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A Review of the $\mathcal{N}$=4 Super Yang-Mills/Type IIB AdS/CFT Correspondence

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Abstract

The original AdS/CFT correspondence relating $\mathcal{N} = 4$ super Yang-Mills to Type IIB string theory in $AdS_5 \times S^5$ is discussed. The necessary background is first reviewed, with the relevance to the conjecture emphasised throughout. The correspondence is then motivated via two arguments (the large N limit of gauge theories and the decoupling argument), and stated in three different forms of varying strength. A precise mapping between the observables of the two theories is then provided, and some simple checks of the weakest form of the correspondence (relating classical supergravity to strongly-coupled gauge theory) are discussed. Finally, extensions of the correspondence beyond the original case are briefly considered.

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Declaration

The following dissertation is my own work; the structure and manner in which concepts are explained is my own, though numerous resources have been used in forming that understanding, and references are given where appropriate. Some sections follow closely the work of others, and are always indicated as such.

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CHAPTER 1

Introduction

The *Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence*, otherwise known as the *gauge/gravity duality*, is one of the major breakthroughs to arise from string theory in recent years. The correspondence is significant from both a conceptual and practical point of view; not only does it shed valuable physical insight into both sides of the correspondence, but it also provides new ways of performing calculations where more conventional methods are intractable.

The correspondence, roughly speaking, states the equivalence between a string theory containing gravity living in a certain geometry, and a gauge theory living on the boundary of that geometry. More precisely, the strongest form of the original correspondence due to Maldacena [MAL] states that the 10-dimensional Type IIB superstring theory on the product space $AdS_5 \times S^5$ (with 5-form flux N) is equivalent to $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with gauge group $SU(N)$, living on the flat 4-dimensional boundary of AdS_5. What one means by 'equivalence' in this context is something that will be clarified throughout this dissertation; essentially it means that there is a one-to-one correspondence between all aspects of the theories including the global symmetries, observables, and correlation functions. The theories are thus considered to be *dual* descriptions of each other; this notion of duality is an interesting one because it turns out that the regimes within which it is possible to perform calculations easily do not coincide on the two sides of the correspondence. Indeed, the correspondence comes in several forms of different strengths, related to which restrictions are imposed on the various parameters in the theories; depending on the form of the correspondence, calculations on either side are possible to differing extents. No form of the correspondence has been proven in a rigorous manner (leading it to be known also as the AdS/CFT *conjecture*), though considerable evidence has been offered in their support, some of which we shall discuss in the present dissertation.

As we shall see, the correspondence can be motivated by an argument which itself rests fundamentally on a duality; namely, a dual interpretation of objects in string theory known as *D-branes*. On the one hand, these objects are considered to be dynamical hyperplanes upon which the endpoints of open strings are fixed (but are free to move parallel to the brane); such objects arise naturally in the analysis of the open string as we shall see. On the other hand, D-branes can be considered as background solutions (with particular symmetries) to the low energy effective spacetime theory of string theory known as *supergravity*; one can then consider closed strings propagating in such a background. That these points of view are equivalent is of great importance, since by considering a particular physical set-up from each in turn, we shall see that (in certain limits) there are two decoupled theories in both interpretations; by recognising a common theory present, we are then led to identify the other two theories as equivalent or dual descriptions, which is exactly the AdS/CFT correspondence mentioned above. This *decoupling argument* will be described in detail in chapter 6.

The fact that the information of the 10-dimensional dynamics (compactified onto a 5-dimensional space) can somehow be encoded in a 4-dimensional theory has led to the conjecture being known as the *holographic principle*, in analogy to the way in which conventional holograms encode the information about a 3-dimensional object in a 2-dimensional surface. Indeed, the question of whether holography may play a role in a theory of quantum gravity has been entertained for sometime, originating in the crucial result that black holes have an entropy proportional to the *area* of their horizon [BEK]; this is in contrast to familiar thermodynamic systems for which the entropy is an extensive property that scales with the volume of the system. Since the original Maldacena conjecture the AdS/CFT correspondence has been extended to other cases, containing, for example, field theories with less supersymmetry or no conformal symmetry on the gauge side, and different string theories and geometries on the gravity side; in all cases there remains this holographic aspect, equating two theories in spacetimes of different dimensions. Furthermore, recently there has been considerable work into investigating a possible *de Sitter/Conformal Field Theory (dS/CFT) correspondence* [STR], something that would perhaps attract even more interest considering the positiveness of the experimentally observed cosmological constant.

In addition to these conceptual curiosities, the correspondence is computationally very powerful by virtue of the fact that non-perturbative problems in super Yang-Mills theory can be studied using perturbative string theory. In fact, in certain limits (to be discussed in more detail later) Type IIB string theory reduces to a classical supergravity theory, and so one may use the correspondence to study strongly-coupled gauge theories simply using classical gravity theory. Although the gauge theories in question (e.g. $\mathcal{N} = 4$ SYM with a large number of colours N) are quite remote from those we believe to be realised in nature (e.g. QCD which is neither supersymmetric nor conformal and has $N = 3$), the correspondence is continually being extended to new cases, and valuable general properties of strongly-coupled gauge theories are being learnt from these studies. The correspondence has also recently been greatly used in the field of condensed matter physics, and has led to the creation of a subject now known as the *Anti-de Sitter/Condensed Matter Theory (AdS/CMT) correspondence* (see [HAR]). We shall not discuss such applications in the present dissertation however, and our focus shall be predominantly on the original Maldacena correspondence; it is nevertheless interesting to note that a correspondence that appears so fundamental has found application to the physics of large systems.

The AdS/CFT correspondence brings together many areas of modern theoretical physics. For example, on one side of the correspondence one has the $\mathcal{N} = 4$ SYM theory; this is a supersymmetric but also a conformal field theory, which in fact combine to form a larger superconformal symmetry. On the other side of the correspondence one must study superstring and supergravity theory (including the properties of D-branes), in addition to the properties of a maximally symmetric spacetime known as anti-de Sitter space. In Part 1 of this dissertation I will thus introduce the different components of the correspondence, all of which must be adequately tackled in order to understand the conjecture; considerable time will thus be devoted to this purpose, and the relevance to the correspondence will be emphasised throughout. In Part 2 I will then motivate (via two arguments) and state the original AdS/CFT conjecture in its different forms; I will then describe the precise mapping between the observables on the two sides, and perform some simple checks of the correspondence. Finally, in Part 3 I will briefly describe some extensions of the correspondence to cases other than the original Maldacena conjecture.

Part I

Ingredients of the Correspondence

Conformal Field Theory

The study of conformal field theory (CFT) will be crucial in the following since, in addition to the fact that string theory can be described as a 2-dimensional CFT, the gauge theory in the correspondence ($\mathcal{N} = 4$ SYM) exhibits conformal invariance. A CFT is simply a quantum field theory (QFT) that has conformal invariance; however, this turns out to be a very strict condition, greatly restricting the QFT and its correlation functions, and requiring the introduction of new concepts not present in other field theories. In this chapter we will first introduce conformal transformations and the conformal algebra, and then proceed to review the key features of CFT relevant for the AdS/CFT correspondence.

2.1 Conformal Transformations

A *conformal transformation* in $\mathbb{R}^{1,d-1}$ is a local transformation $x_\mu \to \tilde{x}_\mu(x)$ such that the line element changes by a scaling:

$$\eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \Omega(x)^2 \eta_{\rho\sigma} dx^\rho dx^\sigma \tag{2.1}$$

for some function $\Omega(x)$ i.e. angles, but not necessarily distances, are preserved by the transformation. From the chain rule $d\tilde{x}^\mu = (\partial\tilde{x}^\mu/\partial x^\sigma)dx^\sigma$ one then trivially derives:

$$\eta_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\sigma} \frac{\partial \tilde{x}^\nu}{\partial x^\rho} = \Omega(x)^2 \eta_{\rho\sigma} \tag{2.2}$$

and we see clearly that Poincaré transformations, for which $\Omega(x) = 1$ (forming the isometry group of $\mathbb{R}^{1,d-1}$), are a subset of conformal transformations. By considering infinitesimal transformations of the form $\tilde{x}_\mu = x_\mu + v_\mu(x)$ and $\Omega(x) = 1 + \omega(x)$ it is easy to derive from (2.2), by working to first order in v_μ and ω, the equation:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\omega(x)\eta_{\mu\nu} \tag{2.3}$$

Taking the trace one finds that $\omega(x) = (\partial \cdot v(x))/d$ and thus we obtain the *conformal Killing equation*:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}(\partial \cdot v)\eta_{\mu\nu} \tag{2.4}$$

Note that in a general spacetime this equation changes by replacing $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial \to \nabla$, though our case of interest will be that of flat spacetime above.

One must solve (2.4) for $v(x)$ to obtain the infinitesimal conformal transformations. For $d = 2$, it is very easy to show by considering the different possible values of μ and ν that equation (2.4) is equivalent to the Cauchy-Riemann equations, and thus conformal transformations are generated by all holomorphic functions $v(x) \equiv v_1(x) + iv_2(x)$. For $d > 2$ the general solution to (2.4) is (see [GOM]):

$$v_\mu(x) = a_\mu + \omega_{\mu\nu}x^\nu + \lambda x_\mu + b_\mu x^2 - 2(b \cdot x)x_\mu \tag{2.5}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ but a_μ, b_μ and λ are arbitrary. The parameters $a_\mu, \omega_{\mu\nu}, \lambda$ and b_μ correspond to translations, rotations, scale transformations (or *dilatations*), and *special conformal transformations* respectively, giving a total of $d + d(d-1)/2 + 1 + d = (d+1)(d+2)/2$ parameters.

In addition to the above continuous transformations generated by infinitesimal Killing vectors, one can consider a discrete conformal transformation, known as *inversion*, defined by:

$$x^\mu \to \tilde{x}^\mu = x^\mu / x^2 \tag{2.6}$$

which is clearly a conformal transformation with $\Omega(x) = 1/x^2$. Note that the term 'conformal transformations' usually refers to the continuous transformations generated by infinitesimal conformal Killing vectors as in (2.5) and so does not include inversions (much in the same way that the term 'Lorentz transformations' usually only refers to the proper (orthochronous) subgroup of the Lorentz group). Importantly, the special conformal transformations can be constructed by performing a translation, preceded and proceeded by an inversion:

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + b^\mu \to \frac{\frac{x^\mu}{x^2} + b^\mu}{(\frac{x^\mu}{x^2} + b^\mu)^2} = \frac{x^\mu + b^\mu x^2}{1 + b^2 x^2 + 2b \cdot x} =_{b_\mu \to 0} \; x^\mu + (b^\mu x^2 - 2\,(b \cdot x) x^\mu) \tag{2.7}$$

which is exactly the infinitesimal form of a special conformal transformation as contained within equation (2.5).

2.2 The Conformal Algebra

The conformal transformations form a group known as the *conformal group*. Denoting the generators of translations, rotations, scale transformations, and special conformal transformations respectively as $P_\mu, M_{\mu\nu}, D$ and K_μ, we may use the infinitesimal transformations contained in (2.5) to easily find the following differential operator representations:

$$P_\mu \equiv -i\partial_\mu \tag{2.8}$$
$$M_{\mu\nu} \equiv -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.9}$$
$$D \equiv i x^\mu \partial_\mu \tag{2.10}$$
$$K_\mu \equiv -i(x^2 \partial_\mu - 2x_\mu x \cdot \partial) \tag{2.11}$$

We can then easily calculate the commutators, giving the *conformal algebra* as [GOM]:

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D + M_{\mu\nu}) \tag{2.12}$$
$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \tag{2.13}$$
$$[M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \tag{2.14}$$
$$[D, P_\mu] = -iP_\mu \tag{2.15}$$
$$[D, K_\mu] = iK_\mu \tag{2.16}$$
$$[M_{\mu\nu}, D] = 0 \tag{2.17}$$
$$[P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \tag{2.18}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i\,(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\nu\rho} M_{\mu\sigma}) \tag{2.19}$$

There are several interesting observations to make regarding this algebra. First, the Poincaré algebra is clearly contained within it (in (2.13), (2.18) and (2.19)), as expected. Second, equation (2.14) states that K_μ is a Lorentz vector, whilst (2.17) states that D is a Lorentz scalar. Third, equation (2.12) shows that dilatations may be obtained simply from combining Poincaré transformations and special conformal transformations (and thus the entire conformal algebra can be generated by Poincaré transformations and inversion alone, following the discussion at the end of section 2.1). Finally, equations (2.15)-(2.16) show that P_μ and K_μ are raising and lowering operators respectively for the dilatation operator D, which proceeds in direct analogy with the algebra for the harmonic oscillator (and which we discuss further in section 2.3). One can also interpret D as reading off the *length dimension* (not to be confused with its inverse, the mass dimension) of the other operators (as is appropriate for a scaling operator) from (2.15-2.17), since P_μ, K_μ and $M_{\mu\nu}$ have length dimensions $-1, +1$ and 0 respectively, as is clear from (2.8)-(2.11).

The algebra above turns out to be isomorphic to $SO(2,d)$ (including inversions one in fact has the full orthogonal group $O(2,d)$), the dimension of which agrees with the number of parameters calculated in section 2.1. This can be seen more explicitly by defining antisymmetric generators L_{MN} $(M = 0, 1...d+1)$ as the following linear combinations of conformal generators (see [GOM]):

$$L_{\mu\nu} \equiv M_{\mu\nu} \qquad L_{d,d+1} \equiv D \qquad L_{\mu d} \equiv \frac{1}{2}(P_\mu + K_\mu) \qquad L_{\mu,d+1} \equiv \frac{1}{2}(P_\mu - K_\mu) \tag{2.20}$$

where $\mu = 0, 1, ...d-1$. Using the conformal algebra, one can then straightforwardly show that the generators L_{MN} do indeed satisfy the $SO(2,d)$ algebra. This will be important later in the AdS/CFT correspondence, where we shall see an identification between the conformal group in 4-dimensions and the isometry group of AdS_5 (see section 6.4).

2.3 Aspects of Conformal Field Theory

One of the central premises of relativistic quantum field theory is that the field operators transform under a representation of the Poincaré group. Under a Poincaré group transformation $x \to \tilde{x} = g \cdot x$, a field operator $\phi^A(x)$ transforms as (following [GOM]):

$$\phi^A(x) \to \tilde{\phi}^A(\tilde{x}) = R^A_B(g)\phi^B(x) \tag{2.21}$$

or equivalently:

$$\tilde{\phi}^A(x) = R^A_B(g)\phi^B(g^{-1} \cdot x) \tag{2.22}$$

where $R^A_B(g)$ is a representation of the group element g (for example, $R^A_B(g) = 1$ for all g for a scalar field). We see that in addition to the transformation of the field argument, there is additional information specified about how the internal or 'spin' index A of the field transforms; this additional information is only relevant for Lorentz transformations, as the translations act only on the argument of the field. For a conformal field theory, one must specify one further bit of information, namely how a field operator transforms under a scale transformation. Under a dilatation $x \to \tilde{x} = \lambda x$ we have:

$$\tilde{\phi}^A_\Delta(x) = \lambda^{-\Delta}\phi^A_\Delta(\lambda^{-1}x) \tag{2.23}$$

where Δ is known as the *dimension* of the operator ϕ_Δ. The dimension of ϕ^A_Δ can be defined equivalently as:

$$[D, \phi^A_\Delta] = -i\Delta\phi^A_\Delta \tag{2.24}$$

i.e. as an eigenvalue of the dilatation operator.

We define in general a *primary operator* [GOM] to be one that transforms as a *tensor density* under general conformal transformations. For example, a scalar $\phi(x)$ transforms as:

$$\tilde{\phi}(x) = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{\Delta/d} \phi(g^{-1} \cdot x) \tag{2.25}$$

The *descendants* are then obtained from the primaries by taking derivatives; these do *not* transform as tensor densities and so cannot be primaries. These concepts can also be introduced in an alternative way. We mentioned previously that P_μ and K_μ act as raising and lowering operators for the dilatation operator D. We can see this by considering an operator ϕ_Δ of dimension Δ and finding the dimension of $[P_\mu, \phi_\Delta]$. Using the Jacobi identity we have;

$$[D, [P_\mu, \phi_\Delta]] = -[P_\mu, [\phi_\Delta, D]] - [\phi_\Delta, [D, P_\mu]] \tag{2.26}$$

and thus using the conformal algebra we find:

$$[D, [P_\mu, \phi_\Delta]] = -i(\Delta + 1)[P_\mu, \phi_\Delta] \tag{2.27}$$

showing that $[P_\mu, \phi_\Delta]$ has dimension $\Delta + 1$ as claimed. An analogous proof shows that $[K_\mu, \phi_\Delta]$ has dimension $\Delta - 1$. For a representation to be unitary the conformal dimensions must be positive, and thus there must be an operator in the representation of lowest dimension (i.e. that is annihilated by K_μ), since otherwise one can continuously generate lower-dimensional operators. These lowest-dimensional operators are the primary operators, now defined by the condition $[K_\mu, \phi^A] = 0$. A unitary representation of the conformal group is then given by a single primary operator ϕ^A, together with the set of descendants of this primary which are obtained by application of the translation generator P_μ. We will see a generalisation of this structure when we consider the superconformal algebra in section 3.3.

