit is nontrivial, even though the dual coordinates were necessary for the formulation of closed string field theory, as they represent physical degrees of freedom (see [6]).

These first ideas were not developed for quite some time, until the interest in the double coordinates revived. This happened partially due to the will for a deeper understanding of non-geometric backgrounds which were introduced in the previous decade (see [7] and [8]). Heuristically, these are backgrounds that cannot be defined as ordinary riemannian manifolds, since a change of coordinates can also include a string duality transformation. In this way we are trying to make the dualities a property of the geometry of spacetime, which can give us a deep insight into the mathematics of string theory. In particular, such backgrounds involving T-duality, called T-folds, can be studied if we formulate string theory as a sigma-model with a doubled space as target space, which is parameterised by both the x and the $\tilde{x}$ coordinates. T-duality is then equivalent to a change of the subset of the doubled space on which the string coordinates depend.

Then, the foundations of double field theory were laid ([9], [10], [11] and [12]). Double field theory is a (generally successful) attempt to formulate a field theory which is manifestly invariant under the T-duality group $O(D, D)$. In order for this to be achieved, we must find a consistent truncation of string field theory which is not a low energy limit, since we have to keep all the momentum and winding modes for T-duality to hold. These four foundational papers were only concerned with the bosonic supergravity fields g, b and ϕ, which were now allowed to depend on the coordinates of the doubled spacetime. However, this doubling of degrees of freedom must come with a constraint to reduce them again. This constraint can be seen to originate from an analogous one in string field theory, called the level matching constraint. As we will see explicitly, these considerations are enough to construct an action to cubic order in the fields, which can be seen to have the correct supergravity limit ([9]). In particular, the gauge transformations of double field theory should reproduce those of supergravity, namely the diffeomorphisms and B-field gauge transformations, which indeed is the case.

A more complete investigation of the gauge algebra of double field theory was carried out in [10], where it was seen that the gauge parameters do not satisfy the usual diffeomorphism algebra, but the so called C-algebra, determined by the C-bracket. We should note that in general the algebra is much more complicated, but the work was carried out assuming a stronger version of the constraint described before. This strong constraint implies that all fields and gauge parameters depend on the same null subspace of the doubled space and so, there is always a T-duality transformation that can eliminate the $\tilde{x}$ dependence. This means that the theory is not a fundamentally double field theory, yet a doubled formalism can be developed, since we need not make the choice of subspace explicit from the beginning.

In fact, this strong constraint is very helpful: it was used in [11] to construct a complete to all orders background independent action and in [12] to finally construct an action which is manifestly invariant under T-duality transformations, with all the objects transforming linearly under $O(D,D)$. Specifically, the coordinates are organised in X^M and the Einstein metric and Kalb-Ramond B-field give rise to a generalized metric $\mathcal{H}_{MN}$, with both objects belonging to representations of $O(D,D)$ ($M, N, \ldots$ are vector $O(D,D)$ indices).The dilaton d is a scalar under $O(D,D)$ and thus the tensor density e^{-2d} provides an integration measure. This dependence on the dilaton is also consistent with a string theory theorem which states that the dilaton should determine the string coupling.

In [12] it was also shown that the Lagrangian of the action can acquire the form of a "generalized Ricci scalar". We then arrive at a situation analogous to having the Einstein-Hilbert action. A very important question is if there is a "generalized" analogue of the Riemann curvature tensor. If this is the case, then most likely double geometry has a deeper, rigorous mathematical formulation. This must mean that the $O(D,D)$ tensor algebra derived from the gauge transformations has an invariant, geometric formulation. In the final subsection we will discuss the current state of this area of research.

A very interesting area of mathematics was developed independently a while ago, called generalized geometry. It was based on ideas introduced in [13] and formulated in [14]. Generalized geometry studies structures on the "generalized tangent space" $TM \oplus T^*M$ of a manifold M and it can be thought of as containing complex and symplectic geometry as two extremal cases. What makes it interesting from a physical point of view is that the symmetries of the generalized tangent bundle are precisely the symmetries of the Einstein plus B-field supergravity theory. In fact, there are attempts to formulate supergravity entirely in terms of generalized geometry, see for example [15]. In addition, it provides a very convenient setting for the study of flux compactifications (see [16] for a review). We are interested in it because, imposing the strong constraint in double field theory means that we half the coordinate dependence, leaving a tangent space identical to the one considered in generalized geometry and all the structures defined there are inherited, with the additional property of having a rigorous mathematical formulation. In that sense double field theory is more general than generalized geometry, but both areas have striking similarities.

2 Generalized Geometry

2.1 Preliminaries

We begin our exposition of generalized geometry by explaining some general concepts on fibre bundles and complex geometry that will be required as we move on. The discussion of this subsection is based on [17], [18] and [19].

Roughly speaking, a vector bundle is a family of vector spaces E_x, parametrized by a smooth manifold M (i.e. by points x such that $x \in M$). More precisely,

Definition 2.1.1 (Vector Bundle of rank n) *Let M be a smooth manifold. A* vector bundle of rank n *is a manifold E, together with a projection $\pi : E \to M$ such that:*

1. *The fibre E_x at each point $x \in M$, defined by $E_x := \pi^{-1}(x)$, is a n-dimensional real vector space.*

2. (Local Triviality) *For every $x \in M$ there exists an open neighbourhood U of x and a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ whose restriction to E_y is a vector space isomorphism (i.e. a linear map) onto $\{y\} \times \mathbb{R}^n$ for every $y \in U$.*

Note that for a point $x \in U_\alpha \cap U_\beta$ there are two such diffeomorphisms ψ_α and ψ_β. Then, for every $x \in U_\alpha \cap U_\beta$, the map $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ is a linear map, taking a vector $v^i \in \mathbb{R}^n$ to another vector $w^j \in \mathbb{R}^n$. Thus, it can be represented by a $n \times n$ matrix $(m)^i_j \in GL(n, \mathbb{R})$ acting as $w^i = m^i_j v^j$ and we can write

$$\psi_\alpha \circ \psi_\beta^{-1} : (x, v^i) \to (x, m_{\alpha\beta}(x)^i_j v^j).$$

The maps $m_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ are called transition functions and they contain all the information about the topology of the bundle. More intuitively, they determine how the fibres are patched together to form the bundle.

We can easily check that consistency requires:

- $m_{\alpha\alpha} = I$ on U_α
- $m_{\alpha\beta} = m_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta$
- $m_{\alpha\beta} \cdot m_{\beta\gamma} = m_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

We note that given a local trivialization (U_α, ψ_α) and the transition functions $m_{\alpha\beta}$, the bundle E is determined up to bundle isomorphism (see definition 2.1.2).
We can view the above construction as the free and transitive action of a Lie group G on the fibres of the bundle E. We will call G the *structure group* of E. For a general vector bundle of rank n, $G = GL(n, \mathbb{R})$ but, as we will see, we can also consider reductions to subgroups $G \subset GL(n, \mathbb{R})$.
This point of view induces the following natural generalisation: if the fibres E_x at each point x of M are general manifolds (of the same dimension), then we get the notion of a general fibre bundle, requiring also that there exists a Lie group G acting freely and transitively on the fibres.

We proceed by giving some more definitions.

Definition 2.1.2 (Bundle Isomorphism) *Let E and $\tilde{E}$ be two vector bundles over M. A* vector bundle isomorphism *is an invertible smooth map $f : E \to \tilde{E}$ which takes the fiber E_x of E over x to the fiber $\tilde{E}_x$ of $\tilde{E}$ over x (restricting to a* linear map *on them, $f_x : E_x \to \tilde{E}_x$) and induces a diffeomorphism when restricted to M.*

Definition 2.1.3 (Subbundle) *Let E be a vector bundle over M. A* subbundle *(also called* distribution*) of E is a vector bundle F over M map such that F_x is a vector subspace of E_x for every $x \in M$.*

Definition 2.1.4 (Section) *A* section *of a fibre bundle (E, π) is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. The space of all sections of E is denoted by $C^\infty(E)$.*

If E and $\tilde{E}$ are two vector bundles over M, we can naturally construct the dual vector bundle E^*, the complexification $E_{\mathbb{C}}$ (see definition 2.1.5), the (Whitney) sum bundle $E \oplus \tilde{E}$, the direct product bundle $E \otimes \tilde{E}$ and other tensor powers such as $\bigotimes^k E$, $Sym^k E$, and $\bigwedge^k E$, by performing the corresponding operations on the fibres, while keeping the same base space M.

The archetypal example of a vector bundle is the tangent bundle TM of a manifold M. The fibre at each point $x \in M$ is the tangent space at that point T_xM and the structure group is $GL(n, \mathbb{R})$, where n is the dimension of the manifold. We see that the dimension of TM is $2n$.

Then, we can regard the various tensor fields on M as sections of the appropriate vector bundles, e.g. if ω is a k-form we have $\omega \in \bigwedge^k(T^*M)$ and, for g a riemannian metric, we have $g \in Sym^2(T^*M)$.

We also introduce the concept of the frame bundle, which consists of a manifold M, with fibres at each point the set of ordered bases of the tangent space at that point, i.e., locally it is of the form (x, e_α), where $x \in M$ and $e_\alpha, \alpha = 1, 2, ..., n$ is a set of vectors that form a basis of T_xM. The structure group is again $GL(n, \mathbb{R})$, acting naturally on e_α. Actually, the fibres are isomorphic to the structure group $GL(n, \mathbb{R})$ and we have constructed a $GL(n, \mathbb{R})$-*principal bundle*. Conversely, given a G-principal bundle and a particular representation of G on a vector space V, we can construct its *associated* vector bundle identifying the fibres with V and assuming the natural action of G on V induced by the particular representation.

Now, a reduction of the structure group G can be described by globally defined, nondegenerate tensors. If we have one such tensor S, we can choose frames e_α at each patch such that S has the same form everywhere, and then require that the action of a transition function leaves this form invariant. Thus, in general, we will only allow actions by a subgroup $G' \subseteq G$. The most interesting examples are provided by an orientation, reducing the structure group to $SL(n, \mathbb{R})$ and a riemannian metric g, reducing it to $O(n,\mathbb{R})$. If we consider both g and its natural volume form, the structure group reduces to $SO(n,\mathbb{R})$.

More concretely, we can introduce a basis of vector fields orthonormal with respect to the metric g ("frames"), such that $g = ee^T$ or $g = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \delta_{\alpha\beta} e^\alpha_\mu e^\beta_\nu \,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, using the *vielbein* e^μ_α and the inverse vielbein $e^{*\alpha}_{\ \mu}$ (sometimes we may omit the '*' and write it as e^{-1}). Similarly we can write any tensor with "flat" indices $\alpha, \beta, \ldots$ instead of "curved" $\mu, \nu, \ldots$. Then, $GL(n,\mathbb{R})$ acts naturally on the flat indices as a transformation of basis, and tensors can be thought of as representations of $GL(n,\mathbb{R})$.

If the metric is defined in an overlap region $U_{(1)} \cap U_{(2)}$ and we require that it has the same form in both coordinate systems we can write

$$g = \delta_{\alpha\beta} e^\alpha_{(1)\mu} e^\beta_{(1)\nu} \,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \delta_{\alpha\beta} e^\alpha_{(2)\mu} e^\beta_{(2)\nu} \,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu .$$

Then, the existence of transition functions imply that, at each point in $U_{(1)} \cap U_{(2)}$, $e^\alpha_{(1)\mu} = \Lambda^\alpha_\beta e^\beta_{(2)\mu}$ for $\Lambda^\alpha_\beta \in GL(n,\mathbb{R})$. Thus, switching to matrix notation, we find that

$$\Lambda\Lambda^T = I,$$

i.e. $\Lambda \in SO(n,\mathbb{R})$.

The above considerations show that in general, a riemannian metric g parameterises the coset space $GL(n,\mathbb{R})/O(n,\mathbb{R})$, since every transformation in $O(n,\mathbb{R})$ preserves its form.

Finally, the generalisation to complex vector bundles is immediate if we replace $\mathbb{R}$ with $\mathbb{C}$ everywhere in the above.

Next, we proceed to introduce the basics of complex linear algebra and complex differential geometry.

Let V be a finite dimensional real vector space.

Definition 2.1.5 (Complexification) *The complexification of the real vector space V is $V \otimes \mathbb{C}$ and it will be denoted by $V_{\mathbb{C}}$.*

Definition 2.1.6 (Almost Complex Structure (on vector spaces)) *An endomorphism $J : V \to V$ with $J^2 = -\mathrm{I}$ is called an almost complex structure on V.*

We can easily see that if V is the real vector space underlying a complex vector space, then $J(v) = i \cdot v, \forall v \in V$ defines an almost complex structure on V. Conversely, by defining $(a + ib) \cdot v = a \cdot v + b \cdot J(v)$, we see that V admits the structure of a complex vector space, i.e. for vector spaces almost complex structures are equivalent to complex structures, while, as we will mention later, this is not the case for the analogous structures on manifolds. This equivalence implies that V must be even-dimensional, since $V \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$.[1] In the following, we will also denote by J the $\mathbb{C}$-linear extension $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ of the almost complex structure J.

Next we mention some results that can be easily verified.

Lemma 2.1.7 *If $V^{1,0}$ and $V^{0,1}$ are the $\pm i$-eigenspaces of J, $V^{1,0} = \{v \in V_{\mathbb{C}} | J(v) = i \cdot v\}$ and similarly for $V^{0,1}$, then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

and $V^{1,0} \cong V^{0,1}$.

Lemma 2.1.8 *The dual space V^* has a natural almost complex structure given by $J(f)(v) = f(J(v))\ \forall f \in V^*$. In addition, we have that*

$$(V^*)^{1,0} = (V^{1,0})^*$$

$$(V^*)^{0,1} = (V^{0,1})^*$$

[1] This can also be seen by: $0 \leq (\det J)^2 = \det J^2 = det(-1) = (-1)^k \Rightarrow k \in 2\mathbb{N}$.

We can define similar gradings of $\bigwedge^k V^*$, but we will not analyse this point further.

Next, we transfer the almost complex structure to manifolds.

Definition 2.1.9 (Almost Complex Structure (on manifolds)) *An* almost complex structure J *on a manifold M is an endomorphism $J : TM \to TM$, or equivalently a section $J \in C^\infty(TM \otimes T^*M)$ squaring to the identity $J^2 = -\mathrm{I}$ and respecting the bundle structure, i.e. such that $\pi(Jv) = \pi(v)$, $\forall v \in TM$.*

We understand that at each point x of the n-dimensional manifold M we can apply the above results for the linear spaces. Specifically, the manifold M must be even-dimensional and the complexified tangent bundle $TM \otimes \mathbb{C}$ decomposes as $TM \otimes \mathbb{C} = T^{(1,0)} \oplus T^{(0,1)}$, where $T^{(1,0)}$ and $T^{(0,1)}$ are the $\pm i$ eigenbundles of J. Then, the structure group reduces to $GL(\frac{n}{2}, \mathbb{C})$.

At any point x of an even-dimensional manifold M, we can find two bases of vector fields $\{\dfrac{\partial}{\partial z^\alpha}\}$ and $\{\dfrac{\partial}{\partial \bar{z}^\alpha}\}$, $\alpha = 1, 2, ..., \frac{n}{2}$ spanning $T_x^{(1,0)}$ and $T_x^{(0,1)}$ respectively. These will generally not be defined in a neighbourhood of x. *If* they can be defined in a local neighbourhood of any point in M, the almost complex structure J can be made to have the canonical form $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ with respect to the above basis. Then we say that the distributions $T^{(1,0)}$ and $T^{(0,1)}$ are integrable (the integrability of $T^{(1,0)}$ implies the integrability of $T^{(0,1)}$, as they are related by complex conjugation) or equivalently that J is integrable and, in that case, M admits a complex structure and becomes a complex manifold in the usual sense. A classic theorem of Frobenius states that:

Theorem 2.1.10 (Frobenius) *A distribution L is integrable if and only if it is invlolutive, i.e. $[X, Y] \in C^\infty(L)$, $\forall X, Y \in C^\infty(L)$, where $[\cdot, \cdot]$ is the Lie bracket.*

The Newlander-Nirenberg theorem states that:

Theorem 2.1.11 (Newlander-Nirenberg) *J is integrable if and only if the* Nijenhuis tensor $N_J = J[JX,Y] + J[X,JY] - [JX,JY] - J^2[X,Y]$ *vanishes for all* $X,Y \in C^\infty(TM)$.

We note here that elements of $\bigwedge^k T^*M$ are called complex differential forms.

As we noted in the introduction, generalized geometry unifies complex and symplectic geometry, so we also need to introduce the latter concept.

Definition 2.1.12 (Symplectic Manifold) *A manifold M is called* symplectic *if it is equipped with a nondegenerate, closed 2-form ω.*

This in particular implies that M is even-dimensional. A very important theorem holds for symplectic manifolds:

Theorem 2.1.13 (Darboux) *Let (M,ω) be a symplectic manifold of dimension $2n$. Then it is locally diffeomorphic to the symplectic manifold $(\mathbb{R}^{2n},\omega_0)$, with $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$, with (x^i,y^i), $i = 1,\ldots,n$ the canonical coordinates on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.*

Finally, we introduce some more concepts that will lead us to the definition of a Kähler manifold.

Definition 2.1.14 (Almost Hermitian Manifold) *An* almost hermitian manifold *is an almost complex manifold equipped with a* hermitian metric, *i.e. a riemannian metric g invariant under the almost complex structure J in the sense that*

$$g(Jv,Ju) = g(v,u), \ \forall v,u \in TM.$$

We observe that every almost complex riemannian manifold with riemannian metric g admits the hermitian metric $h(v,u) = g(v,u) + g(Jv,Ju)$, $\forall v,u \in TM$.

Definition 2.1.15 (Fundamental 2-form) *Let (M,J,g) be an almost hermi-*

tian manifold. The fundamental 2-form *is defined by*

$$\omega(v,u) = g(v, Ju),$$

which is also invariant under J and nondegenerate.

Definition 2.1.16 (Kähler Manifold) *If the fundamental 2-form ω of an almost hermitian manifold (M, J, g) is closed, i.e. it is a symplectic structure on M, the hermitian metric g is called* Kähler metric *and the manifold M is called* almost Kähler*. If in addition J is integrable, M is a* Kähler manifold.

Then, the structures (J, g, ω) form a *compatible triple*, since any two of them determine the third, if we also require that the relevant compatibility conditions (such as those in the definitions above) hold.

2.2 Fundamentals of Generalized Geometry

In the rest of this section, we will closely follow the presentation of [14]. Another useful reference for generalized geometry and applications is [20].

We begin our analysis of generalized geometry by studying the structures on the direct sum $V \oplus V^*$ of an m-dimensional vector space V and its dual V^*. Any element belonging in $V \oplus V^*$ can be written as $v = X + \xi$, with $X \in V$ and $\xi \in V^*$. We can define a natural inner product $\langle \cdot, \cdot \rangle$ in $V \oplus V^*$ by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)), \tag{2.1}$$

where $X, Y \in V$ and $\xi, \eta \in V^*$. If we introduce a matrix notation by setting

$$v = \begin{pmatrix} X \\ \xi \end{pmatrix}$$

we can write the left-hand side of (2.1) as

$$\langle X+\xi, Y+\eta\rangle = (X \quad \xi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Y \\ \eta \end{pmatrix} \tag{2.2}$$

and we can easily see that $\langle\cdot,\cdot\rangle$ is nondegenerate, symmetric and has signature (m,m). Thus, the linear transformations in $GL(2m,\mathbb{R})$ preserving the inner product actually belong to $O(m,m)$.[2] $V \oplus V^*$ is obviously orientable and possesses a canonical orientation which reduces the Lie group of symmetries to $SO(m,m)$. We also note here that $SO(m,m)$ has two connected components, contrary to the usual orthogonal groups $O(m)$ (heuristically speaking, you can reverse the orientation of both V and V^* or of none and you still keep the same orientation of $V \oplus V^*$).

We move on to examine the Lie algebra $\mathfrak{so}(m,m)$, using the splitting $V \oplus V^*$. We can easily check that it contains elements T of the form

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix} \tag{2.3}$$

where $A \in \mathrm{End(V)}$, $B : V \to V^*$, $\beta : V^* \to V$ with B and β skew, i.e. $B^* = -B$ and $\beta^* = -\beta$ (recall that the transpose of a map $B : V \to V^*$ is the map $B^* : V \to V^*$ such that $Bv(w) = Bw(v)\ \forall v, w \in V$). If we consider the maps as matrices, these relations become $B^T = -B$ and $\beta^T = -\beta$. Equivalently, by viewing the actions of B and β as $B(X) \equiv i_X B \in V^*$ and $\beta(\xi) \equiv i_\xi \beta \in V$ we can regard B as a 2-form in $\wedge^2 V^*$ and β as a bivector in $\wedge^2 V$. Thus, we find the decomposition $\mathfrak{so}(m,m) = \mathrm{End(V)} \oplus \wedge^2 V^* \oplus \wedge^2 V$.

By exponentiating the above infinitesimal elements we can find the corresponding group transformations:

[2]It would be more accurate to write $GL(V)$, $O(V \oplus V^*)$... instead of $GL(m)$, $O(m,m)$..., but there are natural isomorphism between the corresponding groups, so we will not distinguish them in the following.

