IMPERIAL COLLEGE LONDON

Department of Theoretical Physics

---

# Higher Derivative Theories of Gravity

---

Simon Arthur Woolliams

September 20, 2013

# Contents

# 1 Introduction

It is well known that any field theory can be described using a variational principle. In particular, Einstein's general theory of relativity can be described by promoting the metric tensor $g_{\mu\nu}$ to a dynamical field variable and using it to characterise the gravitational field. The action used in this variational formulation of the theory is traditionally the Einstein-Hilbert action, which is second order in its derivatives of the metric. However, the Einstein-Hilbert action is the simplest such action that can replicate the Einstein field equations that govern general relativity and other, more complicated actions involving higher order derivatives might be possible.

These 'higher derivative' theories are usually dismissed because of the fact that there are serious constraints on their physical viability. In particular, they are frequently subject to Ostrogradsky's Instability Theorem, a result which classifies all nondegenerate higher derivative theories as unstable (in the Lyapunov sense). This is a major affliction because such unstable theories usually possess negative energy states of the sort that are strenuously banished from quantum field theories. In spite of this, however, there has been a recent flurry of interest in higher derivative theories because they are renormalisable, meaning that their study might provide vital clues as to how to quantize gravity.

This project is divided into several parts. In the initial chapter, the Einstein field equations are reviewed from a variational perspective, thus demonstrating the viability of the Einstein-Hilbert action. Objections to the addition of higher derivatives to the action are then discussed at a classical level: these include the derivation of Ostrogradsky's Instability Theorem and the introduction of negative energy modes in linearised gravitational perturbations. However, the project will then show that adding higher derivatives to the Einstein-Hilbert action promotes gravity to a renormalisable theory and concludes by discussing recent efforts to develop a physically acceptable higher derivative theory of gravity using asymptotic safety.

# 2 Einstein-Hilbert Action

## 2.1 Variational Formalism

To derive the equation that yields the Einstein field equations as its equations of motion, we initially consider the form that the action must take, namely the integral of a scalar Lagrange density. Since derivatives lower the order on the field upon which it acts by one, this Lagrange density should contain at least two derivatives of the metric to ensure that the equation of motion for the metric field - which is what we are ultimately interested in when trying to find the dynamics of spacetime curvature - is at least linear. [I]Since any non-trivial tensor made from the metric and its derivatives can be expressed in terms of the metric and the Riemann tensor [31], the only independent scalar that can be constructed from the metric that is no higher than second order in its derivatives is the Ricci scalar (as this is the unique scalar that we can construct from the Riemann tensor that is itself made from second derivatives of the metric). This led Hilbert to suggest that the simplest form of the action for gravity must be:

$$S = \int d^4x \sqrt{-g} R$$

Here, the integral is over curved, rather than flat, spacetime, hence the factor of $\sqrt{-g}$ where $g = \det g_{\mu\nu}$ is the determinant of the metric tensor.

We can infintesimally vary this action by treating the metric tensor $g^{\mu\nu}$ as the dynamical field variable:

$$\delta S = \int d^4x [\delta(\sqrt{-g})R + \sqrt{-g}\delta R] = \int d^4x \sqrt{-g} \left[ \frac{1}{\sqrt{-g}} \left( \frac{1}{2} \frac{1}{\sqrt{-g}} (-\delta g) \right) R + \frac{\delta R}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \right]$$

We proceed by using **Jacobi's Rule** for evaluating the derivative of determinants:

$$\delta g \equiv \delta(\det g) = g g^{\mu\nu} \delta g_{\mu\nu} \tag{1}$$

---

[I]Else it would be constant and therefore not very dynamical!

Suppose that we now want to express the variation of the covariant metric tensor in terms of the variation of the contravariant metric tensor. To this end, we consider rewriting it in the equation:

$$g^{\mu\nu}\delta g_{\mu\nu} = g^{\mu\nu}\delta\left(g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}\right) = g^{\mu\nu}\left(\delta g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma} + g_{\mu\rho}\delta g_{\nu\sigma}g^{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}\right)$$

Multiplying out the brackets and using $g^{\mu\nu}g_{\mu\rho} = \delta_\rho^\nu$ on the first and second terms on the R.H.S.:

$$g^{\mu\nu}\delta g_{\mu\nu} = g^{\mu\nu}\delta_\nu^\rho\delta g_{\mu\rho} + g^{\mu\nu}\delta_\mu^\sigma\delta g_{\nu\sigma} + g^{\mu\nu}g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} = g^{\mu\rho}\delta g_{\mu\rho} + g^{\sigma\nu}\delta g_{\nu\sigma} + g^{\mu\nu}g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$$

We can rearrange the terms using dummy indices and the symmetry of $g_{\mu\nu}$ to get:

$$g^{\mu\nu}g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} = g^{\mu\nu}\delta g_{\mu\nu} - g^{\mu\nu}\delta g_{\mu\nu} - g^{\mu\nu}\delta g_{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$$

Since the indices on the R.H.S. are already summed, we cannot simply cancel the $g^{\mu\nu}$ terms by simply multiplying both sides by $g_{\mu\nu}$ (as this would be repeating dummy indices more than twice), but we can equate its coefficients instead:

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} \tag{2}$$

In a similar way, we can also derive:

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$$

This means that Jacobi's Rule becomes:

$$\delta g \equiv \delta(\det g) = gg^{\mu\nu}\delta g_{\mu\nu} = -gg^{\mu\nu}g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} = -g\delta_\rho^\nu g_{\nu\sigma}\delta g^{\rho\sigma} = -gg_{\rho\sigma}\delta g^{\rho\sigma}$$

This can be written more compactly using dummy indices as:

$$\delta g \equiv \delta(\det g) = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu} \tag{3}$$

Substituting (3) into our previous expression:

$$\delta S = \int d^4x\sqrt{-g}\left[\frac{-R}{-2g}(gg^{\mu\nu}\delta g_{\mu\nu}) + \frac{\delta R}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right] = \int d^4x\sqrt{-g}\left[\frac{R}{2}g^{\mu\nu}(-g_{\mu\alpha}\delta g^{\alpha\beta}g_{\beta\nu}) + \frac{\delta R}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right]$$

$$= \int d^4x\sqrt{-g}\left[-\frac{R}{2}\delta_\mu^\mu\delta_\alpha^\nu\delta g^{\alpha\beta}g_{\beta\nu} + \frac{\delta R}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right] = \int d^4x\sqrt{-g}\left[-\frac{R}{2}\delta g^{\nu\beta}g_{\beta\nu} + \frac{\delta R}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right]$$

After rewriting the first term using dummy indices and the fact that the metric tensor is symmetric, we have:

$$\delta S = \int d^4x\sqrt{-g}\left[-\frac{R}{2}g_{\mu\nu} + \frac{\delta R}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu} \overset{!}{=} 0 \text{ by the Principle of Least Action}$$

Because an integral always vanishing can only be achieved by the vanishing of the integrand, we conclude that the equation of motion for the gravitational field is:

$$-\frac{R}{2}g_{\mu\nu} + \frac{\delta R}{\delta g^{\mu\nu}} = 0 \tag{4}$$

Let us now examine the second term of (4) further. It involves the derivative of the Ricci scalar, which we find by contracting the Ricci tensor, itself a contraction of the **Riemann tensor** which is given by:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{5}$$

Taking the variation of this Riemann tensor yields:

$$\delta R^\rho_{\sigma\mu\nu} = \partial_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \partial_\nu(\delta\Gamma^\rho_{\mu\sigma}) + \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma} \tag{6}$$

Now, $\delta\Gamma^\alpha_{\beta\gamma}$ is the infintesimal variation of two metric connections and while the metric connection itself does not constitute a tensor, the difference of two metric connections does.[19] Because of its tensorial transformation properties, there will be a covariant derivative associated with this variation. Noting that the covariant derivatives of a covariant (0,1) field and a (0,1) contravariant field are, respectively:

$$\nabla_\mu \phi_\nu = \partial_\mu \phi_\nu - \Gamma^\rho_{\mu\nu}\phi_\rho \qquad \nabla_\mu \phi^\nu = \partial_\mu \phi^\nu + \Gamma^\nu_{\mu\rho}\phi_\rho$$

we have the expression for a mixed (1,1) field:

$$\nabla_\mu \phi^\rho_\nu = \partial_\mu \phi^\rho_\nu - \Gamma^\alpha_{\mu\nu}\phi^\rho_\alpha + \Gamma^\rho_{\beta\mu}\phi^\beta_\nu$$

The prescription is clear: to the ubiquitous partial derivative, we add an additional Christoffel symbol for each upper index and subtract one for each lower index; working from left to right across the indices, we shift the index encountered onto the field multiplying the Christoffel symbol and replace it with a dummy index. Doing this for the (1,2) tensor field $\delta\Gamma^\rho_{\mu\nu}$:

$$\nabla_\sigma(\delta\Gamma^\rho_{\mu\nu}) = \partial_\sigma(\delta\Gamma^\rho_{\mu\nu}) + \Gamma^\rho_{\alpha\sigma}\delta\Gamma^\alpha_{\mu\nu} - \Gamma^\beta_{\sigma\mu}\delta\Gamma^\rho_{\beta\nu} - \Gamma^\rho_{\sigma\nu}\delta\Gamma^\rho_{\mu\gamma} \tag{7}$$

We can see that the structure of (6) roughly matches the structure of the difference of two of the above terms, albeit with two less Christoffel symbols. Let us then match the indices of the partial derivative term of (7) with the first two terms of (6):[II]

$$\nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) = \partial_\mu(\delta\Gamma^\rho_{\nu\sigma}) + \Gamma^\rho_{\alpha\mu}\delta\Gamma^\alpha_{\nu\sigma} - \Gamma^\beta_{\mu\nu}\delta\Gamma^\rho_{\beta\sigma} - \Gamma^\gamma_{\mu\sigma}\delta\Gamma^\rho_{\gamma\nu}$$

$$\nabla_\nu(\delta\Gamma^\rho_{\mu\sigma}) = \partial_\nu(\delta\Gamma^\rho_{\mu\sigma}) + \Gamma^\rho_{\delta\nu}\delta\Gamma^\delta_{\mu\sigma} - \Gamma^\epsilon_{\nu\mu}\delta\Gamma^\rho_{\epsilon\sigma} - \Gamma^\phi_{\nu\sigma}\delta\Gamma^\rho_{\phi\mu}$$

Subtracting these:

$$\nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\sigma}) = \partial_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \partial_\nu(\delta\Gamma^\rho_{\mu\sigma}) + \Gamma^\phi_{\nu\sigma}\delta\Gamma^\rho_{\phi\mu} + \Gamma^\rho_{\alpha\mu}\delta\Gamma^\alpha_{\nu\sigma} - \Gamma^\gamma_{\mu\sigma}\delta\Gamma^\rho_{\gamma\nu} - \Gamma^\rho_{\delta\nu}\delta\Gamma^\delta_{\mu\sigma}$$
$$- \Gamma^\beta_{\mu\nu}\delta\Gamma^\rho_{\beta\sigma} + \Gamma^\epsilon_{\nu\mu}\delta\Gamma^\rho_{\epsilon\sigma}$$

As hoped for, the terms on the second line cancel by dummy indices because, at least for applications concerning General Relativity, the Christoffel symbol is summetric on its lower two indices. Comparing

---

[II]These would have to match in order for this strategy to work in any case!

this with (6), we see that they are identical aside from some differing dummy indices and inconsequential reorderings of Christoffel symbols (the above has been arranged so that its nth term corresponds to the nth term of (6)). Hence, we can write:

$$\delta R^{\rho}_{\sigma\mu\nu} = \nabla_{\mu}(\delta\Gamma^{\rho}_{\nu\sigma}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\sigma})$$

The **Ricci tensor** is formed by contracting the first and third indices on the Riemann tensor:

$$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu} \Leftrightarrow g^{\alpha\rho}R_{\alpha\mu\rho\nu} \tag{8}$$

This means that the variation of the Ricci tensor is given by:

$$\delta R^{\mu}_{\sigma\mu\nu} \equiv \delta R_{\sigma\nu} = \nabla_{\mu}(\delta\Gamma^{\mu}_{\nu\sigma}) - \nabla_{\nu}(\delta\Gamma^{\mu}_{\mu\sigma}) \tag{9}$$

This result is sometimes referred to as the **Palatini Identity**. For future applications, we note here that it can be recast in the form[27]:

$$\delta R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}[\nabla_{\alpha}\nabla_{\nu}\delta g_{\mu\beta} + \nabla_{\alpha}\nabla_{\mu}\delta g_{\nu\beta} - \nabla_{\mu}\nabla_{\nu}\delta g_{\alpha\beta} - \nabla_{\alpha}\nabla_{\beta}\delta g_{\mu\nu}] \tag{10}$$

The **Ricci scalar** is formed by contracting the two indices of the Ricci tensor using :

$$R \equiv R^{\mu}_{\mu} \Leftrightarrow g^{\mu\nu}R_{\mu\nu} \tag{11}$$

So, using (9), its variation will be given by:

$$\delta R = \delta(g^{\sigma\nu}R_{\sigma\nu}) = \delta g^{\sigma\nu}R_{\sigma\nu} + g^{\sigma\nu}\delta R_{\sigma\nu} = \delta g^{\sigma\nu}R_{\sigma\nu} + g^{\sigma\nu}[\nabla_{\mu}(\delta\Gamma^{\mu}_{\nu\sigma}) - \nabla_{\nu}(\delta\Gamma^{\mu}_{\mu\sigma})]$$

To proceed, we note that (by the product rule):

$$\nabla_{\mu}(g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma}) = \nabla_{\mu}(g^{\sigma\nu})\delta\Gamma^{\mu}_{\nu\sigma} + g^{\sigma\nu}\nabla_{\mu}(\delta\Gamma^{\mu}_{\nu\sigma}) = g^{\sigma\nu}\nabla_{\mu}(\delta\Gamma^{\mu}_{\nu\sigma})$$

$$\nabla_{\nu}(g^{\sigma\nu}\delta\Gamma^{\mu}_{\mu\sigma}) = \nabla_{\nu}(g^{\sigma\nu})\delta\Gamma^{\mu}_{\mu\sigma} + g^{\sigma\nu}\nabla_{\nu}(\delta\Gamma^{\mu}_{\mu\sigma}) = g^{\sigma\nu}\nabla_{\nu}(\delta\Gamma^{\mu}_{\mu\sigma})$$

where the first term vanishes in both cases because, by definition of the Levi-Civita connection, the covariant derivative of the metric vanishes (a result known as **metric compatability**)[III]:

$$\nabla_{\mu}g^{\nu\rho} \equiv 0 \tag{12}$$

This means that we can rewrite the above in terms of total derivatives:

$$\delta R = \delta g^{\sigma\nu}R_{\sigma\nu} + \nabla_{\mu}(g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma}) - \nabla_{\nu}(g^{\sigma\nu}\delta\Gamma^{\mu}_{\mu\sigma}) = \delta g^{\sigma\nu}R_{\sigma\nu} + \nabla_{\mu}(g^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\sigma} - g^{\sigma\mu}\delta\Gamma^{\nu}_{\nu\sigma})$$

---

[III]The general result that the covariant derivative of any second order, covariant tensor vanishes provided that the connection used in the covariant derivative is the Christoffel symbols of the 2nd kind and the tensor is both symmetric and positive definite is known as **Ricci's Lemma**.[25]

Although the second and third terms form a total derivative and so will not contribute to the equations of motion of the gravitational field $g^{\mu\nu}$, we would be overly presumptious to discard it immediately because, as it turns out, they can make a contribution to the action provided the the first derivative of the metric is allowed to vary. Technically, this arises from the fact that the total derivative over the spacetime region can be transformed into an integral over the boundary of this spacetime region using the generalised form of Stokes' Law; normally, it is enough to assume that $\delta g_{\mu\nu} = 0$ on the boundary, but this particular integral is non-vanishing unless $\delta \nabla_\rho g_{\mu\nu} = 0$ also, However, this unwanted boundary term can be removed by a trivial redefinition of $S_{EH}$ and so we do not concern outselves with it for the remainder of this project.[IV] As a result, we neglect the total derivative, yielding:

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} \implies \frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{13}$$

Substituting this result back into (4), we define the **Einstein tensor**:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 0$$

This is the Einstein equation for a vacuum with a vanishing cosmological constant. It can be modified in several ways:

1. The inclusion of a matter field in the action:

$$S = \int d^4x \sqrt{-g}(R + L_M)$$

gives rise to the field equation of motion:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\ nu} = -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}}$$

In analogy with the conventional theory of general relativity, we know by Noether's Theorem that there must be a conserved current, so we define the **Hilbert stress-energy tensor**:

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \tag{14}$$

Of course, for Noether's Theorem to apply, we need a symmetry and in this case, the relevant isometry (symmetry of the metric) is diffeomorphism invariance.[3]

2. We can also add an overall multiplicative constant:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R$$

---

[IV]See Appendix E.1 of [26] for further details.

where the form of the constant is purely conventional. The value of the constant is determined by the requirement that the general relativistic theory should reduce to the Newtonian theory once the effects of the gravitational field become very weak, conditions known as the **weak field limit**. The result is:

$$\kappa = \frac{8\pi G}{c^4} \tag{15}$$

3. Finally, we can add a non-zero cosmological constant:

$$S = \int d^4x \sqrt{-g}(R - 2\Lambda)$$

where the factor of 2 is conventional (and can be reabsorbed into the definition of $\Lambda$). The resulting field equation would be:

$$R_{\mu\nu} - \frac{R}{2} + \Lambda g_{\mu\nu} = 0$$

This constant $\Lambda$ would be zeroth order in derivatives, which does not be itself lead to any dynamics, but the fact that it is multiplied by $\sqrt{-g}$, which does vary, means that it cannot be neglected entirely. Furthermore, if $\Lambda$ has a non-zero value, then the general relativistic theory does not exactly coincide with Newtonian theory in the weak field limit, but if $\Lambda \approx 0$, then the deviations should be negligible (see Section 5.2 of [26]). Finally, because of metric compatability, it is automatically conserved since $\nabla_\mu \Lambda \overset{!}{=} 0$ and so can be thought of as the energy-momentum tensor of the the vacuum.[3]

Because of the linearity on the underlying action, the previous results can be combined to give the **Einstein-Hilbert action**:

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R - 2\Lambda + L_M) \tag{16}$$

giving rise to the **Einstein equation**:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \implies G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{17}$$

This formula is identical to the relation found heuristically by Einstein himself based on considering the covariant ways that one could couple mass and energy (c.f. $T_{\mu\nu}$) to curvature (c.f. $G_{\mu\nu}$). In fact, (17) is the most general form of the equation of motion that a second-order derivative theory can produce: it can be shown[11] that $g_{\mu\nu}$ and $G_{\mu\nu}$ are the only two-index tensors that are both symmetric and divergence-free (both of which are properties of $T_{\mu\nu}$ on the R.H.S.) that can be built from the metric tensor and its first two derivatives in 4 dimensions, so the most general expression satisfying these criteria must be a linear sum of these. It is worth noting, however, that this result, known as **Lovelock's Theorem**[12], that the independent (0,2) tensors that satisfy the aforementioned criteria depend on the number of dimensions of the theory, so we should not necessarily expect the above conclusion to hold in the higher dimensional theories that we consider later in the project.

