Time in Quantum Mechanics

Stephanie Greis

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1 Introduction - A Historical Perspective on Time in Quantum Mechanics

The question of time in quantum mechanics has a long and controversial history, spanning approximately a century from Bohr's introduction of the interaction of radiation with atoms to more recent work on the duration of quantum tunnelling processes. As Schulman puts it: "It is ironic that experimentally, time is the most accurately measured physical quantity, while in quantum mechanics one must struggle to provide a definition of so practical a concept as time-of-arrival." [22]

This paper will give an overview of four main aspects of temporal phenomena in the quantum realm: dwell time - or: how long does it take a particle to tunnel through a barrier?, the Zeno effect - or: does a continuously observed arrow ever hit the target?, arrival time - or: when does a particle cross a boundary?, and finally backflow - or: how can something that moves forward end up behind? Following descriptions of each of these phenomena, some ways of measuring them will be explored.

1.1 The History of Time in Quantum Mechanics

Early 20^{th} Century Historically, the question of time in quantum mechanics first arose in 1913 when Bohr introduced the interaction of radiation with atoms via instantaneous transitions called "quantum jumps" (Muga, 2008). However, he did not provide a mechanism for timing these jumps, and a contradiction between theory and reality could be observed in the fact that spectral lines do exhibit a finite, albeit narrow, width, indicating that these jumps are not instantaneous but do take a finite amount of time (Muga, 2008). During the 1920's the question of the meaning of a time duration was explored by both scientists and philosophers of science, with Heisenberg indicating that only a "rough" description of such a duration might be possible, while matrix theorists doubted that a particle's position in time could ever be given with any accuracy [15].

A few years later, in 1927, Bohr suggested the resolution of a wavefunction into Fourier components of the type $\exp\{i\,(px - Et)\,/\hbar\}$, and then examining the time-dependence of the interference of Fourier components at a spatial point, such as the origin $x = 0$. The spread of possible times is given as $\Delta\tau$, giving rise to the well-known time-energy relation of $\Delta\tau \cdot \Delta E \gtrsim \hbar$. But how is this duration related to the wavefunction? (Allcock 1, 1969) In his 1969 paper, Allcock indicates that a theory of time measurement is required to allow answers to such questions [3], while Muga admits that there appears to have been much confusion about the meaning of $\Delta\tau$ in the uncertainty relations [15].

A different uncertainty relation was developed by Mandelstam and Tamm who, using $\dot{Q}$ as the rate of change of the observable quantity Q, as well as applying Schwartz' inequality (which states that $\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle \geq$

$|\langle\psi_1|\psi_2\rangle|^2$[29]), arrived at the relation $(\Delta Q)_{rms}\,(\Delta E)_{rms} \geq \frac{\hbar}{2}|\langle\dot{Q}\rangle|$. This gives the characteristic time as the "time taken for mean motion of system to produce in the mean value of Q a change large enough to stand out against the mean deviation of Q", or $T_Q \equiv \frac{(\Delta Q)_{rms}}{|\langle\dot{Q}\rangle|}$ [3]. It should be noted, however, that this does not refer to the time of occurrence but to the times of average expectation values [15].

1930's In 1932 von Neuman criticises that, unlike the recently developed theory of general relativity, quantum mechanics does not treat time and position on the same footing. This was in turn criticised by Hildevoort who argued that von Neuman was confusing the position of a particle with its coordinates in space, and that thus equal treatment was not justified (Muga, 2008). The measuring process which von Neuman had suggested collapses the particle's wave function, i.e. $|\psi\rangle \to |\psi_i\rangle$, a process which can be taken as a quantum jump, albeit a different one to the one originally theorised by Bohr [15].

In 1933 Pauli published a paper including a footnote which would prove to be fateful in the investigation of time in quantum mechanics. In the footnote, he put forward the argument that if there existed a self-adjoint time operator, it would lead to the energy spectrum extending continuously from negative infinity to positive infinity, and suggested that the search for such an operator be abandoned. Due to this footnote time was for many years treated as "only a parameter" [15, 21].

More Recent Developments It appears that much time was lost due to both Pauli's footnote and the mistaken understanding of the scientific community that the search for time operators, or the investigation of time durations in quantum mechanics, would be futile, and it was not until the 1960's that the question was investigated again; c.f. [3, 1, 2].

Since the beginning of the 1990's, the subject has received increased interest as can be seen, for example, in [13].

One area in which gaining a better understanding of time in quantum mechanics became important was in chemical processes, where growing evidence suggested the ubiquity of resonances which have an influence on the reaction dynamics [16].

Many processes which have been considered as occurring "instantaneously" might in fact take a finite amount of time. An interesting question which arises out of this is whether the advancement of technology might shed light on questions such as the instantaneous appearance of electrons in the photoelectric effect, or the decay of metastable elements, as the current technology tends to consider any process which takes less than about a nanosecond to be instantaneous [22].

1.2 Defining Time in Quantum Mechanics

One problem when investigating time in quantum mechanics is the lack of a consistent definition of time. One possible definition given by Allcock is the time "at which the spatial disposition of a quantal system undergoes some definite and measurable change", however there are many similar or different definitions. It is also worth noting that time appears in two ways in quantum mechanics, namely as parametric time (e.g. the time-derivative in the Schrödinger equation), and as a dynamic variable, such as arrival time [3].

Two concepts of time will be explored in this paper: arrival time, which can be defined as the time at which a particle is detected at the point $x = 0$, and dwell or transit time, which gives the time it takes a particle to traverse a region of space (and can thus also be called traversal time) (Allcock 1, 1969). One specific example of the latter, which will be explored in more depth here, is tunnelling time which seeks to find the duration of a tunnelling process through a potential barrier. Since dwell or traversal time registers both the particle's arrival time *and* allows it to continue its motion, its measurement has to be non-destructive [3].

While most students first learn about the time-independent Schrödinger equation, it is interesting to realise that all quantum phenomena can, at least theoretically, be described using the time-dependent equation, while this is not the case for the time-independent one. The time-independent treatment of quantum mechanics does not allow for the probability density of a particle to change over time, effectively fixating the particle in place[27]

Looking at nature, it does not appear as though quantum systems distinguished between different time directions, while the Schrödinger equation, containing first order differentials $\psi(q,t) \neq \psi(q,-t)$, indicates that a direction of time does need to be defined, ideally via macroscopic processes [23]. Questions like these have become ever more pressing since the development of single atom manipulation techniques and cooling methods of atoms in recent years [4].

1.3 Overview of the Paper

In the following chapters we will start by looking more deeply at the question which has historically been of more interest due to its relevance in atomic processes: the tunnelling time, an example of dwell time. The topic of superluminal propagation, also known as the Hartman effect, will be mentioned, and different approaches to dwell time will be covered. These approaches can be split into two broad areas, the clock model and the kinematical approaches. Examples of both of these will be given, followed by a brief look at the possibility of physically measuring tunnelling time.

In Chapter 3 we will briefly turn to the Zeno effect and its relevance in time measurements in quantum mechanics. Since it postulates that a system which is continuously observed is found to never decay, the Zeno effect becomes relevant when making frequent measurements of a quantum system.

In Chapter 4 we will give an overview of arrival time, including different approaches to it.

Following on, we will explore the backflow phenomenon in Chapter 5. In this phenomenon a wave packet with purely positive momentum initially in the negative x-region, can be found to have moved towards the *more* negative region over time. In this chapter we will look at the link between decoherence and backflow, as well as the naive classical limit of letting $\hbar$ approach zero.

Finally in Chapter 6, we will turn our gaze towards possible or proposed ways of detecting these phenomena, before concluding with Chapter 7.

Time has been transformed, and we have changed; it has advanced and set us in motion;
it has unveiled its face, inspiring us with bewilderment and exhilaration.

Khalil Gibran

2 Tunnelling Time

2.1 A Short History of Tunnelling in Nuclear Physics, and the Investigation of Tunnelling Time

The study of tunnelling began with the discovery of α-decay in the early 20^{th} century when it was realised that the α-particle, originally trapped within the nucleus' potential barrier, can escape by tunnelling through the barrier [18]. The theory of quantum mechanics implies a non-vanishing probability for a freely moving particle encountering a potential barrier higher than the particle's energy to cross this barrier. Trying to define a duration of this event, and thus an average speed through the barrier, is the objective of the study of tunnelling time [20].

The problem of a definition for tunnelling time was first mentioned in the early 1930's by MacColl and Condon [18, 20, 31], with MacColl noting that the "approximate wave-packet analysis of the time-dependent Schrödinger equation ... suggested that tunnelling takes not appreciable time" [31].

In the 1950's tunnelling experiments, e.g. involving tunnelling junctions, tunnelling diodes, or tunnelling microscopes became more prevalent, and started to be studied theoretically [18]. During the 50's and 60's, the problem of timing the duration of a quantum collision emerged, a question which had been largely ignored since the 1920's due to Pauli's footnote suggesting the futility of studying time concepts in quantum mechanics [18]. A few decades later, in 1982, Tinkerer and Landauer proposed to measure the tunnelling time by modulating the height of the potential barrier through which the particle tunnels. In this scenario the tunnelling particle can no longer adiabatically follow the oscillating potential. They refer to this duration as "traversal time" [31].

More recently the advent of high-speed electronic devices reignited an interest in tunnelling time, extending it from the realm of nuclear physics (such as α-radiation, fission, fusion and the like) to tunnelling processes (such as in the scanning tunnelling microscope) [18, 20]. Also, in the early 1990's, experiments based on electromagnetic analogues of quantum mechanical tunnelling provided support for group delay and phase time (which will be discussed below) though seemingly implying superluminal group velocities [31]. This superluminal propagation will be examined more closely in the section on the Hartman effect below.

2.2 Informal Definition

2.2.1 No Universal Definition

There is currently no one universally accepted definition of "dwell time", "tunnelling time" or "traversal time". As Landauer puts it: "There is no copyright on the expressions traversal time and tunnelling time; each author can choose an interpretation. If an investigator wants to associate it with the time required to write the Bardeen tunnelling Hamiltonian on the blackboard, we cannot say that is wrong."[13] However, it is also mentioned in Damborenea et al. that "dwell time" is distinct from "traversal time", "delay time" or "reflection time" [4], which appears to complicate the situation further.

Some reasons for this confusion are suggested by Olkhovsky, Recami, and Jakiel in their 2004 paper [18]. One reason might be that tunnelling time is related to a quantum-collision duration, and thus closely connected with the difficulty of time sometimes being viewed as a parameter and other times as a (quantum) physical observable. There is also no direct classical limit to the quantum phenomenon of a particle's motion within a potential barrier, and various definitions have assumed different conditions, which have not yet been sufficiently analysed [18]. However, Olkhovsky, Recami, and Jakiel also claim that using the "framework of conventional quantum mechanics", all known definitions of tunnelling time can be unified as being either a particular case of the more general definition, or a definition for a process which accompanies tunnelling.

In this paper we have thus decided to use the words "dwell time", "traversal time" and "tunnelling time" interchangeably, while also acknowledging that there might be slight differences between the situations in which each concept becomes relevant.

2.2.2 General Definition of Tunnelling Time

A dwell-time distribution gives the probability that a particle spends a time interval $\mathrm{d}\tau$ in a defined spatial interval [32]. Dwell-time, or tunnelling time, is a conserved quantity which bypasses Pauli's theorem [4], and characterises such phenomena as the duration of collision processes, the lifetime of unstable systems, a system's response to perturbations, or properties of chaotic scattering [17]

According to Jaworski and Wardlaw, the mean dwell time for a particle in the region Ω during the time interval (t_1, t_2) is defined as the quantity

$$\tau(\Omega, t_1, t_2; \psi) = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Omega} |\left[\exp(-itH)\psi\right](x)|^2 \tag{1}$$

where ψ is the wave function describing the initial state of the particle, H is the Hamiltonian, and $\hbar$ has been set to 1. They also note that, if ψ is an eigenstate of the Hamiltonian, then

$$\tau(\Omega, t_1, t_2; \psi) = (t_2 - t_1) \int_{\Omega} |\psi(x)|^2 \, \mathrm{d}x \tag{2}$$

[12].

Treating ψ as a scattering state and Ω as a bounded region containing the scattering centre, then

$\tau(\Omega, -\infty, \infty; \psi)$ gives a measure of the duration of the collision. The delay time can then be defined as the difference between two dwell times: one for the interacting particle and the other for the free (reference) particle. In two-body scattering, the delay is averaged over all scattering angles [12].

Jaworski and Wardlaw also note that the dwell time occurs naturally in first-order perturbation theory for a barrier perturbed by a finite-range potential, where the effect of the perturbation on the observable being changed due to scattering are - to first-order approximation - expressible in terms of dwell time operators [12].