There is an important result that one can derive immediately for field theories that have (classical) conformal invariance. Noether's theorem proves the existence and provides the construction of a conserved current for every continuous symmetry of the action. In the case of Poincaré invariance, one obtains the familiar currents given by the energy-momentum tensor $T^{\mu\nu}$ as (see [ERD]):

$$j^\mu \equiv T^{\mu\nu} v_\nu \tag{2.28}$$

where v_ν is a Killing vector generating Lorentz transformations or translations, and the currents are conserved in the sense that $\partial_\mu j^\mu = 0$. From translation and Lorentz symmetry respectively one finds the conditions $\partial_\mu T^{\mu\nu} = 0$ and $T_{\mu\nu} = T_{\nu\mu}$ (sometimes after requiring 'improvement' of the energy-momentum tensor). Interestingly, it is possible to show that the currents that arise due to full conformal invariance have the same form as in (2.28), where now v_μ can represent a general conformal Killing vector. Conformal invariance means that the associated current should be conserved and so we find:

$$\partial_\mu j^\mu = 0 = \partial_\mu T^{\mu\nu} v_\nu + T^{\mu\nu} \partial_\mu v_\nu = T^{\mu\nu} \frac{1}{2}(\partial_\mu v_\nu + \partial_\nu v_\mu) \tag{2.29}$$

where in the last equality we used the conservation and symmetry of $T^{\mu\nu}$, and thus using the conformal Killing equation (2.4) we find:

$$0 = T^{\mu\nu} \frac{1}{2}\frac{2}{d} \eta_{\mu\nu} \partial \cdot v = T^\mu_\mu \frac{\partial \cdot v}{d} \tag{2.30}$$

which implies that the energy-momentum tensor is traceless i.e. $T^\mu_\mu = 0$. Classically, a field theory is conformally invariant if there are no dimensionful couplings in the action (e.g. mass terms); this is intuitive, since a dimensionful coupling sets a scale, thereby breaking scale invariance. Upon quantization however, conformal invariance may be broken in the form of *anomalies* arising from loop corrections. The form of these anomalies is sometimes in the non-vanishing expectation value of T^μ_μ, conflicting with the classical conformal invariance condition we derived in (2.30). In fact, a necessary condition for a theory to be conformally invariant quantum mechanically (see [GOM] for a discussion) is the vanishing of the renormalisation-group *beta functions* [PES]:

$$\beta_g(\mu_s) \equiv \mu_s \frac{\partial g}{\partial \mu_s} \tag{2.31}$$

where g is a coupling in the theory and μ_s is the renormalisation scale, since this quantity directly measures the scale-dependence of the couplings in the theory. This fact will be very important in chapter 4 when we discuss the effective spacetime actions for string theories.

The key observables that one wants to calculate in any quantum field theory are the n-point correlation functions. Conformal invariance turns out to provide very strict conditions on the forms of the n-point functions (for small values of n). The statement of conformal invariance for the n-point correlation function of primary operators $\theta_i^{A_i}$ (from which one can derive correlation functions involving descendants) is:

$$\langle \theta_1^{A_1}(x_1)...\theta_n^{A_n}(x_n)\rangle = \langle \tilde{\theta}_1^{A_1}(x_1)...\tilde{\theta}_n^{A_n}(x_n)\rangle \tag{2.32}$$

where $\tilde{\theta}_i^{A_i}$ are the transformed operators. We shall illustrate the power of the restrictions that conformal invariance imposes for the simplest case of scalar primary operators, following [GOM].

- Consider the 1-point function of a scalar primary operator of dimension Δ. Translation invariance clearly fixes:

  $$\langle \theta_\Delta(x)\rangle = C \tag{2.33}$$

  for a constant C. Scale invariance under dilatations $\tilde{x} = \lambda x$ then imposes $\langle \theta_\Delta(\tilde{x})\rangle = \langle \tilde{\theta}_\Delta(\tilde{x})\rangle$ from (2.32), and since $\tilde{\theta}_\Delta(\tilde{x}) = \lambda^{-\Delta}\theta_\Delta(x)$ (from (2.23)) we have $C = \lambda^{-\Delta}C$, which forces C and thus the 1-point function to zero unless $\Delta = 0$ in which case the operator is the identity. We thus have the result:

  $$\langle \theta_\Delta(x)\rangle = \delta_{\Delta,0} \tag{2.34}$$

- A less trivial example is given by the 2-point function. Recall that in general quantum field theories the 2-point functions can be very complicated, and are usually only accessible via the machinery of perturbation theory. Here we shall see that their form is entirely determined by conformal invariance. By translation and Lorentz invariance we see that the 2-point function of scalar primary operators must take the form:

  $$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle = f(|x_1 - x_2|) \tag{2.35}$$

  for some function $f(x)$. Scale invariance then fixes $\langle \theta_{\Delta_1}(\tilde{x}_1)\theta_{\Delta_2}(\tilde{x}_2)\rangle = \langle \tilde{\theta}_{\Delta_1}(\tilde{x}_1)\tilde{\theta}_{\Delta_2}(\tilde{x}_2)\rangle$ and thus using (2.35) together with $\tilde{x} = \lambda x$ and $\tilde{\theta}_\Delta(\tilde{x}) = \lambda^{-\Delta}\theta_\Delta(x)$ we have:

  $$f(\lambda|x_1 - x_2|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1 - x_2|) \tag{2.36}$$

  from which one can inspect the solution:

  $$f(|x_1 - x_2|) = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.37}$$

  for some constant C. We are not quite finished however, since we must still impose invariance under special conformal transformations. Given the discussion at the end of section 2.1, it is much easier to achieve this by instead imposing invariance under inversion.

Under inversion $\tilde{x}^\mu = x^\mu/x^2$ we have $\tilde{\theta}_\Delta(\tilde{x}) = \theta_\Delta(x)/(\tilde{x}^{2\Delta})$ and thus using (2.32) and (2.37) we see that inversion invariance imposes:

$$\frac{1}{|\tilde{x}_1 - \tilde{x}_2|^{\Delta_1+\Delta_2}} = \frac{1}{\tilde{x}_1^{2\Delta_1}} \frac{1}{\tilde{x}_2^{2\Delta_2}} \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.38}$$

Using the definition of inversion $\tilde{x}^\mu = x^\mu/x^2$, straightforward algebra shows that:

$$\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1 - \tilde{x}_2|^2} = \frac{1}{|x_1 - x_2|^2} \tag{2.39}$$

and thus combining this with (2.38) we obtain the equality:

$$\frac{\tilde{x}_1^{2\Delta_1} \tilde{x}_2^{2\Delta_2}}{|\tilde{x}_1 - \tilde{x}_2|^{\Delta_1+\Delta_2}} = \left[\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1 - \tilde{x}_2|^2}\right]^{\frac{\Delta_1+\Delta_2}{2}} \tag{2.40}$$

from which we can infer the condition $\Delta_1 = \Delta_2$. The form of the 2-point function is thus fixed as:

$$\langle \theta_{\Delta_1}(x_1) \theta_{\Delta_2}(x_2) \rangle = \frac{C \delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \tag{2.41}$$

for some constant C (which can be set to 1 with appropriate field redefinition). We see that the 2-point functions in a conformal field theory are thus entirely determined by the *spectrum* $\{\Delta\}$.

- One can proceed in a similar fashion to the 2-point function above to see what conditions conformal invariance places on higher-point functions. We shall not repeat the analysis here, but we state that the 3-point function of scalar primary operators is restricted to take the form:

$$\langle \theta_{\Delta_1}(x_1) \theta_{\Delta_2}(x_2) \theta_{\Delta_3}(x_3) \rangle = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}} \tag{2.42}$$

  where C is a constant that, unlike the 2-point function case, cannot be removed by field redefinition and is in fact theory-dependent. We see that, although the overall normalisation is not, the spacetime dependence of the 3-point function is still entirely fixed by conformal invariance.

- For n-point functions with $n \geq 4$ the spacetime dependence is no longer entirely fixed either. This is due to the existence of conformally invariant *cross-ratios* of coordinates which the correlation function can thus depend on arbitrarily. We shall not need to discuss n-point functions with $n \geq 4$ any further in the following however.

Although the above analysis was for scalar primary operators, similar results follow through for higher-rank tensor operators. For example, the 2-point function of two vector primary operators can be derived in an analogous (and only slightly more complicated) way to the scalar case and is given by (see [GOM]):

$$\langle V^\mu_{\Delta_1}(x_1) V^\nu_{\Delta_2}(x_2) \rangle = \frac{\mathcal{V}^{\mu\nu}(x_1 - x_2) \delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \tag{2.43}$$

where $\mathcal{V}^{\mu\nu}(x) \equiv \eta^{\mu\nu} - 2x^\mu x^\nu / x^2$. We thus see that conformal invariance imposes great restrictions on the correlation functions in the field theory, and CFTs are thus considerably simpler than generic field theories. We shall mention these structures again in chapter 7 when we discuss the prescription for mapping correlation functions in the AdS/CFT correspondence.

On the gauge theory side of the correspondence one finds $\mathcal{N} = 4$ super Yang-Mills which, in addition to exhibiting conformal symmetry as described in the previous chapter, is also a maximally supersymmetric theory. In this chapter, the notion of supersymmetry (SUSY) is first introduced, followed by a review of the essential features of $\mathcal{N} = 4$ SYM, including its superconformal symmetry which arises from the non-trivial combination of supersymmetry and conformal symmetry.

3.1 The Super-Poincaré Algebra and its Representations

The familiar Poincaré algebra may be extended (in a way that circumvents the famous *Coleman-Mandula theorem* [COL]) by promoting it to a *graded Lie algebra* or *superalgebra*, and including spinor *supercharges* Q^i_α where α is the spinor index (which may be Weyl, Majorana or both depending on the spacetime dimension) and $i = 1....\mathcal{N}$, with $\mathcal{N}$ being known as the *degree of supersymmetry*. The supercharges transform under a spinor representation of the Lorentz group and commute with translations, and in 3+1 dimensions they further obey the structure relations [BAL]:

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{3.1}$$

$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon_{\alpha\beta} Z^{ij} \tag{3.2}$$

where P_μ is the translation generator as in chapter 2, we define $\bar{Q}_{\dot{\beta}j} \equiv (Q_{\beta j})^\dagger$, σ^μ are the usual Van-der Waerden matrices, and Z^{ij} are the antisymmetric *central charges* which commute with all generators. The central charges automatically vanish for $\mathcal{N} = 1$ but may be non-zero for $\mathcal{N} > 1$. There is an automorphism symmetry group of the supersymmetry algebra known as the *R-symmetry* [FRE]. Indeed, the algebra is invariant under a global $U(1)_R$ symmetry which causes the supercharges to change by a phase rotation (the subscript R is simply notational). Furthermore, for $\mathcal{N} > 1$ there is in fact a non-abelian $SU(\mathcal{N})_R$ symmetry, rotating the different supercharges into one another. Thus, for $\mathcal{N} = 4$ the R-symmetry group is $SU(4)$, which will be important later (see section 6.4) as it corresponds to the isometry group of S^5 since $SU(4) \cong SO(6)$ (see [ZHO]).

To construct representations of the supersymmetry algebra one proceeds in a similar fashion to the Poincaré case, by first transforming to a particularly simple Lorentz frame (see [BAL] for a discussion). One must distinguish the massless and massive cases separately (for non-zero central charges the unitary representations are necessarily massive), and representations are then again labelled by the helicity or spin respectively (and, of course, the number of supersymmetries $\mathcal{N}$); one finds that there are an equal number of bosonic and fermionic states in a given representation, and that the masses of all states in a representation must be the same. In accordance with CPT invariance, representations (or so-called *supermultiplets*) that are not self-conjugate are taken together with the direct sum of their conjugate. Let us consider, for example, a pure gauge theory that contains helicities ± 1 but no higher; in the process of constructing supermultipets, one finds (schematically) that each non-zero supercharge $\bar{Q}^i$ raises the helicity by $1/2$, and thus the maximal supersymmetry must be $\mathcal{N} = 4$ (as can be seen by starting from the minimum helicity state -1, and acting with the 4 different non-zero conjugated supercharges to reach the maximum helicity $+1$). The $\mathcal{N} = 4$ theory that appears in the AdS/CFT correspondence (see section 3.2) is thus a *maximally supersymmetric* gauge theory.

Let us as an example briefly discuss the specific case of massive representations with non-zero central charges, since this introduces an important concept that is later generalised in the superconformal case. To study massive representations we transform to the Lorentz rest frame defined by $P^\mu = (M, 0, 0, 0)$ which one can easily show reduces the structure relation (3.1) to:

$$\{Q^i_\alpha, (Q^j_\beta)^\dagger\} = 2M\delta^i_j\delta_{\alpha\beta} \tag{3.3}$$

since σ^0 is the identity matrix. Since Z is an antisymmetric matrix it can be brought to block diagonal form consisting of 2×2 antisymmetric matrices. The real, positive skew eigenvalues are then denoted by $Z_{\bar{a}}$ where $\bar{a} = 1...r$ and r is defined by $\mathcal{N} = 2r$ or $\mathcal{N} = 2r + 1$ depending on whether $\mathcal{N}$ is even or odd. Defining a particular linear combinations of supercharges (see [KIR] for details) denoted by $\mathcal{Q}^{\bar{a}}_{\alpha\pm}$, one finds that the only non-vanishing structure relations are then given by:

$$\{\mathcal{Q}^{\bar{a}}_{\alpha\pm}, (\mathcal{Q}^{\bar{b}}_{\beta\pm})^\dagger\} = \delta^{\bar{a}}_{\bar{b}}\delta^\beta_\alpha(M \pm Z_{\bar{a}}) \tag{3.4}$$

where the $\pm$ are correlated throughout. Clearly, for a unitary representation the operator on the LHS must be positive-definite, and so one derives the following bound on the mass:

$$M \geq |Z_{\bar{a}}| \tag{3.5}$$

for each value of $\bar{a}$, which is known as the *BPS bound*. There will be partial saturation of the bound whenever $M = |Z_{\bar{a}}|$ for a particular value of $\bar{a}$; we then see from (3.4) that $\mathcal{Q}^{\bar{a}}_{\alpha\pm}$ must vanish for either $+$ or $-$. Since one or more of the supercharges vanishes, the representation will be smaller than a generic representation (since there will be fewer creation operators), and the shortened multiplet is known as a *BPS multiplet*. If the bound is saturated for $r_o < r$ of the $\bar{a}$, then the representation is known as a $1/2^{r_o}$ *BPS multiplet* and its dimension is reduced to $2^{2\mathcal{N}-2r_o}$. We will describe a generalisation of this concept in section 3.3, which plays an important role in the AdS/CFT correspondence.

3.2 The $\mathcal{N} = 4$ Super Yang-Mills Theory

For any $1 \leq \mathcal{N} \leq 4$ there exists a *gauge multiplet* which transforms under the adjoint representation of a gauge group. It turns out that for $\mathcal{N} = 1, 2$ there exist other multiplets which can be considered as matter multiplets, whereas for $\mathcal{N} = 4$ the gauge multiplet is the only possible multiplet. This $\mathcal{N} = 4$ gauge multiplet is given by [BAL]:

$$(A_\mu, \lambda^a_\alpha, X^i) \tag{3.6}$$

where A_μ is a spin-1 gauge field, λ^a_α ($a = 1, ...4$) are Weyl spinors, and X^i ($i = 1, ...6$) are real scalars. Under the R-symmetry group these transform as a singlet, a vector, and a rank-2 antisymmetric tensor respectively; the a and i indices on the spinors and scalars respectively are these R-symmetry indices.

For $\mathcal{N} = 4$ one unfortunately cannot appeal to the power of the off-shell *superfield* formalism that is so valuable for $\mathcal{N} = 1$ (see [BAL]). Nevertheless, one can work in terms of components, and the Lagrangian for the so-called $\mathcal{N}=4$ *super Yang-Mills theory* (with field content (3.6)) is given by [FRE]:

$$\mathcal{L}=\mathrm{Tr}\bigg(-\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu}+\frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}-i\sum_a\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a-\sum_i D_\mu X^i D^\mu X^i \\ +g_{YM}\sum_{a,b,i}C_i^{ab}\lambda_a[X^i,\lambda_b]+g_{YM}\sum_{a,b,i}\bar{C}_{iab}\bar{\lambda}^a[X^i,\bar{\lambda}^b]+\frac{g_{YM}^2}{2}\sum_{i,j}[X^i,X^j]^2\bigg) \tag{3.7}$$

where g_{YM} is the coupling constant, θ_I is the so-called *instanton angle*, $F_{\mu\nu}$ is the usual field-strength of the gauge field, D_μ is the usual gauge-covariant derivative, $\tilde{F}$ is the *Hodge dual* of F, and C_i^{ab} are the structure constants of $SU(4)_R$. The trace is over the gauge indices (which are suppressed in (3.7)) and is to ensure gauge invariance of the action. The action given by (3.7) is invariant under the supersymmetry transformations (where, for clarity, we write the indices explicitly) given by [ERD]:

$$(\delta X^i)^a_\alpha=[Q^a_\alpha,X^i]=C^{iab}\lambda_{\alpha b} \tag{3.8}$$
$$(\delta\lambda_{\beta b})^a_\alpha=\{Q^a_\alpha,\lambda_{\beta b}\}=F^+_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}\delta^a_b+[X^i,X^j]\epsilon_{\alpha\beta}(C_{ij})^a_b \tag{3.9}$$
$$(\delta\bar{\lambda}^b_{\dot\beta})^a_\alpha=\{Q^a_\alpha,\bar{\lambda}^b_{\dot\beta}\}=C_i^{ab}\sigma^\mu_{\alpha\dot\beta}D_\mu X^i \tag{3.10}$$
$$(\delta A^\mu)^a_\alpha=[Q^a_\alpha,A^\mu]=\sigma^\mu_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta\alpha} \tag{3.11}$$

where F^+ is the self-dual part of the field-strength, and the constants $(C_{ij})^a_b$ are related to bilinears in Clifford Dirac matrices of $SO(6)_R$.

This theory is classically conformally invariant; indeed, with the standard mass-dimensions of the fields given by $[A_\mu]=[X^i]=1$ and $[\lambda^a_\alpha]=3/2$, it is easy to see from (3.7) that the single coupling constant has dimension $[g_{YM}]=0$ (and $[\theta_I]=0$) since the Lagrangian must have $[\mathcal{L}]=4$ in natural units and in 4-dimensions. The theory is thus scale invariant, which together with Poincaré invariance forms full conformal invariance. More strikingly, upon quantisation one finds that the theory is UV finite; since no renormalisation scale is introduced one thus finds that the β-function vanishes to all orders of perturbation theory (or at least is believed to), and thus the theory remains conformally invariant at the quantum level as discussed in [FRE].