Of course, (16) is the simplest action that could have produced a dynamical gravitational field in that we insisted that it be second order in derivatives of the metric. That said, (17) has been tested extensively in numerous settings, so any prospective alternate action would have to reproduce (17); the easiest way to ensure this is simply to keep the Einstein-Hilbert action as it stands and add on additional derivative terms, expecting the extra terms to only cause small, possibly high energy corrections to the experimental predictions in a similar fashion to the addition of the non-zero cosmological constant. However, while most introductory General Relativity textbooks cover Einstein's heuristic derivation of the field equations, and sometimes also the Einstein-Hilbert formalism, none mention higher derivative theories of gravity in any great detail. The reason for this is because there are several flaws and complications in higher-derivative theories that are not present in the lower-derivative counterpart, the details of which comprise the next chapter.

# 3 Problems With Classical Higher Derivative Gravity

## 3.1 Introduction

In the Einstein-Hilbert action, we explicitly considered all possible second-order derivative terms and came to the conclusion that only the Ricci and cosmological constants could act as suitable terms in the action. We can perform a similar analysis on the fourth order derivative terms, and it turns out that, again, there is only a limited number of scalars that we can construct:

1. $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

2. $R_{\mu\nu}R^{\mu\nu}$

3. $R^2 \Leftrightarrow R^\mu_\mu R^\nu_\nu$

Furthermore, we can eliminate one of these terms by expressing it in terms of the other two. We do this by exploiting the **Gauss-Bonnet Theorem**, which states that for a compact, two-dimensional Riemannian manifold M with boundary $\delta M$, then the Euler charateristic of M, $\chi(M)$, is given by:

$$\int_M K dA + \int_{\delta M} k_g ds = 2\pi \chi(M)$$

In the above formula, $K$ is the Gaussian curvature and $k_g$ is the geodesic curvature. Intuitively, this relation is responsible for the sum of the interior angles $\theta_i$ of a triangle being 180 degrees in Euclidean space, since a corollary of this result is:

$$\int_\Delta K dA = \pi - (\theta_1 + \theta_2 + \theta_3)$$

Since Euclidean space is flat, the curvature $K = 0$ and so the L.H.S. vanishes and $\sum_{i=1}^3 \theta_i = \pi$. This result is likewise responsible for the fact that in spherical trigonometry, the sum of interior angles exceeds $\pi$ since the curvature of a 3-sphere of radius r, $K = \frac{1}{r^2}$, is always positive.

However, although the Gauss-Bonnet Theorem only applies in two-dimensions, it can be generalised to give the **Chern-Gauss-Bonnet Theorem**, which holds for any compact, orientable, 2n-dimensional Riemannian manifold without boundary. In particular, for $n = 2$, we have the result:

$$\chi(M) = \frac{1}{32\pi^2} \int_M \left( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \right)$$

For our purposes, we can use the above result to write the following[24]:

$$\sqrt{-g} \left( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \right) = T \tag{18}$$

Here, T stands for a topological invariant quantity by virtue of the fact that the Euler charateristic $\chi(M)$ is likewise topologically invariant. It is a constant whose value does not concern us because we will be substituting it into the higher-derivative action and constant terms in a Lagrangian will not affect the

equations of motion. This means that we can eliminate one of the previous higher-derivative scalars; because the Riemann tensor is the most complicated of these terms, we discard that one. Since the most general action is a linear superposition of all permissible terms, our higher-derivative action is:

$$S = -\int d^4x \sqrt{-g}\left(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \gamma\kappa^{-2}R\right) \tag{19}$$

Note that the signs and forms of the dimensionless constants is a matter of convention. In the above action, we have redefined the constant $\kappa$ from satisfying (15) to instead having the value:

$$\kappa^2 = 32\pi G \tag{20}$$

We have also kept the value of $\gamma$ arbitrary for the time being so as to leave open the possiblity of higher-derivative corrections modifying the taking of the Newtonian limit.

Alternatively, we can use the **Weyl tensor**, which is defined for $d > 2$ dimensions as:[31]

$$C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} + \frac{1}{d-2}\left(g_{\mu\sigma}R_{\rho\nu} + g_{\nu\rho}R_{\sigma\mu} - g_{\mu\rho}R_{\sigma\nu} - g_{\nu\sigma}R_{\rho\mu}\right) + \frac{1}{(d-1)(d-2)}\left(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}\right)R$$

By re-expressing the Riemann tensor using the Weyl tensor, we can rewrite (19) as:[14]

$$S = \int d^4x \sqrt{g}\left(-\frac{1}{\kappa^2}R + \frac{2}{s}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - \frac{\omega}{3s}R^2\right) \Leftrightarrow \int d^4x \sqrt{g}\left(-\frac{1}{\kappa^2}R + \frac{2}{s}C^2 - \frac{\omega}{3s}R^2\right) \tag{21}$$

In what follows, we are not going to be working exclusively in 4 dimensions, in which case we need to add an additional integrand of the Gauss-Bonnet term (in 4D, it is negligible, hence its previous omission), denoted E. We can also include a cosmological constant term, $\tilde{\Lambda}$, yielding (after an absorption of a factor of 4 into the definition of s):[V]

$$S = \int d^4x \sqrt{g}\left(\tilde{\Lambda} - \frac{1}{\kappa^2}R + \frac{1}{2s}C^2 - \frac{\omega}{3s}R^2 + \frac{\theta}{s}E\right)$$

If we use the following parameterisations:

$$\Lambda = \frac{\kappa^2 \tilde{\Lambda}}{2} \qquad x \equiv \frac{1}{s}\left[\theta + \frac{1}{2}\right] \qquad y \equiv -\frac{2}{s}\left[2\theta + \frac{1}{d-2}\right] \qquad z \equiv \frac{1}{s}\left[\theta - \frac{\omega}{3} + \frac{1}{(d-1)(d-2)}\right]$$

we can rewrite the above action as:[13]

$$S = -\int d^4x \sqrt{g}\left[\frac{1}{\kappa^2}\left(2\Lambda - R\right) + xR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + yR_{\mu\nu}R^{\mu\nu} + zR^2\right] \tag{22}$$

---

[V]Note that we have used the fact that $\kappa^2 \overset{!}{=} c_d G_N$, which is derived by comparing the coefficients of R in (21) and the first line of (2.14) of [13]

## 3.2 Equations of Motion For Higher Derivative Gravity

We now proceed to find the equations of motion for the higher-derivative action. We start by taking the variation of (19):

$$\delta S = \int d^4 x \, \delta \left[ \sqrt{-g} \left( \alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + \gamma \kappa^{-2} R \right) \right]$$

$$= \int \frac{d^4 x}{2\sqrt{-g}} (-\delta g) \left( \alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + \gamma \kappa^{-2} R \right) + \int d^4 x \sqrt{-g} \left[ \alpha \left( \delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \delta R^{\mu\nu} \right) - \beta \delta R^2 + \gamma \kappa^{-2} \delta R \right]$$

Now:

$$\delta R^2 = \delta (R_\mu^\mu R_\nu^\nu) = \delta (g^{\mu\rho} R_{\mu\rho} g^{\nu\sigma} R_{\nu\sigma}) = (\delta g^{\mu\rho} R_{\mu\rho} + g^{\mu\rho} \delta R_{\mu\rho}) g^{\nu\sigma} R_{\nu\sigma} + g^{\mu\rho} R_{\mu\rho} (\delta g^{\nu\sigma} R_{\nu\sigma} + g^{\nu\sigma} \delta R_{\nu\sigma})$$

$$= (\delta g^{\mu\rho} R_{\mu\rho} + g^{\mu\rho} \delta R_{\mu\rho}) R + R(\delta g^{\nu\sigma} R_{\nu\sigma} + g^{\nu\sigma} \delta R_{\nu\sigma}) = 2R(R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}) \text{ by dummy indices}$$

It is worth noting explicitly that naive application of the chain rule, $\delta R^2 = 2R\delta R$, would only provide the first term. Similarly:

$$\delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \delta R^{\mu\nu} = \delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \delta (g^{\mu\rho} g^{\nu\sigma} R_{\rho\sigma})$$

$$= \delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \delta g^{\mu\rho} g^{\nu\sigma} R_{\rho\sigma} + R_{\mu\nu} g^{\mu\rho} \delta g^{\nu\sigma} R_{\rho\sigma} + R_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} \delta R_{\rho\sigma}$$

$$= \delta R_{\mu\nu} R^{\mu\nu} + R_{\rho\sigma} g^{\rho\mu} g^{\sigma\nu} \delta R_{\mu\nu} + R_{\mu\rho} \delta g^{\mu\nu} g^{\rho\sigma} R_{\nu\sigma} + R_{\sigma\nu} g^{\sigma\rho} \delta g^{\nu\mu} R^{\rho\mu}$$

$$= 2(R^{\mu\nu} \delta R_{\mu\nu} + g^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} \delta g^{\mu\nu}) \text{ by dummy indices}$$

We therefore rewrite the above variation as:

$$\delta S = - \int \frac{d^4 x}{2\sqrt{-g}} \frac{\sqrt{-g}}{\sqrt{-g}} \left( \alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + \gamma \kappa^{-2} R \right) \delta g$$

$$+ \int d^4 x \sqrt{-g} \left[ 2\alpha (R^{\mu\nu} \delta R_{\mu\nu} + g^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} \delta g^{\mu\nu}) \right] - 2\beta R \left[ R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right] + \gamma \kappa^{-2} \delta R$$

Grouping together terms yields:

$$\delta S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2g} \left( \alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + \gamma \kappa^{-2} R \right) \delta g + \gamma \kappa^{-2} \delta R \right]$$

$$+ 2 \int d^4 x \sqrt{-g} \left[ (\alpha R^{\mu\nu} - \beta R g^{\mu\nu}) \delta R_{\mu\nu} + (\alpha g^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} - \beta R R_{\mu\nu}) \delta g^{\mu\nu} \right]$$

We now substitute (3), (10) and (13) for the variations:

$$\delta S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2g} \left( \alpha R_{\rho\sigma} R^{\rho\sigma} - \beta R^2 + \gamma \kappa^{-2} R \right) (-g g_{\mu\nu} \delta g^{\mu\nu}) + \gamma \kappa^{-2} R_{\mu\nu} \delta g^{\mu\nu} \right]$$

$$+ 2 \int d^4 x \sqrt{-g} \left( \alpha R^{\mu\nu} - \beta R g^{\mu\nu} \right) \left( \frac{1}{2} g^{\alpha\beta} [\nabla_\alpha \nabla_\nu \delta g_{\mu\beta} + \nabla_\alpha \nabla_\mu \delta g_{\nu\beta} - \nabla_\mu \nabla_\nu \delta g_{\alpha\beta} - \nabla_\alpha \nabla_\beta \delta g_{\mu\nu}] \right)$$

$$+ 2 \int d^4 x \sqrt{-g} \left( \alpha g^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} - \beta R R_{\mu\nu} \right) \delta g^{\mu\nu}$$

After some more rearranging, we gain the intermediate result:

$$\delta S = \int d^4 x \sqrt{-g} \left[ -\frac{1}{2} \left( \alpha R_{\rho\sigma} R^{\rho\sigma} - \beta R^2 + \gamma \kappa^{-2} R \right) g_{\mu\nu} + 2 \left( \alpha g^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} - \beta R R_{\mu\nu} \right) + \gamma \kappa^{-2} R_{\mu\nu} \right] \delta g^{\mu\nu}$$

$$+ \int d^4 x \sqrt{-g} \left( \alpha R^{\mu\nu} - \beta R g^{\mu\nu} \right) \left[ \nabla^\beta \{ \nabla_\nu \delta g_{\mu\beta} + \nabla_\mu \delta g_{\nu\beta} \} - g^{\alpha\beta} \nabla_\mu \nabla_\nu \delta g_{\alpha\beta} - \nabla^\beta \nabla_\beta \delta g_{\mu\nu} \right] \tag{23}$$

The first line is in the form that we desire, so the next task is to rework the second line to match it. We do this by initially swapping dummy indices to form $\delta g_{\mu\nu}$ on each term:

$$\int d^4x \sqrt{-g} \left( \alpha R^{\mu\rho} - \beta R g^{\mu\rho} \right) \nabla^\nu \nabla_\rho \delta g_{\mu\nu} + \int d^4x \sqrt{-g} \left( \alpha R^{\rho\mu} - \beta R g^{\rho\mu} \right) \nabla^\nu \nabla_\rho \delta g_{\mu\nu}$$
$$- \int d^4x \sqrt{-g} \left( \alpha R^{\rho\sigma} - \beta R g^{\rho\sigma} \right) g^{\mu\nu} \nabla_\rho \nabla_\sigma \delta g_{\mu\nu} - \int d^4x \sqrt{-g} \left( R^{\mu\nu} - \beta R g^{\mu\nu} \right) \Box \delta g_{\mu\nu}$$

Exploiting the fact that both $R_{\mu\nu}$ and $g_{\mu\nu}$ are symmetric, we can write:

$$2 \int d^4x \sqrt{-g} \left( \alpha R^{\mu\rho} - \beta R g^{\mu\rho} \right) \nabla^\nu \nabla_\rho \delta g_{\mu\nu} - \int d^4x \sqrt{-g} \left( \alpha R^{\rho\sigma} - \beta R g^{\rho\sigma} \right) g^{\mu\nu} \nabla_\rho \nabla_\sigma \delta g_{\mu\nu}$$
$$- \int d^4x \sqrt{-g} \left( \alpha R^{\mu\nu} - \beta R g^{\mu\nu} \right) \Box \delta g_{\mu\nu}$$

We now use integration by parts to shift around the covariant derivatives (subject to the previous caveat concerning the vanishing nature of the boundary terms). Note that, although there is no overall sign change - the factor of $-1$ from each integration by parts is nullified by the fact that we perform the integration twice on each term - the order of differentiation on each term changes:

$$\int d^4x \sqrt{-g} \left[ 2\nabla_\rho \nabla^\nu \left( \alpha R^{\mu\rho} - \beta R g^{\mu\rho} \right) - g^{\mu\nu} \nabla_\sigma \nabla_\rho \left( \alpha R^{\rho\sigma} - \beta R g^{\rho\sigma} \right) - \Box \left( \alpha R^{\mu\nu} - \beta R g^{\mu\nu} \right) \right] \delta g_{\mu\nu}$$

We have invoked metric compatability (12) to move the metric tensor out of the covariant derivative on the second term.

