# Mirror Symmetry in 3d supersymmetric gauge theories


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"Oh, Kitty! how nice it would be if we could only get through into Looking- glass House! I'm sure it's got, oh! such beautiful things in it! Let's pretend there's a way of getting through into it, somehow, Kitty. Let's pretend the glass has got all soft like gauze, so that we can get through. Why, it's turning into a sort of mist now, I declare! It'll be easy enough to get through."

L. Carrol, *Through the Looking Glass*

# Abstract

An introduction to studies of moduli spaces of vacua for the purposes of mirror symmetry in 3d $\mathcal{N} = 4$ supersymmetric gauge theories is presented. We first consider the established techniques of calculating the Hilbert series, which is the partition function counting gauge invariant BPS operators at all orders in the fields, as they are applied to the Higgs branch of quiver gauge theories. The Higgs branch does not receive quantum corrections and computations of the partition function can be approached classically by considering F-terms and D-terms of the superpotential for the $\mathcal{N} = 2$ theory. Unlike the Higgs branch, the Coulomb branch receives quantum corrections and it's notoriously difficult to characterise. We introduce a new technique which exploits monopole operators in order to compute the Hilbert series on the Coulomb branch of quiver gauge theories. We are then able to test mirror symmetry by comparing the partition functions of the Higgs branch and Coulomb branch of predicted dual theories.

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# Chapter 1

# Introduction

## 1.1 In the web of dualities: mirror symmetry

Although the path towards a construction of a fundamental and encompassing theory seems to some to have only accidentally come across string theory, one cannot say the same for the peculiar and unique feature that string theory displays: the phenomenon of dualities. These seem to pervade every aspect of modern superstring theory; they can not be avoided and should not be. Reporting, even briefly, on the set and subsets of dualities in string theory is not for an introduction, nor for a whole dissertation. Here we limit ourselves to dealing with one particular duality which was suggested about fifteen years ago and has recently seen a revival. This duality has an unfortunate name: mirror symmetry. It is unfortunate because another, much more known, investigated and longer-lived duality exists which also bear the name mirror symmetry. Tis second type of mirror symmetry relates two geometrically different Calabi Yau 3-folds which turn out to yield the same theory when they serve as the compactified space of string theory. Let us be clear that this second type of mirror symmetry is *not* the topic of this dissertation. Even though connections are known to arise between our mirror symmetry and this one, we will ignore these as well.

In order to avoid confusion the duality at hand is always addressed in the literature as 3d mirror symmetry. It relates pairs of supersymmetric gauge theories in three dimensions. A sufficient amount of supersymmetry is required: it was first proposed for $\mathcal{N} = 4$ by Seiberg and Intriligator in their 1996 paper [25] and then argued for $\mathcal{N} = 2$ by [1] and [12]. Here we focus exclusively on the former. 3d mirror symmetry predicts the existence of pairs of theories that flow to the same conformal theory in the infrared. Since the gauge coupling in three dimensions has positive scaling with mass, the theory is strongly coupled in the infrared and a perturbative analysis is not appropriate. Like for many other dualities in field theory and string theory, systematic proofs are absent. Signs of the conjectures can nonetheless be observed by studying the so called moduli spaces of the dual gauge theories. With the term moduli spaces, or sometimes moduli, one refers to the geometrical space parametrised by the expectation value of the scalar fields present in the theory. In light of this the moduli space of vacua is sometimes called the scalar manifold. The key feature of the moduli space is its dimension, since this enumerates the various scalars that can take non-zero expectation value in the theory. As a very simple example one can consider a theory which contains only one scalar. Let this scalar take only positive values, for example by being expressed as an exponential. Then the manifold that this scalar field parametrises is just $\mathbb{R}^+$.

Moduli spaces of vacua in gauge theories arise when the scalar potential vanishes. The vacuum expectation values of the scalars are precisely the parameters or "moduli" of the space. Usually these scalar manifolds are constructed by means of disconnected pieces which intersect at a point or at a submanifold. In particular for 3d supersymmetric gauge theories, the scalar manifold is made up of two pieces which join at the origin. Mirror symmetry arguments imply that the two pieces of the moduli space of dual theories effectively swap roles. We explain this in what follows.

Consider a string theory background with 8 supercharges. Massless states arise which saturate 4 bosonic and 4 fermionic degrees of freedom (d.o.f). The maximal spacetime dimension allowed is 5+1 and such an amount of supersymmetry implies, excluding a gravity multiplet, the existence of a vector multiplet and a hypermultiplet.

The vector multiplet in 5+1 d with 8 supercharges contains a gauge field $A_\mu$(4 bosonic d.o.f.) and a spinor field (4 fermionic d.o.f). Hypermultiplets always consists of just scalar fields and fermions: they are the matter sector of the theory. Since the bosonic degrees of freedom come only from the scalars, the hypermultiplet in 5+1 d will have: 4 scalars (4 bosonic d.o.f.) and again a spinor field.

Dimensional reduction is successively performed on both multiplets in order to write down the two multiplets in 2+1 dimensions. Since dimensional reduction of fermions never yields scalars, these are uninteresting here and one can focus on the bosonic sector. The 2+1 d vector multiplet with 8 supercharges contains 4 bosonic d.o.f. coming from the 5+1 d vector $A_\mu$ which reduces to

- a gauge field $a_\mu$, which in 3d is dual to a compact scalar field
- three non-compact scalars

The vector multiplet now contributes four scalars to the moduli space. The space of vacua parametrised by the scalar fields in the vector multiplet is called the *Coulomb branch.*

The 2+1 d bosonic sector of the hypermultiplet reduces to 4 scalar fields. The space of vacua parametrised by these scalars is known as the *Higgs branch.* One should notice that the present Coulomb and Higgs branches are both 4 dimensional spaces and that the Coulomb branch should display one compact dimension $S^1$ to account for the parametrisation of the compact scalar. Moreover the scalars are usually combined into complex ones, so that effectively the spaces will be complex manifolds, with other features that specify the geometry further.

Mirror symmetry predicts that the Coulomb branch of a certain gauge theory corresponds to the Higgs branch of a dual theory and vice versa. For a theory with a given gauge group, the vector multiplet always transforms in the adjoint representation, whereas the hypermultiplet can be in any representation. Like all couplings, the gauge coupling can be thought of as a background field and, as such, this field will transform in the vector multiplet. By means of a 3d non-renormalisation theorem, which applies when extended supersymmetry is present, the vector multiplet and the hyper multiplet don't mix. Since the gauge coupling dictates quantum corrections, the vector multiplet moduli space is affected by these whereas the Higgs branch is not. The power of mirror symmetry lies precisely in the fact that when the Coulomb branch of a given theory is difficult to compute because of quantum corrections, one can appeal to the dual theory and compute its Higgs branch which is classically fully determined and in so doing obtain an exact answer for the original, hard to compute Coulomb branch. In light of this, one should recognise the surprising predictions of mirror symmetry.

Our discussion of mirror symmetry will focus on comparing the Higgs branch and the Coulomb branch of mirror theories, thus the reader should get acquainted with how moduli spaces are analysed. We provide a gentle introduction in the following subsection.

## 1.2 A simple example of moduli space

Consider a manifold $\mathcal{M}$ acted upon by a finite group $G$. This means that a point $p \in \mathcal{M}$ will be subject to the action of a finite group. Under such an action $p$ will be mapped to a point $p'$. The orbit of each point $p$ under $G$ is the set of points $p'$ that $p$ can reach when acted upon by $G$. An orbifold is a space constructed by "folding along orbits" of $G$, i.e. it is constructed as a quotient $\frac{\mathcal{M}}{G}$ of the manifold $\mathcal{M}$ by a smooth action of the finite group $G$ . The singularities of the orbifold arise precisely by identifying points in the orbit of $G$. In algebraic geometry the study of such spaces has often the aim of classifying and resolving the singularities so as to map singular spaces to smooth ones. In string theory the study of orbifolds arises in the context of moduli spaces of vacua. Moduli spaces that are orbifolds allow for a richer structure than moduli spaces which are simply manifolds since the singularities of the orbifolds correspond physically to points where extra massless states arise. Identifying moduli spaces of supersymmetric gauge theories as orbifolds is not usually easy. It might thus be easier to identify them algebraically rather than geometrically. As a simple example consider a supersymmetric theory in 4d as in [37] with a superpotential given simply by the product of three complex uncharged chiral superfields $X$, $Y$, $Z$, each of which contains a complex scalar field, a Weyl fermion and a complex auxiliary field, e.g. $x$, $\psi_X$, $F_X$ respectively for $X$, and similarly labelled for the other two.

The moduli space $\mathcal{M}$ of this theory is a complex space of which we seek to find the algebraic description, i.e we seek a complex function $f(x, y, z) = 0$. Let the holomorphic superpotential be $\mathcal{W}(X, Y, Z) = XYZ$. Supersymmetry dictates that the potential for the scalar fields $x$, $y$, $z$ be:

$$V(x, y, z) = \left| \frac{\partial W(X, Y, Z)}{\partial X}\bigg|_{X\to x} \right|^2 + \left| \frac{\partial W(X, Y, Z)}{\partial Y}\bigg|_{Y\to y} \right|^2 + \left| \frac{\partial W(X, Y, Z)}{\partial Z}\bigg|_{Z\to z} \right|^2 \tag{1.1}$$

The vacuum conditions $V = 0$, i.e. $\partial_{X\to x,Y\to y,Z\to z}\mathcal{W} = 0$ yield three equations on the three complex variables $x$, $y$, $z$ :

$$yz = 0;\ xz = 0;\ yx = 0 \tag{1.2}$$

They are satisfied in the three regions where the scalar fields vanish in pairs:

$$\begin{aligned} \langle x\rangle &\neq 0; y = z = 0 \\ \langle y\rangle &\neq 0; x = z = 0 \\ \langle z\rangle &\neq 0; y = x = 0 \end{aligned} \tag{1.3}$$

The expectation values of the scalar fields are called moduli, or parameters for the vacuum space. The moduli space $\mathcal{M}$ is then a space made of three pieces and described exactly by (1.3). In Fig. 1.1 each nonvanishing complex line is represented as a cone, so that, visually, this is a faithful representation of the algebraic description (1.3). Note that the three pieces are connected at the point where the expectation value of all three of the scalar fields vanish. Here extra massless states arise by construction. This moduli space is a singular space with a singularity arising at its origin.

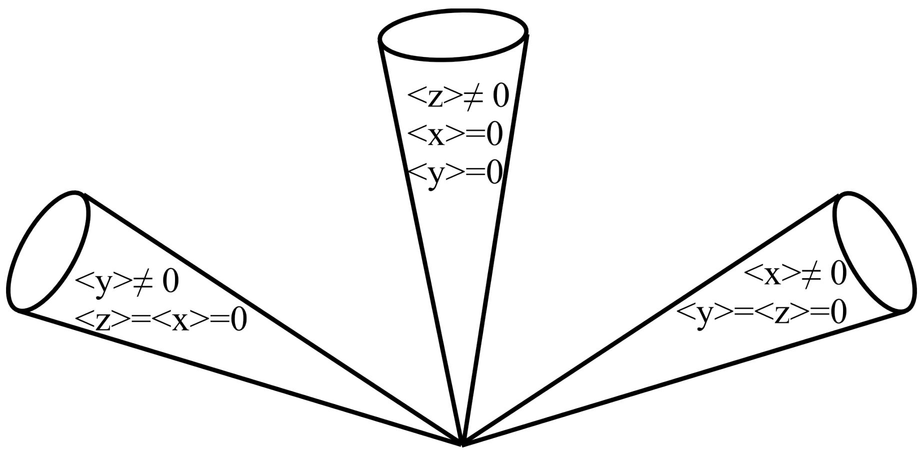



Figure 1.1: The moduli space for the XYZ model is made of three pieces which join at the origin

The XYZ moduli space displays two peculiar aspects that will be encountered again for other spaces: it is made of pieces that join at the origin and it has a singularity. These two elements alone give rise to a wealth of interesting features underlying the theories that have such moduli spaces. Our study is heavily focussed on disquisitions of such matters.

## 1.3 Thesis Overview

Mirror symmetry will be introduced in steps, each section providing a necessary background for a check of the duality. In particular this work is organised as follows:

- In section 2, we equip ourselves with some algebraic tools needed to analyse the moduli space of certain gauge theories. We succinctly introduce the concepts that are used for the definition and computation of the Hilbert series as the generating function for the chiral ring of gauge invariant operators. Furthermore, we exploit the power of algebraic methods to describe orbifolds. A class of gauge theories found on the worldvolume of branes plays a fundamental role in mirror symmetry: they are known as quiver gauge theories. We lay the details of their construction and show a map from a quiver theory with $\mathcal{N} = 2$ supersymmetry to the corresponding theory with $\mathcal{N} = 1$, by modifying the quiver and introducing a superpotential. This allows us to describe the Higgs branch of the quiver gauge theory by means of a Hilbert series. The calculations in this section are mostly taken from the work found in [4, 17, 5, 23] with some minor editing.

- In section 3, supersymmetric gauge theories in three dimensions are introduced. We present the symmetries of the theory and analyse the field content in terms of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ multiplets. Their transformation under the global symmetry and gauge symmetry are made explicit and tabulated for clarity. It is then explained how the gauge field in the vector multiplet has the unusual feature of being dual to a scalar. Thereon we move to the most important method of this section, which is how to count dimensions for the Coulomb branch and the Higgs branch. It is stressed in this section how mirror symmetry is a duality, like most, which arises at low energies. Moreover we introduce the idea of hidden symmetries in the gauge theory, an important ingredient in the development of mirror symmetry.

- Before analysing and demonstrating some of the statements of mirror symmetry, we take a little and very pleasurable mathematical detour on the McKay correspondence in section 4. This is an ingenious equivalence between the ADE Dynkin diagrams and graphs which encode information about representations of finite subgroups of $SU(2)$. The McKay correspondence elucidates and anticipates some "duality" features that are of interest to us. This section will give some slightly more formal statements about the idea of algebraic curves for rings of invariants. At the end it is noted how the ADE classification is ubiquitous in string theory.

- Section 5 is dedicated to the original work of this thesis. The technique implemented to analyse the algebraic form of the Coulomb branch by means of Hilbert series is presented. The calculations are performed by the author, although intensive collaboration with A. Zaffaroni, S. Cremonesi and especially A. Hanany are profusely acknowledged: extensive checks of these calculations have been performed by comparing with their more comprehensive results. Monopole operators are the mediators of this new technique. An introduction to the nature of these operators that don't appear in the Lagrangian is provided, although this will mainly occur as a tool to set the scene. Most interest will be taken to write a formula that captures the conformal dimension of these operators, which depends among other things on a set of hidden symmetries which we specify. Using this conformal dimension the Coulomb branch for certain abelian quiver gauge theories can be encoded in a generating function. We then move on to non abelian examples where care has to be taken to account for the Weyl group of symmetry of the gauge group. We introduce the concept of the classical dressing function, which mods out terms which are not invariant under the Weyl group. Some extra features, like explicit symmetry enhancement, that arise from this approach are presented.

- Finally, after all the hard work has been done by computing the Higgs branch and the Coulomb branch, mirror symmetry takes the stage in section 6. It is stated in its original form as it was presented by [25]. However the often laconic language of that paper should have received plenty of footnotes and elucidations in the form of the previous sections of this work. Mirror dual theories are stated and the cross matching of the Higgs branch and Coulomb branch are made explicit by means of the Hilbert series. This will amount to a check of mirror symmetry in the context of matching moduli spaces.

- The last chapter before the concluding remarks is dedicated to the string theory description of mirror symmetry. We give detail of the $NS5 - D5 - D3$ brane constructions which reproduce a mirror transformation. The material is taken from [24] and should constitute a fun exercise in the brane realisation of supersymmetric gauge theories. We also hope to make some of the diagrams in the original paper clearer.

The work of this dissertation is, apart from some computations in chapter 5, not an original contribution. To the author's knowledge, a review of $3d$ mirror symmetry doesn't not exist; although this makes sourcing the information harder, we have strived to provide all the references which were drawn upon.

# Chapter 2

# Higgs Branch

## 2.1 Algebraic description of moduli spaces

Let us commence by constructing the simplest two complex dimensional orbifold. We take the complex plane $\mathbb{C}^2$ and construct the quotient of this by the central symmetry of the origin. The orbifold is thus $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. Note that this space is not a Riemannian manifold and is homeomorphic to a cone in $\mathbb{CP}^1$, the cone point failing to have a neighbourhood that locally looks like $\mathbb{C}^2$.To identify the algebraic description of this quotient orbifold let $z_1$ and $z_2$ be coordinates of $\mathbb{C}^2$ and the action of the parity group $\mathbb{Z}_2$ be:

$$(z_1, z_2) \longleftrightarrow (-z_1, -z_2)$$

In matrix notation, the (diagonal) action of $\mathbb{Z}_2$ is simply

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{2.1}$$

Let us identify all the monomial functions of $z_1$ and $z_2$ that are invariant under the action of parity. In this case these are very easy to find:

$$\begin{gathered} f(z_1, z_2) = z_1^i z_2^j \\ i - j = 0 \bmod 2 \end{gathered} \tag{2.2}$$

The three $\mathbb{Z}_2$ invariant monomials of degree two are

$$\begin{gathered} X \equiv z_1^2 \\ Y \equiv z_2^2 \\ Z \equiv z_1 z_2 \end{gathered} \tag{2.3}$$

$X, Y, Z$ are three complex variables which we use to describe the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ algebraically. They are called *generators* of the (infinitely many) $\mathbb{Z}_2$ invariant polynomials and they are related to each other by the constraint

$$XY = Z^2 \tag{2.4}$$

This relation is just an algebraic curve in $\mathbb{C}^3$ and fully encapsulates the description of the orbifold. Note that if one regards the variables $X$,$Y$,$Z$ as real coordinates (2.4) is simply the equation of a cone. The orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ is often called a "complex cone" and the singularity of this space is referred to as a *conical singularity.*

Let us consider the next natural example of $\frac{\mathbb{C}^2}{\mathbb{Z}_3}$ . The action of $\mathbb{Z}_3$ on $\mathbb{C}^2$ can be identified[1] through a generator $\omega$, which obeys the usual condition $\omega^3 = 1$. Powers of $\omega$ are irreducible representations of $\mathbb{Z}_3$. The two dimensional (reducible) diagonal representation can be written:

$$\begin{pmatrix} \omega^{a_1} & 0 \\ 0 & \omega^{a_2} \end{pmatrix}, \; \omega^{a_1} = 1, \; \omega^{a_2} = 1 \tag{2.5}$$

The action of $\mathbb{Z}_3$ on $(z_1, z_2) \in \mathbb{C}^2$ is then

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} \omega^{a_1} & 0 \\ 0 & \omega^{a_2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{2.6}$$

$$a_1 + a_2 = 0 \bmod 3 \tag{2.7}$$

(2.7) can be rewritten as

$$a_1 = -a_2 \bmod 3 \tag{2.8}$$

and we can choose $a_1 = 1$, and thus $a_2 = -1$. Then

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{2.9}$$

and orbits of $\mathbb{Z}_3$ on $\mathbb{C}^2$ are $(z_1, z_2) \sim (\omega z_1, \omega^{-1} z_2)$, with $\omega^3 = 1$.

We now look for invariant polynomials under this action, in order to describe this orbifold algebraically. By inspection

$$W \equiv z_1 z_2 \tag{2.10}$$

$$X \equiv z_1^3 \tag{2.11}$$

$$Y \equiv z_2^3 \tag{2.12}$$

$W$, $X$, $Y$ are the lowest degree polynomials of order 2, 3 and 3 respectively - these are the generators for this space and all other invariants can be built out of them. They are constrained by one relation, since $XY = W^3$ . Hence we can write the algebraic description of the space, also called *defining equation* as

$$\frac{\mathbb{C}^2}{\mathbb{Z}_3} := \left\{ X, Y, W \in \mathbb{C}^3 \mid XY = W^3 \right\} \tag{2.13}$$

[1]The action of $\mathbb{Z}_3$ on $\mathbb{C}$ is implemented by a rotation of size $\omega$, a cube root of unity; orbits of $\mathbb{Z}_3$ are points at 120° in the Argand diagram, thus $\frac{\mathbb{C}^2}{\mathbb{Z}_3}$ is constructed by identifying such points

## 2.2 Hilbert series of Moduli Spaces

The two examples encountered so far highlight the key properties of moduli spaces we are interested in: the *dimension*, the *number of generators* and the number and the form of the constraining *relations*. By combining basic invariants one could then construct infinitely many others, as can be seen by inspection in the previous examples. The next question is whether these infinitely many invariants can be collected by a grading of the polynomial vector space and whether they could be nicely encapsulated in some function. Indeed this is what the so called *Hilbert series* does. This is a generating function which counts the number of polynomial invariants under the finite group action at a given degree. In gauge theories, the Hilbert series will count the chiral gauge invariant operators at a given degree. We proceed to construct simple Hilbert series by examples and introduce minimal mathematical machinery to keep the present work as concise as possible. The definitions used here can be found in a series of papers [4, 17, 23]. As extensive use of it will occur, we introduce here a function that counts symmetric products of its arguments, called the plethystic exponential (PE).

**Definition (Plethystic Exponential):** For a multivariable function $f(t_1, ..., t_n)$ with $f(0, .., 0) = 0$, define:

$$PE\left[f\left(t_1, ..., t_n\right)\right] := \exp\left(\sum_{r=1}^{\infty} \frac{f\left(t_1^r, ..., t_n^r\right)}{r}\right) \tag{2.14}$$

E.g. for $f\left(t\right) = t$, the PE is:

$$PE\left[t\right] = \exp(\sum_{r=1}^{\infty} \frac{t^r}{r}) = \exp(-\ln(1-t)) = \frac{1}{1-t} \tag{2.15}$$

and for a power series in $t$, $f\left(t\right) = \sum_n a_n t^n$

$$PE\left[\sum_n a_n t^n\right] = \frac{1}{\prod_n (1-t^n)^{a_n}} \tag{2.16}$$

where one notices that $PE[f+g] = PE[f]PE[g]$.

The plethystic exponential can of course be used for symmetric products of *any* function, but for our purposes the functions at hand will be characters of representations $R_i$ of Lie groups. These are expressed as polynomials in $(x_1, ..., x_p)$ where $p$ is the rank of the group. Evaluation of the plethystic exponential yields a series whose coefficients are symmetric products of these characters.

In order to count gauge invariant quantities, these representations must be restricted to the subspace, or subrepresentation, which is invariant under the action of the gauge group $G$. One can do so by integrating the series over the gauge group itself, with the integration measure the group Haar measure $\int_G d\mu_G$

$$\int_G d\mu_G = \frac{1}{(2\pi i)^r} \oint_{|z_1|=1} \cdots \oint_{|z_r|=1} \frac{dz_1}{z_1} \cdots \frac{dz_r}{z_r} \prod_{\alpha^+} \left(1 - \prod_{j=1}^{r} z_j^{\alpha^+}\right) \tag{2.17}$$

where $r$ is the rank of $G$ and $\alpha^+$ are the positive roots for the lie algebra $\mathfrak{g}$. The projection over singlets of the gauge group is known as the Molien-Weyl formula and it gives the Hilbert series for the algebra of group invariants:

$$I^G = \int_G d\mu_G PE[R_1(x_1, ..., x_r) + R_2(x_1, ..., x_r) + ...] \tag{2.18}$$

### 2.2.1 Simple Hilbert series

Consider first the simple case of a moduli space $\mathbb{C}^n$, with variables $z_i$, $i = 1, .., n$ . A natural $U(n)$ symmetry acts on $\mathbb{C}^n$. The Cartan subgroup is $U(1)^n$, the $i^{th}$ $U(1)$ acting on each $z_i$. We now introduce a parameter $t_i$ for each $U(1)$, which counts each $z_i$; these parameters are called fugacities. Since there is no group of symmetry to quotient over, the Hilbert series just counts monomials of degree k in the variables $z_i$: $t_1^{i_1} t_2^{i_2} ... t_n^{i_n}$, for any positive integer $i_1, i_2, ..., i_n$. The Hilbert series is then the generating function:

$$HS(t_1, t_2, ..., t_n; \mathbb{C}^n) = \sum_{i_1,..,i_n=0}^{\infty} t_1^{i_1} t_2^{i_2} ... t_n^{i_n} = \prod_{i=1}^{n} \frac{1}{1 - t_i} \tag{2.19}$$

$$= PE\left[\sum_{i=1}^{n} t_i\right] \tag{2.20}$$

Let us now perform a change of variables from $t_1, ..., t_n$ to $y_1, .., y_{n-1}, t$ with map:

$$t_1 = ty_1,\ t_2 = t\frac{y_2}{y_1},\ \cdots,\ t_n = t\frac{1}{y_{n-1}} \tag{2.21}$$

Then

$$HS(t_1, t_2, ..., t_n; \mathbb{C}^n) = PE\left[(y_1 + \frac{y_2}{y_1} + ... + \frac{1}{y_{n-1}})t\right] \tag{2.22}$$

$$= PE\left[\chi\left([1, 0, ..0]_{SU(n)}\right) t\right] \tag{2.23}$$

$$= \sum_{k=0}^{\infty} \chi\left([k, 0, .., 0]_{SU(n)}\right) t^k \tag{2.24}$$

where we have used Dynkin label $[n_1, ..., n_r]$ to denote representations of simple Lie algebras and $\chi([n_1, ..., n_r])$ the corresponding character. We see that the Hilbert series for $\mathbb{C}^n$ corresponds to the function that generates $k^{th}$ rank symmetric products of the fundamental representation of $SU(n)$. Set now $t_i = t$ in (2.20) to obtain the so called unrefined Hilbert series:

$$HS(t;\mathbb{C}^n) = \frac{1}{(1-t)^n} \tag{2.25}$$

$$= \sum_{k=0}^{\infty} \left( \begin{array}{c} n+k-1 \\ k-1 \end{array} \right) t^k \tag{2.26}$$

$$= \sum_{k=0}^{\infty} dim([k,0,..,0]_{SU(n)}) t^k \tag{2.27}$$

(2.25) gives us information on the moduli space. Indeed the order of the pole of the Hilbert series is the dimension of the underlying moduli space. Here the trivial example of $HS(t,\mathbb{C}^n)$ displays a divergence of order $n$, signifying the dimension of the space itself.