In addition to superconformal symmetry (to be described in section 3.3), $\mathcal{N}=4$ SYM exhibits a further symmetry (see [FRE]), most easily expressed by first combining the coupling constant and instanton angle as:

$$\tau\equiv\frac{\theta_I}{2\pi}+\frac{4\pi i}{g_{YM}^2} \tag{3.12}$$

Although the quantised theory is already invariant under $\tau\to\tau+1$, the *Montonen-Olive conjecture* [MON] promotes this symmetry to a full $SL(2,\mathbb{Z})$ symmetry group, known as *S-duality* and realised as:

$$\tau\to\frac{a\tau+b}{c\tau+d} \tag{3.13}$$

where $ad-bc=1$ and $a,b,c,d\in\mathbb{Z}$. This symmetry will feature later in the AdS/CFT correspondence in the context of mapping the global symmetries on the two sides (see section 6.4).

3.3 The Superconformal Group $SU(2,2|4)$ and its Representations

The presence of both supersymmetry and conformal symmetry in $\mathcal{N}=4$ SYM in fact leads to an even larger symmetry group of the theory, due to the fact that supersymmetry and special conformal transformations do not commute, and thus their commutator gives a new symmetry generator. The full group is known as the *superconformal group* and is given by the supergroup $SU(2,2|4)$, where the notation labels the components of the bosonic subgroup $SU(2,2)\times SU(4)_R$. We briefly sketch the different components leading to this full global continuous symmetry group of $\mathcal{N}=4$ SYM as in [FRE]:

- **Conformal Symmetry**: This forms the subgroup $SO(2,4)\cong SU(2,2)$ and is generated by $P_\mu, M_{\mu\nu}, D$ and K_μ, with algebra given as in section 2.2.
- **R-symmetry**: This forms the subgroup $SO(6)_R\cong SU(4)_R$ and is generated by T^A with $A=1,2...,15$. Note that this commutes with the conformal symmetry subgroup.
- **Poincaré Supersymmetry**: This is generated by the spinor supercharges Q^a_α and their conjugates, with algebra given as in section 3.1. This does *not* commute with the entire conformal symmetry subgroup.
- **Conformal Supersymmetry**: This is generated by $S_{\alpha a}$ and their conjugates $\bar{S}^{\dot{\alpha}a}$, which arise because of the non-commutativity between supersymmetry and special conformal transformations. They satisfy the following structure relations:

$$\{S_{\alpha a}, S_{\beta b}\} = \{Q^a_\alpha, \bar{S}^b_{\dot{\beta}}\} = 0 \tag{3.14}$$

$$\{S_{\alpha a}, \bar{S}^b_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} K_\mu \delta^b_a \tag{3.15}$$

$$\{Q^a_\alpha, S_{\beta b}\} = \epsilon_{\alpha\beta}(\delta^a_b D + T^a_b) + \tfrac{1}{2}\delta^a_b M_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} \tag{3.16}$$

Representations of the superconformal algebra are built in a similar way to representations of the conformal and supersymmetry algebras (c.f. sections 2.3 and 3.1). We wish to construct gauge invariant operators which are polynomials in the elementary fields; the gauge invariance is necessary for the operators to be physical observables, and the polynomial condition means that the operators have a definite dimension as is required to form a representation of the conformal group. One defines a *superconformal primary operator* $\mathcal{O}$ by:

$$[S, \mathcal{O}\} = 0 \tag{3.17}$$

which means that $\mathcal{O}$ is the lowest dimensional operator in the representation, the existence of which is again required by unitarity; the conformal supercharges S have dimension $[S] = -1/2$ and so successive operation of these supercharges lowers the dimension. The notation $[,\}$ denotes a commutator or anti-commutator for bosonic or fermionic $\mathcal{O}$ respectively. Note that this definition encompasses (from the superconformal algebra relation (3.15)), but is *not* equivalent to, the definition of a conformal primary operator, given previously in section 2.3 as $[K_\mu, \mathcal{O}] = 0$. One can then define the other operators $\mathcal{O}'$ in the superconformal multiplet as the *superconformal descendants* of the superconformal primary operator:

$$\mathcal{O}' = [Q, \mathcal{O}\} \tag{3.18}$$

where the scaling dimensions are clearly related by $\Delta_{\mathcal{O}'} = \Delta_{\mathcal{O}} + 1/2$ since $[Q] = 1/2$. In analogy to the conformal multiplet case, a superconformal descendant is never a superconformal primary operator, since there is always an operator of lower dimension. A superconformal multiplet then consists of a single superconformal primary operator and its descendants.

With this condition in mind, one can show as described in [FRE] that the gauge invariant superconformal primary operators in $\mathcal{N}=4$ SYM are given by the scalars only, albeit in a symmetrised manner; one shows this essentially by using the SUSY transformations (3.8)-(3.11) and the fact that a superconformal primary can never be a Q-(anti)commutator of another operator, since it would then be a superconformal descendant. The simplest such operators are the so-called *single trace operators* (where the trace is to ensure the operator is gauge invariant) defined as [FRE]:

$$\mathcal{O}_n \equiv \text{Tr}[X^{(i_1}X^{i_2}...X^{i_n)}] \tag{3.19}$$

where we see that the $SO(6)_R$ indices are symmetrized in the trace. This is in fact generally a reducible representation of the R-symmetry, and one may further decompose it into a trace and a traceless symmetric part. As mentioned previously, all fields in the $\mathcal{N}=4$ gauge multiplet transform in the adjoint representation of the gauge group, and are thus traceless hermitian matrices. One thus has $\text{Tr}[X^i]=0$ and so the simplest irreducible operators one can form are given by [FRE]:

$$\begin{aligned} &\text{Konishi Multiplet:} && \text{Tr}[X^iX^i] \\ &\text{Supergravity Multiplet:} && \text{Tr}[X^{\{i}X^{j\}}] \end{aligned} \tag{3.20}$$

where summation over i is implied in the first expression, and $\{ij\}$ denotes the traceless part in the second (with symmetrisation true automatically by virtue of the cyclic property of the trace). The latter name is pre-emptive of the field/operator map in the AdS/CFT correspondence that we will discuss in section 7.1.

The supergravity multiplet is the simplest example of a superconformal 1/2-BPS multiplet, so-called because they are annihilated by half of the supercharges and thus are shortened multiplets, in analogy with the supersymmetric BPS multiplets mentioned in section 3.1. BPS multiplets play a very important role in testing the AdS/CFT correspondence and so it is worth unpacking this analogy a bit. Representations of the superconformal algebra are labelled by their Lorentz quantum numbers and scaling dimension, as for the conformal case, but are also labelled by the Dynkin labels $[r_1, r_2, r_3]$ of the R-symmetry group $SU(4)_R$ discussed previously; they are thus labelled fully by the quantum numbers of the bosonic subgroup. In the same way that unitarity led to the BPS bound in section 3.1, unitarity here requires that the conformal dimension is bounded from below by the spin and R-symmetry quantum numbers (see [FRE]); considering only the primaries (since these have the lowest dimension anyway), we mentioned before that these are scalars and hence have vanishing Lorentz quantum numbers, meaning that the conformal dimension is bounded below by the R-symmetry charges only. When this bound is saturated one again has shortened multiplets (i.e. the primary, known in this case as a *chiral primary*, is annihilated by some of the supercharges), with the conformal dimension related directly to the R-symmetry charges. These are known in this context as *BPS multiplets* and are very important since the conformal dimension is *protected* by the representation theory and thus does not receive quantum corrections; this is useful for testing the AdS/CFT correspondence since, as we shall see, when one considers the classical supergravity regime on the string side (which is the easiest for calculations) one simultaneously has the strong-coupling regime on the gauge theory side, for which it is not possible to calculate quantum corrections using perturbation theory. Protected or unrenormalised quantities are thus particularly important for testing the correspondence. We illustrate these concepts by briefly summarising some properties of superconformal multiplets below as in [FRE]:

Operator	$SU(4)_R$ primary	Dimension
1/2 BPS	$[0,k,0]$, $k \geq 2$	k
1/4 BPS	$[l,k,l]$, $l \geq 1$	$k+2l$
1/8 BPS	$[l,k,l+2m]$	$k+2l+3m$, $m \geq 1$
Non-BPS	Any	Unprotected

Superstrings and Supergravity

The AdS/CFT correspondence states the equivalence between the $\mathcal{N} = 4$ SYM theory described in chapter 3, and Type IIB superstring or supergravity theory (depending on the regime) defined on $AdS_5 \times S^5$. In this section we provide a brief review of string theory up to the point of being able to discuss the essential features of Type IIB superstring theory (such as its field content and symmetries), as well as its supergravity limit. Properties of D-branes will also be discussed, which play a crucial role in the AdS/CFT correspondence.

4.1 Review of the Bosonic String

The bosonic string, although fundamentally incomplete, is important as many of its features still play a role in superstring theory. It is thus worth devoting some time to understand its analysis. The string action for bosonic string theory in d-dimensional flat spacetime is given by the *Polaykov action* [POL]:

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{4.1}$$

Some clarifications of (4.1) are in order: the *fields* $X^\mu(\tau, \sigma)$ are the embedding of the 2-dimensional string *worldsheet* (with coordinates $\sigma^\alpha = (\tau, \sigma)$) in spacetime, $h_{\alpha\beta}$ is the worldsheet metric, and α' is a constant known as the *slope parameter*, related to the *string tension* by $T = 1/(2\pi\alpha')$. This is known as the first-order action, and varying with respect to $h_{\alpha\beta}$ and using the resulting equations of motion gives the second-order *Nambu-Goto action* [POL]:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\text{det}[\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}]} \tag{4.2}$$

which is simply the *proper area* of the worldsheet (in analogy to the action for the relativistic point particle which is given by the proper length of the worldline).

The Polyakov action is more desirable than the Nambu-Goto action for several reasons; the lack of the square-root allows for quantisation more easily, but furthermore (4.1) exhibits a symmetry not present in (4.2). Although both actions exhibit manifest spacetime Poincaré invariance and worldsheet diffeomorphism invariance $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^\beta)$, (4.1) also has worldsheet *Weyl* or conformal invariance (see [POL]):

$$h_{\alpha\beta}(\tau, \sigma) \to e^{\omega(\tau,\sigma)} h_{\alpha\beta}(\tau, \sigma) \tag{4.3}$$

for any $\omega(\tau, \sigma)$. Worldsheet diffeomorphism together with Weyl invariance allows us to fix the 3 independent components of the 2D worldsheet metric, and using the so-called *conformal gauge*, where $h_{\alpha\beta} = \eta_{\alpha\beta}$, the action (4.1) reduces to:

$$S_C = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\dot{X}^2 - X'^2) \tag{4.4}$$

where $\dot{X}^2 \equiv \partial_\tau X^\mu \partial_\tau X_\mu$ and $X'^2 \equiv \partial_\sigma X^\mu \partial_\sigma X_\mu$. The equations of motion for X^μ derived from (4.4) are simply the 2D wave equations, but one must not forget that there are additional constraints associated with the fact that there is a gauge symmetry (which was exploited in deriving (4.4)). These *Virasoro constraints* [CAC] are simply given by the equations of motion for the worldsheet metric, which can also be expressed as:

$$\frac{\delta S_P}{\delta h^{\alpha\beta}} \propto T_{\alpha\beta} = 0 \tag{4.5}$$

i.e. the worldsheet energy-momentum tensor must vanish. Altogether, one must then solve the set of equations:

$$\begin{aligned} (\partial_\tau^2 - \partial_\sigma^2) X^\mu &= 0 \\ (X' \pm \dot{X})^2 &= 0 \end{aligned} \tag{4.6}$$

In addition to the equations of motion (and the constraints), in the variation of (4.4) one encounters a boundary term:

$$\delta S_C|_{boundary} \propto \int_{-\infty}^{\infty} d\tau \left(X' \cdot \delta X|_{\sigma=\pi} - X' \cdot \delta X|_{\sigma=0} \right) \tag{4.7}$$

which must also vanish. There are multiple ways of ensuring (4.7) vanishes:

- **Open String**: If the endpoints of the string ($\sigma = 0, \pi$) are distinguished, then (4.7) may be made to vanish by taking:

$$X'^\mu(\tau, \sigma^*) = 0 \qquad \text{or} \qquad X^\mu(\tau, \sigma^*) = x^\mu_{\sigma^*} \tag{4.8}$$

  for some fixed $x^\mu_{\sigma^*}$, and where σ^* represents a string endpoint ($\sigma = 0$ or π). These are known as *Neumann* and *Dirichlet* boundary conditions respectively. Dirichlet boundary conditions (for which the endpoint is fixed) lead naturally to the concept of *D-branes*, which are objects upon which open strings end; these will be discussed in greater detail in section 4.4.

- **Closed String**: If the endpoints of the string are to be identified, then (4.7) will vanish with the *periodic* boundary conditions:

$$X^\mu(\tau, \pi) = X^\mu(\tau, 0) \tag{4.9}$$

One then proceeds in a similar way to standard QFT i.e. construct solutions to (4.6) via mode expansions, ensuring that they are consistent with the boundary conditions chosen. As an example, the mode expansion [CAC]:

$$X^\mu(\tau, \sigma) = x^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{m \neq 0} \frac{\alpha_m^\mu}{m} e^{-im\tau} \cos(m\sigma) \tag{4.10}$$

is a solution for the open string with Neumann boundary conditions at both endpoints, where x^μ is some fixed number. One then quantises the theory in the usual way, by imposing the equal-time commutation relations:

$$[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = 0 \tag{4.11}$$

$$[X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = \eta^{\mu\nu} \delta(\sigma - \sigma') \tag{4.12}$$

where $P_\mu \equiv \partial\mathcal{L}/\partial\dot{X}^\mu$ is the canonical momentum conjugate to the field X^μ. Using these together with the mode expansion for X^μ, one can then find the implied commutation relations for the expansion coefficients α_m^μ, which are now operators in the quantum theory. One finds (see [ZWE]) an infinite set of harmonic oscillators for the rescaled expansion coefficients $a_m^\mu \equiv \alpha_m^\mu/\sqrt{m}$ and their conjugates, which may thus be interpreted as creation and annihilation operators.

The spectrum can then be constructed in the usual way; define a *vacuum state* $|0\rangle$ by the condition $a_m^\mu\,|0\rangle = 0$ for all values of m, and then construct the rest of the spectrum as the states $a_m^\mu a_n^\nu ...\,|0\rangle$. The exact procedure is known as *light-cone gauge quantisation* and makes use of the worldsheet symmetries to remove negative norm states which threaten unitarity (see [ZWE] for details). The spectrum is built level-by-level in the *number operator* $N \equiv \sum_{m=1}^{\infty}\sum_{i=1}^{D-2} m a_m^{\dagger i} a_m^i$, where the upper bound $(D-2)$ on the i-summation is a result of removing redundant gauge degrees of freedom in the light-cone gauge quantization procedure. One finds the following particle content (where M represents the mass of the state) in the lower levels of the open and closed string spectra (see [ZWE]):

	Open String	Closed String
$N = 0\ (M^2 < 0)$	T	T
$N = 1\ (M = 0)$	A_μ	$g_{\mu\nu}$, $B_{\mu\nu}$, Φ

where the *tachyon* T has negative mass, A_μ is a vector, $g_{\mu\nu}$ is a 2nd-rank symmetric traceless tensor (*graviton*), $B_{\mu\nu}$ is a 2-form, and Φ is a scalar (*dilaton*). The higher level states are all massive, and will generally be ignored throughout; we mention in passing however that the masses depend inversely on α' and thus become very large in the limit $\alpha' \to 0$. Note also that, although in principle problematic, the tachyons are in fact projected out in the full superstring theory (to be discussed later), via the so-called *GSO projection* (see [POL]).

It is worth noting that, as is clear from the stated form of the number operator N, manifest Lorentz invariance is lost in the light-cone gauge quantization procedure (this is not so in the more complicated *covariant quantisation* procedure [ZWE], but we will not discuss this further). The requirement that Lorentz invariance be preserved actually leads to a remarkable prediction. One can construct worldsheet currents associated with Lorentz transformations in the usual way, and the associated conserved charges are given in the string theory by [CAC]:

$$\begin{aligned} J^{\mu\nu} &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma(\dot{X}^\mu X^\nu - \dot{X}^\nu X^\mu) \\ &= x^\mu P^\nu - x^\nu P^\mu + i\sum_{m\neq 0}\frac{1}{m}(\alpha_{-m}^\mu\alpha_m^\nu - \alpha_{-m}^\nu\alpha_m^\mu) \end{aligned} \tag{4.13}$$

Using the commutation relations for the oscillators α_m^μ, one can then check whether the $J^{\mu\nu}$ obey the usual Lorentz algebra. It turns out that they do indeed satisfy the Lorentz algebra provided that $D = 26$, known as the *critical dimension*; in this manner, the requirement that the theory be Lorentz invariant essentially predicts the dimensionality of spacetime. Since we seem to observe $D = 4$, we are led to the idea of *Kaluza-Klein reduction* [FRE], which involves compactifying certain spatial dimensions on compact spaces such as circles or tori, and then letting the size of these compactified dimensions go to zero. In this way, one can derive a lower dimensional theory from a higher dimensional one. We shall mention this procedure again in chapter 7.

4.2 Coupling the Bosonic String to a Background and Effective Spacetime Actions

The natural generalisation of the previous section is to consider the string propagating in a background of its own massless modes (for the graviton this corresponds to the string propagating in a curved spacetime, for example). For the closed string this can be achieved via the action (see [CAC]):

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) \\ -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X)-\sqrt{-h}\alpha' R^{(2)}\Phi(X)\right) \end{aligned} \tag{4.14}$$

where $R^{(2)}$ is the Ricci scalar on the 2D worldsheet, and $\epsilon^{\alpha\beta}$ is the 2D antisymmetric symbol. This is known as the *non-linear sigma model*; note that the first term is just the Polyakov action (4.1) but with a general target space metric $g_{\mu\nu}(X)$, and the other two terms are appropriate generalisations to the other massless fields of the closed string spectrum. The quantity:

$$\frac{1}{4\pi}\int d^2\sigma\sqrt{-h}R^{(2)} = \chi \equiv 2-2g \tag{4.15}$$

is a topological invariant known as the *Euler characteristic* [FRA], where g is the *genus* of the worldsheet i.e. the number of holes. In string perturbation theory, one calculates amplitudes by considering a path integral of the form (see [VIE]):

$$\text{String amplitude} = \sum_g (g_s)^{2g-2}\int_{\Sigma_g}\mathcal{D}X e^{-S[X]}V_1...V_n \tag{4.16}$$

where g is the genus as above, Σ_g represents all surfaces of genus g, g_s is the *string coupling constant*, and V_i are so-called *vertex operators*. The string coupling constant can be understood as follows. If the dilaton acquires a non-zero vacuum expectation value, $\Phi\to\phi+\Phi$, then we see from above that the path integral includes a multiplicative factor of $e^{-\chi\phi}=e^{\phi(2g-2)}$. From (4.16) we thus see that it is natural to identify the (closed) string coupling constant as $g_s=e^\phi$; increasing the genus by 1, and thus adding a hole to the worldsheet, then contributes a factor of $e^{2\phi}=g_s^2$. For the open string one has instead $g_s=e^{\phi/2}$. These facts will be important when we state the AdS/CFT correspondence in chapter 6.