In his 1994 paper, Landauer mentions that the reduction due to the barrier having an imaginary component in the potential is proportional to the dwell time, rather than the traversal time [13], while Sokoloski and Connor suggest in their 1993 paper that the solution to the tunnelling time problem might not have an "ultimate" candidate for τ, but rather that answers might be found by investigating the way the classical time parameters are quantised [25]. Sokolovski and Connor conclude that the classical concept of time scales cannot simply be applied to the quantum case, particularly with regards to tunnelling time.

2.3 Attempt at a More Formal Definition of Tunnelling Time

In order to give a more formal definition of tunnelling time, and to explore different approaches to it, in this section we will first give a short review of tunnelling before briefly introducing several different approaches to understanding dwell time. Apparently superluminal propagation of electromagnetic pulses, a phenomenon which can be explained by the Hartman Effect, will be looked at in greater detail below. This will lead on to an introduction to Aharonov's weak measurements. Following this, we will find an equation for the mean tunnelling time, as well as look at different solutions to the Schrödinger equation.

The following subsections will then delve deeper into the different approaches to tunnelling time, such as the dwell time operator approach, two approaches using clocks, as well as two kinematical path approaches and their pitfalls. The last part of this chapter will provide a brief overview of potential techniques to measure tunnelling time.

2.3.1 Brief Review of Tunnelling

Labelling the region in front of the barrier I, the barrier region II (for a barrier of width a), and the region into which the particle exists the barrier by III, the wavefunction can be written in the following way for the three regions. For a particle with momentum $\hbar k$, mass m - and thus an energy given by $E = \hbar^2 k^2/2m$, the wavefunction in these three regions is:

$$\psi(x;k) = \begin{cases} \psi_I = \mathrm{e}^{ikx} + R(k)\mathrm{e}^{-i(kx-\beta)} & x \le 0 \\ \psi_{II} = \chi(x;k) & 0 \le x \le a \\ \psi_{III} = T(k)\mathrm{e}^{i(kx+\alpha)} & x \ge a \end{cases},$$

where $R(k)$ and $T(k)$ are the reflection and transmission amplitudes (with phase delays $\beta = \beta(k)$ and $\alpha = \alpha(k)$), respectively, with

$$R(k) = \sqrt{1 - T(k)^2}.$$

[20]

The barrier height is assumed to be V_0, while $\chi(x;k)$, $R(k)$, $T(k)$, $\alpha(k)$ and $\beta(k)$ are analytically known [20].

Thus,

$$\chi(x;k) = \begin{cases} A(k)\mathrm{e}^{-\kappa x} + B(k)\mathrm{e}^{\kappa x}, & E < V_0 \\ A(k)\mathrm{e}^{-i\kappa x} + B(k)\mathrm{e}^{i\kappa x}, & E > V_0 \end{cases},$$

where

$$\kappa = \begin{cases} \sqrt{2m(V_0 - E)}/\hbar, & E < V_0 \\ \sqrt{2m(E - V_0)}/\hbar, & E > V_0 \end{cases}.$$

2.3.2 Short Introductions to Different Dwell Time Approaches

Several authors acknowledge three main approaches to the traversal time question [13, 18].

The first approach is one in which the incident wave packet is followed both onto and through the barrier. It is worth realising that in this case the incoming peak does *not* turn into the transmitted peak since the higher-energy components of the incident packet arrive at the barrier before the less energetic ones. This leads to the higher-energy components being transmitted more effectively, resulting in the peak of the transmitted packet leaving the barrier long before the peak of the incident wave packet has arrived, thereby leading to a lack of causal relationship [13, 20]. These emergent wave packets also have a higher mean velocity, which means that by introduction of a barrier into e.g. an electron accelerator, the particles' propagation can in fact be sped up [13].

Landauer noted in 1989 that none of the performed or proposed experiments can measure electron wave packet delays. This is because measuring the arrival time of a wave packet just before the barrier constitutes an invasive process which would not leave the packet unaltered. A solution might be to a have a reservoir which periodically releases wave packets, and subsequently measuring the arrival at the far side of the barrier.

This would overcome the problem of altering the wave packet just before tunnelling. However, it is also worth noting that the behaviour of electrons differs from that of photons, as cited by Landauer from an earlier work [13].

A related approach which will not be covered further here, but is worth mentioning, is the Olkhovsky-Recami (OR) approach. It gives a generalisation of time durations and is defined for atomic and nuclear collisions. Similarly to the dwell time operator approach described below, the OR approach makes use of a quantum operator for time [18].

The second approach mentioned here is to determine a set of dynamic paths through the barrier and ask how long each of these paths spends within the barrier. Averaging over this set gives the tunnelling time. Such sets can be found through the Feynman path-integral formulation, via the Bohm approach or by using the Wigner distribution [13]. The prior two will be described in more detail below. According to Landauer, it is still an open question whether traversal time is best viewed as a distribution or as a single time scale indicator. The kinematical path approaches yield a distribution of times, and not a single time scale [13, 20], however they also have pitfalls when restricting the paths a particle can take [11].

The third and final approach to be described in this paper is the clock approach which uses the system's degrees of freedom to determine the time elapsed during tunnelling [13, 20]. One example of such a clock approach is given by Landauer as presented in the paper by Peres in 1980. Landauer also notes that "clocks can be chosen so as to be minimally invasive", and while not all possible clocks yield the same outcome, there is a lot of overlap and similarity between the different clock models [13].

2.3.3 Superluminality - the Hartman Effect, and Weak Measurements

General Definition of Hartman Effect An interesting phenomenon in quantum tunnelling is that "the mean tunnelling time $\langle \tau_{tun}(0, a) \rangle$ does not depend on the barrier width a for sufficiently large a" [18]. According to Sokolovski's 2003 paper it was notes in 1932 by MacColl that quantum tunnelling "takes no time" as though the time spent inside the barrier did not exist. Due to this it can appear as though a pulse propagates through the barrier at superluminal speeds. This is known as the "Hartman effect". All "physically reasonable definitions of tunnelling time" imply "the existence of Hartman-type effects, which have been confirmed by experiments" [20]. Similarly, experiments in microwave tunnelling have confirmed that the Hartman effect would vanish if the barrier's absorption was high enough [18].

Also, since there is only a very weak dependence of the tunnelling time on a, the tunnelling time can assume negative values [18].

Aharonov's Weak Measurement The superluminal appearance of the transmitted pulse is most easily explained through Aharonov's weak measurements, which according to Sokolovski appear to be the best explanation, as he indicates that it is "unlikely that a deeper or more 'classical' explanation could be found for the apparent superluminality observed in quantum tunnelling" [24].

The weak measurement theory was introduced in 1988 by Aharonov, Albert and Vaidman starting from the "classical" measurement theory by von Neumann [20, 28]. It states that under certain conditions quantum measurements can yield "weak values" which may be different from the eigenvalues or allowed outcomes according to standard quantum mechanics [28].

The weak value of an observable A is defined for the two state vector $\langle\psi_2|\psi_1\rangle$ as

$$A_w \equiv \frac{\langle\psi_2|A|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle} \tag{3}$$

while the expectation value of A for a state $|\psi\rangle$ is a particular case of a weak value when $|\psi_1\rangle = |\psi_2\rangle = |\psi\rangle$ [28].

These weak values emerge only when the coupling between the system and measuring device is both bounded and (generally) weak, hence the name.

A weak value might be far away from the range of the system's eigenvalues; e.g. the kinetic energy of a system might be negative, and in general the weak value is a complex number.

According to Vaidman, "weak values emerge in procedures which are very close to the standard quantum measurements", such as an adiabatic measurement on a single system with a non-degenerate energy spectrum, or an adiabatic measurement on a single system described by a non-hermitian Hamiltonian (in which case the weak value might lie outside the range of eigenvalues) [28].

Causality? The question of causality arises when contemplating superluminal propagation through any medium.

Olkhovsky, Recami and Jakiel note that this "un-causal" time-advance is due to "interference between incoming waves and waves reflected by the barrier forward edge". They also mention that the larger the barrier width, the more incoming wavepackets will be extinguished by reflected waves, up to a saturation level which is independent of the barrier width a. This effect is noticeable even before the barrier front, i.e. even near the front wall of the barrier, as the interference between the incoming and reflected waves spreads backwards [18].

Sokolovski argues that all tunnelling speeds inside the barrier are below the speed of light c, thereby conforming to the speed limit set by special relativity [24], while Olkhovsky, Recami and Jakiel note that the non-negativity of the mean duration of a tunnelling event is a sufficient but not necessary causality condition [18].

Privitera et al. also indicate that superluminal velocities could be acceptable in an “extended relativity”, such as one which includes tachyons, thereby suggesting that such velocities might not be entirely unreasonable [20].

No Iinformation Transfer One reason why superluminal propagation is difficult to reconcile with the standard laws of physics is the notion of a universal speed limit for information transfer which is given by the speed of light. However, the issue of superluminal propagation is at least partly resolved when it is realised that quantum tunnelling cannot be used to transmit information, and is thus not necessarily contradicting relativity. Rather than enabling faster-than-light information transfer, the barrier filters and rearranges the amplitudes into an apparently “superluminal” single transmitted peak [24].

Contrary to Sokolovski, Winful claims that no measurement to date (as of 2006) has actually measured a “superluminal group velocity in barrier tunnelling”. Instead it is found that “even the theoretical predictions of superluminal tunnelling group velocity are based on an untested and unspoken assumption: that the group delay in barrier tunnelling is a traversal time” [31]. Winful also claims that the assumption of tunnelling time being a traversal time is false and that instead of being transit times, they are in fact lifetimes [31].

The Hartman Effect in Different Approaches to Tunnelling Time The Hartman Effect is manifested differently in the different approaches to tunnelling time.

In the mean dwell time approach, the mean Larmor time approach, and in the real part of the complex time obtained via the Feynman path-integral method, the independence of the barrier width on the tunnelling time can be seen for quasi-monochromatic particles tunnelling through an opaque rectangular barrier. All of these tunnelling times become equal to ${}^{\hbar k}/_{\kappa V_0}$, i.e. proportional to the particle’s momentum, and are thus independent of the barrier width [20].

However, for the “second Larmor time” (τ_z^{La}), the Büttiker-Landauer time, and the imaginary part of the complex tunnelling time obtained from the Feynman approach this is not the case as these dwell times are given by ${}^{ma}/_{\hbar\kappa}$ and are thus proportional to the barrier width a. This means that the Hartman effect is not valid for these approaches. The reason for this is that these approaches, instead of being mean times, are in fact standard deviations (“mean square fluctuations”) of the tunnelling time. Instead of being related to the peak (or group) velocity of the particles, they represent the spread of tunnelling velocity distributions [20].

2.3.4 No Superluminality

A different interpretation of seemingly superluminal propagation through a potential barrier is given by Winful. He indicates that it is wrong to simply take the length of the barrier and divide it by the delay time, i.e. that $v = L/\tau$ does not give an accurate tunnelling time, and argues that there is in fact no superluminal propagation through the barrier [30].

Winful mentions that "a number of experiments were initiated in the early 1990's using electromagnetic waves which can tunnel as evanescent waves through forbidden regions in a manner analogous to that of quantum wavepackets", which all showed that "the delay for the tunnelling pulse is shorter than that of a non-tunnelling pulse traversing the same distance" [30].

According to Winful, the measured and theorised group delays, instead of being transit times, are "lifetimes of stored energy or integrated probability density leaking out of both ends of the barrier. In other words the barrier acts as an evanescent mode cavity with a finite lifetime" [30]. In cases where most of a wavepacket is reflected, he argues that the group delay cannot be used to define a group velocity. This group delay, or alternatively know as phase time or Wigner time, is the energy derivative of the transmission phase shift [30].

According to Winful, the dwell time is identical to the group delay for electromagnetic tunnelling, which he gives as a weighted average of transmission and reflection: $\tau_g = |T|^2 \tau_{gt} + |R|^2 \tau_{gr}$ [31].

He concludes that both the Hartman and the reshaping theories are flawed, and gives experimental evidence of the latter by citing experiments using narrowband pulses which showed "no pulse reshaping or pulse narrowing" [30].

2.3.5 Mean Tunnelling Time

If restricted to the one-dimensional case of a particle moving along the x-axis, with a time-independent barrier located at $(0, a)$, then the mean value $\langle t_\pm(x) \rangle$ of the time t at which the particle crosses position x is given by

$$\langle t_\pm(x) \rangle \equiv \frac{\int_{-\infty}^{\infty} t \, J_\pm(x,t) \, \mathrm{d}t}{\int_{-\infty}^{\infty} J_\pm(x,t) \, \mathrm{d}t} \tag{4}$$

while the variance to this time distribution is

$$Dt_\pm(x) \equiv \frac{\int_{-\infty}^{\infty} t^2 \, J_\pm(x,t) \, \mathrm{d}t}{\int_{-\infty}^{\infty} J_\pm(x,t) \, \mathrm{d}t} - \left[\langle t_\pm(x) \rangle\right]^2 \tag{5}$$

where $J_\pm(x)$ represents the positive and negative values of the probability flux density of a wavepacket $\Psi(x,t)$ evolving in time respectively, given by

$$J(x,t) = \mathrm{Re}\left[\frac{\mathrm{i}\hbar}{m}\Psi(x,t)\frac{\partial \Psi^\dagger(x,t)}{\partial x}\right] \tag{6}$$

[18].