Let us go back to the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. The natural $U(2)$ action on the parent space $\mathbb{C}^2$, with Cartan subalgebra $U(1)^2$, allows us to arrange coordinates $z_i$ as a doublet [1] of $SU(2)$ with charge +1 under $U(1)$: $z_i \to [1]_{+1}$. However the weights of the doublet $z_1$, $z_2$ are not invariants under the symmetry so [1] cannot be a generator for all $\mathbb{Z}_2$ invariant polynomials. The first invariant monomials were at order 2 and they were 2-nd rank symmetric products of $z_1$, $z_2$. In group theoretic language this is $Sym^2[1] = [2]$, i.e the second rank symmetric representation of $SU(2)$, which has dimension 3. These represent the three $\mathbb{Z}_2$ invariant monomials of (2.3). Notice that there are no monomials with odd numbers of weights of [1]. Let us then redefine the order of invariants by taking the monomials a, b, c in (2.3) to be order 1. All higher order invariants will be symmetric products of these, e.g.

$$Sym^2[2] = [4] + [0] \tag{2.28}$$

$$Sym^3[2] = [6] + [2] \tag{2.29}$$

$$etc.$$

(2.28) and (2.29) also encapsulate the relations between the generators. We know that the first relation among the generators is at order 2 in the products of generators from (2.4). This is precisely the role of the singlet in (2.28). Proceeding with higher symmetric products, we will find more invariants (the highest weight $[2k]$ in the decomposition) and more relations (all the other representations in the decomposition). The Hilbert series is then easy to write as a summation. Choosing fugacities $t_1$,$t_2$:

$$HS(t_1,t_2;\frac{\mathbb{C}^2}{\mathbb{Z}_2}) = \sum_{\substack{i,j=0 \\ i-j=0 \bmod 2}}^{\infty} t_1^i t_2^j \tag{2.30}$$

$$= 1 + t_1^2 + t_2^2 + t_1 t_2 + ... \tag{2.31}$$

Introduce the fugacity map $t_1 = tx$ and $t_2 = \frac{t}{x}$, then simple substitution[2] shows that

$$HS(x,\, t^2; \frac{\mathbb{C}^2}{\mathbb{Z}_2}) = \sum_{k=0}^{\infty} \chi\left([2k]\right) t^{2k} \tag{2.32}$$

where the fugacity $t^2$ rather than $t$ reflects the degree of the generators. If we unrefine the series by setting the fugacity $x = 1$, it follows that $\chi([2k]) \to dim\chi([2k]) = 2k+1$ and (2.32) becomes

$$HS(t^2; \frac{\mathbb{C}^2}{\mathbb{Z}_2}) = \sum_{k=0}^{\infty} (2k+1) t^{2k} = \frac{1-t^4}{(1-t^2)^3} \tag{2.33}$$

$$= (1-t^4) PE[3t^2] \tag{2.34}$$

$$= (1-t^4) PE[dim(\chi([2]))t^2] \tag{2.35}$$

and since dimensions match we can now refine the series again and write

$$HS(x,\, t^2; \frac{\mathbb{C}^2}{\mathbb{Z}_2}) = \sum_{k=0}^{\infty} \chi\left([2k]\right) t^{2k} \tag{2.36}$$

$$= (1-t^4) PE[(\chi([2]))t^2] \tag{2.37}$$

$$= \frac{1-(t^2)^2}{(1-x^2t^2)(1-t^2)\left(1-\frac{t^2}{x^2}\right)} \tag{2.38}$$

(2.38) is the closed formula for the Hilbert series that represents $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. The three factors in the denominator signify that the ring of monomials invariant under $\mathbb{Z}_2$ is generated by three monomials. The numerator encodes the relation between these generators: the power of $t$ signifies that this relation is at order 4, or order 2 in the generators. As mentioned above, the pole of the Hilbert series gives the dimension of the space. We simplify (2.33), as much as possible and notice that there is a pole of order 2, i.e. the moduli space has complex dimension 2, as expected from the parent space $\mathbb{C}^2$. Note that the dimension of the space in this case is $d = g - r$, where $g$ is the number of generators and $r$ is the number of relations. A moduli space whose dimension obeys this relation is called a complete intersection[3]. Spaces that are complete intersections enjoy some nice properties: there is a finite number of relations between the three generators and these relations are indeed the defining equation of the underlying algebraic space; the Hilbert series can be written simply and information about the generators and their relations straightforwardly extracted. We refer the reader to [22] for a classification of complete intersection spaces and their associated Hilbert series. The majority of the moduli spaces encountered in this dissertation will be complete intersections.

[2]The characters of representations $[n](x)$ of $SU(2)$ are $\chi([n]) = \sum_{m=-\frac{n}{2}}^{m=\frac{n}{2}} x^{2m}$, where $x^2 = \exp(i\theta)$

[3]If r=0, the moduli space is said to be freely generated

## 2.3 Quiver gauge theories

In the next sections a special class of gauge theories will be investigated, called quiver gauge theories. These are SYM theories whose gauge group and matter content can be encoded in certain graphs. The amount of supersymmetry dictates the information encoded in a given graph. The importance of quiver gauge theories was first noted by Douglas and Moore in [15], where it was realised that these theories arise on the worldvolume of Type II $Dp$-branes in a background of $D(p+4)$ branes probing a certain singularity of the space.

### 2.3.1 Simple Quivers

In this subsection we review briefly $\mathcal{N}=2$ $d=4$ quiver gauge theories. The structure of these quiver diagrams applies without any change to $\mathcal{N}=4$ $d=3$ theories, with which we will be concerned later. Quiver diagrams consist of *nodes*, to which we assign vector multiplets transforming in the gauge group, and *links*, to which we assign hyper multiplets. Consider the quiver gauge theories as represented in Fig. 2.1 .

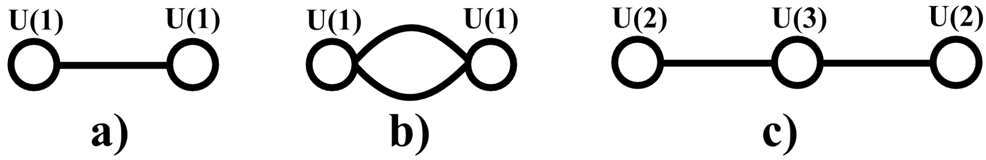



Figure 2.1: Some simple quiver gauge theories

The quiver shown in **a)** represents a theory with gauge group $U(1)\times U(1)$ with one hypermultiplet transforming in the bifundamental $(+1,-1)$ representation. The quiver in **b)** has the same gauge group but two hyper multiplets, one in the bifundamental and one in the conjugate bifundamental, i.e $(-1,+1)$. The quiver in **c)** represents a theory with gauge group $U(2)\times U(3)\times U(2)$ with 12 hypermultiplets transforming in the representation $(\mathbf{2},\bar{\mathbf{3}},\mathbf{1})\oplus(\mathbf{1},\mathbf{3},\bar{\mathbf{2}})$ which is written more commonly as $(\mathbf{2},\bar{\mathbf{3}})\oplus(\mathbf{3},\bar{\mathbf{2}})$, with the understanding that $(\mathbf{p},\mathbf{q})$ is the representation for adjacent gauge group factors. One can also introduce a different type of node to represent a global flavour group. Typically this is done by means of a square node. For example the theory with gauge group $U(K)$ and $N$ flavours, i.e. $SU(N)$ flavour symmetry is shown in Fig. 2.2.


Figure 2.2: The quiver $U(k)$ with $N$ flavours symmetry

The matter content is summarised by the link: $N$ hypermultiplets, each in the fundamental representation, $\mathbf{k}$, of $U(k)$, adding up to a total of $kN$ hypermultiplets.

### 2.3.2 From $\mathcal{N} = 2$ to $\mathcal{N} = 1$ quivers

In order to compute the partition function for the gauge invariant ring on the Higgs branch of quiver theories, one needs to map from a $4d$ quiver with $\mathcal{N} = 2$ to a quiver with $\mathcal{N} = 1$. Again this will apply unmodified to three dimensional theories in going from an $\mathcal{N} = 4$ to an $\mathcal{N} = 2$ quiver. Theories with $\mathcal{N} = 1$ supersymmetry are specified by giving the gauge group, the hypermultiplets representations and in addition a *superpotential* which encodes the interaction term.

A $\mathcal{N} = 2$ vector multiplet decomposes into a $\mathcal{N} = 1$ vector multiplet and a chiral multiplet in the adjoint of the gauge group; hence a node translates to a node with loop. A $\mathcal{N} = 2$ hypermultiplet decomposes in two chiral multiplets which are in complex conjugate representations of the gauge group; thus the line becomes bidirectional. We show the map in Table 2.1.

| $\mathcal{N} = 2$ | | $\mathcal{N} = 1$ | | Summary |
|---|---|---|---|---|
| ○, Node | Vector Multiplet | ○↺, | Node+ Adjoint Chiral Multiplet | $V_2 = V_1 + \chi_1$ |
| —, Line | Hyper Multiplet | ⟷, | Bidirectional line two chiral multiplets | $H_2 = \chi_1 + \widetilde{\chi}_1$ |

Table 2.1: Mapping of the elements of a $\mathcal{N} = 2$ quiver to those of a $\mathcal{N} = 1$ quiver. The mapping should also include a superpotential

**Superpotential**

The superpotential can be read off from the $\mathcal{N} = 1$ quiver. Let us consider again the $\mathcal{N} = 2$ quiver in Fig. 2.1a and map it to a $\mathcal{N} = 1$ quiver using our prescription as shown in Table 2.1. $\phi_1$ and $\phi_2$ are the chiral multiplets that arise from decomposing each $U(1)$ vector multiplet, whereas $A$, $B$ are the chiral multiplets arising from decomposing the bifundamental hypermultiplet. A line between two nodes enters in two cubic terms of the superpotential with opposite sign contributions.

The superpotential for the quiver in Fig. 2.3a) is:

$$W = B\phi_1 A - A\phi_2 B$$

There are no indices since the fields are in this case abelian. If the nodes stand for two non abelian groups with fundamental representations $\mathbf{n_1}$ and $\mathbf{n_2}$, the fields are matrix valued in the bifundamental: $A \longrightarrow A_{n_2 \times n_1}$ and $B \longrightarrow B_{n_1 \times n_2}$ so that a trace is always implicit in the superpotential terms. In Fig. 2.3b) we show how to map the second quiver in Fig. 2.1 from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ and read off the associated superpotential.

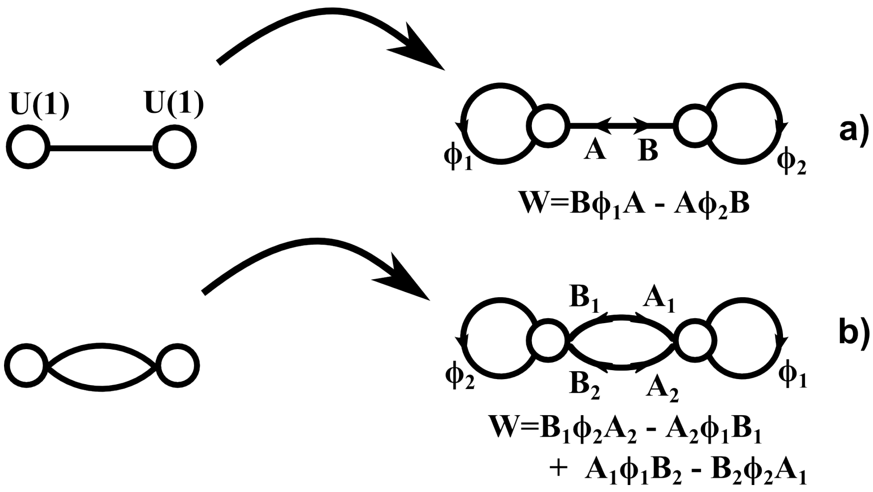



Figure 2.3: The $\mathcal{N}=2$ quivers of Fig. 2.1 a) and b) are mapped using the prescription provided to $\mathcal{N}=1$ quivers, which must include a superpotential

## 2.4 Higgs branch

Equipped with the simple machinery of this chapter so far, we can now proceed to evaluate the Higgs branch of certain quiver gauge theories. In the following subsections we follow [5] closely as the results found in this paper will be of use for nontrivial checks of mirror symmetry.

### 2.4.1 1-SU(N) instanton

We consider a $D2$-brane probing a background of $N$ $D6$-branes. The theory living on the $D2$-brane is a $\mathcal{N}$ =4 $3d$ supersymmetric theory with gauge group $U(1)$. Strings connecting the $D6$-branes to the $D2$-branes are interpreted as $N$ massive hypermultiplets, with mass proportional to the length of the strings. When the $D6$-branes are stacked together, the hypermultiplets masses are all equal and can be set to zero by a shift of the origin. This $U(N)$ enhanced background gauge symmetry becomes a global flavour symmetry on the worldvolume of the $D2$-brane. Moreover the centre of $U(N)$ can be absorbed by the $U(1)$ gauge symmetry living on the $D2$-brane, hence the theory becomes that of $U(1)$ with $SU(N)$ flavour symmetry shown in Fig. 2.2 for $k=1$.

Recall that the quivers introduced above describe $\mathcal{N}=2$ or $\mathcal{N}=1$ supersymmetric gauge theory in $4d$, but they apply for the same theories in $3d$ with respectively $\mathcal{N}=4$ and $\mathcal{N}=2$ supersymmetry. Consequently we can map from our $\mathcal{N}=4$ quiver to the corresponding $\mathcal{N}=2$ one using the prescription of the previous subsection: this is shown in Fig. 2.4.

The node $U(1)$ becomes a node with an adjoint chiral multiplet $\Phi$, whereas the link between the gauge group and the flavour group, which signifies bifundamental hypermultiplets in the $\mathcal{N}=4$ quiver, becomes a bidirectional line representing two $\mathcal{N}=2$ chiral multiplets $\chi$ and $\xi$, which transform in the bifundamental of the

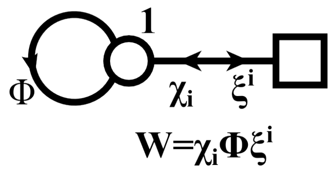



Figure 2.4: The $\mathcal{N}=2$ quiver obtained by mapping the quiver in Fig. 2.2 which has $\mathcal{N}=4$ in $3d$ using the prescription of the previous subsection. This allows for the superpotential to be easily written down.

$U(1)_{gauge} \times SU(N)_{flavour}$ and are complex conjugate representations. We show their representation and associated fugacity in Table 2.2. The superpotential can be read off the $\mathcal{N}=2$ quiver as

$$\mathcal{W} = \chi_i \Phi \xi^i \tag{2.39}$$

where the Latin indices label the gauge representation, the fundamental being an upper index, and the conjugate fundamental a lower index.

| Field | $U(1)_{gauge}$ | $U(N)_{flavour}$ | |
|---|---|---|---|
| | | $SU(N)$ | $U(1)$ |
| Fugacity | $z$ | $x_1,...,x_{N-1}$ | $q$ |
| $\Phi$ | 0 | $[0,...,0]$ | 0 |
| $\chi$ | 1 | $[0,...,0,1]$ | -1 |
| $\xi$ | -1 | $[1,0,...,0]$ | 1 |

Table 2.2: Representation of fields for the $U(1)$ theory with $N$ flavours. The label $[n_1,...,n_r]$ will always represent the Dynkin highest weight

The F-terms, obtained by letting the derivative of the superpotential with respect to the field multiplets vanish, are

$$\frac{\partial \mathcal{W}}{\partial \chi_i} = \Phi \xi^i = 0 \tag{2.40}$$

$$\frac{\partial \mathcal{W}}{\partial \xi^i} = \chi_i \Phi = 0 \tag{2.41}$$

$$\frac{\partial \mathcal{W}}{\partial \Phi} = \chi_i \xi^i = 0 \tag{2.42}$$

So either $\chi_i$, $\xi^i = 0$ and $\langle \Phi \rangle = any$, which would correspond to the Coulomb branch where the expectation value of the scalars in the vector multiplet take non-zero value, or $\Phi = 0$, $\chi_i = any$, $\xi^i = any$ with $\chi_i \xi^i = 0$, which corresponds to the Higgs branch where scalars in the hyper multiplet take non-zero expectation value. However one cannot calculate the Coulomb branch in this manner since it receives quantum corrections, i.e. the superpotential would need loop renormalisation and instanton corrections. The Higgs branch on the other hand is not renormalised, hence this classical computation is valid and gives an exact result. We will focus on this.

The space of solutions $\Phi = 0$, $\chi_i$, $\xi^i = any$ with $\chi_i \xi^i = 0$ is called the F-flat space.

We denote it by $\mathcal{F}^b$,

$$\mathcal{F}^b \equiv \left\{\Phi = 0, \langle\chi_i\rangle \neq 0, \langle\xi^i\rangle \neq 0 \mid \chi_i\xi^i = 0\right\} \tag{2.43}$$

We wish to select gauge invariant operators: since the group is abelian $\chi_i$, $\xi^i$ transform just by a local phase. The first gauge invariant operator is to be found at order 2 and it is $A^i_j \equiv \xi^i\chi_j$. In group theoretic language we write

$$[1,0,..,0]_{SU(N)} \otimes [0,0,..,1]_{SU(N)} = [1,0,..,0,1]_{SU(N)} \oplus [0,..,0]_{SU(N)} \tag{2.44}$$

but $TrA = A^i_i = \xi^i\chi_j = 0$, hence the singlet above corresponds to a condition on the generator $A^i_j$. Moreover one can notice that $A^i_j A^j_k = \xi^i\chi_j\xi^j\chi_k = 0$. Therefore the restriction of the F-flat space $\mathcal{F}^b$ to the gauge invariant operators $\mathcal{M}^{\mathcal{H}}$ can be described algebraically as

$$\mathcal{M}^{\mathcal{H}} = \left\{A \in GL(N,\mathbb{C}) \mid TrA = 0,\ A^2 = 0\right\} \tag{2.45}$$

It is often impractical or cumbersome to find gauge invariant operators and their relations by inspection especially if they occur at higher order. Moreover we would like to characterise the space by means of the Hilbert series, as discussed above. We present here the procedure, as it appeared in [5].

The generating function encoding the F-flat space $\mathcal{F}^b$, (2.43), is

$$g^{\mathcal{F}^b}_{1,N}(t, x_1, ..., x_{N-1}, q, z) = \left(1 - t^2\right) PE\left([1,0,...,0]_{SU(N)}\, wt + [0,...,1]_{SU(N)}\, \frac{1}{w}t\right) \tag{2.46}$$

where

1. $w \equiv \frac{z}{q}$ is the redefined fugacity to absorb the $U(1)$ factor of $U(N)$, with fugacity $q$, into the gauge $U(1)$ with fugacity $z$
2. $PE\left([1,0,...,0]_{SU(N)}\, wt\right)$ signifies the symmetric products of $\xi^i$ and $PE\left([0,...,1]\,\frac{1}{w}t\right)$ the symmetric products of $\chi_i$
3. the prefactor in front of the $PE$ represents the relation for the generators occurring at order 2

In the generating function (2.46) gauge invariance hasn't yet been imposed. Let the characters of the representations be

$$[1,0,...,0] = x_1 + \frac{x_2}{x_1} + ... + \frac{x_{N-1}}{x_{N-2}} + \frac{1}{x_{N-1}} = x_1 + \frac{1}{x_{N-1}} + \sum_{k=2}^{N-1}\frac{x_k}{x_{k-1}} \tag{2.47}$$

$$[0,...,0,1] = \frac{1}{x_1} + \frac{x_1}{x_2} + ... + \frac{x_{N-2}}{x_{N-1}} + x^{N-1} = \frac{1}{x_1} + x_{N-1} + \sum_{k=2}^{N-1}\frac{x_{k-1}}{x_k} \tag{2.48}$$

then (2.46) can be written as a rational function

$$g^{\mathcal{F}^b}_{1,N}(t, x_1, ..., x_{N-1}, w) = \frac{(1-t^2)}{(1-x_1 wt)\left(1-\frac{1}{x_{N-1}}wt\right)\prod_{k=2}^{N-1}\left(1-\frac{x_k}{x_{k-1}}wt\right)} \times \frac{1}{\left(1-\frac{1}{x_1}\frac{t}{w}\right)\left(1-x_{N-1}\frac{t}{w}\right)\prod_{k=2}^{N-1}\left(1-\frac{x_{k-1}}{x_k}\frac{t}{w}\right)} \quad (2.49)$$

(2.49) can be projected out so as to extract only the gauge invariant part of the series. This is performed by integrating $g^{\mathcal{F}^b}_{1,N}$ over the Haar measure for the gauge group $U(1)$. The contour integration over an appropriately closed path will pick up contributions from the poles of the function. Using the Molien-Weyl formula

$$HS^H_{1,SU(N)}(t, x_1, ..., x_{N-1}) = \frac{1}{2\pi i}\int_{|w|=1}\frac{dw}{w}g^{\mathcal{F}^b}_{1,N}(t, x_1, ..., x_{N-1}, w) \quad (2.50)$$

where we restrict to the circle $| w |= 1$ since the radius of convergence for $t$ is 1 and thus only poles within it must be considered:

$$w = t\frac{1}{x_1}, t\frac{x_1}{x_2}, ..., t\frac{x_{N-2}}{x_{N-1}}, tx_{N-1} \quad (2.51)$$

After the contour integration (2.50) the following generating function is obtained

$$HS^H_{1,SU(N)}(t, x_1, ..., x_{N-1}) = \sum_{p=0}^{\infty}[p, 0, ..., 0, p]_{SU(N)}\, t^{2p} \quad (2.52)$$

Recall that $[p, 0, ..., 0, p]_{SU(N)}$ stands for the *character* of the representation, hence a function of $(x_1, ..., x_{N-1})$. By setting $x_i = 1$, one obtains the dimension corresponding to this representation and thus the unrefined Hilbert series which counts the gauge invariant operators at a given degree. After making use of the Weyl dimension formula, we arrive at.

$$HS^H_{1,SU(N)}(t, x_1, ..., x_{N-1}) = \frac{\sum_{p=0}^{N-1}\begin{pmatrix} N-1 \\ p \end{pmatrix}^2 t^{2p}}{(1-t^2)^{2(N-1)}} \quad (2.53)$$

Notice that the order of the pole at $t = 1$ is $2(N-1)$ complex dimensional, i.e $N-1$ in quaternionic units.

### 2.4.2 $1 - SO(2N)$ instanton

We proceed to analyse the quiver in Fig. 2.5.

Recall that the algebra associated to $Sp(1)$ is isomorphic to the algebra associated to $SU(2)$: $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$, thus we are effectively studying a gauge theory of $SU(2)$ with $N$ quarks. The global symmetry arises as the orthogonal group $SO(2N)$, since


Figure 2.5: The quiver gauge theory with gauge group $Sp(1)$ and $N$ flavours, which arise as a global symmetry $SO(2N)$

the fundamental representation of $SU(2)$ is pseudoreal. The brane picture behind this quiver theory is similar to the one for the $1 - SU(N)$ theory. Consider a $D2$ brane in a background of $N$ $D6$ branes and the $N$ images about the orientifold $O6^-$. The presence of the orientifold has a two-fold effect:

- the background gauge group for the $D6$ becomes $SO(2N)$
- the gauge group on the $D3$ is projected to $Sp(1) \cong SU(2) \subset U(1)$

Hence on the worldvolume of the $D2$-brane an observer will experience an $SU(2)$ gauge theory with $SO(2N)$ flavour symmetry.

Note that in the quiver of Fig.2.5 the matter is in so called half-hyper multiplets. They are still in the bifundamental of the two adjacent symmetry groups: $SU(2)_{gauge} \times SO(2N)_{flavour}$, which add up to a total of $2 \times 2N$ half-hyper multiplets[4] or alternatively $\frac{1}{2}(2 \times 2N) = 2N$ hypermultiplets. Consequently, in going to the $\mathcal{N} = 2$ quiver the link doesn't become bidirectional, since one gets, not two, but one chiral multiplet, which we call $Q$. We show the mapping in Fig. 2.6.

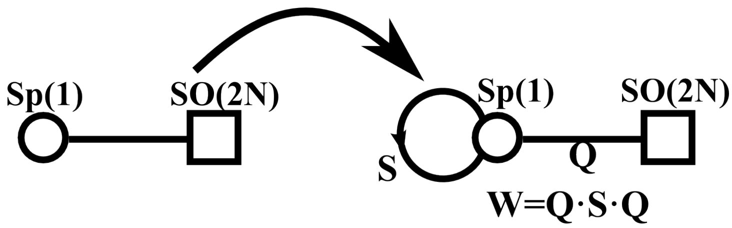



Figure 2.6: From the $\mathcal{N} = 4$ to the $\mathcal{N} = 2$ quiver. The field multiplets for $\mathcal{N} = 2$ are: a vector multiplet, the node $Sp(1)$ itself, a chiral multiplet $S$ in the adjoint of $Sp(1)$, arising from the vector multiplet for $\mathcal{N} = 4$, and one chiral multiplet $Q$ in the bifundamental of $Sp(1) \times SO(2N)$, whose indices have been suppressed, arising from one bifundamental half-hypermultiplet.

The transformation for the $\mathcal{N} = 2$ fields is:

- $S_{bc}$ transforms in the adjoint representation [2] of $SU(2)$ and we assign fugacity $z$
- $Q^i_a$, with $i$ the flavour index and $a$ the gauge index, transforms in the bifundamental $\left[1_{SU(2)}; 1, 0, ..., 0_{SO(2N)}\right]$ and we again assign fugacity $z$ for the $SU(2)$ character and $x_1, ..., x_N$ for the $SO(2N)$ representation character.

[4]For $Sp(n)$ the fundamental representation is $2n$ dimensional

The superpotential can be read off the $\mathcal{N}=2$ quiver using the prescription explained previously

$$W = Q \cdot S \cdot Q \tag{2.54}$$
$$= Q_a^i \varepsilon^{ab} S_{bc} \varepsilon^{cd} Q_d^i \tag{2.55}$$

where $\varepsilon^{ab}$ is the invariant tensor of $SU(2)$.

The F-terms are obtained by taking derivatives with respect to the (scalars in the) multiplets. The Higgs branch occurs when the scalars coming from the $\mathcal{N}=4$ vector multiplet vanish whilst the ones from the hyper multiplet take nonzero value.

$$\frac{\partial W}{\partial Q_f^i} = 2\varepsilon^{fb}\varepsilon^{cd} S_{bc} Q_d^i \tag{2.56}$$
$$\frac{\partial W}{\partial S_{bc}} = \varepsilon^{ab}\varepsilon^{cd} Q_a^i Q_d^i \tag{2.57}$$
$$= Q_b^i Q_c^i + Q_c^i Q_b^i \tag{2.58}$$

where the last line comes about since the two $\varepsilon$'s make the expression symmetric in $(b,c)$. Hence the F-flat space $\mathcal{F}^b_{1,SO(2N)}$ is the space of solutions

$$\mathcal{F}^b_{1,SO(2N)} \equiv \left\{ S=0, \langle Q_a^i \rangle \neq 0, \mid Q_a^i Q_b^i + Q_b^i Q_a^i = 0 \right\} \tag{2.59}$$

The condition on the $Q_a^i$ is that the $2^{nd}$ symmetric product vanishes, i.e the relation transforms as $[2]_{SU(2)}$, with the fugacity $t$ to the power 2 to signify that the relation is square in the fields. The character of this representation will appear as the prefactor of the plethystic exponential. The argument of the latter is instead the character of the bifundamental chiral multiplet $Q_a^i$, hence

$$g^{\mathcal{F}^b}_{1,SO(2N)}(t, x_1, ..., x_N, z) = \left(1 - z^2 t^2\right)\left(1 - t^2\right)\left(1 - \frac{t^2}{z^2}\right) PE\left[ [1,0,...,0]_{SO(2N)} \left(z + \frac{1}{z}\right) t \right] \tag{2.60}$$

The character for the fundamental representation of $SO(2N)$ can be written as

$$[1,0,...,0] = \sum_{a=1}^{N} \left(x_a + \frac{1}{x_a}\right) \tag{2.61}$$

and inserting this into (2.60) we can write the generating function $g^{\mathcal{F}^b}_{1,SO(2N)}$ as a rational function:

$$g^{\mathcal{F}^b}_{1,SO(2N)} = \left(1 - z^2 t^2\right)\left(1 - t^2\right)\left(1 - \frac{t^2}{z^2}\right) PE\left[ \left(z + \frac{1}{z}\right) \sum_{a=1}^{N} \left(x_a + \frac{1}{x_a}\right) t \right] \tag{2.62}$$

$$= \frac{\left(1 - z^2 t^2\right)\left(1 - t^2\right)\left(1 - \frac{t^2}{z^2}\right)}{\prod_{a=1}^{N} \left(1 - z x_a t\right)\left(1 - \frac{x_a}{z} t\right)\left(1 - \frac{z}{x_a} t\right)\left(1 - \frac{1}{z x_a} t\right)} \tag{2.63}$$

The Hilbert series, i.e. the projection of (2.63) onto the space of gauge invariant operators, can be obtained by contour integrating with the $SU(2)$ Haar measure

$$d\mu_{SU(2)} = \oint dz \frac{1-z^2}{2\pi i z} \tag{2.64}$$

Thus we evaluate the following integral and obtain the result

$$\begin{aligned} HS^{H}_{1,SO(2N)}(t, x_1, ..., x_N) &= \frac{1}{2\pi i}\oint dz \left(\frac{1-z^2}{z}\right) g^{\mathcal{F}^b}_{1,SO(2N)} & (2.65)\\ &= \sum_{p=0}^{\infty} [0, p, ..., 0]_{SO(2N)}\, t^{2p} & (2.66)\end{aligned}$$

recalling that $[0, 1, ..., 0]_{SO(2N)}$ is the adjoint representation of $SO(2N)$.