In making the transition from the action (4.1) to (4.14), it is important that we maintain worldsheet Weyl or conformal invariance. As mentioned in section 2.3, this can be expressed as the vanishing of the renormalisation-group beta functions; there will be three independent beta functions here, as $g_{\mu\nu}, B_{\mu\nu}$ and Φ can each be interpreted as a coupling in the string action (4.14). To leading order in α' (which corresponds to the *low energy limit* i.e. $\alpha'\to 0$), one finds the beta functions (see [CAC]):

$$\begin{aligned} \beta^{(g)}_{\mu\nu} &= \alpha'\left(R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-\frac{1}{4}H_{\mu\lambda\rho}H_\nu^{\lambda\rho}\right) \\ \beta^{(B)}_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu}+\nabla^\lambda\Phi H_{\lambda\mu\nu}\right) \\ \beta^{(\Phi)} &= \alpha'\left(-\frac{1}{2}\nabla^2\Phi+\nabla_\mu\Phi\nabla^\mu\Phi-\frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) \end{aligned} \tag{4.17}$$

where $H_3\equiv dB_3$ is the 3-form field-strength associated with the 2-form B_2, and $R_{\mu\nu}$ is the spacetime Ricci tensor associated with the graviton $g_{\mu\nu}$. Clearly, the vanishing of the beta-functions in (4.17) looks like a set of field equations for the massless spacetime fields of the string spectrum. The natural question arises as to whether these fields equations may be derived from an action principle; this is a non-trivial statement since the existence of an associated action in general depends on certain integrability conditions. Remarkably, the field equations can be derived from the following spacetime action [CAC]:

$$S_{eff.} = \int d^{26}x\sqrt{-\text{det}g}e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\nabla_\mu\Phi\nabla^\mu\Phi\right) \tag{4.18}$$

which is known as the *effective spacetime action* of the (closed) bosonic string theory. This procedure of deriving an effective spacetime field theory by demanding conformal invariance of the worldsheet theory is crucial, and in the superstring case leads to the derivation of supergravity theory as the low energy approximation of string theory.

4.3 Superstrings and Type IIB Theory

The bosonic string, although illustrative, is incomplete for several reasons. Perhaps most importantly, the spectrum does not contain any fermions, and so there is no hope of somehow recovering the standard model of particle physics. Furthermore, both the open and closed string spectra contain a problematic tachyon state, and the critical dimension $D = 26$ is somewhat far from the 4-dimensions we appear to live in.

Following the *RNS formalism*, we thus consider a supersymmetric completion of the Polyakov action by introducing *worldsheet fermions* (which, note, are still spacetime *vectors*) Ψ^μ via the action (expressed in conformal gauge) [CAC]:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X_\mu\partial^\alpha X^\mu + i\bar{\Psi}^\mu\rho^\alpha\partial_\alpha\Psi_\mu\right) \tag{4.19}$$

where $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$ and $\bar{\Psi} \equiv i\Psi^\dagger\rho^0$ (we use the notation ρ^α to emphasise that these are not the familiar 4-dimensional γ-matrices). Decomposing the worldsheet fermion into Weyl spinors $\Psi^\mu = (\psi^\mu_-, \psi^\mu_+)$, choosing a particularly simple representation for ρ^α, and using worldsheet light-cone coordinates $\sigma^\pm \equiv \tau \pm \sigma$, the fermionic part of the action can then be written as [CAC]:

$$S_\Psi = -\frac{1}{2\pi\alpha'}\int d^2\sigma\left(\psi^\mu_-\partial_+\psi_{\mu-} + \psi^\mu_+\partial_-\psi_{\mu+}\right) \tag{4.20}$$

One can then vary this action to obtain the equations of motion:

$$\partial_-\psi^\mu_+ = \partial_+\psi^\mu_- = 0 \tag{4.21}$$

and so there are separate right and left-moving waves, as in the bosonic case (c.f. solutions to the wave equation in (4.6)). There are again additional constraints associated with the gauge invariance, which are still expressed as $T_{\alpha\beta} = 0$, though the introduction of the fermionic action now changes the form of the worldsheet energy-momentum tensor.

Furthermore, in varying the action (4.20), one encounters a new boundary term:

$$\delta S_\Psi|_{boundary} \propto \int d\tau\left(\psi^\mu_-\delta\psi_{-\mu} - \psi^\mu_+\delta\psi_{+\mu}\right)|^{\sigma=\pi}_{\sigma=0} \tag{4.22}$$

which must be made to vanish (note that the first-order nature of the fermionic action has led to there being no derivatives in this boundary term, in contrast to the bosonic sector). For the open string (which has distinct endpoints and thus the $\sigma = 0, \pi$ contributions must independently vanish) this can be achieved in two ways:

- **Ramond Sector (R)**: $\psi_+^\mu(\tau,\pi) = +\psi_-^\mu(\tau,\pi)$.
- **Neveu-Schwarz Sector (NS)**: $\psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi)$.

where, note, there is also a freedom in choosing $\psi_+^\mu(\tau,0) = \pm\psi_-^\mu(\tau,0)$, but these are redundant (i.e. not physically different) and so one imposes $\psi_+^\mu(\tau,0) = +\psi_-^\mu(\tau,0)$ by convention (see [CAC]). For the closed superstring, the vanishing of (4.22) must occur in a different manner i.e. by cancelling the contribution $\psi_\pm^\mu \delta\psi_{\pm\mu}|_{\sigma=\pi}$ with $\psi_\pm^\mu \delta\psi_{\pm\mu}|_{\sigma=0}$, where the $\pm$ are correlated throughout. There are 4 different ways of achieving this:

$$\begin{aligned}\psi_+^\mu(\tau,\sigma) &= \pm\psi_+^\mu(\tau,\sigma+\pi)\\ \psi_-^\mu(\tau,\sigma) &= \pm\psi_-^\mu(\tau,\sigma+\pi)\end{aligned} \tag{4.23}$$

leading to 4 sectors of the closed superstring theory: **R-R**, **R-NS**, **NS-R**, and **NS-NS**, where R refers to periodic and NS to anti-periodic boundary conditions.

One then proceeds as in the bosonic string case, by considering mode expansion solutions of the equations of motion consistent with the relevant boundary conditions, and quantizing by promoting the fields to operators; the expansion coefficients of the worldsheet fermions now become operators satisfying certain *anti*-commutation relations. One uses again the light-cone gauge quantization procedure, now appealing to the *superconformal symmetry* of the theory to remove negative-norm states and preserve unitarity. There is a different vacuum state for the R and NS sectors; the NS vacuum state is a tachyon whereas the R vacuum state is a *spacetime spinor* (we thus see that spacetime fermions have indeed arisen from the inclusion of a worldsheet fermionic action). The GSO projection mentioned previously projects out the tachyon, as well as one of the chiralities of the R vacuum. For the closed string (which, recall from above, is like a tensor product of open strings), it turns out there are then only 2 inequivalent choices of vacuum, which give rise to different closed superstring theories known as *Type IIA/B*, both with critical dimension $D = 10$ (see [POL] for details). Our focus for the AdS/CFT correspondence will be on Type IIB, which has the following massless spectrum [CAC]:

RR	A_0, A_2, A_4^+
R-NS	Ψ_+^1, χ_-^1
NS-R	Ψ_+^2, χ_-^2
NS-NS	$\Phi, B_2, g_{\mu\nu}$

where A_n is an n-form (and A_4^+ is self-dual), Ψ_+^I $(I = 1,2)$ are right-handed *dilatini*, χ_-^I $(I = 1,2)$ are left-handed *gravitini*, and the NS-NS sector is just the massless sector of the closed bosonic string spectrum that we have seen previously. We see that the theory is *chiral* since the 2 dilatini and the 2 gravitini have the same chirality.

As in the bosonic string case, Type IIB string theory has a low energy effective spacetime theory, which remarkably turns out to be a *supergravity theory* (i.e. a supersymmetric theory of gravity, here with $\mathcal{N} = 2$). It is not possible to write a complete action for the theory due to the presence of the self-dual 4-form A_4^+, but one can write an action that includes both dualities, and then supplement this with the self-duality condition $*F_5 = F_5$ where F_5 is the 5-form $F_5 \equiv dA_4$ and $*$ is the Hodge dual. The bosonic part of this action then has the form [FRE]:

$$\begin{aligned} S_{IIB} = \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-\det g}\, e^{-2\Phi}\left(2R + 8\nabla_\mu\Phi\nabla^\mu\Phi - |H_3|^2\right)\\ -\frac{1}{4\kappa^2}\int d^{10}x\left[\sqrt{-\det g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + A_4^+\wedge H_3\wedge F_3\right] + S_{\text{fermions}}\end{aligned} \tag{4.24}$$

where κ is the coupling constant, $F_n \equiv dA_{n-1}$ is the n-form field-strength, H_3 is the 3-form $H_3 \equiv dB_2$, $\tilde{F}_3 \equiv F_3 - A_0 H_3$ and $\tilde{F}_5 \equiv F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. The Type IIB action exhibits a non-compact $SU(1,1) \cong SL(2,\mathbb{R})$ symmetry (see [FRE]), most manifest in the *Einstein frame* which is obtained from the *string frame* above as:

$$g_{\mu\nu} \to e^{-\Phi/2} g_{\mu\nu} \tag{4.25}$$

The Einstein frame is so-called because, as is clear, the Type IIB action then contains the usual Einstein-Hilbert action familiar from general relativity. Combining the *axion* A_0 and the dilaton into a complex scalar $\tau \equiv A_0 + ie^{-\Phi}$, the symmetry transformation is then given by:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{4.26}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Note that in the quantum theory there is a quantization condition $\tau \cong \tau + 1$, and thus the symmetry group reduces to the subgroup $SL(2,\mathbb{Z})$. This symmetry will feature later in the AdS/CFT correspondence (see section 6.4).

4.4 Branes in Supergravity and Superstring Theory

We have seen previously that objects known as D-branes arise in string theory as hyperplanes upon which open string endpoints with Dirichlet boundary conditions are fixed. D-branes can in fact be seen to arise from a different point of view; indeed, this dual interpretation is of great significance in motivating the AdS/CFT correspondence as we shall see in section 6.2. We now discuss this alternative point of view by considering particular solutions to supergravity theory following [FRE], and then mention how the two points of view converge. We will then briefly discuss the important subject of gauge theories living on the worldvolumes of branes.

It is natural to consider a $(p+1)$-form A_{p+1} as coupling to an object Σ_{p+1} of dimension $p+1$ because one can construct the action:

$$S_{p+1} \propto \int_{\Sigma_{p+1}} A_{p+1} \tag{4.27}$$

which is diffeomorphism invariant since the form is being integrated over a manifold of dimension equal to the form's rank (see [NAK]). The action (4.27) is also invariant under the gauge transformation $A_{p+1} \to A_{p+1} + d\rho_p$, and we note that the field-strength $F_{p+2} \equiv dA_{p+1}$ is clearly gauge invariant since the exterior derivative is nilpotent (the field-strength also has a conserved flux). We can then define a *p-brane* to be a solution of the supergravity field equations that has a non-zero charge associated to the gauge field A_{p+1}. The possible brane solutions in a given supergravity theory are thus limited by what p-forms are present in the field content. Type IIB supergravity, for example, contains the following brane solutions:

Brane	**Couples to**:	**Dual Brane**
D(-1)	τ	D7
F1	B_2	NS5
D1	A_2	D5
D3	A_4^+	D3

Some clarifications of nomenclature are in order:

- A p-brane associated with a gauge field A_{p+1} that is in the R-R sector is known as a *Dp-brane*. The similarity of name with the aforementioned D-branes is not an accident, and will be explained later in this section.

- The 1-brane that couples to the NS-NS field B_2 is known as the *fundamental string* F1; this is intuitive, given the coupling of the original string to this 2-form in the action (4.14). The prefix 'NS' in NS5 simply labels the fact that the 2-form B_2 is an NS-NS field.

- The D(-1) brane is localised not only in all spatial directions but also in time, and is thus an *instanton*.

- The *dual* of a p-brane (coupled to a $(p+1)$-form) is the $(D-4-p)$-brane coupled to the $(D-3-p)$-form that is the Poincaré dual of the $(p+1)$-form, defined by:

$$dA^{dual}_{D-3-p} \equiv *dA_{p+1} \tag{4.28}$$

We will focus here on Dp-branes, and in particular D3-branes (which, note, are self-dual), since these are of most relevance for the AdS/CFT correspondence.

In $D = 10$, a p-brane solution to supergravity theory has symmetry group $\mathbb{R}^{p+1} \times SO(1,p) \times SO(9-p)$ i.e. the solution contains a flat hypersurface of dimension $(p+1)$ which has Poincaré invariance $\mathbb{R}^{p+1} \times SO(1,p)$, and the $(9-p)$-dimensional transverse space has maximal rotational invariance $SO(9-p)$. Each brane solution breaks half of the supersymmetries of the supergravity theory [FRE] i.e. it is an example of a 1/2 BPS solution (c.f. section 3.1). Denoting the coordinates parallel to the brane as x^μ and those perpendicular to the brane as y^i, an ansatz that has the above symmetry and satisfies the supergravity field equations is given by:

$$ds^2 = \frac{1}{\sqrt{H(\vec{y})}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(\vec{y})}d\vec{y}^2 \tag{4.29}$$

where, furthermore, $e^\Phi = [H(\vec{y})]^{(3-p)/4}$ and $H(\vec{y})$ must be a *harmonic function* of $\vec{y}$. Requiring that flat space be recovered far away from the brane (i.e. in the limit $y \equiv \sqrt{\vec{y}\cdot\vec{y}} \to \infty$) fixes the function $H(\vec{y})$ to take the form:

$$H(\vec{y}) = 1 + \left(\frac{L}{y}\right)^{D-p-3} \tag{4.30}$$

where L is some scale factor; we write D for generality, but recall that the case of interest is $D = 10$. Of particular importance to the AdS/CFT correspondence will be the solution for a stack of N coincident Dp-branes, for which one finds:

$$L^{D-p-3} = Ng_s(4\pi)^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right)\alpha'^{(D-p-3)/2} \tag{4.31}$$

where the factor of N comes from N units of *5-form flux* sourced by the N branes. We will use the solution (4.29)-(4.31) in chapter 6 when we discuss the decoupling argument.

In addition to the supergravity limit of superstring theory ($\alpha' \to 0$) discussed previously, another well-defined limit of string theory is the *weak coupling limit* $g_s \to 0$, which is the string perturbation theory regime (c.f. equation (4.16)). It turns out that Dp-branes also have a $g_s \to 0$ limit, whereas other p-branes do not. Indeed, we see from (4.29)-(4.31) that in this limit $H(\vec{y}) \to 1$ and thus the metric becomes flat everywhere except at $y = 0$; on this $(p+1)$-dimensional hypersurface the metric is in fact singular. Thus we see that, in the weak coupling limit, Dp-brane solutions to supergravity become localised defects in spacetime; as discussed in [FRE], it turns out that the interaction with a string propagating in such a background is described entirely by the boundary conditions of the string on the brane, which turn out to be Neumann conditions parallel to the brane and Dirichlet conditions perpendicular to it. In this manner, we see that the Dp-branes introduced as solutions of supergravity are precisely the D-branes introduced earlier in open string theory.

To conclude this chapter we discuss the important subject of how gauge theories arise on the worldvolumes of D-branes; this is of utmost relevance to the AdS/CFT correspondence, and so we shall review the topic but not discuss the exact details of open string quantization in the presence of D-branes (see [ZWE] for details). Let us first consider an open string with both endpoints fixed on a single D-brane. Clearly, the length of the string may become arbitrarily short in which case there is no tension; the string mode must then be massless, and it is possible to show that it is in fact a vector A^μ in the worldvolume i.e. there is a $U(1)$ gauge theory on the worldvolume of the brane. If we consider instead a stack of N *coincident* branes then we must further label the string states by indices which denote which brane the endpoints lie on. These are known as *Chan-Patton factors* [ZWE], and for N branes there will be $N^2 - N$ possible string configurations leading to $N^2 - N$ massless modes $[A^\mu]_{ij}$ on the worldvolume of the branes, where $i, j = 1...N$. One then finds a $U(N)$ gauge theory on the brane worldvolume; in fact, the factor of $U(1) = U(N)/SU(N)$ corresponds to the overall position of the branes in spacetime, and thus is not relevant for considering the dynamics on the brane worldvolume itself, which is then described by an $SU(N)$ gauge theory (separating the branes would give the modes mass and thus correspond to spontaneously breaking the gauge symmetry, but we shall not need such mechanisms in the following). As mentioned previously, each brane solution breaks half of the Poincaré superymmetries and so, in particular, for a stack of N D3-branes, the brane dynamics is described by 4-dimensional $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ (see [FRE]). This fact will be very important in chapter 6 when we discuss the decoupling argument.

Anti-de Sitter Space

The Type IIB theory on the gravity side of the correspondence is taken to be in an $AdS_5 \times S^5$ background. In this chapter we define the anti-de Sitter space AdS_d and discuss its essential features including isometries, important coordinate systems and the conformal boundary.