Two Equivalent Forms of Mean Dwell Time Olkhovsky, Recami and Jakiel mention two equivalent forms of dwell time in their 2004 paper:

$$\langle \tau^{dw}(x_i,x_f)\rangle = \frac{\int_{-\infty}^{\infty}\mathrm{d}t\,\int_{x_i}^{x_f}|\psi(x,t)|^2\mathrm{d}x}{\int_{-\infty}^{\infty}J_{in}(x_i,t)\mathrm{d}t} \tag{7}$$

where J_{in} is the incident probability flux, and

$$\langle \tau^{dw}(x_i,x_f)\rangle = \frac{\int_{-\infty}^{\infty} tJ(x_f,t)\,\mathrm{d}t - \int_{-\infty}^{\infty} tJ(x_i,t)\,\mathrm{d}t}{\int_{-\infty}^{\infty}J_{in}(x_i,t)\mathrm{d}t} \tag{8}$$

with $-\infty < x_i \leq 0$ and $a \leq x_f < \infty$, where a is width of the potential barrier [18].

Notice also that

$$\int_{-\infty}^{\infty} J_{in}(x_i,t)\,\mathrm{d}t = \int_{-\infty}^{\infty}|\psi(x,t)|^2\,\mathrm{d}x\,, \tag{9}$$

which follows from the continuity equation

$$\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0\,. \tag{10}$$

[18].

Probability of Finding Particle Within Certain Spatial Interval The probability of finding the particle inside the spatial interval (x_1,x_2) at time t is given by

$$P(x_1,x_2;t) = \frac{\int_{x_1}^{x_2}|\psi(x,t)|^2\mathrm{d}x}{\int_{-\infty}^{\infty}|\psi(x,t)|^2\mathrm{d}x} \tag{11}$$

[18].

Probability Flux The probability flux at the incident boundary of the barrier can be separated into incident, reflected and interference parts:

$$J(x_i, t) = J_{in}(x_i, t) + J_R(x_i, t) + J_{int}(x_i, t) \tag{12}$$

while the outgoing flux consists entirely of transmitted flux $J(x_f, t) = J_T(x_f, t)$ [18].

Olkhovsky, Recami and Jakiel obtain in their 2004 paper the following weighted average relation for a system with no interference:

$$\langle \tau^{dw}(x_i, x_f) \rangle = \langle T \rangle_E \langle \tau_T(x_i, x_f) \rangle + \langle R \rangle_E \langle \tau_R(x_i, x_f) \rangle \tag{13}$$

where $\langle T \rangle_E + \langle R \rangle_E = 1$ [18].

Transparent Barrier When the barrier is transparent the mean dwell time is equal to the transmitted time, i.e. $\langle \tau^{dw}(x_i, x_f) \rangle = \langle \tau_T(x_i, x_f) \rangle$ [18].

The dwell time can also be defined as

$$\tau_D = \frac{1}{j} \int_a |\psi^2| \, \mathrm{d}x \, , \tag{14}$$

where j is the incident flux. This gives the time it takes for a given flux to create an accumulated particle storage in the barrier [13, 20], meaning that this dwell time is the "ratio between the probability density in the tunnelling region and the incoming flux entering the barrier j" [20]. This definition does not distinguish between transmitted and reflected components [20].

One can also define a time $\tau'_D = T\tau_T + R\tau_R$ which averages over the weighted reflection and transmission times [13]. Another version of an equation linking the transmission and reflection channels is given by Privitera as $\tau^D = |T(k)|^2\tau_T + |R(k)|^2\tau_R$. This is, however, not enough to uniquely determine τ_T and τ_R [20].

Equivalence Between τ_D and τ'_D? The question of the equivalence of τ_D and τ'_D defined above arises.

Landauer indicates that they are often assumed to be equivalent, and provides several examples of authors who assumed this in his 1994 paper, such as Leavens and Aers (1987, 1990), Golub et al. (1990), Jonson (1991) and Jauho (1992) [13]. The Bohm approach is also said to give results consistent with $\tau_D = \tau'_D$ according to Leavens and Aers, as mentioned by Landauer [13].

However, Landauer does question the equivalence relation between dwell time and the weighted reflection and transmission time, in a similar way to, as he cites, Olkhovsky and Recami (1992), and Brouard et al.

(1993) [13]. He also doubts that $\tau_D = \tau_D'$ should hold in the quantum mechanical case. If it was true, then $\int_a \mathrm{d}x\, |\psi^2| = jT\tau_T + jR\tau_R$ meaning that under the barrier the integrated density would be made up of additive contributions from both the transmitted and reflected beams. However, in quantum mechanics, it is not the particle densities that are added, but the complex wave functions, which then give rise to interference effects. The absorption of the barrier acts on all particles within it, and thus within the barrier the particles cannot be separated into transmitted or reflected components [13].

2.3.6 Different Pulse Solutions

A pulse-like wavefunction propagates through a classically forbidden region very much how a light pulse travels through a dispersive and attenuating medium [26].

Brillouin Pulse The usual way to solve a problem like the tunnelling time would be to find solutions to the Schrödinger equation by separating the variables, which leads one to find eigenstates and eigenfunction of the Hamiltonian. However, these eigenfunctions cannot be normalised. This problem is circumvented by finding wave packet solutions which are normalisable, and which can be Gaussian or other shapes. One such other shape could be the Brillouin pulse. This pulse is localised in either space or time, of a uniform amplitude which in a dispersive system develops fore- and after-runners as it propagates. The velocity of a Brillouin pulse is taken to be the signal velocity, which is the velocity at which the front of the main part of the pulse propagates [26]. This signal velocity continues to be meaningful when the wave packet enters a classically forbidden region, making it a good candidate for studying tunnelling properties.

According to Stevens, this pulse propagates more like one would expect from classical mechanics: for a plane wave described as $\exp \mathrm{i}(\omega \mathrm{t} - \mathrm{kx})$, the group velocity is defined as $\frac{\mathrm{d}\omega}{\mathrm{d}k}$ which describes the velocity of the wave packet's peak [26].

It is interesting to note that uniform amplitude pulses and Gaussian wave packets propagate differently through barriers. They have different velocity characteristics, with the uniform pulse propagating at a constant velocity, unlike the Gaussian one. This can be seen by breaking the uniform amplitude pulse into line contributions: the fore- and after-runners, and pole contributions: the pulse's main part. These contributions give different results from a Gaussian wave packet when tunnelling through a uniform barrier. A propagating Gaussian pulse remains a Gaussian pulse with the velocity of the peak being independent of the barrier thickness [26]. This leads to the traversal time of the barrier time being independent of the thickness of the barrier, a phenomenon already encountered in the Hartman Effect. It was also noted by Sokolovski that the barrier reshapes the wavepacket by "propagating components of the incident pulse with purely subluminal velocities before reassembling them with complex weights determined by the potential"

[24].

An interesting question arises from these observations: is it possible to generate pulses of different shapes, i.e. with specific boundary conditions, in order to observe whether their tunnelling behaviour will be different? It is hoped that this question will be explored elsewhere.

2.4 Dwell Time Approach

The dwell time approach uses an operator $\hat{T}_D$ to give a particle's evolution under a Hamiltonian. This can subsequently be used to calculate the probability distribution and dwell time of the particle.

2.4.1 Relation to Dwell Time Operator

According to Damborenea et al., the average value of dwell-time operator observable over a given state involves both reflection and transmission contributions, as well as interference terms [4].

2.4.2 Operator or Wave Function?

There appears to be an argument regarding the nature of the dwell time operator. Landauer argues that "interesting" quantities do not need to be described by a wave function, and gives the example of the spatial distribution of a current near a scanning tunnelling microscope tip as a quantity which does not correspond to a clearly defined operator [13].

Sokolovski and Connor, on the other hand, suggest that "quantally the traversal time, like every other quantity, should be described by a wave function, i e., the amplitude distribution for its possible values τ." [25] (also quoted in [13])

2.4.3 Dwell Time Operator $\hat{T}_D$

For a particle evolving under a Hamiltonian $\hat{H}$ both Munoz et al., and Damborenea et al. give the dwell-time operator as

$$\hat{T}_D = \int_{-\infty}^{\infty} \mathrm{d}t\, \hat{\chi}_D(t) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i\hat{H}t/\hbar} \chi_D(\hat{x})\, \mathrm{e}^{-i\hat{H}t/\hbar}\,, \tag{15}$$

where $\hat{\chi}_D(t)$ is the Heisenberg projector onto the one-dimensional region of interest $D = \{x : x_1 \leq x \leq x_2\}$ and $\chi_D(\hat{x}) = \hat{\chi}_D(0) = \int_{x_1}^{x_2} \mathrm{d}x\, |x\rangle\langle x|$. $\hat{T}_D$ is self-adjoint, and positive definite [4, 17]. It can be seen that $\hat{T}_D$ commutes with the Hamiltonian:

$$\begin{aligned} \hat{T}_D\, \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar} &= \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\hat{H}\tau/\hbar} \chi_D\left(\hat{x}\right) \mathrm{e}^{-\mathrm{i}\hat{H}(\tau+t)/\hbar} \\ &= \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\hat{H}(\tau-t)/\hbar} \chi_D\left(\hat{x}\right) \mathrm{e}^{-\mathrm{i}\hat{H}\tau/\hbar} = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\, \hat{T}_D \end{aligned} \tag{16}$$

[17].

The dwell-time for the quantum state $|\psi\rangle = |\psi(t=0)\rangle$ is given by $\Pi_\Psi(\tau) = \langle\psi|\delta(\hat{T}_D - \tau)|\psi\rangle$ [17], while the probability distribution is $\Pi(t) = \langle\psi_0|\delta(t - \hat{T}_D)|\psi_0\rangle$ [32].

Yearsley finds that in the limit where $|p|L \gg 1$ the dwell-time operator can be approximated by

$$\hat{T}_D \approx 2mL/|\hat{p}| \tag{17}$$

which results in the expected semiclassical form of the probability distribution

$$\Pi(t) = \langle\psi_0|\delta\left(t - \frac{2mL}{|\hat{p}|}\right)|\Psi_o\rangle \tag{18}$$

[32].

2.4.4 Degeneracy of Dwell Time Operator

One peculiarity of the quantum dwell time operator is that diagonalisation of T at a given energy generally results in two distinct eigenvalues, even where only a single classical time exists - "the dwell-time operator is multiply degenerate" [17]. This leads to a broader variance of quantum dwell-time compared to classical dwell time [4, 17].

This degeneracy can been seen in the dwell time equation derived by Munoz et al. which clearly shows the two distinct values

$$t_\pm(k) = \frac{mL}{k\hbar}(1 \pm \frac{1}{kL}\sin kL) \tag{19}$$

[17].

For a free particle, it is also worth noting that $t_+(k)$ grows without bound when $k \to 0$[4], while $t_-(k)$ tends to 0 as $k \to 0$ [17].

Conversely, for a constrained particle facing a barrier, both eigenvalues are bounded, thus there is a maximum dwell time for quantum particle in a barrier. This is in sharp contrast to classical dwell-time where the dwell time of a particle which has enough energy to overcome the barrier can be made as large as desired by smoothly reducing the particle's energy [4].

2.5 Clock Approaches to Tunnelling Time

Another approach to tunnelling time is the clock approach which uses additional degrees of freedom of the system as "clocks". This approach is of particular interest as it constitutes a thought experiment which might be experimentally realisable in the future [18]. When a system is treated as a clock, it measures the mean values of the dwell time operator or the real parts of some of its matrix elements [12].

In this approach the clock position corresponds to the measured time. Measurement models for both arrival and dwell times lead to distributions which depend on both the initial state of the particle and the details of the clock [32]. The probability distribution is then given by

$$\Pi_C(t) = \int_{-\infty}^{\infty} \mathrm{d}s \, R(t,s) \, \Pi(s) \,, \tag{20}$$

where $\Pi(t)$ is one of the ideal distributions and $R(t,s)$ is some function of the clock variables (e.g. a convolution). On coarse-grained scales, the resolution function R will drop out if the interval of interest is much larger than the time scale associated with R, so that

$$p(t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t \, \Pi_c(t) \approx \int_{t_1}^{t_2} \mathrm{d}t \, \Pi(t) \,. \tag{21}$$

[32]. The system's total Hamiltonian is given by Hamiltonian of the particle in addition to an interaction Hamiltonian, such as $\lambda\chi(\hat{x})H_C$ where χ is the characteristic function of the region where the clock is running, e.g. $\chi(x) = \theta(x)$ for the arrival-time problem, or $\chi(x) = \theta(x+L)\,\theta(L-x)$ for the dwell-time problem (where the spatial interval of interest is taken to be $[-L, L]$). Also, the Hamiltonian for the clock is assumed to be self-adjoint, such as $H_C = \int \mathrm{d}\epsilon \, \epsilon \, |\epsilon\rangle\langle\epsilon|$, where $|\epsilon\rangle$ form an orthonormal basis for Hilbert space of clock [32].

