N=4 Gauge Theories in 3D, Hilbert Series, Mirror Symmetry and Enhanced Global Symmetries

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Abstract

Brane constructions of 3D $N = 4$ supersymmetric gauge theories are revised in order to explain mirror symmetry. After introducing quiver notation and the definitions of Higgs branch, Coloumb branch and Hilbert series, a new method for computing the Hilbert series of the Coloumb branch is explained and applied to many gauge theories. Comparing the Higgs branch of one theory with the Coloumb of the mirror, mirror symmetry is confirmed. Also, enhancements of the global symmetry of the Coloumb branch are identified from the results for the Hilbert series.

Contents

1 Introduction

In the study of supersymmetric theories the existence of dualities can be very useful for the calculation of quantities that may be difficult to obtain in one theory but easier in its dual. In Ref [1] a new duality was proposed by K. Intriligator and N. Seigberg known as three dimensional mirror symmetry. Such duality provides two gauge theory descriptions in the UV of the same $\mathcal{N} = 4$ superconformal field theory. These theories have a moduli space that splits into a Higgs branch and a Coloumb branch that intersect in the origin. The Higgs branch receives no quantum corrections, but the Coloumb branch is modified by quantum effects in the infrared. Mirror symmetry exchanges both branches. For the computation of the Coloumb branch, mirror symmetry provides an alternative to the computation of the quantum corrections of the metric of the moduli space; it is only necessary to compute the Higgs branch of the mirror theory, which might be easier to do.

With the help of sting theory and brane engineering it is fairly easy to find many pairs of mirror theories. In [2] the authors find the mirror of $U(k)$ and $SU(k)$

theories with m flavors and in [3], using orientifold planes, for $Sp(N)$ and $SO(N)$. However, the gauge group of many mirror theories is too large even for computing its Higgs branch, in that case mirror symmetry is not useful for finding the Coloumb branch of the original theory using the mirror.

Very recently, a new method for computing the Hilbert series of the Coloumb branch has been proposed in [4]. The Hilbert series is a generating function that counts chiral operators and is graded according to their dimension, in this case the conformal dimension. The set of chiral operators to be considered are the Casimir invariants of the unbroken gauge group of the theory and Weyl invariant combinations of monopole operators dressed by the classical fields.

The purpose of this work is firstly providing an introduction to brane engineering of gauge theories, with and without orientifold planes, using $D3-$, $D5-$ and $NS5-$branes, and then giving the rules for computing mirror pairs. Afterwards, quiver notation of gauge theories is reviewed as a tool for summarising the information of a gauge theory in a compact and practical way. Higgs branches are defined and a method for finding their Hilbert series using the Molien-Weyl integral is explained with detail. Then , the new method for computing Hilbert series of Coloumb branches is presented with a computation of the G_2 case as an example. Finally, many different theories are studied, using the new formula for comparing the Higgs branch of one theory with the Coloumb branch of its mirror in order to confirm mirror theory, as well as finding the Hilbert series of the Coloumb branch of a theory whose mirror theory has a gauge group too complicated for being able to find the Hilbert series of its Higgs branch. Also, monopole operators will give rise to an enhancement of the global symmetry of the Coloumb branch, which will be seen from the result for the Hilbert series.

2 Brane Constructions and Mirror Symmetry

This chapter revises the methods of construction of brane configurations, starting with the most basic setups in the first subsection, then explaining the rules for the Hanany-Witten transition introduced in [2], followed with an example where mirror symmetry is studied in detail. The last two subsections are devoted to brane constructions using O3-planes, which generates theories with non-unitary groups, revision based on the reference [3].

2.1 Simple Brane Constructions

The massless limit of string theory is known to be a supergravity. For type IIA we may find the field content by dimensional reduction of the multiplet for M theory in 11 dimensions, but we are interested in type IIB, where the massless fields can be found by tensor product of the vector multiplet of N=2 supersymmetry in 10 dimensions by itself, finding: the graviton, the 2-form, the dilaton, 2 gravitinos of same chirality, 2 gravifermions of same chirality, a 4-form, another 2-form, and the axion, a compact scalar. The variation of the action for the p-forms $C^{(p)}$ gives Maxwell equations for each of them of the form:

$$\begin{aligned} d * G^{(p+1)} &= \delta^{(D-p)} \\ dG^{(p+1)} &= \delta^{(p+2)} \end{aligned} \tag{1}$$

where $G^{(p+1)} = dC^{(p)}$ and D is the dimension of spacetime. For $D = 10$ and $p = 0, 2, 4$ we find that the sources of the form fields are objects localized in 10,8,6 dimensions, respectively, solving the electric equation or localized in 2,4,6, respectively, solving the magnetic equation. We call them Dp-branes: $D(-1), D1, D3$ and $D7, D5, D3$ (coincides with the previous $D3$ since the 4-form is selfdual), respectively, each one extended in p+1 dimensions. Taking into account that we have two 2-forms, there will be two 1-branes and two 5-branes, call the extra pair F1 and NS5.

These objects can be understood as charged objects that may fill spacetime, just like the usual particle-like charges that we have in 3+1 dimensional spacetime. Therefore, we can have many different configurations of branes with properties to be studied. Since supergravity in 10 dimensions has $N = 2$ supersymmetry with 32 supercharges, and we want to keep only 8 supercharges ($N = 4$ in 3 dimensions, or $\mathcal{N} = 2$ in 4 dimensions), we need to find a configuration that breaks 3/4 of the supersymmetry. Consider first a D5-brane localised in 4 dimensions, without loss of generality we can take $x^6 = x^7 = x^8 = x^9 = 0$, this breaks half of the supersymmetry. An NS5-brane localised at the same point would break the other half of supersymmetry generators, so if we want to combine both types of branes they cannot be both located at definite values of the same coordinates. If instead we place a D5-brane at definite values of x^3, x^4, x^5, x^6 and an NS5-brane at definite values of x^6, x^7, x^8, x^9 then 1/4 of the supersymmetry is preserved, 8 supercharges giving $N = 4$ in 3D. This configuration still allows D3-branes localised in $x^3, x^4, x^5, x^7, x^8, x^9$ while preserving the same amount of supersymmtry.

Let us call $\vec{m} = (x^3, x^4, x^5)$ and $\vec{w} = (x^7, x^8, x^9)$, then the position of an NS5-brane is given by $t = x^6$ and $\vec{w}$, that of a D5-brane by $z = x^6$ and $\vec{m}$, and a D3-brane by $\vec{x} = \vec{m}$ and $\vec{y} = \vec{w}$. With this notation one possible configuration of branes is the following:

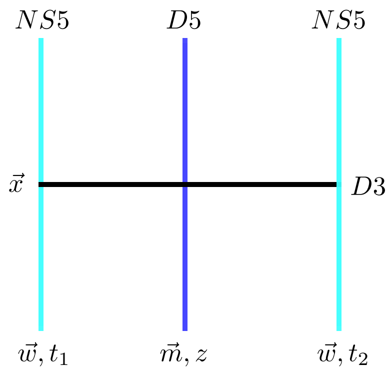


Figure 1: U(1) theory with one electron

Note that for D3 and NS5 to intersect it must be that $\vec{w}_1 = y$ and $\vec{w}_2 = y$ and so $\vec{w}_1 = \vec{w}_2 = \vec{w}$. If instead D3 connects two D5-branes then it must be that $m_1 = m_2 = \vec{x}$. The last possible case is when a D3-brane stretches between an

NS5-brane and a D5-brane, then both values of $\vec{x}$ and $\vec{y}$ are set by $\vec{x} = \vec{m}$ of D5 and $\vec{y} = \vec{w}$ of NS5.