5.1 Definition of Anti-de Sitter Space

Let us first define the concept of a *maximally symmetric space* of d-dimensions [ZEE], meaning that it has the maximum number of Killing vectors possible for a d-dimensional manifold, namely $d(d+1)/2$ (corresponding locally to d translations and $d(d-1)/2$ rotations). A maximally symmetric space can be understood intuitively as one that is homogeneous and isotropic at some (and thus every) point, meaning it looks the same in all directions and at all positions. Maximal symmetry provides $d(d-1)/2$ constraints on the Riemann curvature tensor, which turn out to be enough to fix its form uniquely as (see [ZEE]):

$$R_{\mu\nu\rho\sigma} = C(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{5.1}$$

for some constant C, and thus one finds by contracting that $R_{\mu\nu} = (d-1)Cg_{\mu\nu}$ and $R = d(d-1)C$ i.e. maximally symmetric spaces have constant curvature scalars.

The *anti-de Sitter space* (AdS) is a space of Lorentzian signature and (as we shall see) constant negative curvature. In a similar fashion to other constant curvature spaces (e.g. the sphere), AdS space may be defined as an embedding in a higher-dimensional space. If we consider a flat embedding space $\mathbb{R}^{2,d-1}$ with coordinates X_a $(a = 0...d)$ and metric:

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2 \tag{5.2}$$

then we may define AdS_d as the set of solutions of:

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = L^2 \tag{5.3}$$

where L is known as the *AdS radius*. We also mention briefly that *Euclidean* AdS_d may be defined in an analogous way, but embedded instead in $\mathbb{R}^{1,d}$ and with the defining equation:

$$X_0^2 - X_d^2 - \sum_{i=1}^{d-1} X_i^2 = L^2 \tag{5.4}$$

Indeed, the Euclidean version of AdS will in fact be used later in chapter 7 when we discuss tests of the correspondence, and the results discussed below translate simply.

It is obvious from the defining equations that the isometry group of AdS_d is $O(2, d-1)$ (or $O(1, d)$ for the Euclidean case); restricting to our case of interest, AdS_5, we thus see that the isometry group is $O(2,4)$, a group we encountered previously in section 2.2 as the (extended) conformal group in 4-dimensional Minkowski space. The space AdS_d can also be expressed as the coset manifold $SO(2,d-1)/SO(2,d-2)$ i.e. the isometry group minus (by quotient) the subgroup that leaves a point in the space invariant (in the same way that $S^2 \cong SO(3)/SO(2)$). Since the dimension of $O(2,d-1)$ is $d(d+1)/2$ we see that AdS_d is indeed a maximally symmetric space, and thus should have constant curvature scalars.

By eliminating the final coordinate via $(X^d)^2 = L^2 + \eta_{\mu\nu}X^\mu X^\nu$, where $\mu = (0, 1...d-1)$ and $\eta_{\mu\nu}$ is the d-dimensional Minkowski metric, we may provide a set of coordinates for AdS_d and write the metric as:

$$ds^2 = \left(\eta_{\mu\nu} - \frac{\eta_{\mu\lambda}\eta_{\nu\rho}X^\lambda X^\rho}{X \cdot X + L^2}\right) dX^\mu dX^\nu \tag{5.5}$$

where in an obvious notation $X \cdot X \equiv \eta_{\mu\nu}X^\mu X^\nu$. One can then calculate the Riemann tensor from this (bearing in mind that, since this is a maximally symmetric space, only the constant C in (5.1) needs to be fixed) and one finds [ZEE] that $C = -1/L^2$. From the expressions for $R_{\mu\nu}$ and R underneath (5.1) we thus see that AdS_d has constant negative curvature scalar and that:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{(d-1)(d-2)}{2L^2}g_{\mu\nu} \tag{5.6}$$

meaning that (in $d > 2$) AdS_d is a solution to the vacuum Einstein field equations with a *negative cosmological constant*.

5.2 Coordinate Systems on AdS_d

Let us for convenience now set $L = 1$ in (5.3). We may introduce a set of coordinates on AdS_d by writing:

$$\begin{aligned} X_0 &= \tilde{r}\text{cos}t \\ X_d &= \tilde{r}\text{sin}t \\ X_i &= rx_i \end{aligned} \tag{5.7}$$

where $\sum_{i=1}^{d-1} x_i^2 = 1$, and the other coordinates range over $\tilde{r}, r > 0$ and $t \in [0, 2\pi)$. The defining equation (5.3) then clearly implies $\tilde{r}^2 - r^2 = 1$. Finding the differentials dX_μ of the coordinates (5.7) and substituting into the metric (5.2) (together with the fact that $\sum_i x_i^2 = 1 \to \sum_i x_i dx_i = 0$) we thus find after simple algebra that:

$$ds^2 = -d\tilde{r}^2 - \tilde{r}^2dt^2 + dr^2 + r^2d\Omega_{d-2}^2 \tag{5.8}$$

Using the constraint $\tilde{r}^2 - r^2 = 1$ we find that $d\tilde{r}^2 = \frac{r^2}{\tilde{r}^2}dr^2$ and thus more simple algebra gives the metric:

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2d\Omega_{d-2}^2 \tag{5.9}$$

We have thus eliminated $\tilde{r}$ and now have a set of d coordinates for AdS_d. We see from (5.9) that t acts as a time coordinate, yet from it's definition in (5.7) this coordinate appears to be periodic. As discussed in [ZEE], to avoid the existence of closed timelike curves and casual inconsistencies, we thus unwrap the time coordinate (technically, we move to the *universal cover*) and simply define the space AdS_d by equation (5.9) (which is, after all, a solution to the Einstein field equations) for $t \in \mathbb{R}$. Note interestingly that the metric (5.9) for AdS_d has the same form as the Schwarzchild metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \tag{5.10}$$

but here $f(r) = 1 + r^2 > 0$, and thus we see that the anti-de Sitter space does not have an event horizon, unlike the Schwarzchild spacetime for which the horizon is defined by the coordinate singularity when $f(r) = 0$.

We now make a further coordinate transformation in (5.9) given by $r = \sinh\rho$ for $\rho > 0$. Since $dr = \cosh\rho d\rho$ and $1 + r^2 = \cosh^2\rho$ we easily find:

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-2}^2 \tag{5.11}$$

These are known as *global coordinates* [ZEE] (so-called because they cover the entire AdS space). We can make instead a different coordinate substitution in (5.9) given by $r = \tan\beta$ for $\beta \in [0, \pi/2)$. Since $dr = \sec^2\beta d\beta$ and $1 + r^2 = \sec^2\beta$ we easily find the metric:

$$ds^2 = \frac{1}{\cos^2\beta}\left(-dt^2 + d\beta^2 + \sin^2\beta d\Omega_{d-2}^2\right) = \frac{1}{\cos^2\beta}\left(-dt^2 + d\Omega_{d-1}^2\right) \tag{5.12}$$

where the second equality follows by the general relation $d\Omega_{p+1}^2 = d\beta^2 + \sin^2\beta d\Omega_p^2$ with $\beta \in [0, \pi]$ (we return to this matter in section 5.3). These are known as *conformal coordinates* [ZEE], so-called because we see manifestly from (5.12) that AdS_d is conformally equivalent to the cylinder $\mathbb{R} \times S^{d-1}$ with metric $ds^2 = -dt^2 + d\Omega_{d-1}^2$ (note that flat space $\mathbb{R}^{1,d-1}$ is also conformally equivalent to this cylinder; we return to this in section 5.4).

5.3 The Conformal Boundary of AdS_d

We now introduce the important notion of the *conformal boundary* of AdS_d space, a concept that may be visualized in a number of ways. We introduce it as in [ZEE] by building on the material in the previous section, and in particular the set of conformal coordinates that defined the metric (5.12).

The coordinate β in the first half of (5.12) clearly plays the role of a latitude; strangely, however, we saw from its definition that it ranges over the values $\beta \in [0, \pi/2)$ rather than the usual $\beta \in [0, \pi]$. Thus, identifying $d\Omega_{d-1}^2 = d\beta^2 + \sin^2\beta d\Omega_{d-2}^2$ in equation (5.12), although true locally, is somewhat misleading since the spatial part of AdS_d really only covers the northern hemisphere of S^{d-1} and *not* the full sphere. Thus, we really have a hemisphere (after a conformal transformation, of course) for the spatial part of AdS_d, with boundary at the equator; topologically this is equivalent to the ball B^{d-1} (e.g. a hemisphere of S^2 is topologically equivalent to the disk B^2, as seen via a direct projection onto the plane). We thus see that the spatial sections B^{d-1} of AdS_d are bounded by S^{d-2} (since $\partial B^d = S^{d-1}$), which we commonly associate with $\mathbb{R}^{d-2}$ with spatial infinity identified as a single point (c.f. the Riemann sphere [FRA]). Taking the time coordinate into account we thus arrive at the important result that AdS_d is bounded by Minkowski space $\mathbb{R}^{1,d-2}$:

$$\partial(AdS_d) = \mathbb{R}^{1,d-2} \tag{5.13}$$

This result is of crucial importance in the AdS/CFT correspondence, and is at the heart of its holographic nature.

5.4 Poincaré Coordinates

It will be instructive to introduce one further set of coordinates for AdS_d which are particularly useful in the AdS/CFT correspondence. We will here restore L since we shall refer back to the Poincaré form of the metric when we consider the decoupling argument in chapter 6. We introduce for AdS_d the coordinates $y > 0$ and (t,$\vec{x}$) $\in \mathbb{R}^{d-1}$ via:

$$\begin{aligned} X_0 &= \frac{1}{2y}(1 + y^2(L^2 + \vec{x}^2 - t^2)) \\ X_d &= Lyt \\ X_{d-1} &= \frac{1}{2y}(1 - y^2(L^2 - \vec{x}^2 + t^2)) \\ X_i &= Lyx_i \end{aligned} \tag{5.14}$$

where $(i = 1...d-2)$ and $\vec{x}^2 = \sum_{i=1}^{d-2} x_i^2$. It is simple to check that the coordinates in (5.14) do indeed satisfy (5.3). Straightforward algebra then gives the metric in these coordinates as [ERD]:

$$ds^2 = \frac{L^2}{y^2}dy^2 + \frac{y^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{5.15}$$

where $x^\mu = (t, \vec{x})$. We shall see this metric arise in section 6.2 as the near-horizon limit of a stack of D-branes discussed in section 4.4.

Making the coordinate substitution $u = L^2/y$ we have $dy^2 = L^4/u^4 du^2$ and $L^2/y^2 = u^2/L^2$ and thus we find:

$$ds^2 = \frac{L^2}{u^2}\left(du^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \tag{5.16}$$

which is the metric in *Poincaré coordinates*. From this form of the metric we see that AdS_d is conformally equivalent to Minkowski space $\mathbb{R}^{1,d-1}$; this is not surprising since we saw in section 5.2 that AdS_d is conformally equivalent to the cylinder $\mathbb{R} \times S^{d-1}$, as is Minkowski space $\mathbb{R}^{1,d-1}$. We also notice that the slices of constant u are copies of Minkowski space $\mathbb{R}^{1,d-2}$. In particular, the conformal boundary discussed in section 5.3 is given by the slice $u = 0$ (i.e. $y = \infty$ in (5.15)); this can be seen, for example, by recognising (5.16) as a Minkowski version of the Poincaré half-plane [ZEE], with spatial boundary at $u = 0$. In Poincaré coordinates one can also see the manifest isometry given by:

$$\begin{aligned} u &\to \lambda u \\ x^\mu &\to \lambda x^\mu \end{aligned} \tag{5.17}$$

for any $\lambda \in \mathbb{R}$. We saw such transformations previously in section 2.1; they are conformal transformations known as dilatations or scale transformations. That they form an isometry of AdS space can be see trivially from the metric (5.16), since the scaling of the denominator u^2 cancels any scaling in the numerator. This isometry will play an important role in section 7.1 when we discuss the field/operator map in the AdS/CFT correspondence.

Part II

The $\mathcal{N} = 4$ SYM/Type IIB AdS/CFT Correspondence

Motivating the AdS/CFT Correspondence

In this chapter we motivate the AdS/CFT correspondence via two arguments. The first argument suffers a shortcoming in that, although it motivates a duality between string theory and gauge theories in a certain limit, it fails to provide more concrete information about the precise correspondence. The second argument, due to Maldacena, is a more specific situation and directly motivates the duality between $\mathcal{N}=4$ SYM and Type IIB string theory on $AdS_5 \times S^5$. After these motivations we will state the correspondence precisely, and perform the first basic check by mapping the global symmetries on both sides of the correspondence, relying heavily on the material discussed previously.

6.1 Motivation: The Large N Limit of Gauge Theories

We sketch here an illuminating argument to motivate the claim that string theory (as defined by a perturbative expansion) can be considered a dual description of non-abelian gauge theories when the number of 'colours' N is very large, following closely the logic of [VIE]. For simplicity (and since it captures all the essential features) we consider a 0-dimensional quantum field theory, otherwise known as a *matrix model* (the discussion can indeed be generalised to higher-dimensional quantum field theories).

Consider a 0-dimensional non-abelian gauge theory of $N \times N$ hermitian matrices. The n-point correlation functions of operators are defined by the usual expression:

$$\langle \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n \rangle \equiv \frac{\int \mathcal{D}M e^{-S[M]} \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n}{\int \mathcal{D}M e^{-S[M]}} \tag{6.1}$$

where $\mathcal{D}M$ is the path integral measure, integrating over all $N \times N$ hermitian matrices (which have no spacetime dependence since we are in 0-dimensions), given explicitly as:

$$\mathcal{D}M \equiv \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d(\mathrm{Re}[M_{ij}]) \prod_{i<j} d(\mathrm{Im}[M_{ij}]) \tag{6.2}$$

where we need only integrate over $i \leq j$ because the matrices are hermitian. The action $S[M]$ takes the general form:

$$S[M] = -\frac{1}{2g_{YM}^2} \mathrm{Tr}\left[M^2 + V(M)\right] \tag{6.3}$$

where $\mathrm{Tr}[M^2]$ is the kinetic term, and $V(M) = \sum_{i>2} a_i M^i$ gives the potential term for some constants a_i. We note that this theory has the non-abelian *gauge symmetry* given by $M \to UMU^{-1}$ (for some group with representation matrices U) by virtue of the cyclic property of the trace.

We wish to compute the propagator $\langle M_{ij} M_{kl} \rangle$, which involves using (6.1) for the case of the free action (i.e. $\mathrm{Tr}[M^2]$ term only). Recall from quantum field theory that for a free theory with

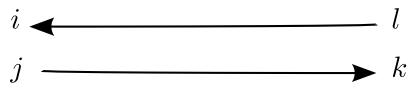

Figure 6.1: Double-line notation for the propagator $\langle M_{ij} M_{kl} \rangle = g_s \delta_{il} \delta_{jk}$.

partition function:

$$Z[J] \equiv \frac{\int \mathcal{D}x e^{-\frac{1}{2}x \cdot A \cdot x + J \cdot x}}{\int \mathcal{D}x e^{-\frac{1}{2}x \cdot A \cdot x}} = e^{\frac{1}{2}J \cdot A^{-1} \cdot J} \tag{6.4}$$

we have $\langle x_a x_b \rangle = A^{-1}_{ab}$ and for higher-point functions we obtain Wick's theorem, expressing them as a sum over all pairings of 2-point functions [PES]. Since Tr$[M^2] = \sum_i (M^2)_{ii} = \sum_{ij} M_{ij} M_{ji}$, for our case we have $x \cdot A \cdot x \to \frac{1}{g_{YM}^2} M_{ij} \delta^{il} \delta^{jk} M_{kl}$ (using the summation convention) and thus we have:

$$\langle M_{ij} M_{kl} \rangle = g_{YM}^2 \delta_{il} \delta_{jk} \tag{6.5}$$

We may represent this diagrammatically with so-called *double-line notation*, as in Figure 6.1. This notation will be important as, in a certain limit, it will in a sense begin to resemble string perturbation theory.

Consider, for example, using Wick's theorem to compute $\left\langle \frac{1}{g_{YM}^2} \text{Tr} M^4 \right\rangle$ in the free theory. We have:

$$\left\langle \frac{1}{g_{YM}^2} \text{Tr} M^4 \right\rangle = \left\langle \frac{1}{g_{YM}^2} \sum_{ijkl} M_{ij} M_{jk} M_{kl} M_{li} \right\rangle \tag{6.6}$$

and since the trace is cyclic we see that contracting the first M with the 2nd or the 4th will produce the same result (since they both involve contracting nearest neighbours), and thus using Wick's theorem we obtain:

$$\left\langle \frac{1}{g_{YM}^2} \text{Tr} M^4 \right\rangle = \frac{1}{g_{YM}^2} \sum_{ijkl} \left(2 \langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle + \langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle \right) \tag{6.7}$$

and so using (6.5) we get:

$$\left\langle \frac{1}{g_{YM}^2} \text{Tr} M^4 \right\rangle = \frac{1}{g_{YM}^2} (g_{YM}^2)^2 \sum_{ijkl} \left(2 \delta_{ik} \delta_{jj} \delta_{ki} \delta_{ll} + \delta_{il} \delta_{jk} \delta_{ji} \delta_{kl} \right) = g_{YM}^2 (2N^3 + N) \tag{6.8}$$

upon performing all delta index contractions. We notice that the first contribution is clearly dominant in the large N limit. The interest arises when we start considering diagrammatic expansions for these contributions using the double line notation. The operator we are considering is given diagrammatically in Figure 6.2, and the two types of contraction we found can be drawn as in the same figure. The first diagram (which, recall, is the dominant one in the limit of large N) is known as a *planar graph*, so-called because it can be drawn on the sphere. The second graph cannot be drawn on a sphere, though it can be drawn on a torus; it is an example of a *non-planar graph*. Note that the first graph has 3 closed loops whereas the second has only 1, corresponding to the powers of N^3 and N that these two diagrams contribute. This example

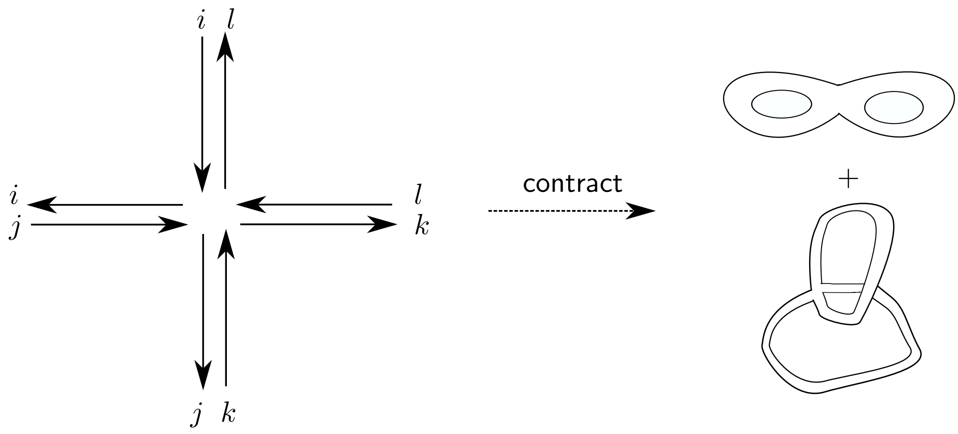

Figure 6.2: Vertex Tr$[M^4]$ and two different contractions

gives the first hint that, in the limit of large N, it may be possible to organise the perturbation theory into a topological expansion based on the topology of the manifolds that the corresponding diagrams can be embedded in.