Yearsley notes that this approach is one which has been studied by several authors, such as Peres, Aharanov, Hartle, and Mayato.

Example: Atoms in Lattice As an example of a physical clock, Landauer mentions the possibility of comparing the speed at which two simultaneous processes take place and using one as a clock. One example mentioned in Landauer's 1994 paper (taken from an example given by from Zener in 1948) uses atoms in a lattice which can jump between two positions. By applying an oscillatory small stress to the atom which favours one of the two positions at a particular part of the cycle, this arrangement can be used as a clock. When the forcing frequency is low compared to the jump rate, it can be treated as a static stress, and thermal equilibrium between the two sites will be established. At high frequencies compared to the jump rate, the atom cannot respond to the oscillations, while at intermediate frequencies the adjustment of the atom's position will lag behind and the jump will be a source of energy dissipation [13].

Criticisms There are, however, some criticisms of the clock model for measuring tunnelling time. Privitera et al. note that not all clock models are equivalent, with some giving more approximate results than others.

Also, since clocks make use of a system's additional degrees of freedom, they change the effective number of degrees of freedom. As this is an invasive process it may affect experimental outcomes [20].

Measuring tunnelling time using clocks can take two approaches: one using Larmor clocks, the other one using Büttiker-Landauer clocks.

2.5.1 Larmor Clock

In the Larmor case, the clock makes use of the Larmor precession and spin-flip phenomena in a weak homogeneous field which coincides with the barrier region or region of interest. After tunnelling, the spin of a particle whose initial spin was entirely in the x-direction, is found to have acquired small y- and z-components. The Larmor time in then defined by the ratio of the spin-rotation angles, which in turn are defined by the acquired y- and z-components, to the precession and rotation frequency. Thus, the dwell time for an opaque barrier which satisfies $\kappa a \gg 1$, where $\kappa \equiv [2m(V_0 - E)]^{1/2} / \hbar$ is given by the Larmor time in the y direction:

$$\langle \tau^{La}_{y,tun} \rangle = \langle \tau^{dw}(x_i, x_f) \rangle = \langle {}^{\hbar k}/_{\kappa V_0} \rangle_E \tag{22}$$

and the tunnelling time in the z direction is given by

$$\langle \tau^{La}_{z,tun} \rangle = \langle {}^{ma}/_{\hbar k} \rangle_E . \tag{23}$$

The latter does not in fact relate to a rotation, but to a spin-flip jump ("spin-up" or "spin-down"), in addition to an energy-level Zeeman splitting [18].

Landauer mentions a clock approach which measures the precession of the spin of a tunnelling particle due to a uniform infinitesimal magnetic field associated with the barrier region. Due to Zeeman splitting, the spin component parallel to the magnetic field corresponds to a higher tunnelling energy and hence tunnels preferentially. During the tunnelling process, the spin acquires a component along the direction of motion as well as one along the direction of the magnetic field. As quoted by Landauer, Tinker argues in his 1983 paper that the tunnelling time is specified by the total rotation of the spin in three dimensions, giving rise to the total traversal time of $\tau^L_T = \sqrt{\left(\tau^L_y\right)^2 + \left(\tau^L_z\right)^2}$ [13].

2.5.2 Büttiker-Landauer Clock

The Büttiker-Landauer clock model describes an oscillatory tunnelling barrier [18]. In this model, small oscillations are added to the height of the originally static potential barrier through which the particle tunnels. At very low modulation frequencies the incoming particle perceives an effectively static barrier, while components of the wave packet which arrive later encounter a slightly different barrier height. With increasing frequency, the particle might be affected by a substantial part of the oscillation cycle, or even several cycles. An indication of the length of time the particle interacts with the barrier can be gleaned by observing at which frequency a substantial deviation from the adiabatic approximation occurs. It is, however, important to note that this only gives an approximate indication of the tunnelling time scale, and is not an eigenvalue of the Hamiltonian [13].

This clock model can also be adapted to different situation, such as instead of varying the barrier's height, it is expanded and contracted, or the barrier is moved back and forth [13].

2.6 Kinematical Path Approaches to Tunnelling Time

The kinematical paths approach to dwell time uses a particle's trajectory through spacetime to calculate the duration of the tunnelling process. The two methods presented here are the Feynman path-integral approach, as well as the Bohm approach. Despite their success at calculating the tunnelling time, there are, however, some pitfalls which will also be explored below.

Complex Times When the tunnelling particle has above-barrier energies, the time taken to cross the barrier is $\tau_T = a/v$ where v is the particle's velocity: $v = \hbar\sqrt{k^2 - \varepsilon^2}$, taking $\varepsilon = 2mV_0/\hbar$. However, for $E < V_0$ the wave-vector becomes imaginary, resulting in an imaginary velocity [20].

In a semi-classical approximation it is possible to have complex trajectories while the times and velocities are real, leading to

$$\tau_T^S = \frac{a}{v} = \frac{ma}{\hbar\kappa} \tag{24}$$

which gives unphysical results as it diverges for $k = \varepsilon$ [20].

2.6.1 Feynman Path-Integral Approach

In nonrelativistic quantum mechanics, the amplitude $g_\Delta\left(\mathbf{x}_f, t_f|\mathbf{x}_0, t_f\right)$ for a particle evolving from spacetime point $(\mathbf{x}_0, t_0)$ via the spatial region Δ to the final spacetime point of $(\mathbf{x}_f, t_f)$ is given by

$$g_\Delta\left(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0\right) = \int_\Delta \mathcal{D}\mathbf{x}(t) \exp\left(i\int_{t_0}^{t_f} \mathrm{d}t \, \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U\left(\mathbf{x}\right)\right]\right) \tag{25}$$

[11].

Time Spent in Space Region Assuming a classical particle is emitted at $\mathbf{x}_0$ at time t_0 and, after moving along the trajectory $\mathbf{x}(t)$ inside a potential $U(\mathbf{r})$, detected at $\mathbf{x}_f$ at time t_f. Then the time spent in the space region of interest is given by

$$\tau_{cl}^{\Omega} = \int_{t_1}^{t_2} \mathrm{d}t\, \Theta_\Omega(\mathbf{x}(t))\,, \tag{26}$$

where $\Theta_\Omega(\mathbf{x}(t))$ is 1 when inside the region of interest Ω and 0 otherwise. Using the Feynman path-integral method then gives

$$\tau^{\Omega}\left(\mathbf{x}_0, t_0; \mathbf{x}_f, t_f; k\right) = \langle \tau_{cl}^{\Omega}\left[x(\cdot)\right]\rangle_{path} \tag{27}$$

where $x(\cdot)$ is an arbitrary path between $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}_f, t_f)$. Generally, τ^{Ω}will be complex [20].

This tunnelling time concept allows the probing of the Copenhagen view of quantum mechanics, where a decaying particle was considered as undergoing a quantum jump when transitioning from one state to another. This jump might be analysed by ascribing to the particle a trajectory through the tunnelling barrier - a method for which many authors have used the Feynman path-integral approach [22].

2.6.2 Pitfalls of Path-Integral Approaches

Despite the success of path integrals in defining quantum mechanical amplitudes for particles propagating through spacetime, there are, according to Halliwell and Yearsley, significant complications to them. This is due to the fact that "concrete implementation of the restrictions on the paths over an interval of time corresponds, in an operator language, to sharp monitoring at every moment of time in the given time interval" [11]. Such monitoring leads to the Zeno effect which will be discussed in greater detail in Section 3. In short, the Zeno effect suggests that a particle which is continuously monitored is found to never decay. Halliwell and Yearsley argue that this may be avoided by "implementing the restrictions on the paths in the path integral in a "softer" way" [11].

As seen above, the amplitude for a particle propagating through a spatial region Δ is given by

$$g_\Delta\left(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0\right) = \int_\Delta \mathcal{D}\mathbf{x}(t) \exp\left(i\int_{t_0}^{t_f} \mathrm{d}t\, \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U\left(\mathbf{x}\right)\right]\right)\,. \tag{28}$$

It can similarly be seen that the amplitude $g_r\left(\mathbf{x}_f, t_f|\mathbf{x}_f, t_f\right)$ of the particle never entering this region is given by

$$g_r\left(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0\right) = \int_r \mathcal{D}\mathbf{x}(t) \exp\left(i\int_{t_0}^{t_f} \mathrm{d}t\, \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U\left(\mathbf{x}\right)\right]\right)\,, \tag{29}$$

where $g_r\left(\mathbf{x}_f, t_f|\mathbf{x}_f, t_f\right)$ is the restricted propagator given by a sum over paths which are restricted to always lie outside of Δ. It is thus clear that the usual propagator summing over all paths from initial to final point is given by

$$g\left(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0\right) = g_\Delta\left(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0\right) + g_r\left(\mathbf{x}_f, t_f|\mathbf{x}_f, t_f\right)\,. \tag{30}$$

By using a time-slicing procedure in which the time interval is divided into n equal parts of size ε, such that $t_f - t_0 = n\varepsilon$, and with slices labelled as $t_k = t_0 + k\varepsilon$, where $k = 0, 1 \cdots n$, the restricted propagator Eq.29 is defined as the limit

$$\begin{aligned} g_r\left(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0\right) &= \lim_{\varepsilon \to 0,\, n \to \infty} \int_r \mathrm{d}^d x_1 \cdots \int_r \mathrm{d}^d x_{n-1} \\ &\quad \times \prod_{k=1}^{n} g\left(\mathbf{x}_k, t_k | \mathbf{x}_{k-1}, t_{k-1}\right) , \end{aligned} \tag{31}$$

where $\mathbf{x}_n = \mathbf{x}_f$, and the integrals are over the region restricted to being outside of Δ.

For small times, the propagator can be approximated by

$$g_r\left(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0\right) = \left(\frac{m}{2\pi i \varepsilon}\right)^{d/2} \exp\left(iS\left(\mathbf{x}_k, t_k | \mathbf{x}_{k-1}, t_{k-1}\right)\right) , \tag{32}$$

where the exponent is the “action between the indicated initial and final points and the limit is taken in such a way that $n\varepsilon$ is fixed” .

Writing this in operator form,

$$g_r\left(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0\right) = \langle \mathbf{x}_f | \hat{g}_r\left(t_f, t_0\right) | \mathbf{x}_0 \rangle , \tag{33}$$

the probabilities for a particle whose initial state is given by $|\psi\rangle$ to enter or not enter the region Δ during the time interval $[t_0, t_f]$ are given by

$$p_\Delta = \langle \psi | \hat{g}_\Delta\left(t_f, t_0\right)^\dagger \hat{g}_\Delta\left(t_f, t_0\right) | \psi \rangle , \tag{34}$$

$$p_r = \langle \psi | \hat{g}_r\left(t_f, t_0\right)^\dagger \hat{g}_r\left(t_f, t_0\right) | \psi \rangle . \tag{35}$$

One would naturally assume that these probabilities obey the sum rule,

$$p_\Delta + p_r = 1 , \tag{36}$$

but using $\hat{g} = \hat{g}_\Delta + \hat{g}_r$ it can be seen that this is not generally the case unless there is no interference between the two types of paths, which means that

$$Re\langle \psi | \hat{g}_r\left(t_f, t_0\right)^\dagger \hat{g}_\Delta\left(t_f, t_0\right) | \psi \rangle = 0 . \tag{37}$$

Even though this condition can hold for certain situations, there is "no guarantee that those states are physically interesting ones" [11].

It is interesting to note that despite these shortcomings path integrals have been used extensively to study time, and that Feynman's original paper on the topic was in fact entitled "Space-time approach to non-relativistic quantum mechanics", suggesting that path integrals should be used to study spacetime features [11].

The problem with path integrals, however, is that their properties can be very different from those intuitively expected. "The consequence", Halliwell and Yearsley write, "is that the amplitude of [Eq. 29], or equivalently $\hat{g}_r(t_2, t_1)$, actually describes *unitary* propagation on the Hilbert space of states with support only in the region outside Δ and therefore gives probability $p_r = 1$ for any incoming state. This then means that either the sum rule is not satisfied, in which case the probabilities are not meaningful, or that it is satisfied but $p_\Delta = 0$, which means that any incoming state aimed at Δ has probability zero for entering that region, a physically nonsensical result" [11]. This suggests that setting the restrictions on the path integrals effectively results in reflecting boundaries.