The field content on a single infinite D3-brane is an $\mathcal{N} = 4$ vector multiplet in 4 dimensions transforming under a U(1) gauge group. When a D3-brane ends on a 5-brane, half of the supercharges are broken and the vector multiplet decomposes into an $\mathcal{N} = 2$ vector multiplet and a hypermultiplet. Boundary conditions set either the vector multiplet or the hypermultiplet to zero, depending on whether the D3-brane ends on a D5 or NS5 brane. Since in our case one dimension of the D3-brane is finite, the effective field theory on the world-volume of the brane is 2+1 dimensional. The bosonic massless modes will be either $\vec{x}$ plus the vector a_μ of the vector multiplet, in the case when D3 ends on NS5 and the boundary conditions set the hypermultiplet to 0, or $\vec{y}$ plus the fourth scalar b of the hypermultiplet, when D3 ends on D5 and boundary conditions set the vector multiplet to 0. Note that in three dimensions a vector is dual to a scalar and therefore we can consider the scalar dual to a_μ. The vacua of the field theory is parametrized by the scalars of the vector and hyper multiplets. In three dimensions this moduli space splits into two branches, two submanifolds, the Coloumb branch and the Higss branch, the former spanned by the scalars belonging to vector multiplets and the latter by scalars belonging to hypermultiplets.

We know from supersymmetry that $N = 4$ in three dimensions has an $SO(4) = SU(2)_V \times SU(2)_H$ R-symmetry. From our brane construction we see that the Lorentz group $SO(1,9)$ is broken down to $SO(1,2) \times SO(3) \times SO(3)$ where the first factor acts on x^0, x^1, x^2, the second on $\vec{m} = (x^3, x^4, x^5)$ and the third on $\vec{w} = (x^7, x^8, x^9)$. Calling $SO(3)_V$ the second factor and $SO(3)_H$ the third one, we can identify the R-symmetry with the double covers of these groups. $SU(2)_V$ is a symmetry of the Coloumb branch and $SU(2)_H$ of the Higgs branch.

Now we are ready to introduce mirror symmetry. An $SL(2,\mathbb{Z})$ transformation exchanges D5 and NS5. Combining this with a rotation that maps x^j to x^{j+4} and x^{j+4} to x^j for $j = 3, 4, 5$, while leaving the other coordinates invariant, the net effect is that our brane configuration is mapped to itself, except for D5 and NS5 branes, that are exchanged. This is what is called mirror symmetry. Therefore, mirror symmetry exchanges vector multiplets with hypermultiplets, i.e. $\vec{x}$ with $\vec{y}$ and b with the scalar dual to a_μ.

So far we have interpreted the position of D3-branes as the scalars in the vector multiplets or the hypermultiplets. What about the other parameters of the configuration? The positions t_1 and t_2 give the coupling constant of the electric gauge group by

$$\frac{1}{g_e^2} = \mid t_1 - t_2 \mid \tag{2}$$

up to a multiplicative constant. There is a global symmetry of the theory living on the D3-branes given by the symmetry group of the set of parallel NS5-branes. Two NS5-branes give $U(1) \times U(1)$ global symmetry that becomes $U(1)_{ns}$ after factoring out the center, this symmetry is enhanced to $SU(2)_{ns}$ when the coupling constant becomes infinite, $g_e = \infty$, i.e. the two NS5-branes are coincident. The same applies to the position x^6 of D5-branes, the distance between two D5-branes corresponds to the magnetic gauge coupling

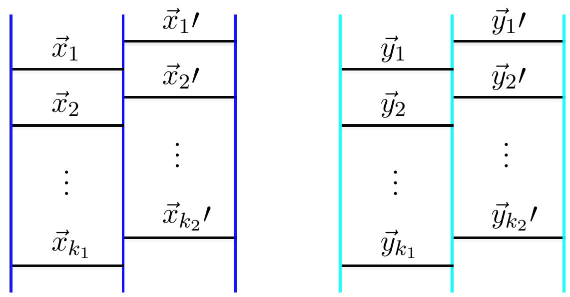
$$\frac{1}{g_m^2} = \mid z_1 - z_2 \mid \tag{3}$$

and the global symmetry of the set of two D5-branes, $U(1)_d$, is enhanced to $SU(2)_d$ when $g_m = \infty$.

On the other hand we have the positions w_1 and w_2 of the NS5-branes. We have mentioned that it must be that $w_1 = w_2$ for a supersymmetric configuration to be possible. However, we can make a transition to a Higgs branch where the two NS5-branes are not connected directly with a D3-brane; in that case it is possible to have $\vec{D} = \vec{w}_1 - \vec{w}_2 \neq 0$. We interpret $\vec{D}$ as a Fayet-Iliopoulos parameter, corresponding to the U(1) factor that the gauge group always has for the case of a stack of D3-branes. The position m of D5-branes will be interpreted later as a mass of a hypermultiplet.

In the general case we will have many D3-branes stretched between D5 and NS5 branes. For the case where n_v parallel D3-branes end on a pair of NS5-branes, we get n_v vector multiplets that transform under $U(1)^{n_v}$ gauge group. When the parallel branes coincide the gauge symmetry is enhanced to $U(n_v)$ by Chan-Paton factors. A good explanation on how this happens may be found in [5]. Another case is when n_h D3-branes stretch between a pair of D5-branes, in that case we get n_h hypermultiplets. Performing a mirror duality the hypermultiplets become vectors that transform under $U(n_h)$ when the branes coincide. The former gauge group is called electric gauge group, the latter, magnetic gauge group. Finally, when D3-branes end on a D5-brane in one side and on an NS5-brane in the other, due to boundary conditions there are no massless modes.

The next question is: what happens when we add more 5-branes in our configuration? Two possibilities would be:



(a) $U(k_1) \times U(k_2)$ electric gauge group

(b) $U(k_1) \times U(k_2)$ magnetic gauge group

Figure 2

In both cases strings stretched between branes i and j, $i = 1, \ldots, k_1$, $j = 1, \ldots, k_2$, give a hypermultiplet the mass of which is proportional to $\mid \vec{x}_i - \vec{x}_{j\prime} \mid$ (or $\mid \vec{y}_i - \vec{y}_{j\prime} \mid$). Hence, when a brane from the left meets a brane from the right, we get a massless hypermultiplet. We have a total of $k_1 k_2$ hypermultiplets transforming as $(k_1, \bar{k}_2)$ of $U(k_1) \times U(k_2)$. The four real scalars in the hypermultiplets transform under the R-symmetry group $SU(2)_V \times SU(2)_H$ as (1,2) in the case of the electric theory and as (2,1) for the magnetic theory.