Substitution of (2) into our previous expression yields:

$$- \int d^4x \sqrt{-g} \left[ 2 g_{\mu\alpha} g_{\nu\beta} \nabla_\rho \nabla^\nu \left( \alpha R^{\mu\rho} - \beta R g^{\mu\rho} \right) - g_{\mu\alpha} g_{\nu\beta} g^{\mu\nu} \nabla_\sigma \nabla_\rho \left( \alpha R^{\rho\sigma} - \beta R g^{\rho\sigma} \right) - g_{\mu\alpha} g_{\nu\beta} \Box \left( \alpha R^{\mu\nu} - \beta R g^{\mu\nu} \right) \right] \delta g^{\alpha\beta}$$

After using (12) to bring the metric tensor inside the derivatives on the first and third terms and $g_{\mu\alpha} g_{\nu\beta} g^{\mu\nu} = g_{\mu\alpha} \delta^\mu_\beta = g_{\beta\alpha}$ on the second term we have:

$$- \int d^4x \sqrt{-g} \left[ 2\nabla_\rho \nabla_\beta \{ g_{\mu\alpha} \left( \alpha R^{\mu\rho} - \beta R g^{\mu\rho} \right) \} - g_{\beta\alpha} \nabla_\sigma \nabla_\rho \left( \alpha R^{\rho\sigma} - \beta R g^{\rho\sigma} \right) - \Box \{ g_{\mu\alpha} g_{\nu\beta} \left( \alpha R^{\mu\nu} - \beta R g^{\mu\nu} \right) \} \right] \delta g^{\alpha\beta}$$

We can now use $g_{\mu\alpha} g^{\mu\rho} = \delta^\rho_\alpha$ on the first term, the symmetry of the metric tensor on the second term and $g_{\mu\alpha} g_{\nu\beta} g^{\mu\nu} = g_{\mu\alpha} \delta^\mu_\beta = g_{\beta\alpha}$ on the final term:

$$- \int d^4x \sqrt{-g} \left[ 2\nabla_\rho \nabla_\beta \left( \alpha R^\rho_\alpha - \beta R \delta^\rho_\alpha \right) \} - g_{\alpha\beta} \nabla_\sigma \nabla_\rho \left( \alpha R^{\rho\sigma} - \beta R g^{\rho\sigma} \right) - \Box \left( \alpha R_{\beta\alpha} - \beta R g_{\beta\alpha} \right) \} \right] \delta g^{\alpha\beta}$$

Rearranging this equation with the help of (12):

$$- \int d^4x \sqrt{-g} \left[ \alpha \left( 2\nabla_\rho \nabla_\beta R^\rho_\alpha - g_{\alpha\beta} \nabla_\sigma \nabla_\rho R^{\rho\sigma} - \Box R_{\alpha\beta} \right) - \beta \left( 2\nabla_\alpha \nabla_\beta R - g_{\alpha\beta} g^{\rho\sigma} \nabla_\sigma \nabla_\rho R - g_{\beta\alpha} \Box R \right) \right] \delta g^{\alpha\beta}$$

Recognising the penultimate term contains $g^{\rho\sigma} \nabla_\sigma \nabla_\rho R = \nabla^\rho \nabla_\rho R \equiv \Box R$, we can perform a final switching of dummy indices and order of indices through the symmetry of $g_{\mu\nu}$:

$$- \int d^4x \sqrt{-g} \left[ \alpha \left( 2\nabla_\rho \nabla_\nu R^\rho_\mu - g_{\mu\nu} \nabla_\sigma \nabla_\rho R^{\rho\sigma} - \Box R_{\mu\nu} \right) - 2\beta \left( \nabla_\mu \nabla_\nu R - g_{\mu\nu} \Box R \right) \right] \delta g^{\mu\nu}$$

13

To proceed, we first make use of the **Second Bianchi Identity**:

$$R_{abmn;l} + R_{ablm;n} + R_{abnl;m} = 0 \qquad (24)$$

Using this, we can multiply both sides twice by the metric tensor and exploit the antisymmetries of the Riemann tensor:

$$g^{bn}g^{am}\left(R_{abmn;l} + R_{abnl;m} + R_{ablm;n}\right) = g^{bn}\left(R^m{}_{bmn;l} + R^m{}_{bnl;m} + R^m{}_{blm;n}\right)$$
$$= g^{bn}\left(R^m{}_{bmn;l} - R_b{}^m{}_{nl;m} - R^m{}_{bml;n}\right)$$

Realising that the contractions of the Riemann and Ricci tensors bring about the Ricci tensor and scalar respectively:

$$g^{bn}g^{am}\left(R_{abmn;l} + R_{abnl;m} + R_{ablm;n}\right) \equiv g^{bn}\left(R_{bn;l} - R_b{}^m{}_{nl;m} - R_{bl;n}\right) = R^n{}_{n;l} - R^{nm}{}_{nl;m} - R^n{}_{l;n}$$
$$\equiv R_{;l} - R^m{}_{l;m} - R^n{}_{l;n} = R_{;l} - 2R^m{}_{n;m} \overset{!}{=} 0$$

In the second line, we have recognised that the two negative terms are identical via dummy indices. Rearranging this gives us the **Contracted Bianchi Identity**:

$$R^m{}_{n;m} = \frac{1}{2}R_{;l} \Leftrightarrow \nabla_m R^m_l = \frac{1}{2}\nabla_l R \qquad (25)$$

Through the use of metric compatability, we can apply the Contracted Bianchi Identity to simplify:

$$\nabla_\sigma \nabla_\rho R^{\rho\sigma} = \nabla_\sigma \nabla_\rho \left(g^{\sigma\alpha}R^\rho_\alpha\right) = g^{\sigma\alpha}\nabla_\sigma\nabla_\rho R^\rho_\alpha = \nabla^\alpha\left(\frac{1}{2}\nabla_\alpha R\right) = \frac{1}{2}\Box R$$

This means that we can rewrite the second term of our previous expression. The integral becomes:

$$-\int d^4x\sqrt{-g}\left[\alpha\left(2\nabla_\rho\nabla_\nu R^\rho_\mu - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu}\right) - 2\beta\left(\nabla_\nu\nabla_\mu R - g_{\mu\nu}\Box R\right)\right]\delta g^{\mu\nu}$$

We now move onto the first term, which can be simplified by using the commutator of covariant derivatives, for which the general expression is given by (3.2.12) in [26][VI]:

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)T^{c_1\ldots c_k}{}_{d_1\ldots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i}T^{c_1\ldots e\ldots c_k}{}_{d_1\ldots d_l} + \sum_{j=1}^{l} R_{abd_j}{}^e T^{c_1\ldots c_k}{}_{d_1\ldots e\ldots d_l} \qquad (26)$$

In this case, we only have the mixed (1,1) tensor $R^\rho_\mu$; because of the symmetry of the Ricci tensor, we need not go through the rigmarole of raising and lowering using the metric tensor and can instead apply (26) immediately:

$$\left(\nabla_\rho\nabla_\nu - \nabla_\nu\nabla_\rho\right)R_\mu{}^\rho = \left(-R_{\rho\nu e}{}^\rho R^e{}_\mu + R_{\rho\nu\mu}{}^e R^\rho{}_e\right)$$
$$= R_{\rho\nu}{}^\rho{}_e R^e_\mu + R_{\rho\nu\mu}{}^e R^\rho{}_e \text{ by antisymmetry of Riemann tensor}$$
$$\equiv R_{\nu e}R^e_\mu + R_{\rho\nu\mu}{}^e R^\rho{}_e \Leftrightarrow R_{\nu e}R^e_\mu + R_{\rho\nu\mu e}R^{\rho e}$$

---

[VI]See also (3.19) of [22]

We can also use (25) once more to write:

$$\nabla_\rho \nabla_\nu R_\mu{}^\rho = \nabla_\nu \nabla_\rho R_\mu^\rho + R_{\rho\nu\mu e}R^{e\rho} + R_\nu^e R_{\mu e} = \frac{1}{2}\nabla_\nu \nabla_\mu R + R_{\rho\nu\mu e}R^{e\rho} + R_\nu^e R_{\mu e}$$

Substituting this into our integral gives:

$$-\int d^4x\sqrt{-g}\left[\alpha\left(2\left[\frac{1}{2}\nabla_\nu\nabla_\mu R + R_{\rho\nu\mu e}R^{e\rho} + R_\nu^e R_{\mu e}\right] - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu}\right) - 2\beta\left(\nabla_\nu\nabla_\mu R - g_{\mu\nu}\Box R\right)\right]\delta g^{\mu\nu}$$

$$= -\int d^4x\sqrt{-g}\left[\alpha\left(\nabla_\nu\nabla_\mu R + 2R_{\rho\nu\mu\sigma}R^{\sigma\rho} + 2R_\nu^\sigma R_{\mu\sigma} - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu}\right) - 2\beta\left(\nabla_\nu\nabla_\mu R - g_{\mu\nu}\Box R\right)\right]\delta g^{\mu\nu}$$

Having simplified as much as we can, we now substitute our result back into (23):

$$\delta S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\left(\alpha R_{\rho\sigma}R^{\rho\sigma} - \beta R^2 + \gamma\kappa^{-2}R\right)g_{\mu\nu} + 2\left(\alpha g^{\rho\sigma}R_{\mu\rho}R_{\nu\sigma} - \beta RR_{\mu\nu}\right) + \gamma\kappa^{-2}R_{\mu\nu}\right]\delta g^{\mu\nu}$$

$$- \int d^4x\sqrt{-g}\left[\alpha\left(\nabla_\nu\nabla_\mu R + 2R_{\rho\nu\mu\sigma}R^{\sigma\rho} + 2R_\nu^\sigma R_{\mu\sigma} - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu}\right) - 2\beta\left(\nabla_\nu\nabla_\mu R - g_{\mu\nu}\Box R\right)\right]\delta g^{\mu\nu}$$

Collecting together similar terms, we can write:

$$\delta S = \int d^4x\sqrt{-g}\left[\alpha\left(-\frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + 2g^{\rho\sigma}R_{\mu\rho}R_{\nu\sigma} - \nabla_\nu\nabla_\mu R - 2R_{\rho\nu\mu\sigma}R^{\sigma\rho} - 2R_\nu^\sigma R_{\mu\sigma} + \frac{1}{2}g_{\mu\nu}\Box R + \Box R_{\mu\nu}\right)\right]\delta g^{\mu\nu}$$

$$+ \int d^4x\sqrt{-g}\left[\beta\left(\frac{1}{2}R^2 g_{\mu\nu} - 2RR_{\mu\nu} - 2\nabla_\nu\nabla_\mu R + 2g_{\mu\nu}\Box R\right) + \gamma\left(-\frac{1}{2}\kappa^{-2}R + \kappa^{-2}R_{\mu\nu}\right)\right]\delta g^{\mu\nu}$$

The second and fifth terms on the first line cancel after switching dummy indices $\mu$ and $\nu$ and utilising the symmetry of the metric tensor. This leaves us with:

$$\delta S = \int d^4x\sqrt{-g}\left[\alpha\left(-\frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} - \nabla_\nu\nabla_\mu R - 2R_{\rho\nu\mu\sigma}R^{\sigma\rho} + \frac{1}{2}g_{\mu\nu}\Box R + \Box R_{\mu\nu}\right)\right]\delta g^{\mu\nu}$$

$$+ \int d^4x\sqrt{-g}\left[\beta\left(\frac{1}{2}R^2 g_{\mu\nu} - 2RR_{\mu\nu} - 2\nabla_\nu\nabla_\mu R + 2g_{\mu\nu}\Box R\right) + \gamma\left(-\frac{1}{2}\kappa^{-2}R + \kappa^{-2}R_{\mu\nu}\right)\right]\delta g^{\mu\nu}$$

$$\stackrel{!}{=} 0 \text{ by the Principle of Least Action}$$

As before, the integral always vanishing is equivalent to the vanishing of the integrand and the equation of motion is:

$$\alpha\left(-\frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} - \nabla_\nu\nabla_\mu R - 2R_{\rho\nu\mu\sigma}R^{\sigma\rho} + \frac{1}{2}g_{\mu\nu}\Box R + \Box R_{\mu\nu}\right)$$

$$+ \beta\left(\frac{1}{2}R^2 g_{\mu\nu} - 2RR_{\mu\nu} - 2\nabla_\nu\nabla_\mu R + 2g_{\mu\nu}\Box R\right) + \gamma\left(-\frac{1}{2}\kappa^{-2}R + \kappa^{-2}R_{\mu\nu}\right) = 0 \tag{27}$$

## 3.3 Linearised Analysis

The equations of motion for the higher-derivative action are very complicated, so we begin by considering the solution for a static, spherically symmetric field. The metric describing such a field can be written using Schwarzschild co-ordinates as:

$$ds^2 = A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - B(r)dt^2$$

We calculate the components of the Riemann and Ricci tensors, along with the Ricci scalar, in Section 6.1 so as not to obscure the following argument with calculational detail. To further simplify the equations, we linearise the metric by using the functions:

$$A(r) = 1 + w(r) \qquad B(r) = 1 + v(r)$$

Here, we are assuming that both $w(r)$ and $v(r)$ are infintesimal in magnitude so that we need only consider terms that in linear in w or v. Because of this, we can calculate their inverses relatively easily by calculating the Taylor series expansion:

$$A^{-1}(r) = (1 + w(r))^{-1} \approx 1 - w(r) \qquad B^{-1}(r) = (1 + v(r))^{-1} \approx 1 - v(r)$$

We also assume that this linearisation criterion applies to their derivatives with respect to r, the nth order of which we denote by suffixing n dashes.

Using these definitions, the non-zero Riemann tensor components calculated in Section 6.1 simplify to:

1. $R^t_{rtr} \approx -\frac{v''}{2}(1-v) - \frac{v'}{4}(1-v)[w'(1-w) + v'(1-v)] \approx -v'' - \frac{v'}{4}(w'+v') \approx -v''$

2. $R^t_{\phi t\phi} \approx r\sin^2\theta(1+w)\frac{v'}{2}(1-v) \approx \frac{r\sin^2\theta}{2}v'$

3. $R^t_{\theta t\theta} \approx r(1-w)\frac{v'}{2}(1-v) \approx \frac{r}{2}v'$

4. $R^r_{\phi r\phi} \approx -\frac{r\sin^2\theta}{2}w'(1-2w) \approx -\frac{r\sin^2\theta}{2}w'$

5. $R^r_{\theta r\theta} \approx -\frac{rw'}{2}(1-2w) \approx -\frac{r}{2}w'$

6. $R^\theta_{\phi\theta\phi} \approx \sin^2\theta[-1 + (1-w)] = \sin^2\theta w$

The non-zero components of the Ricci tensor, meanwhile, are:

1. $R_{tt} \approx -\frac{v''}{2}(1-w) + \frac{v'}{4}(1-w)[w'(1-w) + v'(1-v)] - \frac{v'}{r}(1-w) \approx -\frac{v''}{2} - \frac{v'}{r}$

2. $R_{rr} \approx -\frac{w'}{r}(1-w) + \frac{v''}{2}(1-v) - \frac{v'}{4}(1-v)[v'(1-v) + w'(1-w)] \approx \frac{w'}{r} + \frac{v''}{2} - \frac{v'}{4}[v'-w'] \approx \frac{w'}{r} + \frac{v''}{2}$

3. $R_{\theta\theta} \approx -1 + (1-w) + \frac{r}{2}(1-w)[-w'(1-w) + v'(1-v)] \approx -w + \frac{r}{2}(1-w)(-w'+v') \approx -w + \frac{r}{2}(-w'+v')$

4. $R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \approx \sin^2\theta \left[-w + \frac{r}{2}(-w'+v')\right]$

16

Finally, the Ricci scalar will be:

$$R \approx -\frac{2}{r^2} + \frac{2}{r^2}(1-w) + w''(1-v)(1-w) + \frac{v'}{2}(1-v)(1-w)[w^2(1-w) + v'(1-v)] - \frac{2}{r}(1-w)[w'(1-w) - v'(1-v)]$$

$$\approx -\frac{2w}{r^2} - \frac{2}{r}(w'-v') + \frac{v'}{2}[w'+v'] - \frac{2}{r}(1-w)[w'-v'] \approx -\frac{2w}{r^2} - \frac{2}{r}(w'-v') - \frac{2}{r}(w'-v') = -\frac{2w}{r^2} - \frac{4}{r}(w'-v')$$

We check that if we switch off the gravitational perturbation completely by setting $w = v = 0$, then the components of the Riemann and Ricci tensors and the Ricci scalar of the resulting spacetime vanish, meaning that the spacetime around which we are expanding is flat Minkowski. This acts as a useful check of the algebra.