For $N = 4$, the $1-SO(8)$ instanton, the coefficient of (2.66) is just $[0, p, 0, 0]$ as a function of $x_1, x_2, x_3, x_4$. For example $[0, 1, 0, 0]$ is the adjoint of $SO(8)$ but also happens to be the $2^{nd}$ rank antisymmetric representation and the character can be found by antisymmetrising the character for $[1, 0, 0, 0]$ , thus giving

$$[0, 1, 0, 0] = \sum_{a<b} x_a x_b + x_a x_b^{-1} + x_b x_a^{-1} + x_a^{-1} x_b^{-1} \tag{2.67}$$

Setting all the $x_i$ to one, we obtain the unrefined series, which just *counts* chiral operators at a given degree

$$\begin{aligned} HS^{H}_{1,SO(8)}(t) &= \sum_{p=0}^{\infty} dim\left([0, p, 0, 0]_{SO(8)}\right) t^{2p} & (2.68)\\ &= \frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}} & (2.69)\end{aligned}$$

The pole at $t = 1$ is of order 10, hence the moduli space has 5 quaternionic dimensions.

### 2.4.3 The exceptional instantons

A Lagrangian description for a theory with $E_6$ global symmetry is not known. It has been argued in [20] that such a theory can be realised by means of three $M5$-branes that wrap a sphere with three punctures. Each puncture with a wrapped $M5$ realises an $SU(3)$ global symmetry. The $E6$ global symmetry arises by enhancement of the $SU(3)^3$ symmetry.

Similarly unknown are the theories with $E_7$ and $E_8$ as flavour symmetries. We sketch the quiver of such theories by means of a shaded circular node, which represents the unknown theory and in particular its gauge symmetry, and a square $E_n$, $n = 6, 7, 8$ flavour node, in Fig. (2.7).

The conjectured Hilbert series are generalised from comparison with the $1-SU(N)$ and $1-SO(2N)$ instanton moduli spaces whose Hilbert series have been demonstrated

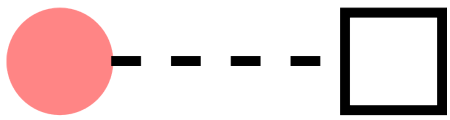



Figure 2.7: The quiver for the unknown theory that has one of the three $E$ exceptional groups as global symmetry

in the previous subsection to have the form

$$HS^{H}_{1,G}\left(t,x_1,...,x_r\right)=\sum_{p=0}^{\infty}\chi[Adj^p]_G t^{2p} \tag{2.70}$$

where $Adj^k$ is the irreducible representation constructed by replacing the entries $\theta$ in the Dynkin label of the adjoint representation of $G$ by $\theta k$. Hence for the exceptional groups $E$ this generalises to

$$HS^{H}_{1,E_6}\left(t,x_1,...,x_6\right) = \sum_{p=0}^{\infty}[0,p,0,0,0,0]_{E_6}\, t^{2p} \longrightarrow 1+78t+2430t^2+.. \tag{2.71}$$
$$HS^{H}_{1,E_7}\left(t,x_1,...,x_7\right) = \sum_{p=0}^{\infty}[p,0,0,0,0,0,0]_{E_7}\, t^{2p} \longrightarrow 1+133t+7371t^2+ \tag{2.72}$$
$$HS^{H}_{1,E_8}\left(t,x_1,...,x_8\right) = \sum_{p=0}^{\infty}[0,0,0,0,0,0,0,p]_{E_8}\, t^{2p} \longrightarrow 1+248t+... \tag{2.73}$$

where the arrows show the terms in the *unrefined* series up to order two for $E_6$ and $E_7$ and order 1 for $E_8$. In our subsequent check of mirror symmetry these terms will be checked against, whilst the higher order ones require substantial computational effort.

The expressions for the Hilbert series for the $1-G$ instanton moduli space has been dealt with in this section. The reader should bear in mind that this has not been done for mere computational sake. The *Higgs branch* of the quiver theories studied above *is* the $1-G$ *instanton moduli space.* This Higgs branch will be reappear again in a new fashion: it will coincide with the Coulomb branch of a dual theory. In order for us to reach the statement of mirror symmetry, we sketch the salient features of supersymmetric theories in three dimensions.

# Chapter 3

# $N = 4$ supersymmetric gauge theories in 3d

## 3.1 Symmetries

Consider first a $\mathcal{N} = 1$ supersymmetric gauge theory in six dimensions. The R-symmetry group is $SU(2)_R$ which rotates the fermions. The $6d$ vector multiplet in representations of $SO(4)_{little} \times SU(2)_R$ is made up of a gauge field $[1; 1; 0]$, and a right Weyl spinor field $[1; 0; 1]$. Performing dimensional reduction to $3d$, one can obtain a three dimensional supersymmetric gauge theory with 8 supercharges, i.e. $\mathcal{N} = 4$.

The $3d$ vector multiplet is made up of a vector, 3 scalar fields $\phi_1$, $\phi_2$, $\phi_3$ and 4 fermionic d.o.f arranged in a way which we explain below. The appearance of an extra symmetry group is disclosed by the multiplicity of the scalars: $\phi_1$, $\phi_2$, $\phi_3$ can be arranged as a triplet of $SO(3)$, this symmetry corresponding to rotations in the three reduced dimensions. We denote this $SO(3)_L$ and its double cover $SU(2)_L$. Hence the R-symmetry of $\mathcal{N} = 4$ $3d$ gauge theories is $SU(2)_L \times SU(2)_R \cong SO(4)$ [1] with supercharges transforming in the $[1; 1]$ representation.

## 3.2 Fields

$\mathcal{N} = 4$ Super Yang-Mills (SYM) is a highly constrained theory. It is convenient to work with $\mathcal{N} = 2$ multiplets as these embody the building blocks for $\mathcal{N} = 4$ ones. Note that $\mathcal{N} = 2$ $3d$ multiplets can themselves be obtained by dimensional reduction of $\mathcal{N} = 1$ $4d$ gauge and chiral multiplets.

The $\mathcal{N} = 2$ vector multiplet is made up of a gauge field $A_\mu$, a two-component complex Dirac spinor $\lambda_\alpha$ and a real scalar $\eta$, which corresponds to the component of the vector field in the reduced direction. There is also a so called half-hypermultiplet[2] made up of a complex scalar and a Dirac fermion. This half-hypermultiplet is the dimensional reduction of the 4d $\mathcal{N} = 1$ hypermultiplet.

[1] There is also a Lorentz symmetry $SL(2, \mathbb{R})$, which we don't write explicitly

[2] Sometimes we will call it a chiral multiplet

We now construct a $\mathcal{N}=4$ vector multiplet by combining a $\mathcal{N}=2$ vector multiplet $V$ and a half-hypermultiplet $\Phi$. Hence a $\mathcal{N}=4$ vector multiplet will consist of the vector field $A_\mu$, the Dirac spinor $\lambda_\alpha$, the real scalar $\eta$, a complex scalar $\varphi$ and finally another Dirac spinor $\xi_\alpha$. All of these fields will transform in the adjoint representation of the gauge group. The $\mathcal{N}=4$ action is

$$S_{gauge} = \frac{2\pi}{e^2}\int d^3x Tr\left\{ -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\eta D^\mu\eta + 2i\overline{\lambda}\not{D}\lambda + 2\overline{\lambda}\left[\eta,\lambda\right] \right.$$
$$+ 2D_\mu\varphi^\dagger D^\mu\varphi + 2i\overline{\xi}\not{D}\xi + \left|\left[\varphi,\varphi^\dagger\right]\right|^2 + \left|\left[\eta,\varphi\right]\right|^2$$
$$\left. + 2\sqrt{2}i\left(\left[\varphi^\dagger,\xi\right]\lambda + \overline{\lambda}\left[\varphi,\overline{\xi}\right]\right) + 2\overline{\xi}\left[\eta,\xi\right]\right\} \tag{3.1}$$

where the first line is the $\mathcal{N}=2$ vector multiplet action and the remaining lines represent gauge couplings, Yukawa couplings and potential terms for the half-hypermultiplet added to construct the $\mathcal{N}=4$ vector multiplet. The action (3.1) can be made more compact by using superspace notation and superfields. Letting $\Sigma$ be the linear multiplet that contains the field strength $F_{\mu\nu}$ we rewrite (3.1) as:

$$S_{gauge} = \frac{2\pi}{e^2}\int d^3x d^2\theta d^2\bar{\theta}\left(\frac{1}{4}\Sigma^2 - \Phi^\dagger e^{2V}\Phi\right) \tag{3.2}$$

Let us analyse the fields' transformation under the global symmetry $SU(2)_L \times SU(2)_R$. The fermions transform as a doublet of $SU(2)_L$ and $SU(2)_R$, hence as a vector of the global $SO(4)$, $[1;1]$. More importantly for our purposes we can now take the scalars $\phi_1$, $\phi_2$, $\phi_3$ mentioned in the previous subsection to be $\phi_3 \equiv \eta$ and $\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \equiv \varphi$ so that $\vec{\phi} = (\phi_1,\phi_2,\phi_3)$ is the triplet of real scalars under $SU(2)_L$. We summarise the field content of the vector multiplet in Table 3.1 , which is similar to the one in [28].

| $\mathcal{N}=4$ | $\mathcal{N}=2$ | Field type | Label | $SU(2)_L \times SU(2)_R$. |
|---|---|---|---|---|
| Vector multiplet | Vector multiplet $(V)$ | gauge<br>Dirac spinor<br>real scalar | $A_\mu$<br>$\lambda_\alpha$<br>$\eta$ | $(\lambda_\alpha,\xi_\alpha) \longrightarrow [1;1]$<br>$(\eta, Re\varphi, Im\varphi) \longrightarrow [2;0]$ |
| | Chiral multiplet $(\Phi)$ | complex scalar<br>Dirac spinor | $\varphi$<br>$\xi_\alpha$ | |

Table 3.1: Field content for the $\mathcal{N}=4$ vector multiplet, obtained by combining $\mathcal{N}=2$ multiplets. All of the fields are in the adjoint of the gauge group $G$

By rewriting the potential for the scalars in the action, which are the last two terms in the second line of (3.1), in terms of $\phi_1$, $\phi_2$, $\phi_3$ we obtain the scalar potential

$$V = \frac{2\pi}{e^2}\sum_{i<j} Tr\left[\phi^i,\phi^j\right]^2 \tag{3.3}$$

where $i,j = 1,2,3$ . This potential indicates that a supersymmetric vacuum exists, since flat directions $V = 0$ can be achieved by a set of commuting $\phi^i$. Therefore the scalars must take values in the Cartan subalgebra of the gauge group. For a gauge group $G$ of rank $r$ $U(1)^r \subset G$ is the Cartan subalgebra and hence fields which take value in $U(1)^r$ can be written

$$\phi^i = diag\left(x_1^{(i)}, ..., x_r^{(i)}\right) \tag{3.4}$$

This means that along flat directions the scalars acquire a nonzero vacuum expectation value (VEV): the gauge group $G$ is broken by the adjoint Higgs mechanism to its maximal torus $U(1)^r$. We refer to this as complete Higgsing. The acquisition of a VEV for the scalars is precisely the statement that there is moduli space for the vector multiplet: the Coulomb branch.

The most crucial point for the discussion of SYM in $3d$ is related to the nature of the vector field. $A_\mu$ transforms non linearly in the adjoint of $G$. However when the gauge group is broken to its maximal torus $U(1)^r$, and this occurs at a generic point of the Coulomb branch by complete Higgsing, one is left with $r$ $U(1)$ massless[3] gauge fields (photons): $A_\mu^{(j)}$ $j = 1, ...r$. In $3d$ *only* a gauge field is dual to a scalar field. Let $F_{\mu\nu}^{(j)}$ be the field strength associated to $A_\mu^{(j)}$ and write:

$$F_{\mu\nu}^{(j)} = \varepsilon_{\mu\nu\sigma}\partial^\sigma\gamma^{(j)} \tag{3.5}$$

where $\gamma^{(j)}$ are thus the scalars dual to $A_\mu^{(j)}$. Since $A_\mu^{(j)}$ are one-forms, the $\gamma^{(j)}$ must be zero-forms. Zero-forms are *compact* scalar fields which means take value in $S^1$. Hence a generic point of the Coulomb branch will be a parameter value of $3r$ non-compact scalars $\phi_i^{(j)}$ $i = 1, 2, 3$ and $r$ compact scalars $\gamma^{(j)}$ $j = 1, ..., r$, which adds to a total of $4r$ scalars arising solely from the vector multiplet. The Coulomb branch $\mathcal{M}_\mathcal{C}$ has thus real dimension $4r$ or $2r$ complex dimension or $r$ in quaternionic units. We summarise by writing

$$dim(\mathcal{M}_\mathcal{C}) = r \tag{3.6}$$

The Coulomb branch has the right dimension and indeed, because of $\mathcal{N} = 4$ supersymmetry, the right structure to be a hyperKähler manifold (of quaternionic dimension $r$). A simple definition of such a class of spaces is given in the Appendix. The classical Coulomb branch is the moduli space

$$\mathcal{M}_C = \left(\mathbb{R}^3 \times S^1\right)^r \tag{3.7}$$

where the $\phi_i^{(j)}$ parametrise the non-compact space $\mathbb{R}^{3r}$ and the dual photons $\gamma^{(j)}$ parametrise the compact space $S^1$. The metric describing this space is a flat hyper-Kähler metric

$$ds^2 = \sum_{i=1}^{r}\left(\frac{1}{e^2}\,d\vec{x}_i^{\,2} + e^2 d\theta_i^2\right) \tag{3.8}$$

where the vector $\vec{x}_i$ is a triplet for each $i = 1, ...r$ and $\theta_i$ are coordinates on the $r$

[3] The remaining gauge bosons acquire a mass

sphere $S^r$ . This metric has been written by purely classical considerations. Since the Coulomb branch is quantum mechanically corrected by loop corrections and instanton effects, the metric needs modifications to allow for this. A classical treatment of $3d$ $\mathcal{N} = 4$ theories with mathematical derivations of one-loop corrected metrics on the Coulomb branch can be found in [34, 35].

## 3.3 The Higgs branch and an IR duality

Matter couplings can of course be provided to the pure $\mathcal{N} = 4$ super yang Mills described above; one does so by adding hypermultiplets to the theory. The $\mathcal{N} = 4$ hypermultiplets are constructed by combining two $\mathcal{N} = 2$ half-hypermultiplets, which we name after the complex scalar they contain. Thus each $\mathcal{N} = 4$ hypermultiplet consists of two Dirac spinors $\psi_\alpha$, $\vartheta_\alpha$, which transform as a doublet of $SU(2)_L$ and $SU(2)_R$, and two complex scalars $A$, $B$ whose real and imaginary parts can be arranged as doublets of one of the two $SU(2)$ as shown in Table **??**. The two half-hypermultiplets, which together are used to make the $\mathcal{N} = 4$ hypermultiplet, transform in conjugate representations of the gauge group $G$.

| $\mathcal{N} = 4$ | $\mathcal{N} = 2$ | Field type | Label | $SU(2)_L \times SU(2)_R$. | $G$ |
|---|---|---|---|---|---|
| Hyper multiplet | Chiral multiplet $(A)$ | complex scalar | $A$ | $(\psi_\alpha, \vartheta_\alpha) \longrightarrow [1;1]$ | $R$ |
| | | Dirac spinor | $\psi_\alpha$ | $\left(A, B^\dagger\right) \longrightarrow [1;0]$ | |
| | Chiral multiplet $(B)$ | complex scalar | $B$ | $\left(A^\dagger, B\right) \longrightarrow [0;1]$ | $R^*$ |
| | | Dirac spinor | $\vartheta_\alpha$ | | |

Table 3.2: Field content for the $\mathcal{N} = 4$ hypermultiplet, obtained by combining two $\mathcal{N} = 2$ chiral multiplets.

The action for the $\mathcal{N} = 4$ hypermultiplet in superspace language is

$$S_{hyper} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_{hyper's} \left(A^\dagger e^{2V} A + B^\dagger e^{2V} B\right) \tag{3.9}$$

There can also exist a holomorphic superpotential constrained by $\mathcal{N} = 4$ to be

$$S_{sp} = -i\sqrt{2} \int d^3x d^2\theta d^2\bar{\theta} \sum_{hyper's} B\Phi A + c.c. \tag{3.10}$$

where recall that $\Phi$ is the adjoint $\mathcal{N} = 2$ chiral multiplet that enters the $\mathcal{N} = 4$ vector multiplet. The sum is restricted to hypermultiplets which are charged under the gauge group for which $\Phi$ is in the adjoint. Further there is a trace implicit for the tensor product of $R^* \otimes Adj \otimes R$ for the cubic terms in the superpotential.

The VEV of the scalars in the hypermultiplet also parametrise a hyperKähler space, the *Higgs branch*. Note that non-zero VEV for the hypermultiplets means the gauge group is broken completely. Consequently the dimension of this space is given by the number of $\mathcal{N} = 4$ hypermultiplets, $dim(R)$, minus the number of gauge fields that become massive due to complete Higgsing. This is only valid when no superpotential is present, that is, for $\mathcal{N} = 4$ in 3d. If $U(k)$ is the gauge group, all $k^2$ generators are broken, i.e. $k^2$ d.o.f. become massive and need to be subtracted from the hyper d.o.f. For a general group $G$ we have:

$$dim(\mathcal{M}_{\mathcal{H}}) = dim(R) - |G| \tag{3.11}$$

Since supersymmetry invariance dictates that no other cross terms between scalars in the vector multiplet and scalars in hypermultiplet may show in the Lagrangian, the total moduli space will be a product $\mathcal{M}_{\mathcal{C}} \times \mathcal{M}_{\mathcal{H}}$.

The Kähler form associated to the Higgs branch transforms in the adjoint of the global symmetry of the Higgs branch and trivially under $SU(2)_L$, which we can write as$[2;0]_{SU(2)_L \times SU(2)_R}$ using Dynkin labels. Conversely the Kähler form associated to the Coulomb branch transforms oppositely as $[0;2]_{SU(2)_L \times SU(2)_R}$. The Higgs branch, unlike the Coulomb branch, gets no quantum corrections: classical computations suffice to describe it. This difference between these two sections of the moduli space depends on the transformation properties of the gauge coupling constant.

All the coupling constants should really be thought of as VEV of non-dynamical background superfields [33]. As such, one can identify their transformation properties under the global symmetry. In particular, after being promoted to superfield, the gauge coupling constant transforms in the $[2+0;0]_{SU(2)_L \times SU(2)_R}$. Since it transforms trivially under the symmetry group of the Higgs branch, $SU(2)_R$, the scalars in this background superfield can only be moduli of the Coulomb branch. The gauge coupling controls quantum loop corrections, so the Coulomb branch is subject to these whilst the Higgs branch is exempt. This is an instance of non-renormalisation theorem.

Extra terms can be added to the Lagrangian which preserve supersymmetry: mass terms for the hypermultiplets and Fayet-Iliopoulos (FI) terms associated to the $U(1)$ vector fields. When such terms are included, the two types of couplings, mass and FI parameters, can again be promoted to background superfields: after promotion, mass parameters transform as a triplet of $SU(2)_L$ and are invariant under $SU(2)_R$. Since they transform in $[2;0]_{SU(2)_L \times SU(2)_R}$, they can only affect the Coulomb branch (they actually deform the Coulomb branch by resolving the singularities of the space); on the other hand Fayet-Iliopoulos couplings transform in the $[0;2]_{SU(2)_L \times SU(2)_R}$ and, as such, are a deformation parameter for the Higgs branch. [4]

Supersymmetric gauge theories in $3d$ are super-renormalizable. The Lagrangian description applies to finite energies and the the theory is free in the ultraviolet. It is also known to flow in the infrared (IR) to a non-trivial superconformal fixed point. It is here that mirror symmetric pairs flow to and become dual faces of one theory.

[4]The effect of these two terms on the underlying algebraic space is that of resolving the singularities. In very rough terms, the singularity are smeared by for example decreasing the order of the pole. We will not consider this in our work.

Most importatnly the scaling dimension of the fields at the superconformal fixed point is dictated by how they transform under the infrared superconformal R-symmetry, which is $Spin(4) \cong SU(2) \times SU(2)$. If this R-symmetry coincides, and it's not always the case, with the Lagrangian R-symmetry, $SU(2)_V \times SU(2)_R$, then the fields conformal dimensions are well-defined: scalars in the hypermultiplet have IR scaling dimension 1/2, while the scalars in the vector multiplet and the gauge field have IR scaling dimension 1. We will always consider such theories.

This treatment serves as a background for the discussion of mirror symmetry. Let us present qualitatively the claims of this duality for the case of $U(1)$ with $n+1$ electrons in light of what has been exposed. The flavour symmetry of the Lagrangian is $SU(n+1) = A_n$. The Higgs branch of this theory is the moduli space of $A_n$ instantons. The Coulomb branch is a one-dimensional hyperkähler manifold with an $A_n$ singularity. In [25] it is claimed that, when this theory flows in the IR to the non-trivial superconformal fixed point, these two moduli spaces become respectively the Coulomb branch and the Higgs branch of another abelian gauge theory. In order for this to happen this duality must exchange the $SU(2)_L$ and $SU(2)_R$ R-symmetries, and Fayet-Iliopoulos terms and mass terms if they are present. Again, surprisingly, one-loop and instanton corrections on Coulomb branch arise classically in the Higgs branch of the mirror theory.

## 3.4 Hidden Symmetries

A particular feature of gauge theories in $3d$ arises by virtue of the scalars $\gamma^{(j)}$ dual to the photons. Suppose the gauge group is $U(1)$. The field strength $F^{(2)}$ has a Hodge dual $\star_{2+1} F^{(2)} = J^{(1)}$, a current. Alternativelywe can write this in coordinates notation

$$J^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} \tag{3.12}$$

$J^{(1)}$ is a current in that it is topologically conserved by the Bianchi identity, $d\star_{2+1} F^{(2)} = 0$. This topologically conserved current presupposes, by Noether theorem, the existence of a global $U(1)_J$ symmetry which is not explicit in the Lagrangian. The charge associated to this $U(1)_J$ is called vortex charge or topological charge. Fundamental fields in the theory have zero vortex charge but we will construct operators with non-zero vortex charge; these are the so called monopole operators. For a general gauge group $G$ of rank $r$, the theories thus hide a global $U(1)^r_J$, which is not explicit in the Lagrangian.

By promoting mass and FI parameters to background superfield, $\mathcal{N}=4$ supersymmetry constrains the mass terms, which transform in the adjoint of any global flavor symmetry, to be in the Cartan subalgebra of the latter; the FI terms are on the other hand free and not associated to any global symmetry. Since mirror symmetry exchanges mass and FI parameters one imagines that a hidden global symmetry must be arise by enhancement; in the dual theory the visible flavour symmetry of original theory is exchanged with the hidden. What is this hidden global symmetry then?

Here is where the $U(1)_J^r$ shows its nature. The conserved current $J^{(1)}$ can be coupled to a background gauge field $a_\mu$ via $a \wedge F$. We then let $a$ become a vector superfield and give VEV to the scalar in this vector multiplet. This is a FI term and its coupling to a global symmetry $U(1)_J$ is now manifest. In fact the topological global $U(1)_J^r$ is the maximal torus of a hidden, non abelian global symmetry of rank $r$.

Before further aspects of 3d $\mathcal{N} = 4$ gauge theories, and in particular mirror symmetry can be discussed, one must first take a detour on the so called ADE classification. We will return to analysing features that emerge from the above discussion in section 5.

# Chapter 4

# The ADE classification and the McKay correpondence

The ADE classification refers to a subset of root systems such that roots can only form angles of 90° or 120°. The Dynkin diagrams are consequently simply laced, i.e. there are no double links between nodes of the diagram. Such diagrams represent the root systems $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, hence the name. These ADE root systems represent classes of several objects in Mathematics. In particular they can be used to classify ADE semisimple Lie algebras and binary polyhedral groups, which are the finite subgroups of $SU(2)$. There is an exact correspondence between these two, known as McKay correspondence.

## 4.1 Finite subgroups of $SU(2)$

Let us begin with examining finite subgroups of $SU(2)$. Take the usual homomorphism $\rho : SU(2) \twoheadrightarrow SO(3)$. Then $ker(\rho) = \{\pm 1\}$, with $-1$ being the only element with even order. Let $G$ be a finite subgroup of $SU(2)$. If $G$ has even order, it is the inverse image $\rho^{-1}(H)$ of a subgroup $H$ of $SO(3)$; if it has odd order it is isomorphic to a subgroup of $SO(3)$ of odd order, the only one of which is the cyclic group of odd order. Since subgroups of $SO(3)$ are classified, the following list of subgroups of $SU(2)$ is exhaustive[1]:

- The cyclic group of order $n+1$: $\mathbb{Z}_{n+1}$
- The binary dihedral group of order $4n$: $Dic_n$
- The binary tetrahedral group of order 24: $\mathbb{BT}$
- The binary octahedral group of order 48: $\mathbb{BO}$
- The binary icosahedral group of order 120: $\mathbb{BI}$

[1]The symmetry group of tetrahedron (self-dual), the octahedron (and its dual the cube), the dodecahedron (and its dual the icosahedron), also known as the Platonic solids, are, together with $\mathbb{Z}_n$ and $Dic_n$, the symmetry groups of $SO(3)$.

### 4.1.1 Invariant theory for the subgroups of $SU(2)$

In discussing orbifolds as spaces constructed by quotients of manifolds over finite groups, a prominent role is played by invariant polynomial functions under the action of the group. This is the realm of invariant theory of finite groups. The interested reader is referred to an accesible work by Stanley [36]. We discuss elementary results for the subgroups of $SU(2)$, $\Gamma_{SU(2)}$.

Take $W$ to be a finite dimensional complex vector space. Let $f$ be a polynomial function in the coordinates of a given basis for $W$. Let $\mathbb{C}[W]$ denote the ring of polynomial functions in the coordinates $\{z_i\}_{i=1}^n$with respect to a basis $\{e_i\}_{i=1}^n$ of $W$. The $\{z_i\}$ form the basis for the dual vector space $W^\star$, defined so that $z_i(e_j) = \delta_{ij}$, and the ring of polynomials is $\mathbb{C}[W] = \mathbb{C}[z_1, .., z_n]$.