Another interesting case is that of Figure 1. We have a string stretched between the D5-brane and the D3-brane giving a hypermultiplet of mass $\mid \vec{x} - \vec{m} \mid$, which becomes massless when both branes coincide. The scalars in this hypermultiplet transform as $(1, 2)$ under the R-symmetry $SU(2)_V \times SU(2)_H$. Performing a mirror symmetry we also have the case depicted in the following figure

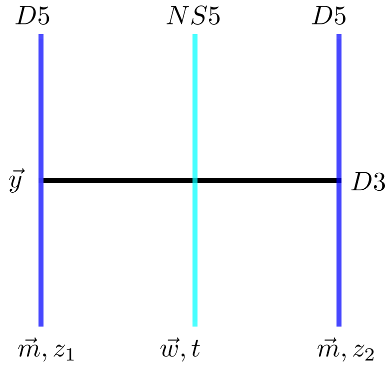


Figure 3: U(1) magnetic theory with one hypermultiplet

with a hypermultiplet of mass $\mid \vec{y} - \vec{w} \mid$ and scalars transforming as (2,1). Adding more branes we find the general case:

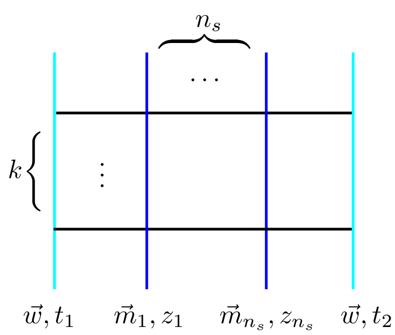


Figure 4: Electric hypermultiplets transforming as n_s fundamentals of $U(k)$

where for each D5-brane we get k hypermultiplets transforming as the fundamental representation of $U(k)$. The fundamental hypermultiplet j^{th}, $j = 1, \ldots, n_s$, has bare mass $\vec{m}_j$. Thus, we see as we said before how the $\vec{m}$ parameter of D3-branes must be interpreted as a mass.

2.2 Hanany-Witten Transition

Consider the configuration of Figure 1 and see what happens when the position z of D5 increases: at $z = t_2$ the D5-brane coincides with the right NS5-brane and then, for $z > t_2$, their positions are exchanged. However, in the original configuration we now know we have a hypermultiplet that becomes massless when $\vec{m} = \vec{x}$, but for the new phase we do not have any hypermultiplet! Something is wrong. In reference [2] a solution was proposed. Define the total magnetic charge measured on an NS5-brane (or its linking number) as

$$L_{NS} = \frac{1}{2}(r - l) + (L - R) \tag{4}$$

where r is the number of D5-branes to the right of NS5, l the number to the left, R the number of D3-branes to the right of NS5 ending on it and L the number ending to the left. Analogously we define

$$L_D = \frac{1}{2}(r - l) + (L - R) \tag{5}$$

It can be shown that those quantities must be preserved under any transition of branes. For the brane configuration of Figure 1 the linking numbers are $L_{NS_l} = -\frac{1}{2}$ for the NS5-brane at the left, $L_{NS_r} = \frac{1}{2}$ for the right one and $L_D = 0$. Moving the D5-brane so that $z > t_2$

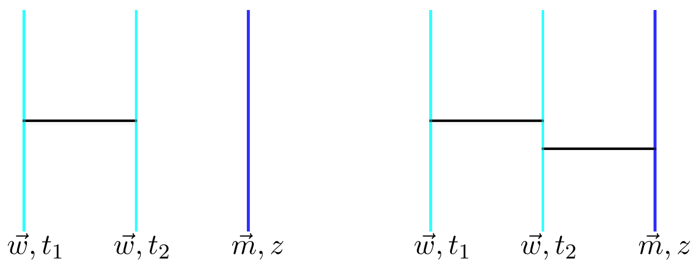
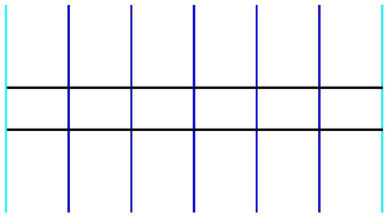


Figure 5: Two possibilities for the phase $z > t_2$ of U(1) theory with one electron

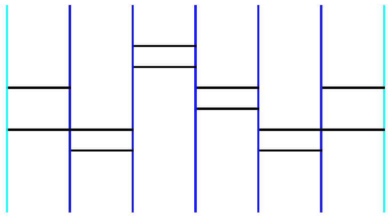
we find $L_{NS_l{\prime}} = -\frac{1}{2}$, $L_{NS_r{\prime}} = \frac{3}{2}$ and $L_{D{\prime}} = -1$!! We can fix this if a D3-brane is created when NS5 and D5 cross each other, then the linking numbers match properly.

2.3 Example of Mirror Symmetry

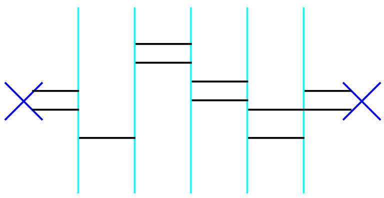
As an example let us consider the case of the $U(2)$ theory with 5 flavors. The brane picture of this theory is represented in figure (a) of the following figure:



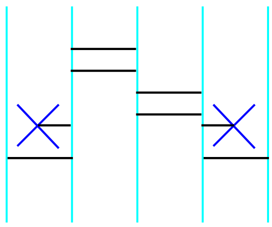
(a) Brane picture: Coloumb branch



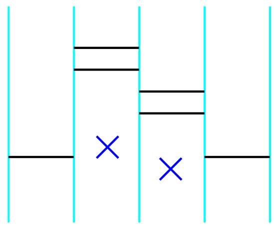
(b) Another phase: Higgs branch



(c) Mirror theory



(d) Another phase of the mirror



(e) Simplest phase of the mirror

The Coloumb branch is spanned by the scalars of two vector multiplets, so it has quaternionic dimension $d_V = 2$. The Higgs branch is spanned by the scalars in the hypermultiplets we get from each D5-brane, 5 hypermultiplets transforming as fundamentals of $U(2)$, this is 10 hypers, giving $d_H = 6$ after substracting four hypermultiplets that make the vector multiplets become massive by the Higgs mechanism. We can move to a Higgs branch connecting the D3-branes in another way such that the free parameters are the positions $\vec{y}$ instead of the positions $\vec{x}$ of the D3-branes. This is depicted in the figure (b).