One solution to this problem, suggested by Halliwell and Yearsley, was to suppose that $\hat{g}_\Delta$ is related to a measurement scheme, thereby avoiding the requirement to satisfy the probability sum rule. In this case, the probability of entering the region of interest is given by p_Δ. A more detailed formula for g_Δ using the path decomposition expansion becomes relevant. This formula is given by partitioning the paths according to "the time t and location $\mathbf{y}$ at which they cross the boundary Σ of Δ",

$$\begin{aligned} g_\Delta\left(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0\right) &= \frac{i}{2m} \int_{t_0}^{t_f} \mathrm{d}t \int_\Sigma \mathrm{d}^{d-q} y \, g\left(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0\right) \\ & \quad \times \mathbf{n} \cdot \nabla_{\mathbf{x}} g_r\left(\mathbf{x}, t | \mathbf{x}_0, t_0\right)|_{\mathbf{x}=\mathbf{y}} \end{aligned} \tag{38}$$

where $\mathbf{n}$ is the outward pointing normal to Σ.

2.6.3 Bohm Approach

Another kinematical approach to quantum tunnelling time is the Bohm approach, named after David Bohm. It employs semi-classical trajectories which can be used to calculate the average tunnelling time. By using a set of equations equivalent to the Schrödinger equation, the Bohm method gives a "classical" interpretation of quantum mechanics [20].

Also, the Bohm approach works by matching the phase with the classical action while ignoring any presence of a particle's spin [20].

Madelung Equations for Probabilistic Fluid The Madelung equations for a probabilistic fluid are useful in kinematical path approaches. They are found by separating the real and imaginary parts when the Schrödinger equation is applied to a general scalar wave function $\psi \in \mathbb{C}$, such as $\psi = \sqrt{\rho}\exp\left[i\frac{\varphi}{\hbar}\right]$ where $\rho(\mathbf{x},t)$ and $\varphi(\mathbf{x},t) \in \mathbb{R}$:

$$\partial_t\varphi + \frac{1}{2m}\left(\nabla\varphi\right)^2 + \frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2 - \frac{\Delta\rho}{\rho}\right] + U = 0 \tag{39}$$

where

$$\frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2 - \frac{\Delta\rho}{\rho}\right] \equiv -\frac{\hbar^2}{2m}\frac{\Delta|\psi|}{|\psi|} \tag{40}$$

is often called the "quantum potential" , and

$$\partial_t\rho + \nabla\cdot\left(\rho\nabla\varphi/m\right) = 0 \tag{41}$$

[20]. These equations constitute the "hydrodynamic" formulation of the Schrödinger theory, and upon defining boundary conditions, yield the semiclassical phase φ and the probability density ρ. A particle's "local" velocity can be found by matching this quantum phase with the classical action, and integrating the thus-found velocity field, allows the computation of the semiclassical Bohm trajectories. From these trajectories, semiclassical tunnelling times can be derived [20].

2.7 Physically Measuring Tunnelling Time

2.7.1 Difficulties in Measuring Tunnelling Time

The difficulty in measuring tunnelling time is due to the collapse of the wavefunction upon observation. A sufficiently noninvasive measurement technique would need to be discovered which overcomes the problem posed by the wavefunction collapsing when the particle is detected at the entrance of the region of interest, which in turn renders further measurement at the exit of the region impossible, and thus does not produce an ideal dwell-time distribution [17]. This obstacle is similar to the impossibility of determining which slit a particle might have travelled through in a double slit experiment without destroying the interference pattern [25].

Due to the uncertainty principle the actual amount of time that the particle spends in the barrier region cannot be determined unless the paths are restricted to a known traversal time, such as

$$t_{ab}^{cl}\left[x(t)\right] = \int_{t_1}^{t_2} \theta_{ab}(x(t))\,\mathrm{d}t\,, \tag{42}$$

where θ is a step function indicating the barrier region, and $x(t)$ is the particle's trajectory. Knowing this traversal time, however, would perturb the motion of the particle [25], and would thus not lead to the same dwell time one would have measured in an unperturbed case.

Sokolovski and Connor ([25]) indicate that by making the interaction between the particle and the measuring device arbitrarily small, an indirect measurement can be taken. However, this results in general in a complex valued time, rather than a real time interval.

2.7.2 Proposed Experiments

Despite these difficulties in measuring tunnelling time, some experiments have been proposed, which will be covered in greater detail in Section 6 on experiments.

According to Winful, the only type of experiment on tunnelling times that has been undertaken has focused on group delays, while "no one has done an experiment with a modulated barrier or measured the Larmor times associated with a tunnelling particle in the presence of a magnetic field" [31]. Similarly, Privitera et al. noted that the question of how much time a particle takes to cross a potential barrier "still remains debated" [20].

There does appear to be experimental evidence supporting the definition of phase time (essentially a measurement in the change of the wave function's phase) [5], while in some cases the results indicate group-velocities larger than the speed of light in vacuum. These superluminal velocities are, however, no cause of worry, as both superluminal and negative group-velocities have also been observed in classical optics where the group-velocity may exceed c or become negative in regions with strong anomalous dispersion near a resonance. Privitera notes that it has been proposed that in those cases the group-velocity may lose its ordinary meaning, resulting in no signal being transmitted by the medium at a faster-than-light velocity, and hence the "universal speed limit" still being obeyed. Another explanation for this superluminal propagation may be the assumption that the barrier reshapes the incoming signal (by attenuation of the less energetic, and thus slower, components) which results in the outgoing signal seemingly having travelled at velocities higher than the velocity at which the more energetic parts actually propagated [20].

3 Zeno Effect

3.1 Definition of the Zeno Effect

Setting the Scene for the Zeno Paradox - Continuous Measurement Consider a system which is prepared in the undecayed state. The Zeno paradox occurs when one asks a question about the probability of decay during a given time interval, with the system being observed continuously throughout this interval, e.g. the probability that the system prepared in the undecayed state ρ will be found to be in the undecayed state throughout $[0, t_1] \equiv \Delta$, but will subsequently be found to decay at some point during the interval $[t_1, t] \equiv \Delta_2,\ 0 < t_1 < t$ [14].

It is important to note that the system is not left unobserved after the initial preparation of the state but that a continuous measurement is taking place; such continuous observation is, of course, an idealisation.

Consider furthermore the time interval between successive measurements approaching zero. This appears to be an admissible process in quantum theory as there is nothing that forbids the duration of a measurement or the time between measurements being arbitrarily small. The concept of continuous observation hinges on the assumption that there is no "elementary and indivisible unit of time", which despite being acknowledged as an "exciting possibility" is argued by Misra and Sudarshan in their 1977 paper ([14]) to not be part of the then accepted theories of quantum mechanics.

Example of Physical Continuous Measurement One example of a continuous measurement could be the tracks of unstable charged particles in bubble chambers where the tracks amount to more or less continuous detection of the existence of the unstable particle [14].

Continuously Observed System Does Not Decay An unstable particle which is continuously observed is found to *never* decay. This evokes the famous paradox of Zeno "denying the possibility of motion to a flying arrow" and Misra and Sudarshan thus chose to "call this result the Zeno's paradox in quantum theory" [14].

However, Schulman wonders in this context if this would not lead to a situation in which nothing ever decays. In his 2001 paper he argues that most measurements involving a detector could be described as being "continuous" measurements since the silence of the measuring device can be taken as no event taking place. He suggests that the "unwavering attention of the apparatus should act like a continuous check - effectively with a zero time interval between measurements - that no decay has taken place". He also mentions several papers in which "adding an apparatus-like term" to the system's Hamiltonian could "stop or slow the decay" [22].

In his 2001 work, Schulman indicates that the criterion for determining whether or not a measurement is in fact continuous, is given by the comparison between two quantities: the response time of the apparatus and the time the system needed to transition between different states. The essential idea was that since no physical apparatus can take truly instantaneous measurements, the observation was not in fact continuous, and thus the system does decay despite being observed [22].

3.2 Mathematical Background

We denote the probability of the system, which is prepared in the undecayed state ρ at time 0, to be found to decay during the time interval $\Delta = [0, t]$ by $P(0, t; \rho)$, and the probability of it not decaying during this interval by $Q(0, t; \rho)$. The probability that the system which is initially in state ρ is found to be undecayed throughout the interval $[0, t_1]$, but decayed sometime during the following period $[t_1, t]$ is denoted by $R(0, t_1, t; \rho)$ [14].

The relations between the three probability functions are given by

$$P(0, t; \rho) + Q(0, t; \rho) = 1 \tag{43}$$

and

$$R(0, t_1, t; \rho) = Q(0, t_1; \rho) P(0, t - t_1; \rho_1) \tag{44}$$

where ρ_1 is the state in which the system (initially prepared to be in state ρ) finds itself at t_1, having been continuously observed and found to be undecayed throughout the time interval leading up to t_1. Given these relations, it is sufficient to concentrate on Q and ρ_1. A complete theory of quantum mechanics has to be able to provide an algorithm for calculating $P(0, t; \rho)$, $Q(0, t; \rho)$, and $R(0, t_1, t; \rho)$ [14].

Incremental Measurements Assuming that the system is originally in the state ρ, a series of $n + 1$ idealised instantaneous measurements is made. The probability that each of these measurements finds an undecayed state is given by $Q(\Delta, n; \rho)$. The limit as $n \to \infty$ then gives $Q(\Delta; \rho) \equiv Q(0, t; \rho)$. The collapsed state after a measurement has been taken - with the conclusion that at that moment the system had not decayed - does not uniquely determine the measured observable, but depends on the details of the measuring apparatus. The state thus collapses to the new unnormalised state ρ' of the form

$$\rho' = \sum_j A_j^* \rho A_j \tag{45}$$

where

$$\sum_j A_j^* A_j = E \tag{46}$$

and the orthogonal projection onto the subspace of the undecayed states of the system E [14].

3.3 Conclusions

According to Misra and Sudarshan's 1977 paper, a complete theory of quantum mechanics should provide an algorithm for computing the probabilities of continuously observed quantities which are not constants of motion. However, such algorithms are lacking, suggesting the incompleteness of quantum mechanics. This lack of a reliable algorithm for calculating the probabilities of an event happening during a specified time interval, such as the decay of an initially undecayed system within the interval $[0, t]$, is intimately linked with the difficulties of defining an operator of arrival or transition time in quantum theory.

It is argued that the tracks of particles in a bubble chamber or photographic emulsion appear to be in contradiction to Zeno's paradox. To overcome this, four possible views are suggested by Misra and Sudarshan, together with their objections to each [14]:

1. The probabilities for continuous measurements have no operational meaning as there is a fundamental principle in quantum mechanics which denies this possibility.

    - Objection: No such principle has been either derived or incorporated into quantum mechanics.

2. Contrary to being ideal measurements on which the Zeno effect is based, the measurement causes a wave function to collapse and thus the assumption of ideal measurements needs to be abandoned

    - Objection: This has the side effect that the observed lifetime on an unstable particle is dependent not only on properties intrinsic to the particle itself, but also to details of the measurement process. At the time of writing of the Misra and Sudarshan paper in 1976 there was no indication for this being true, and we are unaware of any more recent research suggesting this phenomenon.

3. "The track of a particle is not a continuous observation" of the particle's not having decayed but rather constitutes a discrete sequence of observations. Zeno's paradox only applies to the limit of continuous measurements.

    - Objection: This would imply that a sufficiently frequently observed particle would exhibit a lengthened lifetime. This has not been observed.

4. Abolishing the notion that the temporal evolution of a quantum system can be thought of as linear and seeing it only as a "persistent interaction between the quantum system and the classical measuring apparatus". The Zeno effect is then understood as the effect of the interaction with the measuring device.

    - Objection: No detailed theory describing the coupling between the quantum system and a classical apparatus has been published (as of 1976).

The conclusion by Misra and Sudarshan is that there does not appear to be a clear cut resolution of the Zeno's paradox [14].

4 Time of Arrival

4.1 Definition of Arrival Time

Measuring arrival time seeks to answer the question: "What is the probability that a particle enters a region of space for the first time during a given time interval?" However, the notion of crossing a specific surface, such as the origin, is not well defined in quantum mechanics. Some notions that are well defined are those of the first and last crossing of a wave packet [9]. Thus, before measuring the arrival time of a particle, the concept of "arriving" needs to be more clearly defined.

According to Allcock, the scene is even more bleak, as he suggests that "it is very unlikely indeed that quantum mechanics admits any ideal arrival-time concepts" [2].

Despite these reservations, arrival time has been studied in depth by several authors, and a number of approaches will be described in this paper. Before exploring further the probability distributions of arrival time, the decoherent histories approach or the concept of Positive Operators Valued Measures (or POVMs), we take a quick look at how periodic projections can shed light on the arrival time problem.