A polynomial function $f \in \mathbb{C}[W]$ is said to be homogenous of degree $d$ if $f(\alpha w) = \alpha^d f(w)$ for all $\alpha \in \mathbb{C}$, $w \in W$ . We then can write $\mathbb{C}[W]$ as a graded $\mathbb{C}$-ring. Let $\mathbb{C}[W]_d$ be the subspace of homogeneous polynomial of degree $d$. Then

$$\mathbb{C}[W] = \bigoplus_d \mathbb{C}[W]_d \tag{4.1}$$

The ring of polynomials of degree 1 corresponds to the dual space $\mathbb{C}[W]_1 = W^\star$. More generally one can identify $\mathbb{C}[W]$ with the symmetric algebra of $W^\star$, $S(W^\star)$, itself graded with degree $d$, i.e.

$$S(W^\star) = \bigoplus_d S_d(W^\star) = \mathbb{C}[W] \tag{4.2}$$

Now consider a subgroup $G$ of $GL(W)$, the general linear group of linear transformations on $W$, embedded by a suitable homomorphism. $G$ acts on $W$ in the usual way: $G \times W \to W$, $(g, w) \longmapsto gw$. We define invariants in two ways:

- A polynomial function $f \in \mathbb{C}[W]$ is $G$-invariant if $f(gw) = f(w)$ $\forall g \in G$ and $w \in W$. Alternatively $f$ is G-invariant if it is constant along all orbits of $G$. The $G$-invariant polynomial functions form a subalgebra, called the *invariant ring*. This is labeled $\mathbb{C}[W]^G$. It is by inheritance a graded algebra:

$$\mathbb{C}[W]^G. = \bigoplus_d \mathbb{C}[W]_d^G \tag{4.3}$$

- Consider the action of $G$ on $\mathbb{C}[W]$ and hence on the symmetric algebra $S$: $gf(w) = f(g^{-1}w)$ with $f \in S$, $w \in W$. Then the subalgebra of $S$ which is $G$-invariant is defined by

$$S^G = \mathbb{C}[W]^G = \{f \in S(W^\star)\,|gf = f\ \forall g \in G\} \tag{4.4}$$

It is a theorem due to Hilbert that the invariant ring is finitely generated, i.e. one can find elements in $\mathbb{C}[W]^G$ and relations among them from which the whole ring can be constructed. The question of what these generators and relations are is the subject of classic invariant theory.

The invariant ring for $\Gamma_{SU(2)}$ is generated by three elements, $X, Y, Z$ which obey a single relation $\varrho(X, Y, Z) = 0$, i.e.

$$\mathbb{C}[W]^G = \mathbb{C}[z_1, z_2] = \frac{\mathbb{C}[X, Y, Z]}{\langle \rho \rangle} \subset \mathbb{C}^3 \tag{4.5}$$

Moreover the isomorphism

$$\chi : \mathbb{C}^2 \rightarrow \mathbb{C}^3 \tag{4.6}$$
$$(z_1, z_2) \rightarrow (X(z_1, z_2), Y(z_1, z_2), Z(z_1, z_2)) \tag{4.7}$$

preserves the quotient structure and we can write the algebraic ring in terms of the quotient space constructed by letting $\mathbb{C}^2$ being acted upon by $\Gamma$ .

$$\frac{\mathbb{C}[X, Y, Z]}{\langle \rho \rangle} \cong \frac{\mathbb{C}^2}{\Gamma} \tag{4.8}$$

In table 4.1 the algebraic relation between $X, Y, Z$ for each of the five subgroups $\Gamma$ of $SU(2)$ is listed.

| $\Gamma$ | $\rho(X, Y, Z) = 0$ |
|---|---|
| $\mathbb{Z}_{n+1}$ | $X^2 + Y^2 + Z^{n+1}$ |
| $Dic_{n-2}$ | $X^2 + Y^2 Z + Z^{n-1}$ |
| $\mathbb{BT}$ | $X^2 + Y^3 + Z^4$ |
| $\mathbb{BO}$ | $X^2 + Y^3 + YZ^3$ |
| $\mathbb{BI}$ | $X^2 + Y^3 + Z^5$ |

Table 4.1: Shows the relation between $X, Y, Z \in \mathbb{C}^3$, for each of the five finite subgroups $\Gamma$ of $SU(2)$. The $\Gamma$-invariant ring of polynomials is algebraically described by $\mathbb{C}[W]^\Gamma = \frac{\mathbb{C}[X,Y,Z]}{\langle \rho \rangle}$

We showed in section 1 how to obtain the algebraic description of the orbifold $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ which were described by $X, Y, Z$ subject to the relation (2.4). The extension to $\frac{\mathbb{C}^2}{\mathbb{Z}_n}$ is straightforward: for three generators $W$, $U$, $Z$, the relation $WU = Z^n$ can be obtained by identical computations that led to (2.4) and with a change of variables $W = X + iY$ and $U = X - iY$, one arrives at $\rho(X, Y, Z) = 0$ in the first line of Table 4.1. The remaining four defining equations can be worked out by similar, if slightly more involved, means. The details of the calculations are found in the excellent notes by [14].

## 4.2 The McKay correspondence

### Mckay graph

McKay graphs are oriented graphs that encode the irreducible representations of a given finite groups $G$. Here we consider McKay graphs for the finite subgroups $\Gamma$ of $SU(2)$. This graphs are unoriented, unlike more general groups.

Let $R$ be the faithful representation obtained by the embedding $\Gamma \hookrightarrow SU(2)$ and let $\chi_i$ , $i = 1, ..., d$ be characters of the irreducible representations $\{R_i\}$ of $\Gamma$ . The McKay graph for $\Gamma$ is defined to be a quiver diagram with $d$ vertices, one for each $R_i$, and $n_{ij}$ lines from $R_i$ to $R_j$ where $n_{ij}$ is the number of times $R_j$ appears in the decomposition

$$R \otimes R_i = \bigoplus_j n_{ij} R_j \tag{4.9}$$

For the subgroups of $SU(2)$,$\Gamma_{SU(2)}$, it holds that $n_{ij} = n_{ji}$ and the graph is unoriented.

### Mckay correspondence

There is a one-to-one correspondence between McKay graphs for subgroups of $SU(2)$ and the affine Dynkin diagrams of simply laced semisimple Lie groups.

Rigorous proofs of this statement exist. Otherwise one can inspect the claim on a case by case basis. Here we show how the latter is done for the simplest case of $\Gamma = \mathbb{Z}_n$. The faithful completely reducible 2 dimensional representation obtained by embedding $\mathbb{Z}_n \hookrightarrow SU(2)$ is:

$$R_{\mathbb{Z}_n} := \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \ \omega^n = 1 \tag{4.10}$$

The one dimensional irreducible representations $\{R_i\}$, $i = 0, ..n-1$ are $\omega^i$. Then the tensor product decomposition (4.9) is

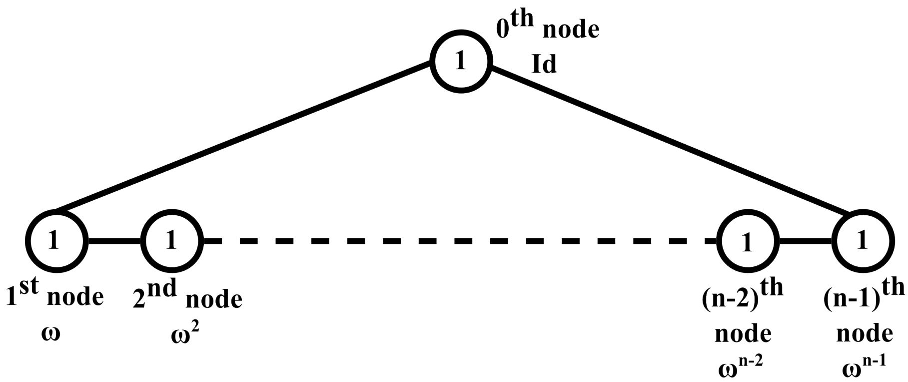
$$R_{\mathbb{Z}_n} \otimes \omega^i = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \otimes \omega^i = \begin{pmatrix} \omega^{i+1} & 0 \\ 0 & \omega^{i-1} \end{pmatrix} \tag{4.11}$$

The McKay graph can thus be constructed straightforwardly. There are $n$ nodes representing the irreducible representations and the $i^{th}$ node is connected to the $(i+1)^{th}$ and $(i-1)^{th}$, the $0^{th}$ node being connected to the $1^{st}$and the $(n-1)^{th}$. It is depicted in Fig. 4.1a .

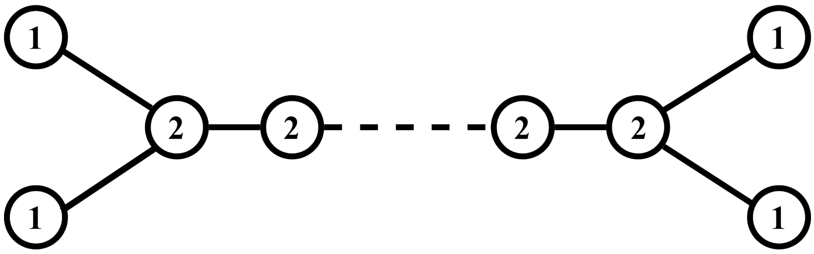
The reader should recognise that this is precisely the affine Dynkin diagram for the lie groups $\hat{A}_{n-1} = SU(n)$, where the zero node corresponds to the extended node, proving the McKay statement for the cyclic subgroup of $SU(2)$.

Consider now $\Gamma = Dic_n$. There are 4 one dimensional representations and $(n-1)$ two dimensional representations[2] . The McKay graph is shown in Fig. 4.1b . As the

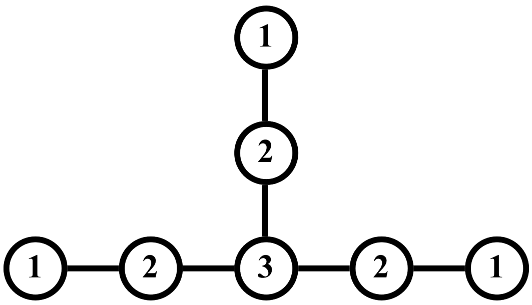
[2] A quick check is always: $\sum_i k_i^2 = |G|$. Here $4 + 4(n-1) = 4n = |Dic_n|$.



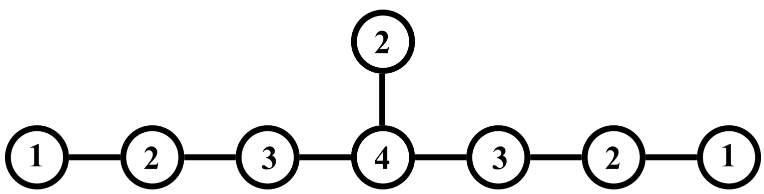

(a) McKay graph for $\mathbb{Z}_n$. All representations are one dimensional



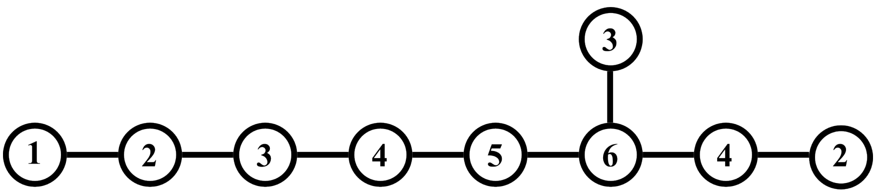

(b) McKay graph for $Dic_n$. There are 4 one dimensional representations and $(n-1)$ two dimensional ones.




(c) McKay graph for $\mathbb{BT}$. The dimensions of the irreducible representations are inside the nodes.




(d) McKay graph for $\mathbb{BO}$




(e) McKay graph for $\mathbb{BI}$

Figure 4.1: The McKay graphs for the subgroups of $SU(2)$. Inside each node is the dimension of the corresponding representation.

reader will have noticed, this is equivalent to the affine Dynkin diagram of $\hat{D}_{n+2} = SO(2n+4)$.

In Fig. 4.1c we show the Mckay graph for the binary tetrahedral group which is identical to the affine Dynkin diagram of the exceptional Lie algebra associated to $\widehat{E}_6$; 4.1d is the McKay graph for the binary octahedral group, which corresponds to the affine Dynkin diagram of $\widehat{E}_7$ and finally 4.1e is the McKay graph for the binary icosahedral group which corresponds to the affine Dynkin diagram of $\widehat{E}_8$.

## 4.3 Hilbert series for du Val singularities

Eq. 4.8 relates the ring of $\Gamma_{SU(2)}$-invariant polynomial to to the corresponding orbifold $\frac{\mathbb{C}^2}{\Gamma}$ . The isolated singularity of this special class of quotient spaces is called *du Val singularity.* The spaces themselves are instead known in the physics literature as **ALE** spaces (Asymptotically Locally Euclidean).

The defining algebraic equations that characterizes **ALE** spaces of Table 4.1 contains all the needed information to write down the unrefined Hilbert series. Let us show how to this for the case of $\mathbb{Z}$, $Dic_n$, $\mathbb{BT}$, the other two case being analogously written.

### $\mathbb{Z}_n$

The algebraic curve in $\mathbb{C}^3$ is $\varrho(X, Y, Z) = X^2+Y^2+Z^{n+1} = 0$. The three generators of invariants $X, Y, Z$ are found at different degrees. In order to have integers appearing as powers of the fugacity $t$, let us *choose* $Z$ to be of degree 2, and consequently, in order for the polynomial relation to be homogeneous, choose $X$ and $Y$ to be of order $n+1$. The relation occurs then at order $2n+2$. The Hilbert series can be written at once:

$$HS(t; \frac{\mathbb{C}^2}{\mathbb{Z}_n}) = \frac{1 - t^{2n+2}}{(1 - t^{n+1})(1 - t^{n+1})(1 - t^2)} \tag{4.12}$$

### $Dic_{n-2}$

The algebraic curve in $\mathbb{C}^3$ is $\varrho(X, Y, Z) = X^2 + Y^2 Z + Z^{n-1} = 0$. Choose $Z$ to be of degree 2, $Y$ of degree $n-2$ and thus $X$ of degree $n-1$. Then the relation is of order $2n-2$ and the Hilbert series is:

$$HS(t; \frac{\mathbb{C}^2}{Dic_{n-2}}) = \frac{1 - t^{2n-2}}{(1 - t^{n-2})(1 - t^{n-1})(1 - t^2)} \tag{4.13}$$

### $\mathbb{BT}$

The algebraic relation is $\varrho(X, Y, Z) = X^2 + Y^3 + Z^4 = 0$. Choose $X$ to be of degree 6, and consequently, $Y$ of order 4 and $Z$ of order 3. The relation occurs then at order

12. The Hilbert series is:

$$HS(t; \frac{\mathbb{C}^2}{\mathbb{BT}}) = \frac{1-t^{12}}{(1-t^3)(1-t^4)(1-t^6)} \tag{4.14}$$

It is similarly straightforward to write the Hilbert $HS(t^2; \frac{\mathbb{C}^2}{\Gamma})$ for the other two subgroups of $SU(2)$, and in general for any algebraic space once the defining equation of the space is known.

## 4.4 ADE classification in string theory

In string theory **ALE** spaces appear when extra dimensions are compactified. For example, one can have Type II theories on $\mathbb{R}^{1,5} \times \frac{\mathbb{C}^2}{\Gamma}$. $Dp$-branes, where $-1 \leq p \leq 5$, can exist in such a background. Moreover the gauge theory on such a $Dp$-brane will no longer be a simple $U(k)$ theory but instead will be an ADE-quiver gauge theory, i.e a gauge theory where the gauge group is identified with the Dynkin diagram corresponding to the choice of $\Gamma$. The gauge group will be simply a product of $U(k_i)$ where $k_i$ is the dimension of the representation of the $i^{th}$ node. For example a $Dp$-brane with $-1 \leq p \leq 5$ in a spacetime $\mathbb{R}^{1,5} \times \frac{\mathbb{C}^2}{\mathbb{Z}_n}$ carries on its worldvolume the quiver gauge theory shown in Fig. 4.1a where the gauge group is $U(1) \times ... \times U(1) = U(1)^n$. For the reasons aforementioned $Dp$-branes are often called *spacetime probes.*

This is another nuance of the correspondence: from the purely group theoretic theorem we are able to identify a special class of gauge theories, known as ADE-quiver gauge theories. Schematically the correspondence is on three levels:

McKay graph for $SU(2)$ subgroups

$\rightleftharpoons$ADE Affine Dynkin Diagram

$\rightleftharpoons$ADE quiver gauge theories

This connection is fascinating and the results non trivial. As a further example consider the theory whose quiver is 4.1a, where the gauge group at each node is $U(1)$. The moduli space of vacua of such a theory has two branches: a Higgs Branch whose moduli space is the ALE space with $A_n$ du Val singularity and a Coulomb branch whose moduli space corresponds to the $1 - A_n$ instantons. In principle such a quiver gauge theory would have nothing to do at all with the Lie algebra $\mathfrak{su}(n)$ and yet, because of the McKay correspondence, the latter arises in different guises in both spaces. Let us summarise this section with the following:

- ADE groups are algebro-geometrically deeply connected to du Val singularities.
- ADE-quiver gauge theories are the field-theoretic manifestation of the mathematical correspondence.

# Chapter 5

# Coulomb branch

## 5.1 Monopole operators in 3d

In order to understand the Coulomb branch we need to introduce the concept of monopole operators. This is intrinsically tied to the existence of hidden symmetries of the action. The work of Kapustin, Strassler, Witten, Gaiotto, Kim and collaborators in [2, 3, 6, 7, 21, 27, 26, 29, 28, 30]has in the past decade shed light on the crucial role of these objects on certain 3d gauge theories, including those with $\mathcal{N} = 4$ supersymmetry.

As we have mentioned in section 3, a certain hidden symmetry arises in presence of a gauge group that has $U(1)$ factors. For each $U(1)$, one can construct a current $J^\mu$ as defined in 3.12 which is conserved topologically by Bianchi identity. This conserved current presumes the the existence of a global symmetry $U(1)_J$ (not visible in the action, hence hidden). Fields in the Lagrangian are neutral under this symmetry but there exist special operators which carry non-zero charge under $U(1)_J$ . We call its associated charge the *topological charge* and we label it using Latin letters from the beginning of the alphabet $a, b, c, \ldots$ .

The hidden global symmetry can be enhanced to a non abelian symmetry by means of monopole operators which i. These are disorder operators which are inserted at a certain spatial point such that the gauge field has a Dirac magnetic monopole singularity there. This in turns introduces a non-zero magnetic flux through a sphere surrounding the point. Consider for example a theory in $\mathbb{R}^{2,1}$ with gauge group $U(1)$. The gauge field with such a singularity at the origin, with $m$ magnetic charge, is:

$$A^{N,S}(\vec{r}) = \frac{m}{2}(\pm 1 - \cos\theta)\mathrm{d}\varphi \tag{5.1}$$

where the opposite signs corresponds to opposite hemispheres of the $S^2$ that surround the insertion point. The magnetic charge $m$ is subject to the usual Dirac quantisation condition. In order for gauge fields to have such singularities, operators $V_m$, which carry magnetic charge $m$ must be inserted at the singularity point. If they carry magnetic charge $\pm 1$ under $U(1)_J$, they are called vortex creating operators. Note that for theories with both abelian and nonabelian gauge group we will always talk about *monopole operators*. These also carry 'charges', which in general are not conserved. Strictly speaking vortex creating operators are different in nature from monopole operators, since their charge is a globally conserved charge. However we will often fall into the habit of naming them all monopole operators.

We state here and refer the reader to the literature that if these operators are to be BPS operators and thus preserve supersymmetry, matter fields will also display a singularity at the spatial intersection point. To obtain information about monopole operators, authors resort to a technique known as radial quantization[1]: a superconformal theory on $\mathbb{R}^3$ can be "radially quantized", i.e written as a theory on $\mathbb{R} \times S^2$.

Through this procedure local operators on $\mathbb{R}^3$are mapped to states on $\mathbb{R} \times S^2$. Thus monopole operators of the original theory carrying magnetic charge $m$ are in a one-to-one mapping to states on the radially quantized theory with flux $m$ through the sphere. Crucially the *conformal dimension of the monopole operators* corresponds to the *energy of the states*.

The energy for the states in the energy spectrum for abelian and non abelian gauge theories on $\mathbb{R} \times S^2$ and thus of the vacuum state were calculated in [6]. It depends only on the magnetic charge $m$. For example, for $U(1)$ with $N_f$ flavours they show that $E = \frac{N_f}{2} |m|$ for a vacuum state with given magnetic flux $m$. Moreover since the vacuum is rotationally invariant, its spin is 0. It is also invariant under the gauge transformation, i.e. it is a color singlet. By the operator-state correspondence we have obtained monopole operators with conformal dimension $\Delta = E$ which are gauge invariant chiral scalars.

**Non-abelian gauge groups** For a non abelian group $G$ of rank $r$ one simply embeds $U(1) \hookrightarrow G$ and the image of the $U(1)$ monopole under this homomorphism defines the $G$ monopole charges: $e^{im} \to e^{iH}$, where $H = diag(m_1, ..., m_r)$, and the $m_i$ are integers known as magnetic charges. Then the gauge field with a non-abelian magnetic singularity is:

$$A^{N,S}(\vec{r}) = \frac{H}{2}(\pm 1 - \cos\theta) \tag{5.2}$$

In other words $H$ is an element of the Cartan subalgebra of $\mathfrak{g}$. $M$ must also obey a generalised Dirac quantisation condition which, in group theoretic language, means that $M$ can only take value in the weight lattice restricted to the Weyl chamber of the gauge group[2]. We refer to [26] and to the last chapter of [19] for a general introduction on the dual group and to

The energy of the flux carrying vacuum and thus the conformal dimension $\Delta(H)$ of the monopole operators depend on the magnetic charges. The expression we will use is derived in [3]. Let $\alpha$ be the positive roots of the root system $\Xi^+$ and $\rho_i$ the

[1]The procedure is not overly complicated. One writes the action of the superconformal theory being studied, Wick rotates one coordinate to have an Euclidean action and introduces a dimensionless parameter to replace the radial coordinate by a non-compact (time) direction. The fields are also rescaled so as to become dimensionless.

[2]This is actually not correct since the Dirac quantisation condition forces $M$ to lie in (the closure of) the Weyl chamber of the *dual gauge group*. We have avoided such a complication by choosing gauge groups which are self-dual.

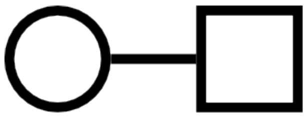



Figure 5.1: The quiver for $U(1)$with $N$ flavours

weights of the matter field representation $R_i$ where $i = 1, ..., n$ is the number of hypermultiplets . Then

$$\Delta(H) = -\sum_{\alpha \in \Xi^+} |\alpha(H)| + \frac{1}{2} \sum_{i=1}^{n} \sum_{\varrho_i \in R_i} |\varrho_i(H)| \tag{5.3}$$

where the negative contribution comes from the vector multiplet and the positive contribution from the matter fields. We will encounter only one non-unitary gauge group, $G_2$, where the above form of this formula will become essential. In the case of unitary groups (5.3) can be written without thinking about roots and weights in a manner that we show in our examples below.

Once the conformal dimension of monopole operators in a given gauge theory is computed, a generating function that encodes the chiral ring for the Coulomb branch can be written. We will study the Coulomb branch of some quiver gauge theories, including the ADE types. For the quivers representing these gauge theories it always holds that, for each node, the number of flavours is twice the number of colours $n_f = 2n_c$. These quivers are called balanced. For balanced quivers gauge theories the monopole operators always have $E \geq 1$. This preserves unitarity which in turns imply that the $R-$symmetry of the $IR$ theory coincides with the $R$-symmetry at the $UV$. This is essential if one is to make any statement at all about the conformal dimension of monopole operators.

The concrete examples that we present in the next section will greatly clarify the use of monopole operators in the description of the Hilbert series of the Coulomb branch of quiver gauge theories. This new method has been developed in the past two months by S. Cremonesi, A. Hanany and A. Zaffaroni and is not yet published [8]. The author reproduces here her calculations and her understanding of the excitement of the past few weeks.

## 5.2 Hilbert series for the Coulomb branch: abelian theories

### 5.2.1 $U(1)$ with $N$ flavours

Consider the simple gauge theory of $U(1)$ with $N$ flavours. The quiver is in Fig. 5.1 .

The Coulomb branch is one dimensional in quaternionic units since there is only one vector multiplet. The Higgs branch has dimension[3] $d = N - 1$.

We would like to characterise the Coulomb branch through the Hilbert series, again the generating function for the chiral ring of gauge invariant operators, and possibly closed forms thereof. Since the Coulomb branch receives quantum corrections it is not possible to approach the issue as we did for the Higgs branch, by finding the F-terms for the superpotential and projecting onto the gauge invariant section. The Hilbert series for the Coulomb branch can be obtained by finding the Hilbert series for the Higgs branch of the mirror theory. It is through this that we know that the Coulomb branch is in fact the ALE-space with $A_{N-1}$ singularity. However, we want to be able to derive the algebraic description independently. The procedure is as follows.

The Coulomb branch has a global symmetry $SU(2)_L \times U(1)_J$. The topological charge under the $U(1)_J$ is labelled $m$. We start by writing a formula for the scaling dimension of operators $V_m$ that carry non-zero magnetic charge $m$. For a theory with an abelian gauge group there is no contribution to the scaling dimension from the vector multiplet. The matter contribution is instead $\Delta_{matter} = \frac{N}{2}|m|$, i.e each flavour gives a contribution of 1/2 magnetic charge. Hence the conformal dimension for monopole operators with charge $m$ is

$$\Delta(m) = \Delta_{matter} = \frac{N}{2}|m| \tag{5.4}$$

The generating function for gauge invariant operators on the Coulomb branch of $U(1)$ with $N$ flavours is written by choosing a fugacity $t$, which counts operators with a given dimension $\Delta(m)$

$$\begin{aligned} HS^{C}_{1,N}(t) &= \frac{1}{1-t}\sum_{m=-\infty}^{\infty} t^{\Delta(m)} \\ &= \frac{1+t^{N/2}}{(1-t)(1-t^{N/2})} \qquad (5.5) \\ &= \frac{1-t^{N/2}}{(1-t)(1-t^{N/2})(1-t^{N/2})} \qquad (5.6) \end{aligned}$$

Note that the factor in front of the summation reminds us that there is the aforementioned complex scalar in the vector multiplet which does not enter in the construction of the monopole operators.

We can refine the Hilbert series by introducing a fugacity $z$ for the topological charge $m$

[3] Recall that the quaternionic dimension of the Coulomb branch is given by the rank of the gauge group, i.e. a vector multiplet for each $U(1)$ in the Cartan subalgebra.

$$HS^{C}_{1,N}(z,t) = \frac{1}{1-t}\sum_{m=-\infty}^{\infty} z^m t^{\Delta(m)} \tag{5.7}$$

$$= \frac{1-t^{N/2}}{(1-t)\,(1-zt^{N/2})\,(1-z^{-1}t^{N/2})} \tag{5.8}$$

The space is a complete intersection and from the Hilbert series we can read off that there are three generators: the complex scalar $\Phi$ which corresponds to the first factor in the denominator and is neutral under the $U(1)_J$ as expected, a monopole operator $V_{+1}$ of magnetic charge $+1$ with fugacity $zt^{N/2}$ corresponding to the second factor in the denominator and a monopole operator $V_{+1}$ of magnetic charge $-1$ with fugacity $z^{-1}t^{N/2}$ corresponding to the last factor in the denominator. They obey a relation $V_{+1}V_{-1} = \Phi^N$

Note that the fugacity $t$ can be replaced by, say, a fugacity $s^2$ to cast the generating function in a suitable form. This is a useful fact to keep in mind when we deal with mirror symmetry.