So far we have been representing NS5-branes with a light-blue line and D5-branes with dark-blue. However, D5-branes span different directions than NS5-branes, so we can think of them as coming out from the paper and therefore represent them as a cross, that way there won't be any confusion on whether a D3-brane intersects them or not. Figure (c) is the mirror of figure (b) where now D5-branes are crosses.

Note that we have not connected the left NS5-brane with the first from the left D5-brane with two D3-branes. Configurations with NS5-branes and D5-branes connected with more than one D3-branes are called s-configurations. For consistency with the field theory, s-configurations should be considered to break supersymmetry and hence we should not allow them. Next we move to another phase in figure (d) performing a brane transition, for this a D3-brane is annihilated when D5 crosses NS5 in order for the linking numbers to be conserved. With another transition we get to the brane picture of figure (e), where we read off the theory as a $U(1) \times U(2) \times U(2) \times U(1)$ gauge theory with hypermultiplets transforming as $(1,2) \oplus (2,2) \oplus (2,1)$ plus another 2 flavor hypermultiplets transforming as the fundamental of each $U(2)$ factor. Counting dimensions we find for the Coloumb branch $d_V = 6$ and for the Higgs $d_H = 2 + 2 \times 2 + 2 + 2 + 2 - 10 = 2$, exactly the opposite of the original theory, as it should be.

Consider now $SU(2)$ gauge group instead of $U(2)$. The Coloumb branch decreases in one quaternionic dimension and the Higgs branch increases in one. Hence, we want for the mirror to increase the Coloumb branch in one dimension and decrease the Higgs in one. It can be shown that this is achieved by gauging the two U(1) flavor groups.

2.4 Brane Constructions with O3 planes

We are now interested in brane constructions for gauge groups Sp(k). For this we will need to use orientifold planes and hence need to know how they change under mirror symmetry, that we recall is basically an S-duality transformation. In the following table we summarize the four different kinds of orientifold planes $O3$, their charges, the gauge group they generate and their S-dual:

O-plane	Charge	Gauge group	S-dual
$O3^-$	$-\frac{1}{4}$	$SO(2n)$	$O3^-$
$\widetilde{O3}^-$	$+\frac{1}{4}$	$SO(2n+1)$	$O3^+$
$O3^+$	$+\frac{1}{4}$	$Sp(n)$	$\widetilde{O3}^-$
$\widetilde{O3}^+$	$+\frac{1}{4}$	$Sp\prime(n)$	$\widetilde{O3}^+$

Table 1: Kinds of O3-planes and their S-duals.

When an O3-plane passes through a 1/2NS5-brane or 1/2D5-brane it changes its type. The different possible cases are exemplified in the following figure:

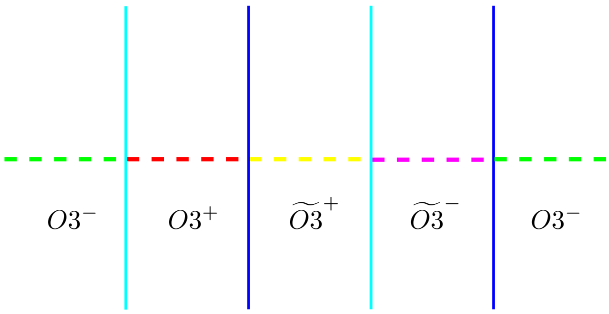


Figure 6: Change of O3 plane when passing through a 5-brane.

Applying S-duality we see that the rule is consistent:

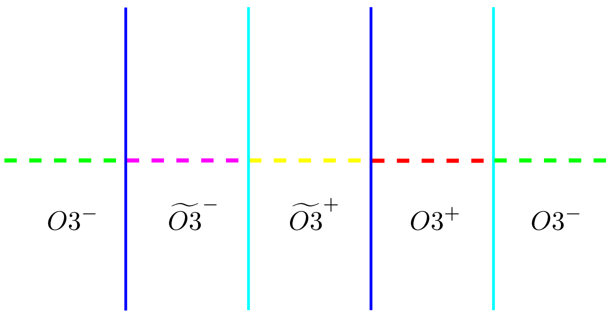
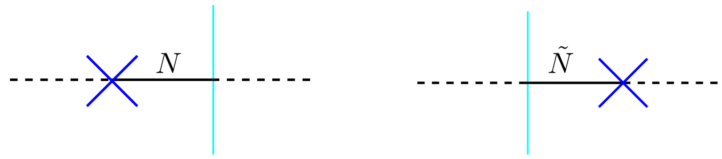


Figure 7: S-dual of the previous figure.

Now we need to know what happens with D3-branes stretched between 1/2NS5-branes and 1/2D5-branes when the half-5-branes cross each other. What we want to know is the relation between the number N of D3-branes stretched between the half-5-branes before the transition and the number $\tilde{N}$ of D3-branes after the transition.

The process is depicted in the following figure:

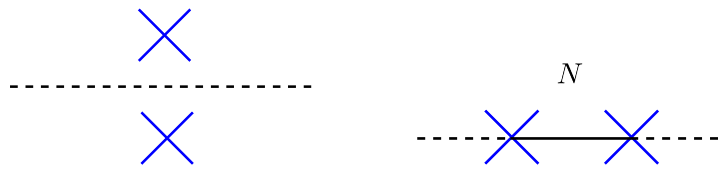


(a) Before transition **(b)** After transition

Figure 8: Transition of half-5-branes with O3 planes

Using the conservation of the linking number for the 1/2NS5-brane and the 1/2D5-brane (half-5-branes have charge 1/2) it is straightforward to check that when the charges of the O3 planes at the left of the left half-5-brane and at the right of the right half-5-brane are the same, there is either creation or annihilation of a D3-brane in the transition satisfying $N + \tilde{N} = 1$, and when the charges are different then there cannot be any D3-brane stretched between the 5-branes, i.e. $N = \tilde{N} = 0$.

Lastly, it remains to be analysed the splitting of D5-branes. That is: the mirror theory is given by D3-branes ending on 1/2NS5-branes, which are the S-duals of 1/2D5-branes, hence we need to split the D5-branes into two independent 1/2D5-branes that move freely on the O3-plane. This process is depicted in the following figure:



(a) D5-brane before splitting (a 1/2D5-brane and its image) **(b)** Two independent 1/2D5-branes after splitting

Figure 9: Splitting of a D5-brane

The value N of D3-branes between the pair of 1/2D5-branes depends on the type of O3-plane. Looking for the configuration with minimum tension, which is proportional to charge/linking number in the case of Dp-branes, we find that $N = 0$ for the orientifolds $O3^+$, $\widetilde{O3}^+$, $O3^-$, and $N = 1$ for $\widetilde{O3}^-$. For the last case, consistency with the dimension of the Higgs branch must also be taken into account.

For a general number of D5-branes, starting with linking numbers 0 for all D5-branes the rule is: $L = 0$ for $O3^+$ and $\widetilde{O3}^+$, $L = \pm\frac{1}{2}$ for $O3^-$ and $\widetilde{O3}^-$ alternating signs and starting with a minus sign from the left. The consequence of this is that a D3-brane will be created between each pair of splitted 1/2D5-branes only in the case of the $\widetilde{O3}^-$-plane, which is represented in the following figure.