Periodic Projection The question to be answered in the arrival time problem is what happens to an initial state $|\psi\rangle$ when acted upon by a series of periodic projection operators P while evolving under unitary time evolution:

$$|\psi_P(\tau)\rangle = \mathrm{e}^{-\mathrm{i}H\epsilon} P \mathrm{e}^{-\mathrm{i}H\epsilon} \cdots P \mathrm{e}^{-\mathrm{i}H\epsilon} |\psi\rangle, \tag{47}$$

i.e. n projections, giving the total time as $\tau = (n+1)\epsilon$ [10].

Zeno Time It is worth noting that for small ϵ the wave packet will stay in the original region, and as $\epsilon \to 0$, the Zeno limit is approached, leading to the packet being totally reflected at the barrier. When observing a wave packet through frequent projections, it is important to determine the timescale at which the total reflection due to the Zeno effect becomes significant. For a state with an energy E this timescale is given by the Zeno time: $t_Z = \frac{1}{E}$, while for a wave packet of momentum p and with spatial width σ, the Zeno time is given by the timescale on which the wave packet crosses the origin, and is of the order $\frac{m\sigma}{p}$. This is also the timescale at which the state changes significantly from a previous state through unitary evolution. Even though the Zeno time is a classical timescale, the reflection occurs due to the increase in uncertainty in momentum when the position projection takes place - an entirely quantum process [10].

The reason for using periodic projections onto the x-axis is that the state obtained through such projections is approximately equivalent to the state obtained by evolving the system in the presence of a complex potential. For sufficiently small ϵ, Eq. 47 can be used as a candidate for the measurement of the amplitude remaining in the its original region (such as $x > 0$ if the wave packet originate in the positive x-region) during the time interval $[0, \tau]$. However, as mentioned above, $\epsilon \to 0$ leads to the Zeno limit and should thus be avoided. Hence, a balance has to be found so that the system is monitored sufficiently frequently to allow detection in the region of interest, but not so much that the incoming state is reflected at the origin due to the Zeno effect [10].

Path Decomposition Expansion The idea of a "path decomposition expansion" or PDX is introduced by Halliwell and Yearsley in their 2010 paper. In this expansion the propagators are factored across the surface at the origin, thereby reducing the problem of proving the approximate equivalence of

$$\mathrm{e}^{-iH\varepsilon}P\mathrm{e}^{-iH\varepsilon}\cdots P\mathrm{e}^{-iH\varepsilon}|\psi\rangle \approx \exp\left(-iH\tau - V_0\theta(-x)\tau\right) \tag{48}$$

to simply showing that it holds for the propagation between points at the origin at different times [10].

4.2 Probability Distribution

The arrival time problem seeks to find the probability $p(t_1, t_2)$ of a free particle in initial state $|\psi\rangle$situated in $x > 0$ and consisting of purely negative momentum to cross the origin during the time interval $[t_1, t_2]$. One candidates, among many approaches, is

$$p(t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t\, J(t) = \langle\psi|C|\psi\rangle \tag{49}$$

where

$$C = \int_{t_1}^{t_2} \mathrm{d}t\, \frac{(-1)}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}) = \theta(\hat{x}(t_1)) - \theta(\hat{x}(t_2))\,, \tag{50}$$

i.e. the probability for crossing the origin during a given time interval is the current integrated over this time interval [9].

Classically, there is no difference between C and C^2, so one could replace $\langle C\rangle$ with $\langle C^2\rangle$ in the crossing probability. The difference may then be expressed as $\langle\psi|C^2|\psi\rangle - \langle\psi|C|\psi\rangle = -\langle\psi|C(1-C)|\psi\rangle$, where the right-hand side gives a superposition between the state $C|\psi\rangle$ representing a crossing during the time interval, and $(1-C)|\psi\rangle$ representing no crossing during this interval (and thus crossing at another time), i.e. the difference represents the interference between crossing and not crossing, with no interference given by $\langle C\rangle = \langle C^2\rangle \geq 0$ [9].

Another probability distribution is given by Yearsley as

$$\begin{aligned}\Pi(t) &= J(t) = \frac{(-1)}{2m}\langle\psi_t|\,[\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}]\,|\psi_t\rangle \\ &= \frac{i}{2m}\left(\psi * (0,t)\frac{\partial\psi(0,t)}{\partial x} - \frac{\partial\psi * (0,t)}{\partial x}\psi(0,t)\right)\end{aligned} \tag{51}$$

which is normalised to one when integrated over all time, but is not necessarily positive; this is related to the phenomenon of quantum backflow discussed below [32].

One can normalise the distribution by dividing by the probability of the particle ever being detected:

$$\begin{aligned}\Pi_N(t) &= \frac{\Pi(t)}{\int_0^\infty \mathrm{d}s\,\Pi(s)} \\ &= \frac{1}{m|\langle p\rangle|}\langle\psi_t|\hat{p}\delta(\hat{x})\hat{p}|\psi_t\rangle\end{aligned} \tag{52}$$

where $\langle p\rangle$ is the average momentum of initial state. This is independent of detector details [32].

According to Allcock, the probability current is $j = -\frac{1}{2}i\left(\psi * \frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi *}{\partial x}\right)$, where ψ denotes the Schrödinger amplitude. This may give the arrival probability at $x = 0$, i.e. $j(t, 0)$ (Allcock 3)

One short-coming of j is that "it is not always positive even for unidirectional wave packets of positive energy", making it impossible to be treated as a probability [2]. This observation is, again, related to the backflow phenomenon.

The Kijowski Probability Densities Distribution for Arrival Times of Free Particles One arrival time probability distribution which is often cited is that of Kijowski. It gives a probability density for arrival times of free particles, which can be measured using fluorescence photons. Kijowski's operator can be identified with Aharonov and Bohm's operator in a free particle case. In the standard quantum mechanics interpretation, observable quantities are only realised when a measurement is performed. This is important to keep in mind to avoid confusion about a zero probability density at a point when the probability density of arrivals is non-zero at this point [6].

4.3 Decoherent Histories Approach

The decoherent histories approach introduced by Halliwell in 2009 suggests that "a homogenous history (the simplest type of history) is represented by a class operator $C_\alpha$which is a time-ordered string of projections $C_\alpha = P_{a_n}(t_n)\ldots P_{a_1}(t_1)$". It is closely linked to path integral methods for dealing with arrival time [9].

Class Operators The class operators representing a homogeneous history satisfy

$$\sum_{\alpha} C_{\alpha} = 1\,, \tag{53}$$

with the probabilities assigned as

$$p(\alpha) = \mathrm{Tr}\left(C_{\alpha}\rho C_{\alpha}^{\dagger}\right)\,. \tag{54}$$

Due to quantum interference, this can mean that the probabilities do not necessarily obey the sum rules. To overcome this, it is necessary to introduce a decoherence functional, such as

$$D(\alpha,\beta) = \mathrm{Tr}\left(C_{\alpha}\rho C_{\beta}^{\dagger}\right)\,, \tag{55}$$

which satisfies the decoherence condition of

$$D(\alpha,\beta) = 0,\ \alpha \neq \beta\,. \tag{56}$$

[9].

Using class operators allows us to show that the assumptions of quasi-probability are robust arguments: consider the case in which the particle crosses the origin during the large time interval $[0,\tau]$, and split this interval to introduce discrete time in the form $t_k = k\epsilon$ where $k = 0, 1\cdots n$, and thus $\tau = n\epsilon$. The projection operators for a particle originally in the positive x-region are $P = \theta(\hat{x})$ and $\bar{P} = 1 - P = \theta(-\hat{x})$. The class operator for not crossing during this time interval is then given by $C_{nc} = P(t_n)\cdots P(t_2)P(t_1)$ which indicates that at each discrete time the particle was in the $x > 0$ region, while not specifying its location at intermediate times. By taking the limit $\epsilon \to 0$, one again encounters the Zeno effect as $C_{nc} \to \mathrm{e}^{iH\tau}g_r(\tau,0)$ where $g_r(\tau,0) = P\exp(-iPHP\tau)$ is a restricted operator. This means that the state never leaves its original region, but instead undergoes total reflection at the origin. To avoid the Zeno effect, the separation of projections has to remain sufficiently large, i.e. ϵ has to remain finite. The important timescale associated with the Zeno effect is the Zeno time $t_z = \frac{1}{H}$ which is inversely proportional to the particle's energy. In order to avoid significant reflection, the projection intervals have to be larger than the Zeno time, i.e. $\epsilon > t_z$ [9].

Quasi-Probability In order to satisfy the probability sum rules, Halliwell defines the quasi-probability $q(\alpha) = \mathrm{Tr}(C_{\alpha}\rho)$ which is linear in C_{α}. However, this in not generally a real number, but can be related to

$p(\alpha)$and $D(\alpha,\beta)$ in the following way:

$$q(\alpha) = p(\alpha) + \sum_{\beta,\beta\neq\alpha} D(\alpha,\beta) \tag{57}$$

Thus when there is decoherence, the probabilities are equivalent, i.e. $p(\alpha) = q(\alpha)$, also ensuring that $q(\alpha)$ is a real and positive number, which seems to indicate that the probabilities in the standard arrival time formula (49)

$$p(t_1,t_2) = \int_{t_1}^{t_2} dt\, J(t) = \langle\psi|C|\psi\rangle$$

are in fact quasi-probabilities, explaining why 49 gives reasonable answers in some circumstances but not in others [9]

4.4 Positive Operators Valued Measures and Aharonov-Bohm Time Operator

What are POVMs? According to the spectral theorem, the moments of a probability distribution for measurement of an observable coincide with the operator moments, given the associated operator is self-adjoint. "Each observable is associated to a positive operator valued measure (POVM)", thus indicating a mapping from the "subsets of the set of positive values to the space of positive operators". These POVMs "can be understood as associated to 'unsharp' or 'nonideal' measurements", and it is also worth noting that there are measurements which are "intrinsically associated with POVMs", such as phase observables [6]

Aharonov-Bohm Time Operator Aharonov and Bohm introduced a "clock" which measures time from the position and momentum of a free particle, whose operator can be obtained via symmetrising the classical expression my/p_y in which y and p_y correspond to the particle's position and momentum respectively. Symmetrising the classical arrival time expression for arrival at the origin of a particle with position x and momentum p, $t = -mx/p$ gives

$$\hat{T}_{AB} := -\frac{m}{2}\left(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x}\right) \tag{58}$$

which has the correct commutation relation with the Hamiltonian $\hat{H}_0 = \hat{p}^2/2m$,

$$\left[\hat{H}_0, \hat{T}_{AB}\right] = i\hbar\,. \tag{59}$$

A reversal relative to parametric time can be observed by noting that in the Heisenberg picture

$$\mathrm{d}\hat{T}_{AB}(t)/\mathrm{d}t = -1 \tag{60}$$

[6].

Also, from Pauli's theorem, $\hat{T}_{AB}$ cannot be a self-adjoint operator [6].

4.5 Measuring Arrival Time

Proposed Detector Models Several detector models have been proposed to measure arrival time; three will be covered in greater detail below in Ch.6:

The atom-laser model uses lasers to illuminate the region of interest. The first photon emitted by the atom indicates the atom's arrival time in this region [21]. Another detector sweeps the region with photographic plates or scintillation counters and registers at which sweep the particle is detected. Hence the arrival time is taken to be between this and the previous sweep [1]. The third model describes a detector coupled to a particle which "undergoes a transition when the coupling is switched on" [7], such as a two-level system which changes state upon detection of the arrived particle.

Difficulties There are however several difficulties in physically measuring arrival time.

The particle's position needs to be measured at two moments in time; just before entering a given spatial region a measurement of t outside the region of interest needs to be made, followed by a further measurement immediately after t when the particle has entered the region. A further problem arises when calculating the probabilities for entering the region of interest: the amplitude for crossing or not not crossing the origin into the region of interest are found by summing the paths which always or never enter this region, and the probabilities are, as usual, obtained by squaring these amplitudes. However, due to interference between the paths, these probabilities do not add up to one and can thus not be regarded as true probabilities. Also, due to the non-commutativity of position and time in quantum mechanics, it is expected that no single hermitian operator can be associated with the arrival time of a particle [7].

5 Backflow

A "striking but little-appreciated phenomenon in quantum mechanics is the backflow effect" [33], which has not yet received as much attention as other quantum effects [19].

5.1 Definition

Backflow is a phenomenon where, for a free particle characterised by a wavefunction centred in the negative x-region, i.e. $x < 0$, and consisting entirely of positive momenta, a negative current is exhibited. This means that the probability is flowing in the opposite direction to the momentum, indicating that the probability for remaining in the negative x-region may, contrary to classical expectations, *increase* over time [34, 33]. This effect, though first noted by Allcock in the 1960's, was not studied in detail until the mid-90's when Bracken and Melloy noted that the probability backflow can be no greater than the dimensionless number c_{bm} which is approximately equal to 0.04. Despite its curious independence of $\hbar$, which will be seen below, they declared this to be a "new quantum number" [19, 33].