Having calculated these linearised metric components, we can obtain the linearised equations of motion by manipulating the equations of motion using the metric tensor, substituting the above results and then simplifying the resulting expressions. This involves a lot of tedious, but straightforward computation, the details of which we omit and simply state the answer[20]:

$$H_{rr}^L = -(\alpha - 4\beta)r^{-1}v''' - 2(\alpha - 4\beta)r^{-2}v'' + 2(\alpha - 4\beta)r^{-3}v' - \gamma\kappa^{-2}r^{-1}v'$$

$$+ (3\alpha - 8\beta)r^{-2}w'' - 2(3\alpha - 8\beta)r^{-4}w + \gamma\kappa^{-2}r^{-2}w \qquad (28)$$

$$H_{\theta\theta}^L = -\frac{1}{2}(\alpha - 4\beta)r^2 v'''' - \frac{3}{2}(\alpha - 4\beta)rv''' + (\alpha - 4\beta)v'' - (\alpha - 4\beta)r^{-1}v' - \frac{1}{2}\gamma\kappa^{-2}r^2 v'' - \frac{1}{2}\gamma\kappa^{-2}rv'$$

$$+ \frac{1}{2}(3\alpha - 8\beta)rw''' - (3\alpha - 8\beta)r^{-1}w' + 2(3\alpha - 8\beta)r^{-2}w + \frac{1}{2}\gamma\kappa^{-2}rw' \qquad (29)$$

$$H_{tt}^L = (\alpha - 2\beta)v'''' + 4(\alpha - 2\beta)r^{-1}v''' - (\alpha - 4\beta)r^{-1}w''' - (\alpha - 4\beta)r^{-2}w''$$

$$+ 2(\alpha - 4\beta)r^{-3}w' - 2(\alpha - 4\beta)r^{-3}w' - 2(\alpha - 4\beta)r^{-4}w - \gamma\kappa^{-2}r^{-1}w' - \gamma\kappa^{-2}r^{-2}w \qquad (30)$$

We now calculate the linear combinations:

$$H_{rr}^L + 2r^{-2}H_{\theta\theta}^L \pm H_{tt}^L = -(\alpha - 4\beta)r^{-1}v''' - 2(\alpha - 4\beta)r^{-2}v'' + 2(\alpha - 4\beta)r^{-3}v' - \gamma\kappa^{-2}r^{-1}v'$$

$$+ (3\alpha - 8\beta)r^{-2}w'' - 2(3\alpha - 8\beta)r^{-4}w + \gamma\kappa^{-2}r^{-2}w$$

$$+ 2r^{-2}\left[-\frac{1}{2}(\alpha - 4\beta)r^2 v'''' - \frac{3}{2}(\alpha - 4\beta)rv''' + (\alpha - 4\beta)v'' - (\alpha - 4\beta)r^{-1}v' - \frac{1}{2}\gamma\kappa^{-2}r^2 v'''\right]$$

$$+ 2r^{-2}\left[-\frac{1}{2}\gamma\kappa^{-2}rv + \frac{1}{2}(3\alpha - 8\beta)rw''' - (3\alpha - 8\beta)r^{-1}w' + 2(3\alpha - 8\beta)r^{-2}w + \frac{1}{2}\gamma\kappa^{-2}rw'\right]$$

$$\pm \left[(\alpha - 2\beta)v'''' + 4(\alpha - 2\beta)r^{-1}v''' - (\alpha - 4\beta)r^{-1}w''' - (\alpha - 4\beta)r^{-2}w''\right]$$

$$\pm \left[2(\alpha - 4\beta)r^{-3}w' - 2(\alpha - 4\beta)r^{-3}w' - 2(\alpha - 4\beta)r^{-4}w - \gamma\kappa^{-2}r^{-1}w' - \gamma\kappa^{-2}r^{-2}w\right]$$

$$= \left[-(\alpha - 4\beta) \pm (\alpha - 2\beta)\right]v'''' + \left[-4(\alpha - 4\beta) \pm 4(\alpha - 2\beta)\right]r^{-1}v''' + \gamma\kappa^{-2}\left[-v^2 - 2r^{-1}v'\right]$$

$$+ \left[-2(\alpha - 4\beta) + 2(\alpha - 4\beta)\right]r^{-2}v^2 + 2\left[(\alpha - 4\beta) - (\alpha - 4\beta)\right]r^{-3}v'$$

$$+ \left[(3\alpha - 8\beta) \mp (\alpha - 4\beta)\right]r^{-1}w''' + \left[(3\alpha - 8\beta) \mp (\alpha - 4\beta)\right]r^{-2}w'' + \left[\gamma\kappa^{-2} \mp \gamma\kappa^{-2}\right]r^{-1}w'$$

$$+ \left[-2(3\alpha - 8\beta) \pm 2(\alpha - 4\beta)\right]r^{-3}w' + \left[\gamma\kappa^{-2} \mp \gamma\kappa^{-2}\right]r^{-2}w + \left[2(3\alpha - 8\beta) \mp 2(\alpha - 4\beta)\right]r^{-4}w$$

All of the terms on the second line cancel (regardless of whether we choose '+' or '-'). We are left with:

$$
\begin{aligned}
H_{rr}^L + 2r^{-2}H_{\theta\theta}^L \pm H_{tt}^L &= [-(\alpha - 4\beta) \pm (\alpha - 2\beta)]\left(v'''' + 4r^{-1}v'''\right) - \gamma\kappa^{-2}\left[v^2 + rr^{-1}v'\right] \\
&\quad + [(3\alpha - 8\beta) \mp (\alpha - 4\beta)]\,r^{-1}w''' + [(3\alpha - 8\beta) \mp (\alpha - 4\beta)]\,r^{-2}w'' \\
&\quad - 2\,[(3\alpha - 8\beta) \mp (\alpha - 4\beta)]\,r^{-3}w' + 2\,[(3\alpha - 8\beta) \mp (\alpha - 4\beta)]\,r^{-4}w \\
&\quad + \left[\gamma\kappa^{-2} \mp \gamma\kappa^{-2}\right]\left(r^{-1}w' + r^{-2}w\right) \\
&= [-(\alpha - 4\beta) \pm (\alpha - 2\beta)]\left(v'''' + 4r^{-1}v'''\right) - \gamma\kappa^{-2}\left[v^2 + rr^{-1}v'\right] \\
&\quad + [(3\alpha - 8\beta) \mp (\alpha - 4\beta)]\left(r^{-1}w''' + r^{-2}w'' - 2r^{-3}w' + 2r^{-4}w\right) \\
&\quad + \gamma\kappa^{-2}\left[1 \mp 1\right]\left(r^{-1}w' + r^{-2}w\right)
\end{aligned}
\tag{31}
$$

To simplify this result further, we use the results for the Laplacian in spherical co-ordinates:

$$
\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}
$$

Since $v = v(r)$, the angular derivatives vanish and we are left with:

$$
\nabla^2 v = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) = \frac{1}{r^2}\left(2r\frac{dv}{dr} + r^2\frac{d^2 v}{dr^2}\right) = 2r^{-1}\frac{dv}{dr} + \frac{d^2 v}{dr^2}
\tag{32}
$$

We can also take the Laplacian of this result again:

$$
\begin{aligned}
\nabla^4 v \equiv \nabla^2(\nabla^2 v) \equiv \nabla^2 E &= 2r^{-1}\frac{dE}{dr} + \frac{d^2 E}{dr^2} \equiv 2r^{-1}\frac{d}{dr}\left(2r^{-1}\frac{dv}{dr} + \frac{d^2 v}{dr^2}\right) + \frac{d^2}{dr^2}\left(2r^{-1}\frac{dv}{dr} + \frac{d^2 v}{dr^2}\right) \\
&= 2r^{-1}\left(-2r^{-2}\frac{dv}{dr} + 2r^{-1}\frac{d^2 v}{dr^2} + \frac{d^3 v}{dr^3}\right) + \frac{d}{dr}\left(-2r^{-2}\frac{dv}{dr} + 2r^{-1}\frac{d^2 v}{dr^2} + \frac{d^3 v}{dr^3}\right) \\
&= -4r^{-3}\frac{dv}{dr} + 4r^{-2}\frac{d^2 v}{dr^2} + 2r^{-3}\frac{d^3 v}{dr^3} + 4r^{-3}\frac{dv}{dr} - 2r^{-2}\frac{d^2 v}{dr^2} - 2r^{-2}\frac{d^2 v}{dr^2} + 2r^{-1}\frac{d^3 v}{dr^3} + \frac{d^4 v}{dr^4} \\
&= \frac{d^4 v}{dr^4} + 4\frac{d^3 v}{dr^3} \equiv v'''' + 4r^{-1}v'''
\end{aligned}
\tag{33}
$$

Furthermore, inspired to substitute for the last term of (31), we define:

$$
Y \equiv r^{-2}(rw)' = wr^{-2} + w'r^{-1}
\tag{34}
$$

Now:

$$
\begin{aligned}
\frac{dY}{dr} &= \frac{d}{dr}(wr^{-2} + w'r^{-1}) = w'r^{-2} - 2wr^{-3} + w''r^{-1} - r^{-2}w' = -2wr^{-3} + w''r^{-1} \\
\frac{d^2 Y}{dr^2} &= \frac{d}{dr}(-2wr^{-3} + w''r^{-1}) = 6wr^{-4} - 2w'r^{-3} + w'''r^{-1} - w''r^{-2}
\end{aligned}
$$

Substituting these into (32) with $v \leftrightarrow Y$ yields:

$$
\begin{aligned}
\nabla^2 Y &= 2r^{-1}(-2wr^{-3} + w''r^{-1}) + 6wr^{-4} - 2w'r^{-3} + w'''r^{-1} - w''r^{-2} \\
&= 2wr^{-4} - 2w'r^{-3} + w'''r^{-1} + w''r^{-2}
\end{aligned}
\tag{35}
$$

Inserting (32), (33), (34) and (35) into (31), we get:

$$H_{rr}^L + 2r^{-2}H_{\theta\theta}^L \pm H_{tt}^L = [-(\alpha - 4\beta) \pm (\alpha - 2\beta)]\nabla^4 v - \gamma\kappa^{-2}\nabla^2 v + [(3\alpha - 8\beta) \mp (\alpha - 4\beta)]\nabla^2 Y + \gamma\kappa^{-2}[1 \mp 1]Y$$

Writing these equations out explicitly, we see that they match exactly with (3.5) of [20]:

$$\begin{aligned}
H_{rr}^L + 2r^{-2}H_{\theta\theta}^L + H_{tt}^L &= 2\beta\nabla^4 v - \gamma\kappa^{-2}\nabla^2 v + (2\alpha - 4\beta)\nabla^2 Y \\
&= 2\beta\nabla^4 v - \gamma\kappa^{-2}\nabla^2 v + 2(\alpha - 2\beta)\nabla^2 Y
\end{aligned} \tag{36}$$

$$\begin{aligned}
H_{rr}^L + 2r^{-2}H_{\theta\theta}^L - H_{tt}^L &= (-2\alpha + 6\beta)\nabla^4 v - \gamma\kappa^{-2}\nabla^2 v + (4\alpha - 12\beta)\nabla^2 Y + 2\gamma\kappa^{-2}Y \\
&= 2(3\beta - \alpha)\nabla^4 v - \gamma\kappa^{-2}\nabla^2 v + 4(\alpha - 3\beta)\nabla^2 Y + 2\gamma\kappa^{-2}Y
\end{aligned} \tag{37}$$

The following substitution presents itself as a useful simplification:

$$X(r) \equiv (\nabla^2 v)(r) \tag{38}$$

Using this variable, the above equations become:

$$H_{rr}^L + 2r^{-2}H_{\theta\theta}^L + H_{tt}^L = 2\beta\nabla^2 X - \gamma\kappa^{-2}X + 2(\alpha - 2\beta)\nabla^2 Y \tag{39}$$

$$H_{rr}^L + 2r^{-2}H_{\theta\theta}^L - H_{tt}^L = 2(3\beta - \alpha)\nabla^2 X - \gamma\kappa^{-2}X + 4(\alpha - 3\beta)\nabla^2 Y + 2\gamma\kappa^{-2}Y \tag{40}$$

Despite their relatively simple appearance, (39) and (40) are two coupled, differential equations which are very difficult to solve, even in the simplest possible case where $H_{\mu\nu}^L = 0$ and so, since all of the summands are zero, the L.H.S. of (39) and (40) both vanish. If this is the case, then we can rearrange (39) as:

$$\nabla^2 Y = \frac{1}{2(\alpha - 2\beta)}\left[-2\beta\nabla^2 X + \gamma\kappa^{-2}X\right]$$

Substituting this into (40):

$$H_{rr}^L + 2r^{-2}H_{\theta\theta}^L - H_{tt}^L = 2(3\beta - \alpha)\nabla^2 X - \gamma\kappa^{-2}X + \frac{4(\alpha - 3\beta)}{2(\alpha - 2\beta)}\left[-2\beta\nabla^2 X + \gamma\kappa^{-2}X\right] + 2\gamma\kappa^{-2}Y \overset{!}{=} 0$$

After some small rearrangements:

$$-2\gamma\kappa^{-2}Y = 2(3\beta - \alpha)\left[1 + \frac{2\beta}{\alpha - 2\beta}\right]\nabla^2 X - \gamma\kappa^2\left[1 + \frac{2(3\beta - \alpha)}{(\alpha - 2\beta)}\right]X = \frac{2(3\beta - \alpha)\alpha}{\alpha - 2\beta}\nabla^2 X - \gamma\kappa^2\frac{(-\alpha + 4\beta)}{(\alpha - 2\beta)}X$$

This means that we can put the equation into the form:

$$\nabla^2 X - \gamma\kappa^{-2}\frac{(-\alpha + 4\beta)}{2\alpha(3\beta - \alpha)}\frac{(\alpha - 2\beta)}{(\alpha - 2\beta)}X = -\frac{2\gamma\kappa^{-2}(\alpha - 2\beta)}{2\alpha(3\beta - \alpha)}Y$$

There are some internal cancellations and we are left with:

$$\nabla^2 X - \gamma\kappa^{-2}\frac{(4\beta - \alpha)}{2\alpha(3\beta - \alpha)}X = -\frac{\gamma\kappa^{-2}(\alpha - 2\beta)}{\alpha(3\beta - \alpha)}Y$$

This is in the form of a **screened Poisson equation**:

$$\left[\nabla^2 - \lambda^2\right] X(\underline{r}) = -F(\underline{r}) \text{ with } \lambda \equiv \sqrt{\frac{\gamma(4\beta - \alpha)}{2\kappa^2\alpha(3\beta - \alpha)}} \text{ and } F(\underline{r}) = \frac{\gamma\kappa^{-2}(\alpha - 2\beta)}{\alpha(3\beta - \alpha)}Y(r) \tag{41}$$

To solve the screened Poisson equation, we use the method of Green's functions, which are defined as the function $G(\underline{r})$ which satisfies:[VII]

$$\left[\nabla^2 - \lambda^2\right] G(\underline{r}) = -\delta^3(\underline{r})$$

The translational invariance of the Laplacian suggests that a continuous 3D Fourier transform would be useful:

$$G(\underline{k}) = \int dV\, e^{-i\underline{k}\cdot\underline{r}} G(\underline{r})$$

Doing this will mean that the spatial derivatives merely bring down a factor of $-i\underline{k}$, so the Laplacian yields a factor of $-\underline{k}\cdot\underline{k} = -k^2$. Dividing through by the negative sign, we are left with:

$$[k^2 + \lambda^2]G(\underline{k}) = 1 \implies G(\underline{k}) = \frac{1}{k^2 + \lambda^2}$$

To get $G(\underline{r})$ back, we perform the inverse 3D Fourier transform:

$$
\begin{aligned}
G(\underline{r}) &= \frac{1}{(2\pi)^3} \int d^3k\, e^{i\underline{k}\cdot\underline{r}} G(\underline{k}) = \frac{1}{(2\pi)^3} \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi\, k^2 \sin\theta\, \frac{e^{i\underline{k}\cdot\underline{r}}}{k^2 + \lambda^2} \\
&= \frac{1}{(2\pi)^2} \int_0^\infty dk \int_0^\pi d\theta\, k^2 \sin\theta\, \frac{e^{ikr\cos\theta}}{k^2 + \lambda^2}
\end{aligned}
$$

Defining the variable $K \equiv kr\cos\theta$ and substituting for $\theta$ puts the integral in the form:

$$
\begin{aligned}
G(\underline{r}) &= \frac{1}{(2\pi)^2} \int_0^\infty dk \int_{kr}^{-kr} k^2 \left(-\frac{dK}{kr}\right) \frac{e^{iK}}{k^2 + \lambda^2} = \frac{1}{(2\pi)^2 r} \int_0^\infty dk\, k \int_{-kr}^{kr} dK\, \frac{e^{iK}}{k^2 + \lambda^2} \\
&= \frac{1}{(2\pi)^2 r} \int_0^\infty dk\, \frac{k}{k^2 + \lambda^2} \frac{e^{ikr} - e^{-ikr}}{i} = \frac{1}{2\pi^2 r} \int_0^\infty dk\, \frac{k \sin(kr)}{k^2 + \lambda^2}
\end{aligned}
\tag{42}
$$

In the last line, we have used Euler's formula to subsitute for the difference of to imaginary exponentials.

The above integral can be evaluated by using complex analysis. Initially, we note that the integrand has poles at $k = \pm i\lambda$, so we extend the range of integration from $\{0, \infty\}$ to $\{-\infty, \infty\}$ to use a semi-circular contour whose diameter spans the k-axis, meaning that the only pole inside our contour is at $i\lambda$. We now evaluate the residue of this pole for the function $\frac{ke^{ikr}}{k^2 + \lambda^2} = \frac{ke^{ikr}}{(k+i\lambda)(k-i\lambda)}$:

$$a_{-1} = \lim_{k\to i\lambda} \frac{(k - i\lambda)}{(k - i\lambda)} \frac{ke^{ikr}}{(k + i\lambda)} = \frac{i\lambda e^{i(i\lambda)r}}{2i\lambda} = \frac{e^{-\lambda r}}{2}$$

Now, by the **residue theorem**:

$$\int_C \frac{ke^{ikr}\, dk}{k^2 + \lambda^2} = 2\pi i \left(\frac{e^{-\lambda r}}{2}\right)$$

---

[VII]Technically, Green's functions are like the Dirac-Delta functions in that they should strictly be called distributions.

Since the semi-circular arc contribution $\Gamma \to 0$ by **Jordan's Lemma**[VIII], we have:

$$\int_{-\infty}^{\infty} \frac{k e^{ikr} dk}{k^2 + \lambda^2} = \pi i e^{-\lambda r}$$

We now equate the imaginary components and use the fact that the resulting integrand is even overall to get:

$$\int_{-\infty}^{\infty} \frac{k \sin(kr) dk}{k^2 + \lambda^2} = 2 \int_{0}^{\infty} \frac{k \sin(kr) dk}{k^2 + \lambda^2} = \pi e^{-\lambda r}$$

Having evaluated the integral, we substitute back into (42):

$$G(\underline{r}) = \frac{1}{2\pi^2 r} \left( \frac{\pi e^{-\lambda r}}{2} \right) = \frac{e^{-\lambda r}}{4\pi r} \tag{43}$$

This is of the form of the **Yukawa potential**.