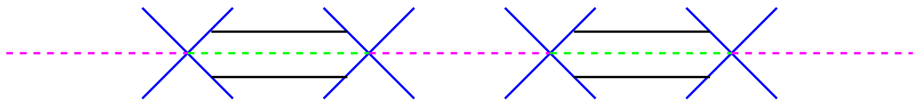


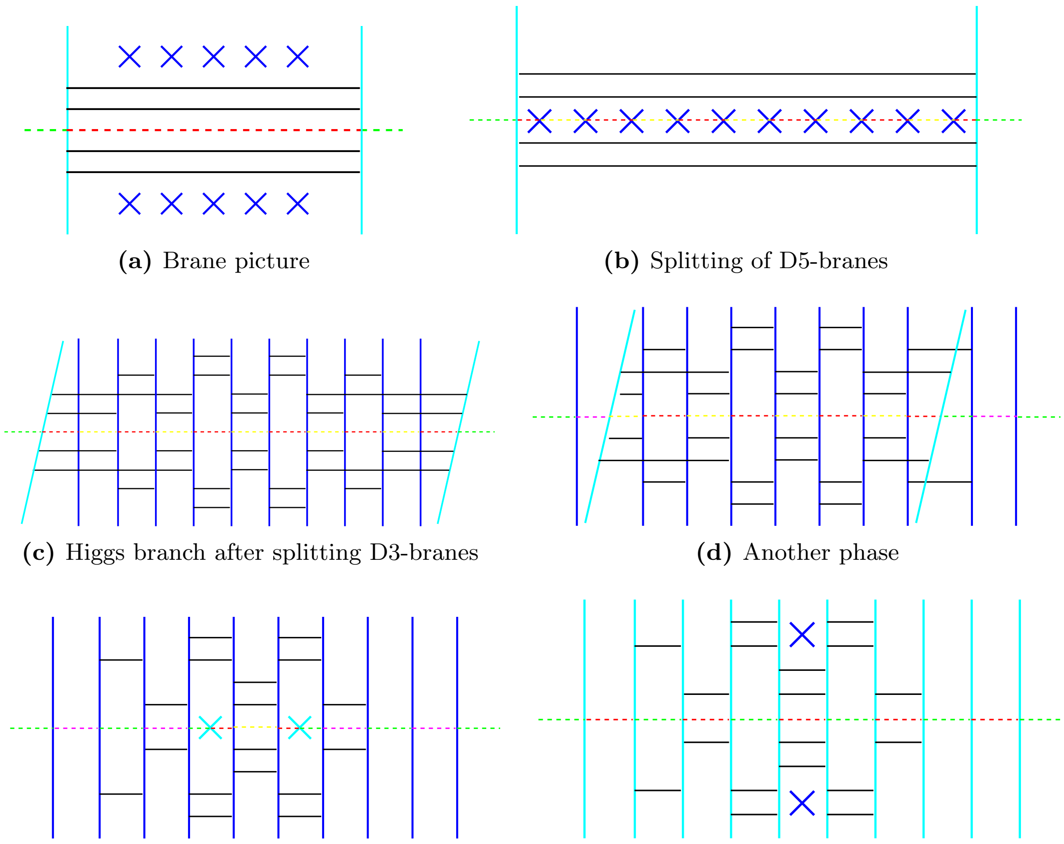
Figure 10: Splitting of two D5-branes.

However, we will only use O3-planes for dealing with Sp(k) theories and hence there won't be any D3-brane created when splitting D5-branes.

2.5 Example of Mirror symmetry with O3-planes

As an example of brane construction and mirror theory with O3 planes we are going to show the case of $Sp(2)$ gauge group with $N_f = 5$ flavors. The brane picture for this theory is depicted in (a) of the following Figure 11.

We begin by splitting all five D5-branes on the orientifold plane, this is diagram (b). Then, for simplicity, we perform a rotation on the diagram so that in (c) the 1/2D5-branes are vertical lines and the 1/2NS5-branes come out from the paper (but we don't use light blue crosses because we need to attach many D3-branes on them), and then split all D3-branes getting a Higgs branch. Note that due to the rules for D3-branes stretched between a 1/2D5-brane and a 1/2NS5-brane we cannot put a D3-brane between the 1/2NS5-branes and the closest to them 1/2D5-branes. Further to this, the second D3-brane that ends on 1/2NS5-brane skips again a 1/2D5-brane. We see the reason for this in diagram (d). With the left 1/2NS5-brane we show a first transition that doesn't change the number of D3-branes. With the right 1/2NS5-brane we perform a second transition, annihilating a D3-brane and reaching a distribution of branes where it is clear that the second D3-brane ending on 1/2NS5-brane must skip also a 1/2D5-brane.



(a) Brane picture
(b) Splitting of D5-branes
(c) Higgs branch after splitting D3-branes
(d) Another phase
(e) Another phase
(f) Mirror theory

Figure 11: Sp(2) with 5 flavors and mirror theory

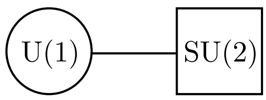
After some more transitions we reach diagram (e), where we can observe that, due to the rules already mentioned, one more transition for each 1/2NS5-brane is possible without neither creation nor annihilation of D3-branes, reaching finally diagram (f) after performing a mirror duality and joining both 1/2D5-branes so that they can leave the orientifold plane. From the last diagram we can read off the gauge group, which is $O(2) \times Sp(1) \times O(4) \times Sp(2) \times O(4) \times Sp(1) \times O(2) \times$, with half-hypermultiplets transforming in the bifundamental of each pair of adjacent factors, and a full hypermultiplet (since the two 1/2D5-branes have joined) transforming in a flavor group $O(2)$.

3 Quivers and Hilbert series of the Higgs branch

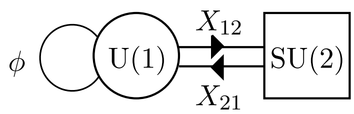
An $\mathcal{N} = 2$ theory in four dimensions is fully specified with the gauge group and the matter fields transforming as hypermultiplets in a certain representation. A quiver encodes this information using nodes for the gauge groups and edges for the hypermultiplets. From an $\mathcal{N} = 2$ quiver we can find the corresponding $\mathcal{N} = 1$ quiver which must be oriented in order to encode also the superpotential. For this we need to do three steps:

- Each gauge group transforms in an $\mathcal{N} = 2$ adjoint vector multiplet, which decomposes in an adjoint vector multiplet and an adjoint chiral multiplet. We add an edge starting and ending at the node corresponding to the gauge group in order to account for this chiral multiplet.
- Each hypermultiplet decomposes into two chiral multiplets, hence substituting each edge for two oriented edges (or one bi-directional edge).
- We read off the superpotential summing up all possible gauge invariant terms, i.e. closed paths of edges in the quiver, that contract properly.