5.2 Formulation of the Problem

5.2.1 The Flux

For a free particle with initial wave function $\psi(x)$ centred in $x < 0$ region and consisting entirely of positive momenta, the probability flux $F(t_1, t_2)$ of crossing the origin during a time interval $[t_{1,}t_2]$ is found by comparing the probabilities at the two times, or by integrating the current density over the time interval:

$$\begin{aligned} F(t_1,t_2) &= \int_{-\infty}^{0} \mathrm{d}x\, |\psi(x,t_1)|^2 - \int_{-\infty}^{0} \mathrm{d}x\, |\psi(x,t_2)|^2 \\ &= \int_{t_1}^{t_2} \mathrm{d}t\, J(t) \end{aligned} \tag{61}$$

where $J(t)$ is the usual quantum mechanical current at the origin

$$J(t) = -\frac{i\hbar}{2m}\left(\psi^*(0,t)\frac{\partial\psi(0,t)}{\partial x} - \frac{\partial\psi^*(0,t)}{\partial x}\psi(0,t)\right) . \tag{62}$$

[33]

The flux equation (61) is positive when these probabilities behave in a classical way in which the probability of remaining in the initial state (here: in $x < 0$) decreases over time [33].

In the quantum-mechanical case, however, there can be negative flux. This can be explained by the Wigner function which need not be positive for a general state, or by realising that while both $\hat{p}$ and $\delta(\hat{x})$

are non-negative, the current operator $\hat{J}$ is not necessarily positive due to the non-commutativity of $\hat{p}$ and $\delta(\hat{x})$ [33].

The flux may be thus rewritten in terms of the Wigner function at time t, given by $W_t(p,q)$:

$$F(t_1,t_2) = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}p\,\mathrm{d}q \frac{p}{m}\delta(q) W_t(p,q) \tag{63}$$

[33]

5.2.2 Projection Operators

Projection operators, similar to the ones introduced in the discussion on arrival time, can be used to project onto the positive or negative x-axis, given by $P = \theta(\hat{x})$ and $\bar{P} = 1 - P = \theta(-\hat{x})$ respectively. The flux can then be written in terms of the operator $\hat{F}(t_1,t_2)$:

$$\begin{aligned} \hat{F}(t_1,t_2) &= P(t_2) - P(t_1) = \int_{t_1}^{t_2} \mathrm{d}t\,\dot{P}(t) \\ &= \int_{t_1}^{t_2} \mathrm{d}t \frac{i}{\hbar}\left[H, \theta(\hat{x})\right] = \int_{t_1}^{t_2} \mathrm{d}t\,\hat{J}(t) \end{aligned} \tag{64}$$

where now the current is given by $\hat{J} = \frac{1}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p})$, so that Eq.61 may be written as

$$F(t_1,t_2) = \langle \hat{F}\,(t_1,t_2)\rangle = \int_{t_1}^{t_2} \mathrm{d}t\,\langle\psi|\hat{J}(t)|\psi\rangle \tag{65}$$

[33]

5.2.3 Eigenvalue Problem

Yearsley and Halliwell indicate that they used a technique from Bracken and Melloy's 1994 paper in which the spectrum of the flux operator is investigated to learn about the backflow effect [33]. This means looking at the eigenvalue equation

$$\theta(\hat{p})\hat{F}(t_1,t_2)|\Phi\rangle = \lambda|\Phi\rangle \tag{66}$$

where $|\Phi\rangle$ are states consisting entirely of positive momenta. Here, the backflow states will have $\lambda < 0$ and the most negative value of $F(t_1,t_2)$ is given the most negative eigenvalue of Eq. 66. By defining suitable variables Yearsley and Halliwell give a rescaled eigenvalue equation:

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}v \frac{\sin(u^2 - v^2)}{(u-v)}\phi(v) = \lambda\phi(u) \tag{67}$$

where $\phi(x) = \left(\frac{m\hbar}{4T}\right)^{\frac{1}{4}} \Phi(p)$ is a dimensionless function. It can be seen that all physical constants have dropped out of this equation resulting in λ being both dimensionless and independent of $\hbar$, m and T. The independence of T indicates that a period of backflow can have an arbitrary duration, while the independence of $\hbar$ suggests that a naive classical limit may not be appropriate for backflow [33]. We will look at the classical limit in more detail below.

Yearsley and Halliwell found that the eigenvalues of Eq. 67 lie within the range $-c_{bm} \leq \lambda \leq 1$ where it was computationally found that $c_{bm} \approx 0.038452$ [33].

5.3 Backflow for a Superposition of Gaussians

According to Yearsley and Halliwell in their 2013 paper, it might be experimentally realisable to observe backflow for a superposition of Gaussian states. If and when this becomes experimentally possible, it is important to verify that the backflow in the Gaussian states is not due to their having an initial negative momentum, but purely due to the backflow effect. "It can be shown that the probability that a measurement of the momentum of this state would yield a negative answer is of order 10^{-10}", indicating that the negative backflow is "entirely due to the backflow effect" [33]. It is worth noting however, that the size of the backflow effect in Gaussians is significantly smaller than the theoretical maximum [33].

Yearsley and Halliwell also mentions that Bracken and Melloy demonstrated in their 1994 paper that quantum backflow may be observed in the superposition of plane waves. It should be possible to turn this into a physical state by replacing the plane waves with Gaussians with a very narrow spread in momentum [33].

A normalised state made up of two Gaussian wave packets both with spatial with σ and evolving for a time t is given by

$$\psi(x,t) = \sum_{k=1,2} A_k \frac{1}{\sqrt{4\sigma^2 + 2it}} \exp\left(ip_k \left(x - p_k t\right) - \frac{\left(x - p_k t\right)^2}{4\sigma^2 + 2it} \right) \tag{68}$$

As $\sigma \to \infty$, the sum of two plane waves is recovered. Yearsley and Halliwell investigated solving this equation with a set of parameters (in their case $p_1 = 0.3$, $p_2 = 1.4$, $\sigma = 10$, $A_1 = 1.8$, and $A_2 = 1$) which showed several intervals of negative current when plotted. This effect was robust under light perturbation of the parameters [33].

5.4 States Exhibiting Backflow

In their recent paper, Halliwell, Gillman, Lennon, Patel and Ramirez claim to have found an "essentially exhaustive class of states with backflow, which in momentum space, have the general form

$$\phi(p) = N\theta(p)(a-p)f(p)\,, \tag{69}$$

where $f(p)$ is a general complex function of momentum subject only to some simple restrictions" [8]. These restrictions are on the first three moments of the function $f(p)$ and on the constant a. They claim that, "in particular, for any complex function $f(p)$ which is such that the current exists and is non-zero, these states are always backflow states for some values of the complex constant a". Thus, they conclude that "all backflow states must be expressible in this form (Eq. 69) and satisfy the conditions" [8].

5.5 Decoherent Histories Approach for Backflow

According to Halliwell and Yearsley, the equations

$$\begin{aligned} p\left(t_1, t_2\right) &= \int_{t_1}^{t_2} \mathrm{d}t\, J(t) \\ &= \langle\psi|C|\psi\rangle \end{aligned}$$

where

$$\begin{aligned} C &= \int_{t_1}^{t_2} \mathrm{d}t \frac{(-1)}{2m}\left(\hat{p}\delta\left(\hat{x}\right) + \delta\left(\hat{x}\right)\hat{p}\right) \\ &= \theta\left(\hat{x}(t_1)\right) - \theta\left(\hat{x}(t_2)\right) \end{aligned}$$

seen previously in Section 4.2, are found to not hold in situations which exhibit backflow [9].

In a case without interference, it can be found that

$$\langle C\rangle = \langle C^2\rangle \geq 0\,, \tag{70}$$

however, "when there is backflow $\langle C\rangle < 0$", indicating that "backflow is strongly linked to interference effects" [9].

5.6 Classical Limit of Backflow

The naive classical limit of taking $\hbar \to 0$ does not work in the case of backflow since, as we saw above in Eq. 67, the eigenvalues of the flux operator are independent of $\hbar$ [33]. This would suggest that backflow should be observable at all scales, and not be restricted to the quantum realm. However, it is clearly a quantum phenomenon, and thus an explanation for the classical limit needs to be found. Yearsley and Halliwell note that the independence of the eigenvalues of the flux operator is due to the impossibility of constructing a dimensionless number from the parameters in the problem: $\hbar$, m and T [33].

Taking the classical limit as meaning $\hbar \to 0$ is, as should be assumed, an oversimplification, and Yearsley et al. suggest two more realistic ways in their 2012 paper [34].

The first proposed solution makes use of the Wigner function of the particle, which, for a system as that described above, "will become positive after a short period of time". This indicates that the backflow vanishes in the approach to the classical limit since the flux will then be positive [34].

Another approach is to realise that the exact projectors are in fact an idealisation of more realistic quasi-projectors. Using such quasi-projectors seems sensible given that any measurement of the particle is inevitably imprecise due to its quantum nature. An example for a quasi-projector is given by Yearsley et al. as

$$Q = \int_0^\infty \mathrm{d}y \, \delta_\sigma(\hat{x} - y) \tag{71}$$

where $\delta_\sigma(\hat{x} - y)$ is a smoothed out δ-function

$$\delta_\sigma(\hat{x} - y) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(\hat{x} - \hat{y})^2}{2\sigma^2}\right) \tag{72}$$

which goes to the usual δ-function as $\sigma \to 0$. In that case $Q \to \theta(\hat{x})$ as expected. Replacing the projection operators P by the quasi-projection operators Q, the flux equation is recovered, however with a different current operator:

$$\hat{J} = \frac{1}{2m} \left(\hat{p}\delta_\sigma(\hat{x}) + \delta_\sigma(\hat{x})\hat{p}\right) . \tag{73}$$

From this it can be shown that the flux can be rewritten as

$$F(-T/2, T/2) = \frac{1}{\pi} \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \, \phi * (u) \frac{\sin\left(u^2 - v^2\right)}{(u - v)} \mathrm{e}^{-a^2(u-v)^2} \phi(v) \tag{74}$$

where a is a dimensionless number given by $a^2 = 2m\sigma^2/\hbar T$. This results in the eigenvalues now depending upon a, which means that they are no longer independent of $\hbar$. Thus, in the limit $\hbar \to 0$ becomes more meaningful, indicating the regime where $a \gg 1$ or, equivalently, $\hbar \ll 2m\sigma^2/T$ [34].

6 Detection Methods, Applications, Experiments

6.1 Possible Detection Methods of Dwell Time

6.1.1 Using Flouresence

One possible detection method for dwell time makes use of the bimodal dwell-time distribution. It was proposed by Damborenea et al. and uses flourescence as a proxy for the dwell-time distribution it aims to measure [4].

In the proposed detector the region of interest is illumined by laser which is resonant with an internal transition property of the particle. This region is perpendicular to the particle's initial direction of motion, and any motion which is transversal to the particle's classical path is minimised. In order to measure the dwell-time distribution, the number of resonance photons is measured in each run of the experiment. A single photon emission per atom is needed to give an accurate distribution, which is achieved when the average time the particle spends in the region of interest is much less than the delay. The distribution of flourescence photons is expected to give a proxy for the dwell-time distribution and should be bimodal. However, due to reflection and detection delays due to the laser, this set-up is unlikely to produce bimodality in practice as the total signal and the signal discrimination would be too poor to adequately observe it. This is because the bimodality of dwell-time can only be observed if the characteristic interval between modes is greater than the characteristic interval between successive emissions of flourescence photons - but in order to avoid reflection, the particle's energy has to be much larger than the characteristic Rabi energy. In the procedure proposed by Damborenea et al. both of these conditions cannot be met simultaneously [4].

6.1.2 Using Clocks

Another method for measuring traversal time is to use both direct and indirect measurements using Larmor clocks discussed in Chapter 2, as proposed by Sokolovski and Connor in 1993 [25].

In this method, which was described by Baz' and later also by other authors, the spin of the particle is initially aligned with the x-axis, while the z-axis component is assumed to be arbitrarily small. This set-up leads to an indirect measurement of traversal time and produces a complex valued average time.

Experiments related to Clocks (Dwell Time) Landauer mentions several experiments related to measurements of the dwell time: he indicates that in (Gueret et al., 1988a, 1988b) the tunnelling current through a long low hetero-junction is measured, while an experiment mentioned in (Steinberg et al., 1993) observed the delay of a photon passing through an evanescent wave region with the effective velocity in the middle of the stop band exceeded the velocity of light in vacuum [13].

6.1.3 Conclusion - Dwell Time Detection

It is hoped that both of the methods for dwell time detection introduced above should be experimentally realisable within the near future. The experiments already undertaken on related questions, such as those mentioned by Landauer, give further hope that a measurement of the tunnelling time might take place soon.

6.2 Possible Detection Methods of Arrival Time

6.2.1 Atom-Laser Model

To measure arrival time, the atom-laser model was proposed by Ruschhaupt in 2009 [21].