#### 5.2.1.1 $U(1)$ with 2 flavours

For this special case, we have a set of three chiral fields $\{\Phi, \Lambda_{\pm}\}$which have the same scaling dimension. Indeed, since $\Delta(m) =\mid m \mid$, there are only two operators that have $\Delta = 1$, the ones carrying charge $m = \pm 1$, i.e. precisely the vortex-creating operators defined above.

The most general monomial formed out of the three chiral operators is $\Phi^n\Lambda^m$ where $n$ is a non-negative integer and $m$ is any integer. The the scaling dimension $\varsigma$ of a general monomial will be $\varsigma = n + |m|$. For example monomials with dimension $\varsigma = 1$ are $\{\Phi, \Lambda_+, \Lambda_-\}$. Monomials with dimension $\varsigma = 2$ are $\{\Phi^2, \Lambda_+^2, \Lambda_-^2, \Phi\Lambda_+, \Phi\Lambda_-\}$.

Since $\{\Phi, \Lambda_{\pm}\}$ have the same scaling dimension and are complex they form a natural triplet [2] of an $SU(2)$. The three chiral fields transformation under $U(1)_L \times U(1)_J$ , where $U(1)_L$ is the Cartan subalgebra of the $R-symmetry\ SU(2)_L$, can be written as

$$\begin{aligned} \Lambda_+ &\to (1,2) \\ \Phi &\to (1,0) \\ \Lambda_- &\to (1,-2) \end{aligned} \tag{5.9}$$

where the first number in the bracket is the scaling dimension and the second is their charge under $U(1)_J$ with fugacity $q^2$.

By means of symmetric product of [2] we can obtain all monomials with scaling dimension $\varsigma$, e.g $Sym^2[2] = [4] + [0]$ gives the operators at order 2. From this we infer that a relation must exist between these operators ([4] is the 5d representation of $SU(2)$, which completes the count of operators with dimension $\varsigma = 2$). The singlet must then represent a relation at degree 2 which can only be: $\Phi^2 = \Lambda_+\Lambda_-$, in order for the relation to be neutral under $U(1)_J$. The Hilbert series can be written at once as we showed in (2.33); however we note that since the generators are here at order 1, we take fugacity $t$ and not $t^2$. Then

$$HS^C(t,x) = (1-t)^2 PE[dim\,([2])\,t] = \frac{1-t^2}{(1-t)^3} \tag{5.10}$$

Alternatively let $t$ count $\Delta(n,m)$ and let $x$ count the weights of highest weight representation of symmetric products of [2], then the generating function of monomials with charge $\Delta$ is:

$$g(t,x) = \frac{1}{1-t}\sum_{m=-\infty}^{\infty} t^{|m|}q^{2m} \tag{5.11}$$

$$= \frac{1}{1-t}\left(\frac{1}{1-tq^2} + \sum_{m=-\infty}^{-1} t^{|m|}q^{2m}\right) \tag{5.12}$$

$$= \frac{1-t^2}{(1-t)(1-tq^2)(1-tq^{-2})} \tag{5.13}$$

which shows an enhancement of the symmetry $U(1)_J$ to $SU(2)$. This enhancement is unique to $U(1)$ with 2 flavours, since the theory happens to be self-mirror. We will show this using branes in chapter 7.

### 5.2.2 Coulomb branch for the $\hat{A}_2$ quiver gauge theory

Consider now the $\hat{A}_2$ quiver gauge theory as shown in Fig. **??** on the left. It consists of a gauge group $U(1)^2$ and three hypermultiplets in the bifundamental representation of adjacent $U(1)$ factors. We can open this quiver, turn one gauge node into a flavour node and add an extra flavour node to obtain an equivalent gauge theory shown in Fig. **??** on the right.

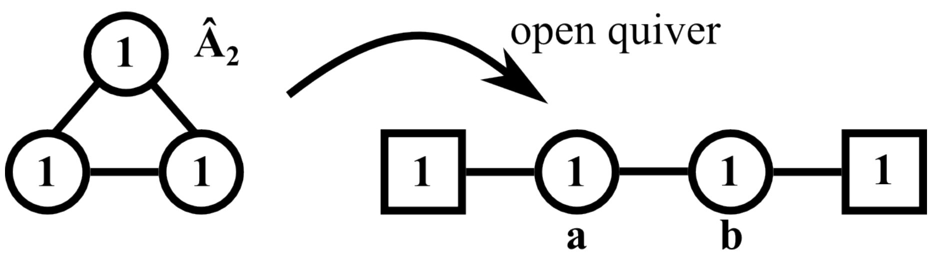



Figure 5.2: $\hat{A}_3$ quiver diagram on the left. Its open version on the right

The dimension of the Coulomb branch is $dim(\mathcal{M}_C) = 3 - 2 = 1$ quaternionic. Since the gauge group is $U(1)^2$, a topological global symmetry $U(1)^2_J$ arises, with conserved charges $a$, $b$. Monopole operators $\Lambda = \Lambda(a,b)$ have conformal dimension dependent on these magnetic charges. The matter sector contributes to $\Delta$ as follows:

$$\Delta_{matter} = \frac{1}{2}\left(|a| + |b| + |a-b|\right) \tag{5.14}$$

and since the gauge sector gives vanishing contribution, the monopole operators $\Lambda$ have conformal dimension $\Delta(a,b) = \Delta_{matter}$ and are gauge invariant as previously discussed. Recall that monopole operators are constructed using the dual photon and one of the scalars which appears in the triplet $\overrightarrow{\phi}$ of $SU(2)_V$ . The remaining two real scalars, $\phi_1$, $\phi_2$ are also gauge invariant since the gauge group is abelian. These contribute to the chiral ring by means of nonnegative powers of them. Hence a function that generates gauge invariant monomials is:

$$HS^H_{\widehat{A}_3}(t) = \sum_{n_1,n_2=0}^{\infty} \sum_{a,b=-\infty}^{\infty} t^{n_1+n_2+\Delta(a,b)} \tag{5.15}$$

$$= \frac{1}{(1-t)^2} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} t^{\frac{1}{2}(|a|+|b|+|a-b|)} \tag{5.16}$$

$$= \frac{1+4t+t^2}{(1-t)^4} \tag{5.17}$$

$$= \sum_{k=0}^{\infty} (k+1)^3 t^k \tag{5.18}$$

Notice that $g(t) = \sum_{k=0}^{\infty} dim[k,k]_{SU(3)} t^k$ . This nonabelian symmetry enhancement can be made more explicit by refining the Hilbert series. We introduce fugacities $z_1$, $z_2$, one for each topological charge $a$, $b$ and modify (5.15) as follows

$$HS^H_{\widehat{A}_3}(t;\, z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} t^{n_1+n_2+\Delta(a,b)} z_1^a z_2^b \tag{5.19}$$

$$= 1 + (2 + \frac{1}{z_1} + z_1 + \frac{1}{z_2} + z_2 + \frac{1}{z_1 z_2} + z_1 z_2)t + ... \tag{5.20}$$

By using the fugacity map $z_1 = \frac{y_2}{y_1^2}$, $z_2 = y_1 y_2$ we can recognise that the coefficient of the linear term is the character of the adjoint of $SU(3)$. This fugacity map can be extended to all coefficients to arrive at (5.23):

$$HS^H_{\widehat{A}_3}(t;\, y_1, y) = 1 + \left( y_1 y_2 + \frac{y_2^2}{y_1} + \frac{y_1^2}{y_2} + \frac{y_1}{y_2^2} + \frac{y_2}{y_1^2} + \frac{1}{y_1 y_2} + 2 \right) t + .. \tag{5.21}$$

$$= 1 + [1,1]_{SU(3)} t + .. \tag{5.22}$$

$$= \sum_{k=0}^{\infty} [k,k]_{SU(3)} t^k \tag{5.23}$$

The current theory fully displays an $SU(3)$ symmetry enhancement through the use of monopole operators. This is a type of hidden symmetry that we had hinted at and its appearance will become even more clear when we discuss mirror symmetry in the next chapter.

## 5.3 Coulomb branch for non abelian gauge groups: the classical dressing

Here we need to introduce a modification to the computation of the Hilbert series since the gauge group is nonabelian and we need to account for the presence of a Weyl group of symmetry. Our formula so far for the computation of the unrefined Hilbert series for the Coulomb branch of a theory with abelian gauge group $G$ with rank $r$ has been:

$$HS^C(t) = \frac{1}{(1-t)^r} \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_r=-\infty}^{\infty} t^{\Delta(\vec{m})} \tag{5.24}$$

Nonetheless for nonabelian gauge group the set of integers $\{m_i\}$ that solve $\Delta(\vec{m}) = j$ with $j \in \mathbb{N}$ is larger than it should be, since the action of the Weyl group is not taken into account. In order to understand this issue we analyse the quiver gauge theory of $\hat{D}_4$ as shown in Fig. 5.3. The gauge group is $U(1)^3 \times U(2)$. There are 4 hypermultiplets, each in the fundamental of $U(2)$ and charged under the relevant $U(1)$.

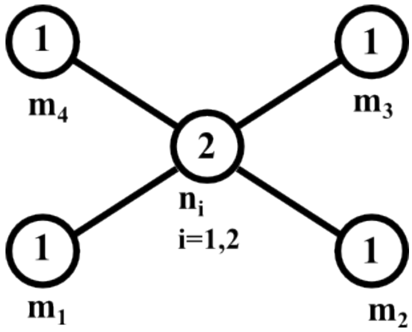



Figure 5.3: The gauge theory constructed out of $\hat{D}_4$. The nodes labelled 1 correspond to $U(1)$ factors and the central node labelled 2 corresponds to a $U(2)$ factor. The hypermultiplets are represented by the links between the nodes and are in the bifundamental of the nodes they join. The letters under the $U(1)$ factors $m_1$, $m_2$, $m_3$, $m_4$ signify four topological charges. The letters under the $U(2)$ node $n_1$, $n_2$ signify the monopole charges: a linear combination of these $n_1 + n_2$ is a global conserved charge, corresponding to the centre $U(1)$ of $U(2)$. The remaining independent linear combination is not conserved and it is associated to the maximal torus of $SU(2)$.

We assign magnetic charges $m_1$, $m_2$, $m_3$, $m_4$ to the $U(1)^4_J$ topological hidden symmetry and charges $n_1$, $n_2$ to the maximal torus of $U(2)$. Since $U(2)$ has a $U(1)$ factor, we can extrapolate another topologically conserved charge: $n_1 + n_2$. The conformal dimension $\Delta$ of monopole operators $\Lambda$ is a sum of a positive contribution coming from the hypermultiplets and a negative contribution from the vector multiplet. Firstly we need to set one of the topological charges to zero, i.e working modulo $U(1)_J$. To understand the reason behind this procedure one has to realize that the energy $\Delta(m_i)$ is invariant under a rigid shift of the magnetic charges. If we didn't fix one of the topological charges, there would be a continuous of solution for the conformal dimension. Using the formula 5.3 we can write the conformal dimension:

$$\Delta_{unfixed} = \frac{1}{2} \sum_{i=1,2} \sum_{j=1,2,3,4} |m_j - n_i| - |n_1 - n_2| \tag{5.25}$$

Let $\vec{m}^1 = (m_1^1, m_2^1, m_3^1, m_4^1, n_1^1, n_2^1)$ be one solution of $\Delta = 1$. Since the energy is invariant under a shift $m_i \longrightarrow m_i + \kappa$,$n_i \longrightarrow n_i + \kappa$ for any$\kappa$, we have just found a continuum of solutions.To avoid this we set $m_4 = 0$ and the formula for the conformal dimension becomes

$$\Delta_{\hat{D}_4} = \frac{1}{2} \left( \sum_{i=1,2} \sum_{j=1,2,3} |m_j - n_i| + |n_1| + |n_2| \right) - |n_1 - n_2| \tag{5.26}$$

The monopole operators set is $\{\Lambda_{\overrightarrow{m}}\}$ , where $\overrightarrow{m}$ is the 5-vector $(m_1, m_2, m_3, n_1, n_2)$ with nonnegative integral dimension $\Delta$. In order to count the number of primary monopole operators we find solutions to the equation $\Delta = 1$. The results are summarised in Table 5.1 .

| $m_1 m_2 m_3$; $n_1 n_2$ | $m_1 m_2 m_3$; $n_1 n_2$ | $m_1 m_2 m_3$; $n_1 n_2$ |
|---|---|---|
| 1 0 0 ; 0 0 | 0 0 0 ; 1 0 | 0 0 0 ; 0 1 |
| 0 1 0 ; 0 0 | 1 1 1 ; 1 0 | 1 1 1 ; 0 1 |
| 0 0 1 ; 0 0 | 1 0 0 ; 1 0 | 1 0 0 ; 0 1 |
| 1 1 1 ; 1 1 | 0 1 0 ; 1 0 | 0 1 0 ; 0 1 |
| | 0 0 1 ; 1 0 | 0 0 1 ; 0 1 |
| | 1 1 0 ; 1 0 | 1 1 0 ; 0 1 |
| | 1 0 1 ; 1 0 | 1 0 1 ; 0 1 |
| | 0 1 1 ; 1 0 | 0 1 1 ; 0 1 |

Table 5.1: Shows all solutions to the equation $\Delta(\overrightarrow{m}) = 1$. The total number solutions is twice the one in the table as we can take the same vectors with negative entries. Note that the solutions with $n_1 \neq n_2$ are not invariant under the Weyl group.

Notice how in the first colum $n_1$ and $n_2$ are equal and thus this subset of solutions is invariant under the $\mathbb{Z}_2$ Weyl group of $U(2)$, whose action is just to permute the charges. In the second column instead $n_1 \neq n_2$ and thus the vector $\overrightarrow{m}$ is not $\mathbb{Z}_2$ invariant. The third column is identical to the second but for the fact that the values of $n_1$ and $n_2$ are swapped. We can construct sum and differences of the vectors in the second and third column to obtain $\mathbb{Z}_2$ even (i.e. invariant) vectors and $\mathbb{Z}_2$ odd ones. We discard the latter to obtain a set of 12 possible solutions which are $\mathbb{Z}_2$ invariant. There will also be 12 other solutions by taking the negative of these vectors, which are still solutions. Hence altogether out of an initial set of 40 solutions we select only 24 which are invariant under the Weyl group of the nonabelian factor group $U(2)$.

In fact, what is happening is that we have accounted for the quantum "corrections", without having established a classical background. The magnetic flux $m$ carried by the monopole operators breaks the symmetry group $G$ to a residual gauge group $H_m$.

Gauge invariants under $H_m$ are accounted for by a the so called "classical dressing function", which we define below. We will find the residual group of symmetry for a gauge group $G$ on a case by case basis.

Let $G$ be a non abelian gauge group with rank $r$. Assign the usual monopole charges $m_1, ..., m_r$. The "classical dressing" function is a piecewise constant function defined as follows:

$$P(\overrightarrow{m_i}; t) = \prod_i (1 - t^{\alpha_i})^{-1} \tag{5.27}$$

where $\{\alpha_i\}$ is the set of degrees of the Casimir operators of the residual groups of symmetry of $H_m$. The classical dressing is just a factor which counts the Casimir invariants of the residual gauge group. These Casimir invariants are constructed using the complex scalar $\Phi$ in the vector multiplet. The piecewise linearity becomes explicit in the following examples.

### 5.3.1 Coulomb branch for the $\hat{D}_4$ quiver gauge theory

Let us return to the quiver gauge theory of $\hat{D}_4$ as show in Fig. 5.3 . The nonabelian factor is $H = U(2)$ with assigned monopole charges $n_1$, $n_2$. The group $U(2)$ has two residual symmetry groups depending on whether it is broken or not:

- $\tilde{H}_1 = U(2)$ itself which has two Casimir operators with degrees $\{\alpha_i\} = \{1, 2\}$. This corresponds to $n_1 = n_2$
- $\tilde{H}_2 = U(1)^2$ which has two Casimir operator of degree 1, i.e $\{\alpha_i\} = \{1, 1\}$. This corresponds to $n_1 \neq n_2$

Hence the classical dressing function is:

$$P_2\left(n_1, n_2; t\right) = \begin{cases} \frac{1}{(1-t)(1-t^2)} & if\ n_1 = n_2 \\ \frac{1}{(1-t)(1-t)} & if\ n_1 \neq n_2 \end{cases} \tag{5.28}$$

This classical dressing function can be used as a weight on the Hilbert series and we thus write

$$\begin{aligned} HS^C_{\hat{D}_4}(t) &= \frac{1}{(1-t)^3} \sum_{n_1 \le n_2} \sum_{\substack{m_1, m_2, m_3 \\ \in \mathbb{Z}}} P_2\left(n_1, n_2; t\right) t^{\Delta_{\hat{D}_4}(\overrightarrow{m})} \\ &= \frac{1}{(1-t)^3} \sum_{\substack{m_1, m_2, m_3 \\ n_1 \le n_2 \\ \in \mathbb{Z}}} P_2\left(n_1, n_2; t\right) t^{\frac{1}{2}\left(\sum_{i,j} |m_j - n_i| + |n_1| + |n_2|\right) - |n_1 - n_2|} \end{aligned} \tag{5.29}$$

where the factor of $\frac{1}{(1-t)^3}$ accounts for the remaining three complex scalars $\Phi_1$, $\Phi_2$, $\Phi_3$, each associated with one of the $U(1)$'s. Notice that there also is a complex scalar

$\varphi$ in the adjoint of $U(2)$. The way to think about this is to diagonalise it and consider its eigenvalues: $\varphi = diag(\varphi_1, \varphi_2)$. Alternatively one can choose use the basis of the $Tr(\varphi)$ and $Tr(\varphi^2)$.The Weyl group acts on $\varphi$ by permuting the eigenvalues, hence $Tr(\varphi)$ and $Tr(\varphi^2)$ are Weyl invariant quantities. The classical dressing takes care of both accounting for the Weyl invariance of the monopole operators and for the Weyl invariance of the operators constructed out of the classical fields $Tr(\varphi)$ and $Tr(\varphi^2)$. In order to see the contributions that come only from the classical fields, i.e the fields that appear in the Lagrangian description , it is sufficient to set $\overrightarrow{m} = 0$ and observe how the Hilbert series reduces to that of four scalars at order 1 and one at order 2, corresponding to $\Phi_1$, $\Phi_2$, $\Phi_3$, $Tr(\varphi)$, $Tr(\varphi^2)$. When the Hilbert series is evalauted the result is:

$$HS^C_{\hat{D}_4}(t) = 1 + 28t + 300t^2 + ... \quad (5.30)$$

$$= \sum_{k=0}^{\infty} dim\left([0, k, 0, 0]_{SO(8)}\right) t^k \quad (5.31)$$

which shows the enhancement of the global symmetry to $SO(8)$. To make this enhancement more evident we refine the Hilbert series by introducing fugacities $z_i$ $i = 1, ..4$ for the *conserved* $U(1)_J$ charges. Recall that there are four of these $m_1$, $m_2$, $m_3$ and $(n_1 + n_2)$. The $z_i$ provide a counter for these. Hence we rewrite (5.29) as

$$HS^C_{\hat{D}_4}(t;\, z_1, z_2, z_3, z_4) = \frac{1}{(1-t)^3} \sum_{n_1 \le n_2} \sum_{\substack{m_1, m_2, m_3 \\ \in \mathbb{Z}}} P_2(n_1, n_2; t)\, t^{\Delta_{\hat{D}_4}(\overrightarrow{m})} z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{n_1+n_2} \quad (5.32)$$

We use the Cartan matrix for $D_4$ in order to find a fugacity map between the $z_1$, $z_2$, $z_3$, $z_4$ and the fugacities $y_1$, $y_2$, $y_3$, $y_4$ which give the character of $D_4$ representations in the basis of the fundamental weights, i.e.

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad (5.33)$$

where we read this matrix notation as follows: $z_1 = y_1^2 y_2^{-1}$,$z_2 = y_1^{-1} y_2^2 y_3^{-1} y_4^{-1}$, $z_3 = y_2^{-1} y_3^2$, $z_4 = y_2^{-2} y_4^2$, i.e. th entries of the Cartan matrix correspond to the power of the fugacity $y_i$. After making use of the just found fugacity map, (5.32) yields a series for the characters $[0, k, 0, 0]$ in $y_1, y_2, y_3, y_4$ and thus we can write the fully refined series

$$HS^C_{\hat{D}_4}(t;\, y_1, y_2, y_3, y_4) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{SO(8)} t^k \quad (5.34)$$

which encompasses the symmetry enhancement to $SO(8)$.

### 5.3.2 Coulomb branch for the $\hat{D}_k$ quiver gauge theory

We will now generalise the case of $\hat{D}_4$ to the the quiver gauge theory underlying the affine Dynkin diagram $\hat{D}_k$. The gauge group is the product: $U(1)^4 \times U(2)^{k-3}/U(1)$. A global shift symmetry $U(1)_J^k$ is generated. We assign topological charges $a, b, c, d$ to the the four $U(1)_J$ associated to the legs of the diagrams and monopole charges $n_i, m_i$, $i = 1, ..., k-3$ to the Cartan subalgebra of the nonabelian factors of the gauge group as shown in Fig. 5.4. Note the topological charge $d$ will be set to zero, as we work modulo $U(1)$. Moreover we can construct the additional $k-3$ conserved charges by the centre $U(1)$ of each $U(2)$, i.e. $n_i + m_i$.

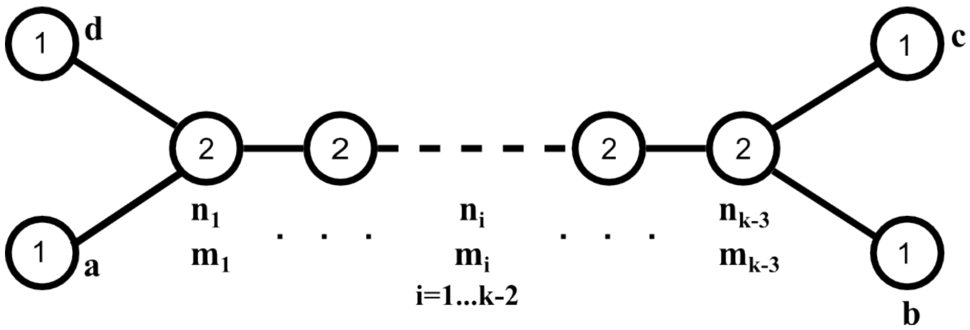



Figure 5.4: The quiver gauge theory associated to $\hat{D}_N$ . To the Cartan subalgebra of each $i^{th}$ factor $U(2)$ monopole charges $m_i, n_i$ are assigned, for $i = 1, ..., k-3$. The $U(1)_J^4$ is assigned topological charges $a, b, c, d$. The remaining conserved topological charges are $n_i + m_i$

Again, for a non abelian gauge group the conformal dimension $\Delta$ will receive contribution both from the vector multiplets and the hyper mutltiplets.

$$\Delta_{gauge} = -\sum_{i=1}^{k-3} |n_i - m_i| \tag{5.35}$$

and

$$\begin{aligned}\Delta_{hyper} = \frac{1}{2}\Bigg\{ &\sum_{i=1}^{k-4} |n_i - n_{i+1}| + |n_i - m_{i+1}| + |m_i - m_{i+1}| + |m_i - n_{i+1}| \\ &+ |b - n_{k-3}| + |b - m_{k-3}| + |c - n_{k-3}| + |c - m_{k-3}| \\ &|a - n_1| + |a - m_1| + |n_1| + |m_1| \Bigg\}\end{aligned} \tag{5.36}$$

where in the last two terms we have set $d = 0$. The conformal dimension is given by the sum of these two expression $\Delta_{\hat{D}_k}(a, b, c, n_i, m_i) = \Delta_{gauge} + \Delta_{hyper}$. The classical dressing function includes a factor (5.28) for each $U(2)$, hence

$$P = \prod_{i=1}^{k-3} P_2(n_i, m_i) \tag{5.37}$$

We assign fugacities $z_1$to the magnetic charge labelled by $a$; $z_{i+1}$ $i = 1, ..., k-3$ to the $k-3$ fugacities labelled by $n_i + m_i$; $z_{k-1}$ to the magnetic charge labelled by $b$ and $z_k$ to the magnetic charge labelled by $c$. The refined HIlbert series is then

$$HS^{C}_{\hat{D}_k}(t;\ z_j) = \frac{1}{(1-t)^3} \sum_{\substack{m_i \leq n_i \\ i=1,...,k-3}} \sum_{\substack{a,b,c \\ \in \mathbb{Z}}} t^{\Delta_{\hat{D}_k}} \prod_{i=1}^{k-3} P_2(n_i, m_i)\, z_1^a z_{i+1}^{m_i+n_i} z_{k-1}^b z_k^c$$

where we recall that the factor in front of the summation corresponds to the polynomial which encodes the three adjoint scalar field associated to the three $U(1)$. The Cartan matrix for $D_k$ can again be used be used to find a map between the $z_i$ to the $y_i$ which are the fugacities used in the to write the character of representations of $D_k$ in the highest weight basis and we obtain the generalisation of (5.34)

$$HS^{C}_{\hat{D}_k}(t;\ y_i) = \sum_{p=0}^{\infty} [0, p, 0, ...]_{SO(2k)}\, t^k$$

with an explicit $SO(2k)$ symmetry enhancement brought about by the monopole operators.

### 5.3.3 Coulomb branch for $U(K)$ with $N$ flavours

The theory is shown in the Fig. 5.5. We assign magnetic charges $m_i$ $i = 1, ..., K$ to the maximal torus of $U(K)$. There is one topologically conserved charge, $\sum_i m_i$ , corresponding to the centre of $U(1)$, and $K-1$ linearly independent monopole charges. The conformal dimension in terms of these is again a sum of the hyper multiplets positive contribution and the vector multiplets negative contribution

$$\Delta = \frac{N}{2} \sum_{i=1} |m_i| - \sum_{i<j} |m_i - m_j| \tag{5.38}$$

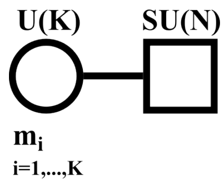



Figure 5.5: The quiver for $U(K)$ with $N$ flavours. The global symmetry is $SU(N)$ since the fundamental representation of a unitary groups is complex. Under the gauge group we have indicated the $K$ monopole charges $m_i$ , $i = 1, ..., K$, where $\sum_i m_i$ is a conserved topological charge.