Let us exemplify this with the U(1) theory with two flavors. The global symmetry group is SU(2) and hence the $\mathcal{N} = 2$ quiver is:



Then, the $\mathcal{N} = 1$ quiver is:



where the vector multiplet has decomposed into a vector and a chiral multiplet Φ, and the bifundamental hypermultiplet has decomposed into the two chiral multiplets X_{12} and X_{21}. Hence, the superpotential is

$$\mathcal{W} = Tr\left[X_{21} \cdot \phi \cdot X_{12}\right] \tag{6}$$

The moduli space of vacuum expectation values of scalar fields a priori, without any restriction, would be $\mathbb{C}^E$, where E is the number of complex scalar fields. There are two types of restriction, the existence of a superpotential $\mathcal{W}$ gives the F-terms, and the Fayet-Iliopoulos the D-terms. We define the *Master Space* to be the subspace defined by the F-terms:

$$\mathcal{F}^b = \mathbb{C}^E / < F - terms > \tag{7}$$

being the F-term condition:

$$\frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} = 0 \tag{8}$$

and the *Mesonic Moduli Space* restricting further to combinations of fields that are gauge invariant, i.e. closed paths of edges on the quiver:

$$\mathcal{M}^{mes} = \mathcal{F}^b /\!/ U(1)^G \tag{9}$$

for the case when the gauge group is a product of G $U(1)$ factors. In the case of gauge group $U(N)^G$ we have

$$\mathcal{M}^{mes}_N = \mathcal{F}^b /\!/ (SU(N)^G \times U(1)^G) = \mathcal{F}^b_N /\!/ U(1)^G = (\mathcal{M}^{mes})^N / S_N \tag{10}$$

where we have defined

$$\mathcal{F}^b_N = \mathcal{F}^b /\!/ SU(N)^G \tag{11}$$

The same definition applies to any non-unitary gauge group.

In the previous example, the F-term condition that defines the Higgs branch is

$$\frac{\partial \mathcal{W}(\Phi)}{\partial \phi} = X_{21} \cdot X_{12} = 0 \tag{12}$$

and the scalars from vector multiplets are set to 0, i.e. $\phi = 0$. We can understand this condition better writing explicitly all the indices. Define $A_i = X_{21}$, $B^j = X_{12}$, then:

$$\mathcal{W} = Tr\left[A_i \Phi^i_j B^j\right] \qquad 0 = \frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} = Tr\left[A_i B^j\right] = A_i B^i \tag{13}$$

Defining the matrix $M_i^j = A_i B^j$ this is equivalent to requiring

$$Tr\,[M] = 0 \qquad M^2 = A_i B^j \cdot A_j B^k = 0 \tag{14}$$

the first equation being the F-term condition and the second a consequence of the definition of M.
For any number of flavors N_f, and gauge group U(1), we find analogously:

$$\mathcal{M}^{mes} = \{M_{N_f \times N_f} \mid TrM = 0, M^2 = 0\} \tag{15}$$

We need now a way to characterize the moduli spaces so that we can compare and classify them easily. We use for this the Hilbert series of the moduli space, which encodes the algebraic structure of the space. The *Hilbert series* is a partition function counting gauge invariant operators in the chiral ring. If we have a set of basis operators, say $\{a, b, c\}$, the complete set of operators made out of them is

$$\left\{1, a, b, c, a^2, ab, ac, b^2, bc, c^2, a^3, a^2b, a^2c, ab^2, ac^2, abc, b^3, b^2c, bc^2, c^3, a^4, ...\right\} \tag{16}$$

and we can count them keeping track of the degree of each operator using the Plethystic Exponential. Using t_i as dummy variable for the degree of each of the basis operators, the Hilbert series is

$$HS = PE[t_1 + t_2 + t_3] = PE[t_1]PE[t_2]PE[t_3] = \frac{1}{1-t_1}\frac{1}{1-t_2}\frac{1}{1-t_3} \tag{17}$$

and using just one variable for the degree of the operators, $t = t_1 = t_2 = t_3$,

$$HS = \frac{1}{(1-t)^3} \tag{18}$$

Furthermore, the generators may be in a certain representation of a certain group, defined by a character. Take for instance a,b,c to be in the adjoint of $SU(2)$, whose character is $x^2 + 1 + x^{-2}$. Then we can keep track of the representation of all the operators of the Hilbert series in which they transform just taking into account this fugacity. In this case:

$$HS = PE[x^2 t + t + x^{-2} t] = \frac{1}{(1 - x^2 t)(1 - t)(1 - t/x^2)} \tag{19}$$

Finally, the generators may satisfy a certain relation. Let us return to the example of U(1) with 2 flavors. Here we have three operators M_{11}, M_{12}, M_{21} of order 2, the three independent components of the matrix M, order 2 because each of them comes from $M_i^j = A_i B^j$, and we take order 1 to be the order of the complex scalars X_{12} and X_{21}. These operators satisfy the relation $M^2 = 0$, which translates into $M_{11} M_{22} = M_{12} M_{21}$, i.e. detM=0, that transforms as a singlet. Hence, the Hilbert series takes the form

$$HS = (1 - t^4) PE[x^2 t^2 + t^2 + x^{-2} t^2] = \frac{1 - t^4}{(1 - x^2 t^2)(1 - t^2)(1 - t^2/x^2)} \tag{20}$$

where the first factor keep track of the relation, and now we have set the order of the generators to be 2.

The calculation of the Hilbert series of the Higgs branch can be done in a more systematic way. We define the F-flat Hilbert series the partition function that takes into account the F-term conditions but not the gauge invariance of the operators. Starting from the operators A_i and B_j for any number N_f of flavors, they transform as the fundamental and the antifundamental, respectively, and have charges -1,+1 under the gauge group U(1). The relation $Tr\left[A_i B^j\right]$ is of order two and transforms as a singlet. Then,

$$g^{\mathcal{F}^b} = (1 - t^2) PE\left[[1, 0, \ldots, 0]_{SU(N_f)} tz + [0, \ldots, 0, 1]_{SU(N_f)} \frac{t}{z}\right] \tag{21}$$

which, setting the fugacities for the representations of $SU(N_f)$ to 0, for the sake of simplifying the calculations, gives:

$$g^{\mathcal{F}^b}(t, z) = \frac{1 - t^2}{(1 - tz)^{N_f} (1 - t/z)^{N_f}} \tag{22}$$

Now we can project onto the set of gauge invariant operators by performing a Molien-Weyl integral:

$$HS = g^{Higgs}(t) = \frac{1}{2\pi i}\oint d\mu_G(z) g^{\mathcal{F}^b}(t,z) \tag{23}$$

where $d\mu_G(z)$ is the Haar measure of the gauge group. In the case $N_f = 2$:

$$g^{Higgs}(t) = \frac{1}{2\pi i}\oint \frac{dz}{z}\frac{1-t^2}{(1-tz)^2(1-t/z)^2} \tag{24}$$

which can be easily solved by the residue theorem:

$$\begin{aligned} g^{Higgs}(t) &= \underset{z=t}{Res}\left[\frac{1}{z}g^{\mathcal{F}^b}(t,z)\right] = (1-t^2)\frac{d}{dz}\left[\frac{1}{z}\frac{(z-t)^2}{(1-tz)^2(1-t/z)^2}\right] = \\ &= \cdots = \frac{1+t^2}{(1-t^2)^2} \end{aligned} \tag{25}$$

and coincides with the result that we already got if we set the fugacity $x = 0$.