$GL(m)$ **action:**

$$g_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} + ... = \begin{pmatrix} \exp A & 0 \\ 0 & (\exp A^*)^{-1} \end{pmatrix} \tag{2.4}$$

Since $\det \exp A > 0$ and $\det g_A = (\det \exp A)^2 > 0$, this defines is an embedding of the identity component $GL^+(m) = \{g \in GL(m) | \det g > 0\}$ of $GL(m)$ into the identity component of $SO(m,m)$. We can see that, by extending the above mapping according to

$$T \to \begin{pmatrix} T & 0 \\ 0 & (T^*)^{-1} \end{pmatrix} \tag{2.5}$$

for all $T \in GL(m)$, we can map $GL^-(m)$ to the second connected component of $SO(m,m)$. Its action on elements $X + \xi$ of $V \oplus V^*$ is: $X + \xi \to G(X) + (G^*)^{-1}\xi$.

B**-transform:** Following the same steps for the B-transformations, we find:

$$g_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \tag{2.6}$$

Its action on elements $X + \xi$ of $V \oplus V^*$ is: $X + \xi \to X + \xi + i_X B$. In the bibliography, this transformation is usually denoted by e^B and is called *B-transformation*.

β**-transform:** Similarly we compute:

$$g_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

This β-transform acts on elements $X+\xi$ of $V \oplus V^*$ as: $X+\xi \to X+i_\xi\beta+\xi$.

Having completed the presentation of the symmetries of $V \oplus V^*$ equipped with the inner product $\langle \cdot, \cdot \rangle$, we move on to introduce the concept of isotropic subspaces.

Definition 2.2.1 (Isotropic Subspace) *A subspace $L < V \oplus V^*$ is called* isotropic *when $\langle v, w\rangle = 0,\ \forall v, w \in L$. In particular, setting $w = v$, we see that every element of L has zero norm with respect to $\langle \cdot, \cdot \rangle$. Since the metric has signature (m, m), the maximal possible dimension of an isotropic subspace is m and we call such subspaces* maximal isotropic subspaces*.*

The most trivial examples of maximal isotropics are V and V^*. More generally, any subspace of the form $L(E, \epsilon) \equiv \{X + \xi \in E \oplus V^* |\ \xi_{|E} = \epsilon(X)\}$, where $E < V$ and $\epsilon \in \bigwedge^2 E^*$, is a maximal isotropic subspace, as can be easily verified. A little less trivial is to prove that *every* maximal isotropic L can be written in the form $L(E, \epsilon)$. E is of course the projection $E = \pi(L)$ with π the natural projection $\pi : V \oplus V^* \to V$ and the appropriate 2-form ϵ on E^* can be constructed uniquely. It is a simple matter now to extend the above constructions to the complexified space $(V \oplus V^*) \otimes \mathbb{C}$. The inner product $\langle \cdot, \cdot \rangle$ can be naturally extended and a maximal isotropic is now a complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$, isotropic with respect to $\langle \cdot, \cdot \rangle$ and characterised by a complex subspace $E < V \otimes \mathbb{C}$ together with a complex 2-form $\epsilon \in \bigwedge^2 E^*$.
We also need to define the real index:

Definition 2.2.2 (Real Index) *Let $L < (V \oplus V^*) \otimes \mathbb{C}$ be a maximal isotropic subspace. Then $L \cap \bar{L}$ is real, i.e. the complexification of a real space: $L \cap \bar{L} = K \otimes \mathbb{C}$, for $K < V \oplus V^*$.*[3] *The real index r of L is defined by*

$$r = \dim_{\mathbb{C}} L \cap \bar{L} = \dim_{\mathbb{R}} K.$$

Now, we can transfer the above structures to a general differentiable manifold M of dimension m. If TM is the tangent bundle and T^*M is the cotangent bundle, we can construct the sum $TM \oplus T^*M$. Then all the above considerations

[3]This can be easily proven as follows: a general vector in L has the form $z \equiv (a_1 + ia_2)(v + iw)$, for $a_1, a_2 \in \mathbb{R}$ and $v, w \in (V \oplus V^*)$. Requiring that it also belongs in $\bar{L}$ we find that $a_2 v + a_1 w = 0$, so $z = cv$, where $\mathbb{C} \ni c = (a_1^2 + (a_2^2)/a_1 + i(a_1 a_2 - (a_2^2)/a_1))$.

hold at each point $x \in M$. The bundle $TM \oplus T^*M$ has the same inner product $\langle \cdot, \cdot \rangle$ and natural orientation as $V \oplus V^*$. So, we can consider as structure group the group $O(m, m)$ or $SO(m, m)$, acting naturally on the fibres while preserving the inner product.

Before we conclude this subsection, we should note that we can define the Clifford algebra $CL(V \oplus V^*)$ and the spin group $Spin(V \oplus V^*)$ in the usual way and they can be represented on the space of polyforms $\bigwedge^\bullet V^* := \sum_{k\in\mathbb{N}} \bigwedge^k V^*$. In addition, we can always lift the structure group $SO(m, m)$ of $TM \oplus T^*M$ to a $Spin(m, m)$ structure. Then we can prove the very interesting fact that a maximal isotropic subbundle of $TM \oplus T^*M$ can be represented by a line subbundle of $\bigwedge^\bullet V^*$ and we can translate most of the results in generalized geometry in corresponding statements about these line subbundles. However, this interesting analogy will not be developed here, as it is not directly relevant to the rest of the thesis.

2.3 The Courant Bracket

We continue our presentation with a few important definitions. A Lie algebroid can be seen as a generalisation of the tangent bundle:

Definition 2.3.1 (Lie Algebroid) *A* Lie Algebroid *is a vector bundle* (L, M, π) *equipped with a Lie bracket* $[\cdot, \cdot]$ *on* $C^\infty(L)$ *and a smooth bundle map* $a : L \to TM$ *called the* anchor*. The anchor must induce a Lie algebra homomorphism* $a : C^\infty(L) \to C^\infty(TM)$*, i.e.*

$$a([X, Y]) = [a(X), a(Y)], \quad \forall X, Y \in C^\infty(L),$$

such that the following Leibnitz rule is satisfied:

$$[X, fY] = f[X, Y] + (a(X)f)Y, \quad \forall X, Y \in C^\infty(L), f \in C^\infty(M). \tag{2.8}$$

Note that the condition (2.8) corresponds to the similar property for vector fields: $[X, fY] = f[X, Y] + X[f]Y, \quad \forall X, Y \in C^\infty(TM), f \in C^\infty(M)$.

A *complex* Lie algebroid is defined analogously, with L a complex vector bundle and $a : L \to T \otimes \mathbb{C}$ a complex map, satisfying the analogous complexified conditions. A simple example of a complex Lie Algebroid is the $+i$ subbundle $T_{1,0}$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ of a manifold M endowed with a complex structure J (from theorem (2.1.11) we see that an *almost* structure is not enough, since $T_{1,0}$ is closed under the Lie bracket if and only if J is an actual complex structure), with a the inclusion map.

We can construct an analogue of the exterior derivative and the Lie derivative on Lie algebroids:

Definition 2.3.2 (Schouten Bracket) *The* Schouten bracket *acting on sections $X_1 \wedge \ldots \wedge X_p \in C^\infty(\bigwedge^p L)$, $Y_1 \wedge \ldots \wedge Y_q \in C^\infty(\bigwedge^q L)$ of a Lie algebroid L is*

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \hat{X}_i \ldots \hat{Y}_j \ldots \wedge Y_q, \quad (2.9)$$

where, on the right-hand side, $[\cdot, \cdot]$ is the Lie bracket on L and the hat indicates a term missing from the exterior product. The Schouten bracket can be extended to act on functions f on M as

$$[X, f] = -[f, X] = a(X)f, \quad (2.10)$$

where $X \in C^\infty(L)$ and $f \in C^\infty(M)$.

Definition 2.3.3 (Lie Algebroid Derivative) *The* Lie algebroid derivative *is a linear operator $d_L : C^\infty(\bigwedge^k L^*) \to C^\infty(\bigwedge^{k+1} L^*)$ defined by:*

$$\begin{aligned} d_L \sigma(X_0, \ldots, X_k) = & \sum_i (-1)^i a(X_i) \sigma(X_0, \ldots, \hat{X}_i, \ldots, X_k) \\ & + \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \end{aligned} \quad (2.11)$$

where $\sigma \in C^\infty(\bigwedge^k L^*)$ *and* $X_i \in C^\infty(L)$*.*

The Jacobi identity for $[\cdot,\cdot]$ implies that $d_L^2 = 0$. Then, if we define the interior product i_X as a -1 degree derivation on $C^\infty(\bigwedge^\bullet L^*)$ by $i_X\sigma = \sigma(X, \ldots)$, we can also define a Lie derivative $\mathcal{L}_X^L$ by

$$\mathcal{L}_X^L = d_L i_X + i_X d_L, \tag{2.12}$$

similarly to the form of the usual Lie derivative when acting on exterior forms. Another concept that we will use later on is the Lie bialgebroid, which is a special pair of Lie algebroids:

Definition 2.3.4 *A Lie algebroid* L *and its dual bundle* L^* *form a* Lie bialgebroid (L, L^*) *if* L^* *is also a Lie algebroid and they are "compatible" with the exterior derivative* d_L *in the sense that:*

$$d_L[X,Y] = [d_L X, Y] + [X, d_L Y].$$

If we consider the tangent bundle TM of a manifold M with the usual Lie bracket and the identity anchor, and the cotangent bundle T^*M with the zero bracket and anchor, we can easily check that the pair (TM, T^*M) forms a Lie bialgebroid.

Now, we move on to the main definitions of this subsection.

Definition 2.3.5 (Dorfman Bracket) *The* Dorfman bracket $\circ$ *is defined on sections* $X+\xi, Y+\eta$ *of* $TM \oplus T^*M$ *by:*

$$(X+\xi)\circ(Y+\eta) = \mathcal{L}_X(Y+\eta) - i_Y d\xi \tag{2.13}$$

A simple calculation shows that the Dorfman bracket is not skew-symmetric, but it satisfies a Leibnitz rule similar to the Jacobi identity:

$$A\circ(B\circ C) = (A\circ B)\circ C + B\circ(A\circ C), \tag{2.14}$$

for all $A, B, C \in C^\infty(TM \oplus T^*M)$. Thus, it is also termed "Dorfman derivative". The symmetrisation of the Dorfman bracket is an exact form,

$$((A, B)) \equiv \frac{1}{2}(A \circ B + B \circ A) = d\langle A, B\rangle \tag{2.15}$$

for all $A, B \in C^\infty(TM \oplus T^*M)$. This fact will be useful later on.

Definition 2.3.6 (Courant Bracket) *The antisymmetrisation of the Dorfman bracket is called the* Courant bracket*:*

$$\begin{aligned} [[A, B]] &\equiv \frac{1}{2}(A \circ B - B \circ A) = [X + \xi, Y + \eta]_C \\ &= [X, Y] + \mathcal{L}_X \eta + \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) \end{aligned} \tag{2.16}$$

for all $A = X + \xi, B = Y + \eta \in C^\infty(TM \oplus T^*M)$. We will usually omit the "C" *when denoting the Courant bracket. It should be clear from the context whether it is the Courant or the Lie bracket (generally no confusion will be caused, since, when acting on vector fields, the Courant bracket reduces to the Lie bracket).*

The Courant bracket is not a Lie bracket despite being skew-symmetric, because it does not satisfy the Jacobi identity. However, it fails to do so in a interesting way:

Proposition 2.3.7

$$\mathrm{Jac}(A, B, C) = d(\mathrm{Nij}(A, B, C)), \tag{2.17}$$

for all $A, B, C \in C^\infty(TM \oplus T^*M)$, where we have defined the Jacobiator *by*

$$\mathrm{Jac}(A, B, C) = [[A, B], C] + \text{ cyclic permutations} \tag{2.18}$$

and the Nijenhuis operator *by*

$$\mathrm{Nij}(A, B, C) = \frac{1}{3}(\langle [A, B], C\rangle + \text{ cyclic permutations}). \tag{2.19}$$

This proposition can be proved by brute force computation, using equation (2.15) and the relations $\mathcal{L}_X = i_X d + d i_X$ (when acting on forms), $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$, $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$.
Thus, the Courant bracket does not make the vector bundle $TM \oplus T^*M$ into a Lie algebroid. Additionally, axiom (2.8) is not satisfied, but instead

$$[A, fB] = f[A, B] + (\pi(A)f)B - \langle A, B\rangle df, \tag{2.20}$$

where $A, B \in C^\infty(TM \oplus T^*M)$, $f \in C^\infty(M)$ and π is the natural projection to the tangent bundle $\pi : TM \oplus T^*M \to TM$. A similar direct computation shows that

$$\pi(A)\langle B, C\rangle = \langle [A, B] + d\langle A, B\rangle, C\rangle + \langle B, [A, C] + d\langle A, C\rangle\rangle. \tag{2.21}$$

We can define a generalisation of a Lie algebroid, called *Courant algebroid*, by relaxing some of the defining conditions in definition 2.3.1 as follows:

Definition 2.3.8 (Courant Algebroid) *A* Courant algebroid $(E, \langle,\rangle, [,], \pi)$ *is a vector bundle E over M with an inner product $\langle\cdot,\cdot\rangle$, a skew-symmetric bracket $[\cdot,\cdot]$ and an anchor $\pi : E \to TM$ if the following conditions hold:*

- $\pi([A, B]) = [\pi(A), \pi(B)]$
- $\mathrm{Jac}(A, B, C) = \mathcal{D}(\mathrm{Nij}(A, B, C))$
- $\pi(A)\langle B, C\rangle = \langle [A, B] + \mathcal{D}\langle A, B\rangle, C\rangle + \langle B, [A, C] + \mathcal{D}\langle A, C\rangle\rangle$

for all $A, B, C \in C^\infty(E)$. Here we denote by $\mathcal{D}$ the differential operator $\mathcal{D} : C^\infty(M) \to C^\infty(E)$ defined by $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\pi(A)f$, $\forall f \in C^\infty(M)$, $A \in C^\infty(E)$.

If we set $E = TM \oplus T^*M$ and $[\cdot,\cdot]$ the Courant bracket, we can see that $\mathcal{D}$ is just the differential d of functions on the manifold M and then $(TM \oplus T^*M, \langle,\rangle, [,], \pi)$ is a Courant algebroid.

Next, we are interested in finding the symmetries of the Courant bracket. First of all, we can compute the effect of a B-transformation (see equation (2.6)) on the Courant bracket:

$$\begin{aligned}&[e^B(X+\xi), e^B(Y+\eta)] = [X+\xi+i_XB, Y+\eta+i_YB] = \dots\\&= e^B([X+\xi, Y+\eta]) + i_Yi_XdB\end{aligned} \tag{2.22}$$

for all $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M),\ B \in C^\infty(\bigwedge^2 T^*M)$. Thus, we get the lemma:

Lemma 2.3.9 *The B-transformation e^B is an automorphism of the Courant bracket if and only if B is closed. In that case, we refer to the transformation as a B-*field transformation.

The B-field transformation is important because of the two following propositions:

Proposition 2.3.10 *Let F be an automorphism of the tangent bundle TM (see definition (2.1.2)), restricting to the diffeomorphism f on M. If F is a symmetry of the Lie bracket in the sense that $F([X,Y]) = [F(X), F(Y)],\ \forall X, Y \in C^\infty(TM)$, then F must be the pushforward of f, i.e. $F = f_*$. So, the group of Lie automorphisms of TM is just Diff(M).*

Proposition 2.3.11 *Let F be an* orthogonal *(i.e. preserving the inner product $\langle,\rangle$) automorphism of $TM \oplus T^*M$, restricting to the diffeomorphism f on M. If F is a symmetry of the Courant bracket in the sense that $F([A,B]) = [F(A), F(B)],\ \forall A, B \in C^\infty(TM \oplus T^*M)$, then F must be the composition of the "pushforward" of f and a* closed *B-field transformation. The group of Courant automorphisms of $TM \oplus T^*M$ is Diff$(M) \ltimes \Omega^2_{closed}(M)$.*

Here, by "pushforward" we actually mean the orthogonal transformation $f_C = \begin{pmatrix} f_* & 0 \\ 0 & (f^*)^{-1} \end{pmatrix}$, which obviously preserves the Courant bracket.

The next key concept we will introduce is that of a *Dirac structure*:

Definition 2.3.12 (Dirac Structure) *A real, maximal isotropic subbundle L of a Courant algebroid $(E, \langle,\rangle, [,], \pi)$ is called an* almost Dirac structure. *We will say that an almost Dirac structure is integrable to a* Dirac structure *if L is involutive, i.e. closed under the Courant bracket. Similarly we define the* complex *Dirac structure.*

The following proposition is particularly useful, as it connects some properties of $L < TM \oplus T^*M$ with its involutivity.

Proposition 2.3.13 *Let L be a (possibly complex) almost Dirac structure in $TM \oplus T^*M$. Then the following conditions are equivalent:*

$$L \text{ is involutive} \Leftrightarrow \text{Nij}|_L = 0 \Leftrightarrow \text{Jac}|_L = 0. \tag{2.23}$$

We saw previously that a Courant algebroid failed to be a Lie one due to some terms involving the inner product $\langle,\rangle$. Then, for a Dirac subbundle $L < TM \oplus T^*M$ these terms vanish and $(L, [,], \pi)$ is a proper Lie algebroid. Also, any transformation of L in $Diff(M) \rtimes \Omega^2_{closed}(M)$ will still be a Lie algebroid, since both $\langle,\rangle$ and $[,]$ will be preserved.
The above definitions enable us to understand a deeper connection between Lie bialgebroids and Courant algebroids, first noted in [21], where the next two theorems were proven.

Theorem 2.3.14 *Let (L, L^*) be a Lie bialgebroid. Then we can define the inner product $\langle,\rangle$ on $L \oplus L^*$:*

$$\langle A + \alpha, B + \beta \rangle = \frac{1}{2}(\alpha(B) + \beta(A)), \quad \forall A, B \in L, \ \alpha, \beta \in L^*$$

and the skew-symmetric bracket

$$\begin{aligned} [A + \alpha, B + \beta] &= [A, B] + \mathcal{L}_\alpha B - \mathcal{L}_\beta A - \frac{1}{2} d_{L^*}(i_A\beta - i_B\alpha) \\ &+ [\alpha, \beta] + \mathcal{L}_A \beta - \mathcal{L}_B \alpha + \frac{1}{2} d_L(i_A\beta - i_B\alpha), \end{aligned} \tag{2.24}$$

with the exterior and Lie derivatives defined on Lie algebroids as in definition (2.3.3) and equation (2.12). If we also define the map

$$\pi = a + a_* : L \oplus L^* \to TM,$$

where a and a_* are the anchors of L and L^* respectively, we see that $(L \oplus L^*, [,], \langle,\rangle, \pi)$ is a Courant algebroid. Then we also have $\mathcal{D} = d_L + d_L^*$.

The following theorem describes the converse construction:

Theorem 2.3.15 *Let $(E, [,], \langle,\rangle, \pi)$ be a Courant algebroid and $L, L' < E$ be Dirac subbundles such that $L \oplus L' = E$. Then, using the inner product $\langle,\rangle$ we can identify $L' = L^*$ and (L, L') is a Lie algebroid.*

We note that theorem (2.3.14) treats L and L^* symmetrically. However, in the Courant algebroid $TM \oplus T^*M$ we studied above, the Courant bracket is very asymmetric in sections of TM and T^*M. We now see that this is because the Lie algebroid structure of TM and T^*M is very different: TM has the usual exterior and Lie derivative, but the corresponding operations in T^*M vanish (see definition (2.3.4) and the following discussion). Then the bracket (2.24) reduces to the Courant bracket (2.16).

Finally, we can define a twisted Courant bracket on $TM \oplus T^*M$ by a 3-form H as:

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + i_Y i_X H \tag{2.25}$$

Then, if we set $A = X + \xi,\ B = Y + \eta,\ C = Z + \zeta$, the corresponding Jacobiator and Nijenhuis tensor are

$$\mathrm{Nij}_H(A, B, C) = \mathrm{Nij}(A, B, C) + H(X, Y, Z) \tag{2.26}$$

and

$$\mathrm{Jac}_H(A, B, C) = d(\mathrm{Nij}_H(A, B, C)) + i_Z i_Y i_X dH. \tag{2.27}$$

Thus, we see that $[,]_H$ defines a Courant algebroid structure on $TM \oplus T^*M$ (using the same inner product and anchor) if and only if $dH = 0$. In addition, an easy computation shows that:

$$[e^b(A), e^b(B)]_H = b^b([A,B])_{H+db}, \tag{2.28}$$

for all $A, B \in C^\infty(TM \oplus T^*M)$ and $b \in \bigwedge^2(T^*M)$. So, e^b is a symmetry of the H-twisted Courant bracket $[,]_H$ if and only if $db = 0$, i.e. b must be a B-field transformation.

Using equation (2.28) we can see that a subbundle L is involutive for $[,]_H$ if and only if e^{-b} is involutive for $[,]_{H+db}$ and so the twist actually depends only the cohomology class $[H] \in H^3(M, \mathbb{R})$ of the closed 3-form H.[4]

2.4 Generalized Complex Structures

In order to study the main concept of generalized geometry, namely generalized complex structures, we need to return to studying structures in vector spaces and first introduce the linear generalized complex structure.