Having worked out the Green's function for the problem, we can use it to find the full solution:

$$X(\underline{r}) = \int d^3 R \; G(\underline{r} - \underline{R}) \; F(\underline{R})$$

Of course, since $F(\underline{R})$ depends on $Y(r)$, which we still have not determined yet, this is only of limited utility; worse, even if we were to have obtained $Y(r)$, we still have to substitute for (34) and (38) to find the original functions $v(r)$ and $w(r)$. However, the problem is not intractable, merely one of onerous algebra more suited for Mathematica or another mathematical computation package. In lieu of this, we will content ourselves with having demonstrated the salient point of the calculation, namely the appearance of the Yukawa potential, and simply quote the solutions[20]:

$$v(r) = C + \frac{C^{2,0}}{r} + C^{2+} \frac{e^{m_2}}{r} + C^{2-} \frac{e^{-m_2}}{r} + C^{0+} \frac{e^{m_0}}{r} + C^{0-} \frac{e^{-m_0}}{r} \tag{44}$$

$$\begin{aligned} w(r) = &-\frac{C^{2,0}}{r} - C^{2+} \frac{e^{m_2}}{r} - C^{2-} \frac{e^{-m_2}}{r} + C^{0+} \frac{e^{m_0}}{r} + C^{0-} \frac{e^{-m_0}}{r} \\ &+ \frac{1}{2} C^{2+} m_2 e^{m_2 r} - \frac{1}{2} C^{2-} m_2 e^{-m_2 r} - C^{0+} m_2 e^{m_0 r} + C^{0+} m_2 e^{m_0 r} \end{aligned} \tag{45}$$

In the above equations, $C^a$ are constants and $m_i$ are the masses in the Yukawa potential given by:

$$m_2 \equiv \sqrt{\frac{\gamma}{\alpha \kappa^2}} \qquad m_0 \equiv \sqrt{\frac{\gamma}{2(3\beta - \alpha)\kappa^2}} \tag{46}$$

Comparing (45) with (41) shows the origin of the denominator factors of $m_0$ and $m_2$, further exonerating our cursory analysis.

---

[VIII]See Section 6.2 for a justification of this procedure.

### 3.3.1 Results of Linearised Analysis For a Point Particle

In the case of the point particle, the Hilbert stress-energy tensor (14) takes the form $\delta_\mu^0 \delta_\nu^0 M d^3(x)$ and the gravitational field from (44) is:

$$v(r) = -\frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \tag{47}$$

This potential appears pathological at $r = 0$, but we can see that this is not a true singularity by Taylor expanding the exponential:

$$\begin{aligned}
v(0) &= -\frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma r}(1 - m_2 r + O(r^2)) - \frac{\kappa^2 M}{24\pi\gamma r}(1 - m_0 r + O(r^2)) \\
&= \left[-\frac{3\kappa^2 M}{24\pi\gamma} + \frac{4\kappa^2 M}{24\pi\gamma} - \frac{\kappa^2 M}{24\pi\gamma}\right]\frac{1}{r} + \left[-\frac{4\kappa^2 M m_2}{24\pi\gamma} + \frac{\kappa^2 M m_2}{24\pi\gamma}\right]\frac{r}{r} + O(r^2/r) \\
&= \frac{\kappa^2 M}{24\pi\gamma}(4m_2 - m_0) + O(r)
\end{aligned}$$

Hence, the singularity at the origin vanishes entirely at lowest order and since the above value is perfectly finite, there are no further regularity problems in this case. On the other hand, in the limit $r \to \infty$, the Yukawa potentials become negligible and we are left with the first term. By imposing the weak field limit, we insist that as $r \to \infty$, the above potential reduces to its Newtonian counterpart, $v(r) = -\frac{GM}{2r}$; because the Yukawa potentials will be negligible as $r \to \infty$, we are left after substituting (20) with the following:

$$v(r) = -\frac{\kappa^2 M}{8\pi\gamma r} = -\frac{32\pi GM}{8\pi\gamma r} = -\frac{4GM}{\gamma r} \stackrel{!}{=} -\frac{GM}{2r} \implies \gamma \stackrel{!}{=} 2 \tag{48}$$

If we want our higher derivative theory to reduce to conventional General Relativity in the weak field limit, we must therefore set $\gamma = 2$.

This raises the question as to whether there are any restrictions on the values of $\alpha$ and $\beta$. From the denominators of (46), we can see that there are two problematic values, $\alpha = 0$ and $\alpha = 3\beta$, that will cause $m_2$ and $m_0$ to respectively diverge.[IX] Noting that a sensible Newtonian limit for (47) can only be recovered if $m_i \in \mathbb{R}$, these potentials must correspond to exponential growth/decay, rather than oscillatory solutions; as a result, having $m_2$ or $m_0$ diverge in (47) is equivalent to removing the associated massive field entirely. Examining the form of (47) also leads to another startling conclusion: the overall potential for $m_2$ is positive in sign (and hence repulsive), whereas the other two potentials are negative (and hence attractive). The fact that the $m_2$ potential is repulsive means that its energy is negative and since the existence of negative energy excitations results in the failure of causality,[1] such a theory cannot describe our universe. The only way to avoid these undesirable, negative energy solutions is to banish them by setting $\alpha = 0$, because this will ensure the divergence of the decaying Yukawa potential.

---

[IX]As $\kappa^2$ is a constant, it has no effect on the asymptotic behaviour.

What we have observed in the preceding analysis is a recurring obstacle in considering higher derivative theories of gravity: for example, in Section 5 of [20], the dynamics of the linearised model are considered[X] and the problem of negative energy modes arises once more. In fact, this is a manifestation of a more general result that we discuss in the next section.

---

[X]It is proven here that the spins of $m_2$ and $m_0$ are 2 and 0 respectively, hence the initially puzzling choice of indices.

## 3.4 Ostrogradsky Instability

Having exhaustively considered the possible terms that could make up our higher derivative action in Section 3.1, it is sobering to realise that they cannot be added in a haphazard manner. In fact, due to a general result known as the **Ostrogradsky Instability Theorem**, it could have been anticipated that setting a non-zero value of $\alpha$ would lead to the appearance of negative energy solutions.

To see this, it will be instructive to briefly review the construction of the Hamiltonian formulation of Classical Mechanics. For the familiar Lagrangian $L(q, \dot{q}; t)$, we define the canonical co-ordinates:

$$P_1 \equiv \frac{\partial L}{\partial \dot{q}} \qquad Q_1 \equiv q$$

The Hamiltonian is then obtained by using a Legendre transform on $\dot{q}$ to change the variables involved in the Lagrangian to the canonical ones defined above:

$$H = \sum_i P_i q^{(i)} - L \tag{49}$$

Here, the 'i' index on $q^{(i)}$ refers to the ith temporal derivative of q. The Euler-Lagrange equations of motion are then recovered by **Hamilton's Equations**:

$$\dot{Q}_i = \frac{\partial H}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial H}{\partial Q_i} \tag{50}$$

In higher derivative theories, although Hamilton's Equations remain identical, there are more canonical co-ordinates to take into account.[XI] For example, given the Lagrangian $L(q, \dot{q}, \ddot{q}; t)$, we can construct the following canonical co-ordinates[17]:

$$P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \quad P_2 \equiv \frac{\partial L}{\partial \ddot{q}} \quad Q_1 \equiv q \quad Q_2 \equiv \ddot{q}$$

Using these co-ordinates, the higher-derivative Hamiltonian can be constructed via canonical transformations to be of the following form:

$$H = Q_1 P_2 + h(Q_1, Q_2, P_1, P_2) \tag{51}$$

Of crucial importance is the fact that the function $h(Q, P)$ does not contain any linear terms, so it is at least quadratic in $Q_i$ and $P_i$. The upshot of this is that near the origin of phase space, the first term will be the leading order contribution:

$$H \approx Q_1 P_2 \; \forall \; P_i \approx Q_i \approx 0$$

The relevant Hamilton's equations for this approximate Hamiltonian are:

$$\frac{\partial H}{\partial P_1} = 0 \overset{!}{=} \dot{Q}_1 \qquad \frac{\partial H}{\partial P_2} = Q_1 \overset{!}{=} \dot{Q}_2 \tag{52}$$

We now demonstrate the following result, known as **Cheta(y)ev's Theorem**...

---

[XI]The formerly extranneous subscripts on the previous equations were added in anticipation of this fact.

**Theorem 1** *Let $\underline{x} = \underline{0}$ be an equilibrium for the system*

$$\dot{\underline{x}} = f(\underline{x}) \text{ with } \underline{x}(t = 0) = \underline{x}_0$$

*Furthermore, let $V(\underline{x})$ be a continuously differentiable function satisfying $V(\underline{x}_0) > 0$ for some $\underline{x}_0$ arbitrarily close to the equilibrium point $\underline{x} = \underline{0}$. Let $B_r$ be the closed ball for some $r > 0$:*

$$B_r \equiv \{\underline{x} \in \mathbb{R}^n \mid ||\underline{x}|| \leq r\}$$

*Defining U as the set of all $\underline{x}$ in the closed ball of radius $r$ such that the fuction $V(\underline{x}) > 0$ (such a point will always exist because $V(\underline{x}_0) > 0$ for some $\underline{x}_0$):*

$$U \equiv \{\underline{x} \in B_r \mid V(x) > 0\}$$

*Then if $\dot{V} > 0$ in U, then the origin $\underline{x} = 0$ is unstable.*

**Proof:** To prove this, we begin by noting that if the trajectory starts in the region $U - \{\underline{0}\}$, then the trajectory cannot leave through the boundary $V = 0$: by definition, U is the set of all points such that $V > 0$ and since $\dot{V} > 0$ by stipulation, if $V > 0$ initially, then it will continue to be so for all time. Geometrically, this is stating that any trajectory that escapes the region $B_r$ must do so by passing through the boundary $x = ||\underline{r}||$, rather than through $V = 0$.

What we have left to prove, therefore, is that every trajectory that begins within U leaves U. Since $\underline{x}_0 \in U$, $V(\underline{x}_0) \geq 0$ and so the trajectory starting at $\underline{x}(t = 0) = \underline{x}_0$ must leave U To see this, note that if if $V(t = 0) = a$ then $V \geq a$ since $\dot{V} \geq 0$ in U. Now let $\gamma$ be the minimum value of the rate of change of V in this region:

$$\gamma \equiv \min\{\dot{V} \mid \underline{x} \in U \text{ and } V \geq a\}$$

As $\dot{V} > 0$, $\gamma > 0$. Hence, because $\gamma$ is the minimum value of the rate of change, which must be a constant:

$$V(\underline{x}(t)) = V(\underline{x}_0) + \int_0^t \dot{V}(\underline{x}(s))ds = a + \int_0^t \dot{V}(\underline{x}(s))ds \geq a + \int_0^t \gamma ds = a + \gamma \int_0^t ds = a + \gamma t$$

Furthermore, $\gamma > 0$ means that $\gamma t$ grows in time and since $V(\underline{x}) \overset{!}{\geq} a + \gamma t$, then $V(\underline{x})$ must likewise grow in time. Hence, $\underline{x}$ MUST leave U at some point and since we've already established that it cannot do so by going through $V = 0$, it must go through $\underline{x} = ||\underline{r}||$. Hence, by definition, $\underline{x} = \underline{0}$ is unstable.

Having now proved Chetayev's Theorem, let us consider the function $V(Q_1, Q_2, P_1, P_2) = Q_1 Q_2$. Anticipating our interest in the higher-derivative Hamiltonian, we calculate its time derivative using (52):

$$\dot{V} = \dot{Q}_1 Q_2 + Q_1 \dot{Q}_2 = (0)Q_2 + Q_1(Q_1) = Q_1^2 \overset{!}{\geq} 0$$

This potential function satisfies the conditions of Chetayev's Theorem and so the origin must be unstable. But by appealing to the well-known result that translations in phase space are canonical[7], we conclude that

we can equally well recover a Hamiltonian of the form (51) at any point in phase space simply by altering the exact canonical transformations that we apply to our Hamiltonian in order to get it in the standard form (51). As a result, we conclude that this theory must be unstable everywhere.

This problem is not encountered in the familiar $L(q, \dot{q}; t)$ because in this case, there are only two canonical co-ordinates, $P_1$ and $Q_1$, and so it is impossible to construct a Hamiltonian of the form:

$$H = Q_i P_j + f(Q, P) \; ; \; i \neq j$$

In other words, we cannot form a Hamiltonian that is linear in one of the canonical momenta because Hamilton's equations mean that we can re-express $\dot{q}$ as a function of $q$ and $P$, $\dot{q} = v(q, P)$. This means that the lower-derivative Hamiltonian can be put in the form:

$$H(Q, P) = P\dot{q} - L = Pv(q, P) - L \Leftrightarrow Pv(Q, P) - L$$

By inspection, we can see that the Hamiltonian is non-linear in $P$, so the previous argument cannot be applied. On this basis, it would appear that the higher derivative terms in the Lagrangian are the source of this instability, but it is not apriori inconceivable that adding higher-still derivatives might solve the problem. However, this is not the case: as demonstrated in [30], the resulting Hamiltonian can always be cast in a form similar to (51) with a $Q_i P_j$ leading order term, meaning that we can always find a pathological Chetayev function that ensures the instability of the theory.

It is worth noting, however, that we have implicitly assumed throughout that the phase space transformation associated with the definition of canonical co-ordinates is invertible, so that for a given $P(q, \dot{q})$ and $Q(q, \dot{q})$, we can calculate $q(Q, P)$ and $\dot{q} = (Q, P)$ (see, for instance, (52). This condition is known as **nondegeneracy**; its higher-derivative generalisation is the assumption that the definition of the highest conjugate momentum can be inverted to solve for the highest order derivative. It is this nondegeneracy that is at the root of the Ostrogradsky Instability: any higher derivative theory whose conjugate momenta depends nondegenerately on its highest derivative is necessarily unstable.

To understand the problems associated with a theory possessing an Ostrogradsky instability, we first have to appreciate the physical meaning of the Hamiltonian. Recall that if the equations defining the generalised co-ordinates $q$ have no explicit temporal dependence and the potential V does not depend on the generalised velocities, then the Hamiltonian is the energy of the system.[7] Because of the linearity in one of the conjugate momenta, the energy of the field can be changed by assuming arbitrarily positive or negative values of this conjugate momentum, thereby ensuring the existence of both positive and negative energy solutions. As already mentioned, this leads to the breakdown of causality.[1] Furthermore, if the theory allows interactions between particles, then the Second Law of Thermodynamics drives the system towards the state of maximum

entropy, which consists of a cascade of positive and negative particles undergoing as many instantaneous self-interactions as possible.[30] Such a theory cannot physically describe our universe, where such pandemonium is not observed.

The only way to evade this conclusion is to violate the premises necessary to ensure it, namely the nondegeneracy of the Lagrangian. In this case, the theory is degenerate and there will therefore be associated continuous symmetries. These symmetries can sometimes impose couplings between the canonical variables that stabilise the system, the details depending on the exact theory under consideration. In particular, it can be shown that a fourth order higher-derivative theory for which there is no $R^{\mu\nu}R_{\mu\nu}$ term (i.e. $\alpha = 0$) is capable of having solely positive energy soluions.[23] In fact, this result can be generalised to even higher derivative models: theories whose Lagrangians consist of functions of the Ricci scalar only do not suffer from Ostrogradsky instabilities.[16] This means that generalisations of the Einstein-Hilbert action (16) are stable if they are of the following form :

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}[f(R) - 2\Lambda + L_M]$$ (53)

Such theories are known as $f(R)$ theories of gravity and have been studied in various cosmological and general relativistic models (see, for instance, [16] and [17]).

If higher-derivative theories with $\alpha \neq 0$ are inherently classically unstable, it could be questioned whether there was any virtue in pursuing such theories further. In a purely classical theory, there seems little point in considering such models as the difficulties associated with them seem insurmountable. However, in a quantum theoretical context, the picture is very different, as the inclusion of the $R^{\mu\nu}R_{\nu\mu}$ can have unexpected and desirable consequences as we will see in the next chapter.

# 4 Quantum Aspects of Higher Derivative Gravity

## 4.1 Introduction

It is well-known that conventional General Relativity is non-renormalisable, representing a major challenge in quantizing gravity. The root of this non-renormalisability lies in the Einstein-Hilbert action upon which the theory is based. To demonstrate this, consider (16):

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R - 2\Lambda + L_M)$$

Using dimensional analysis on the above action reveals that, due to the fact that it is second order in derivatives, the Ricci scalar $R \sim [L]^{-2}$ (or, alternatively, $[m]^2$ since we are using natural units), whereas the integral measure $d^4x \sim [L]^4$ (the square root of the metric does not contribute). Because the overall action needs to be dimensionless, $S \sim [L]^0$, we must have $\kappa^{-1} \sim [L]^{-2}$ or $\kappa \sim [L]^2$. Recall that power counting arguments can result in one of three outcomes for the dimension of the coupling constant $\lambda^{\text{XII}}$:

- $\lambda < 0$: Super-renormalisable

- $\lambda = 0$: Renormalisable

- $\lambda > 0$: Non-renormalisable

For our purposes, it is sufficient to observe that since $[\kappa] > 0$, the Einstein-Hilbert action is non-renormalisable: although t'Hooft and Veltman showed that the theory exhibited a surprising lack of divergences at one-loop level[24], this phenomenon does not seem to be repeated at higher-loop level.[8] Note, however, that in two dimensions, the integral measure becomes $d^2x \sim L^2$, so $[\kappa] = 0$ and the theory becomes renormalisable. We will make use of this fact in Section 4.2.

In contrast, it was argued by Deser and van Nieuwenhuizen[5] that a higher derivative theory of gravity would be renormalisable, but that it would contain **ghosts**, states of negative norm whose existence violates unitarity by allowing probabilities to be negative. Since unitarity is an essential component of any consistent quantum field theory, this would necessarily introduce difficulties in the interpretation of the higher-derivative theory as a step towards a successful quantisation of gravity, but this problem might not be insurmountable. A rigorous proof of this result was provided by Stelle[21], who showed that a higher-derivative theory based on the action (19) was renormalisable to all orders of perturbation theory However, as predicted, the additional renormalisability came with the caveat that unitarity was lost: as noted in [20], this is the quantum field analogue of the negative energy modes that appeared in the classical theories.