With this example in mind, and considering now a general diagram in the full interacting theory, we see that a general diagram will contribute a factor of $(g_{YM}^2)^{E-V}N^F$ where E is the number of edges (number of propagators), V is the number of vertices (both internal and external), and F is the number of faces (number of closed loops). Since $E - V - F = 2g - 2$ by Euler's theorem [FRA], we see that a general diagram contributes $(g_{YM}^2)^{2g-2}(g_{YM}^2 N)^F$, and thus for a given genus g depends only on the *t'Hooft coupling* defined as $\lambda \equiv g_{YM}^2 N$. We thus see that a general observable $\langle ...\rangle$ is given as:

$$\langle ...\rangle = \sum_g (g_{YM}^2)^{2g-2}\mathcal{F}_g(\lambda) = \sum_g N^{2-2g}\tilde{\mathcal{F}}_g(\lambda) \tag{6.9}$$

where $\mathcal{F}_g(\lambda)$ is some function that represents the summation of all diagrams of a given genus g, and $\tilde{\mathcal{F}}$ is clearly a function that is simply related to $\mathcal{F}$. In the *t'Hooft limit*:

$$N \to \infty \qquad g_{YM} \to 0 \qquad \lambda \equiv g_{YM}^2 N \text{ fixed} \tag{6.10}$$

we thus see that the perturbation theory organises itself into a topological expansion in the genus, with small coupling constant $g_{YM}^2 \sim 1/N$, analogous to equation (4.16). It is thus natural to associate the gauge theory and string theory coupling constants as $g_{YM}^2 = g_s$, and $\lambda = g_{YM}^2 N = g_s N$ with the string tension. We see that perturbation theory will be valid for small values of the *t'Hooft coupling* λ.

We have thus seen that, in the t'Hooft limit of non-abelian gauge theories, perturbative string theory seems to provide a dual description of the gauge theory's perturbation expansion (although we only considered the 0-dimensional case, the argument can be generalised to higher-dimensional gauge theories). Although illustrative, this argument falls short in that it does not provide a precise map between a particular gauge theory and a particular string theory, and merely provides a hint of such a duality. Nevertheless, the ideas introduced here are important and indeed we shall see that the t'Hooft limit will reappear when we state a particular form of the AdS/CFT correspondence in the following.

6.2 Motivation: The Decoupling Argument

The original argument providing direct motivation for the AdS/CFT correspondence (as opposed to the mere hint towards a connection between gauge theory and string theory as in section 6.1) is due to Maldacena [MAL] and is known as the *decoupling argument*. The argument does not amount to a proof, but provides motivation which can then be tested using a variety of checks (see for example chapter 7). As we shall see, the argument rests directly on the dual interpretation of D-branes as discussed in section 4.4; by considering the branes in each interpretation, we find (in certain limits) two decoupled theories in each point of view, and are led to identify $\mathcal{N}=4$ SYM in 4D Minkowski space with Type IIB string theory in $AdS_5 \times S^5$. In this section we follow the form of the argument as portrayed in [VIE].

Before discussing the argument itself, it is illustrative to consider an analogy. Consider the interaction of an electron and a heavy proton. Such an interaction may in fact be described using two different interpretations, analogous to the brane case we will consider shortly. On the one hand, one may sum up all Feynman diagrams contributing to such an interaction, including the tree level diagram and the radiative corrections such as Bremsstrahlung, vertex corrections, photon self-energy etc. On the other hand, one may think of these diagrams as providing corrections to the familiar Coulomb potential that arises from the tree level diagram. In other words, one may remove the heavy proton from the picture entirely, and instead consider the electron moving freely (i.e. non-interacting) but now in some non-trivial *background*, represented by the corrected potential.

Turning now to the decoupling argument, the set-up in question is a stack of N D3-branes, as found in the Type IIB theory described in section 4.3. As discussed in section 4.4, we may think of D-branes in two ways (just as we may for the physical situation described above) which is fundamentally linked to an open/closed string duality; on the one hand they are hyperplanes upon which open strings end (or dynamical walls with open string excitations), and on the other hand they are solutions to the supergravity field equations, and thus deform the background and can be considered to emit closed strings (e.g. gravitons). This already indicates that there should be some correspondence between the theory of open string excitations on D-branes (i.e. super Yang-Mills gauge theory) and the theory of closed strings in the bulk i.e. Type IIB string theory. We now consider the set-up from these two points of view in turn.

D-Branes as Dynamical Walls with Open String Excitations

From the open string point of view, the action describing the physical set-up has the form:

$$S_{bulk} + S_{int.} + S_{branes} \tag{6.11}$$

where S_{bulk} is given by 10D supergravity plus massive modes (which scale as α'), S_{int} describes the interaction between the branes and the bulk theory (and scales with Newton's constant $\sqrt{G_N} \sim g_s \alpha'^2$), and S_{branes} is given by $\mathcal{N}=4$ SYM (with gauge group $SU(N)$) plus massive modes (which scale as α'). In the low energy limit $\alpha' \to 0$ we thus see that the interaction term drops out, and the bulk and brane terms simplify, giving us two decoupled theories:

$$(\mathcal{N}=4 \text{ SYM in 4D}) \oplus (\text{Type IIB Supergravity in 10D}) \tag{6.12}$$

Note that, as mentioned in section 6.1, $\mathcal{N}=4$ SYM is a useful description in the regime $\lambda << 1$, since in this regime perturbation theory is valid. We repeat for clarity the fact that gravity becomes free at low energies or large distances; as is well known the coupling G_N is dimensionful and thus the effective dimensionless coupling scales with energy, causing it to vanish in the low energy limit (see [VIE]).

D-Branes as Supergravity Solutions

In parallel to the proton/electron analogy discussed previously, we now no longer think of the branes as physical objects, and instead look for solutions to the supergravity field equations that describe a charged 'heavy body' and create some non-trivial background which closed strings propagate in. In fact, we already discussed such solutions in section 4.4, which are given by equations (4.29)-(4.31). For convenience we express here the solution specifically for the case of N D3-branes, focusing on the gravitational part which is given by:

$$ds^2 = \frac{1}{\sqrt{1+\frac{L^4}{y^4}}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{1+\frac{L^4}{y^4}}d\vec{y}^2 \tag{6.13}$$

where equation (4.31) gives $L^4 = g_s N 4\pi\alpha'^2$. Note that the supergravity description is useful when the curvature (which is set by the scale L) is large compared to the string length l_s since otherwise string effects are important and cannot be ignored. The useful regime is thus given by $L \sim \sqrt{\alpha'}(g_s N)^{\frac{1}{4}} \gg l_s \sim \sqrt{\alpha'}$ and thus we require $\lambda \equiv g_s N \gg 1$. We thus see that this is the opposite regime to which the gauge theory description is useful in.

As discussed in section 4.4, in the limit $y \gg L$ (i.e. far away from the branes) the solution becomes that of 10-dimensional flat Minkowski space, which is easy to see from (6.13) since $\sqrt{1+L^4/y^4} \to 1$. Another interesting limit however, considered by Maldacena, is the *near-horizon limit* $y \ll L$ (i.e. close to the branes). In this limit we have $\sqrt{1+L^4/y^4} \to L^2/y^2$ and thus the metric becomes:

$$ds^2 \to \frac{y^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{y^2}d\vec{y}^2 \tag{6.14}$$

Writing the 6-dimensional Euclidean metric in spherical coordinates as $d\vec{y}^2 = dy^2 + y^2 d\Omega_5^2$ we have in the near-horizon limit:

$$ds^2_{y\to 0} = \left(\frac{y^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{y^2}dy^2\right) + L^2 d\Omega_5^2 \tag{6.15}$$

We recognise, using (5.15), that this is nothing but the metric for the product geometry $AdS_5 \times S^5$, where the corresponding radius for both parts of the geometry is L. The geometry near to the branes is thus regular and highly symmetrical.

We saw that, from the open string point of view, there are two decoupled theories in the low energy limit. As we shall see, a similar decoupling occurs in the present case. Note that, as discussed above, the metric (6.13) becomes flat at $y \to \infty$ and thus the coordinate t is the proper time for an observer at infinity. In contrast, as is familiar from general relativity, the proper time elapsed for an observer at some other spacetime point is given by $\Delta\tau = \sqrt{-g_{tt}}\Delta t$, and correspondingly the energies are related by $E = 1/\sqrt{-g_{tt}}E_\infty$. In particular, close to the branes we have from (6.15) that $E_\infty = yE/L$ and thus, for fixed E, the energy as observed at infinity goes to zero as $y \to 0$. For an observer at infinity in this point of view, there are thus two decoupled low energy regimes:

- 10-dimensional supergravity close to the observer, since gravity becomes free at low energies/large distances.
- Full Type IIB string theory close to the branes (i.e. in the geometry $AdS_5 \times S^5$); everything (i.e. all strings) becomes a low energy effect close to the branes for an observer at infinity, and thus there is no restriction to low energy massless modes.

We thus have the two decoupled theories:

$$(\text{Type IIB String theory on } AdS_5 \times S^5) \oplus (\text{Type IIB Supergravity in 10D}) \tag{6.16}$$

Thus, we finally reach the celebrated AdS/CFT correspondence, by looking at (6.12) and (6.16) and noticing that we can 'cross off' Type IIB supergravity on either side and thus identify:

$$(\text{Type IIB String theory on } AdS_5 \times S^5) \equiv (\mathcal{N} = 4 \text{ SYM in 4D}) \tag{6.17}$$

The fact that D-branes have a dual interpretation has led us to identify these two theories as dual descriptions of each other. Although not generally regarded as a proof (because of, for example, the subtleties to do with the various limits), the decoupling argument provides strong motivation for the above correspondence.

6.3 Statement of the Correspondence

With the correspondence motivated we can now state it in its different forms; these differ in their strengths based on the conditions they impose on the various relevant parameters that appear in the theories. The precise form of the correspondence is given as:

Type IIB string theory on $AdS_5 \times S^5$ (both with radius L) with 5-form flux N and string coupling g_s is equivalent/dual to 4-dimensional $\mathcal{N} = 4$ SYM with gauge group SU(N) and coupling constant g_{YM},

where the couplings are identified in the following way:

$$g_s = g_{YM}^2 \qquad\qquad L^4 = 4\pi g_s N \alpha'^2 \tag{6.18}$$

(and one further identifies the axion expectation value with the instanton angle as $\langle C \rangle = \theta_I$ [FRE]); the motivation for the second equation in (6.18) is clear from the previous section, and the motivation for the first (in addition to the argument presented in section 6.1 concerning the large N limit of gauge theories) comes from the fact that the closed string coupling constant is the square of the open string coupling constant (see section 4.2). This is referred to as the *strong form* of the correspondence as it is supposed to hold for all values of the coupling constant $g_s = g_{YM}^2$ and all values of N. The strong form is extremely difficult to check however, namely because the theory of string quantization on general curved manifolds such as $AdS_5 \times S^5$ is not yet adequately developed. We will thus shortly state two progressively weaker forms of the correspondence which admit more tangible checks.

Before doing so however, it is instructive to make contact with the holographic nature of the correspondence alluded to previously. We saw in section 5.3 that AdS_5 has a boundary given by Minkowski space $\mathbb{R}^{1,3}$, and we have now seen that on one side of the correspondence we have $\mathcal{N} = 4$ SYM in $\mathbb{R}^{1,3}$. Indeed, it is in fact possible to consider this gauge theory as living on the boundary of AdS_5. We saw previously that rather than having branes at the origin, as in the open string picture, we may replace them with $AdS_5 \times S^5$ space, whose Minkowski boundary will then patch onto the full solution. Clearly, in the open string picture, it must be the information of the branes themselves that provides the boundary conditions for patching onto the full solution. In this manner it is then natural to identify the branes as being in some sense *on the boundary* of AdS_5, and thus the gauge theory (which lives on the branes) can be said to live on the boundary of AdS_5. This is the sense in which the correspondence is a holographic principle, since the 5-dimensional dynamics of Type IIB theory (after compactification on S^5) can be encoded in a gauge theory living on the 4-dimensional boundary.

Returning now to stating the correspondence, we can state a slightly weaker form, which we shall call the *t'Hooft form*, by going to the t'Hooft limit. Thus, in this form of the correspondence, one claims the above equivalence with the added conditions that $N \to \infty$, $g_{YM} \to 0$ and $\lambda \equiv g_{YM}^2 N$ is fixed. On the gauge theory side this then corresponds to a perturbation theory topological expansion in $1/N$ as we saw in section 6.1, and on the string theory side one has *classical* Type IIB string theory with small expansion parameter $g_s = \lambda/N$, as we also described in the same section; this form of the correspondence thus coincides (albeit in a more precise manner) with that which the argument in section 6.1 pointed towards. Finally, we have the *weak form* of the correspondence which, after taking the t'Hooft limit, involves taking the large λ limit. On the gauge theory side we know from section 6.1 that this corresponds to the strong coupling (i.e. non-perturbative) regime, whereas on the string theory side we have classical Type IIB *supergravity*, with an expansion in small α' (as is clear from (6.18)). Although the weakest statement, this final form of the correspondence turns out to be extremely powerful, since one may use classical gravity to perform calculations in non-perturbative gauge theory (see chapter 7).

6.4 First Check: Correspondence of the Global Symmetries

The first check of the correspondence that we may perform, which is simple but necessary, is to map the global symmetries on the two sides of the correspondence. Luckily, we have already done most of the work for this check in previous sections. We saw in section 3.3 that $\mathcal{N} = 4$ SYM has a global symmetry group given by the superconformal group $SU(2,2|4)$ which arose from the non-trivial combination of supersymmetry and conformal symmetry, together with the R-symmetry group $SO(6)_R \cong SU(4)_R$. The bosonic subgroup of this supergroup is given by the product of the conformal group and the R-symmetry group i.e. $SO(2,4) \times SO(6)_R$. On the string theory side we recognise this bosonic subgroup as the isometry group of the spacetime $AdS_5 \times S^5$, since we saw in section 5.1 that the isometry group of AdS_d is $SO(2, d-1)$ (and of course the isometry group of S^5 is $SO(6)$). Furthermore, as discussed in [FRE] although the D3-brane breaks precisely half of the Poincaré supersymmetries (i.e. 16 of the 32), in the $AdS_5 \times S^5$ near-horizon limit, these are in fact supplemented by a further 16 conformal supersymmetries, enhancing the overall symmetry group to the full $SU(2,2|4)$ in correspondence with the field theory side.

In addition to the global symmetry group $SU(2,2|4)$, we saw in section 3.2 that SYM also has the (conjectured) S-duality symmetry group $SL(2,\mathbb{Z})$. However, we also saw in section 4.3 that Type IIB string theory has an $SL(2,\mathbb{R})$ symmetry that when quantized reduces to $SL(2,\mathbb{Z})$, in accordance with the field theory side. This agreement only holds in the strongest form of the correspondence however, as it is only in this form that we have the full quantum Type IIB string theory. We thus see that, at the level of the global symmetries at least, the correspondence in its strongest form seems to be valid. We shall provide more thorough tests in the next chapter, though for weaker forms of the correspondence as these are easier to check.

The Field/Operator Map and the Witten Prescription

In the previous chapter we saw the motivation of the original AdS/CFT correspondence due to Maldacena, as well as its statement in three forms of differing strength. We also saw a first basic check of the correspondence, which involved mapping the global symmetries of the theories on the two sides of the correspondence. In this chapter we shall investigate the subject of testing the correspondence in more detail. In particular, if the two theories are supposed to be dual, then there should be some mapping between the fundamental observables i.e. the fields/operators, as well as the correlators. We shall thus provide such a field/operator map as well as a prescription, due to Witten, for mapping the correlators between the two theories; we well then discuss some simple checks of the correspondence. We will be working in the classical supergravity limit and thus considering the weak form of the correspondence.

7.1 The Field/Operator Map

In section 6.4 we saw an agreement between the global symmetries on both sides of the correspondence. In addition to this it is important that there exists a correspondence between the specific representations of the symmetry group that the observables of the two theories lie in. In particular, we would like a correspondence between the supergravity fields in AdS_5 and observables in $\mathcal{N}=4$ SYM (which are the local gauge invariant operators described briefly in section 3.3). Consider a free massless scalar field ϕ in $AdS_5 \times S^5$. The Kaluza-Klein procedure of dimensional reduction involves decomposing this field into a basis of 5-dimensional *spherical harmonics* on S^5 as (see [FRE]):

$$\phi(x,y) = \sum_n \phi_n(x) Y_n(y) \tag{7.1}$$

where x are the AdS_5 coordinates and y are the S^5 coordinates. The equation of motion $\Box_{10}\phi = 0$ on $AdS_5 \times S^5$ then implies $(\Box_5 + m^2)\phi_n = 0$ for some mass m on AdS_5. The 10-dimensional massless scalar fields, when compactified on S^5, thus becomes massive scalar fields on AdS_5, with particular masses determined by the original action. We will see now that this mass is in fact related to the scaling dimension of the field, which allows for a correspondence between the scalar fields and the SYM observables (similar arguments hold for higher-spin fields, but we restrict the discussion to the simplest case of scalars).