Recall that a complex structure J on a real vector space V is an endomorphism such that $J^2 = -1$, while a symplectic structure is a nondegenerate skew form $\omega \in \bigwedge^2 V^*$. ω induces an isomorphism $\omega : V \to V^*$ by

$$\omega : v \to i_v \omega, \quad \forall v \in V.$$

Then $i_u(\omega(v)) = i_u i_v \omega = -i_v i_u \omega = -i_v(\omega(u))$, i.e. $\omega^* = -\omega$, where ω^* maps V to V^*.

Naturally, we consider the sum of V and V^* and define the generalized complex structure on V as:

Definition 2.4.1 (Generalized Complex Structure (on V)) *A generalized*

[4] If $\frac{1}{2\pi i}[H]$ is integral, we can interpret geometrically the twist as a nontrivial topological twist of the bundle $(TM \oplus T^*M)$, by using the language of gerbes (see for example [15]), but we will not use this approach.

complex structure $\mathcal{J}$ on V is an endomorphism of $V \oplus V^*$ which is both complex and symplectic, i.e. such that

$$\mathcal{J}^2 = -1 \text{ and } \mathcal{J}^* = -\mathcal{J}. \tag{2.29}$$

We also find that $\mathcal{J}^*\mathcal{J} = 1$.

Actually, the above definition is equivalent to the following statement:

Proposition 2.4.2 *A generalized complex structure $\mathcal{J}$ on V is equivalent to the specification of a maximal isotropic complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ of real index zero, i.e. such that $L \cap \bar{L} = \{0\}$.*

We can easily see that L is (conventionally) the $+i$ eigenspace of $\mathcal{J}$.

In analogy with the complex structure on V, a generalized complex structure exists if and only if the dimensional m of V is even and defines a reduction of the symmetries (structure) to $U(\frac{m}{2}, \frac{m}{2}) = O(m, m) \cap GL(m, \mathbb{C})$. That means that the space of inequivalent generalized complex structures on V is $S_{\mathcal{J}} = O(m, m)/U(\frac{m}{2}, \frac{m}{2})$. Thus we can apply a transformation $g \in O(m, m)$ on $\mathcal{J}$ as

$$\mathcal{J} \rightarrow g\mathcal{J}g^{-1}, \tag{2.30}$$

with the corresponding maximal isotropic subspace mapped to $L \rightarrow g(L)$.

The usual complex J and symplectic ω structures are special cases of generalized complex structures, since we can define

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \tag{2.31}$$

and

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{2.32}$$

and we can check that $\mathcal{J}_J$ and $\mathcal{J}_\omega$ indeed satisfy the conditions of definition (2.4.1). This is the first strong indication that generalized geometry, "interpolates" in a sense between complex and symplectic geometry. The corresponding maximal isotropics are $L_\omega = \{X - i\omega(X) | X \in V \otimes \mathbb{C}\}$ and $L_J = V_{1,0} \oplus V^*_{0,1}$.

We move on to transfer these structures to a manifold M. The definition of a generalized almost complex structure is natural:

Definition 2.4.3 (Generalized Complex Structure (on M)) *A* generalized almost complex structure *on an m-dimensional manifold M is an almost complex structure $\mathcal{J}$ on $(T \oplus T^*)M$, orthogonal with respect to the inner product $\langle , \rangle$, or in other words, a reduction of the structure group of $(T \oplus T^*M)$ from $O(m, m)$ to $U(\frac{m}{2}, \frac{m}{2})$. Equivalently, it is determined by a maximal isotropic subbundle $L < (T \oplus T^*M) \otimes \mathbb{C}$ of real index zero, $L \cap \bar{L} = 0$.*

This means that we have the decomposition $(TM \oplus T^*M) \otimes \mathbb{C} = L \cap \bar{L}$, with L being the $+i$-eigenbundle of $\mathcal{J}$ and $\bar{L}$ identified as the vector bundle dual to L, i.e. $\bar{L} = L^*$.

It can be seen that a generalized almost complex structure exists on M if and only if an almost complex structure exists.

We would like to define an integrability condition, analogous to the one for the almost complex structure (see subsection 2.1). Now, the appropriate notion is that of Courant involutivity, in the sense of subsection 2.3.

Definition 2.4.4 (Integrable Generalized Almost Complex Structure) *We say that the generalized almost complex structure $\mathcal{J}$ on a manifold M is integrable to a* generalized complex structure *if its $+i$-eigenbundle $L < (TM \oplus T^*M) \otimes \mathbb{C}$ is a complex Dirac structure of real index zero.*

In that case, the pair $(L, \bar{L})$ forms a Lie bialgebroid and $L \oplus \bar{L} = (TM \oplus T^*M) \otimes \mathbb{C}$ is naturally a Courant algebroid.

Suppose now that M is a hyperkähler manifold, i.e. it is equipped with 3 distinct complex structures I, J, K, each with its own fundamental 2-form $\omega_I, \omega_J, \omega_K$, such that $IJ = K = -JI$ etc. Then, it can be shown that

$$\mathcal{J}_t = \sin t \mathcal{J}_I + \cos t \mathcal{J}_{\omega_J}, \quad \forall t \in [0, \frac{\pi}{2}] \tag{2.33}$$

is a generalized complex structure that interpolates between a complex and a symplectic one.

We can also prove a generalized version of the Darboux theorem (2.1.13):

Theorem 2.4.5 (Generalized Darboux) *A generalized complex manifold M is locally equivalent, by a diffeomorphism and a B-field transformation, to $A \times B$, where A is an open set $A \subset \mathbb{C}^k$ and B is an open set $B \subset (\mathbb{R}^{2n-2k}, \omega_0)$.*

Finally, we mention that we can define the twisted version of a generalized complex structure:

Definition 2.4.6 *A generalized almost complex structure $\mathcal{J}_H$ is said to be H-*twisted *if its $+i$-eigenbundle $L < (TM \oplus T^*M) \otimes \mathbb{C}$ is involutive with respect to the H-twisted Courant bracket.*

Note that a similar analysis to that at the end of the previous subsection shows that $\mathcal{J}_H$ depends only on the cohomology class $[H] \in H^3(M, \mathbb{R})$.

2.5 The Generalized Metric

It is obvious that the $O(m, m)$ structure of $TM \oplus T^*M$ can be reduced to $O(m) \times O(m)$ by an appropriate transformation, which will change the form of the inner product matrix $\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ to $\eta' = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Then $TM \oplus T^*M$ can be written as $TM \oplus T^*M = C_+ \oplus C_-$, with C_+ and C_- positive and negative definite with respect to the inner product, respectively. Now, we can define a positive definite metric on $TM \oplus T^*M$ as:

$$G = \langle , \rangle|_{C_+} - \langle , \rangle|_{C_-}. \tag{2.34}$$

G can be seen as a symmetric automorphism of $TM \oplus T^*M$ such that $G^2 = 1$, or in other words a symmetric *almost real structure* and it always exists on $TM \oplus T^*M$.

We can check that

Proposition 2.5.1 *A further reduction of the structure group to $U(m/2) \times U(m/2)$ is equivalent to the existence of an generalized almost complex structure $\mathcal{J}_1$ which commutes with G, i.e. $G\mathcal{J}_1 = \mathcal{J}_1 G$. This implies that $\mathcal{J}_2 \equiv G\mathcal{J}_1$ is a generalized almost complex structure, too.*

In fact, any two of the objects $G, \mathcal{J}_1, \mathcal{J}_2$, together with an appropriate compatibility condition similar to $\mathcal{J}_2 = G\mathcal{J}_1$, determines the third.

Definition 2.5.2 (Generalized Kähler Structure) *If $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable and the condition $G = -\mathcal{J}_1\mathcal{J}_2$ holds, we call the pair $(\mathcal{J}_1, \mathcal{J}_2)$* generalized Kähler structure*.*

The prototypical example is that of a Kähler manifold (M, J, g) such that $\omega = gJ$ is closed (see definition (2.1.16)). Then, as we have seen, $\mathcal{J}_J$ and $\mathcal{J}_\omega$ are generalized complex structures (the integrability conditions hold due to corresponding conditions for J and ω) and

$$G = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{2.35}$$

is indeed a positive definite metric on $TM \oplus T^*M$, so $(\mathcal{J}_J, \mathcal{J}_\omega)$ defines a generalized Kähler structure.

Now, any B-field transformation of the generalized complex structures $(\mathcal{J}_J, \mathcal{J}_\omega)$ in the sense of subsection 2.4 defines again a generalized Kähler structure $(g_B \mathcal{J}_J g_B^{-1}, g_B \mathcal{J}_\omega g_B^{-1})$, since all the required properties remain invariant. In that case, G becomes:

$$G^B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}. \tag{2.36}$$

This example is very important because a slight generalisation gives the most general form of the metric G:
It can be seen that a riemannian metric g and a (not necessarily closed) 2-form b fully determine G by

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 0 \end{pmatrix}. \tag{2.37}$$

This can also be stated as the fact that $C_\pm$ is the graph of $b \pm g : TM \to T^*M$.

We end our discussion of generalized geometry by defining the object $\mathcal{H}$, which will be called the *generalized metric* in double field theory:

$$\mathcal{H} = \eta G = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}, \tag{2.38}$$

As can be seen, mH has very similar properties with G.

3 String Theory and T-duality

3.1 String Theory Basics

In order to study double field theory, we first introduce some fundamental concepts of string theory which will be necessary later on. We will mostly follow references [22], [23], [24], [25] and [26], but more details can be found in any string theory textbook or review.

The general idea of the perturbative approach to string theory is to quantise a relativistic string, which is an object with one spatial dimension, tracing a 2-dimensional surface in spacetime, called the worldsheet. We first suppose that the string lives in the Minkowski spacetime $\mathbb{R}^{1,D-1}$.[5] The correct relativistic action for the string is

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \, \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{3.1}$$

where $X^\mu(\tau,\sigma)$ are the coordinates of the string in spacetime (treated as bosonic fields from the point of view of the worldsheet), $\gamma_{\alpha\beta}$ is an auxiliary metric on the worldsheet, α' is the universal Regge slope (related to the fundamental string length by $\alpha' = l_s^2$ and $\mathrm{d}^2\sigma = \mathrm{d}\sigma\,\mathrm{d}\tau$, with σ and τ parameterising the worldsheet. The above "Polyakov action" enjoys 3 different kind of symmetries (we denote σ and τ collectively by σ):

- (Global) D-dimensional Poincaré invariance:

$$X'^\mu(\sigma) = \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu, \; \gamma'_{\alpha\beta}(\sigma) = \gamma_{\alpha\beta}(\sigma)$$

[5] In this thesis we will freely Wick rotate to Euclidean time, so that the spacetime becomes $\mathbb{R}^D$. In particular, we will talk of the Lorentz group as being $O(1,D-1)$ or (D) and we will write the volume elements as $\sqrt{g}\,\mathrm{d}x$ or $\sqrt{-g}\,\mathrm{d}x$, without stating clearly the difference, as no subtleties will arise if we replace the Euclidean metric with the Minkowski one and vice versa anywhere in the following.

- Reparameterization invariance: :

$$X'^{\mu}(\sigma') = X^{\mu}(\sigma,),\ \gamma'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^{\gamma}}{\partial\sigma'^{\alpha}}\frac{\partial\sigma^{\delta}}{\partial\sigma'^{\beta}}\gamma_{\gamma\delta}(\sigma)$$

- (Local) Weyl invariance:

$$X'^{\mu}(\sigma) = X^{\mu}(\sigma,),\ \gamma'_{\alpha\beta}(\sigma) = W(\sigma)\gamma_{\alpha\beta}(\sigma).$$

We can use the reparameterization and Weyl invariance to gauge away 2 out of the 3 degrees of freedom of the auxiliary metric γ by requiring it to be conformally flat. Then we can change to worldsheet light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$ and find the equations of motion for the bosonic fields X^{μ}:

$$\partial_{+}\partial_{-}X^{\mu} = 0. \tag{3.2}$$

Now, there are 2 different kinds of strings, closed ones and open. In this thesis we will only consider closed strings, whose boundary conditions are

$$X^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma + 2\pi). \tag{3.3}$$

The open strings obey the corresponding Neumann or Dirichlet boundary conditions. For the closed string, the solution of equation (3.2) can be written as:

$$\begin{aligned} X^{\mu}(\tau,\sigma) &= X^{\mu}_{L}(\sigma^{+}) + X^{\mu}_{R}(\sigma^{-}),\\ X^{\mu}_{L}(\sigma^{+}) &= \frac{1}{2}X^{\mu}_{0} + \alpha' p^{\mu}_{0}\sigma^{+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}^{\mu}_{n}}{n}e^{-in\sigma^{+}},\\ X^{\mu}_{R}(\sigma^{-}) &= \frac{1}{2}X^{\mu}_{0} + \alpha' p^{\mu}_{0}\sigma^{-} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^{\mu}_{n}}{n}e^{-in\sigma^{-}}. \end{aligned} \tag{3.4}$$

In the above, X^{μ}_{L} and X^{μ}_{R} correspond physically to left-moving and right-moving waves on the string. $\tilde{\alpha}^{\mu}_{n}$ and α^{μ}_{n} are the oscillatory excitation modes, while X^{μ}_{0} and p^{μ}_{0} are the center of mass position and momentum of the closed string.

The equations of motion with respect to $\gamma_{\alpha\beta}$ give rise to the constraints $\tilde{L}_n = L_n = 0$, where

$$\begin{aligned} \tilde{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{\infty} \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m, \\ L_n &= \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}\cdot\alpha_m \end{aligned} \tag{3.5}$$

become the "Virasoro operators" in the quantum theory and we have defined $\tilde{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p_0^\mu$.

These constraints give an infinite number of conserved currents of the 1+1 dimensional theory, corresponding to the conformal field theory living on the worldsheet.

The quantisation of this classical theory is more subtle than expected because there is a residual gauge symmetry even after the choice of the conformally flat worldsheet metric. A careful analysis produces the following results that we will just mention without going into much depth.

The canonical commutation relations $[X^\mu, P_\nu] = i\delta^\mu_\nu$ give rise to

$$[x_0^\mu, p_0^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu}, \tag{3.6}$$

while all other commutators vanish. Thus we can treat $\tilde{\alpha}_{-n}^\mu$ and α_n^μ as (multiples of) raising and lowering operators and build the Hilbert space of states in the usual way, by acting on a vacuum $|k;0\rangle$, which is an eigenstate of the operator p_0^μ with eigenvalue k^μ. In addition, the Virasoro operators L_0 and $\tilde{L}_0$:

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \equiv \frac{1}{2}\alpha_0^2 + N \\ \tilde{L}_0 &= \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n \equiv \frac{1}{2}\tilde{\alpha}_0^2 + \tilde{N} \end{aligned} \tag{3.7}$$

need to be replaced by $L_0 - 1$ and $\tilde{L}_0 - 1$ respectively when quantising any classical expression, due to normal ordering effects. Then the (quantum) Hamiltonian of the Polyakov action (for closed strings) can be written as

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty}(\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \alpha_{-n}\cdot\alpha_n) = \tilde{L}_0 + L_0 - 2 \tag{3.8}$$

The classical constraints $L_0 = \tilde{L}_0 = 0$ now become:

$$\begin{aligned}(L_0 - 1)|\text{phys}\rangle &= 0,\\ L_n|\text{phys}\rangle &= 0, \quad \forall n \in \mathbb{N}^+\end{aligned} \tag{3.9}$$

for all physical states $|\text{phys}\rangle$ of the Hilbert space, and similar for the tilde operators. The conditions involving L_0 and $\tilde{L}_0$ can be written as:

$$\begin{aligned}(L_0 + \tilde{L}_0 - 2)|\text{phys}\rangle &= 0,\\ (L_0 - \tilde{L}_0)|\text{phys}\rangle &= 0.\end{aligned} \tag{3.10}$$

From the first one we can deduce the mass-shell relation

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.11}$$

and from the second the only equation relating the left-moving and the right-moving oscillators:

$$N - \tilde{N} = 0. \tag{3.12}$$

Actually, it can be shown that the worldsheet momentum generating translation of the σ coordinate is proportional to $L_0 - \tilde{L}_0$, so the latter statement reflects the very important fact that the string has no special point. This constraint, usually called "level-matching condition", will play a major role in this thesis.

If we go ahead and examine the spectrum of the closed bosonic string, we find (apart from the ground state $N = 0$ tachyon that will no longer exist in the superstring theories) in the first excited level $N = \tilde{N} = 1$, $(D-2)^2$ massless states, transforming in the symmetric, antisymmetric and trivial representations

of the little group $SO(D-2)$ in D-dimensions. These correspond to the spin-2 graviton $g_{\mu\nu}$, antisymmetric spin-2 "Kalb-Ramond B-field" $b_{\mu\nu}$ and spin-0 "dilaton" ϕ.
We should note here that, since string theory is supposed to be a theory of quantum gravity, the next mass level contains states of mass $m^2 \sim \frac{1}{\alpha'} \approx m_p^2$, where m_p is the Planck mass. These states are so heavy that they are not of direct phenomenological significance and so we are mostly concerned with the massless states.
Finally, requiring that no negative norm states ("ghosts") exist in our Hilbert space restricts the spacetime dimension of the bosonic string theory to $D = 26$.

Superstring theories are not directly relevant to our subject, so we will just mention here that all five of them have a common massless sector with bosonic string theory (the *universal gravitational* sector), i.e. they contain the graviton, B-field and the dilaton, as well as other fermionic and bosonic fields. Another important difference is that they are defined in 10 spacetime dimensions. In the present work, we will only deal with this particular sector, so, while we will examine the bosonic string, everything will remain almost identical if we want to examine superstrings, apart from the spacetime dimensionality. Thus, in general, we will consider an arbitrary number of dimensions D.

Now, if we wish to consider string interactions, the best way to proceed is to define the Feynman path integral

$$Z = \int \mathcal{D}g\mathcal{D}X \; e^{-S_P[X,g]}, \tag{3.13}$$

where S_P is the action (3.1).Here we integrate over all worldsheet metrics $g_{\alpha\beta}$ and embedding coordinates X^μ, and we also have to perform a Fadeev-Popov gauge fixing. Actually, in order to include possible interactions, we should also integrate over metrics describing worldsheets with non-trivial topology. The correct weighting of the perturbation expansion in riemannian surfaces is done by adding to S_P the term

$$\Phi_0 \cdot \chi = \Phi_0 \cdot \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{g} R^{(2)} = \Phi_0 \cdot (2 - 2h - k), \tag{3.14}$$

where $\Phi_0 \in \mathbb{R}$, $R^{(2)}$ is the worldsheet curvature scalar and χ is the Euler number of the Riemann surface with h handles and k boundaries, which is a topological invariant of the surface. Then, the string coupling constant is given by

$$g_s = e^{\Phi_0} \tag{3.15}$$

and we see that as we go to higher loop orders, the weight of the relevant terms is proportional to g_s^2. Thus, for $g_s \ll 1$, we can trust the perturbation expansion and only keep worldsheets of low genus.

Next, we are going to generalise the Polyakov action to the following one:

$$\begin{aligned} S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, [&\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \\ &- \frac{\alpha'}{2}\sqrt{-\gamma}\Phi(X) R^{(2)}] \end{aligned} \tag{3.16}$$

The first term in the action describes a string in a general curved background (called *non-linear sigma model* for historical reasons), with $G_{\mu\nu}(X)$ being the target space metric. However, we know that $G_{\mu\nu}(X)$ should be constructed out of the gravitons that we encountered before, when we quantised the string on a flat spacetime and the gravitons were small perturbations around the background. Indeed, using the graviton vertex operator, we can see that the two pictures are consistent and $G_{\mu\nu}(X)$ is a coherent state of gravitons. In other words, the massless graviton that we found before can be thought of as a fluctuation around a classical value (or the expectation value of the operator): $G_{\mu\nu} = \hat{G}_{\mu\nu} + g_{\mu\nu}$

Similarly, the B-field and the dilaton can form classical condensates and give rise to the other terms in the action (3.16). Now, the B-field is regarded as an analogue of the Maxwell potential A_μ for the electric charge and it describes the coupling of the string to $B(X)$. This is apparent if we write the relevant term as the pullback of the 2-form field $B_{\mu\nu}$ on the worldsheet: $\int \mathrm{d}^2\sigma\ \epsilon^{\alpha\beta} B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \equiv \int_{w.sheet} B$. In this form, it is also clear that the action is invariant under gauge transformations of B:$B \to B + d\Lambda$, where Λ is a one-form. In the context of string theory, we will call these transformations *B-field gauge transformations*, but note that they are not exactly the same as those discussed in section 2, as the previous ones did not have to be exact.

The dilaton $\Phi(X)$ is a function of spacetime, so we take its asymptotic value as the constant mode which defines the string coupling: $\Phi_0 = \lim_{X\to\infty} \Phi(X)$, and we can only trust perturbation theory in the appropriate spacetime regions.

Note that the theory defined by the action (3.16) is an interacting quantum field theory. This can be seen if we perform the α'-expansion by writing $X^\mu(\sigma) = x_0^\mu + \sqrt{\alpha'} Y^\mu(\sigma)$. Now the actual fields are $Y^\mu(\sigma)$ and the theory has an infinite number of coupling constants given by derivatives of $G_{\mu\nu}$. We can see that each $G_{\mu\nu}$ derivative comes with a factor of $\sqrt{\alpha'}$ and thus the field theory is weakly coupled if the radius of curvature r of the metric, where $r \sim G_{\mu\nu,\rho}$, is large compared to $\sqrt{\alpha'}$ (so it is actually an expansion in $\sqrt{\alpha'}/r$). This is equivalent to looking at low-energy physics because, if the wavelength satisfies $\sqrt{\alpha'}/r \ll 1$, there is not enough energy to create states of mass $\sim 1/(\alpha')$ and we are effectively considering only the massless states.