In ref [6] they introduce and apply this method for calculating the Hilbert series of the moduli space of instantons for different gauge groups, which is identified with the Higgs branch of supersymmetric theories where the characteristic group becomes de global symmetry of the theory.

4 Hilbert series of the Coloumb branch and Enhanced Global Symmetries

A hidden symmetry is generated by a set of conserved currents which are not associated to a symmetry of an action. In any 3D gauge theory with a $U(1)$ factor in the gauge group we find a topological conserved current

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} Tr F_{\nu\lambda} \tag{26}$$

In Ref [1] the authors predicted hidden symmetries of the mirror theories of $U(1)$ and $SU(2)$ with N_f flavors, showing explicitly in the original theory as the flavor symmetry, which are generated by the topological currents of each of the U(1) factors of the gauge group and by monopole operators of conformal dimension 1. A nice review on magnetic monopoles may be found in [7], and also a treatment of the metric of $SU(N)$ monopoles in [8]. The chiral scalars, superpartners of the topological currents, to be considered are the Casimir operators of the gauge group; these together with the monopole operators generate the Coloumb branch of the moduli space of the theory, and those of conformal dimension 1 must have topological charges corresponding to the roots of the global hidden symmetry group.

The conformal dimension of a monopole operator equals the energy of a state of a supersymmetric theory on $\mathbb{R} \times S^2$ the states of which are in one-to-one correspondence with the monopole operators of the original theory on $\mathbb{R}^3$. For an arbitrary gauge group of rank r with N_f flavors we have r magnetic charges and the conformal dimension is given by

$$\Delta = -\sum_{r^+} \mid r^+(H) \mid + \frac{1}{2} \sum_{w} \mid w(H) \mid \tag{27}$$

where the first sum runs over all positive roots in a chosen basis and the second over all weights of the representation in which the hypermultiplets transform. H is the embedding that assigns a magnetic charge to each $U(1)$ factor of the gauge group corresponding to each Cartan generator. Examples of this formula for different gauge groups can be found in [9], and more details about its origin in [10], [11].

Let us exemplify how this works with a G_2 theory with hypermultiplets transforming in the fundamental representation. The set of positive roots is

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \qquad W = D_6 \tag{28}$$

and W is the Weyl group. The weights of the fundamental representation of G_2 are

$$\alpha, 2\alpha + \beta, \alpha + \beta, -\alpha, -2\alpha - \beta, -\alpha - \beta, 0 \tag{29}$$

and hence the conformal dimension of monopole operators is given by

$$\Delta = -\left(\mid m_1 \mid + \mid m_2 \mid + \mid m_1 + m_2 \mid + \mid 2m_1 + m_2 \mid + \mid 3m_1 + m_2 \mid + \mid 3m_1 + 2m_2 \mid\right)$$
$$+ \frac{N_f}{2}\left(\mid m_1 \mid + \mid 2m_1 + m_2 \mid + \mid m_1 + m_2 \mid + \mid -m_1 \mid + \mid -2m_1 - m_2 \mid + \mid -m_1 - m_2 \mid + \mid 0 \mid\right)$$

$$= N_f\left(\mid m_1 \mid + \mid 2m_1 + m_2 \mid + \mid m_1 + m_2 \mid\right)$$
$$-\left(\mid m_1 \mid + \mid m_2 \mid + \mid m_1 + m_2 \mid + \mid 2m_1 + m_2 \mid + \mid 3m_1 + m_2 \mid + \mid 3m_1 + 2m_2 \mid\right) \tag{30}$$

The Coloumb branch is the set of chiral operators of the form

$$\{V_{\vec{m}} \phi_1^{n_1} \phi_2^{n_2}\}_{\vec{m} \in \mathbb{Z}^r; n_1, n_2 \in \mathbb{N}} \tag{31}$$

that are invariant under Weyl transformations, where ϕ_1, ϕ_2 are the two diagonal chiral complex scalars of the vector multiplet and V_m is a monopole operator of charge $\vec{m}$. The Hilbert series is given by

$$HS = \sum_{\vec{m}} t^{\Delta} P(\vec{m}) \tag{32}$$

where t is a dummy variable that counts the degree of the operator and P is the product of plethystic exponential of each Casimir invariant of the unbroken gauge group, which is broken by the values of the magnetic charges.

The different possibilities for the unbroken gauge group in the case of the example are summarised in the following table:

$\vec{m}$	Unbroken group	Degrees of Casimirs	$P(\vec{m})$
$(0,0)$	G_2	2,6	$\frac{1}{(1-t^2)(1-t^6)}$
$(m,0),(0,m)$	$U(1)\times SU(2)$	1,2	$\frac{1}{(1-t)(1-t^2)}$
(m_1,m_2)	$U(1)\times U(1)$	1,1	$\frac{1}{(1-t)^2}$

Table 2: Different breakings of the gauge group by the set of values of the magnetic charges.

Since we want only Weyl invariant operators we restrict the sum to:

$$HS = \sum_{m_1, m_2 \in \mathbb{N}} t^{\Delta} P(\vec{m}) \tag{33}$$

and solving the four cases in the table separately using the conformal dimension formula given above we find:

$$HS_{G_2, N_f} = \frac{1 + t^{2N_f - 4} + t^{2N_f - 3} + t^{2N_f - 2} + t^{2N_f - 1} + t^{4N_f - 5}}{(1 - t^2)(1 - t^6)(1 - t^{2N_f - 6})(1 - t^{2N_f - 5})} \tag{34}$$

An algebraic variety is defined by the simultanous vanishing of a set of homogeneous polynomials. When the difference between the number of generators and the number of relations equals the dimension of the space we call the variety a complete intersection. The generators always appear in the Hilbert series as monomials in the denominator, one factor for each generator, the order of the monomial indicating the order of the generator. In the case of a complete intersection we find a factor in the numerator for each relation, the order of which corresponds to the order of the relation.

In the case of G_2 we see that the numerator does not factorise, meaning that the Coloumb branch of G_2 is not a complete intersection. In the following section we will see many cases where we find a complete intersection.