---

XIIThese results only hold for when the reference dimension is length or time; if mass is used as the reference dimension, these inequalities are reversed.

Stelle's proof employed a graviton propagator of the form:[21]

$$
\begin{aligned}
D_{\mu\nu\rho\sigma} &= \frac{1}{i(2\pi)^4} \left[ 2\frac{P^{(2)}_{\mu\nu\rho\sigma}(k)}{k^2(\alpha\kappa^2 k^2 + \gamma)} - \frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{k^2([3\beta - \alpha]\kappa^2 k^2 + \frac{1}{2}\gamma)} + \frac{4P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\kappa^2 k^4} \right] \\
&\quad - \frac{1}{i(2\pi)^4} \frac{\Delta[3P^{(0-s)}_{\mu\nu\rho\sigma}(k) - \sqrt{3}\{P^{(0-sw)}_{\mu\nu\rho\sigma}(k) + P^{(0-ws)}_{\mu\nu\rho\sigma}(k)\} + P^{(0-w)}_{\mu\nu\rho\sigma}(k)]}{\kappa^2 k^4} \\
&= \frac{1}{i(2\pi)^4} \left[ 2\frac{(P^{(2)}_{\mu\nu\rho\sigma}(k) - 2P^{(0-s)}_{\mu\nu\rho\sigma}(k))}{\gamma k^2} - \frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{\gamma(k^2 + \gamma[\alpha\kappa^2]^{-1})} + \frac{4P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\gamma(k^2 + \gamma\{2(3\beta - \alpha)\kappa^2\}^{-1})} \right] \\
&\quad - \frac{1}{i(2\pi)^4} \frac{\Delta[3P^{(0-s)}_{\mu\nu\rho\sigma}(k) - \sqrt{3}\{P^{(0-sw)}_{\mu\nu\rho\sigma}(k) + P^{(0-ws)}_{\mu\nu\rho\sigma}(k)\} + P^{(0-w)}_{\mu\nu\rho\sigma}(k)]}{\kappa^2 k^4}
\end{aligned}
\tag{54}
$$

Here, the $P^{(x)}_{\mu\nu\rho\sigma}(k), x \in \{1, 2, (0 - s)(0 - w)\}$ terms are the projectors for symmetric, rank-two tensors while $P^{(0-sw)}_{\mu\nu\rho\sigma}(k) \text{ and } P^{(0-ws)}_{\mu\nu\rho\sigma}(k)$ are the spin-zero transfer operators. It can be seen from the first line of (54) that the general high energy behaviour of this propagator is $k^{-4}$. However, from the third line of (54), we can see the familiar factors of $\alpha$ and $(3\beta - \alpha)$ appearing as they did in Section 3.3.1. Furthermore, the opposite sign of the $k = \gamma[\alpha\kappa^2]^{-1}$ pole on the third line means that non-zero $\alpha$ is once again responsible for the bad quantum behaviour, namely the possibility of either the norms or the energies being negative.

Despite this, the loss of unitarity might be confined to the extent that the higher-derivative model has some value as an effective field theory. To explain what is meant by an effective theory, consider a field theory that we know is valid up to a certain momentum $k_0$. We can integrate out the high-momentum degrees of freedom to find a new theory that is valid up to a lower momentum $k_1 < k_0$. However, by rescaling our distances and momenta, we can expand its region of applicability such that it covers the old theory's domain (i.e. its validity is now considered up to $k_0$). This process of successively integrating out high k and then 'zooming out' through rescaling is known as **coarse graining** and the resultant model is an **effective field theory**. While it is not the 'true' microscopic field theory, it will give averaged results that can still be used to give important predictions.[XIII]

The effective field theory framework relies on the decoupling assumption that there exists a scale $M_{\text{eff}}$ for which the low energy degrees of freedom ($E \ll M_{\text{eff}}$) obey dynamics that can be approximated by autonomous differential equations.[13] An **autonomous differential equation** is an ODE that does not depend on the independent, parametric variable:

$$
\frac{dx}{dt} = f(x(t)) \neq f(x(t), t)
$$

This assumption may fail when massless degrees of freedom are present. The extent to which the decoupling assumption holds is a dynamical problem that can be investigated by extrapolating the data from high energy theory in the hope of discovering results in experimentally accessible energy regimes. In this framework, we

---

[XIII]An appropriate analogy might be the use of semiclassical mechanics in that both subjects retain 'enough' of the underlying theory to be useful.

can see that the effective theory approach relies on the higher energy results being available for its own validation; this viewpoint also emphasises that the true value of obtaining such a consistent theory of quantum gravity lies not in the calculation of higher energy results, but in being able to compute potential signs of quantum gravity at relatively low energies.

The effective theory also depends on the coarse-graining procedure; this similarly affects the renormalisation properties since each application of coarse graning incrementally alter the action. As to be expected from any dynamical system, this differential change in the action can have fixed points whose local vicinity can be divided into stable and unstable manifolds. These are defined as the set of all points related to the fixed point by the given coarse graining procedure via a smooth flow that either flows towards the fixed point and terminates at it in the case of a **stable manifold** or else flows away from the fixed point in the case of an **unstable manifold**.

We are particularly interested in the unstable manifold since any small perturbation from the fixed point on the unstable manifold will follow a renormalised trajectory and flow away from it: it is these fixed points that will allow a continuous limit to be taken. This is because of the fact that, by definition, a **fixed point** for a given coarse graining procedure does not change under this particular coarse graining procedure (i.e. the location of the fixed point does not change with k); thus, if we have any cutoff $k_0$, moving to a different value, $k_1$ will not change the value of the fixed point. Now, if the fixed point is stable, then any deviation from the value $k_0$ will cause the flow to return to $k_0$: there will be <u>no</u> new physics to extract from the change in cutoff as the flow is not allowed to move away from the fixed point. Hence, in order for there to be any overall renormalisation flow, we need to look at the unstable manifolds.

However, our analysis of the unstable manifold will be complicated if the coupling constants have the potential of diverging at intermediate points - this danger could be realised by the presence of a pathological region which could contain unphysical singularities. To avoid this, we need to stipulate specifically that the manifold around the fixed point partitions into stable and unstable manifolds without including any of the unphysical regions described above.

For a given action with a particular set of couplings, we can make a number of definitions to classify the various couplings:

- An **irrelevant coupling** is a coupling that moves towards the fixed point after a fixed number of coarse grainings and hence lies on the stable manifold of the fixed point.

- A **relevant coupling** is a coupling that moves away from the fixed point after a number of coarse grainings, thereby lying on the unstable manifold of the fixed point.

- The **dimension of the unstable manifold** is hence defined as the number of independent, renormalised trajectories connected to the fixed point (since each renormalised trajectory, by definition, is a flow of a different coupling constant on the unstable manifold).

- An **essential coupling** is a genuine coupling of the field, whereas an **inessential coupling** can be absorbed by a redefinition of the fields involved.

- A **quasi-essential** coupling is a **running coupling** (a coupling whose value depends on the energy scale) that runs to a fixed, positive asymptotic value. If the ith coupling constant $\lambda_i$ runs, then its behaviour when undergoing changes in energy scale will be determined by its **beta functions**, which will always be a differential equation of the form:

$$\beta_i \equiv k\frac{d\lambda_i}{dk} \Leftrightarrow \frac{d\lambda_i}{d\tau} \tag{55}$$

Here, $k$ is the momentum/energy whereas $\tau \equiv \ln(k)$. By definition of the fixed point, these beta functions vanish at the fixed point.

The case could arise that the unstable manifold is infinite-dimensional for a particular description of a theory (for example, Quantum Chromodynamics in lightfront formulation[15]). In such cases, the above geometric picture breaks down. However, the presence of concealed dependencies is conjectured so that the formerly infinite number of independent couplings is actually finite,[XIV] in a similar way that a countably infinite set of integers can be made finite by employing modular arithmetic. However, it is clear that such a reduction of couplings must remain intact under the renormalisation flow, else new couplings would appear as the momentum scale was changed.

Although this infinite dimensional case might appear to have only academic interest, (at least) one fascinating example demonstrates that it must be afforded due consideration. The $k^{-4}$ propagator discussed so far is not the only possible choice of propagator in higher-derivative theories. Gomis and Weinberg have investigated the use of a $k^{-2}$ graviton propagator and found that these do not have the same problems with unitarity as associated with the $k^{-4}$ propagator.[9] However, the $k^{-2}$ propagator suffers from exactly this infinity of unphysical modes described above and is hence called **weakly renormalisable**, to distinguish it from **renormalisable theories** that are realised using a finite number of coupling constants. Perturbation theory then shows that a **Gaussian fixed point** (i.e. a trivial zero) exists for this theory (depending on the coupling flow, this may turn out be a **non-Gaussian fixed point** (i.e. non-trivial) in the full, nonperturbative theory).

---

[XIV]After all, QCD in the lightfront formulation is still QCD, only written in a different way, and the two formulations should be equivalent.

So far we have discussed unstable manifolds, but we have not yet emphasised the necessity of the fixed points themselves. Because of its scale invariance, the fixed point itself must be independent of the value of the ultraviolet cutoff, meaning that we can vary the UV cutoff regulator without changing the value of the fixed point. This is important in the construction of a scale limit: if there was no fixed point, then there would be no reference point that we could use when changing scales, as all points in theory space would have some form of k-dependence.

It should be stressed that a fixed point is only a fixed point with respect to a particular coarse graining operation, and if a different method of momentum scaling is used, then the location of the fixed point will change. However, universality should ensure that all field theories based on fixed points referring to different coarse graining operations have the same asymptotic behaviour, indicating that this behaviour should be considered in terms of an equivalence class of long range limits in which physical quantities are independent of the choice of coarse graining operation. In any case, the scaling limit is found by extrapolating from the fixed point along the renormalised trajectory in the unstable manifold.

Each point on the renormalised trajectory of the unstable manifold is described by using the coupling constants for a given value of momentum, k. If the value of k changes, then a set of new points on the manifold will be mapped out; the fixed point, being scale invariant, won't be affected by this coarse graining procedure. The trajectories of each of the coupling constants can be determined when $k \to \infty$, with the number of independent coupling constants being equivalent to the dimension of the unstable manifold.

However, not all of the paths that these coupling constants follow need be independent: some could be described by the same variables if their paths are described parametrically. Hence, the number of parameters needed to specify a point on a D-dimensional unstable manifold must be $\leq D$, but not necessarily equal to D. Furthermore, each of these parameters can describe a different type of limiting behaviour in the momentum limit, and so the number of parameters needed to specify a point on the unstable manifold gives the number of possible scaling limits. Even if this is infinite, this does not necessarily mean that the theory has no predictive power due to the aforementioned hidden dependencies in the coupling constants.

This leads us to the following definition, first made by Weinberg:[28] an **asymptotically safe theory** is one for which the essential (i.e. relevant) coupling parameters approach a fixed point as the momentum scale of their renormalisation point goes to infinity. In other words, the coupling constant themselves will be finite as the momentum $k \to \infty$, for if this was not true, then it is conceivable that some physically observable quantities would likewise diverge as $k \to \infty$. For example, if the fine structure constant $\alpha(k)$ diverged as $k \to \infty$, then the electric charge would likewise diverge. However, it is not always the case that a

coupling constant is directly related to a physical observable, so it is a meaningful clarification to distinguish between a coupling constant diverging and a physical observable diverging since there is not necessarily a 1:1 correspondence between them.

Although the beta functions for the $k^{-2}$ propagator are not currently known, it is thought that some of the coupling constants would diverge at some finite momentum scale $k_0$ and so be unphysical at any scale greater than this. Thus, at least some of the infinite couplings of the $k^{-2}$ theory will be asymptotically unsafe and hence unphysical. Heuristically, the problem with higher derivative quantum gravity is to reconcile the finite couplings of the Stelle's $k^{-4}$ propagator theory with the unitarity of Weinberg and Gomis's $k^{-2}$ propagator theory. This will presumably be done by discovering hidden dependencies of the infinite couplings on each other, as this would likely show that the set of non-redundant coupling constants in $k^{-2}$ theory is equivalent to the finite set of couplings in $k^{-4}$ theory, thereby directly establishing a correspondence between the theories.

## 4.2  $2 + \epsilon$ Expansion

We now turn to the motivation behind the belief that a gravitational fixed point exists. Although the existence of such a fixed point cannot yet proven with mathematical rigour[XV], a strong hint as to its presence can be gleaned using perturbation theory. The expansion, first performed by Weinberg,[13] is based on the fact that in $d = 2$ dimensions, **Newton's constant** (defined below) becomes dimensionless (see Section 4.1):

$$g_N \equiv 16\pi G \Leftrightarrow 2\kappa \tag{56}$$

As a result, the bare Einstein-Hilbert action in 2d is power-counting renormalisable in perturbation theory. Weinberg then proceeded to use dimensional regularisation to switch from 2d to $(2 + \epsilon)$d:

$$S_{EH} = \frac{1}{g_N} \int d^2x \sqrt{g} R \rightarrow \frac{1}{g_N} \int d^{2+\epsilon}x \sqrt{g} R$$

Weinberg used this to show how the renormalisability of the Einstein-Hilbert action changed with $\epsilon$. The result is a flow equation of the form (55):

$$\mu \frac{dg_N}{d\mu} = \epsilon g_N - \gamma g_N^2 = g_N \left( \epsilon - \gamma g_N \right) \tag{57}$$

There is some choice in the coefficient $\gamma$, its value resulting from the choice of reference against which the flow of $g_N$ is measured. This arises as a result of the fact that in 2d, the Einstein-Hilbert action is topological and so extra kinematical poles of $O(\frac{1}{\epsilon})$ arise in the graviton propagator, in addition to the $O(\frac{1}{\epsilon})$ poles that usually occur from the ultraviolet divergences as a result of dimensional regularisation. Regardless of the numerical value of $\gamma$, however, we can still show the existence of a fixed point for $\gamma \neq 0$. By definition, the R.H.S. of (57) must vanish at a fixed point, which it does if:

$$g_N = 0 \quad \text{or} \quad \epsilon - \gamma g_N = 0 \implies g_N = \frac{\epsilon}{\gamma}$$

In order to apply this result to d $= 4$ dimensions, Weinberg extrapolated his argument to state that for $\epsilon = 2$, there will be a fixed point at $g_N = \frac{2}{\gamma}$. Furthermore, this non-trivial fixed point had a one-dimensional manifold, thereby satisfying the criteria for asymptotic stability. The argument loses some credibility from the fact that $\epsilon$ is meant to be a small parameter to enable the use of perturbation theory techniques necessary to derive (57)[XVI], but it at least provides a plausibility argument that a non-trivial fixed point around which $g_N$ is asymptotically safe in the required 4 dimensions does, indeed, exist.

---

[XV]The rigorous proof of asymptotic freedom for quantum Yang-Mills theory is involved in one of the seven prestigious Millennium Prize problems of mathematics.

[XVI]And if we hope to use the regular Einstein-Hilbert action in string theory, which requires 11 dimensions, the situation is worse still.

## 4.3 Proof of Asymptotic Safety For $k^{-4}$ Propagator

While the previous section provided a motivational demonstration of the existence of a non-trivial fixed point for the Einstein-Hilbert action, we already know that such a theory is power-counting unrenormalisable, meaning that its utility is limited. However, we are now in a position to show that the higher derivative theory based on the $k^{-4}$ is asymptotically safe. To do this, consider the higher-derivative action in the form of (22). The one-loop counterterm arising from this action is given by:[13]

$$\Delta S^{(1)} = \frac{\mu^{d-4}}{(4\pi)^2(d-4)} \int d^d x \sqrt{g}\, 2s \left( \frac{40\omega^2 + 30\omega - 1}{12\omega} - \gamma(\omega) \right) \tilde{\lambda}$$
$$+ \frac{\mu^{d-4}}{(4\pi)^2(d-4)} \int d^d x \sqrt{g} \left[ \left( \frac{s}{\kappa^2} \right)^2 \frac{20\omega^2 + 1}{8\omega^2} + \gamma(\omega)\frac{sR}{\kappa^2} + \frac{133}{20}C^2 \right]$$
$$+ \frac{\mu^{d-4}}{(4\pi)^2(d-4)} \int d^d x \sqrt{g} \left[ \frac{5}{36} \left( 8\omega^2 + 12\omega + 1 \right) R^2 + \frac{196}{45}E \right]$$

Here, $\gamma(\omega)$ is a gauge-dependent function of $\omega$.