Now, requiring that this theory respects conformal (Weyl) invariance, the β-functions for G, B and Φ must vanish. If we compute these equations at one-loop order, we can see that they are the equations predicted by (the corresponding NS-NS sector of) supergravity, and thus we can write a low-energy effective action for the fields $G(X)$, $B(X)$, $\Phi(X)$:

$$(2\kappa^2)S_* = \int \mathrm{d}^D x \sqrt{-G} e^{-2\Phi}[R + (4\partial\Phi)^2 - \frac{1}{12}H^2], \tag{3.17}$$

where R is the Ricci scalar constructed from $G(X)$ and H is the field strength of the 2-form field: $H = dB$.[6] Thus, the equations of motion of G, B and Φ coming from the variation of S_*

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} H_\mu^{\kappa\lambda} H_{\nu\kappa\lambda} + 2\nabla_\mu \nabla_\nu \Phi = 0 \\ \frac{1}{2} \nabla^\kappa H_{\kappa\mu\nu} - H_{\kappa\mu\nu} \nabla^\kappa \Phi = 0 \\ R + 4(\nabla^\kappa \nabla_\kappa \Phi - (\partial\Phi)^2) - \frac{1}{12} H^2 = 0 \end{aligned} \tag{3.18}$$

imply Weyl invariance at one loop quantum level. If we compute these equations to next order, we find α' corrections to the above equations.

Finally, we will explain the general ideas behind string field theory, but we will not discuss any of its highly technical details.

Heuristically, in classical mechanics we examine the motion of a point particle by writing an action for its coordinates $x^\mu(\tau)$, τ being the proper time. Then, we can construct classical field theory by writing an action for fields $\phi(x^\mu)$, without the need to make explicit reference to τ. Similarly, in string theory we would like to go from the "first quantisation" of strings $X^\mu(\sigma, \tau)$ to the "second quantised" *string fields* $\Psi(X^\mu(\sigma))$. So, for closed strings, $\Psi(X^\mu(\sigma))$ is actually a functional in a loop space. Roughly, this functional creates the loop in consideration. Then, a formal path integral

$$Z = \int \mathcal{D}\Psi \; e^{iS[\Psi(X^\mu(\sigma))]}$$

should reproduce the perturbative string expansion and give insights into non-perturbative effects.

A closed string field state $|\Psi\rangle$ in the bosonic case describes the functional $\langle X^\mu(\sigma)|\Psi\rangle = (t(X) + h_\mu(X)\alpha_{-1}^\mu + k_\mu(X)\tilde{\alpha}_{-1}^\mu + l_{\mu\nu}(X)\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu + \ldots)\Psi_0$, where

[6] Actually, the (bosonic NS-NS) supergravity action does not have the form of (3.17) exactly, but is related to it by field redefinitions. However, we will continue to refer to (3.17) as the "standard Einstein plus B-field plus dilaton" action.

$\alpha_n^\mu \Psi_0 = \tilde{\alpha}_n^\mu \Psi_0 = 0, \ \forall n \geq 1$. We need to impose the constraint $(L_0 - \tilde{L}_0)|\Psi\rangle = 0$ (as well as a similar one for the b-ghost field), which removes the $h_\mu, k_\mu \ldots$ fields from the spectrum, leaving the physical ones, i.e. the tachyon $t(X)$, $l_{\mu\nu}$ which contains the graviton, B-field and the dilaton, and so on. We should also note that string field theory encodes the dynamics of strings in a gauge invariant way, since we impose the condition $|\Psi\rangle \sim Q_B|\Lambda\rangle$, with Q_B the BRST operator.

3.2 S^1 Compactification

We are now ready to discuss some basic facts about compactification in string theory, following the classic review [27], as well as [24] and [23]. We will also use [28].

Mathematically, compactness of a metric space (such as a riemannian manifold) is almost an almost identical notion with the finiteness of its volume. So, when we compactify a spacetime dimension we actually make its length finite. This can be done on the real axis by an equivalence relation of the form

$$x \sim x + 2\pi R \cdot \kappa, \quad \forall \kappa \in \mathbb{N} \tag{3.19}$$

Then, the real axis is diffeomorphic to a circle S^1 with radius R.

The need to compactify some of the spacetime dimensions is apparent if we realize that we live in a 4-dimensional space, while string theory in only consistent in 26 or 10 dimensions. The most obvious way to reconcile these two observations is to suppose that our spacetime has some compact dimensions, making up a compact manifold of such small volume that cannot be observed in current experiments. The simplest case is the toroidal compactification $\mathcal{M} = \mathbb{R}^{D-d} \times T^d$, where our spacetime $\mathcal{M}$ is a product of a $(D-d)$-dimensional Euclidean spacetime $\mathbb{R}^{D-d}$ (see footnote (5)) and a d-torus $T^d \equiv \overbrace{S^1 \times S^1 \times \ldots \times S^1}^{d \text{ times}}$. String theory permits more general background solutions of equations (3.18), where the compact space (also called "internal space") is a *Calabi-Yau* manifold, i.e. a complex, Ricci-flat manifold. Other choices of internal space are K3 surfaces,

manifolds of G_2 holonomy or orbifolds. Also, there has been a lot of work recently in *flux compactifications*, i.e. cases with non-trivial p-form fields turned on in the internal space. In this thesis we will always have in mind the toroidal compactification, since T-duality is clear only in this case.
We should note here that, although later we will develop a formalism that will not explicitly distinguish between compact and noncompact directions, we will never consider compactifying the time direction, since this does not have a clear physical meaning.

We begin by compactifying a usual field theory on a circle S^1 of radius R. We make X^D periodic by setting $X^D \sim X^D + 2\pi R$. We also consider indices $a, b, \ldots$ with range $a, b, \ldots = 1, 2, \ldots, D-1$. Then the D-dimensional metric can be written as

$$G^D_{\mu\nu} = \begin{pmatrix} G_{ab} & G_{aD} \\ G_{Db} & G_{DD} \end{pmatrix} \tag{3.20}$$

or, setting $G_{aD} = D_{DD} \cdot A_a$, we have that

$$ds^2 = G_{ab} dX^a dX^b + G_{DD}(dx^D + A_a dX^a)^2. \tag{3.21}$$

We can easily see that A_a is a one-form and, letting $G^D_{\mu\nu}$ depend only on the noncompact coordinates X^a, we find that reparameterizations $X'^D = X^D + \lambda(X^a)$ lead to the gauge transformations $A'_a = A_a - \partial_a \lambda$. This motivated the so called *Kaluza-Klein* construction. If we consider now a general field ϕ depending on all the coordinates (and choose the canonical metric $G_{DD} = 1$), requiring that the wavefunction is single-valued leads to the quantisation of the momentum in the periodic dimension, i.e. $p_D = \frac{n}{R}$. Then, we can expand in Fourier modes:

$$\phi(X^\mu) = \phi_0(X^a) + \sum_{n \neq 0} \phi_n(X^a) e^{inX^D/R} \tag{3.22}$$

The D-dimensional equation of motion for a massless field becomes:

$$\partial_\mu \partial^\mu \phi = 0 \;\Rightarrow\; \partial_a \partial^a \phi_n(X^b) = \frac{n^2}{R^2}\phi_n(X^b), \tag{3.23}$$

so, from the $(D-1)$-dimensional point of view, a massless field gives rise to a tower of massive modes, with mass inversely proportional to the radius R of the circle, as well as a massless zero mode. By setting the fields to be independent of X^D, or equivalently by looking at energies small compared to the radius, we can neglect all the massive modes. Then we say that we have *dimensionally reduced* our theory.

The generalisation of the above considerations to compactification on a d-torus T^d is straightforward. The only thing that we should notice is that now the dimensionally reduced theory will have a $GL(d,\mathbb{R})$ global symmetry acting on the "internal" indices of the resulting fields, which can be thought of as a "remnant" of the higher dimensional diffeomorphisms. Similarly, the inclusion of an antisymmetric B-field leads to a global shift symmetry $\mathbb{R}^{d(d-1)/2}$. Now, it can be shown that the Einstein plus B-field action reduced on T^d has a global $O(d,d,\mathbb{R})$ symmetry (see [24] or the following subsections). So we see that only the $GL(d,\mathbb{R})\ltimes\mathbb{R}^{d(d-1)/2}$ subgroup of $O(d,d,\mathbb{R})$ has a geometrical interpretation.

The situation in string theory is different in one very important aspect: a string can wind around a noncompact direction and this gives rise to massive "winding" modes with no analogue in field theory. Heuristically, we can think of a closed string wrapped around the circle an integer number of times m, with $m \in \mathbb{N}$, and we can understand that its energy will be proportional to m, since we need to "stretch" it in order to make it circle the compact dimension. More mathematically, the topologically distinct ways a circle can wrap another circle are counted by the fundamental group of S^1. We know that $\pi_1(S^1) = \mathbb{Z}$, showing that the maps are characterized by an integer m, called the *winding number*.

We first analyse the simplest situation, namely compactification on S^1, i.e. $\mathcal{M} = \mathbb{R}^{D-1} \times S^1$. In the noncompact directions nothing changes from the previous analysis, so in the following we will drop the superscript D and we will simply denote X^D by X. Since we are studying strings at critical dimensions, we can set $\gamma_{\alpha\beta}(\sigma,\tau) = \eta_{\alpha\beta}$. The action is just

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \; \partial_\alpha X \partial^\alpha X. \tag{3.24}$$

However, the boundary conditions change from (3.3) to

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi R m, \quad m \in \mathbb{N}. \tag{3.25}$$

The equations of motion are (3.2) as before, yet the solutions are not equations (3.4), but

$$\begin{aligned} X^\mu(\tau,\sigma) &= X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-), \\ X^\mu_L(\sigma^+) &= x_L + \sqrt{\frac{\alpha'}{2}} p_L \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{\tilde{\alpha}_k}{k} e^{-ik\sigma^+}, \\ X^\mu_R(\sigma^-) &= x_R + \sqrt{\frac{\alpha'}{2}} p_R \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{\alpha_k}{k} e^{-ik\sigma^-}, \end{aligned} \tag{3.26}$$

where the center of mass is $x = x_L + x_R$ and the (dimensionless) momenta are now

$$\begin{aligned} p_L &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R} n + \frac{R}{\sqrt{\alpha'}} m\right), \\ p_R &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R} n - \frac{R}{\sqrt{\alpha'}} m\right), \end{aligned} \tag{3.27}$$

where, by a similar argument as in field theory compactification, the momentum operator $\frac{\partial}{\partial x}$ has integer eigenvalues $n \in \mathbb{N}$ which enter in the above expressions. We note that now $p_L \neq p_R \Rightarrow \alpha_0 \neq \tilde{\alpha}_0$, since $p_L \equiv \alpha_0$ and $p_R \equiv \tilde{\alpha}_0$.

Of course the constraints (3.10) still hold, but they lead to the modified conditions:

$$m^2 = \frac{1}{\alpha'}(p_L^2 + p_R^2) + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$
$$= \frac{n^2}{R^2} + \frac{R^2 m^2}{(\alpha')^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.28}$$

and

$$\tilde{N} - N = \sqrt{\frac{\alpha'}{2}}(p_R^2 - p_L^2) = mn, \tag{3.29}$$

where in the above equations m^2 is the $(D-1)$-dimensional mass, i.e. $m^2 = -\sum_{a=1}^{D-1} p_a p^a$ and $N, \tilde{N}$ are the usual number operators $N = \sum_{n=1}^{\infty}\sum_{\mu=1}^{D} \alpha_{-n}^{\mu}\alpha_{n\mu}$ and similarly for $\tilde{N}$. We thus see that both momentum and winding modes contribute to the mass spectrum. If we go ahead and investigate the spectrum, we find that, for $m = n = 0$ we have the following massless fields: a graviton, a B-field, a dilaton, two one-forms and another scalar, as can be easily deduced by performing dimensional reduction of the NS-NS sector. Thus, dimensional reduction is equivalent to throwing away all momentum and winding modes. The two vector fields generate the group $U(1)_L \times U(1)_R \supset O(1,1,\mathbb{R})$. For the special value $R = \sqrt{\alpha'}$ we find additional massless fields: four vector fields and four scalars and the gauge group gets enhanced to $SU(2)_L \times SU(2)_R$, a purely stringy phenomenon with no field theoretical analogue.

We can see now the simplest realisation of T-duality: suppose we make the transformation:

$$X = X_L + X_R \to \tilde{X} \equiv X_L - X_R \tag{3.30}$$

This transformation is equivalent to transforming:

$$\frac{R}{\sqrt{\alpha'}} \to \frac{\sqrt{\alpha'}}{R}, \quad m \leftrightarrow n, \quad \alpha_n \to -\alpha_n, \quad \tilde{\alpha}_n \to \tilde{\alpha}_n, \tag{3.31}$$

including the zero modes $\alpha_0 = p_L$ and $\tilde{\alpha}_0 = p_R$ (and ignoring x_L and x_R). We can easily see that L_0 and $\tilde{L}_0$ remain invariant and thus the physical conditions (3.28) and (3.29) remain invariant, too. This is an indication that the physics is invariant under the above transformations, though not a proof. However, it can actually be shown (see [27]) that the partition function does not change to all orders in loop expansion (if, at the same time, the dilaton transforms as $\Phi \to \Phi + 2\log\frac{R}{\sqrt{\alpha'}}$), so that the transformations are an actual duality of the theory.

In order to explain roughly what a duality is, we should first note that, although there is "only one" string theory, from the point of view of the $2d$ σ-model we have different conformal field theories according to the expansion about different backgrounds (see equation (3.16)). In that way we create a "space of theories", which may be spanned by a continuous group $\mathcal{G}$. If $\mathcal{G}$ has a discrete subgroup $\mathcal{G}_d$ that is an actual symmetry of the physical theory, we say that we have a $\mathcal{G}_d$-duality of our theory. In our case, we will treat $X^\mu(\tau,\sigma)$ and $P_\mu(\tau,\sigma)$ as universal objects , with different representations according to the various possible backgrounds. Thus, the oscillators α_n and $\tilde{\alpha}_n$ will be background dependent.

The above T-duality $R/\sqrt{\alpha'} \to \sqrt{\alpha'}/R$ has a deep physical meaning: a direction forming a circle of radius $R/\sqrt{\alpha'} \gg 1$ is completely equivalent to one with radius $R/\sqrt{\alpha'} \ll 1$ or, in other words, for every small radius (compared to the string length $\sqrt{\alpha'}$), there exists a large dual one. Thus, strings cannot really probe distances smaller than $\sqrt{\alpha'}$, indicating that this is a kind of physical "cutoff" for spacetime. Stringy geometry is one of the most exciting aspects of string theory!

Before we move on to general T^d compactifications, we should observe that $\tilde{X}(\tau,\sigma)$ is also a solution of the equations of motion (3.2), like $X(\tau,\sigma)$. However, X satisfies the boundary condition (3.25), while $\tilde{X}$ satisfies the similar one

$$X(\tau,\sigma+2\pi) = X(\tau,\sigma) + 2\pi\frac{\alpha'}{R}n. \tag{3.32}$$

We can interpret this as a coordinate on a circle of the dual radius α'/R. These *dual coordinates* will be very important when we introduce double field theory. Their necessity can also be understood as follows:

In our analysis, the commutation relations $[X, P] = i$ give among others

$$[x_L, p_L] = [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}}, \tag{3.33}$$

the factor $\sqrt{\alpha'}/2$ coming from the one in (3.26). This gives for the zero modes x and $p = \frac{1}{\sqrt{2\alpha'}}(p_L + p_R)$:

$$[x, p] = i. \tag{3.34}$$

The operator p is then realised as $p = (-i)\partial/\partial x$, with integer eigenvalues n. Similarly, the integers m should correspond to an operator $w = \frac{1}{\sqrt{2\alpha'}}(p_L - p_R)$, which is conjugate to $\tilde{x} = x_L - x_R$. Now, $\tilde{x}$ is the zero mode of $\tilde{X}$, so that we can write $w = (-i)\partial/\partial\tilde{x}$. $\tilde{x}$ is conjugate to w in the sense that:

$$[\tilde{x}, w] = i. \tag{3.35}$$

In the following we will use interchangeably the quantised operators p, w and their eigenvalues n, m, but the meaning will always be clear.

3.3 d-Dimensional Toroidal Compactification

In the rest of this section we will analyse T-duality in the most general setting of compactification on T^d, based on [27], [23], [24], [6] and [29].

The above T-duality is actually a $\mathbb{Z}_2$ symmetry, so naively one would expect the corresponding group for the d-torus T^d to be $\mathbb{Z}_2^d$. Indeed, it is not hard to extend the above formalism to the case of d compact dimensions. However, the large diffeomorphisms of the torus enhance the $\mathbb{Z}_2^d$ symmetry. We will discuss these issues in the even more complex case of compactification is the presence of a background involving the metric $G_{\mu\nu}$ as well as the antisymmetric B-field $B_{\mu\nu}$, both taking constant values in the compactified dimensions.

It is convenient to first lighten our notation by redefining the coordinates $X^i \to \frac{R_i}{\sqrt{\alpha'}}X^i$ (no summation), where R_i is the radius of the i^{th} direction and $i, j, \ldots$ run over the d compact dimensions. Then the coordinates X^i as well as the metric G_{ij} and the B-field B_{ij} are all dimensionless and so X^i satisfy

$$X^i \sim X^i + 2\pi. \tag{3.36}$$

The periodicity conditions now become

$$X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma) + 2\pi m^i, \tag{3.37}$$

with m^i indicating the number of times that the string winds around the i^{th} circle. There is a deeper reason for the position of the index 'i' of m^i than just the simple observation that the two hand-sides of the periodicity conditions should have the same structure; we will come to this shortly.

We restrict the action (3.16) to the compact dimensions, as we have seen what happens in the noncompact ones in section (3.1). Ignoring the dilaton which plays no role in the following analysis, we start with the action

$$S = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \; [\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij}(X) + \epsilon^{\alpha\beta} B_{ij}(X) \partial_\alpha X^i \partial_\beta X^j]. \tag{3.38}$$

The fields G_{ij} and B_{ij} depend nontrivially on the noncompact dimensions X^a, but we take them to be constant in X^i. In fact, the metric G_{ij} is constrained to describe the geometry of a d-torus, but other than that it is arbitrary and the B-field is a 2-form on T^d. Note that the number of degrees of freedom describing the geometry is $\frac{d(d+1)}{2} + \frac{d(d-1)}{2} = d^2$ and we can combine them all in the matrix $E_{ij} = G_{ij} + B_{ij}$, which naturally encodes the full information about the background.[7]

Now, we observe that the term with the B-field can be written as a total derivative

$$\epsilon^{\alpha\beta} B_{ij}(X) \partial_\alpha X^i \partial_\beta X^j = \partial_\alpha (\epsilon^{\alpha\beta} B_{ij}(X) X^i \partial_\beta X^j) \tag{3.39}$$

[7]The role of the dilaton is different, as it determines the string coupling constant.

and so it does not affect the equations of motion, whose solution is:

$$\begin{aligned} X^i(\tau,\sigma) &= x^i + m^i\sigma + \tau G^{ij}(n_j - B_{jk}m^k) \\ &\quad + \frac{i}{\sqrt{2}}\sum_{k\neq 0}\frac{1}{k}[\alpha_k(E)e^{-ik(\tau-\sigma)} + \tilde{\alpha}^i_k(E)e^{-ik(\tau+\sigma)}]. \end{aligned} \tag{3.40}$$

The zero mode oscillators can be found by the usual splitting of X^i in X^i_L and X^i_R:

$$\begin{aligned} \alpha^i_0 &= \frac{1}{\sqrt{2}}G^{ij}(n_j - B_{jk}m^k - G_{jk}m^k) = \frac{1}{\sqrt{2}}G^{ij}(n_j - E_{jk}m^k) \\ \tilde{\alpha}^i_0 &= \frac{1}{\sqrt{2}}G^{ij}(n_j - B_{jk}m^k + G_{jk}m^k) = \frac{1}{\sqrt{2}}G^{ij}(n_j + E_{kj}m^k). \end{aligned} \tag{3.41}$$

Note that n_i and m^j are the eigenvalues of the operators $p_i = \partial/\partial x^i$ and $w^i = \partial/\partial\tilde{x}_i$. Again notational consistency fixes the position of the index of the (zero mode of) dual coordinates.