The classical dressing function is given by (5.27). In order to find the residual symmetry group we can appeal to the brane realisation of symmetry breaking. $U(K)$

is realised by a stack of $Dp$-branes coincident at the same place in spacetime. Pulling branes away one by one, we can get different configurations where the symmetry group is broken as

$$\begin{aligned} U(K) \quad &\longrightarrow \quad \prod_i U(k_i) \\ with \quad & \quad \sum_i k_i = K \end{aligned} \tag{5.39}$$

**K=3** Let us first consider the case of $U(3)$ with $N$ flavours. Then the classical dressing function, $P_3\left(m_1, m_2, m_3; t\right)$ is

$$P_3\left(m_1, m_2, m_3; t\right) = \begin{cases} \frac{1}{(1-t)^3} & m_1 \neq m_2 \neq m_3 \\ \frac{1}{(1-t)(1-t)(1-t^2)} & m_1 \neq m_2 = m_3 \text{ orcyclic} \\ \frac{1}{(1-t)(1-t^2)(1-t^3)} & m_1 = m_2 = m_3 \end{cases} \tag{5.40}$$

which enacts the fact that $U(3)$ can be unbroken ($m_3 = m_1 = m_2$), or break to $U(2) \times U(1)$, when two out of the three $m_i$ are coincident and the other is different, or break to $U(1)^3$ when the $m_i$ are all different. Again we need to project onto the invariant Weyl chamber and we do so by restricting the summations to $m_3 \leq m_1 \leq m_2$. The Hilbert series is then

$$HS^C_{3,N}(t) = \sum_{\substack{m_1 \geq m_2 \geq m_3 \\ \in \mathbb{Z}}} t^{\frac{N}{2}\left(\sum_i |m_i|\right) - \left(\sum_{i<j} |m_i - m_j|\right)} P_3\left(m_1, m_2, m_3; t\right) \tag{5.41}$$

where $i, j = 1, 2, 3$. The Hilbert series was evaluated and found to have the features of a complete intersection moduli space, with the numerator of the rational function to be factorisable

$$HS(\mathcal{M}_C(U(3));\ t) = \frac{\left(1-t^N\right)\left(1-t^{N-1}\right)\left(1-t^{N-2}\right)}{(1-t)(1-t^2)(1-t^3)(1-t^{\frac{N}{2}})^2(1-t^{\frac{N}{2}-1})^2(1-t^{\frac{N}{2}-2})^2} \tag{5.42}$$

Note that the first three generators correspond to the Casimir operators of degree $1, 2, 3$ and the remaining are monopole operators.

### 5.3.4 Coulomb branch for the $\hat{E}_6$ quiver gauge theory

The theory with gauge group given by the affine quiver $\hat{E}_6$ is shown in Fig. 5.6 . We have assign magnetic charges to each factor group. These are all unitary groups $U(K_i)$ with $K_i$ as specified in each node. The gauge group is taken modulo the sum of the $U(1)$ generators, hence $U(1)^2 \times U(2)^3 \times U(3)$ with the hypermultiplets in the bifundamental of adjacent factors.

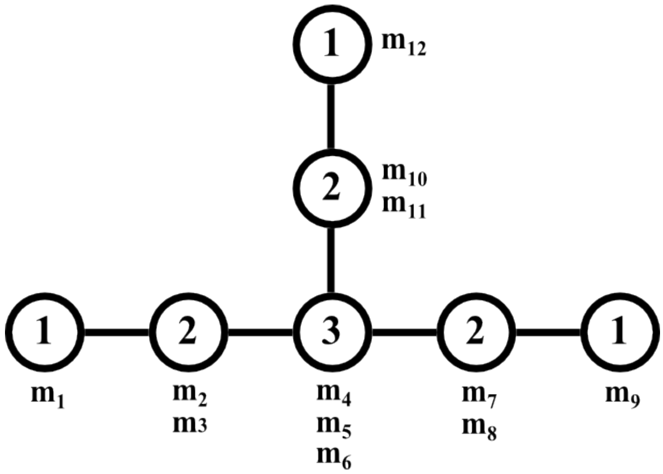



Figure 5.6: The quiver theory with affine $E_6$ diagram. Under each gauge factor we show the magnetic charges assigned. Note that $m_{12}$ is set to zero. Moreover $m_1$, $m_2$, $m_4$, $m_7$, $m_9$, $m_{10}$ are topological and thus conserved charges. The rest are monopole charges.

Let us write the contribution from the hyper multiplets to the conformal dimension. We do so by taking the absolute values of the difference of magnetic charges separated by a link. We then get:

$$\begin{aligned}\Delta_{hyper} = \frac{1}{2}\big(&|m_1-m_2|+|m_1-m_3|+\\ &|m_2-m_4|+|m_2-m_5|+|m_2-m_6|+\\ &|m_3-m_4|+|m_3-m_5|+|m_3-m_6|+\\ &|m_7-m_4|+|m_7-m_5|+|m_7-m_6|+\\ &|m_8-m_4|+|m_8-m_5|+|m_8-m_6|+\\ &|m_7-m_9|+|m_8-m_9|+\\ &|m_{10}-m_4|+|m_{10}-m_5|+|m_{10}-m_6|+\\ &|m_{11}-m_4|+|m_{11}-m_5|+|m_{11}-m_6|+\\ &|m_{10}|+|m_{11}|\big)\end{aligned} \tag{5.43}$$

where in the last line we have taken $m_{12}$ to vanish. The negative contribution to the conformal dimension from the V-plet arises through the nonabelian gauge factors. For each nonabelian factor we take differences of its assigned magnetic charges. Thus

$$\Delta_{gauge} = -\left(|m_2-m_3|+|m_4-m_5|+|m_4-m_6|+|m_5-m_6|+|m_7-m_8|+|m_{10}-m_{11}|\right) \tag{5.44}$$

and the conformal dimension is just the sum

$$\Delta_{\hat{E}_6}(m_i) = \Delta_{hyper} + \Delta_{gauge}$$

The classical dressing function is the product of the classical dressing function for each nonabelian gauge factor. For each of the three$U(2)$ factors it is Eq. 5.28

where $n_1$and $n_2$ must be changed according to the magnetic charges at hand. For the $U(3)$ factor, the classical dressing function is Eq.(5.40) with $m_1 \longrightarrow m_4$, $m_2 \longrightarrow m_5$, $m_3 \longrightarrow m_6$. Then

$$P(m_i) = P_2(m_2, m_3)P_3(m_4, m_5, m_6)P_2(m_7, m_8)P_2(m_{10}, m_{11}) \tag{5.45}$$

For each summation over the magnetic charges of nonabelian factors one needs to restrict to the sector where the magnetic charges are ordered, so as not to double count solutions. The Hilbert series for the Coulomb branch of the affine $E_6$quiver gauge theory is then

$$HS\left(\mathcal{M}_C(K_{\hat{E}_6});\, t\right) = \sum_{\substack{m_1,\ldots,m_{11} = -\infty \\ m_3 \geq m_2 \\ m_6 \geq m_5 \geq m_4 \\ m_8 \geq m_7 \\ m_{11} \geq m_{10}}}^{\infty} t^{\Delta(m_i)} P(m_i) \tag{5.46}$$

The eleven nested sums in the expression above were evalauted up to and including order $t^3$

$$\begin{aligned} HS\left(\mathcal{M}_C(K_{\hat{E}_6});\, t\right) &= 1 + 78t + 2430t^2 + 43758t^3 + \ldots && (5.47) \\ &= \sum_{k=0}^{\infty} dim\left([0,k,0,0,0,0]_{E_6}\right) t^k && (5.48) \end{aligned}$$

which allows us to demonstrate an enhancement of the global symmetry of the Coulomb branch to $E_6$.

### 5.3.5 Coulomb branch for $G_2$ with $N$ flavours

In this subsection we present an example of how to compute the Coulomb branch of gauge theories when the nonabelian gauge group is not a unitary group. Let us in particular consider $G_2$ with $N$ flavours, whose quiver we show in Fig. ??.

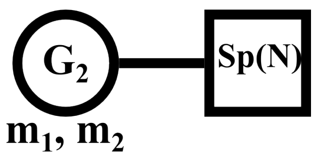



Figure 5.7: The quiver theory of $G_2$with $N$ flavours. The fundamental representation of $G_2$is real, hence the global symmetry is $Sp(N)$. The rank of $G_2$ is two, thus there are two monopole charges, $m_1$ and $m_2$.

Let $H_1$, $H_2$ be two Cartan generators of $G_2$. Choose the Cartan subalgebra$\mathfrak{h} = \mathcal{L}\left(U(1)^2\right)$ in $G_2$ through an embedding$\exp(i\alpha) \hookrightarrow \exp(i\alpha H_1)$ and $\exp(i\beta) \hookrightarrow \exp(i\beta H_2)$

where the$\alpha$, $\beta$ are the simple roots of $G_2$, i.e they are the basis elements of the dual space $\mathfrak{h}*$. Consider an arbitrary element of the Cartan subalgebra and write it as a linear combination of the basis $\{H_1, H_2\}$

$$H = mH_1 + nH_2 \tag{5.49}$$

The negative contribution from the vector multiplet is the sum of the roots in terms of $H$, $\alpha_i(H)$

$$\Delta_{V-plet} = -\sum |\alpha_i(H)| \tag{5.50}$$

For $G_2$ a set of roots is $\alpha$, $\beta$, $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, $3\alpha+2\beta$. Then

$$\Delta_{V-plet} = -\left(|m| + |n| + |m+n| + |2m+n| + |3m+n| + |3m+2n|\right) \tag{5.51}$$

The contribution from the matter depends on which representation of the gauge group the hyper multiplet is in, which here is the fundamental. For each of the $2N$ flavours, there are $dim(fundam)$ terms, one for each weight, and an overall factor of 1/2:

$$\Delta_{h-plet} = \frac{(2N)}{2} \sum |w_i(H)| \tag{5.52}$$

Let us first find the fundamental weights of $G_2$. From the inverse Cartan matrix

$$A^{-1}_{ij} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \tag{5.53}$$

we find the fundamental weights $\Lambda_i = A^{-1}_{ij}\alpha_j$, $\Lambda_1 = 2\alpha + \beta$ and $\Lambda_2 = 3\alpha + 2\beta$. $\Lambda_1$ is the highest weight of the fundamental representation of $G_2$, which is seven dimensional. We find the weights of this representation by acting with the roots on the highest weight: $w_i = \{2\alpha+\beta, \alpha+\beta, \alpha, -2\alpha-\beta, -\alpha-\beta, -\alpha, 0\}$. Then the hyper contribution is

$$\Delta_{V-plet} = 2N\left(|2m+n| + |m+n| + |m|\right) \tag{5.54}$$

and then the conformal dimension is thus a sum of the vector multiplet and hyper-multiplet contribution

$$\begin{aligned} \Delta_{G_2}(m,n) &= 2N\left(|2m+n| + |m+n| + |m|\right) \\ &\quad - \left(|m| + |n| + |m+n| + |2m+n| + |3m+n| + |3m+2n|\right) \end{aligned} \tag{5.55}$$

Let us now proceed to compute the classical dressing function $P(m, n)$. Recall that this is intimately related to the Weyl symmetry of the gauge group. For $G_2$, $W = D_6$, the dihedral group of order $|W| = 12$. The two Casimir operators of $G_2$ have order $2, 6$. In Fig. 5.8 we show the root lattice and the dual lattice for $G_2$.

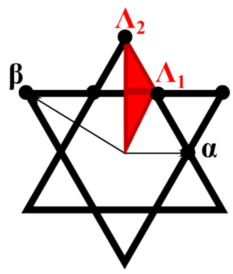



Figure 5.8: The roos system for $G_2$. $\alpha, \beta$ are the two simple roots, which identify the root lattice. The black dots identify the six positive roots of $G_2$. $\Lambda_1, \Lambda_2$ are the fundamental weights, i.e. the basis for the dual lattice. The region shaded in red is the Weyl chamber. One can notice that the Weyl group acts on it by rotating/reflecting it to equivalent chambers.

The fundamental weights delimitate the Weyl chamber, i.e the region the span the whole root system when acted upon by the Weyl group. The boundaries and the interior of the Weyl chamber are the "locations" of symmetry breaking/enhancement:

- in the interior of the Weyl chamber $G_2$is maximally broken $\Longrightarrow$ residual symmetry group is $U(1)^2$ with two Casimir operators of degree $\{1, 1\}$
- at the two boundaries of the Weyl chamber $G_2 \Longrightarrow$ residual symmetry group is $U(2)$ with two Casimir operators of degree $\{1, 2\}$
- at the boundary of the boundaries (the centre) of the Weyl chamber $G_2$is unbroken $\Longrightarrow$ residual symmetry group is $G_2$ with two Casimir operators of degree $\{2, 6\}$

The classical dressing function is then

$$P_{G_2}(m,n;t) = \begin{cases} \frac{1}{(1-t)^2} & m>0,\ n>0 \\ \frac{1}{(1-t)(1-t^2)} & m>0,\ n=0 \ or \\ & n>0,\ m=0 \\ \frac{1}{(1-t^2)(1-t^6)} & m=n=0 \end{cases} \tag{5.56}$$

where we have chosen one of the twelve identical Weyl chamber so that the boundaries occur at the chosen values. The Hilbert series is then

$$HS(\mathcal{M}_\mathcal{C}(G_2);\ t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} t^{\Delta_{G_2}(m,n)} P_{G_2}(m,n;t) \tag{5.57}$$

$$= \frac{1+t^{4n-5}+t^{2n-4}+t^{2n-3}+t^{2n-2}+t^{2n-1}}{(1-t^2)(1-t^6)(1-t^{2n-5})(1-t^{2n-6})} \tag{5.58}$$

### 5.3.6 Coulomb branch for $SU(2)$ with $N$ flavours

Finally we compute the Hilbert series for the Coulomb branch of $SU(2)$ with $N$ flavours, since the result will be used in the next section. The formula for the conformal dimension of the monopole operators, $\Delta$, can be written by taking into account contributions from the gauge sector and the matter sector.

The gauge contribution is as follows. Consider a gauge group$U(2)$ instead of $SU(2)$. Assign monopole charges $m_1$, $m_2$ to the Cartan subalgebra $U(1)^2$of $U(2)$. The contribution to the conformal dimension from the gauge sector for $U(2)$ is then: $-|m_1 - m_2|$. Since the gauge group is $SU(2)$,the traceless condition requires that $m_1 + m_2 = 0$. Set $m_1 = m$ and get a contribuiton:. $\Delta_{gauge} = -2|m|$. The matter sector gives a positive contribution to the conformal dimension [4]. Each flavour contributes a factor of $\frac{1}{2}$. Thus for the case of $SU(2)$the matter contribution is: $\Delta_{matter} = N|m|$ . Hence the scaling dimension of monopole operators is

$$\Delta_{SU(2)} = \Delta_{matter} + \Delta_{gauge} = (N-2)|m| \tag{5.59}$$

Since $SU(2)$ is nonabelian, we need to implement the classical dressing function to account for the residual group of symmetry after breaking of $SU(2)$. $SU(2)$ can only break to $U(1)$ when $m \neq 0$ with Casimir invariant of degree $\alpha = 1$, or remain unbroken with Casimir invariant of degree $\alpha = 2$ , hence we can write the dressing function

$$P(m;t) = \begin{cases} \frac{1}{(1-t^2)} & if\ m = 0 \\ \frac{1}{(1-t)} & if\ m \neq 0 \end{cases} \tag{5.60}$$

Moreover one needs to account for the $\mathbb{Z}_2$ Weyl group in the summation. The projection from $m \in \mathbb{Z}$ to $m \in \mathbb{N}$ takes care of this. Thus the Hilbert series is:

$$\begin{aligned} HS(\mathcal{M}_C(SU(2));t) &= \sum_{m=0}^{\infty} P(m;\,t)\,t^{\Delta(m)} && (5.61) \\ &= \frac{1}{(1-t^2)} + \frac{1}{(1-t)}\sum_{m=0}^{\infty} t^{(N-2)m} && (5.62) \\ &= \frac{1-t^{2N-2}}{(1-t^2)(1-t^{N-1})(1-t^{N-2})} && (5.63) \end{aligned}$$

where in the last line we have cast the rational function in a way which will be useful in the next section.

[4]Recall that for a gauge symmetry whose fundamental representation is pseudoreal (e.g. $SU(2)$) the global symmetry for $N_f$ flavours is $SO(2N_f)$

# Chapter 6

# Mirror Symmetry for $\mathcal{N} = 4$ gauge theories in $3d$

## 6.1 The Kronheimer gauge theories

Seiberg and Intriligator illustrated $\mathcal{N}$ =4 $3d$ mirror symmetry for a special class of gauge theories named after Kronheimer who proposed them in [31]. The Kronheimer gauge group $K_G$ is defined as:

$$K_G \equiv \frac{\prod_{i=0}^{r} U(n_i)}{U(1)} \tag{6.1}$$

where $G$ is an ADE-group, $i$ labels the node of the extended Dynkin diagram of $G$ and $n_i$ is the Dynkin index of the $i^{th}$ node where $n_0 = 1$ corresponds to the extended node. These products of $U(n_i)$ gauge theories are exactly the ADE-quiver gauge theories discussed at the end of the chapter 4, with the sole exception that the $U(1)$ factor corresponding to the sum of all the $U(1)'s$ in the $U(n_i)$ is gauged away. The matter fields are taken to be bifundamental of adjacent gauge group factors:

$$\bigoplus_{ij} a_{ij}(\mathbf{n_i},\mathbf{n_j}) \tag{6.2}$$

where

$$a_{ij} = \begin{cases} 1 & i,j \text{ are connected} \\ 0 & else \end{cases} \tag{6.3}$$

and $(\mathbf{n_i},\mathbf{n_j})$ denotes the bifundamental representation of $U(n_i)\times U(n_j)$, with dimension $n_i n_j$.

The relation between these gauge theories, simply laced extended Dynkin diagrams and ALE spaces was explained previously. In particular Kronheimer argues that the Higgs branch, $\mathcal{M}_H$ , of an ADE-quiver is an ALE space with an ADE singularity, $\frac{\mathbb{C}^2}{\Gamma}$. The Higgs branch can then be described in terms of the Hilbert series of section 4.3.

### Dimension counting

Firstly, we expect from the claim above that the Higgs branch be 2 complex dimensional, i.e.one quaternionic dimensional, since ALE spaces are. Using Eq. 3.11, we compute $dim(\mathcal{M}_H)$ for each $G = ADE$ and we show the result in the first column of Table 6.1 . Dimension counting will be the only check in this dissertation that Kronheimer claim is true. For an advanced mathematical treatment the reader is referred to the original paper [31].

The Coulomb branch is instead given by counting the number of vector multiplets as we have previously stated, i.e $dim(\mathcal{M}_\mathcal{C}) = rank(K_G)$. For the Kronheimer groups is also true that

$$rank(K_G) = C_2(G) - 1 \tag{6.4}$$

where $C_2(G)$ is the dual Coxeter number of the group $G$. Then the Coulomb branch is classically $(\mathbb{R}^3 \times S^1)^{C_2(G)-1}$. We list the dimension of the Coulomb branch in the second column of Table 6.1 .

| Quiver | $dim(\mathcal{M}_H)$ | $dim(\mathcal{M}_C)$ |
|---|---|---|
| $A_n$ | $[n-1] - n = 1$ | $n-1$ |
| $D_n$ | $[2 \times 4 + 4(n-4)] - [3 + 4(n-3)] = 1$ | $2n-3$ |
| $E_6$ | $[3 \times 2 + 3 \times 6] - [1 + 4 + 9 + 4 + 1] = 1$ | 11 |
| $E_7$ | $[2(2 + 6 + 12) + 8] - [2(4 + 9) + 4 + 16 + 1] = 1$ | 17 |
| $E_8$ | $[2 + 6 + 12 + 20 + 30 + 18 + 24 + 8] - [4 + 9 + 16 + 25 + 36 + 16 + 4 + 9] = 1$ | 29 |

Table 6.1: In the first column we check that $dim(\mathcal{M}_H)$ for each $G = ADE$ is one dimensional. In the second column we write $dim(\mathcal{M}_C)$ , which is just the rank of $K_G$

The Coulomb branch gets quantum corrections. [35] argued that such corrections modify $(\mathbb{R}^3 \times S^1)$ to become the moduli space of the corresponding $G$-instanton. In a sense this is a statement of mirror symmetry since the underlying assumption is that the theory is dual to another whose Higgs branch *is* the moduli space of $G$-instanton. We will explain this more carefully by means of the two canonical examples presented in [25]. There are many more mirror symmetric dual theories which have been found by brane constructions and discussed in **(author?)** [11, 10] but we will not be considered here.

## 6.2 Mirror Symmetry $K_{SU(n)}$

Our analysis of mirror pairs commences by considering theories with gauge group $K_{SU(n)} = U(1)^{n-1}$and $n$ hypermultiplets $Q_i$. These have charge +1 under $U(1)_i$ and charge -1 under $U(1)_{i+1}$, $i = 0, ..n-1$ where $U(1)_0 \equiv U(1)_n$. We call this the $A$-theory. We can also turn on FI terms (which recall are triplets of $SU(2)_R$) , one for each $U(1)$of the gauge group, thus in this instance $n-1$ of them: $\vec{\zeta}_i$, $i = 0, ..., n-1$ with $\sum_i \vec{\zeta}_i = 0$. We can also add mass terms $\vec{m}_j$ (triplets again but of $SU(2)_L$) for the

$n$ hypermultiplets. Linear combinations of these masses can nonetheless be removed by shifting the Coulomb branch for each $U(1)$ gauge factor. We are left thus with only one $\vec{m}$. For the time being we don't turn on any of these. From table 6.1 we know that

$$dim(\mathcal{M}_H^A) = 1 \quad (6.5)$$

$$dim(\mathcal{M}_C^A) = n - 1 \quad (6.6)$$

**Claim**

At the IR non-trivial superconformal fixed point $U(1)^{n-1}$ with $n$ hypermultiplets is dual to $U(1)$ with $n$ flavours, which we name the $B$-theory.

**Dimension checking**

$U(1)$ with $n$ flavours has:

$$\begin{aligned} dim(\mathcal{M}_H^B) &= n - 1 \quad (6.7) \\ dim(\mathcal{M}_C^B) &= 1 \quad (6.8) \end{aligned}$$

and thus dimensions cross-match. Moreover $U(1)$ with $n$ flavours allows for one FI term $\vec{\zeta'}$ and $n-1$ mass terms $\vec{m_i'}$ , $i = 0, ..n-1$ with $\sum_i \vec{m_i'} = 0$. Thus mirror symmetry when FI and/or mass terms are turned on claims that

$$\vec{m} \Longleftrightarrow \vec{\zeta'} \quad (6.9)$$

$$\vec{\zeta_i} \Longleftrightarrow \vec{m_i'} \quad (6.10)$$

At this point the paper of Seiberg and Intriligator proceeds by analysing the metric on the Higgs branch and Coulomb branch of both theories. They then cross-identify the metrics on the spaces in the IR, i.e. in the limit of strong-coupling. Here we don't follow this approach relying instead on the more modern technique of identifying the generating function for the ring of gauge invariant operators for the Higgs branch and the Coulomb branch of both theories and show cross-matching. We make use of unrefined Hilbert series for simplicity

Consider the duality for $n = 3$, i.e. our proposed mirror theories are:

- theory A: $U(1)^2$ with 3 hypermultiplets
- theory B: $U(1)$ with 3 flavours

The Higgs branches are known to the reader:

- $\mathcal{M}_{\mathcal{H}}^{\mathcal{A}}$ is the ALE space $\frac{\mathbb{C}^2}{\mathbb{Z}_3}$ as suggested by Kronheimer and the Hilbert series is Eq. 4.12 with $n = 2$

$$HS(\frac{\mathbb{C}^2}{\mathbb{Z}_3}; t) = \frac{1 - t^6}{(1 - t^3)(1 - t^3)(1 - t^2)} \quad (6.11)$$

- $\mathcal{M}_{\mathcal{H}}^{\mathcal{B}}$ is the moduli space of $1-SU(3)$ instanton and the Hilbert series is Eq. () as computed in section [2]

$$HS_{1-SU(3)}(t) = \sum_{k=0}^{\infty} dim[k,k]_{SU(3)} t^k \tag{6.12}$$

The Coulomb branches of both theories are also known to the reader since they were computed in section 5 using monopole operators:

- the Hilbert series of $\mathcal{M}_{\mathcal{C}}^{\mathcal{A}}$ was computed in section 5 subsection [] and we notice that it is indeed the $1-SU(3)$ instanton moduli space, i.e the Higgs branch of $U(1)$ with 3 flavours

$$HS_C^A(t) = \sum_{k=0}^{\infty} dim[k,k]_{SU(3)} t^k = HS_{1,SU(3)}(t) = HS_H^B(t) \tag{6.13}$$

- the Hilbert series for $\mathcal{M}_{\mathcal{C}}^{\mathcal{B}}$ was also computed in section 5 subsection [] and we notice that it is identical to the Hilbert series for the ALE space $\frac{\mathbb{C}^2}{\mathbb{Z}_3}$, i.e. the Higgs branch the $A$-theory

$$HS_C^B(t) = \frac{1-t^6}{(1-t^3)(1-t^3)(1-t^2)} = HS(\frac{\mathbb{C}^2}{\mathbb{Z}_3};t) = HS_H^A(t) \tag{6.14}$$

Matching of the chiral rings by means of comparing the Hilbert series for the Coulomb branch and the Higgs branch confirms the statements of mirror symmetry, at least as far as the moduli space of vacua of dual theories is concerned.

## 6.3 Mirror Symmetry $K_{SO(2N)}$

We now examine the theory with Kronheimer gauge group $K_{D_N}$. The quiver diagram for this gauge theory is the affine Dynkin diagram $\hat{D}_N$ as shown in Fig. 4.1b where the $r_i$ indices label the dimensions of $Dic_N$ by the McKay correspondence. Mapping to quiver gauge theory, each node with label $r_i$ corresponds to a factor $U(r_i)$ in the gauge group. Hence

$$K_{SO(2N)} = \frac{U(1)^4 \times U(2)^{n-3}}{U(1)} = U(1)^3 \times U(2)^{n-3} \tag{6.15}$$

The matter content is as follows:

- 2 doublets of the leftmost $U(2)$, one with charge $+1$ under the top left $U(1)$ and the other with charge $U(1)$ under the bottom left $U(1)$
- 2 doublets of the rightmost $U(2)$, one with charge $+1$ under the top right $U(1)$ and the other with charge $U(1)$ under the bottom right $U(1)$

- $(n-3)$ bifundamentals $(2,\bar{2})$of neighbouring $U(2)$'s

The dimensions for the Higgs and the Coulomb branch were computed in Table 6.1 and are respectively $dim(\mathcal{M}_H) = 1$ and $dim(\mathcal{M}_C) = 2n-3$. This is the $A$-theory.

**Claim**

In the IR $K_{SO(2N)}$ flows to a non-trivial fixed point where it is dual to the theory $SU(2)$ with $N$ flavours.