Although $\gamma(\omega)$ depends on the gauge, the 1-loop flow equations resulting from this counterterm for s, $\omega$ and $\theta$ and universal and given by:

$$\mu\frac{ds}{d\mu} = -\frac{133s^2}{160\pi} \overset{!}{=} 0 \text{ @ fixed points} \implies s \overset{!}{=} 0 \text{ @ fixed points} \tag{58}$$

$$\mu\frac{d\theta}{d\mu} = -\frac{7}{1440\pi^2}(171\theta - 56)s \overset{!}{=} 0 \text{ @ fixed points} \implies s = 0 \text{ or } 171\theta - 56 = 0 \text{ @ fixed points} \tag{59}$$

$$\mu\frac{d\omega}{d\mu} = -\left( \frac{200\omega^2 + 1098\omega + 25}{960\pi^2} \right) s \overset{!}{=} 0 \text{ @ fixed points} \implies s = 0 \text{ or } 200\omega^2 + 1098\omega + 25 = 0 \text{ @ fixed points} \tag{60}$$

From (58), we deduce that $s \overset{!}{0}$ at any fixed point. From (59) and (60), we can see that the full fixed points consist of $\theta$ and $\omega$ either being constant or the solutions of:

$$171\theta - 56 = 0 \implies \theta = \frac{56}{171}$$

$$200\omega^2 + 1098\omega + 25 = 0 \implies \omega = \frac{-1098 \pm \sqrt{1,205,604 - 20,000}}{400} = \frac{-1098 \pm \sqrt{1,185,604}}{400}$$
$$= \frac{-1098 \pm \sqrt{196}\sqrt{6049}}{400} = \frac{-549 \pm \sqrt{49}\sqrt{6049}}{200} = \frac{-549 \pm 7\sqrt{6049}}{200}$$

The fixed points are given by:

$$(s, \theta, \omega) = \left[ (0, \text{constant}, \text{constant}) , \left( 0, \frac{55}{171}, \frac{-549 \pm 7\sqrt{6049}}{200} \right) \right] \tag{61}$$

The flows of the dimensionless Newton and cosmological constants are more complicated in form, but they both have the same structure:[13]

$$\mu\frac{du_i}{d\mu} = k_i u_i + \frac{1}{(4\pi)^2} s u_i X_i(\gamma(\omega))$$

Here, $k_i = \pm 2$ and $X_i(\gamma(\omega))$ is an extranneous function of the gauge-dependent $\gamma(\omega)$. The reason that $X_i$ does not concern us is because we are only interested in finding the fixed points of the above equation; we proved in (61) that for any fixed point, s must vanish and so the $X_i$ will likewise also vanish. The vanishing of s at the fixed points also ensures the vanishing of any product of terms, collectively called a, with s that is differentiated:

$$\frac{d(as)}{d\mu} = \frac{da}{d\mu}s + a\frac{ds}{d\mu}$$

By definition, $\frac{ds}{d\mu}$ vanishes at the fixed point and, as we have shown, s also vanishes at the fixed point, so the entire R.H.S. vanishes. For the practical purpose of finding fixed points, we therefore take:

$$\mu\frac{dg_N}{d\mu} = 2g_N \tag{62}$$

$$\mu\frac{d\lambda}{d\mu} = 2\lambda \tag{63}$$

We will extensively use these results in what follows.

Assuming that $\theta$ and $\omega$ remain at their fixed point values, the most general field redefinition that we can make at one-loop order are given by:

$$\tilde{g}_N = g_N + \hbar\left(c_1 g_N^2 + c_2 g_N s + c_3 s^2\right) + O(\hbar^2)$$
$$\tilde{\lambda}_N = \lambda + \hbar\left(d_1 g_N + d_2 s\right) + O(\hbar^2)$$

This is because each loop adds on either a power of $g_N$ or s. Here, $\hbar$ counts the number of loops and is not Planck's constant: we will set it to unity at the end of the calculation, but retain it as a mathematical book-keeping devide to keep track of the order of the number of loops. Now, as before, because we will ultimately be seeking the fixed points, we discard any terms formed from products with s:

$$\tilde{g}_N \approx g_N + \hbar c_1 g_N^2 \implies \tilde{g}_N^2 + O(\hbar^2)$$

$$\tilde{\lambda} = \lambda + \hbar d_1 g_N$$

Inverting these:

$$\tilde{g}_N \approx g_N + \hbar c_1 \tilde{g}_N^2 \implies g_N \approx \tilde{g}_N - \hbar c_1 \tilde{g}_N^2 \tag{64}$$

$$\lambda \approx \tilde{\lambda} - \hbar d_1 g_N = \tilde{\lambda} - \hbar d_1 \tilde{g}_N + O(\hbar^2) \tag{65}$$

We now substitute (64) into (62):

$$\mu\frac{d}{d\lambda}\left(\tilde{g}_N - \hbar c_1 \tilde{g}_N^2\right) = \mu\frac{d\tilde{g}_N}{d\mu} - \hbar 2c_1 \mu\tilde{g}_N\frac{d\tilde{g}_N}{d\mu} = \mu\frac{d\tilde{g}_N}{d\mu}\left(1 - 2\hbar c_1\tilde{g}_N\right) \overset{!}{=} 2\left(\tilde{g}_N - \hbar c_1\tilde{g}_N^2\right)$$

After rearranging this expression, we can Taylor expand the first brack up to $O(\hbar)$:

$$\mu\frac{d\tilde{g}_N}{d\mu} = \left(1 - 2\hbar c_1\tilde{g}_N\right)^{-1} 2\left(\tilde{g}_N - \hbar c_1\tilde{g}_N^2\right) = \left(1 + 2\hbar c_1\tilde{g}_N + O(\hbar^2)\right) 2\left(\tilde{g}_N - \hbar c_1\tilde{g}_N^2\right) = 2\tilde{g}_N + \hbar 2c_1\tilde{g}_N^2 + O(\hbar^2)$$

To complete the proof, we set $\hbar$ to unity:

$$\mu \frac{d\tilde{g}_N}{d\mu} \to 2\tilde{g}_N \left(1 + c_1 \tilde{g}_N\right) \tag{66}$$

In a similar manner, we substitute (65) into (63):

$$\mu \frac{d}{d\mu}(\tilde{\lambda} - \hbar d_1 g_N) = \mu \frac{d\tilde{\lambda}}{d\mu} - \hbar \mu d_1 \frac{dg_N}{d\mu} \overset{!}{=} -2\left(\tilde{\lambda} - \hbar d_1 \tilde{g}_N\right)$$

To simplify this expression further, we substitute (62):

$$\mu \frac{d\tilde{\lambda}}{d\mu} = \hbar \mu d_1 \frac{dg_N}{d\mu} - 2\left(\tilde{\lambda} - \hbar d_1 \tilde{g}_N\right) = -2\tilde{\lambda} + \hbar\left(d_1 \mu \frac{dg_N}{d\mu} + 2d_1 \tilde{g}_N\right) = -2\tilde{\lambda} + \hbar\left(d_1[2g_N] + 2d_1 \tilde{g}_N\right)$$

Inserting (64) yields:

$$\mu \frac{d\tilde{\lambda}}{d\mu} = -2\tilde{\lambda} + \hbar 2 d_1 \left([\tilde{g}_N - \hbar c_1 \tilde{g}_N^2] + \tilde{g}_N\right) = -2\tilde{\lambda} + \hbar 4 d_1 \tilde{g}_N + O(\hbar^2)$$

Finally, we set $\hbar$ to unity, which gives the final expression:

$$\mu \frac{d\tilde{\lambda}}{d\mu} \to -2\tilde{\lambda} + 4d_1 \tilde{g}_N \tag{67}$$

The flow equation (66) has fixed points:

$$2\tilde{g}_N(1 + \tilde{g}_N c_1) = 0 \implies g_N = 0 \text{ or } 1 + \tilde{g}_N c_1 = 0 \implies \tilde{g}_N = -\frac{1}{c_1}$$

We can now substitute these separately into (67):

$$\tilde{g}_N = 0 \implies -2\tilde{\lambda} = 0 \implies \tilde{\lambda} = 0$$
$$\tilde{g}_N = -\frac{1}{c_1} \implies -2\tilde{\lambda} - \frac{4}{c_1}d_1 = 0 \implies \tilde{\lambda} = -\frac{2}{c_1}d_1$$

These results agree with (2.26) of [13] upon the assumption that $\frac{\partial d_1}{\partial \tilde{\lambda}}$ vanishes. Although there is no obvious reason that $\frac{\partial d_1}{\partial \tilde{\lambda}}$ should vanish, this term should not strictly appear in (2.25) of [13] either. Its presence is due to an error in keeping track of the order of $\hbar$'s: the second term on the R.H.S. of $\mu \frac{dg_N}{d\mu}$ had had a power of $\hbar$ set to unity, but it still contributes to the loop calculation and drops out at $O(\hbar^2)$.

The motivation behind recasting (62) and (63) as (66) and (67) is to draw comparisons with other ways of evaluating these fixed points. A recent study using heat kernel asymptotics confirms the above results for (66) and (67) for particular choices of $c_1$ and $d_1$. The results for (66) and (67) are also strengthened by matching with the equivalent $(g_N, \lambda)$ flow defined via the truncated effective action. We will briefly describe these alternative methods in Section 5

# 5    Conclusion

Whilst the Einstein-Hilbert action is the most general action that can be constructed from two-index, symmetric, divergenceless tensors in 4 dimensions using only up to second order in derivatives in the metric, more general actions can be built by considering higher order derivatives. However, these higher order terms only make small changes to Einstein's equation and can be associated with the introduction of pathological negative energy solutions. We have demonstrated both of these facts explicitly in the linearised higher-derivative theory of the point particle, showing not only the introduction of the negative energy mode for $\alpha \neq 0$, but also that the Newtonian limit only involved the original Einstein-Hilbert term. The latter problem was then shown to be a pervasive difficulty in higher derivative theories through Ostrogradsky's Instability Theorem a general result that limits the physical applicable actions to $f(R)$ theories, at least at a classical level.

The primary reason of continued interest in non-$f(R)$ higher derivative theories stems from their quantum properties, specifically that they are renormalisable. Because of this, renormalisation group techniques involving the location and nature of fixed points from the beta functions of the coupling constants can be used to probe its field theoretical properties. In particular, Weinberg's notion of asymptotic safety - the necessity of the relevant couplings to tend towards a fixed point as the energy scale diverges - is now used as the outstanding criterion of renormalisation viability.

We considered three main theories of gravity, each with an associated physical problem when it came to quantization:

1. Einstein-Hilbert Gravity:

    (a) Non-renormalisable

    (b) Physical, finite couplings

    (c) Unitary

2. Higher derivative gravity with $k^{-4}$ propagator:

    (a) Renormalisable

    (b) Physical, finite couplings

    (c) Non-unitary

3. Higher derivative gravity with $k^{-2}$ propagator:

    (a) Renormalisable

    (b) Unphysical, infinite couplings

(c) Unitary

As an illustration of the concepts mentioned in the previous paragraph, we showed the asymptotic safety of both the Einstein-Hilbert action and higher derivative theory of gravity with a $k^{-4}$ graviton propagator. The beta functions of $k^{-2}$ propagator higher derivative theories are currently unknown, but it is thought that some of the infinite couplings would diverge at finite momentum, leading to this theory being asymptotically unsafe.

There are several different areas of ongoing research in these topics that this project did not consider in detail, particularly with regards to the quantum properties of higher derivative theories. A relatively fertile area concerns the application of the **Exact Renormalisation Group Equation (ERGE)** which was first developed by Wetterich[29] (in which context the ERGE was known as the Wetterich equation) and adapted for gravitational use by Reuter[18]. It has the advantage of being non-perturbative in nature, meaning that it can be used to demonstrate non-perturbative renormalisability even if the theory turns out to be ultimately non-perturbatively renormalisable. The ERGE is difficult to solve, but can be dealt with by using an effective action truncation, whereupon it can be expanded using heat-kernel asymptotics.[XVII][4] However, higher-derivative theories have been studied in cosmological and string-theoretical contexts.[6]

An interesting topic for higher derivative gravity at a classical level involves the Palatini formulation of General Relativity. It has been shown[17] that Ostrogradsky's Instability Theorem can be circumvented for a (Born-Infeld) modification to the action beyond the aforementioned $f(R)$ theories provided that the resulting Lagrangian is varied in the Palatini formalism. Furthermore, it has been shown[2] that there can be solutions (of the equation of motion) found using the Palatini formalism that cannot necessarily be found using the metric formalism discussed in this project. It might therefore be a useful venture to analyse the solutions of the equations of motion found by the Palatini formalism that are unique to the Palatini formalism, with the view of finding a previously neglected solutions that might offer new insights into the renormalisation process.

---

[XVII]These are also referred to as Hessian methods.

# 6 Appendices

## 6.1 Appendix 1: Riemann & Ricci Tensors For a Static, Spherically Symmetric Metric

Following the analysis of [20], the metric describing such a field can be written using Schwarzschild coordinates as:

$$ds^2 = A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - B(r)dt^2$$

This means that the metric elements are:

$$g_{rr} = A(r) \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2\sin^2\theta \qquad g_{tt} = -B(r)$$

Correlating this with (8.1.7) of [27][XVIII], we immediately infer that:

$$g^{rr} = A^{-1}(r) \qquad g^{\theta\theta} = r^{-2} \qquad g_{\phi\phi} = r^{-2}\sin^{-2}\theta \qquad g_{tt} = -B^{-1}(r)$$

and the non-vanishing Christoffel symbols will be given by (8.1.11) of [27]:

$$\Gamma^t_{tr} = \frac{B'}{2B} \qquad \Gamma^r_{rr} = \frac{A'}{2A} \qquad \Gamma^r_{\theta\theta} = \frac{r}{A} \qquad \Gamma^r_{\phi\phi} = \frac{r\sin^2\theta}{A} \qquad \Gamma^r_{tt} = \frac{B'}{2A}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^\phi_{\phi r} = \frac{1}{r} \qquad \Gamma^\phi_{\phi\theta} = \cot\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

In the above and successive formulae, the dashes indicate differentiation with respect to r.

In order to minimise the computation in the next section, we will need to compute the Riemann tensor components for this metric.[XIX] The Riemann tensor is given by (6.1.5) of [27]:

$$R^\alpha_{\beta\gamma\delta} = \partial_\delta\Gamma^\alpha_{\beta\gamma} - \partial_\gamma\Gamma^\alpha_{\beta\delta} + \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta} - \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma}$$

Let us now consider how many independent components we have to calculate. Ostensibly for an n-dimensional theory, since the Rieman tensor has 4 indices, each of which can take one of n values, then there are $4^n$ different combinations, but the symmetries heavily restrict this value. Firstly, the antisymmetry means that any Riemann tensor with 3 or 4 repeated indices vanishes, so we need only consider those with 2 or less repeated indices. There will be $\binom{n}{2}$ ways of choosing two distinct indices out of the n available.

Furthermore, there will be $\binom{n}{3}$ ways of choosing 3 distinct indices, with the repeated index being any one of these 3; this means that there will be $3\binom{n}{3}$ components with one repeated index. Finally, for 4 distinct

---

[XVIII]Note that [27] uses the $(+, -, -, -)$ metric, whereas [20] uses the $(-, +, +, +)$ metric.

[XIX]Despite the applicability of these results, the author was unable to locate them in any textbooks or papers, so they will be derived here.

indices, there will be $n(n-1)(n-2)(n-3)$ different arrangements, but the symmetry and antisymmetries reduce this number by a factor of $2^3 = 8$. Finally, the **first Bianchi identity**:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} \Leftrightarrow R_{\alpha[\beta\gamma\delta]} = 0 \qquad (68)$$

ensures that if we know two values of $R_{\alpha\beta\gamma\delta}$, then the third is already determined. This means that we must multiply this final value by a factor of $\frac{2}{3}$.

Putting this altogether, the total number of independent components of the Riemann tensor is:

$$\binom{n}{2} + 3\binom{n}{3} + \frac{2}{3}\frac{n(n-1)(n-2)(n-3)}{8} \Leftrightarrow \frac{n^2(n^2-1)}{12} \qquad (69)$$

The first, second and third terms correspond to the number of 2, 3 and 4 distinct indices respectively; the final term is the usual formula that is quoted for this result. In our case, $n = 4$, so we will have $\binom{4}{2} = 6$ components with 2 distinct indices, $3\binom{n}{3} = 12$ components with 3 distinct indices and $\frac{2}{3}\frac{4\times3\times2\times1}{8} = 2$ components with 4 distinct indices, or 20 in total. It might be surprising at first glance that there are many more components with repeated indices than there are with no repeated indices, but it is exactly in the latter case that there is the most freedom in index choice before the symmetries are imposed and therefore the case for which the symmetries have the most restrictive power.