The actual dual coordinates are:

$$\begin{aligned} \tilde{X}_i(\tau,\sigma) &= \tilde{x}_i + n_i\sigma + \tau[(G_{ij} - B_{ik}G^{kl}B_{lj})m^j + B_{ik}G^{kj}n_j] \\ &\quad + \frac{i}{\sqrt{2}}\sum_{k\neq 0}\frac{1}{k}[\alpha_k(E)e^{-ik(\tau-\sigma)} + \tilde{\alpha}^i_k(E)e^{-ik(\tau+\sigma)}]. \end{aligned} \tag{3.42}$$

The non-zero commutation relations naturally read

$$\begin{aligned} &[x^i, p_j] = i\delta^i_j, [\tilde{x}_i, w^j] = i\delta^j_i, \\ &[\alpha^i_n(E), \alpha^j_m(E)] = [\tilde{\alpha}^i_n(E), \tilde{\alpha}^j_m(E)] = mG^{ij}\delta_{m,-n}, \end{aligned} \tag{3.43}$$

coming from the more general ones

$$[X^i(0,\sigma), P_j(0,\sigma')] = i\delta^i_j\delta(\sigma-\sigma'), \tag{3.44}$$

where the canonical momentum P_i can be found from the Lagrangian density of the action (3.38):

$$\begin{aligned} 2\pi P_i(\tau,\sigma) &= G_{ij}\dot{X}^j + B_{ij}X^{j\prime} \\ &= n_i + \frac{1}{\sqrt{2}}\sum_{k\neq 0}[E_{ji}\alpha_k^j(E)e^{-ik(\tau-\sigma)} + E_{ij}\tilde{\alpha}_k^j(E)e^{-ik(\tau+\sigma)}]. \end{aligned} \tag{3.45}$$

Then, a short computation gives the Hamiltonian density

$$\begin{aligned} 4\pi H &= (\dot{X}^i\dot{X}^j + X^{i\prime}X^{j\prime})G_{ij} \\ &= (2\pi)^2 P_i G^{ij} P_j + X^{i\prime}(G_{ij} - B_{ik}G^{kl}B_{lj})X^{j\prime} + 4\pi X^{i\prime}B_{ik}G^{kj}P_j \\ &\equiv \begin{pmatrix} X' & 2\pi P \end{pmatrix} \mathcal{H}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \end{aligned} \tag{3.46}$$

where

$$\mathcal{H}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{3.47}$$

is the $2d \times 2d$ (symmetric[8]) *generalized metric*, one of the fundamental constituents of double field theory. We have seen this object in section 2, where it originated from a seemingly completely different approach. In section **??** it will have a somewhat different meaning once again. Some important ideas underlying the unification of these approaches will hopefully be apparent by the end of this work.

Yet another short computation gives the particularly useful form for H:

$$H = \frac{1}{2}Z^T\mathcal{H}(E)Z + N + \tilde{N}, \tag{3.48}$$

where we have set

$$Z = \begin{pmatrix} m^i \\ n_i \end{pmatrix}, \quad N = \sum_{k>0}\alpha_{-k}^i(E)G_{ij}\alpha_k^j(E) \tag{3.49}$$

and similarly for $\tilde{N}$.

[8]Recall that $G^T = G$ and $B^T = -B$.

The full Hamiltonian for noncompact and compact dimensions will enter in the condition $(L_0 + \tilde{L}_0 - 2)|\psi\rangle = 0$, giving the mass from the lower dimensional point of view, i.e.

$$m^2 = \sum_{k=1}^{D-d} p_a p^a = \frac{1}{\alpha'}[Z^T \mathcal{H}(E) Z + 2(N + \tilde{N} - 2)], \tag{3.50}$$

with N and $\tilde{N}$ the full number operators.

Finally, in the same spirit we find that the constraint $(L_0 - \tilde{L}_0)|\psi\rangle = 0$ gives

$$N - \tilde{N} = n_i m^i \equiv \frac{1}{2} Z^T \eta Z, \tag{3.51}$$

where we have defined the $2d \times 2d$ matrix

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3.52}$$

which also plays a fundamental role in double field theory.

3.4 Narain Lattice and T-Duality

Since we have treated $X^i(\tau, \sigma)$ and $P_i(\tau, \sigma)$ as universal objects, we should equate their expansions around different backgrounds E and E'. We then find that they are related by

$$\begin{gathered} \alpha_k(E) - \tilde{\alpha}_k(E) = \alpha_k(E') - \tilde{\alpha}_k(E') \\ E^T \alpha_k(E) + E \tilde{\alpha}_k(E) = E'^T \alpha_k(E') + E' \tilde{\alpha}_k(E') \end{gathered} \tag{3.53}$$

In general these will not be physically equivalent theories. However, it may happen that, for two backgrounds related by a transformation $E' = g(E)$, there is a unitary operator U_g such that $S_E(U_g|\Psi\rangle) = S_{g(E)}(|\Psi\rangle)$ for all string field states $|\Psi\rangle$. Then, the field redefinition $|\Psi\rangle \to U_g|\Psi\rangle$ shows that the two theories are actually equivalent. We will see that U_g form a representation of the group $O(d, d, \mathbb{Z})$.

In order to study the action of T-duality, we must define the Narain lattice, first introduced in [30] and [31]. We start by recalling some basic facts about lattices (see [32]).

Consider the vector space $\mathbb{R}^{p+r}$ equipped with a nondegenerate symmetric bilinear form represented by a matrix S, of signature (p, r), usually denoted by $\mathbb{R}^{p,r}$. A lattice $\Gamma^{p,r}$ is defined by $\Gamma^{p,r} = \{\sum_{i=1}^{p+r} \kappa^i \vec{e}^{\,i} |\ \forall \kappa^i \in \mathbb{N}^i\}$, where $\vec{e}^{\,i},\ i = 1 \ldots, p+r$, is a basis of $\mathbb{R}^{p,r}$. $\Gamma^{p,r}$ naturally inherits the inner product S from $\mathbb{R}^{p,r}$. We can also define a $(p+r) \times (p+r)$ matrix F whose columns are the basis vectors and then, any vector W belonging to $\Gamma^{p,r}$ can be written as $W = FZ$, where Z is a $(p+r)$ column matrix with integer elements.

We say that $\Gamma^{p,r}$ is *even* if $W^T SW \in 2\mathbb{Z},\ \forall W \in \Gamma^{p,r}$. We say that $\Gamma^{p,r}$ is *self-dual* if $\Gamma^{p,r} = (\Gamma^{p,r})^*$, where the dual lattice of $\Gamma^{p,r}$ is $(\Gamma^{p,r})^* = \{x \in \mathbb{R}^{p,r} |\ x^T S y \in \mathbb{Z},\ \forall y \in \Gamma^{p,r}\}$. The basis matrix $\tilde{F}$ of the dual lattice satisfies $\tilde{F}^T SF = I$.

We can define two kinds of transformation of $\Gamma^{p,r}$, a left and a right $GL(p+r, \mathbb{R})$ action on F, i.e. $F \to MF$ and $F \to MF,\ M \in GL(p+r, \mathbb{R})$. Thus, the vector W will be mapped to $W' = MFZ$ or $W' = FMZ$ respectively, which in general do not belong to $\Gamma^{p,r}$, but to different lattices. We will understand the significance of these transformations when we apply our formalism in the specific case that we are studying.

Let us now go back to the compactified string theory on T^d. We define the vielbein $e \equiv e_i^a$ for the metric G_{ij}, i.e. we set $G = e^T e$ and thus for the inverse vielbein $e^* \equiv e_a^i$ we have that $G^{-1} = (e^*)^T e^*$, denoting the "flat" indices of e

by $a, b, \ldots$. [9] We now introduce the momenta

$$
\begin{aligned}
(p_R)_a &\equiv (e^*)^i_a \alpha_{0i} = \frac{1}{\sqrt{2}}(e^*)^i_a(n_i - E_{ik}m^k) \\
(p_L)_a &\equiv (e^*)^i_a \tilde{\alpha}_{0i} = \frac{1}{\sqrt{2}}(e^*)^i_a(n_i + E_{ki}m^k)
\end{aligned}
\tag{3.54}
$$

We can combine them in a single vector W

$$
W = \begin{pmatrix} (p_L)_a \\ (p_R)_a \end{pmatrix} = \frac{1}{\sqrt{2}}e^* \begin{pmatrix} E^T & I \\ -E & I \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \equiv FZ, \tag{3.55}
$$

where

$$
F = \frac{1}{\sqrt{2}}e^* \begin{pmatrix} E^T & I \\ -E & I \end{pmatrix}. \tag{3.56}
$$

Note that (roughly) the first index of F is a flat a index but the second is a "curved" i one.

We define the inner product of two such vectors W and W' to be

$$
W^T S W' = \begin{pmatrix} p_L & p_R \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} p'_L \\ p'_R \end{pmatrix} = p_L p'_L - p_R p'_R, \tag{3.57}
$$

so $S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ is the inner product of the underlying vector space $\mathbb{R}^{d,d}$. Since m and n are integers, we can treat the columns of the $(2d \times 2d)$ matrix F as the basis vectors of a $\Gamma^{d,d}$ lattice, called *Narain lattice*. We can easily verify that $F^T S F = \eta$, where η is defined in (3.52), so that

$$
W^T S W' = Z^T F^T S F Z' = Z^T \eta Z' = m^i {n'}_i + {m'}^i n_i. \tag{3.58}
$$

Specifically, setting $W' = W$, we see that $\Gamma^{d,d}$ is even.

Similarly we can compute that $\tilde{F} = F\eta$, but η is integer valued and invertible

[9] Please do not confuse it with the background field $e_{ij} = g_{ij} + b_{ij}$, which will be extensively used in the next section. The meaning should be clear by the context.

($\eta^{-1} = \eta$), so we understand that it actually describes the same lattice $\Gamma^{d,d}$. This is because they are spanned by the same vectors. Thus $\Gamma^{d,d}$ is also self-dual. In [24] there is another derivation of these properties, relying on the modular invariance of the partition function.

The relation $F^T S F = \eta$ has a very important implication: noting that $S = A^T \eta A$, where $A = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ we find that

$$(F^T A^T)\eta(AF) = \eta, \tag{3.59}$$

which implies that $AF \in O(d,d,\mathbb{R})$, or equivalently, since A is just a constant matrix, $F \in O(d,d,\mathbb{R})$.[10]

Thus, to each background E we can associate an element $AF \in O(d,d,\mathbb{R})$, where F is defined in (3.56). But not all elements of $O(d,d,\mathbb{R})$ correspond to different physical theories. We now move on to study transformations of the basis matrix F.

A simple computations gives us

$$F^T F = \mathcal{H}(E). \tag{3.60}$$

We can say that F is a "vielbein" for the generalized metric $\mathcal{H}(E)$ (or AF, it is the same due to the fact that $A^T A = I$).

Now we rewrite the conditions (3.50) and (3.51) as

$$\begin{aligned} \alpha' m^2 - 2(N + \tilde{N} - 2) = Z^T F^T F Z = W^T W \\ = (p_L)^2 + (p_R)^2 \end{aligned} \tag{3.61}$$

[10] Actually in section (2) we defined $O(d,d,\mathbb{R})$ with respect to the metric $\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$. However, all such groups defined by metrics of the same signature are naturally isomorphic, so will will continue referring to both as $O(d,d,\mathbb{R})$.

and

$$N - \tilde{N} = Z^T \eta Z = W^T (F^T)^{-1} \eta F^{-1} W = W^T (F \eta F^T)^{-1} W \\ = W^T (S)^{-1} W = W^T S W = (p_L)^2 - (p_R)^2. \tag{3.62}$$

Let $GL(2d, \mathbb{R})$ act on F from the left: $F \to MF \quad \Leftrightarrow \quad W \to MW$, $M \in GL(2d, \mathbb{R})$. Requiring that (3.61) is preserved implies that $M \in O(2d, \mathbb{R})$ and requiring that (3.62) is preserved implies $M \in O(d, d, \mathbb{R})$. Thus $M \in O(2d, \mathbb{R}) \cap O(d, d, \mathbb{R}) = O(d, \mathbb{R}) \times O(d, \mathbb{R})$. This is obvious if we note that $O_L(d, \mathbb{R}) \times O_R(d, \mathbb{R})$ rotates p_L and p_R independently and so is does not affect neither (3.61) nor (3.62).
It is a simple matter to check that actually every $F \to MF$ transformation with $M \in O(d, d, \mathbb{R})$ preserves the conditions of evenness and self-duality, and so it leads to a consistent string theory.

Let now $GL(2d, \mathbb{R})$ act on F from the left: $F \to Fh$, $h \in GL(2d, \mathbb{R})$. Then we find that $Z \to h^{-1} Z$ and invariance of (3.62) implies that $h \in O(d, d, \mathbb{R})$. Observing (3.61) we see that $\mathcal{H} \to h^T \mathcal{H} h \neq \mathcal{H}$. So, in order for the spectrum given by (3.61) to be preserved, we should be able to redefine $Z' = hZ \in \mathbb{N}^{2d}$. This holds if both h and its inverse are integer valued, i.e. $h \in O(d, d, \mathbb{Z})$. In that case we have just "reshuffled" the quantum numbers of the states. We call $O(d, d, \mathbb{Z})$ the *T-duality* group. The relation to the transformation found in (3.2) will become apparent shortly.
The fact that the transformation $Z \to hZ \Leftrightarrow F \to Fh$, with $h \in O(d, d, \mathbb{N})$ actually leaves the lattice invariant was also used before, when we proved the self-duality of the Narain lattice.

Using the above arguments, and also conversing some lines of thought, we can show that the moduli space of string theories on T^d is the coset

$$\mathcal{M}_{d,d} = O(d, d, \mathbb{Z}) \backslash O(d, d, \mathbb{R}) / O(d, \mathbb{R}) \times O(d, \mathbb{R}), \tag{3.63}$$

showing the equivalence of the lattices with basis matrices F and MFh, $M \in O(d,\mathbb{R}) \times O(d,\mathbb{R})$ and $h \in O(d,d,\mathbb{Z})$. This is sometimes written as

$$\mathcal{M}_{d,d} = \frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R}) \times O(d,\mathbb{R}) \times O(d,d,\mathbb{Z})}. \tag{3.64}$$

We see that the dimension of the coset space $O(d,d,\mathbb{R})/O(d,\mathbb{R}) \times O(d,\mathbb{R})$ is $\frac{2d(2d+1)}{2} - 2\frac{d(d+1)}{2} = d^2$, which matches exactly the actual degrees of freedom of the background E. The quotient of the above space with $O(d,d,\mathbb{R})$ does not alter the dimension, since it is a discrete group. We should also note here that there are special points of the moduli space corresponding to the self-dual radii $R^i = 1$ or, restoring the dimensions, $R^i = \sqrt{\alpha'}$, where there is an enhancement of the gauge group. However, we will not deal with this case in this work.

From now on we will parameterise $\mathcal{M}_{d,d}$ by

$$AF = \begin{pmatrix} e & 0 \\ -e^* B & e^* \end{pmatrix}. \tag{3.65}$$

If our analysis is not affected by whether we use an element of $O(d,d,\mathbb{R})$ or $O(d,d,\mathbb{Z})$, we will simply write $O(d,d)$.

There was another way to find the above moduli space of toroidal compactification. After defining the Narain lattice, we could have used the fact that all Lorentzian lattices (i.e. with inner product of indefinite signature) can be constructed by an $O(d,d,\mathbb{R})$ transformation Λ of a specific one, say Γ_0. However, transformations Λ' in $O(d,\mathbb{R}) \times O(d,\mathbb{R})$ preserve the physical conditions, so we get $\Lambda\Gamma_0 \sim \Lambda'\Lambda\Gamma_0$. Similarly, actions Λ'' of $O(d,d,\mathbb{Z})$ are symmetries of the lattice Γ_0, so finally $\Lambda\Gamma_0 \sim \Lambda'\Lambda\Lambda''\Gamma_0$, and se we see that $\mathcal{M}_{d,d}$ is indeed the moduli space we were looking for.

Note here that if we restore the R^i dependence by setting $m^i \to m^i R^i$ and

$n^i \to n_i/R^i$ (no summation here), and then look what happens when we take the supergravity limit $R^i \to 0$, we find that T-duality becomes an $O(d,d,\mathbb{R})$ symmetry of the lower dimensional supergravity theory. This can be understood heuristically if we observe the action of $O(d,d,\mathbb{N})$ on Z

$$Z = \left(\begin{array}{c} m^j R^j \\ n_j/R^j \end{array} \right) \to hZ = \left(\begin{array}{cc} a^i{}_j & b^{ij} \\ c_{ij} & d_i{}^j \end{array} \right) \left(\begin{array}{c} m^j R^j \\ n_j/R^j \end{array} \right), \tag{3.66}$$

where a, b, c, d are $d \times d$ matrices. Since we take the limit $R^i \to 0$, it is irrelevant whether a, b, c and d are integer valued or real valued matrices; we can as well take $h \in O(d,d,\mathbb{R})$.

The group $O(d,d)$[11] has some useful properties. First of all, writing h as

$$h = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \tag{3.67}$$

we find the relations

$$a^T c + c^T a = b^T d + d^T b = 0, \quad a^T + c^T d = 1 \tag{3.68}$$

and so

$$h^{-1} = \left(\begin{array}{cc} d^T & b^T \\ c^T & a^T \end{array} \right). \tag{3.69}$$

In addition it is easy to show that

$$h \in O(d,d) \Rightarrow h^T \in O(d,d). \tag{3.70}$$

The element $h^T \in O(d,d)$ (we use the transpose of h just for notational conve-

[11] For the rest of this work, we will consider the definition of $O(d,d)$ with respect to the metric $\eta = \left(\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right)$.

nience) induces the transformation of the generalized metric

$$\mathcal{H}(E) \to h\mathcal{H}(E)h^T. \tag{3.71}$$

Introducing $O(d,d)$ indices $M, N, \ldots = 1, \ldots, 2d$ we can write $\mathcal{H} = (\mathcal{H})_{MN}$ and $h = (h)_M{}^N$, giving

$$\mathcal{H}_{MN} \to h_M{}^P h_N{}^Q \mathcal{H}_{PQ}. \tag{3.72}$$

This linear transformation of $\mathcal{H}$ can be thought as a non-linear transformation of the background E, by

$$\mathcal{H}(E') = h\mathcal{H}(E)h^T. \tag{3.73}$$

In [29] it is shown explicitly that

$$E' \equiv h(E) = (aE+b)(cE+d)^{-1} \tag{3.74}$$

is the required background transformation. The transformation of the metric can be written in two different ways:

$$\begin{aligned} G &= (d - cE^T)^T G'(d - cE^T) \ \equiv MG'M^T \\ G &= (d + cE)^T G'(d + cE) \ \equiv \ \bar{M}G'\bar{M}^T. \end{aligned} \tag{3.75}$$

Returning now to the oscillators, we recall the relations (3.53) holding under a general transformation of the background $E \to E'$. Now, if we want to find a symmetry of the theory, the commutation relations for the oscillators $\alpha(E)$ and $\tilde{\alpha}(E)$, given by (3.43), should be preserved. These are derived from the canonical commutators (3.44), which are preserved if

$$\begin{pmatrix} X' \\ 2\pi P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \tag{3.76}$$

with $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d,\mathbb{R})$, as can be seen using (3.68). However, this gives for the zero modes of X' and $2\pi P$:

$$\begin{pmatrix} m \\ n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}, \tag{3.77}$$

which implies that actually $h \in O(d,d,\mathbb{Z})$, since m^i and n_i should always be integers. Then, the background transformation (3.74) simplifies the transformation of the oscillators (3.53) to

$$\begin{aligned} \alpha_k(E) &\to (M^T)^{-1}\alpha_k(E') \\ \tilde{\alpha}_k(E) &\to (\bar{M}^T)^{-1}\tilde{\alpha}_k(E'), \end{aligned} \tag{3.78}$$

where M and $\bar{M}$ are as in (3.75). We thus observe that the mode numbers of the oscillators do not mix, and also that the number operators

$$N = \sum_{k>0} \alpha^i_{-k}(E) G_{ij} \alpha^j_k(E), \quad \tilde{N} = \sum_{k>0} \tilde{\alpha}^i_{-k}(E) G_{ij} \tilde{\alpha}^j_k(E) \tag{3.79}$$

remain invariant. So, the physical constraints (3.61) and (3.62) are indeed invariant and we are consistent with the previous assumption of neglecting the N and $\tilde{N}$ transformation.

Let us investigate what exactly the T-duality transformation of the background $E = G + B$ correspond to. It can be shown that $O(d,d)$ is generated by the following elements (see [23]):

B-field shift by Θ_{ij}: These are elements of $O(d,d)$ of the form

$$h_\Theta = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}, \tag{3.80}$$

with Θ an antisymmetric $d \times d$ matrix, $\Theta_{ij} = -\Theta_{ji}$. The transformation of E (3.74) gives

$$E' = E + \Theta \tag{3.81}$$

and since Θ is antisymmetric, this is a shift of the B-field: $B \to B + \Theta$. Recalling that the B-term in the action (3.38) is a total derivative, we understand that the shift of B by Θ gives only topological contributions. In case $h_\Theta \in O(d,d,\mathbb{Z})$, the entries of Θ are integers, thus the action is changed by an integer multiple of 2π and finally the path integral $Z \sim e^{iS}$ remains invariant, giving us the same physical theory.

Basis change with $A \in GL(d,\mathbb{R})$: These are elements of $O(d,d)$ of the form

$$h_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \tag{3.82}$$

with $A \in GL(d,\mathbb{R})$. The transformation of E (3.74) gives

$$E' = AEA^T. \tag{3.83}$$

If we consider a lattice $\hat{\Gamma}^d$ such that the spacetime torus T^d is given by $T^d = \mathbb{R}^d/2\pi\hat{\Gamma}^d$, the vielbein vectors e defined by $G = e^T e$ form a basis of $\hat{\Gamma}^d$. Then the above transformation induces a transformation of the basis of $\hat{\Gamma}^d$. If $h_A \in O(d,d,\mathbb{Z}) \Leftrightarrow A \in GL(d,\mathbb{Z})$, then the lattice $\hat{\Gamma}^d$ is mapped to itself and so it induces the *large diffeomorphisms* of the torus T^d, which are indeed symmetries of T^d. The change of the mass condition (3.61) can be undone by a redefinition $m' = A^T m$ and $n' = A^{-1}$, which of course is allowed only when $A \in GL(d,\mathbb{Z})$.