We wish to study this 5-dimensional massive wave equation on AdS_5. Let us in fact consider general *Euclidean* AdS_{d+1} space with unit radius for convenience (there are subtleties with the Minkowski case), in Poincaré coordinates $x = (x_0, \vec{x})$ with metric analogous to (5.16) given by:

$$ds^2 = \frac{dx_0^2 + d\vec{x}^2}{x_0^2} \tag{7.2}$$

where $d\vec{x}^2 = \sum_{i=1}^d dx_i^2$. The Laplacian for a spacetime of metric $g_{\mu\nu}$ is given by [ZEE]:

$$\Box = -\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \tag{7.3}$$

Using this together with the metric (7.2), for AdS_{d+1} one finds:

$$\Box_{d+1} = -x_0^2\partial_0^2 + (d-1)x_0\partial_0 - x_0^2\sum_{i=1}^{d}\partial_i^2 \tag{7.4}$$

We thus have the wave equation $(\Box_{d+1} + m^2)\phi = 0$ with $\Box_{d+1}$ as in (7.4). Let us in particular assume solutions that asymptotically are eigenstates of dilatations in the x_0-direction (this will be important for the correspondence below); in other words, we assume solutions of the form $\phi(x) \sim x_0^k$ for some k. The wave equation then becomes:

$$-x_0^2k(k-1)x_0^{k-2} + (d-1)x_0kx_0^{k-1} - 0 + m^2x_0^k = 0 \tag{7.5}$$

which implies the quadratic equation:

$$k^2 - dk - m^2 = 0 \tag{7.6}$$

The solutions to this equation are found easily as:

$$k = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \tag{7.7}$$

and so such solutions to the wave equation are given by x_0^Δ and $x_0^{d-\Delta}$ where:

$$\Delta \equiv \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \tag{7.8}$$

These are known as *normalizable* and *non-normalizable* modes respectively [FRE], so-called because the latter are not square-integrable, whereas the former are. They are linearly independent modes, and describe the asymptotics at the boundary $x_0 \to 0$ of a general solution to the wave equation. A general solution then asymptotically has the form:

$$\phi_\Delta(x_0, \vec{x}) \sim x_0^\Delta f(\vec{x}) + x_0^{d-\Delta}g(\vec{x}) \tag{7.9}$$

Recall from section 5.4 that AdS space in Poincaré coordinates has a manifest dilatation isometry. Indeed, the metric (7.2) is invariant under the simultaneous scalings $x_0 \to \lambda x_0$ and $\vec{x} \to \lambda\vec{x}$. The scalar field ϕ_Δ should be a representation of the isometry group and thus should be left invariant by this transformation. Given the form (7.9), we thus see that under this dilatation we must have:

$$\begin{aligned} f(\vec{x}) &\to \lambda^{-\Delta}f(\vec{x}) \\ g(\vec{x}) &\to \lambda^{\Delta-d}g(\vec{x}) \end{aligned} \tag{7.10}$$

Recalling (2.23) we see that Δ can thus be interpreted as a conformal dimension for the 4-dimensional field $f(\vec{x})$, and that $g(\vec{x})$ has a related dimension. This observation is crucial in forming a map between observables in the AdS/CFT correspondence as we now discuss. Plugging $d = 4$ into (7.8) for our case AdS_5 we find the following relationship between mass and conformal dimension:

$$m^2 = \Delta(\Delta - 4) \tag{7.11}$$

Since the representations (of the conformal group in SYM and the isometry group of AdS) on both sides of the correspondence should match, we thus associate a field of mass m in AdS_5 with an operator in SYM of conformal dimension Δ, where m and Δ are related as in (7.11). The correspondence is in fact more specific than this; as argued in [KRA], the non-normalizable modes (or, more specifically, their coefficients $g(\vec{x})$ in the solution (7.9)) correspond to *sources* for the associated SYM operators, whereas the normalizable modes correspond to *vacuum expectation values* for the associated operators. Indeed, the normalizable modes define 4-dimensional fields with conformal dimension Δ, whereas the non-normalisable modes define a field $g(\vec{x})$ which is an appropriate source field for a CFT operator $\mathcal{O}$ of dimension Δ since the coupling integral:

$$\int d^d\vec{x}g(\vec{x})\mathcal{O}(\vec{x}) \tag{7.12}$$

that appears in the exponent of the CFT generating functional is then invariant under dilatations:

$$\int d^d\vec{x}g(\vec{x})\mathcal{O}(\vec{x}) \to \int \lambda^d d^d\vec{x}\lambda^{\Delta-d}g(\vec{x})\lambda^{-\Delta}\mathcal{O}(\vec{x}) = \int d^d\vec{x}g(\vec{x})\mathcal{O}(\vec{x}) \tag{7.13}$$

In particular, the non-normalizable modes define associated boundary fields which we now denote as [FRE]:

$$\bar{\phi}_\Delta(\vec{x}) \equiv \lim_{x_0\to 0}\phi_\Delta(x_0,\vec{x})x_0^{\Delta-4} \tag{7.14}$$

These will be important in the following section.

Although the above discussion has been for scalars, similar correspondences exist for higher-spin fields. As discussed in [FRE], one finds that (7.11) also holds for spin-2 fields, and for other spins one has the following relations between mass and conformal dimension:

$$\begin{aligned} \text{Spin } 1/2 \text{ and } 3/2: &\qquad |m| = \Delta - 2 \\ p\text{-form}: &\qquad m^2 = (\Delta-p)(\Delta+p-4) \end{aligned} \tag{7.15}$$

As an example we briefly mention that the supergravity modes (which include both a 5-dimensional supergravity multiplet plus an infinite tower of Kaluza-Klein modes obtained from compactification) correspond to SYM operators given by chiral primaries (such as the $SO(6)$-traceless parts of the single trace operators given in equation (3.19)) and their descendants; for example, the supergravity multiplet defined by equation (3.20) is such a chiral primary (hence its name). The massive Type IIB string modes however are given by non-chiral operators and their descendants, such as the Konishi multiplet Tr$[X^iX^i]$ also introduced in (3.20). A complete mapping between SYM operators and the supergravity modes is provided in Table 7 of [FRE].

7.2 The Witten Prescription for Mapping Correlators

We saw in the previous section a correspondence between the fields on the supergravity side and the operators on the SYM side of the correspondence. However, we would also like a correspondence between the correlation functions on the two sides. Note that, since the field ϕ_Δ has no gauge index, the associated operator $\mathcal{O}_\Delta$ should clearly be gauge invariant (the associated operator must thus be composite since the elementary fields in SYM all have gauge indices; see section 3.2). Considering what we said in section 7.1 regarding non-normalizable modes as defining sources for the associated operator, the SYM generating functional for operators $\mathcal{O}_\Delta$ then has the form [NAS]:

$$Z_{\mathcal{O}}[\bar{\phi}_\Delta] = \int \mathcal{D}[\text{SYM Fields}]e^{-S_{SYM}+\int d^4\vec{x}\mathcal{O}_\Delta(\vec{x})\bar{\phi}_\Delta(\vec{x})} \tag{7.16}$$

which, upon differentiation with respect to $\bar{\phi}_\Delta(\vec{x})$, gives correlation functions for the operators in the usual way:

$$\langle\mathcal{O}(\vec{x}_1)...\mathcal{O}(\vec{x}_n)\rangle = \frac{\delta^n}{\delta\bar{\phi}(\vec{x}_1)...\delta\bar{\phi}(\vec{x}_n)}Z_{\mathcal{O}}[\bar{\phi}]|_{\bar{\phi}=0} \tag{7.17}$$

where we have temporarily dropped the subscript Δ for notational convenience.

Since the correspondence states that SYM is dual to a string theory, there should be some way of computing this correlation function on the string/gravity side. Clearly, the above generating functional (7.16) should be equivalent to some string theory generating functional for the field ϕ (after compactification on S^5) with boundary value $\bar{\phi}$. If we are in the classical supergravity limit however, this generating functional becomes particularly simple since the phase oscillations cancel (i.e. the stationary phase approximation; see [ZEE2]) and thus the only contribution to the path integral comes from the extremum i.e. the classical solution to the equations of motion. Following [NAS], the *Witten prescription* [WIT] for mapping correlators in AdS/CFT (in the classical supergravity limit) is then given as:

$$Z_{\mathcal{O}}[\bar{\phi}_\Delta] = \int \mathcal{D}[\text{SYM Fields}]e^{-S_{SYM}+\int d^4\vec{x}\mathcal{O}_\Delta(\vec{x})\bar{\phi}_\Delta(\vec{x})} = e^{-S_{SUGRA}[\phi[\bar{\phi}]]} \tag{7.18}$$

where $S_{SUGRA}[\phi[\bar{\phi}]]$ is the supergravity action evaluated at the classical solution $\phi[\bar{\phi}]$ which has boundary value $\bar{\phi}$ (note that this supergravity action is obtained by compactification of the Type IIB action on the 5-sphere, and becomes a gauged supergravity theory since it is in an AdS background (see [NAS])). Correlation functions in SYM can then be calculated in classical gravity by using equations (7.17)-(7.18).

An important step in using equation (7.18) to computing correlation functions is to be able to construct the classical bulk solution in AdS_5 space, since this is substituted into the supergravity action on the RHS. To this end, we may use the method of Green's functions familiar from classical field theory. Here there is in fact a slight generalisation since AdS space has a boundary (see [FRE] for a detailed discussion of AdS propagators). We define the *bulk-to-boundary propagator* by:

$$\Box_5 K_B(\vec{x}, x_0; \vec{x}') = \delta^{(4)}(\vec{x} - \vec{x}') \tag{7.19}$$

where $\Box_5$ is as in (7.4), and the RHS clearly represents a delta source on the boundary of AdS space. We then construct the full bulk solution as:

$$\phi(\vec{x}, x_0) = \int d^4\vec{x}' K_B(\vec{x}, x_0; \vec{x}')\bar{\phi}(\vec{x}') \tag{7.20}$$

From (7.19) (generalised to AdS_{d+1}) and using (7.4) one finds the solution:

$$K_B(\vec{x}, x_0; \vec{x}') = \frac{Cx_0^d}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d} \tag{7.21}$$

where C is a normalisation constant. We shall use these expressions in the example discussed in the next section.

7.3 Example: Calculation of the 2-point Function for Scalars

One of the simplest checks we can provide is that of computing the 2-point correlation functions of scalars. This is an illustrative example because we saw in section 2.3 that the 2-point functions of scalars in a CFT are entirely fixed to have the form (2.41). If we are able to derive this form of the correlation function (for an appropriate dimension Δ) from the associated supergravity theory, using the Witten prescription (7.18), then we will have our first check of the correspondence. In this section we follow closely the argument presented in [NAS].

Let us thus consider the 2-pt. function of a scalar CFT operator $\mathcal{O}$ corresponding to a massless scalar ϕ in the gravity theory. From (7.17) and (7.18) we have:

$$\langle\mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\rangle = \frac{\delta^2}{\delta\bar{\phi}(\vec{x}_1)\delta\bar{\phi}(\vec{x}_2)} e^{-S_{SUGRA}[\phi[\bar{\phi}]]}|_{\bar{\phi}=0} \tag{7.22}$$

The 5-dimensional supergravity action, after using (7.20), has the following schematic scalar field contribution:

$$S_{SUGRA}[\phi[\bar{\phi}]] \sim \int d^5x\sqrt{g}\int d^4\vec{x}'d^4\vec{y}'\nabla_\mu K_B(\vec{x},x_0;\vec{x}')\bar{\phi}(\vec{x}')\nabla^\mu K_B(\vec{x},x_0;\vec{y}')\bar{\phi}(\vec{y}')+O(\bar{\phi}^3) \tag{7.23}$$

where, schematically, $\nabla_\mu\nabla^\mu \equiv \Box_5$ is the kinetic operator. From (7.23) we see clearly that:

$$S_{SUGRA}[\phi[\bar{\phi}]]|_{\bar{\phi}=0} = \frac{\delta S_{SUGRA}[\phi[\bar{\phi}]]}{\delta\bar{\phi}}|_{\bar{\phi}=0} = 0 \tag{7.24}$$

since all terms include at least two fields $\bar{\phi}$, and thus non-zero contributions begin at second order derivatives, giving (abbreviating the action as S for convenience):

$$\langle\mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\rangle = \frac{\delta}{\delta\bar{\phi}(\vec{x}_1)}\left(-\frac{\delta S}{\delta\bar{\phi}(\vec{x}_2)}e^{-S}\right)|_{\bar{\phi}=0} = -\frac{\delta^2 S}{\delta\bar{\phi}(\vec{x}_1)\delta\bar{\phi}(\vec{x}_2)}|_{\bar{\phi}=0} \tag{7.25}$$

For the 2-point functions we may thus ignore the interaction terms implicit in (7.23), since we take only 2 derivatives before setting $\phi_0 = 0$; this greatly simplifies the problem, and only applies to the computation of the 2-point function (and not, for example, to the computation we shall discuss in section 7.4). We can thus effectively consider a free massless scalar field with equation of motion $\Box_5\phi = 0$ and action given by the usual expression:

$$S = \frac{1}{2}\int d^5x\sqrt{g}\partial_\mu\phi\partial^\mu\phi \tag{7.26}$$

which upon integration by parts and using the equation of motion to kill one term becomes:

$$S = \frac{1}{2}\int d^5x\sqrt{g}\partial_\mu(\phi\partial^\mu\phi) = \frac{1}{2}\int_{boundary} d^4\vec{x}\sqrt{h}(\phi\vec{n}\cdot\nabla\phi) \tag{7.27}$$

where h is the metric on the boundary, and $\vec{n}\cdot\nabla$ is the component of the gradient normal to the boundary.

Let us unpack each of the terms on the RHS of equation (7.27) individually before taking the boundary limit $x_0 \to 0$ of the full expression. Since the boundary is defined by a slice of constant x_0, from (7.2) we thus have $\sqrt{h} = x_0^{-d}$ and $\vec{n}\cdot\nabla = x_0\partial/\partial x_0$. From (7.27), we see that we also need to investigate $\phi\vec{n}\cdot\nabla\phi = \phi x_0\partial\phi/\partial x_0$ at the boundary. Clearly, by construction, in this limit we let the scalar field become its associated boundary value i.e. $\phi(\vec{x},x_0) \to \bar{\phi}(\vec{x})$. Note that from (7.20) we have:

$$x_0\frac{\partial\phi}{\partial x_0}(\vec{x},x_0) = x_0\int d^d\vec{x}'\frac{\partial K_B}{\partial x_0}(\vec{x},x_0;\vec{x}')\bar{\phi}(\vec{x}') \tag{7.28}$$

and furthermore from (7.21) we have:

$$\frac{\partial K_B}{\partial x_0}(\vec{x},x_0;\vec{x}') = \frac{dCx_0^{d-1}}{(x_0^2+|\vec{x}-\vec{x}'|^2)^d} - \frac{dC2x_0^{d+1}}{(x_0^2+|\vec{x}-\vec{x}'|^2)^{d+1}} \tag{7.29}$$

In the limit $x_0 \to 0$, the factors of x_0 in the denominators in (7.29) become negligible, and furthermore $x_0^{d+1} \to 0$ faster than x_0^{d-1} does. The first term is thus clearly dominant, and we have:

$$x_0\frac{\partial K_B}{\partial x_0}(\vec{x},x_0;\vec{x}') \to_{x_0\to 0} \quad \frac{dCx_0^d}{|\vec{x}-\vec{x}'|^{2d}} \tag{7.30}$$

and so using (7.28) we obtain:

$$x_0 \frac{\partial \phi}{\partial x_0}(\vec{x}, x_0) \rightarrow_{x_0 \rightarrow 0} \quad dC x_o^d \int d^d \vec{x}' \frac{\bar{\phi}(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \tag{7.31}$$

We are now in a position to compute the 2-point correlator. From (7.27) for a general d we have:

$$\begin{aligned} S &= \frac{1}{2} \int_{boundary} d^d \vec{x} \sqrt{h} (\phi \vec{n} \cdot \nabla \phi) \\ &= \lim_{x_0 \rightarrow 0} \frac{1}{2} \int d^d \vec{x} x_0^{-d} \phi(\vec{x}, x_0) x_0 \frac{\partial \phi}{\partial x_0}(\vec{x}, x_0) \\ &= \frac{dC}{2} \int \int d^d \vec{x} d^d \vec{x}' \frac{\bar{\phi}(\vec{x}) \bar{\phi}(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \end{aligned} \tag{7.32}$$

From equations (7.25) and (7.32) we thus find by simple differentiation that the 2-point function is given by:

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = -\frac{Cd}{2} \frac{1}{|\vec{x} - \vec{x}'|^{2d}} \tag{7.33}$$

Comparing back to equation (2.41), we see that this is exactly the form we derived for a 2-point correlation function of scalar primary operators of dimension $\Delta = d$ in a CFT, which is consistent with the fact that here the field in question is massless (c.f. equation (7.11) for general d). We thus have our first check of the AdS/CFT correspondence, since we have derived a CFT correlation function from classical supergravity, and confirmed that it is indeed of the correct form.

7.4 Example: The 3-point Function for the R-Symmetry Currents and its Anomaly

The example provided in the previous section, although illuminating, is somewhat restricted in its scope as evidence for the correspondence, since the calculation was performed at the level of the free theory i.e. the supergravity interaction terms did not contribute to the correlation function. Indeed, notice further that the analysis was in fact entirely done for a general spacetime dimension d, and was thus not specific to the original correspondence. We thus here provide another example, again closely following [NAS]; this example is considerably more involved and we by no means perform the calculation in full, instead being content with briefly highlighting the general structure of the computation. The argument sketched will also illustrate some aspects of the general method used (i.e. Witten diagrams) when calculating CFT correlation functions from the associated supergravity theory (in general, as discussed in section 3.3, correlation functions of BPS operators provide important checks since their conformal dimensions are protected against quantum corrections; details of how to perform such calculations are provided in section 8 of [FRE] and chapter 5 of [ERD]).