We list the 20 components of the Riemann tensor below:

1. $R^t_{rtr} = \partial_r\Gamma^t_{tr} - \partial_t\Gamma^t_{rr} + \Gamma^\mu_{rt}\Gamma^t_{\mu r} - \Gamma^\mu_{rr}\Gamma^t_{\mu t} = \partial_r\left(\frac{B'}{2B}\right) + \Gamma^t_{rt}\Gamma^t_{tr} - \Gamma^r_{rr}\Gamma^t_{tr} = \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right)$

2. $R^t_{\phi t\phi} = \partial_\phi\Gamma^t_{t\phi} - \partial_t\Gamma^t_{\phi\phi} + \Gamma^\mu_{\phi t}\Gamma^t_{\mu\phi} - \Gamma^\mu_{\phi\phi}\Gamma^t_{\mu t} = -\Gamma^r_{\phi\phi}\Gamma^t_{tr} = \frac{r\sin^2\theta}{A}\frac{B'}{2B}$

3. $R^t_{\theta t\theta} = \partial_\theta\Gamma^t_{t\theta} - \partial_t\Gamma^t_{\theta\theta} + \Gamma^\mu_{\theta t}\Gamma^t_{\mu\theta} - \Gamma^\mu_{\theta\theta}\Gamma^t_{\mu t} = -\Gamma^r_{\theta\theta}\Gamma^t_{tr} = \frac{r}{A}\frac{B'}{2B}$

4. $R^r_{\phi r\phi} = \partial_\phi\Gamma^r_{\phi r} - \partial_r\Gamma^r_{\phi\phi} + \Gamma^\mu_{\phi r}\Gamma^r_{\mu\phi} - \Gamma^\mu_{\phi\phi}\Gamma^r_{\mu r} = -\partial_r\left(-\frac{r\sin^2\theta}{A}\right) + \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^r_{rr} = -r\sin^2\theta\frac{A'}{2A^2}$

5. $R^r_{\theta r\theta} = \partial_\theta\Gamma^r_{r\theta} - \partial_r\Gamma^r_{\theta\theta} + \Gamma^\mu_{\theta r}\Gamma^r_{\mu\theta} - \Gamma^\mu_{\theta\theta}\Gamma^r_{\mu r} = -\partial_r\left(\frac{1}{r}\right) + \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^r_{rr} = -\frac{rA'}{2A^2}$

6. $R^\theta_{\phi\theta\phi} = \partial_\phi\Gamma^\theta_{\theta\phi} - \partial_\theta\Gamma^\theta_{\phi\phi} + \Gamma^\mu_{\theta\phi}\Gamma^\theta_{\mu\phi} - \Gamma^\mu_{\phi\phi}\Gamma^\theta_{\mu\theta} = -\partial_\theta\left(-\sin\theta\cos\theta\right) + \Gamma^\phi_{\theta\phi}\Gamma^\theta_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\theta_{r\theta} = \sin^2\theta\left(-1 + \frac{1}{A}\right)$

7. $R^t_{rt\phi} = \partial_\phi\Gamma^t_{tr} - \partial_t\Gamma^t_{r\phi} + \Gamma^\mu_{rt}\Gamma^t_{\mu\phi} - \Gamma^\mu_{\phi r}\Gamma^t_{\mu t} = 0$

8. $R^t_{rt\theta} = \partial_\theta\Gamma^t_{tr} - \partial_t\Gamma^t_{r\theta} + \Gamma^\mu_{rt}\Gamma^t_{\mu\theta} - \Gamma^\mu_{\theta r}\Gamma^t_{\mu t} = 0$

9. $R^t_{\theta t\phi} = \partial_\phi\Gamma^t_{t\theta} - \partial_t\Gamma^t_{\theta\phi} + \Gamma^\mu_{\theta t}\Gamma^t_{\mu\phi} - \Gamma^\mu_{\phi\theta}\Gamma^t_{\mu t} = 0$

10. $R^r_{tr\phi} = \partial_\phi\Gamma^r_{rt} - \partial_r\Gamma^r_{t\phi} + \Gamma^\mu_{tr}\Gamma^r_{\mu\phi} - \Gamma^\mu_{\phi t}\Gamma^r_{\mu r} = 0$

11. $R^r_{tr\theta} = \partial_\theta \Gamma^r_{rt} - \partial_r \Gamma^r_{t\theta} + \Gamma^\mu_{tr}\Gamma^r_{\mu\theta} - \Gamma^\mu_{\theta t}\Gamma^r_{\mu r} = 0$

12. $R^r_{\theta r\phi} = \partial_\phi \Gamma^r_{r\theta} - \partial_r \Gamma^r_{\theta\phi} + \Gamma^\mu_{\theta r}\Gamma^r_{\mu\phi} - \Gamma^\mu_{\phi\theta}\Gamma^r_{\mu r} = 0$

13. $R^\theta_{t\theta r} = \partial_r \Gamma^\theta_{\theta t} - \partial_\theta \Gamma^\theta_{tr} + \Gamma^\mu_{t\theta}\Gamma^\theta_{\mu r} - \Gamma^\mu_{rt}\Gamma^\theta_{\mu\theta} = 0$

14. $R^\theta_{t\theta\phi} = \partial_\phi \Gamma^\theta_{\theta t} - \partial_\theta \Gamma^\theta_{t\phi} + \Gamma^\mu_{t\theta}\Gamma^\theta_{\mu\phi} - \Gamma^\mu_{\phi t}\Gamma^\theta_{\mu\theta} = 0$

15. $R^\theta_{r\theta\phi} = \partial_\phi \Gamma^\theta_{\theta r} - \partial_\theta \Gamma^\theta_{r\phi} + \Gamma^\mu_{r\theta}\Gamma^\theta_{\mu\phi} - \Gamma^\mu_{\phi r}\Gamma^\theta_{\mu\theta} = 0$

16. $R^\phi_{t\phi r} = \partial_r \Gamma^\phi_{\phi t} - \partial_\phi \Gamma^\phi_{tr} + \Gamma^\mu_{t\phi}\Gamma^\phi_{\mu r} - \Gamma^\mu_{rt}\Gamma^\phi_{\mu\phi} = 0$

17. $R^\phi_{t\phi\theta} = \partial_\theta \Gamma^\phi_{\phi t} - \partial_\phi \Gamma^\phi_{t\theta} + \Gamma^\mu_{t\phi}\Gamma^\phi_{\mu\theta} - \Gamma^\mu_{\theta t}\Gamma^\phi_{\mu\phi} = 0$

18. $R^\phi_{r\phi\theta} = \partial_\theta \Gamma^\phi_{\phi r} - \partial_\phi \Gamma^\phi_{r\theta} + \Gamma^\mu_{r\phi}\Gamma^\phi_{\mu\theta} - \Gamma^\mu_{\theta r}\Gamma^\phi_{\mu\phi} = 0$

19. $R^t_{r\theta\phi} = \partial_\phi \Gamma^t_{\theta r} - \partial_\theta \Gamma^t_{r\phi} + \Gamma^\mu_{r\theta}\Gamma^t_{\mu\phi} - \Gamma^\mu_{r\phi}\Gamma^t_{\mu\theta} = 0$

20. $R^t_{\phi\theta r} = \partial_r \Gamma^t_{\theta\phi} - \partial_\theta \Gamma^t_{\phi r} + \Gamma^\mu_{\phi\theta}\Gamma^t_{\mu r} - \Gamma^\mu_{\phi r}\Gamma^t_{\mu\theta} = 0$

From this, we can calculate the components of the Ricci tensor. The non-diagonal components (i.e. $R_{\mu\nu}$ with $\mu \neq \nu$) all vanish because of the vanishing nature of the Riemann tensor with 3 or more distinct indices; the non-diagonal components are:

$$R_{tt} \equiv R^\mu_{t\mu t} = R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t} + R^t_{ttt} = g^{r\rho}R_{\rho trt} + g^{\theta\rho}R_{\rho t\theta t} + g^{\phi\rho}R_{\rho t\phi t}$$

$$= g^{r\rho}R_{t\rho tr} + g^{\theta\rho}R_{t\rho t\theta} + g^{\phi\rho}R_{t\rho t\phi} \text{ by antisymmetry of Ricci tensor}$$

$$= g^{r\rho}g_{t\beta}R^\beta_{\rho tr} + g^{\theta\rho}g_{t\beta}R^\beta_{\rho t\theta} + g^{\phi\rho}g_{t\beta}R^\beta_{\rho t\phi}$$

$$= g^{rr}g_{tt}R^t_{rtr} + g^{\theta\theta}g_{tt}R^t_{\theta t\theta} + g^{\phi\phi}g_{tt}R^t_{\phi t\phi} \text{ since metric is diagonal}$$

$$= -BA^{-1}\left[\frac{B''}{2B} - \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right)\right] - \frac{B}{r^2}\left(\frac{r}{A}\frac{B'}{2B}\right) - \frac{B}{r^2\sin^2\theta}\left(\frac{r\sin^2\theta}{A}\frac{B'}{2B}\right)$$

$$= -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{2Ar} - \frac{B'}{2Ar} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{A}$$

$$R_{rr} \equiv R^\mu_{r\mu r} = R^r_{rrr} + R^\theta_{r\theta r} + R^\phi_{r\phi r} + R^t_{rtr} = g^{\theta\theta}g_{rr}R^r_{\theta r\theta} + g^{\phi\phi}g_{rr}R^r_{\phi r\phi} + R^t_{rtr}$$

$$= r^{-2}A\left(-\frac{rA'}{2A^2}\right) + r^{-2}\sin^{-2}\theta A\left(-\frac{r\sin^2\theta A'}{2A^2}\right) + \left[\frac{B''}{2B} - \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right)\right]$$

$$= -\frac{1}{2r}\frac{A'}{A} - \frac{1}{2r}\frac{A'}{A} + \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) = -\frac{1}{r}\frac{A'}{A} + \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right)$$

$$R_{\theta\theta} \equiv R^\mu_{\theta\mu\theta} = R^r_{\theta r\theta} + R^\theta_{\theta\theta\theta} + R^\phi_{\theta\phi\theta} + R^t_{\theta t\theta} = R^r_{\theta r\theta} + g^{\phi\phi}g_{\theta\theta}R^\theta_{\phi\theta\phi} + R^t_{\theta t\theta}$$

$$= -\frac{rA'}{2A^2} + \frac{r^2}{r^2\sin^2\theta}\sin^2\theta\left(-1 + \frac{1}{A}\right) + \frac{r}{A}\frac{B'}{2B} = -1 + \frac{1}{A} + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right)$$

$$R_{\phi\phi} \equiv R^{\mu}_{\phi\mu\phi} = R^{r}_{\phi r\phi} + R^{\theta}_{\phi\theta\phi} + R^{\phi}_{\phi\phi\phi} + R^{t}_{\phi t\phi} = -r\sin^2\theta\frac{A'}{2A} + \sin^2\theta\left(-1+\frac{1}{A}\right) + \frac{r\sin^2\theta}{A}\frac{B'}{2B}$$

$$= \sin^2\theta\left[-1+\frac{1}{A}+\frac{r}{2A}\left(-\frac{A'}{A}+\frac{B'}{B}\right)\right] = \sin^2\theta R_{\theta\theta}$$

The above calculations for the Ricci tensor agree with (8.1.13) of [27], increasing confidence in our calculation of the components of the Riemann tensor. From this, we can finally calculate the Ricci scalar:

$$R \equiv R^{\mu}_{\mu} = g^{\mu\nu}R_{\nu\mu} = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} + g^{tt}R_{tt} \text{ since metric is diagonal}$$

$$= A^{-1}\left[-\frac{1}{r}\frac{A'}{A}+\frac{B''}{2B}-\frac{B'}{4B}\left(\frac{B'}{B}+\frac{A'}{A}\right)\right] + \frac{1}{r^2}\left[-1+\frac{1}{A}+\frac{r}{2A}\left(-\frac{A'}{A}+\frac{B'}{B}\right)\right]$$

$$+ \frac{1}{r^2\sin^2\theta}\left[\sin^2\theta\left(-1+\frac{1}{A}+\frac{r}{2A}\left(-\frac{A'}{A}+\frac{B'}{B}\right)\right)\right] - B^{-1}\left[-\frac{B''}{2A}+\frac{B'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{1}{r}\frac{B'}{A}\right]$$

$$= -\frac{1}{Ar}\frac{A'}{A}+\frac{B''}{2AB}-\frac{B'}{4AB}\left(\frac{B'}{B}+\frac{A'}{A}\right)+\frac{B''}{2AB}-\frac{B'}{4AB}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{1}{r}\frac{B'}{AB}+\frac{2}{r^2}\left[-1+\frac{1}{A}+\frac{r}{2A}\left(-\frac{A'}{A}+\frac{B'}{B}\right)\right]$$

$$= \frac{B''}{AB}-\frac{B'}{2AB}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{1}{Ar}\left(\frac{A'}{A}-\frac{B'}{B}\right)-\frac{2}{r^2}+\frac{2}{Ar^2}+\frac{1}{Ar}\left(-\frac{A'}{A}+\frac{B'}{B}\right)$$

$$= -\frac{2}{r^2}+\frac{2}{Ar^2}+\frac{B''}{AB}-\frac{B'}{2AB}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{2}{Ar}\left(\frac{A'}{A}-\frac{B'}{B}\right)$$

## 6.2 Appendix 2: Justification of Jordan's Lemma

To be able to use Jordan's lemma in the proof of the Green's function of the screened Poisson equation, we strictly need that for $e^{iz}f(z)$ with:

$$f(z) \leq \frac{M}{R^n}, n \geq 1$$

Only in this case will:

$$\lim_{R\to\infty} \int_\Gamma e^{iz}f(z)dz \to 0$$

We require, therefore, that:

$$\frac{|ke^{ikr}|}{|k^2+\lambda^2|} \equiv \frac{|z/re^{iz}|}{|(z/r)^2+\lambda^2|} = \frac{|zre^{iz}|}{|z^2+r^2\lambda^2|} \leq \frac{M}{R^n}$$

To this end, by inspection:

$$|z^2+r^2\lambda^2| \geq |r^2\lambda^2|-|z^2| \implies \frac{1}{|z^2+r^2\lambda^2|} \leq \frac{1}{|r^2\lambda^2|-|z^2|}$$

Furthermore, on $\Gamma_R$, $|z^2+r^2\lambda^2| \geq R^2-r^2\lambda^2$ since $\max z = R \geq 0$, meaning that:

$$\frac{1}{|z^2+r^2\lambda^2|} \leq \frac{1}{R^2-r^2\lambda^2}$$

Now, let us briefly assume that:

$$2(R^2-r^2\lambda^2) \leq R^2 \implies \frac{1}{R^2}[2(R^2-r^2\lambda^2)] = 2\left(1-\frac{r^2\lambda^2}{R^2}\right) \leq 1$$

This yields $2 \leq 1$ when $R \to \infty$. Since $R^2 \overset{!}{>} 0$, there is no possibility of changing '$\leq$' into '$\geq$' through division of an inequality by a negative number and so, to avoid this absurd conclusion, we must have that as $R \to \infty$:

$$2(R^2-r^2\lambda^2) \geq R^2 \implies R^2-r^2\lambda^2 \geq \frac{R^2}{2} \implies \frac{1}{R^2-r^2\lambda^2} \leq \frac{2}{R^2}$$

Putting this all together yields the required form with $M = n = 2$:

$$\frac{1}{|z^2+r^2\lambda^2|} \leq \frac{2}{R^2} \to 0 \text{ as } R \to \infty$$

# References

[1] Sidney Coleman *Acausality* SubNuclear Phenomena, ed. Zichichi, A. (1970)

[2] Monica Borunda, Bert Janssen and Mar Bastero-Gil *Palatini Versus Metric Formulation in Higher Curvature Gravity*, arXiv: 0804.4440 [hep-th]

[3] Sean Caroll *Spacetime and Geometry: An Introduction to General Relativity* (2003)

[4] Allesandro Codello, Roberto Peracci and Cristoph Rahmede *Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalisation Group Equation* arXiv: 0805.2909v5 [hep-th]

[5] S Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. D 10 401 (1974)

[6] Masafumi Fukuma, So Matsuura and Tadakatsu Sakai *Higher Derivative Gravity and the ADS-CFT Correspondence*, Prog.Theor.Phys. 105 (2001) 1017-1044

[7] Goldstein, Safko and Poole *Classical Mechanics: 3rd Edition* (2002)

[8] Marc Goroff and Augusto Sagnotti *The Ultraviolet Behaviour of Einstein Gravity* Nuclear Physics B266 (1986) 709-736

[9] J. Gomis and S Weinberg *Are Non-Renormalizable Gauge Theories Renormalizable?* Nucl. Phys. B469 473 (1996), arXiv: 9510087 [hep-th]

[10] Hassan Khalili *Non-Linear Systems* (2002)

[11] David Lovelock *The Einstein Tensor and Its Generalizations* J. Math. Phys. 12, 498 (1971)

[12] Alberto Navarro and Jose Navarro *Lovelock's Theorem Revisited* arXiv:1005.2386 [math-ph]

[13] Max Niedermaier *The Asymptotic Safety Scenario in Quantum Gravity - An Introduction*

[14] Max Niedermaier *Gravitational Fixed Points From Perturbation Theory* PhysRevLett.103.101303 (2009)

[15] R Perry and K Wilson *Perturbative Renormalisability With an Infinite Number of Relevant and Marginal Operators* Nucl. Phys. B403, 587 (1993)

[16] Tonguc Rador *Acceleration of the Universe via f(R) Gravities and the Stability of Extra Dimensions* arXiv: 0701267 [hep-th]

[17] Srivatsan Rajagopal and Hauz Kumar *Ostrogradsky Instability and Born-Infeld Modified Cosmology in Palatini Formalism* arXiv: 1303.6026 [gr-qc]

[18] Martin Reuter *Non-Perturbative Evolution Equation For Quantum Gravity* Phys. Rev. D57 (1998) 971-985

[19] David Lovelock and Hanno Rund *Tensors, Differential Forms and Variational Principles* (1975)

[20] K. S. Stelle  *Classical Gravity With Higher Derivatives* Gen. Rel. Grav. 9, 353 (1978)

[21] K. S. Stelle  *Renormalization of higher derivative Quantum Gravity* Phys. Rev. D 16, 953 (1977)

[22] Kumil Srivastava *General Relativity and Cosmology* (2001)

[23] A. Strominger: Phys. Rev. D30, 2257 (1984)

[24] Gerard t'Hooft and Martin Veltman *One-Loop Divergences in the Theory of Gravitation* Annales Poincare Phys.Theor. A20 (1974) 69-94

[25] J R Vanstone *Connections Satisfying a Generalized Ricci Lemma* Canad. J. Math. 16 (1964), 549-560

[26] Robert Wald *General Relativity* (1984)

[27] Steven Weinberg *Gravitation and Cosmology Principles and Applications of the General Theory of Relativity* (1972)

[28] S. Weinberg  *Ultraviolet divergencies in quantum theories of gravitation* General relativity, an Einstein Centenary survey, Eds. S. Hawking, W. Israel, Cambridge University Press, (1979).

[29] Christoph Wetterich *Exact Evolution Equation For the Effective Potential* Phys. Lett. B 301.90

[30] R P Woodard *Avoiding Dark Energy With 1/R Modifications of Gravity* arXiv: 0601672 [astro-ph]

[31] Anthony Zee *Einstein Gravity in a Nutshell* (2013)