**Dimension checking**

For $SU(2)$ with $N$ flavours, our the $B$-theory, the Higgs and Coulomb branch[1] have dimensions:

$$dim(\mathcal{M}_H^B) = rank(SU(2)) = 1 \tag{6.16}$$

$$dim(\mathcal{M}_C^B) = \frac{1}{2}(2\times 2n) - dim(SU(2)) = 2n-3 \tag{6.17}$$

and the dimensions of the two pieces of moduli space of $SU(2)$ with $N$ flavours cross match those of $K_{SO(2N)}$, thus providing dimensional evidence to the claim. Consider the duality for $N=4$; the proposed mirror theories are:

- theory A: $U(1)^4 \times U(2)$ with 8 hypermultiplets according to the prescription above
- theory B: $SU(2)$ with 4 flavours

The Higgs branches are as follows:

- $\mathcal{M}_{\mathcal{H}}^{\mathcal{A}}$ is the ALE space $\frac{\mathbb{C}^2}{Dic_2}$ as suggested by Kronheimer and the Hilbert series is Eq. 4.13 with $n=4$

$$HS(\frac{\mathbb{C}^2}{Dic_2};t) = \frac{1-t^6}{(1-t^2)(1-t^3)(1-t^2)} \tag{6.18}$$

- $\mathcal{M}_{\mathcal{H}}^{\mathcal{B}}$ is the moduli space of $1-SO(8)$ instanton and the Hilbert series is Eq. as computed in section [2]

$$HS_{1,SO(8)}(t) = \sum_{k=0}^{\infty} dim[0,k,0,0]_{SO(8)}t^k \tag{6.19}$$

The Coulomb branches of both theories were computed in section 5 using monopole operators:

[1]The factor of $\frac{1}{2}$ in front of the number of hypermultiplets is due to the fact that these are in fact half-hypermultiplets

- the Hilbert series of $\mathcal{M}_{\mathcal{C}}^{\mathcal{A}}$ was computed in section 5 subsection [], where we had hinted at the $SO(8)$ enhancement; this moduli space corresponds indeed to the $1-S0(8)$ instanton moduli space, i.e the Higgs branch of $SU(2)$ with 4 flavours

$$HS_C^A(t) = \sum_{k=0}^{\infty} dim[0,k,0,0]_{So(8)} t^k = HS_{1,SO(8)}(t) = HS_H^B(t) \tag{6.20}$$

- the Hilbert series for $\mathcal{M}_{\mathcal{C}}^{\mathcal{B}}$ was also computed in section 5 Eq. [] with $N = 4$ where we had cast it in a suitable form: it is indeed identical to the Hilbert series for the ALE space $\frac{\mathbb{C}^2}{Dic_2}$, i.e. the Higgs branch the $A$-theory

$$HS_C^B(t) = \frac{1-t^6}{(1-t^3)(1-t^2)(1-t^2)} = HS(\frac{\mathbb{C}^2}{Dic_2};t) = HS_H^A(t) \tag{6.21}$$

Again, we have shown that the generating function for the chiral rings on the Higgs branch of the $A$-theory match the Hilbert series of the Coulomb branch of the $B$-theory, providing a check of mirror symmetry at the level of the moduli space.

## 6.4 Mirror Symmetry $K_{E_6}$

Our last check of mirror symmetry concerns the theory with Kronheimer gauge group $K_{E_6}$. This is not examined in the original paper and here is the place to truly test our new technique. The quiver diagram for this gauge theory is the affine Dynkin diagram $\hat{E}_6$ as shown in Fig.4.1c where the indices insides the nodes label the dimensions of the binary tetrahedral group $\mathbb{BT}$ by the McKay correspondence. Mapping to the quiver gauge theory, each node with label $r_i$ $i = 1, ..., 7$ corresponds to a factor $U(r_i)$ in the gauge group. Hence

$$K_{E_6} = \frac{U(1)^3 \times U(2)^3 \times U(3)}{U(1)} = U(1)^2 \times U(2)^3 \times U(3) \tag{6.22}$$

This is precisely the affine $\hat{E}_6$ theory whose Coulomb branch we calculated in Chapter 5, the only difference being that here we gauge the sum of the $U(1)$ charges. The matter multiplets are in the bifundamental of adjacent nodes, e.g $(\mathbf{3}, \bar{\mathbf{2}})$ if the left node is $U(3)$ and the right $U(2)$ . The quaternionic dimension of the Higgs branch and Coulomb branch, as computed in Table 6.1, are 1 and 11.

**Claim**
In the IR $K_{E_6}$ flows to a non-trivial fixed point where it is dual to the unknown gauge theory that has $E_6$ as the flavour symmetry, theory $B$.

Since the gauge theory is not known, we cannot cross-match the dimensions of the two branches. However we can rely on the Hilbert series since it encodes information on the moduli spaces. In chapter 2 we wrote down the unrefined Hilbert series for theory $B$:

- $\mathcal{M}_{\mathcal{H}}^{\mathcal{A}}$ is the ALE space $\frac{\mathbb{C}^2}{\mathbb{BT}}$ the Hilbert series is Eq.4.14

$$HS(\frac{\mathbb{C}^2}{Dic_2};t) = \frac{1-t^6}{(1-t^2)(1-t^3)(1-t^2)} \tag{6.23}$$

- $\mathcal{M}_{\mathcal{H}}^{\mathcal{B}}$ is the moduli space of $1-E_6$ instanton moduli space and the Hilbert series was computed in chapter 2 to be

$$HS_{1,E_6}(t) = \sum_{k=0}^{\infty} [0,k,0,0,0,0]_{E_6} t^k \tag{6.24}$$

The Hilbert series $\mathcal{M}_{\mathcal{C}}^{\mathcal{A}}$ for the Coulomb branch of theory $A$ was computed in chapter 5 using monopole operators: it displayed an $E_6$ enhancement. Indeed it corresponds, up to order 2 in the computations of the author, and fully in [8] , to the $1-E_6$ instanton moduli space, that is, the Higgs branch of the unknown gauge theory with $E_6$ as the flavour symmetry.

$$HS_C^A(t) = 1 + 78t + 2430t^2 + ... = HS_{1,E_6}(t) = HS_H^B(t) \tag{6.25}$$

Here we have stopped short of a double cross-matching, since we can not compute the Coulomb branch for theory $A$. We have shown though that the Hilbert series for the Higgs branch of the $A$-theory matches the Hilbert series of the Coulomb branch of the $B$-theory, thus identifying the two pieces of the moduli space. The Coulomb branch for the theory with $E_6$ flavour symmetry should, by mirror symmetry, be identical to the Hilbert series for the ALE space $\frac{\mathbb{C}^2}{\mathbb{BT}}$, i.e. the Higgs branch the $A$-theory.

# Chapter 7

# Brane realisation of mirror symmetry

Soon after the publication of [25] explanations of mirror symmetry by means of string theory settings followed. Two different approaches were introduced to think about the duality using brane construction: on the one hand Porrati and Zaffaroni [32] used brane constructions in type IIA compactified on a singular space, whilst on the other Hanany and Witten [24] described mirror symmetry in terms of brane constructions in type IIB. The two methods are useful in different ways: the IIA constructions are formulated in order to lift mirror duals to M-theory where they become a single theory. The type IIB construction revealed a nice technique to construct mirror duals by means of a brane creation mechanism, now known as Hanany-Witten transition. Since it is actually great fun to play around with branes and realise different gauge theories, we explore this latter realisation in the next subsections.

Type IIB involves, together with the omnipresent NS5-brane, $Dp$-branes for $p$ odd. A system of $D5$, $NS5$ and $D3$-branes is considered and by varying their number several gauge theories can be realised. This section is purposedly light in calculations and should serve as a pedagogical example of how the geometric brane realisation of supersymmetric gauge theories unfolds.

## 7.1 Brane Configurations

We consider type IIB on flat space $\mathbb{R}^{1,9}$, $x^0$ being the time coordinate. This is a maximal supergravity theory, i.e it has 32 supercharges. We will often identify spacetime directions by a sequence of numbers, possibly on top of $x$. So, for example, an object which fills the first four spacetime directions will be labelled as extended in the 0123 or $x^{0123}$ or $x^0x^1x^2x^3$, these notations being used interchangeably to mean the $x^0$, $x^1$, $x^2$, $x^3$directions.

Consider configurations of $NS5$-branes and $D5$-branes. No supersymmetry is preserved at all if both are extended along the same directions, for example in the 012345. In order to preserve some supersymmetry, the $NS5$-brane is placed at definite values of $x^7x^8x^9$, which means it is extended in the 012345 directions, whilst the $D5$-brane is at definite values of $x^3x^4x^5$, i.e. extended in the 012789 directions. Such a system preserves 1/4 of the background supersymmetries, which means that on the

| | $NS5$ | $D5$ | $D3$ |
|---|---|---|---|
| world-volume | 012345 | 012789 | 0126 |
| localised in | 789 | 345 | 345 and 789 |

Table 7.1: The five-branes fill their world-volume directions and are localised in the remaining.

worldvolume of the fivebranes there will be defined supersymmetric gauge theories with 8 supercharges. We show the branes in Fig. 7.1 where the time coordinate is suppressed.

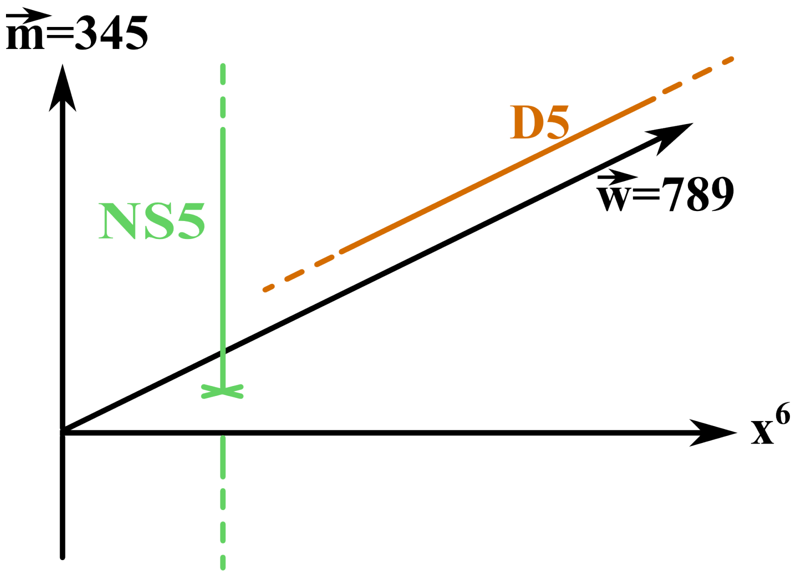



Figure 7.1: The system of $D5$ and $NS5$ that preserves 1/4 of the background supersymmetry. The coordinates $x^0$, $x^1$, $x^2$ are suppressed in this sketch.

Further one can add $D3$ branes extended in the 0126 direction, the only way to preserve the 8 supercharges configuration. We summarise the brane content in Table 7.1.

The Lorentz group of symmetry $SO(1,9)$ breaks as

$$\begin{array}{rcr} SO(1,2) & \times SO(3)_V \times & SO(3)_H \\ 012; & \vec{m} = 345; & \vec{w} = 789 \end{array}$$

The rotation subgroups $SO(3)_V$ and $SO(3)_H$ act as symmetry groups of the reduced coordinates $\vec{m} = 345$ and $\vec{w} = 789$ respectively. The double covers $SU(2)_V$ and $SU(2)_H$ correspond to the symmetries of the Coulomb branch and Higgs branch. The configurations we study involve many $NS5$, $D5$ and $D3$ branes whose positions are as follows:

- the $i^{th}$ $NS5$ will be localised at $x^6 = t_i$ and $\vec{w} = \vec{w}_i$
- the $j^{th}$ $D5$ will be localised at $x^6 = z_j$ and $\vec{m} = \vec{m}_j$

- the $D3$ can have both ends on two $NS5$-branes or on two $D5$-branes or on one $NS5$ and one $D5$ in a way which we specify shortly

Supersymmetry constrains the system even further:

- two $NS5$-branes, $i$ and $i'$, can be connected by a $D3$-brane if and only if they have the same position in the 789 direction, that is $\vec{w}_i = \vec{w}_{i'}$. The $D3$-brane is at some given value of $\vec{m}$, which we name $\vec{x}$. The $\alpha^{th}$ brane will thus be parametrised by $\vec{m} = \vec{x}_\alpha$. This is shown in Fig. 7.2.

- analogously two $D5$-branes, $j$ and $j\prime$ are conneccted by a $D3$-brane if and only if $\vec{m}_j = \vec{m}_{j\prime}$, which can be obtained in Fig. 7.2 by exchanging the directions $\vec{m}$ and $\vec{w}$ and the $NS5$ with the $D5$ label. The $\alpha^{th}$ $D3$-brane is this time at some given value of $\vec{w}$, which we name $\vec{y}_\alpha$.

- a $D3$ brane can connect a $D5$ and a $NS5$ but there is no choice in how to place it in the two sets of transverse directions, 345 and 789. Hence no moduli space for such a configuration arises.

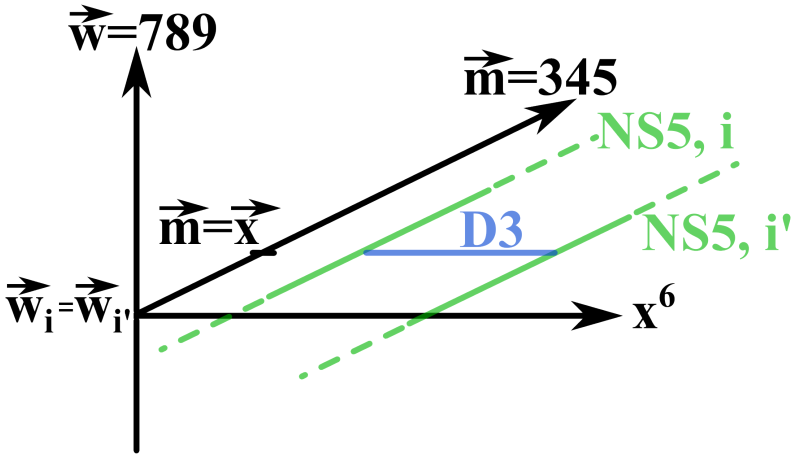



Figure 7.2: The two $NS5$-branes must be at the same position in the 789 directions: $\vec{w}_i - \vec{w}_{i'} = 0$ . Indeed the triplet $\vec{w}_i - \vec{w}_{i'}$ can be thought of as a Fayet-Iliopolous term, which, when present, lifts the Coulomb branch. It must be set to zero for the Coulomb branch to appear.

S-duality is nicely in place in such set-ups. $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$ converts a $D5$ into a $NS5$. Define a rotation $R$ that maps $x^{345}$ to $x^{789}$ and $x^{789}$ to $-x^{345}$. The successive application of these two transformations, $RS$, leaves the system invariant up to exchange of $D5$-branes with $NS5$-branes. We will make use of this $RS$ transformation to find mirror pairs. Indeed this geometrically manifest invariance is what underlies mirror symmetry.

## 7.2 Field Theory on the branes

Why is this brane arrangement useful? To begin with one should notice that the $D3$-brane has only three infinite directions: time $x^0$ and two spatial ones $x^1$ and $x^2$. It is finite along the $x^6$ direction, hence the field theory on its worldvolume is *macroscopically* $2+1$ *dimensional.* It has 8 supercharges, which in $3d$ corresponds to $\mathcal{N}=4$. The potential $Spin(4) \cong SU(2) \times SU(2)$ R-symmetry has been geometrically pointed at above: it is the $SU(2)_V \times SU(2)_H$ corresponding to the (double cover) rotations in the $\vec{w}$ and $\vec{m}$ directions.

Moreover the $D5$ and $NS5$-branes are infintely extended in two extra dimensions than $D3$, they are thus heavier and their parameters are taken fixed: field-theoretically $t$, $\vec{w}$ and $z$, $\vec{m}$ which specify the transeverse positions of the $NS5$ and $D5$ respectively act as *coupling constants for the* $QFT$ *on the* $D3$*-brane.*

The $D3$ positions in the 345 and 789 directions, which recall have been labelled $\vec{x}$ and $\vec{y}$, are instead *dynamical moduli* and they precisely parametrise the moduli space of vacua. We have thus completed the realisation of a $\mathcal{N}=4$ $3d$ supersymmetric theory by means of brane configurations. The fields living on the worldvolume of the $D3$-brane are collected in a vector multiplet and a hypermultiplet. They are not present simultaneously: Neumann (supersymmetry preserving) boundary conditions can be imposed at the ends of the $D3$-brane which force either the vector mulitplet or the hypermulitplet to vanish. The other multiplet will then obey Dirichlet boundary conditions. In particular the $3+1d$ vector $A_\mu$ that lives on an infinite $D3$ reduces in $2+1d$ to a vector $a_\mu$ and a scalar $b$. Neuman boundary conditions on $A_\mu$ imply that $b$ vanish whereas Dirichlet imply that $a_\mu$ vanishes. The aforementioned vector multiplet and hypermultiplet can be constructed geometrically by combining the transverse positions of the $D3$-brane and the fields $a_\mu$ and $b$:

- $a_\mu$ and $\vec{x}$ form the vector multiplet
- $b$ and $\vec{y}$ form the hypermulitplet

When a $D3$-brane ends on a $NS5$-brane, the position $\vec{x}$ in the $\vec{m}$ direction is free to fluctuate whilst Dirichlet boundary conditions enforce the vanishing of $\vec{y}$ and $b$. The massless modes of this configurations are $\vec{x}$ and $a_\mu$. Conversely, when a $D3$-brane ends on a $D5$-brane $\vec{x}$ and $a_\mu$ will be set to zero whilst $\vec{y}$and $b$ become the free parameters.

### 7.2.1 Effective 3d gauge theory on the $D3$-brane

The parameters introduce above provide a geometrical description of the gauge theory living on the $D3$-brane. There are three cases to consider.

#### Case 1

Consider $n_v$ $D3$-branes ending between two $NS5$-branes. The effective theory on the $D3$-branes is a $U(1)^{n_v}$ gauge theory, which follows straightforwardly from the last paragraph in the previous subsection. As the $n_v$ $D3$-branes approach each other and become coincident the $U(1)^{n_v}$ gauge theory is enhanced to $U(n_v)$. In particular this configuration is suitable to analyse the dynamics of the Coulomb branch. Recall that the two $NS5$-branes must be placed at the same value in the 789 direction, i.e. $\vec{w}_1 - \vec{w}_2 = 0$. The separation of the two branes along the $x^6$ direction is interpreted as the gauge coupling:

$$\frac{1}{g_e^2} \propto |t_1 - t_2|$$

This is referred to as the 'electric coupling'.

**Case 2**

Let us now take $n_h$ $D3$-branes ending between two $D5$-branes. This identifies an effective theory on the $D3$-branes with $n_h$ hypermultiplets. An $RS$ transformation can be performed which exchanges the $\vec{m}$ and $\vec{w}$ directions, the $NS5$-brane and $D5$-brane as well as $a_\mu$ and $b$ (since $SL(2,\mathbb{Z})$ implements electric-magnetic duality). A $U(n_h)$ gauge theory is thus realised whose gauge coupling is, along the lines of the previous case,

$$\frac{1}{g_m^2} \propto |z_1 - z_2|$$

To differentiate from $g_e^2$, this is known as the magnetic coupling.

**Case 3**

As previously stated, no moduli exist for a $D3$-brane stretched between a $D5$-brane and a $NS5$-brane since both $\vec{x}$ and $\vec{y}$ are fixed at either eneds: henceforth massless states can not arise.

## 7.3 Singularities

Without going into too many details we state here certain singularities of the brane configurations which lead to extra massless modes. Since these singularities are fairly intutitive a superficial geometrical description will suffice.

### 7.3.1 Singularity of the first kind

The first type of singularity is obtained by letting $D3$-branes on one side of a $NS5$-brane swing close to $D3$-branes on the other side as shown in Fig. 7.3. This is a non-perturbative arrangement. When a left $D3$-brane with $\vec{m} = \vec{x}_L$ is tangent to a right $D3$-brane by $\vec{m} = \vec{x}_R$, i.e. when$|\vec{x}_L - \vec{x}_R| = 0$ a massless hypermultiplet arises. This is due to the string stretching between them becoming lengthless. For $k_1$ left branes and $k_2$ right branes there will be $k_1 k_2$ potential hypermultiplets. Since the enhanced gauge group of such a configuration is $U(k_1) \times U(k_2)$ the hypermultiplets will be in the bifundamental representation $(k_1, \bar{k}_2)$. Furthermore they transform as $(1, 2)$

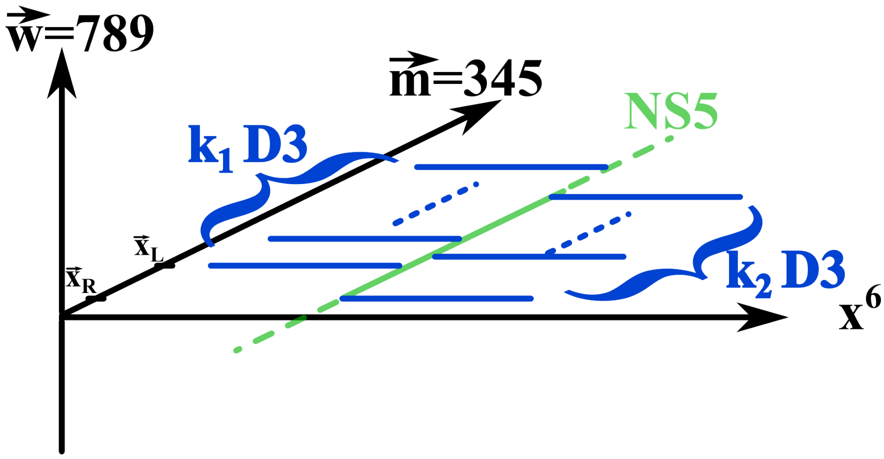



Figure 7.3: Potential singularities arise when the 345 position of a left $D3$-brane coincide with that of a right $D3$-brane. We have marked the position of the “first” left $D3$-brane by $\vec{x}_L$ and the position of the “first” right $D3$-brane by $\vec{x}_R$. When $|\vec{x}_L - \vec{x}_R| = 0$ a massless hypermultiplet arises.

under the R-symmetry $SU(2)_V \times SU(2)_H$. Similar arguments can be exploited for $D3$-branes ending on both sides of a $D5$-brane after application of a $RS$ tranformation, the only difference being that the gauge group is now referred to as magnetic and that the massless hypermultiplets transform as $(2,1)$ under the R-symmetry $SU(2)_V \times SU(2)_H$ .

### 7.3.2 Singularity of the second kind

The second type of singularity occurs when a $D5/NS5$-brane and a $D3$-brane meet in space. For example consider a $D5$-brane sitting at a given $\vec{m} = \vec{m}_{D5}$ and a $D3$-brane at $\vec{m} = \vec{x}$. A string can stretch between the two until $\vec{m}_{D5} = \vec{x}$ at which point a massless hypermultiplet arises, as it is shown in Fig. 7.4. This hypermultiplet transforms in $(1,2)$ under the R-symmetry $SU(2)_V \times SU(2)_H$.

Performing an $RS$ transformation, the $D3$-brane can be replaced by the $NS5$-brane and a similar outcome achieved. This type of singularity can be described perturbatively unlike the singularity of the first kind.

Note that the singularities arising in Fig. 7.3 and Fig. 7.4 are related to each other: if one allows the $D3$-brane to effectively split in two upon encounter with the $D5/NS5$-brane in Fig. 7.4, then the configuration in Fig. 7.3, with one left $D3$-brane and one right $D3$-brane is yielded.

### 7.3.3 From branes to quivers

The discussion so far is sufficient to see how the brane realisation of the general gauge theory of $U(K)$ with $N$ flavours is done: this consists of $K$ parallel and stacked $D3$-branes between two $NS5$-branes which are at positions $t_1$ and $t_2$ along the $x^6$ direction and are located at the same point in the 789 with $\vec{w}_1 = \vec{w}_2$. The flavour group is

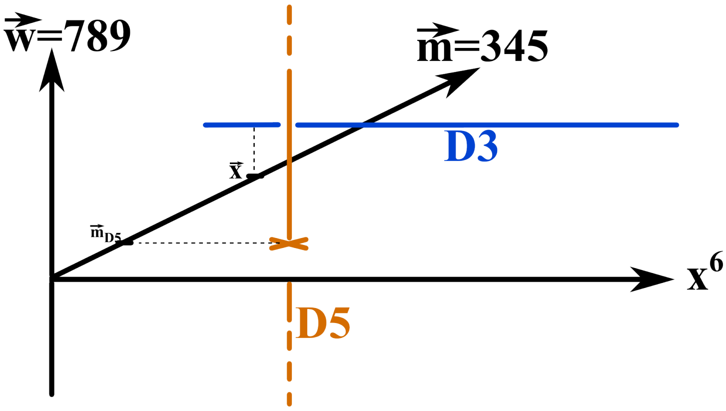



Figure 7.4: The $D3$-brane is "further back" than the $D5$-brane in the 345 direction. When $\vec{m}_{D5}$ and $\vec{x}$ coincide, the string stretching between the two become lengthless and thus the hypermultiplet massless.

realised by stacked $N$ $D5$-branes when they meet the $D3$-brane in spacetime. The $D5$-branes positions $z_i$, for $i = 1, ..., N$, along the $x^6$ direction are bounded by the two $NS5$-brane positions: $t_1 < z_i < t_2$ . The $D5$-branes positions in the 345 direction are labelled $\vec{m}_i$. The set up is shown in Fig.7.5.

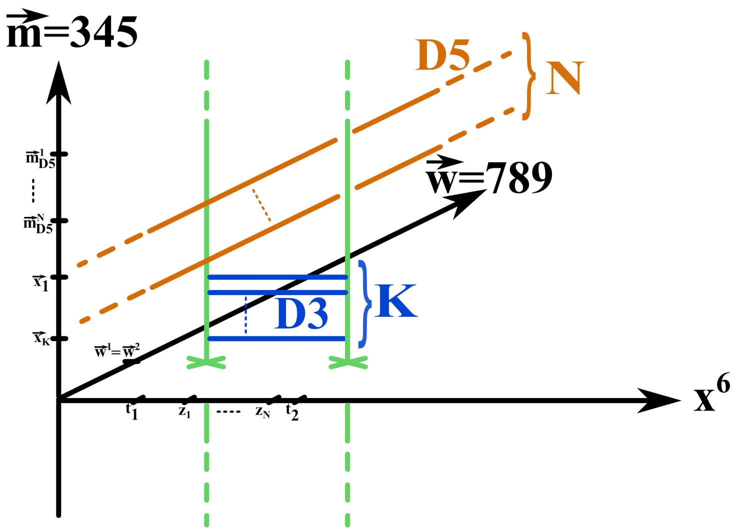
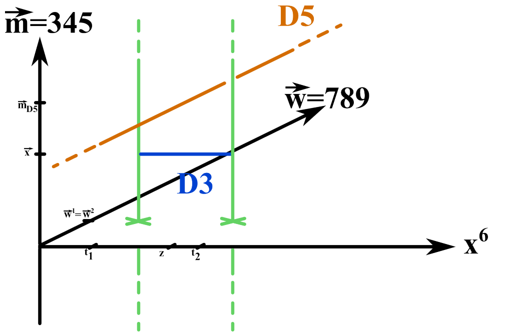



Figure 7.5: the brane realisation of $U(K)$ with $N$ flavours. The $N$ $D5$-branes are "inside" the $NS5$-branes along the $x^6$ direction and have position $\vec{m}^i_{D5}$ in the 345 direction. When the $D3$-branes become coincident and the $D5$-branes are tangent to the stack, the configuration realises a gauge theory $U(K)$ on the $D3$-branes with a global symmetry $U(N)$.

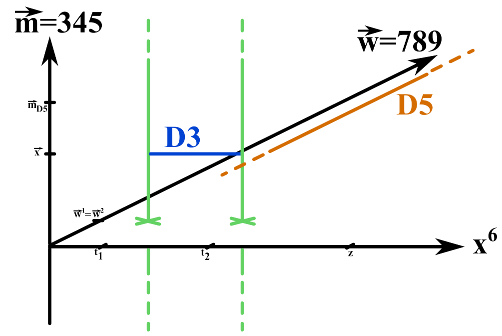
Intersection of $D3$-branes and $D5$-branes gives rise to $N$ hypermultiplets, each of which has a mass triplet $\vec{m}_i$ parametrised by its position along the 345direction. One mass triplet can always be removed by a shift of the origin. When the $D5$-branes meet the $D3$-branes in spacetime, by the second type of singularity, the $N$ hypermultiplets become massless. If the two $NS5$-branes are located at different values along the 789 direction, the nonzero difference $\vec{w}_1 - \vec{w}_2 \neq 0$ acts as a Fayet-Iliopolous term and lifts the Coulomb branch. Thus the brane configurations reproduce every parameter of the supersymmetric gauge theory, including masses and FI terms.