We introduced in section 3.1 the concept of R-symmetry; indeed, the $\mathcal{N} = 4$ SYM theory has an $SU(4)_R$ symmetry and thus will have conserved currents J^a_μ (which are composite, gauge invariant operators) arising from Noether's theorem in the usual way. As mentioned in section 2.3, classical symmetries may sometimes be broken upon quantization giving rise to anomalies; there is such an anomaly for the R-symmetry currents meaning that:

$$\frac{\partial}{\partial x_\mu} \langle J^a_\mu(\vec{x}) J^b_\nu(\vec{y}) J^c_\rho(\vec{z}) \rangle \neq 0 \tag{7.34}$$

and this anomaly is antisymmetric in μ, ν, ρ and appears at 1-loop order only, meaning that it can be calculated exactly i.e. it is an unrenormalised quantity. This is extremely useful as it allows the computation of the anomaly from the AdS side to be directly compared to the known answer computed using CFT (recall the importance of unrenormalised quantities, due to the fact that the classical supergravity regime is the strong-coupling regime on the gauge side as discussed in sections 6.2 and 6.3).

From the Witten prescription we know that the composite, gauge-invariant R-currents J^a_μ will couple to the non-renormalisable modes of some associated supergravity field, and from the index structure this will have the form A^a_μ, with associated boundary value $\bar{A}^a_\mu$. We know that a is an R-symmetry index in the algebra $SU(4) \cong SO(6)$; it turns out that on the supergravity side the fields A^a_μ are then the gauge fields of the 5-dimensional gauged supergravity obtained by compactification of Type IIB supergravity on S^5 (with isometry group $SO(6)$). The Witten prescription, analogous to (7.18), then states the following:

$$Z_J[\bar{A}] = \int \mathcal{D}[\text{SYM Fields}] e^{-S_{SYM}+\int d^4\vec{x} J^a_\mu(\vec{x})\bar{A}^{a\mu}(\vec{x})} = e^{-S_{SUGRA}[A[\bar{A}]]} \tag{7.35}$$

where, again, on the RHS one substitutes the classical solution $A[\bar{A}]$ in S_{SUGRA} with boundary value $\bar{A}$. In analogy to (7.22) and (7.25) one then has:

$$\langle J^a_\mu(\vec{x}) J^b_\nu(\vec{y}) J^c_\rho(\vec{z})\rangle = \frac{\delta^3 e^{-S_{SUGRA}[A[\bar{A}]]}}{\delta\bar{A}^a_\mu(\vec{x})\delta\bar{A}^b_\nu(\vec{y})\delta\bar{A}^c_\rho(\vec{z})}|_{\bar{A}=0} = -\frac{\delta^3 S_{SUGRA}[A[\bar{A}]]}{\delta\bar{A}^a_\mu(\vec{x})\delta\bar{A}^b_\nu(\vec{y})\delta\bar{A}^c_\rho(\vec{z})}|_{\bar{A}=0} \tag{7.36}$$

The bulk solution is then given in terms of the boundary value via the bulk-to-boundary propagator as in (7.20):

$$A^a_\mu(\vec{x}, x_0) = \int d^4\vec{x}' G_{\mu\alpha}(\vec{x}, x_0; \vec{x}')\bar{A}^{a\alpha}(\vec{x}') \tag{7.37}$$

except now the bulk-to-boundary propagator $G_{\mu\alpha}$ is gauge-dependent since A^a_μ is (in actually doing the calculations it is useful to choose a bulk-to-boundary propagator that is conformal on the boundary).

Given (7.36)-(7.37), we see that for the 3-point function we are only interested in terms in S_{SUGRA} that are cubic in the gauge field. The 5-dimensional supergravity action has the following schematic form for A_μ [NAS]:

$$S_{SUGRA}|_{A_\mu} \sim \int \Big[(A)^2 \text{ term} + (A^a_\mu A^b_\nu A^c_\rho) \text{ term} + ...\Big] \tag{7.38}$$

where the quadratic term determines the propagator, and one may then perform classical perturbation theory in the interaction terms using tree-level diagrams known as *Witten diagrams* (the restriction to tree-level comes from the fact that we are in the classical limit; loops would correspond to quantum corrections). Witten diagrams are essentially tree-level Feynman diagrams, but there is an extra feature present due to the fact that AdS space has a boundary; the bulk of the spacetime is then represented as a disc, with the outer circle representing the boundary of the space. Propagators are then represented as usual by lines (and may be either bulk-to-bulk, bulk-to-boundary, or boundary-to-boundary), and the vertex factors are determined in the usual way from the supergravity action. As an example, the only diagram that contributes to the calculation of the 3-point function in question is given in Figure 7.1, which consists of 3 bulk-to-boundary propagators and a single 3-point vertex.

Let us consider specifically the anomaly of the 3-point function. In addition to a term that is cubic in the gauge fields, the contribution from the action should also be antisymmetric in the spacetime indices, since we mentioned previously that the anomaly has these properties. Such

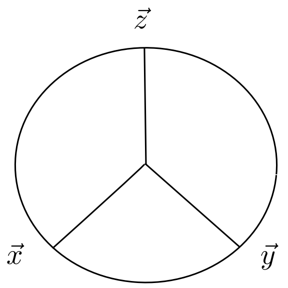

Figure 7.1: Witten diagram contributing to the 3-point R-current function.

a term does appear in the 5-dimensional supergravity action and is known as the *Chern-Simons term* which has the form:

$$S_{CS}(A) = \frac{iN^2}{16\pi^2}\mathrm{Tr}\int d^5x\epsilon^{\mu\nu\rho\sigma\tau}(A_\mu(\partial_\nu A_\rho)\partial_\sigma A_\tau + ...) \tag{7.39}$$

One can determine the R-current anomaly from the gravity side rather neatly in the following way. Under a gauge transformation $\delta A^a_\mu = (D_\mu\Lambda)^a = \partial_\mu\Lambda^a + g_{YM}f^a_{bc}A^b_\mu\Lambda^c$ (where f_{abc} are the structure constants of the gauge group) one finds (see [NAS] for all details):

$$\delta_\Lambda S_{CS} = -\frac{iN^2}{16\pi^2}d_{abc}\int_{boundary} d^4x\epsilon^{\mu\nu\rho\sigma}\Lambda^a\partial_\mu\left(A^b_\nu\partial_\rho A^c_\sigma + \frac{1}{4}f^c_{de}A^b_\nu A^d_\rho A^e_\sigma\right) \tag{7.40}$$

where $d_{abc} = \mathrm{Tr}(T_a\{T_b, T_c\})$, and since the integral is performed on the boundary we may replace $A \to \bar{A}$ throughout. However, from the Witten prescription (7.35) we also have:

$$\delta_\Lambda S_{SUGRA}[A[\bar{A}]] = \delta_\Lambda(-\ln Z[\bar{A}]) = \int d^4x\delta\bar{A}^{\mu a}(x)J^a_\mu(x) = -\int d^4x\Lambda^a(D^\mu J_\mu)^a \tag{7.41}$$

and thus by comparison one finds the following operator equation for the covariant derivative of the R-symmetry current:

$$(D^\mu J_\mu)^a(x) = \frac{iN^2}{16\pi^2}d_{abc}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\bar{A}^b_\nu\partial_\rho\bar{A}^c_\sigma + \frac{1}{4}f^c_{de}\bar{A}^b_\nu\bar{A}^d_\rho\bar{A}^e_\sigma\right) \tag{7.42}$$

It turns out that this agrees exactly with the CFT 1-loop anomaly computation (given via (7.34)), thus confirming the validity of the correspondence in this case.

In addition to the anomalous part, one may proceed to compute the full 3-point function using the method sketchily laid out in this section, and proceeding roughly in a similar fashion to the 2-point function calculation done previously (though in the present case the supergravity interaction terms cannot all be ignored). After some work (again see [NAS] for details), one finds that the function has the correct spacetime dependence for a CFT 3-point function, and furthermore agrees exactly with the result one obtains at 1-loop order in the CFT computation. This latter result is perhaps somewhat surprising, since we mentioned previously that the supergravity limit on the string theory side corresponds to the strong-coupling regime on the gauge theory side; we would thus not expect the supergravity result to precisely agree with the result one obtains from the lower orders of CFT perturbation theory. Clearly then, for the above to be true, there must be a non-renormalisation theorem which means that the full 3-point function (not just the anomaly) is in fact 1-loop exact; as discussed in [NAS], such a theorem was indeed proven, using the superconformal symmetry of the theory. We have thus seen another check of the AdS/CFT correspondence, and indeed one that is more powerful than that in the previous section, since the current check is performed at the level of the full interacting theory. There is, however, a limit to how far the scope of these tests extends related to the fact that they are based on unrenormalised quantities; we shall briefly discuss this issue in section 8.2.

Part III

Conclusion: Beyond the Original Conjecture

CHAPTER **8**

Conclusion

8.1 Extensions of the Correspondence

In this dissertation we have been primarily concerned with the original AdS/CFT correspondence due to Maldacena, relating Type IIB theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ SYM. This required considerable background since we saw, for example, that the gauge theory in the correspondence is both supersymmetric and conformal, leading to a larger superconformal symmetry (which we mentioned has special representations of particular importance in testing the correspondence). These need not be general features of a gauge/gravity duality however. Indeed, the original correspondence has been extended to other cases containing, for example, less supersymmetry and/or no conformal symmetry on the gauge theory side. The original correspondence is particularly powerful since the high levels of symmetry available allow numerous computations to be performed. Conceptually, however, it is fairly remote from the gauge theories we see in nature, which seem to be neither supersymmetric nor conformal. These extensions of the original correspondence are thus important, and are being continually studied in the field.

Before briefly mentioning such examples however, we note that there are in fact other cases which relate a string theory in an AdS background to a gauge theory with conformal symmetry and maximal supersymmetry, much like the original correspondence. The string theory in question is now the somewhat elusive *M-theory* which, although not very well understood, is known to have as its low energy limit the $D = 11$ supergravity theory. In the same way that $AdS_5 \times S^5$ arose as the near-horizon limit to a stack of D3 branes, the maximally supersymmetric backgrounds for $D = 11$ supergravity given by $AdS_4 \times S^7$ and $AdS_7 \times S^4$ can be seen to arise from brane solutions known as M2 and M5-branes. Indeed, in a similar fashion to the decoupling argument discussed in chapter 6, Maldacena motivated [MAL] that M-theory in these backgrounds is in fact dual to particular conformal field theories in 3 and 6-dimensions respectively.

Furthermore, there is a class of correspondences relating a string theory in the background $AdS_{d+1} \times X$ (for some space X) to a gauge theory in d-dimensions with conformal symmetry but less than maximal supersymmetry. Indeed, recall from section 6.4 that the presence of an AdS background relates to the presence of conformal symmetry in the gauge theory, since the isometry group of AdS_{d+1} is the conformal group in $\mathbb{R}^{1,d-1}$. Furthermore, we saw that for $X = S^5$ the isometry group was equivalent to the R-symmetry group of the gauge theory, which is directly related to the degree of supersymmetry $\mathcal{N}$ (see section 3.1). We thus see that we can modify the degree of supersymmetry by modifying this space X. For example, by starting with a maximally supersymmetric theory, we may divide the corresponding space X by some discrete subgroup, thereby reducing its isometry group and thus reducing the number of supersymmetries on the gauge theory side (more details and an example of such a correspondence are provided in [NAS]). In this manner we may produce correspondences with less supersymmetry on the gauge theory side, as is important for the reasons mentioned previously. Such cases may be tested in similar ways to those described in chapter 7 for the original correspondence, though generally fewer tests are possible since the presence of less symmetry means that fewer quantities are protected against quantum corrections. It is also possible to break the conformal invariance of the gauge theory, which is important for practical applications. This may be achieved, for example, by considering the gauge theory at finite temperature; the introduction of a temperature sets an energy scale and thus breaks the scale (and thus conformal) invariance of the theory. The energy scale of the gauge theory then becomes significant since the theory is no longer conformal, and is identified with the additional dimension on the gravity side (see [NAS]) e.g. the radial dimension x_0 or r for AdS_d defined in section 5.4. It turns out that the UV regime of the gauge theory corresponds to the IR (or large distances) regime of the gravity dual, leading to what is sometimes known as the *UV-IR correspondence*.

We briefly summarise some general properties of gauge/gravity dualities relevant to the aspects we have discussed in the present dissertation (a more complete mapping is provided in [NAS]). First, the gauge group (which in addition to $SU(N)$ may be other groups such as $SO(N)$ or $Sp(N)$) has no analogue in the gravity theory (e.g. $SU(N)$ in the original correspondence does not appear on the gravity side); indeed, only gauge invariant observables can be calculated, such as the correlation functions of gauge invariant operators discussed previously. Second, the global symmetries of the gauge theory (e.g. the R-symmetry of $\mathcal{N}=4$ SYM) correspond to gauge symmetries in the compactified gravity theory (e.g. the gauged supergravity one obtains from compactifying Type IIB on S^5); the Noether currents in the former then couple to the gauge fields in the latter, as in the example we saw in section 7.4 (in general, the gravity fields correspond and couple to some gauge invariant operators in the gauge theory, similar to that in section 7.1). Finally, the couplings are always related by $g_s \equiv g_g^2$ where g_g is the gauge theory coupling; this fundamentally arises from the fact that the closed string coupling constant is the square of the open string coupling constant, as discussed in section 4.2. We see that there are thus many structural similarities between different gauge/gravity dualities.

8.2 Closing Remarks

The AdS/CFT correspondence is a continuously growing field of active research. We have seen in some detail how the original correspondence is motivated and formulated, and described how tests of the correspondence are possible. There is however a subtle point worth mentioning regarding these tests as alluded to at the end of chapter 7. The tests we have predominantly described involve unrenormalised quantities on the gauge theory side, which are important since they allow for computations in the CFT that can be compared with the results obtained from the associated gravity theory. However, there is some debate concerning how deep the evidence that these tests provide for the correspondence really runs, since the results are essentially consequences of the symmetries (which we have already checked agree on the two sides) and so do not necessarily prove that the dynamics of the two theories are equivalent in a non-trivial way. There has however been much work in attempting to provide non-trivial checks of the correspondence at the dynamical level, for example by studying integrability in AdS/CFT (see [INT]) and gauge-invariant quantities known as *Wilson loops* (see [GRO] for example). Many such checks have indeed been performed in the literature, providing strong evidence for the conjecture, even if it is yet to be fully proven in a rigorous manner. We have also briefly seen how the correspondence can be extended to other cases; some of these, for example, are of particular interest in applications to condensed matter physics and the study of realistic gauge theories. In addition to these applications, it is of course also of great conceptual interest that the bulk gravitational degrees of freedom may be encoded in a boundary theory via holography; this is an idea that has captured scientists and philosophers alike, and is likely to be one that will be investigated for sometime.

References

[MAL] "The Large N Limit of Superconformal Field Theories and Supergravity". Maldacena. Adv.Theor.Math.Phys.2:231-252. [arXiv:hep-th/9711200] (1997).

[BEK] "Black Holes and Entropy". Bekenstein. Physical Review D 7(8):2333 (1973).

[STR] "The dS/CFT Correspondence". Strominger. JHEP 0110:034. [arXiv:hep-th/0106113] (2001).

[HAR] "Horizons, holography and condensed matter". Hartnoll. [arXiv:1106.4324] (2011).

[GOM] "Lectures on Conformal Field Theory". http://www.perimeterscholars.org/327.html. Gomis. Perimeter Institute (2011).

[ERD] "Introduction to gauge gravity duality". Erdmenger (2010).

[PES] "An Introduction to Quantum Field Theory". Peskin and Schroeder. Westview Press (1995).

[COL] "All Possible Symmetries of the S Matrix". Coleman and Mandula. Physical Review 159 (5): 12511256 (1967).

[BAL] "Supersymmetric Gauge Field Theory and String Theory". Balin and Love. Taylor and Francis (1994).

[FRE] "Supersymmetric Gauge Theories and the AdS/CFT Correspondence". Freedman and d'Hoker. [arXiv:hep-th/0201253] (2002).

[ZHO] "Group Theory for Physicists". Zhong-Qi Ma. World Scientific (2007).

[KIR] "String Theory in a Nutshell". Kiritsis. Princeton University Press (2007).

[MON] C. Montonen and D. Olive, Phys. Lett. B72 (1977).

[POL] "String Theory, Vol. 1 ". Polchinski. Cambridge University Press (2005).

[CAC] "Lectures on String Theory". http://www.perimeterscholars.org/337.html. Cachazo. Perimeter Institute (2011).

[ZWE] "A First Course in String Theory, Second Edition". Zweibach. Cambridge University Press (2009).

[FRA] "The Geometry of Physics, an Introduction, Second Edition". Frankel. Cambridge University Press (2004).

[VIE] "Lectures on Explorations in String Theory". http://www.perimeterscholars.org/341.html. Vieira. Perimeter Institute (2011).

[NAK] "Geometry, Topology and Physics, Second Edition". Nakahara. Taylor and Francis (2003).

[ZEE] "Einstein Gravity in a Nutshell". Zee. Princeton University Press (2013).

[ZEE2] "Quantum Field Theory in a Nutshell, 2nd Edition". Zee. Princeton University Press (2010).

[KRA] "Spacetime and the holographic renormalization group". Balasubramanian and Kraus. Phys. Rev. Lett. 83, 3605. [arXiv:hep-th/9903190] (1999)

[NAS] "Introduction to AdS-CFT". Nastase. arXiv:0712.0689 (2007).

[WIT] "Anti De Sitter Space And Holography". Witten. Adv. Theor. Math. Phys. 2 253-291, [hep-th/9802150] (1998).

[INT] "Review of AdS/CFT Integrability: An Overview". Lett. Math. Phys. 99, 3. arXiv:1012.3982 [hep-th] (2012).

[GRO] "An exact prediction of N = 4 SUSYM theory for string theory". Gross and Drukker. J. Math. Phys. 42, 2896. [arXiv:hep-th/0010274] (2001).