Factorized Duality D_i: These are elements of $O(d,d)$ of the form

$$h_{D_i} = \begin{pmatrix} I - D_i & D_i \\ D_i & I - D_i \end{pmatrix}, \tag{3.84}$$

with $D_i = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$, the '1' being in the i^{th} slot. These transformations are a generalisation of the $R \to \alpha'/R$ duality we found in subsection (3.2), as can be understood if we check how they act on $Z = \begin{pmatrix} m \\ n \end{pmatrix}$. They reproduce the *Buscher rules* which were discovered (from a worldsheet perspective) to be T-duality transformations acting on isometric directions (see [33] and [34]).

In the above analysis we have completely ignored the dilaton. It can be shown (see [27]) that it transforms according to

$$e^{\Phi'}(\det G')^{-1/4} = e^{\Phi}(\det G)^{-1/4}. \tag{3.85}$$

This shows that T-duality relates a weak regime of the theory to another weak regime of the same theory[12] and so it can be analysed using string perturbation tools, as opposed to say S-duality, which is a strong-weak duality.
We observe that it is convenient to define a T-duality invariant dilaton d by

$$e^{-2d} = \sqrt{\det G} e^{-2\Phi}, \tag{3.86}$$

so that $e^{-2d'} = e^{-2d}$.

Finally, there is one other symmetry of the action (3.38): invariance under

$$\sigma \to -\sigma. \tag{3.87}$$

This is equivalent to the transformation

$$B \to -B \;\Leftrightarrow\; E \to E^T. \tag{3.88}$$

It does not belong in the T-duality group $O(d, d, \mathbb{Z})$, since it does not preserve

[12]This is true for the bosonic string. For superstrings, T-duality actually exchanges Type IIA and Type IIB.

the condition (3.62). Thus, there is an extra $\mathbb{Z}_2$ symmetry and, referring to the discussion near the end of subsection (3.2), we identify the group $\mathcal{G}_d$ as

$$\mathcal{G}_d = O(d, d, \mathbb{Z}) \ltimes \mathbb{Z}_2. \tag{3.89}$$

4 Double Field Theory

4.1 Doubled Space and Notation

Having finally set the scene, we are ready to introduce double field theory. In this section we are going to review the foundational papers [9], [10], [11], [12]. The recent reviews [35] and [36], as well as [29], will also be useful.
First, we are going to briefly discuss T-folds. More details can be found in [7] and [8].

Let M be a manifold of dimension D with G a riemannian metric, B a 2-form field and Φ a scalar field on M. The transition functions in the overlap of two coordinate patches involve diffeomorphisms for G, B and Φ, as well as gauge transformations of the B-field $B' = B + d\Lambda$, Λ being a one-form on M. We call (M, G, B, Φ) a *geometric background*. However, the rich symmetries of string theory seems to allow more general backgrounds, called *non-geometric backgrounds*, as we can also consider string dualities relating the different fields in the overlap, apart from the above transition functions. That means that, although locally we will always have a usual spacetime picture for our theory, this may not be the case globally.

Let us restrict our attention to the T-duality we studied in the previous section. Allowing such transformation to play the role of transition functions, we can construct what is called a *T-fold*. Now, supposing as before that $M = N \times T^d$, we can consider M as a d-dimensional torus bundle over the base space N and we can apply the analysis of section (3). If M is a geometric background, the transition functions include the group $GL(d, \mathbb{Z})$ of large diffeomorphisms of T^d. If M is a T-fold, $GL(d, \mathbb{Z})$ combines with the other string symmetries as we saw in subsection (3.4) resulting in the group $O(d, d, \mathbb{Z})$.
Recall that in the case of a rectangular torus with coordinates X^i (i.e. having chosen the canonical diagonal metric, whose entries are just the radii of the

compact dimensions), we could define the dual coordinates $\tilde{X}_i$, parameterising circles of the inverse radius. The dual coordinates can be used to define a *doubled torus* T^{2d} with coordinates $X^i, \tilde{X}_i$. If T^{2d} are considered to be fibres over N we are led to the bigger doubled space $\hat{M} = N \times T^{2d}$. T^{2d} can be thought of as the quotient of $\mathbb{R}^{d,d}$ by the Narain lattice $\Gamma^{d,d}$. T-duality is now of a more "geometrical" nature, as T^{2d} contains the original spacetime torus T^d as well as all other tori related to it by T-duality transformations, with the group $O(d, d, \mathbb{Z})$ acting naturally on T^{2d}. So, in this section, we will let all fields depend on the doubled coordinates. This is something genuinely different from what we have been doing until now, but we will shortly give convincing arguments for this approach. Thus, formally, all the objects we will use will be different from the previous ones. However, the usual theory can be regained by choosing a *polarisation*, i.e. a d-torus $\bar{T}^d \subset T^{2d}$ and so we will use identical symbols for related objects; the distinction should always be clear.Now, T-duality manifests itself as different choices of polarisation. If such a $\bar{T}^d$ fibre can be chosen consistently over all of N, the background is actually a geometric one, but this is not the general case.
It can actually be rigorously shown that such a doubled space is a consistent sigma-model target space.
Double field theory originated as an attempt to understand better the T-folds.

An important thing to keep in mind is that the dual coordinates are necessary in the string field theory for closed strings. More specifically, a state $|\Psi\rangle$ on a d-torus background is of the form:

$$|\Psi\rangle = \sum_I \int \mathrm{d}k \sum_{n_i, m^i} \phi_I(k_a, n_i, m^i) \mathcal{O}^I |k_a, n_i, m^i\rangle, \tag{4.1}$$

where O^I are the appropriate matter and ghost operators, k_a are the momenta in the uncompactified dimensions, n_i, m^i are the usual momentum and winding modes and ϕ_I are the physical (and ghost) fields. A simple Fourier transformation will give the fields $\phi_I(x^a, x^i, \tilde{x}_i)$, depending on the physical coordinates (x^a, x^i) and the dual ones $(\tilde{x}_i)$. In string field theory, the dual coordinates correspond to physical degrees of freedom, the conjugates of the winding modes, and their existence is not a gauge redundancy.

In string field theory, the mass condition (3.50) comes from the equations of motion for the string field, but the level matching condition (3.51) must necessarily be imposed on all fields as well as gauge parameters, since a gauge transformation should result in a consistent field, satisfying the condition.

Double field theory is, in a sense, an effective theory, a restriction of string field theory to the fields that are massless from the full D-dimensional point of view. Recall the general mass formula for a T^d background (3.50)

$$M^2 = \sum_{k=1}^{D-d} p_a p^a - \frac{1}{\alpha'} Z^T \mathcal{H}(E) Z = m^2 - \frac{1}{\alpha'} Z^T \mathcal{H}(E) Z = \frac{2}{\alpha'}(N + \tilde{N} - 2).$$

Here M^2 is the D-dimensional mass. $M^2 = 0$ implies $N = \tilde{N} = 1$. The case $(N, \tilde{N}) = (2, 0)$ or $(0, 2)$ is excluded because the condition (3.51)

$$N - \tilde{N} = n_i m^i$$

would not be satisfied when $n_i = 0$ or $m^i = 0$. That means that we will be concerned with the graviton $h_{ij}(x^a, x^i, \tilde{x}_i)$, the antisymmetric b-field $b_{ij}(x^a, x^i, \tilde{x}_i)$ and the dilaton $d(x^a, x^i, \tilde{x}_i)$, depending on all coordinates x^a, x^i and $\tilde{x}_i$. From now on we will always assume $N = \tilde{N} = 1$ and we will keep in mind that we are considering a physical spacetime $\mathcal{M} = \mathbb{R}^{D-d} \times T^d$.

Keeping full coordinate dependence is equivalent to keeping all momentum and winding modes, which are of course mixed under the action of T-duality symmetry. Thus, double field theory can also be thought of as an $O(d, d, \mathbb{Z})$ covariantisation of supergravity (meaning the usual Einstein plus B-field plus dilaton action (3.17)). In order to find the T-duality action on the doubled coordinates, we note that a transformation

$$Z \to g^T Z \tag{4.2}$$

of the momentum and winding modes $Z = \begin{pmatrix} m^i \\ n_i \end{pmatrix}$ by an element $g \in O(d, d, \mathbb{Z})$ is equivalent to a transformation

$$\mathbb{X} \to g\mathbb{X} \tag{4.3}$$

of the doubled coordinates $\mathbb{X} = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$, since Z and $\mathbb{X}$ are Fourier duals. Assuming that g is of the form $g = \begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1} \end{pmatrix}$, where $a \in GL(d, \mathbb{Z})$, we see that the position of the indices indicates the transformation under the $GL(d, \mathbb{Z})$ subgroup of $O(d, d, \mathbb{Z})$. We say that the coordinates $\mathbb{X}$ form the *vector representation* of $O(d, d, \mathbb{Z})$.

In fact, we can move on to some further generalisations which result in a very convenient notation. Firstly, we introduce dual coordinates $\tilde{x}_a$ for the uncompactified dimensions x^a, too. We can think that we begin with a spacetime $\mathcal{M} = T^D$ with the canonical coordinates, so that the doubled space is $\mathbb{R}^{D,D}$, and then take the decompactification limit for the directions x^a. Doing so, the inverse radius goes to 0, so everything will be independent of $\tilde{x}_a$, as could be intuitively understood from the beginning.
Secondly, we observe that the doubled space $\mathbb{R}^{D,D}$ has an $O(D, D, \mathbb{R})$ global symmetry. Compactification of $2d$ dimensions will result in the space $\mathbb{R}^{D-d,D-d} \times T^{2d}$, whose symmetries are $O(D-d, D-d, \mathbb{R}) \times O(d, d, \mathbb{Z})$. Finally, independence of $\tilde{x}_a$ will break this group to $O(D) \times O(d, d, \mathbb{Z})$, which is just the Lorentz plus the T-duality symmetry.

The above generalisations allow us to formulate double field in a $O(D, D, \mathbb{R})$ covariant way, so that all the cases are covered. That means that our notation will be manifestly $O(D, D)$ covariant, but we understand that the different physical situations will break this symmetry as described in the previous paragraph.
In order to proceed with the formulation, we should consider the D-dimensional versions of the objects introduced in subsections (3.3) and (3.4). For instance, we will work with the full background matrix $E_{\mu\nu}$, but will we remember that if d dimensions are compactified, it will have the form:

$$E_{\mu\nu} = \begin{pmatrix} E_{ij} & 0 \\ 0 & I \end{pmatrix} \tag{4.4}$$

and similarly, everything else that we used, such as the $O(d, d)$ transformations h and the matrices M, $\bar{M}$ defined in (3.75), will formally be $O(D, D)$ objects but actually they will belong in the subgroup $O(d, d) \subset O(D, D)$ preserving the uncompactified directions x^a. Then, the transformations (3.74) and all the relations we found will have exactly the same form.

We now introduce some more formalism and notation, as a natural continuation of subsections (3.3) and (3.4).
Recall that $X^i \sim X^i + 2\pi \Rightarrow x^i \sim x^i + 2\pi$, which in turn implies that $n_i \in \mathbb{Z}$, and similar relations hold for the dual coordinates. We can rewrite the relations (3.41) introducing the derivatives D_i and $\bar{D}_i$:

$$\begin{aligned} \alpha_{0i} &= -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}\right) \equiv -i\sqrt{\frac{\alpha'}{2}} D_i \\ \tilde{\alpha}_{0i} &= -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} + E_{ki}\frac{\partial}{\partial \tilde{x}_k}\right) \equiv -i\sqrt{\frac{\alpha'}{2}} \bar{D}_i. \end{aligned} \tag{4.5}$$

We will write $D^2 = D^i D_i$ and $\bar{D}^2 = \bar{D}_i \bar{D}^i$.[13] We also define the operator Δ as

$$-\frac{\alpha'}{2}\Delta = \frac{1}{2}(\alpha_0^i G_{ij} \alpha_0^j - \tilde{\alpha}_0^i G_{ij} \tilde{\alpha}_0^j). \tag{4.6}$$

A short computation shows that

$$\begin{aligned} \Delta = \frac{1}{2}(D^2 - \tilde{D}^2) &= -\frac{2}{\alpha'} \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \tilde{x}_i} \\ &= \begin{pmatrix} \tilde{\partial}^i & \partial_i \end{pmatrix} \begin{pmatrix} 0 & \delta^{ij} \\ \delta_{ij} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\partial}^j \\ \partial_j \end{pmatrix} \end{aligned} \tag{4.7}$$

and the level matching condition (3.51) becomes (for $N = \tilde{N} = 1$):

$$L_0 - \tilde{L}_0 = -\frac{\alpha'}{2}\Delta = 0. \tag{4.8}$$

Thus, all fields ϕ_I are constrained to satisfy

$$\Delta \phi_I = 0 \Leftrightarrow \partial_i \tilde{\partial}^i \phi_I = 0. \tag{4.9}$$

Similarly we define the operator

$$\Box = \frac{1}{2}(D^2 + \bar{D}^2). \tag{4.10}$$

We also denote the coordinates as

$$X = \begin{pmatrix} \tilde{x}_\mu \\ x^\mu \end{pmatrix} \tag{4.11}$$

and the measure

$$\mathrm{d}X = \mathrm{d}x^a \, \mathrm{d}x^i \, \mathrm{d}\tilde{x}_i, \tag{4.12}$$

since nothing depends on $\tilde{x}_a$.

[13] Of course, until otherwise stated, every index is raised and lowered with the background metric G and the inverse metric G^{-1}.

Let now $\Xi = \begin{pmatrix} \bar{\xi}_\mu \\ \xi^\mu \end{pmatrix}$ be an object transforming in the fundamental representation of $O(D,D)$, i.e.

$$\Xi' = \begin{pmatrix} \bar{\xi}' \\ \xi' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{\xi} \\ \xi \end{pmatrix} = \begin{pmatrix} a\bar{\xi} + b\xi \\ c\bar{\xi} + d\xi \end{pmatrix}, \tag{4.13}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D,D)$. Then we can prove that

$$\begin{aligned} \lambda_\mu &\equiv -\bar{\xi}_\mu + E_{\mu\nu}\xi^\nu \\ \bar{\lambda}_\mu &\equiv \bar{\xi}_\mu + E_{\nu\mu}\xi^\nu \end{aligned} \tag{4.14}$$

transform with the matrices M and $\bar{M}$ as follows:

$$\begin{aligned} \Xi_\mu &= M_\mu{}^\nu \Xi'_\nu \\ \bar{\Xi}_{\bar{\mu}} &= \bar{M}_{\bar{\mu}}{}^{\bar{\nu}} \Xi'_{\bar{\nu}}. \end{aligned} \tag{4.15}$$

or

$$\begin{aligned} \Xi &= M\Xi' \\ \bar{\Xi} &= \bar{M}\Xi'. \end{aligned} \tag{4.16}$$

Then, if we raise the indices with G^{-1} and use the transformation of G (3.75), we can show that

$$\begin{aligned} G^{-1}\Xi &= (M^T)^{-1}G'^{-1}\Xi' \\ G^{-1}\bar{\Xi} &= (\bar{M}^T)^{-1}G'^{-1}\Xi'. \end{aligned} \tag{4.17}$$

We also observe that, although M and $\bar{M}$ multiply the transformed objects, they depend on the background E.

Applying the above to the partial derivative $\partial \equiv \begin{pmatrix} \partial_\mu \\ \tilde{\partial}^\mu \end{pmatrix}$ we find that the derivatives D_μ and $\bar{D}_\mu$ transform as:

$$D_\mu = M_\mu{}^\nu D'_\nu$$
$$\bar{D}_{\bar{\mu}} = \bar{M}_{\bar{\mu}}{}^{\bar{\nu}} \bar{D}'_{\bar{\nu}} \tag{4.18}$$

and any background fluctuation δE as:

$$\delta E_{\mu\bar{\nu}} = M_\mu{}^\rho \bar{M}_{\bar{\nu}}{}^{\bar{\sigma}} \delta E_{\rho\bar{\sigma}}. \tag{4.19}$$

Specifically, this applies to $e_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu}$, so we should consider the second index as barred. However, we will make this distinction only when it is necessary. Recall that, due to equations (3.75), the background metric G can be treated as having two unbarred or two barred indices.

String field theory gives the following action for the fields $e_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu}$ and d, up to cubic order in the fields, where the background is described by the matrix E (setting the constant $(2\kappa)^2 = 1$):

$$\begin{aligned} S(E,e,d) = &\int \mathrm{d}X \mathcal{L}[D_\mu, \bar{D}_\mu, G^{-1}, e_{\mu\nu}(X), d(X)] \\ = &\int \mathrm{d}X [\frac{1}{4} e_{\mu\nu} \Box e^{\mu\nu} + \frac{1}{4}(\bar{D}^\nu e_{\mu\nu})^2 + \frac{1}{4}(D^\mu e_{\mu\nu})^2 - 2d D^\mu \bar{D}^\nu e_{\mu\nu} - 4d\Box d \\ &+ \frac{1}{4} e_{\mu\nu}((D^\mu e_{\rho\sigma})(\bar{D}^\nu e^{\rho\sigma}) - (D^\mu e_{\rho\sigma})(\bar{D}^\sigma e^{\rho\nu}) - (D^\rho e_{\mu\sigma})(\bar{D}^\nu e_{\rho\sigma})) \\ &+ \frac{1}{2} d((D^\mu e_{\mu\nu})^2 + (\bar{D}^\nu e_{\mu\nu})^2 + \frac{1}{2}((D_\rho e_{\mu\nu})^2 + \frac{1}{2}((\bar{D}_\rho e_{\mu\nu})^2 \\ &+ 2e^{\mu\nu}(D_\mu D^\rho e_{\rho\nu} + \bar{D}_\nu \bar{D}^\rho e_{\mu\rho})) + 4e_{\mu\nu} d D^\mu \bar{D}^\nu d + 4d^2 \Box d]. \end{aligned} \tag{4.20}$$

String field theory has a gauge symmetry, which manifests itself in the action

(4.20) as:

$$\delta_\lambda e_{\mu\nu} = D_\mu\bar{\lambda}_\nu + \bar{D}_\nu\lambda_\mu + \frac{1}{2}(\lambda_\rho D^\rho + \bar{\lambda}_\rho\bar{D}^\rho)e_{\mu\nu} + \frac{1}{2}(D_\mu\lambda^\rho - D^\rho\lambda_\mu)e_{\rho\nu} - e_{\mu\rho}\frac{1}{2}(\bar{D}^\rho\bar{\lambda}_\nu - \bar{D}_\nu\bar{\lambda}^\rho)$$
$$\delta_\lambda d = -\frac{1}{4}(D^\mu\lambda_\mu + \bar{D}^\mu\lambda_\mu) + \frac{1}{2}(\lambda_\mu D^\mu + \bar{\lambda}_\mu\bar{D}^\mu)d. \tag{4.21}$$

Here $\lambda_\mu(X)$ and $\bar{\lambda}_\mu(X)$ are independent gauge parameters.

The second line of the action (4.20) is the action to quadratic order

$$S^{(2)} = \int \mathrm{d}X[\frac{1}{4}e_{\mu\nu}\Box e^{\mu\nu} + \frac{1}{4}(\bar{D}^\nu e_{\mu\nu})^2 + \frac{1}{4}(D^\mu e_{\mu\nu})^2 - 2dD^\mu\bar{D}^\nu e_{\mu\nu} - 4d\Box d], \tag{4.22}$$

with the gauge transformations

$$\delta_\lambda e_{\mu\nu} = D_\mu\bar{\lambda}_\nu + \bar{D}_\nu\lambda_\mu\delta_\lambda d \quad = -\frac{1}{4}(D^\mu\lambda_\mu + \bar{D}^\mu\lambda_\mu). \tag{4.23}$$

It is shown in [9] that if we set $\tilde{\partial} = 0$ in (4.22), i.e. if every field is independent of $\tilde{X}$, then we end up with the supergravity action (3.17).

$$S_* = \int \mathrm{d}^D x\sqrt{-G'}e^{-2\Phi'}[R + (4\partial\Phi')^2 - \frac{1}{12}H^2], \tag{4.24}$$

the primes indicating that the above action actually contains the background fields plus the first order fluctuations, for example $G' = G + g$ etc.

The gauge transformations of this action are diffeomorphisms parameterised by a vector field ϵ^μ and B-field gauge transformations parameterised by a one-form $\tilde{\epsilon}_\mu$

$$\delta G' = \mathcal{L}_\epsilon G'$$
$$\delta B' = \mathcal{L}_\epsilon B' + d\tilde{\epsilon} \tag{4.25}$$
$$\delta\Phi' = \mathcal{L}_\epsilon\Phi',$$

which should be included in the double field theory gauge transformations (4.23).