## 7.4 Hanany-Witten transition

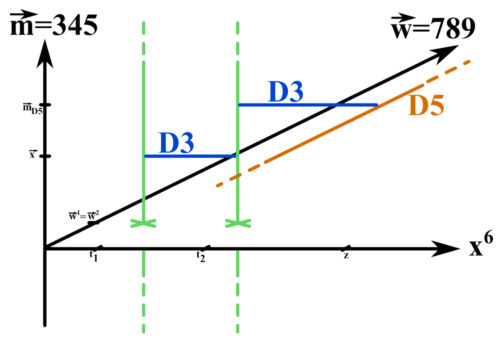
Before turning our attention to how brane configurations encode mirror symmetry, we have to look at the phenomenon of brane creation. We state, without proof, the mechanism for the Hanany-Witten transition. The authors of [24] provide a detailed explananation based on a generalised Gauss' law and the so called linking number but we will neglect these here.




(a) A massive hypermultiplet exists in this set-up since a string can stretch between the $D3$-brane and the $D5$-brane when $z$, the position of the $D5$ along $x^6$ satisfies $t_1 < z < t_2$, where $t_1$ and $t_2$ are the $NS5$ positions on the $x^6$ direction. When $\vec{m}_{D5} = \vec{x}$ the hypermultiplet becomes massless.




(b) Even when $\vec{m}_{D5} = \vec{x}$ there is no hypermultiplet if $t_1 < t_2 < z$. The same parameters are present and physically the $D5$-brane has just shifted along the $x^6$ direction. What happens to the hypermultiplet?




(c) A new $D3$-brane is created when the $D5$-brane crosses the $NS5$-brane. We have restored a massless hypermulitplet when $\vec{m}_{D5} = \vec{x}$ by the singularity of first kind.

Figure 7.6: A $U(1)$ theory living on the $D3$. A massless hypermultiplet can arise in 7.6a but not in 7.6b. In 7.6c the actual outcome of moving the $D5$-brane "outside" the $NS5$ with a new $D3$-brane.

In Fig. 7.6a we show the theory of $U(1)$ with one flavour. The two $NS5$ are at $t_1$, $\vec{w}_1$ and $t_1, \vec{w}_2$. A $D3$-brane connects them and its position along the 345 direction is $\vec{x}$. The $D5$-brane is at $z$ with $t_1 < z < t_2$ and has position $\vec{m}_{D5}$ in the 345 direction. Thus there exists one hypermultiplet arising from a string that connects the $D3$ and $D5$ with mass $\propto \vec{m} - \vec{x}$. When the $D3$-brane and the $D5$-brane are localised in the same point of 345, the hypermultiplet is massless.

Imagine now that the $D5$-brane moves along the $x^6$ direction untill $z > t_2$ as in Fig. 7.6b. The hypermultiplet disappears, both when massive $\vec{m}_{D5} \neq \vec{x}$ and when massless $\vec{m}_{D5} = \vec{x}$. However, to reach this new set-up the $NS5$-brane and the $D5$-brane must cross: to see this it suffices to realise that the $NS5$-brane is localised in 789, where the $D5$-brane is extended, and the $D5$-brane is localised in 345, where the $NS5$-brane is extended.

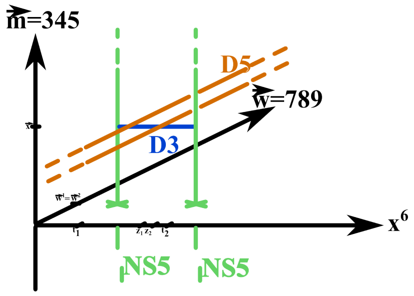
Instead of the one in Fig. 7.6b, the outcome of moving the $D5$-brane past the $NS5$-brane is sketched in Fig. 7.6c where a *new* $D3$-brane stretching along the $x^6$ direction between $t_2$ and $z$ has been created. No moduli exist for the new $D3$-brane since it is between $D5$ and a $NS5$ brane. However from the singularity of the first kind encountered and shown in 7.3, such a configuration can give rise to a massless hypermultiplet. This happens precisely for $\vec{m}_{D5} = \vec{x}$, when the two $D3$ are tangent. Thus the two singularity mechanisms are related by means of a brane creation transition.

## 7.5 Constructing mirror pairs

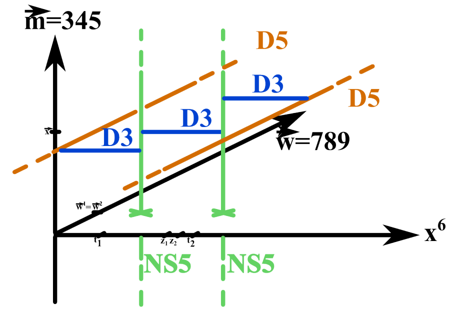
Mirror symmetry can be enacted in brane configurations by exchanging the $\vec{m} = 345$ direction with the $\vec{w} = 789$ direction and concurrently exchanging the $NS5$-brane and the $D5$-brane. We give a few examples of how this is done by drawing the theory and moving to its mirror dual.

### $U(1)$ with 2 flavours

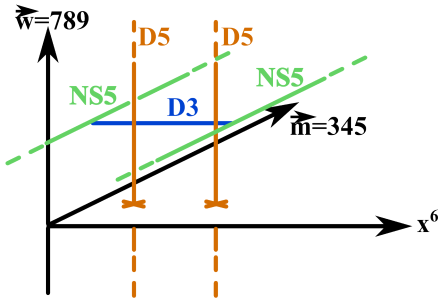
This theory is a special case of $U(1)$ with $N$ flavours whose Coulomb branch was considered in chapter 5. It is special because the theory is self-mirror. The Coulomb branch is $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ and its Higgs branch is the $1 - SU(2)$ instanton moduli space, which coincides with $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$. The brane picture preserves this self-duality in that an RS transformation brings us back to the starting one. In Fig. 7.7a we realise the theory of $U(1)$ with 2 flavours by stretching a $D3$-brane between two $NS5$-branes. The flavour symmetry is implemented by the $D5$-branes since two massless hypermultiplets arise when the $D5$-branes meet the $D3$ along the 345 direction. Fig. 7.7b is the intermediate stage, where the $D5$-branes have been brought at $x^6$ locations $z_1$, $z_2$ such that $z_1 < t_1 < t_2 < z_2$, "outside" the $NS5$-branes. By the Hanany-Witten transition, $D3$-branes are created, which stretch between the $D5$-branes and the $NS5$-branes. In Fig. 7.7c, after performing a $RS$ transformation, we obtain the same set-up of Fig. 7.7a, with the $D3$-branes reconnected in one and stretching between the $NS5$-branes ($U(1)$gauge group) whilst the two $D5$-branes provide matter multiplets.




(a) $U(1)$ with two flavours provided by the $D5$-branes




(b) The $D5$-branes are brought to the outside and two extra $D3$-branes are created

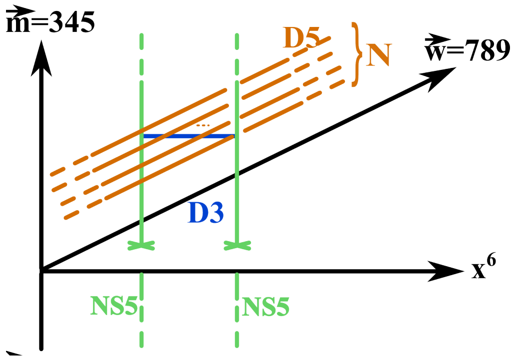



(c) Exchanging $NS5$-branes with $D5$-branes and the 345 directions with the 789, we obtain the mirror theory. In this case is identical to the original theory of $U(1)$ with 2 flavours.

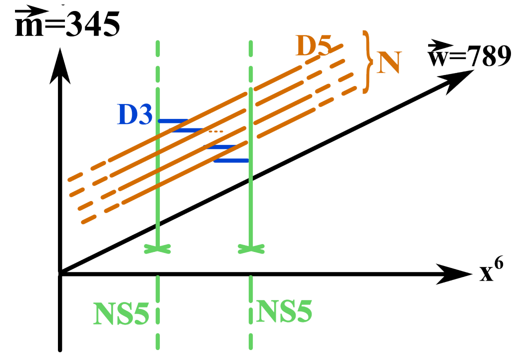
Figure 7.7: A self-dual theory and the brane realisation

## $U(1)$ with $N$ flavours

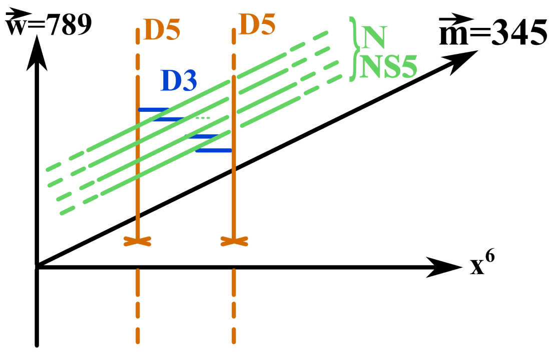
Let us consider again two $NS5$-branes, one $D3$-brane extended between the two which realise a $U(1)$ theory and this time $N$ flavours, provided by $N$ $D5$-branes. The four dimensional Coulomb branch is parametrised by the scalars on the $D3$-brane: geometrically these are given by the location $\vec{x}$ in the 345 direction plus the dual photon. The hypermultiplet masses are the locations $m_i$ of the $D5$-branes. By shifting the $D3$-brane position we can eliminate the centre of mass of the $D5$-branes, hence we are left with $N-1$ mass parameters. The potential separation of the two $NS5$-branes in the 789 direction $\vec{w}_1 - \vec{w}_2$ acts as a FI term, which would lift the Coulomb branch. In Fig. 7.8a we realise the theory. The intermediate stage consists of maximally breaking the $D3$-branes, when they encounter the $D5$-branes. Exchanging the $NS5$-branes and the $D5$-branes in Fig. 7.8c completes the procedure to obtain the mirror theory: there are $N-1$ $D3$ branes, each of which stretches between two $NS5$-branes, giving rise to a $U(1)^{N-1}$ gauge group. The matter multiplets arise by the singularity of first kind and they transform as $(\mathbf{1}, \mathbf{1})$ of adjacent factors. This is precisely the mirror theory that we expect from section 6.




(a) $U(1)$ with $N$ flavours. The $N$ $D5$-branes are "inside" along the $x^6$ direction the $NS5$-branes




(b) The intermediate stage: The $D3$-branes are maximally broken between the $D5$-branes



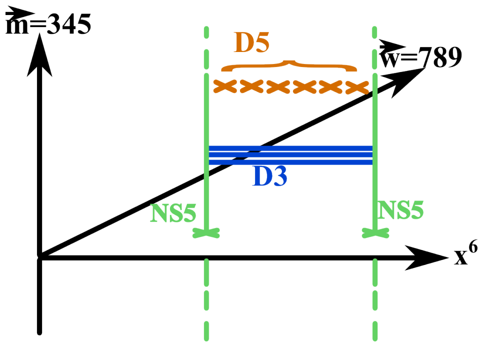

(c) RS symmetry is performed. The $D5$-branes become $NS5$-branes and vice versa. The mirror gauge theory is now $U(1)^{N-1}$ with bifundamental hypermultiplets which recovers what we know from chapter 5.

Figure 7.8: $U(1)$ with $N$ flavours and its dual theory

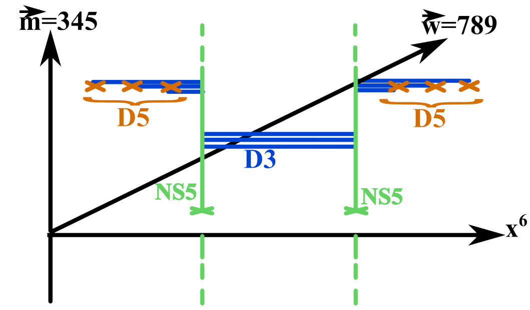
## $U(3)$ with 6 flavours

The last example of how to obtain mirror theories using brane constructions will be a special case of $U(k)$ with $2k$ flavours for $k = 3$. The general case is treated in [24] but it doesn't differ much from the current example. We start in Fig.7.9a with three $D3$-branes streching between two $NS5$. The 6 $D5$-branes provide the flavour symmetry to the gauge theory on the $D3$. We then move the $D5$-branes one by one to the outside, in the $x^6$ direction, of the $NS5$-brane. Each time a $D5$-brane crosses a $NS5$-brane, a $D3$-brane is created. This is shown in Fig.7.9b. By converting the $D5$-branes into $NS5$ and viceversa and maximally breaking the $D3$-branes we obtain the mirror theory, which corresponds to the brane configuration in Fig.7.9c. The gauge theory can be simply read off the brane picture. The gauge group is $U(1)\times U(2)\times U(3)\times U(2)\times U(1)$ with matter multiplets transforming in the $(\mathbf{1},\overline{\mathbf{2}})\oplus(\mathbf{2},\overline{\mathbf{3}})\oplus\mathbf{3}\oplus\mathbf{3}\oplus(\mathbf{3},\overline{\mathbf{2}})\oplus(\mathbf{2},\mathbf{1})$ . This is the quiver gauge theory of Fig.7.9d. In section 5 we computed the Coulomb branch of $U(3)$ with 6 flavours and found that it was a complete intersection. A check that its mirror dual is in fact the quiver gauge theory sketched in Fig.7.9d would be to evaluate the Hilbert series on the Higgs branch of this theory and show it gives the complete intersection moduli space found in section 5.

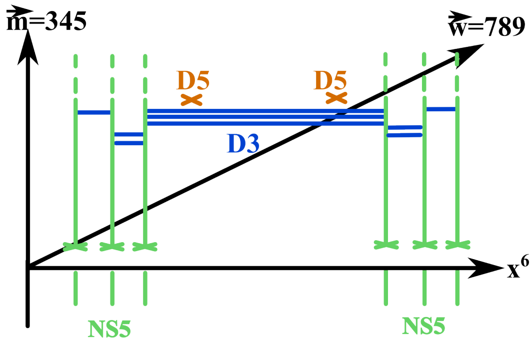
Many other mirror pairs can be constructed using the $NS5/D5/D3$ brane picture. Moreover $Sp(n)$ and $SO(n)$ gauge groups can also be constructed using orientifold hyperplanes $O3^-$ in the background. This is studied in [16] to which we refer the interested reader.



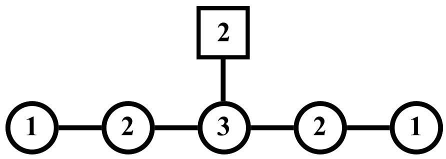

(a) The theory of $U(3)$ with 6 flavours




(b) The $D5$-branes are moved “outside” and at each stage a $D3$-brane is created.




(c) The $RS$ transformation turns the $D5$-branes into $NS5$-branes and viceversa. This is the mirror theory: the gauge group is $U(1) \times U(2) \times U(3) \times U(2) \times U(1)$ as we can read from the number of $D3$-branes stretching between pairs of $NS5$-branes. The gauge theory as matter hypermultiplets as well, which can be read more easily from the corresponding quiver diagram.




(d) The quiver diagram for the mirror theory of $U(3)$ with 6 flavours. Matter hypermultiplets are in the bifundamental of adjacent factors. The square node represents the flavour symmetry that arises because of the two $D5$-branes.

Figure 7.9: $U(3)$ with 6 flavours and its mirror via brane constructions. In order not to clutter the figures we mark $D5$-branes with a cross, but they of course extend in the 789 direction (as well as the suppressed 012)

# Chapter 8

# Conclusion and outlook

Many techniques have been provided throughout this dissertation. We explored some of the features of supersymmetric gauge theories in three dimensions. Phenomena occurring in these theories are far from being fully understood. In particular mirror symmetry is a very surprising duality in that it exchanges the Coulomb branch, which is not classically exact and needs loop corrections, with the Higgs branch, which is protected by quantum corrections. So far, for a given gauge theory, in order to calculate the Coulomb branch one would often resort to the dual mirror theory and evaluate the Higgs branch thereof. In particular, as far as the chiral ring of the moduli space is concerned, calculations on the Higgs branch dominated the scene whilst methods to tackle the Coulomb branch seemed elusive.

Fortunately enough, the role of monopole operators has been gaining momentum. It is fascinating to see how this developed historically: from the first guesses at the poorly understood nature of these operators in [27, 1] to their modern definition and computation of their quantum numbers in [21, 3, 7, 6, 26]. Unfortunately, an encompassing proof of mirror symmetry doesn't yet exist, though one can provide strong arguments on a case by case basis. In this dissertation we have followed relatively simple prescriptions to check the duality:

- an initial measure is to check the dimensions of the Coulomb and Higgs branch and show they cross match with the mirror dual (including counting FI and mass terms)
- matching of the chiral rings for the two branches of the moduli space of mirror duals

A fairly technical approach is to match observables of the dual theories. This amounts to writing the partition function for the 3d gauge theory, implementing mirror symmetry at the level of the partition function and showing that this is equivalent to the partition function of the supposed dual symmetry. This has been done and continues to be studied. In particular the authors of [29, 28] and [13] used the so called localization methods of [18] to write down the partition function for $\mathcal{N}=4$ gauge theories with matter on $S^3$. This approach has not been given a share of this dissertation purely as a matter of taste but it would be very interesting to review mirror symmetry from this point of view.

The aim of this dissertation was not to exploit mirror symmetry but to check its claims for a few pairs of mirror duals. This was accomplished by presenting the previous work on the chiral ring of the Higgs branch and introducing a new technique for the Coulomb branch in order to cross-match the two. In the latter we constructed operators which carry magnetic charges $m_i$. The conformal dimension of these operators depends on these magnetic charges and has positive contribution from the matter sector and negative contribution from the gauge sector, as was detailed in section 5. Once the dimension of these operators is known, a generating function for all monomials made of monopole operators can be written down. Perhaps a little surprising is the idea that one has to include *classical contributions* from the Lagrangian operators that survive the monopoles magnetic flux ex post facto. Usually quantum corrections are precisely this: *corrections* to a classical background. In light of this, the classical contribution was called a "dressing": it consists of classical Casimir operators of the residual gauge group. Once all of this has been accounted for, the Hilbert series take a simple expression: after its evaluation it unambiguously coincides with the Hilbert series of the Higgs branch of the dual theory, thus proving the equivalence of the two moduli spaces in a elegant and neat way. We would like to point out that the new computations for the ring of invariants on the Coulomb branch can sometimes be simpler than calculating the Higgs branch of the dual. This is a hint towards the power of this new method. Much more can be extracted from it, for example by exploring the often difficult gauge theories with exceptional groups.

Lastly, let us stress that the work presented is by no means comprehensive nor does it aim to be. There are several sides, as we have appreciated above, that have not even been mentioned. However we took the view that it was better to instruct the reader in depth on one aspect of $3d$ mirror symmetry than to touch much and leave little.

# References


[1] Ofer Aharony, Amihay Hanany, Kenneth A. Intriligator, N. Seiberg, and M.J. Strassler. Aspects of N=2 supersymmetric gauge theories in three-dimensions. *Nucl.Phys.*, B499:67–99, 1997.

[2] Denis Bashkirov. Examples of global symmetry enhancement by monopole operators. 2010.

[3] Denis Bashkirov and Anton Kapustin. Supersymmetry enhancement by monopole operators. *JHEP*, 1105:015, 2011.

[4] Sergio Benvenuti, Bo Feng, Amihay Hanany, and Yang-Hui He. Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics. *JHEP*, 0711:050, 2007.

[5] Sergio Benvenuti, Amihay Hanany, and Noppadol Mekareeya. The Hilbert Series of the One Instanton Moduli Space. *JHEP*, 1006:100, 2010.

[6] Vadim Borokhov, Anton Kapustin, and Xin-kai Wu. Monopole operators and mirror symmetry in three-dimensions. *JHEP*, 0212:044, 2002.

[7] Vadim Borokhov, Anton Kapustin, and Xin-kai Wu. Topological disorder operators in three-dimensional conformal field theory. *JHEP*, 0211:049, 2002.

[8] Stefano Cremonesi, Amihay Hanany, and Alberto Zaffaroni. Monopole operators and Hilbert series of Coulomb branches of 3d N = 4 gauge theories. *Available online at ARXIV:1309.2657.*

[9] Andrew Dancer. Hyperkahler manifolds. *Surveys in Differential Geometry VI*, Essays on Einstein manifolds:15–38, 1999.

[10] Jan de Boer, Kentaro Hori, Hirosi Ooguri, and Yaron Oz. Mirror symmetry in three-dimensional gauge theories, quivers and D-branes. *Nucl.Phys.*, B493:101–147, 1997.

[11] Jan de Boer, Kentaro Hori, Hirosi Ooguri, Yaron Oz, and Zheng Yin. Mirror symmetry in three-dimensional theories, SL(2,Z) and D-brane moduli spaces. *Nucl.Phys.*, B493:148–176, 1997.

[12] Jan de Boer, Kentaro Hori, and Yaron Oz. Dynamics of N=2 supersymmetric gauge theories in three-dimensions. *Nucl.Phys.*, B500:163–191, 1997.

[13] Anindya Dey and Jacques Distler. Three Dimensional Mirror Symmetry and Partition Function on $S^3$. *Available online at ARXIV:1301.1731.*

[14] Igor V. Dolgachev. Mckay correspondence.

[15] Michael R. Douglas and Gregory W. Moore. D-branes, quivers, and ALE instantons. 1996.

[16] Bo Feng and Amihay Hanany. Mirror symmetry by O3 planes. *JHEP*, 0011:033, 2000.

[17] Bo Feng, Amihay Hanany, and Yang-Hui He. Counting gauge invariants: The Plethystic program. *JHEP*, 0703:090, 2007.

[18] Guido Festuccia and Nathan Seiberg. Rigid Supersymmetric Theories in Curved Superspace. *JHEP*, 1106:114, 2011.

[19] JM Figueroa-O'Farrill. Electromagnetic duality for children. 1998.

[20] Davide Gaiotto. N=2 dualities. *JHEP*, 1208:034, 2012.

[21] Davide Gaiotto and Edward Witten. S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory. *Adv.Theor.Math.Phys.*, 13, 2009.

[22] Amihay Hanany and Noppadol Mekareeya. Complete Intersection Moduli Spaces in N=4 Gauge Theories in Three Dimensions. *JHEP*, 1201:079, 2012.

[23] Amihay Hanany, Noppadol Mekareeya, and Giuseppe Torri. The Hilbert Series of Adjoint SQCD. *Nucl.Phys.*, B825:52–97, 2010.

[24] Amihay Hanany and Edward Witten. Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nucl.Phys.*, B492:152–190, 1997.

[25] Kenneth A. Intriligator and N. Seiberg. Mirror symmetry in three-dimensional gauge theories. *Phys.Lett.*, B387:513–519, 1996.

[26] Anton Kapustin. Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality. *Phys.Rev.*, D74:025005, 2006.

[27] Anton Kapustin and Matthew J. Strassler. On mirror symmetry in three-dimensional Abelian gauge theories. *JHEP*, 9904:021, 1999.

[28] Anton Kapustin, Brian Willett, and Itamar Yaakov. Nonperturbative Tests of Three-Dimensional Dualities. *JHEP*, 1010:013, 2010.

[29] Anton Kapustin, Brian Willett, and Itamar Yaakov. Tests of Seiberg-like Duality in Three Dimensions. 2010.

[30] Seok Kim and Kallingalthodi Madhu. Aspects of monopole operators in N=6 Chern-Simons theory. *JHEP*, 0912:018, 2009.

[31] P. B. Kronheimer. The construction of ALE spaces as hyper-KÃ€hler quotients. *Jour. Differential Geometry*, 29:665–683, 1989.

[32] Massimo Porrati and Alberto Zaffaroni. M theory origin of mirror symmetry in three-dimensional gauge theories. *Nucl.Phys.*, B490:107–120, 1997.

[33] Nathan Seiberg. Naturalness versus supersymmetric nonrenormalization theorems. *Phys.Lett.*, B318:469–475, 1993.

[34] Nathan Seiberg. IR dynamics on branes and space-time geometry. *Phys.Lett.*, B384:81–85, 1996.

[35] Nathan Seiberg and Edward Witten. Gauge dynamics and compactification to three-dimensions. 1996.

[36] Richard P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc.*, 1(3):475–511, 1979.

[37] Matthew J. Strassler. An Unorthodox introduction to supersymmetric gauge theory. pages 561–638, 2003.

# Appendix A

# HyperKähler manifolds

The moduli spaces encountered in this dissertation are a special class of manifolds, known as HyperKähler. This is a very short summary of what the reader needs to know to have an idea of what these spaces are geometrically. For a nice article which introduces the HyperKähler manifolds encountered in this dissertation we refer to [9].

Let us start by considering a complex manifold $\mathcal{M}$ of complex dimension $m$. The almost complex structure of $\mathcal{M}$ is a tensor field $J$ which acts on the complexified tangent space as

$$\begin{aligned} J:\ T_p\mathcal{M}^C &\longrightarrow T_p\mathcal{M}^C \\ Z &\longrightarrow JZ = iZ \end{aligned}$$

where $Z = Z^\mu \frac{\partial}{\partial Z}$ is an element of the complexified tangent space. The condition for $J$ to be a complex structure is that it squares to minus the identity

$$J^2 = -id_{T_p\mathcal{M}}$$

Now let $\mathcal{M}$ be equipped with a riemannian metric $g$. The metric is Hermitian if

$$g_p\left(JX, JY\right) = g_p\left(X, Y\right)$$

for all tangent vectors $X, Y \in T_p\mathcal{M}$ such that $Z = X + iY$.

Let us also introduce a tensor field $\omega$ acting on $T_p\mathcal{M}$ as

$$\omega_p\left(X, Y\right) = g_p\left(JX, Y\right)$$

$\omega$ is by construction antisymmetric and thus defines a two-form: it is known as the Kähler form. In coordinate components

$$\omega = \frac{i}{2} g_{\mu\nu} \mathrm{d}z^\mu \wedge \mathrm{d}\overline{z}^\nu$$

If the Kähler form of an Hermitian manifold is closed, $\mathrm{d}\omega = 0$, the manifold is known as Kähler and $g$ as a Kähler metric.

Consider now a manifold $\mathcal{M}$ on which two Kähler forms $I, J$ exist for the metric $g$ and furthermore that these two anticommute: $IJ = -JI$. Now define $K = IJ$. Then $K$ is also a Kähler form for $g$. Since now $I, J, K$ satisfy the algebra of quaternions $I^2 = J^2 = K^2 = IJK = -1$, the tangent space $T_p\mathcal{M}$ becomes effectively a quaternionic vector space with real dimension $4n$, for $n \in \mathbb{N}$. Such a space space is known as a **HyperKähler manifold**.

The holonomy group of a $4n$-dimensional HyperKähler manifold is $Sp(n)$ which is a subgroup of $SU(2n)$. For $n = 1$, i.e a 4 real dimensional HyperKähler, the holonomy group is $Sp(1) \cong SU(2)$, which is the conditionfor the space to be Ricci flat.

The ALE spaces of chapter 4 are the simplest class of noncompact HyperKahler manifolds.