In this model the region of interest is illumined using a laser, and the first photon emitted by the atom is taken to indicate the atom's arrival time. The laser beam can be modelled as having a sharp boundary at the origin and a semi-infinite field expansion, such as $\Omega(x) = \Theta(x)\Omega$ where $\Theta(x)$ is a step function. Given that both the pumping of the atom to an excited state and the emission of a photon take time, this method does not produce an "ideal" time of arrival. However, it can be assumed that the measured time of arrival distribution $\Pi(t)$ is a convolution of the hypothetical ideal distribution (Π_{id}) with detection probability density $W(t)$, so that $\Pi = \Pi_{id} * W$, where the delay in Π is contained in W. The hypothetical ideal distribution can then be found by making use of this convolution. It should also be noted that as the decay rate approaches infinity, i.e. as the atom's lifetime becomes very short, the ideal arrival time distribution approaches the current density $\Pi_{id}(t) \to J(0,t)$.

Also, it is not possible to increase the laser intensity too much , as a high intensity would cause the atom to be reflected, and would thus not be useful to measure its arrival.

6.2.2 Absorption, Sweeping

Allcock made an early attempt to incorporate the apparatus explicitly in the series of papers published in the late 1960's by introducing a complex localised potential which absorbs the wave packet [21]. He argues that the incident amplitude wavefunction has already been distorted by the apparatus and is thus different from what it would have been in the absence of the measuring device.

The method proposed by Allcock consists in sweeping the region of interest with photographic plates or scintillation counters [1]. Thus, if a particle enters the arrival zone between the n^{th} and $(n+1)^{th}$ sweep, it will be detected on the $(n+1)^{th}$ plate provided the motion of the plates is sufficiently rapid to create ionisation tracks. From this it could be inferred that the arrival must have taken place between the n^{th} and $(n+1)^{th}$ sweep, i.e. the arrival time distribution is considered as being equivalent to the absorption rate. In principle this arrangement is able to measure classical arrival time to arbitrarily high accuracy, while a failure to do so with quantum events might indicate that the classical concept of arrival time is inadequate in the quantum realm. Such a system requires a "big Schrödinger equation" ([1]) in which both the incident particle as well as the internal state of the photographic plates are considered. After a measurement has been made, the system can be readjusted, which involves the rejection of some measurement channels. This is appropriate not only when a measurement has taken place, but also when only a " partially completed measurement is contemplated", according to Allcock [1].

Problems The process described by Allcock is, however, not without shortcomings. Three main objections are raised by Allcock himself: in [2] he argues that the apparatus in fact "destroys the arrival phenomenon it has been designed to detect". This is independent of the particular choice of detector and shows the incompatibility of gleaning precise information regarding the arrival time as well as guaranteeing detection of an incoming particle. Also, in order to distinguish between different arrival times, any detector must have a rapid probability transfer. Otherwise the probability wavelets from different times will spread out and overlap, making a clear arrival time detection impossible [1]. Another disadvantage is that in determining the measurement of a wave packet arriving at a certain point, both the retrospective measurement of its velocity as well as its position are simultaneously required in order to infer when exactly the wave packet arrived. However, since both of these quantities are subject to uncertainty relations and cannot both be determined at the same time, it again leads to the impossibility of determining an exact arrival time [1].

6.2.3 Detector Model due to Halliwell

The detector model introduced by Halliwell in his 1999 paper describes a detector coupled to a particle in the $x < 0$ region which "undergoes a transition when the coupling is switched on" [7]. An example might be a particle coupled with a two-level system where upon detection of the particle, the system goes from one level to the other.

However, this detector model also poses some problems.

Problem of Reversibility A lot of detector models use unitary quantum mechanics which contain the possibility of a reverse transition. This could mean that, even after having detected the particle, the system might revert back to its original (undetected) state through self-interaction, which would lead to an unreliable measurement. Realistic detectors must thus have a large number of degrees of freedom making them effectively irreversible [7].

Other Problems According to Allcock, two other uncertainties arise when attempting to observe arrival time, namely

1. the probabilities which are output by the detector are inconsistent with the treatment of the detector as merely processing "an incoming and apparatus-independent arrival probability distribution". Deviations are introduced by the detector interacting with the arrival probability distribution, thus resulting in a less than ideal output [2]; and

2. "Neither the beginning, the duration, nor the end of a time interval can be specified with complete precision if one wishes to use the measured output to infer an input probability falling within the interval" (Allcock 3).

Also, if the time interval of interest is defined both sharply and to be rather short, the input arrival probability becomes correspondingly uncertain [2]. This is, like many problems encountered in the realm of measuring time in quantum mechanics, due to the complementarity between pairs of dynamical variables.

6.2.4 Conclusion - Arrival Time Detection

There appear to be many problems associated with the physical measurement of an arrival time distribution, and some authors, such as Allcock, even question the possibility of such a measurement. While it appears that an accurate determination of a particle's arrival time might be beyond the scope of physics, due to the fact that the measurement itself takes time (such as the pumping and emission in the first example above), there is hope in realising that ever advancing technology might be able to minimise the time added by the apparatus itself thus coming closer to an ideal measurement.

6.3 Possible Detection Methods for Backflow

As of 2013 there is currently no programme to carry out experiments to determine backflow [19]. Palmero et al. suggest that this might be due to difficulties including "the measurement of current density, and the preparation of states with a detectable amount of backflow" [19].

Despite the lack of research currently being undertaken in this area, a possible detection method has been proposed by Palmero et al. which is presented in a condensed form here.

6.3.1 Using Bose-Einstein Condensates in Harmonic Traps

The detector described by Palmero et al. makes use of Bose-Einstein condensates in harmonic traps, a procedure which should be realisable with technologies currently available. In this set up a state with positive momentum current is prepared. Giving this state a "positive momentum kick" may then lead to a negative current and result in a measurement of backflow. Bose-Einstein condensates (BECs) make ideal objects for this experimental setup as they not only offer a high degree of control and manipulation, but the their very nature means that they are "quantum matter waves" where the "probability density and flux" are a "density and flux of particles", as opposed to a statistical ensemble of single particles [19].

Set up The experiment is set up as a one-dimensional BEC with a narrow positive momentum distribution centred on $\hbar k_1$. A Bragg pulse then transfers momentum $\hbar q > 0$ to part of the atoms, resulting in a state with momentum $\hbar k_2 = \hbar k_1 + \hbar q$. This pulse results in the total wave function of the two momentum states being given by

$$\Psi(x,t) = \psi(x,t)\left(A_1 + A_2 \exp\left[\mathrm{i}qx + \mathrm{i}\varphi\right]\right), \tag{75}$$

where A_1 and A_2 are the positive real amplitudes (with $A_1^2 + A_2^2 = 1$), and φ is an arbitrary phase (which is irrelevant in this set up). All of the parameters (apart from the phase) can be controlled and measured, and $\psi(x,t)$ can be rewritten as $\psi(x,t) = \phi(x,t)\exp\left[\mathrm{i}\theta(\mathrm{x,t})\right]$.

Following some manipulation of the total current density $J_\Psi(x,t) = {}^{\hbar}/_{m}\, \mathrm{Im}\left[\Psi^* \nabla \Psi\right]$, it can be shown that backflow occurs when the density is below a critical value:

$$\rho_\Psi^{crit}(x,t) = \frac{q}{q + 2\nabla\theta(x,t)} |\phi(x,t)|^2 (A_1^2 - A_2^2) \tag{76}$$

[19].

State Preparation The condensate is prepared in a three-dimensional harmonic trap with a tight radial confinement. Initially the BEC is in the ground state of the trap, before the trap is suddenly shifted spatially by a distance d, causing the condensate to perform dipole oscillations. Once the condensate has reached a desired momentum, here $mv_1 = \hbar k_1 = \hbar k(t_1)$ at $t = t_1$, the trap is switched off, leaving the condensate to expand freely. Two possible scenarios can follow: the non-interacting case, or the Thomas-Fermi limit. In both cases $\psi(x,t)$ can be expressed as

$$\frac{1}{\sqrt{b(t)}} \psi_0 \left(\frac{x - v_1 t}{b(t)}\right) \exp\left[\mathrm{i}\frac{m}{2\hbar} x^2 \frac{\dot{b}(t)}{b(t)} + \mathrm{i}k_1 x \left(1 - \frac{\dot{b}}{b} t\right) + \mathrm{i}\beta(t)\right] \tag{77}$$

where $b(t)$ is the scaling parameter, and $\beta(t)$ is an irrelevant global phase. Here, both spatial and time coordinates have been redefined so that at $t = 0$ the condensate is centred at the origin. A Bragg pulse of very short duration with respect to the other timescales involved is applied to the system, resulting in the wave function being described by Eq. 75, while the corresponding critical density for backflow is again given by Eq. 76. The backflow can subsequently be probed by taking a snapshot of the interference pattern shortly after the Bragg pulse, measuring precisely its minimum, and comparing it to Eq. 76 [19].

Quantum Backflow The flux from the current density is found to be proportional to that of the superposition of two plane waves with momenta k_1 and $k_2 = k_1 + q$. This is useful as the probability density for two plane waves is a sinusoidal function, making the critical density a constant. To maximise the backflow effect and its detection, several constraints have to be satisfied: the larger the ratio between the momentum transferred through the Bragg pulse and the initial momentum (i.e. the larger $\alpha = {}^{q}/_{k_1}$), the larger the backflow effect. However, the momentum due to the pulse cannot be arbitrarily large because the wavelength of the density modulations is given by $\lambda = 2\pi/q$ and must be larger than the experimental spatial resolution σ_r, i.e. $\lambda \gg \sigma_r$. Otherwise an experimental detection of backflow might be hindered [19].

6.3.2 Conclusion - Backflow Detection

It is hoped that the experimental set up as described above will soon be realisable. It will be of great interest to investigate both whether backflow can indeed be measured in this way, and furthermore to understand more about the computationally found maximum backflow given by $c_{bm} \approx 0.04$.

7 Conclusion

Questions regarding times and durations of quantum processes have a long history, with a marked increase in interest in the topic over the past twenty or so years. The advancement of technology, bringing with it the ability to manipulate individual atoms, the use of tunnelling processes in microscopes, or the - albeit still futuristic - notion of quantum computers, all make it necessary to acquire a better grasp the measurement of time in quantum processes.

In the present paper we have looked in detail at the problem of determining the time a particle spends tunnelling through a barrier. Unlike in classical physics, there does not appear to be a clear cut solution to this, and even the seemingly superluminal propagation of the particles is being doubted by some authors (such as Winful [31]).

We have explored different approaches to the dwell time, or tunnelling time, problem: in the dwell time approach, an operator is sought which will give a value of the dwell time. This operator is found to be degenerate, leading to two distinct eigenvalues which the dwell time can take on. It is also found that, in contrast to the classical realm, there is a maximum dwell time which a constrained particle can spend inside the barrier.

Another approach involved the introduction of clocks by using the system's degrees of freedom. In the Larmor approach, a change in the particle's spin orientation may indicate how long it was exposed to a potential associated with the barrier, while in the Büttiker-Landauer model the tunnelling barrier is oscillating, allowing the duration of the particle's interaction with the barrier to be found by observing at which frequency a deviation from the adiabatic approximation occurs. It should be noted that while these models give a duration to the tunnelling process, they also constitute invasive processes which may affect experimental outcomes.

A third way of finding the tunnelling time is given by the kinematical path approaches due to Feynman and Bohm. Despite giving ways of calculating the dwell time by restricting the path a particle takes, it has to be noted that there are pitfalls associated with this method: a path which is too restricted will lead to the Zeno effect, effectively making the barrier reflective and not allowing the particle to enter the classically forbidden region. This effect, which arises from continuous or near-continuous measurement, was explored in greater detail in Chapter 3.

Subsequently two further topics in quantum mechanics related to time measurements were discussed: the question of arrival time, and the - rather counterintuitive - phenomenon of the backflow effect. The former seeks to find the time at which a particle crosses a particular point in space. Here, too, the Zeno effect becomes relevant when the measurements occur to frequently, and a balance has to be found between observing the particle in such a way as to make it reflect from the region of interest, while also making sure that it is detected upon entry.

The phenomenon of quantum backflow, discussed in Chapter 5, has only recently begun to be studied. It describes how a wave function with entirely positive momentum, can exhibit negative probability flow. It was found that there is a maximum backflow, given as approximately four percent. This maximum backflow is, curiously, not dependent on $\hbar$, making it difficult to find a naive classical limit to the phenomenon. However, upon realising that the observations of the probability distributions are idealised cases, and that quasi-projectors would be a more suitable way of modelling the observations, a dependence on $\hbar$ is recovered and a classical limit found.

In the preceding chapter, we briefly introduced several proposed methods in which these phenomena could be studied. However, to date there do not seem to be any experiments which can satisfactorily give measurements of such processes as traversal time, tunnelling time, arrival time or quantum backflow. It hoped that at least some of the proposed experiments can soon be realised.

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