Indeed, by writing $\epsilon_\mu \equiv \frac{1}{2}(\lambda_\mu + \bar{\lambda}_\mu)$ and $\tilde{\epsilon}_\mu \equiv \frac{1}{2}(\lambda_\mu - \bar{\lambda}_\mu)$, (4.23) imply that

$$\begin{aligned} \delta g_{\mu\nu} &= \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu & \tilde{\delta} g_{\mu\nu} &= \tilde{\partial}_\mu \tilde{\epsilon}_\nu + \tilde{\partial}_\nu \tilde{\epsilon}_\mu \\ \delta b_{\mu\nu} &= -(\tilde{\partial}_\mu \epsilon_\nu - \tilde{\partial}_\nu \epsilon_\mu) & \tilde{\delta} b_{\mu\nu} &= -(\partial_\mu \tilde{\epsilon}_\nu - \partial_\nu \tilde{\epsilon}_\mu) \\ \delta d &= -\tfrac{1}{2}\partial \cdot \epsilon & \tilde{\delta} d &= \tfrac{1}{2}\tilde{\partial} \cdot \tilde{\epsilon}. \end{aligned} \tag{4.26}$$

We see that by setting $\tilde{\partial} = 0$, we get the infinitesimal version of (4.25):

$$\begin{aligned} \delta g_{\mu\nu} &= \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu & \tilde{\delta} g_{\mu\nu} &= 0 \\ \delta b_{\mu\nu} &= 0 & \tilde{\delta} b_{\mu\nu} &= -(\partial_\mu \tilde{\epsilon}_\nu - \partial_\nu \tilde{\epsilon}_\mu) \\ \delta d &= -\tfrac{1}{2}\partial \cdot \epsilon & \tilde{\delta} d &= 0 \end{aligned} \tag{4.27}$$

with ϵ^μ the vector field parameterising the diffeomorphisms and $\tilde{\epsilon}_\mu$ the one-form gauge parameter of the b-field. Note that all indices are still raised and lowered with $G_{\mu\nu}$ and $G^{\mu\nu}$.

Already by (4.25) the connection of supergravity with generalized geometry is apparent, since we can collectively denote the parameters ϵ^μ and $\tilde{\epsilon}_\mu$ of as an element of $(T \oplus T^*)\mathcal{M}$. We will return to this point in the following subsections.

Finally, we comment on T-duality transformations. (Closed) string field theory is T-duality invariant, as proved in [6]. The fact that the action (4.20) inherits this invariance (which is what we really wanted to achieve from the beginning) can be proven if we check carefully the $O(D,D)$ transformations of the objects $D_\mu, \bar{D}_\mu, G^{-1}, e_{\mu\nu}, d$ and $\mathrm{d}X$, which make up the Lagrangian density $\mathcal{L}$. It can be seen that there are only consistent contraction of upper barred-lowered barred and upper unbarred-lowered unbarred indices, so that the transformations (4.18), (3.75) and (4.19), using (4.16) and (4.17), together with $d(X) = d'(X')$ and $\int \mathrm{d}X = \int \mathrm{d}X'$, imply that the action (4.20) satisfties:

$$S(E, e(e'), d(d')) = S(E', e', d'), \tag{4.28}$$

so T-duality is indeed a symmetry of the cubic double field theory action.

In [6] it was also proven that the $\mathbb{Z}_2$ symmetry (3.88)

$$E \to E^T \tag{4.29}$$

discuss at the end of subsection (3.4) is also a symmetry of closed string field theory. Noting that the above transformation implies the interchange of D and $\bar{D}$: $D \leftrightarrow \bar{D}$ and redefining

$$X' = \begin{pmatrix} \tilde{x}' \\ x \end{pmatrix} = \begin{pmatrix} -\tilde{x} \\ x \end{pmatrix}, \tag{4.30}$$

(which is a natural thing to do if we recall the expressions (3.87) and (3.42)) we can also prove that

$$S(E^T, e, d) = S(E, e, d). \tag{4.31}$$

Thus, the $\mathbb{Z}_2$ is also inherited in the action (4.20).

4.2 Strong Constraint and Gauge Algebra

Recall that the constraint (4.9) must be imposed to all fields and gauge parameters. Let's explore this a little more.

In the "generalized" momentum space the constraint (4.9) takes the form

$$m^i n_i = 0. \tag{4.32}$$

A field $f(x^\mu, \tilde{x}_i)$ can be expanded in momentum modes in the compactified directions by doing a Fourier transform

$$f(x^\mu, \tilde{x}_i) = \sum_{Z \in \mathbb{Z}^{2d}} f(x^a, Z) e^{iZ^T \mathbb{X}}, \tag{4.33}$$

where we recall the definition $Z = \begin{pmatrix} m^i \\ n_i \end{pmatrix}$ and $\mathbb{X} = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$. The level matching condition is now equivalent to the statement that the momentum vector Z is null with respect to the metric $\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, i.e.

$$Z^T \eta Z = 0. \tag{4.34}$$

We can define the projector $[[,]]$ by to the null modes

$$[[A]] = \sum_{Z \in \mathbb{Z}^{2d}} \delta_{Z^T \eta Z, 0} f(x^a, Z) e^{i Z^T \mathbb{X}}, \tag{4.35}$$

so that

$$\Delta [[A]] = 0. \tag{4.36}$$

Now, we can easily see that, while $\Delta A = 0 = \Delta B \Rightarrow \Delta(A+B) = 0$, in general $\Delta A = 0 = \Delta B \nRightarrow \Delta(AB) = 0$, so, the level matching constraint implies that we should use the projector $[[,]]$ to every product of fields or gauge parameters. In particular, the second order terms of the gauge transformations (4.21), i.e actually we should have written:

$$\begin{aligned} \delta_\lambda e_{\mu\nu} &= D_\mu \bar{\lambda}_\nu + \bar{D}_\nu \lambda_\mu \\ &\quad + [[\frac{1}{2}(\lambda_\rho D^\rho + \bar{\lambda}_\rho \bar{D}^\rho) e_{\mu\nu} + \frac{1}{2}(D_\mu \lambda^\rho - D^\rho \lambda_\mu) e_{\rho\nu} - e_{\mu\rho} \frac{1}{2}(\bar{D}^\rho \bar{\lambda}_\nu - \bar{D}_\nu \bar{\lambda}^\rho)]] \\ \delta_\lambda d &= -\frac{1}{4}(D^\mu \lambda_\mu + \bar{D}^\mu \lambda_\mu) + [[\frac{1}{2}(\lambda_\mu D^\mu + \bar{\lambda}_\mu \bar{D}^\mu) d]]. \end{aligned} \tag{4.37}$$

One would expect that we should use the same projectors in the cubic action (4.20), since all the terms are double or triple products of fields. However, the integration over $x^i, \tilde{x}_i$ does automatically the projection: as can be easily checked by using the corresponding Fourier expansions, $\Delta A = 0 \Rightarrow \int \mathrm{d}\mathbb{X} A[[B]] =$

$\int$ d$\mathbb{X}AB$, even if $\Delta B \neq 0$. Thus, although we have $\Delta A = 0$, $\Delta(BC) \neq 0$ and so we should use the projection $[[BC]]$ in the action, we can apply the previous relation and get $\int$ d$\mathbb{X}A[[BC]] = \int$ d$\mathbb{X}ABC$.

However, the analogous statement does not hold for the product of four fields or more. So, it is very difficult to write the quartic terms of the action. There are also some other subtleties, involving the cocycle induced factors, that we need to take into account if we want to proceed (see [9] for further details).

In order to move on our analysis conveniently, we will impose a stronger condition than (4.9): we will assume that all fields and gauge parameters *and* all products of fields and gauge parameters are annihilated by Δ. This means that

$$\partial_i\tilde{\partial}^i(AB) = 0 \;\Leftrightarrow\; \partial_i A\tilde{\partial}^i B + \partial_i B\tilde{\partial}^i A = 0 \tag{4.38}$$

for all fields and gauge parameters A, B which satisfy the level matching condition (4.9).

We will call this new condition the *strong constraint*, while we will refer to (4.9) as the *weak constraint*.

We will now understand better what the strong constraint implies.

As we just saw, the weak constraint implies that a momentum vector $Z = \begin{pmatrix} m^i \\ n_i \end{pmatrix}$ of a field A is null with respect to η. Then, the strong constraint implies that $Z_a^T\eta Z_b = 0$ for all momentum vectors of the same or of different fields. In other words, all momentum vectors must lie in an isotropic subspace of the lattice $\mathbb{Z}^{d,d}$, considered as a vector space with inner product defined by the metric η.

Now we can use some of the tools developed in subsection (2.2). It can be shown that any isotropic subspace is a subspace of a *maximal* isotropic one, i.e. of dimension d. Then, we can consider coordinates $y^j, \tilde{y}_j,\ j = 1, \ldots, d$ with y^j conjugate to the momentum vectors and $\tilde{y}_j$ orthogonal to them, so that the fields depend only on y^i. y^j and $\tilde{y}_j$ are coordinated on the doubled torus T^{2d}. It can also be proven that any maximal isotropic subspace is related by an $O(d,d)$ transformation to any other one. In particular, it is related to the space with momenta of the form $Z_T = \begin{pmatrix} 0 \\ n_i \end{pmatrix}$, which is just the physical spacetime torus T^d, parameterised by x^i.

Thus, imposing the strong constraint means that there is always a T-duality transformation which will bring us back to the *supergravity picture*, i.e. to dependence only on x^i. Then, in this sense, the theory is not truly doubled. The formalism *is* $O(d,d)$ (and $O(D,D)$) covariant, but at the very end we should make a choice of subspace, breaking this symmetry spontaneously.

The rest of the work will be carried out assuming the strong version of the constraint.

We now proceed to understand the gauge algebra of double field theory.

We have found the gauge transformations up to quadratic order (4.21) in the fields. In general they will be very complicated, but, imposing the strong constraints simplifies significantly the situation: assuming some consistency conditions (closure of the gauge algebra, $O(D,D)$ covariance, correct supergravity limit) we find that only one cubic term must be added to (4.21). The full non-linear transformations are now

$$\begin{aligned}
\delta_\lambda e_{\mu\nu} &= D_\mu \bar{\lambda}_\nu + \bar{D}_\nu \lambda_\mu \\
&\quad + \frac{1}{2}(\lambda_\rho D^\rho + \bar{\lambda}_\rho \bar{D}^\rho) e_{\mu\nu} + \frac{1}{2}(D_\mu \lambda^\rho - D^\rho \lambda_\mu) e_{\rho\nu} - e_{\mu\rho}\frac{1}{2}(\bar{D}^\rho \bar{\lambda}_\nu - \bar{D}_\nu \bar{\lambda}^\rho) \\
&\quad - \frac{1}{4} e_{\mu\kappa}(D^\rho \bar{\lambda}^\kappa + D^\kappa \bar{\lambda}^\rho) e_{\rho\nu} \\
\delta_\lambda d &= -\frac{1}{4}(D^\mu \lambda_\mu + \bar{D}^\mu \lambda_\mu) + \frac{1}{2}(\lambda_\mu D^\mu + \bar{\lambda}_\mu \bar{D}^\mu) d
\end{aligned} \tag{4.39}$$

(see [10] for details). We observe here that in fact there is a gauge redundancy: if the gauge parameters can be expressed as $\lambda_\mu = D_\mu \chi$, $\bar{\lambda}_\mu = -\bar{D}_\mu \chi$ with χ a scalar field on the doubled space, then the transformations are trivial.

In general, if we have an algebra with generators $T_A t_\alpha, t_a$ such that t_a annihilate the fields, it can be realised as classical infinitesimal symmetries on some fields only if the action of the Jacobiator, defined in equation (2.18) of any three generators on any field vanishes. This can happen not only when the Jacobiator itself vanishes, but also when it can be expressed as a linear combination of t_a

$$\mathrm{Jac}(T_A, T_B, T_C) = g^a_{ABC} t_a. \tag{4.40}$$

In case of the Einstein-Hilbert action

$$S_{EH} = \int \mathrm{d}^D x \sqrt{-g} R \tag{4.41}$$

the gauge symmetries of the action are diffeomorphisms of the metric field g parameterised by vector fields:

$$\delta g = \mathcal{L}_V g. \tag{4.42}$$

Then, the commutator of two such transformations is another transformation with parameter the commutator of the vector fields

$$[\delta_{V_1}, \delta_{V_2}] g = \mathcal{L}_{[V_1, V_2]} g, \tag{4.43}$$

with "[,]" in the right-hand side being the Lie bracket, which satisfies the Jacobi identity and produces the usual Lie algebra of diffeomorphisms.

The same considerations for the supergravity action (3.17), which includes an antisymmetric 2-form field and a dilaton as well as the metric, lead to a gauge algebra determined by the Courant bracket (2.16). Heuristically, this happens because the parameters of diffeomorphisms (vector fields) and B-field gauge transformations (one-forms) are naturally combined in elements of $(T \oplus T^*)M$. The algebra bracket should be invariant under both x^μ diffeomorphisms and

B-field shifts by a closed 2-form[14] (since they do not the physical field strength $H = dB \to H = d(B + \Theta) = dB$). Indeed, we saw in section 2 that the Courant bracket satisfies these properties. Then, the Jacobiator is an exact one-form (see equation (2.17)), which are trivial B-field gauge transformations: $b \to b + d(dN) = b$. For more details see [29].
There has been interesting progress in trying to fully reformulate supergravity in terms of generalized geometry, see for example [15] and [37].

We seek the analogous formulas for double field theory. First, we are going to introduce for the last time a new notation, that will be important for the next subsection, too.

We define $M, N, \ldots = 1, 2, \ldots, 2D$ to be $O(D, D)$ covariant indices in the following way: let

$$X^M = \begin{pmatrix} \tilde{x}_\mu \\ x^\mu \end{pmatrix} \tag{4.44}$$

So, the first D values of M represent the coordinates $\tilde{x}_\mu$ and the rest represent x^μ. The matrices $h \in O(D, D)$ which act on the coordinates X^M can be denoted as $h^M{}_N$. The $O(D, D)$ metric η has naturally two lower indices, so that the definition of the $O(D, D)$ group is

$$\eta_{NQ} h^M{}_N h^P{}_Q = \eta_{MP}. \tag{4.45}$$

The generalized metric $\mathcal{H}$ comes with two upper indices:

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \equiv \mathcal{H}^{MN}. \tag{4.46}$$

This is a very uniform notation, which makes mane formulas more elegant. For

[14]This is what we called "B-field transformations in section 2.

example, we find that $\mathcal{H}$ satisfies the relation:

$$\eta\mathcal{H}\eta = \mathcal{H}^{-1} \;\Leftrightarrow\; \eta_{MP}\eta_{NQ}\mathcal{H}^{MN} = \mathcal{H}_{PQ}. \tag{4.47}$$

We will always raise and lower the capitalised indices with the constant metric η_{MN} and its inverse η^{MN}.

Of course the coordinates are conjugate to the momenta, so consistency requires that we denote

$$\partial_M = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix} \leftrightarrow \partial^M = \begin{pmatrix} \partial_i \\ \tilde{\partial}^i \end{pmatrix}. \tag{4.48}$$

Finally, the weak constraint can be written as

$$\eta^{MN}\partial_N\partial_M A = 0 \;\Leftrightarrow\; \partial^M\partial_M A = 0 \tag{4.49}$$

and the strong constraint as

$$\partial^M\partial_M(AB) = 0 \;\Leftrightarrow\; \partial^M A\partial_M B = 0, \tag{4.50}$$

for all fields an gauge parameters A, B.

The gauge algebra takes a convenient form if we redefine the parameters according to

$$\Sigma^M = \begin{pmatrix} \tilde{\xi}_\mu \\ \xi^\mu \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -E_{\nu\mu}\lambda^\nu + E_{\mu\nu}\bar{\lambda}^\nu \\ \lambda^\mu + \bar{\lambda}^\mu \end{pmatrix}. \tag{4.51}$$

The relations (4.39) imply that $\lambda_\mu, \bar{\lambda}_\mu$ transform with the matrices M and $\bar{M}$ under $O(D,D)$, so, by (4.14), we see that Σ^M transforms as a vector of $O(D,D)$. In addition, the parameters of the form $\Sigma^M = \eta^{MN}\partial_N\chi$ induce the gauge redundancy described before.

Now, a straightforward computation of the commutator of the transformations of $e_{\mu\nu}$ shows that the bracket satisfies by the double field theory gauge algebra is the so called *C-bracket*:

$$[\delta_{\Sigma_1^M}, \delta_{\Sigma_2^M}] = \delta_{\Sigma_{12}^M} \tag{4.52}$$

with

$$\Sigma_{12}^M = \Sigma_{[2}^N \partial_N \Sigma_{1]}^M - \frac{1}{2} \Sigma_{[2}^N \partial^M \Sigma_{1]N} \equiv -[\Sigma_1, \Sigma_2]_C^M. \tag{4.53}$$

The first term is the Lie bracket analogue for the doubled fields, but the second one describes a correction which involves the metric η in raising and lowering the indices of ∂^M and Σ_{1N}. This does not causes problems because η is constant on all of the doubled space.

For the dilaton d the transformations are simpler. Then, the strong constraint can be used to show that they are consistent with both diffeomorphisms and C-algebra transformations.

If we write the C-bracket more analytically, it has a form very similar to the Lie bialgebroid bracket of equation (2.24):

$$\begin{aligned}[A+\alpha, B+\beta] = [A,B] + \mathcal{L}_\alpha B - \mathcal{L}_\beta A - \frac{1}{2}\tilde{d}(i_A\beta - i_B\alpha) \\ + [\alpha,\beta] + \mathcal{L}_A\beta - \mathcal{L}_B\alpha + \frac{1}{2} d(i_A\beta - i_B\alpha),\end{aligned} \tag{4.54}$$

where the gauge parameters are defined by

$$\Sigma_1 = \begin{pmatrix} \alpha_\mu \\ A^\mu \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \beta_\mu \\ B^\mu \end{pmatrix} \tag{4.55}$$

and the definitions of the various objects of (4.54) are:

$$\begin{aligned}
(\mathcal{L}_\alpha A)^\mu &= \alpha_\nu \tilde{\partial}^\nu A^\mu + A^\nu \tilde{\partial}^\mu \alpha_\nu \\
(\mathcal{L}_A \alpha)_\mu &= A^\mu \partial_\mu \alpha_\nu + \alpha_\nu \partial_\mu A^\nu \\
(\mathcal{L}_A B)^\mu &\equiv [A,B]^\mu = A^\mu \partial_\mu \alpha_\nu + \alpha_\nu \partial_\mu A^\nu \\
(\mathcal{L}_\alpha \beta)_\mu &\equiv ([\alpha,\beta])_\mu = \alpha_\nu \tilde{\partial}^\nu \beta_\mu - \beta_\mu \tilde{\partial}^\mu \alpha_\nu \\
d\chi &= \begin{pmatrix} \partial_\mu \chi \\ 0 \end{pmatrix}, \quad \tilde{d}\chi = \begin{pmatrix} 0 \\ \tilde{\partial}^\mu \chi \end{pmatrix} \\
i_A \alpha &= \alpha_\mu A^\mu,
\end{aligned} \tag{4.56}$$

where χ is a scalar function. We should not be deceived by the apparent similarities with the conventional definitions of the Lie derivative etc. These are all genuinely new objects, since they act on fields depending on the doubled spacetime. However, they have the same form with the usual objects (or a very similar one) and that is why we use the same notation. In the same way, we note that the Lie bialgebroid bracket (2.24) consists of two vector bundles over the *same* base space, while in (4.54) the fields formally depend on all of the doubled coordinates. Thus, only at the very end, when we make a specific choice of a D-dimensional (null) torus $\hat{T}^D \subset T^{2D}$ on which the fields depend, the C-bracket can be identified with the Lie bialgebroid bracket. If in particular we choose the physical spacetime torus, i.e. set $\tilde{\partial} = 0$, something that can always be done as we have seen, then the C-bracket reduces precisely to the Courant bracket and some of the above objects can be identified with the corresponding conventional objects, for instance the Lie derivative. This is also obvious from the discussion in subsection (2.3).

In the next subsection we will write the C-bracket in an even more symmetrical form.

4.3 Background Independence and Generalized Metric Formulation

Heuristically, background independence is the statement that the physics does not depend on the splitting of the full metric $\mathfrak{g}$ into a "background metric" G and a (first order) "fluctuation" g. Actually, as we have discussed before, it is natural to consider as "background" the whole field $E = G + B$, but not the dilaton. Background independence is a very important property that the fundamental theory of quantum gravity should have. It is believed that string theory has a truly background independent formulation, although for the moment it is not completely obvious.

It is particularly interesting to prove that this property is inherited from the full string theory in the cubic action (4.20). More concretely, we should show that any constant shift in the fluctuation field $e_{\mu\nu} \to e_{\mu\nu} + \chi_{\mu\nu}$ can be "absorbed" in the background $E_{\mu\nu}$, or that

$$S(E_{\mu\nu}, e_{\mu\nu} + \chi_{\mu\nu}) = S(E_{\mu\nu} + \chi_{\mu\nu}, {e'}_{\mu\nu}), \tag{4.57}$$

where e' is a redefinition of the fluctuation field e, depending on χ in general, i.e. $e' = f(e, \chi)$.

In [11] it is shown explicitly that this is the case. In fact, something even more remarkable happens: We can write an action depending only on the full background $\mathcal{E}_{\mu\nu} = E_{\mu\nu} + [(1 - \frac{1}{2}e)^{-1}]_\mu{}^\kappa e_{\kappa\nu} \equiv \mathfrak{g}_{\mu\nu} + \mathfrak{b}_{\mu\nu}$ and other background independent objects, which is complete to all orders if we assume that the strong constraint holds.

First, we need to generalise the notion introduced in this section to the corresponding background independent ones. This can be done because we did not use the fact that $E_{\mu\nu}$ is independent of X^M. In particular, we straightforwardly generalise:

$$D_\mu = \partial_\mu - E_{\mu\nu}\tilde{\partial}^\nu \;\rightarrow\; \mathcal{D}_\mu = \partial_\mu - \mathcal{E}_{\mu\nu}\tilde{\partial}^\nu$$
$$M = (d - cE^T)^T \;\rightarrow\; M(X) = (d - c\mathcal{E}^T)^T, \tag{4.58}$$

and similarly $\bar{D}_\mu \rightarrow \bar{\mathcal{D}}_\mu,\ \bar{M} \rightarrow M(\bar{X})$.

We can write a generalized metric without reference to any background field as:

$$\mathcal{H}(\mathcal{E}) = \begin{pmatrix} \mathfrak{g} - \mathfrak{b}\mathfrak{g}^{-1}\mathfrak{b} & \mathfrak{b}\mathfrak{g}^{-1} \\ -\mathfrak{g}^{-1}\mathfrak{b} & \mathfrak{g}^{-1} \end{pmatrix}. \tag{4.59}$$

The $O(D,D)$ transformation

$$\mathcal{H}(\mathcal{E}'(X')) = h\mathcal{H}(\mathcal{E}(X))h^T \tag{4.60}$$

induces the same form of the transformations of $\mathfrak{g}$, $\delta\mathcal{E}(X)$ etc as in equations (3.74), (3.75), (4.19), (4.18), but now with $M(X)$ and $\bar{M}(X)$ depending on X. Thus, as long as there are no second derivatives such as D^2 or $\bar{D}^2$ in the action, a consistent contraction of indices will lead to an $O(D,D)$ invariant action. Note that now we will raise and lower the $\mu, \nu, \ldots$ indices with the full metric $\mathfrak{g}$ and its inverse $\mathfrak{g}^{-1}$.

The action that correctly generalises (4.20) is

$$\begin{aligned} S_{\mathcal{E},d} = \int \mathrm{d}X e^{-2d}[&-\frac{1}{4}\mathfrak{g}^{\mu\lambda}\mathfrak{g}^{\nu\kappa}\mathcal{D}^\rho\mathcal{E}_{\lambda\kappa}\mathcal{D}_\rho\mathcal{E}_{\mu\nu} \\ &+\frac{1}{4}\mathfrak{g}^{\lambda\kappa}(\mathcal{D}^\nu\mathcal{E}_{\mu\lambda}\mathcal{D}^\mu\mathcal{E}_{\nu\kappa} + \bar{\mathcal{D}}^\nu\mathcal{E}_{\lambda\mu}\bar{\mathcal{D}}^\mu\mathcal{E}_{\kappa\nu}) \\ &+(\mathcal{D}^\mu d\bar{\mathcal{D}}^\nu\mathcal{E}_{\mu\nu} + \bar{\mathcal{D}}^\mu d\bar{\mathcal{D}}^\nu\mathcal{E}_{\nu\mu}) + 4\mathcal{D}^\mu d\mathcal{D}_\mu d]. \end{aligned} \tag{4.61}$$

This action can be seen to be $O(D,D)$ invariant, since each term in it is. It is completely fixed by the requirement of gauge invariance and of the correct expansion to cubic order, which must be equation (4.20). It is also invariant under the $\mathbb{Z}_2$ symmetry

$$\mathcal{E}_{\mu\nu} \to \mathcal{E}_{\nu\mu},\ \mathcal{D} \leftrightarrow \mathcal{D},\ d \to d. \tag{4.62}$$

The gauge transformations (4.39), with parameter $\Sigma = \begin{pmatrix} \alpha_\mu \\ A^\mu \end{pmatrix}$ can be written in a more unified way if we supplement the definitions (4.56) with:

$$\tilde{i}_\alpha A = \alpha_\mu A^\mu \tag{4.63}$$

and define the *Lie derivatives with respect to* A^μ *and* α_μ as:

$$\begin{aligned} \mathcal{L}_A &= di_A + i_A d \\ \tilde{\mathcal{L}}_\alpha &= \tilde{d}\tilde{i}_\alpha + \tilde{i}_\alpha \tilde{d}. \end{aligned} \tag{4.64}$$

Now the gauge transformations take the form:

$$\begin{aligned} &\delta_\Sigma e^{-2d} = \partial_M(\Sigma^M e^{-2d}), \\ &\delta_\Sigma \mathcal{E}_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu + \mathcal{L}_A \mathcal{E}_{\mu\nu} + \tilde{\mathcal{L}}_\alpha \mathcal{E}_{\mu\nu} - \mathcal{E}_{\mu\rho}(\tilde{\partial}^\rho A^\lambda - \tilde{\partial}^\lambda A^\rho)\mathcal{E}_{\lambda\nu}. \end{aligned} \tag{4.65}$$

We note in particular that e^{-2d} transforms as a density (more on this in a while), so it is consistent to set it as the integration measure.

By restricting the dependence to x^μ, the $\mathcal{E}$ transformations give:

$$\delta_A \mathcal{E}_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu + \mathcal{L}_A \mathcal{E}_{\mu\nu} + \tilde{\mathcal{L}}_\alpha \mathcal{E}_{\mu\nu}, \tag{4.66}$$

which are exactly the diffeomorphisms and B-field gauge transformations of the supergravity action. The rest of the terms in (4.65) make up for the $O(D, D)$ covariance of the gauge transformation (which is manifest in the transformation of d).

We can proceed even more and construct an action that makes $O(D,D)$ invariance manifest. In order to do this, we will use the generalized metric $\mathcal{H}$, which transforms linearly under $O(D,D)$. In fact, as we saw in subsection 3.4, the matrix $AF = \begin{pmatrix} e & 0 \\ -e^*B & e^* \end{pmatrix}$ parameterises the coset space $O(D,D)/O(D) \times O(D)$. However, due to equation (3.60), we noticed that AF can be described as a "vielbein" for the generalized metric $\mathcal{H}$, and so it contains the same information (AF used the metric vielbein e and B while $\mathcal{H}$ used G and B). For the rest of this work, almost every statement is true up to terms that vanish under imposing the strong constraint and we will assume this is the case.

We consider the background independent definition of the generalized metric (4.59). We can easily check that $\mathcal{H}^{-1} = \mathcal{H}$ and thus

$$\mathcal{H}^{-1} = \eta\mathcal{H}\eta \;\Leftrightarrow\; (\mathcal{H}^{-1})^T\eta\mathcal{H}^{-1} = \eta, \tag{4.67}$$

i.e. $\mathcal{H}^{MN} \in O(D,D)$ (note that it has two upper indices, while the usual $O(D,D)$ parameters $h \in O(D,D)$ were defined with one upper and one lower, $h^M{}_N$). In fact, $\mathcal{H}$ can be thought of

The linear transformation of the generalized metric can also be written as:

$$\mathcal{H}'^{MN}(X') = h^M{}_P h^N{}_Q \mathcal{H}^{PQ}(X). \tag{4.68}$$

Using this convenient notation, we can rewrite the gauge transformations of $\mathcal{E}$, equation (4.65), as a transformation of $\mathcal{H}$ (we also include the dilaton transformation for completeness):

$$\begin{aligned}\delta_\Sigma \mathcal{H}^{MN} &= \Sigma^P \partial_P \mathcal{H}^{MN} + (\partial^M \Sigma_P - \partial_P \Sigma^M)\mathcal{H}^{PN} + (\partial^N \Sigma_P - \partial_P \Sigma^N)\mathcal{H}^{MP} \\ &\equiv \hat{\mathcal{L}}_\Sigma \mathcal{H}^{MN} \\ \delta_\Sigma e^{-2d} &= \partial_M(\Sigma^M e^{-2d}).\end{aligned} \tag{4.69}$$

Here we have defined the very important concept of the *generalized Lie derivative* acting on $\mathcal{H}$. In general the generalized Lie derivative acts on a *generalized tensor* $A_M{}^N$ as

$$\hat{\mathcal{L}}_\Sigma A_M{}^N = \Sigma^P \partial_P A_M{}^N + (\partial_M \Sigma^P - \partial^P \Sigma_M) A_P{}^N + (\partial^N \Sigma_P - \partial_P \Sigma^N) A_M{}^P, \tag{4.70}$$

with an obvious extension to tensors with more indices, as well as to *generalized scalars*, which only contain the first term. Additionally, we can define generalized tensor densities of weight κ by adding to the right-hand side the term

$$\kappa \partial_P \Sigma^P A_M{}^N. \tag{4.71}$$

We see again that there is a correction proportional to the tensor $\eta_{MN}\eta^{PQ}$. This does not cause any problem due to the following properties satisfied by $\hat{\mathcal{L}}$:

- The usual Leibnitz rule:

$$\hat{\mathcal{L}}(AB) = \hat{\mathcal{L}}(A)B + A\hat{\mathcal{L}}(B). \tag{4.72}$$

- $\partial^M \chi$, for χ scalar, are trivial transformations:

$$\hat{\mathcal{L}}_{\partial^M \chi} A = 0. \tag{4.73}$$

- Constancy of the metric η with respect to the generalized Lie derivative as well:

$$\hat{\mathcal{L}}\eta_{MN} = \hat{\mathcal{L}}\eta^{MN} = \hat{\mathcal{L}}\delta_M{}^N = 0 \tag{4.74}$$

- The gauge algebra (4.52) can now be written as:

$$[\hat{\mathcal{L}}_{\Sigma_1}, \hat{\mathcal{L}}_{\Sigma_2}] = -\hat{\mathcal{L}}_{[\Sigma_1,\Sigma_2]_C}, \tag{4.75}$$

  where $[,]_C$ is the Courant bracket of equation (4.53).

In fact, the generalized Lie derivative of two generalized vectors A^M, B^N can be seen as a generalisation of the Dorfman bracket (defined in (2.13)), called *D-bracket*, analogous to the generalisation of the Courant bracket to the C-bracket

$$[A, B]_D^M = (\hat{\mathcal{L}}_A B)^M = [A, B]_C^M + \frac{1}{2}\partial^M(B^N A_N). \tag{4.76}$$

The D-bracket satisfies some properties that the Dorfman bracket does; in particular, it is not skew-symmetric, but it satisfies the Jacobi identity (2.14), contrary to the C-bracket. In addition, it reduces to the Dorfman bracket if we restrict to a null (always with respect to η) subspace of the doubled torus, and all the structures of subsection 2.3 are reproduced.

Now we can write the double field theory action in the generalized metric formulation:

$$\begin{aligned} S_{\mathcal{H}} = \int \mathrm{d}X e^{-2d}\Big(&\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK} \\ &- 2\partial_M d\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d\partial_N d\Big). \end{aligned} \tag{4.77}$$

In [12] it is shown that the above action has all the required properties, i.e. the gauge symmetry (4.69), the $\mathbb{Z}_2$ symmetry $\mathfrak{b} \to -\mathfrak{b}$ and the correct limits, and in fact it is uniquely determined by them.

4.4 Double Geometry

A very interesting active area of research is the attempt to understand and formulate rigorously double geometry. In this subsection we will discuss some of ideas involved, mainly based on [35], [38] and [39].

Firstly, we note that in [12] the action (4.77) was also written in the following form:

$$S_{\mathcal{R}} = \int \mathrm{d}X e^{-2d}\mathcal{R}, \tag{4.78}$$

where

$$\begin{aligned}\mathcal{R}(\mathcal{H},d) = & \, 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d \\ & + 4\partial_M\mathcal{H}^{MN}\partial_M\partial_N d + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} \\ & - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK}.\end{aligned} \tag{4.79}$$

In fact, this action takes the form of (4.77) up to total derivatives and terms that vanish under the imposition of the strong constraint.
The remarkable thing about the object $\mathcal{R}$ is that it transforms as a generalized scalar under the action of the generalized Lie derivative (4.70), i.e. $\delta_\Sigma\mathcal{R} = \hat{\mathcal{L}}_\Sigma\mathcal{R} = \Sigma^M\partial_M\mathcal{R}$, apart from being manifestly $O(D,D)$ invariant. Thus, recalling that e^{-2d} transforms as a generalized tensor density (see equations (4.69) and (4.71)), it is obvious that the action (4.78) is gauge invariant.

Thus, we are tempted to interpret $\mathcal{R}$ as a *generalized Ricci scalar*. A generalized Ricci *tensor* $\mathcal{R}_{MN}$ can be obtained by direct analogy with conventional geometry, as the equations of motion derived from$S_{\mathcal{R}}$ if we vary it with respect to the generalized metric $\mathcal{H}^{MN}$:

$$\delta S_{\mathcal{R}} = \int \mathrm{d}X e^{-2d}\delta\mathcal{H}^{MN}\mathcal{K}_{MN}. \tag{4.80}$$

This gives us $\mathcal{K}_{MN}$ and then, using also the constraints that $\mathcal{H}$ satisfies, we can find $\mathcal{R}_{MN}$ (see [12] for the full expression). The equations of motion are the "generalized Ricci flat" equations $\mathcal{R}_{MN} = 0$.
Then one would expect that, using the $O(D,D)$ tensor calculus we have devel-

oped, they could find a generalized Riemann tensor with four capital indices, such that the contraction of two indices would give $\mathcal{R}_{MN}$. However, as we will see, there has been only partial success in this quest.

Let us try to understand deeper the generalized (or $O(D,D)$) tensors.
The coordinates X^M have been defined with an upper M index, like in the usual riemannian geometry, although they contain "conventional" coordinates with both upper and lower μ indices, depending on how they transform under $GL(d) \subset O(D,D)$. Now, the gauge parameters come also with an upper index:

$$\Sigma^M = \begin{pmatrix} \alpha_\mu \\ A^\mu \end{pmatrix}, \tag{4.81}$$

with α_μ and A^μ depending on the doubled coordinates $X^M = \begin{pmatrix} \tilde{x}_\mu \\ x^\mu \end{pmatrix}$. We know that, restricting the coordinate dependence on just x^μ, $A^\mu(x)$ should give the vector parameterising the diffeomorphisms and $\alpha_\mu(x)$ should give the one-form parameterising the B-field gauge transformations of the conventional supergravity action, see equations (3.17) and (4.27). This is strange; in riemannian geometry, if we make a vector $T^\mu(x^\alpha, x^a)$, $x^\mu \equiv (x^\alpha, x^a)$ independent of the coordinates x^a, it will decompose into a lower dimensional vector $T^\alpha(x^\alpha)$ and a collection of scalar fields $T^a(x^\alpha)$. In addition, the gauge algebra (4.52), (4.53), that is satisfied by the gauge parameters has a gauge redundancy, since $\Sigma^M \sim \Sigma^M + \lambda \partial^M \chi$, with λ and χ functions on the doubled manifold. Thus we understand that, even if we correspond the generalized vector ∂^M to each generalized direction X^M, we cannot define the generalized vector by its action on functions, since it is not unique. In fact, it is very difficult to find an invariant definition of a generalized vector, similar to the one[15] of riemannian geometry.

On the other side, assuming that such a definition exists, the generalized Lie

[15] Actually three, see [40].

derivative along with its properties (4.3), define completely the higher rank tensor fields by their transformation under $\hat{\mathcal{L}}$. A finite form of the infinitesimal generalized Lie derivative transformations was found in [41]: under a generalized coordinate transformation $X \to X'$, a generalized vector transforms as

$$A'_M(X') = \mathcal{F}_M{}^N A_N(X), \tag{4.82}$$

where

$$\mathcal{F}_M{}^N = \frac{1}{2}\left(\frac{\partial X^P}{\partial X'^M}\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\frac{\partial X^N}{\partial X'^P}\right). \tag{4.83}$$

We recall here that all $O(D,D)$ indices are raised and lowered with the constant metric η, so there is no actual distinction between upper and lower indices.
The transformations of higher rank tensors can easily be found by using the above matrix $\mathcal{F}$. In [41] there was also found an analogue of the exponentiated Lie derivative $\exp \mathcal{L}$ acting on vectors: the exponentiated generalized Lie derivative $\exp \hat{\mathcal{L}}$ acting on generalized vectors.

Before [39] there were two main approaches to double geometry: the vielbein formalism, which originated in [4] and [5] and was further developed recently (see for example [42]) and the coordinated basis formalism, which is what we have been developing in this work. In [39] there was achieved a kind of unification, since, by using a general basis of generalized vector fields Z^A, the main results of both approaches were reproduced.
We will continue working in the coordinate basis formalism.

We want to define a generalized connection analogous to the one in riemannian geometry:

$$\nabla_M V^N = \partial_M V^N + \Gamma_{MP}{}^N V^P, \tag{4.84}$$

so that $\nabla_M V^N$ transforms as a generalized tensor.
In conventional geometry, requiring that the torsion of the above connection

vanishes and that it is compatible with the riemannian metric g (i.e. $\nabla_\mu g = 0$), fully determine the connection components in terms of derivatives of the metric (we call it Levi-Civita connection). This is not the case in double geometry, even though we actually require more conditions to hold.

A natural requirement is that the generalized torsion vanishes. The generalized torsion cannot be defined in the usual way, since this does not result in a generalized tensor, so we use the alternative definition:

$$\mathcal{T}_{MN}{}^{Q}\Sigma^M V^N = (\mathcal{L}^{\nabla}_{\Sigma} - \mathcal{L}_{\Sigma})V^Q, \tag{4.85}$$

where by $\mathcal{L}^{\nabla}_{\Sigma}$ we mean that we replace the partial derivatives with covariant derivatives in the definition of the generalized Lie derivative. Then we require that

$$\mathcal{T}_{MN}{}^{Q} = 0. \tag{4.86}$$

We also require that both the $O(D,D)$ invariant metric η and the generalized metric $\mathcal{H}$ are covariantly constant, i.e.

$$\nabla_M \eta^{PQ} = 0 = \nabla_M \mathcal{H}^{PQ}. \tag{4.87}$$

Finally, in order to include the dilaton in the geometry in some way, we require the compatibility condition

$$\int e^{-2d} f \nabla_M V^M = -\int e^{-2d} V^M \nabla_M, \tag{4.88}$$

where f is a function.

These conditions do not determine the connection components in terms of the physical fields, but only some projections (see [35]). Then, neither the generalized Riemann curvature tensor will be completely determined by the physical fields. A solution out of this has not yet been found up to now, so we still cannot give a geometrical meaning to the action (4.77). Some interesting proposals involve [43], where we impose a constraint on our geometric construction, but we partly spoil $O(D, D)$ covariance, and [38], where we can use the Weitzenböck connection, which is flat and has a non vanishing torsion. In the latter case a proper action can be constructed, but, due to the unusual properties of the connection, the physical meaning is yet unclear.

5 Conclusion

Hopefully, by now we have a clear overview of the foundations of double field. We have also seen some interesting questions, mainly regarding the underlying geometry of double field theory, that are still lacking a clear answer. Here, we will briefly discuss some other related areas of research that we did not mention in the thesis. For a much more complete analysis, see the recent review [44].

Recall that, in section 3.2 we saw that a particular subgroup of the $O(d,d,\mathbb{R})$ symmetries of the dimensionally reduced supergravity, namely the geometric subgroup $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2}$, can be interpreted as arising from the global symmetries of the theory before the reduction. So, it was interesting to see what happens in the context of double field theory and study the relation between T-duality and gauge symmetries. This was done in [11] (see also [28]), where it was seen that $O(d,d,\mathbb{R})$ is a remnant of the large diffeomorphisms of the doubled torus only in the case when there is an isometry along the relevant compactified directions. Of course this is not satisfactory, since there is no reason why an isometry should have this result (note that, throughout this work, an isometry was never assumed, in contrast to other T-duality approaches, such as the worldsheet approaches, which can only be pursued only in that specific case). This question is most likely related to the following ones.

Another important issue is whether we can include in double field theory α' corrections of supergravity, predicted by string theory (see subsection 3.1). They contain terms involving the (conventional) Riemann curvature tensor which cannot arise from the generalized metric formulation, so novel structures are required. However, understanding the α' corrections may solve the mystery of the undetermined components of the generalized Riemann tensor, so it is a promising area of research. For a first attempt at this direction, see [45].

Additionally, one would naturally expect to go beyond bosonic string theory. This means that we should also include fermionic fields and supersymmetry. Note that this has partly been done in generalized geometry (spinor bundles are discussed in [14]), but there is still way to go even there.
A similar generalisation would be to account for other dualities and specifically for the U-dualities of M-theory. We need to create an *extended field theory*, which makes manifest the invariance under the exceptional groups. Then, generalized geometry needs to be even more generalized, by adding to the "tangent space" higher powers of the usual tangent and cotangent bundles. A lot of research is carried out in this area. For a review of these approaches, see [36] (original papers can be found therein).

Another very active area of research is concerned with flux compactifications and gauged supergravities. It is believed that they can be naturally studied in the context of double field theory. For more details see [35] and references therein.

Finally, there is the question of foundational importance regarding the relaxation of the strong constraint. Although a constraint should necessarily be imposed in order to restrict the degrees of freedom of the theory, it is clear that the strong constraint is way too strong and a truly double field theory would clarify most of the above questions. For more details on this issue, see [35].

Thus, we understand that double field theory is still young and has a lot more to offer. Even at this stage it looks very promising and we can only believe that its further development will shed light on the deep questions about the geometry of the spacetime we live in.

6 References

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