5 Computations

5.1 $U(1)$ gauge group with N_f flavors

We start with the simplest gauge group, $U(1)$, with N_f hypermultiplets transforming under a global symmetry group $SU(N_f)$. As explained in section 3 the Higgs branch is given by a Molien-Weyl integral:

$$HS = \oint \frac{dz}{z}(1 - t^2) \frac{1}{(1 - tz)^{N_f}(1 - t/z)^{N_f}} \tag{35}$$

which can be solved for every value of N_f. For the Coloumb branch we need the formula for the conformal dimension of monopole operators. In this case there is only one magnetic charge m and the formula is

$$\Delta = \frac{N_f}{2} \mid m \mid \tag{36}$$

The Hilbert series for the Coloumb branch is

$$\begin{aligned} HS_{U(1),N_f} &= \sum_{m=-\infty}^{\infty} t^{\Delta} P(m) q^m = \frac{1}{1-t} \sum_{m=-\infty}^{\infty} t^{\frac{N_f}{2}|m|} q^m \\ &= \frac{1-t^{N_f}}{(1-t^{N_f/2}q)(1-t)(1-t^{N_f/2}/q)} \end{aligned} \tag{37}$$

where the fugacity q keeps track of the conserved charge under the U(1) symmetry of the topological conserved current. Setting $q = 1$ we find the unrefined Hilbert series:

$$HS = \frac{1-t^{N_f}}{(1-t^{N_f/2})^2(1-t)} = PE[t + 2t^{N_f/2} - t^{N_f}] \tag{38}$$

From this result we see there are three generators of the Coloumb branch: the Casimir operator of order 1, call it Φ, and two monopole operators of dimension $N_f/2$ with charges ± 1, $V_{\pm}$. The relation that the generators satisfy which defines the algebraic variety, is

$$V_+ V_- = \Phi^{N_f} \tag{39}$$

Note also that this Coloumb branch is a complete intersection, since the dimension of this moduli space is one quaternionic dimension, which corresponds to two complex dimensions spanned by three generators restricted by one condition.

5.1.1 $N_f = 2$

The special case of $U(1)$ with two flavors is self-dual and exhibits an enhancement of the $U(1)$ global symmetry associated to the topological conserved current to a $SU(2)$ symmetry. The generators of the Coloumb branch transform in the adjoint of this group and we can observe this in the Hilbert series, in this case is

$$HS_{U(1),2} = \frac{1-t^2}{(1-qt)(1-t)(1-t/q)} \tag{40}$$

and we see that the three generators are of order one and have charges corresponding to the adjoint of $SU(2)$, while the relation is at order two. The SU(2) global symmetry shows explicitly in the mirror theory as a square representing a $SU(2)$ flavor group attached to the $U(1)$ node.

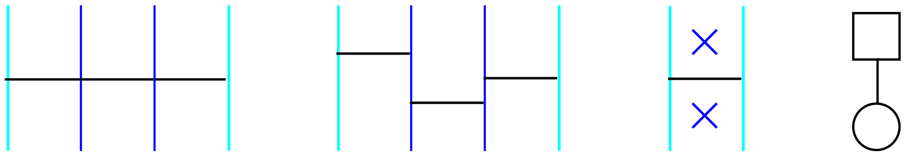
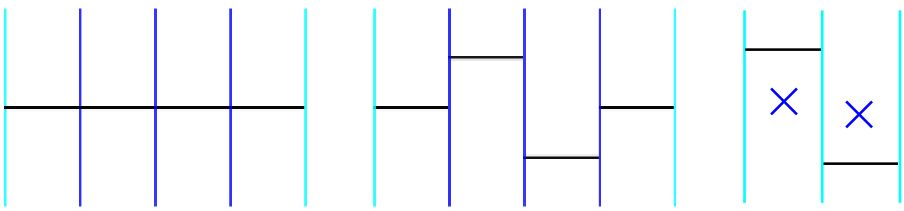


Figure 12: U(1) theory with 2 flavors

Of course, the quiver of the mirror theory is exactly the same as the quiver of the original since, as we already said, this theory is self-dual under mirror symmetry.

5.1.2 $N_f = 3$

This theory has an $SU(3)$ global symmetry of the flavor hypermultiplets (in the brane picture this corresponds to the case when D5-branes are coincident, otherwise it is broken to a subgroup).



(a) Brane picture **(b)** Higgs branch **(c)** Mirror

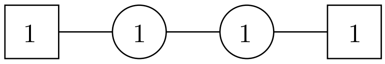
Figure 13: U(1) theory with 3 flavors

For this theory we are going to check that the Higgs branch of the original theory corresponds to the Coloumb branch of the mirror theory. The Hilbert series of the

Higgs branch is given by

$$HS = \oint \frac{dz}{z}(1-t^2)\frac{1}{(1-tz)^3(1-t/z)^3} = (1-t^2)\frac{1}{2}\frac{d^2}{dz^2}\left[\frac{(z-t)^3}{z(1-tz)^3(1-t/z)^3}\right] =$$
$$= \cdots = \frac{1+4t^2+t^4}{(1-t^2)^4} \tag{41}$$

The mirror theory is summarised in the quiver that can be extracted from the above quiver:



all groups being $U(1)$. For this theory we have two magnetic charges m_1, m_2 corresponding to the two $U(1)$ gauge groups, and matter content transforming as a bifundamental hypermultiplet of the two gauge groups and two flavor hypermultiplets, one for each gauge group. Hence, the unrefined Hilbert series is given by:

$$HS = \frac{1}{(1-t)^2}\sum_{m_1,m_2=0}^{\infty} t^{\frac{1}{2}(|m_1|+|m_2|-|m_1-m_2|)} = \cdots = \frac{1+4t+t^2}{(1-t)^4} \tag{42}$$

We see that, after redefining the order of the generators from 1 to 2, we get the same result as the Higgs branch of the original theory.

5.2 $U(2)$ with N_f flavors

Following the procedure detailed in section 4, let us compute the Hilbert series of the Coloumb branch of $U(2)$ with N_f flavors. To begin with we need a set of positive roots for $U(2)$, this is just

$$\Phi^+ = \{e_1 - e_2\}, \tag{43}$$

in the cartesian basis. Using this we find the conformal dimension of monopole operators to be

$$\Delta = \frac{N_f}{2}\left(| \; m_1 \; | + | \; m_2 \; |\right) - | \; m_1 - m_2 \; | \; . \tag{44}$$

The charges of monopole operators give two possibilities for the unbroken gauge group, summarised in the following table:

$\vec{m}$	Unbroken group	Degrees of Casimirs	$P(\vec{m})$
(m,m)	U(2)	1,2	$\frac{1}{1-t}\frac{1}{1-t^2}$
(m_1,m_2)	U(1)xU(1)	1,1	$\frac{1}{1-t}\frac{1}{1-t}$

Table 3: Breaking of $U(2)$ by the set of values of the magnetic charges.

Finally we find the Hilbert series summing over all pairs of magnetic charges satisfying $m_1 \geq m_2$, since for $U(2)$ the Weyl group is the permutations group S_2 that maps the case where $m_2 \geq m_1$ to the former case. After some computations we get

$$\begin{aligned} HS_{U(2),N_f} = \sum_{m_1 \geq m_2 \in \mathbb{Z}}^{\infty} t^{\Delta} P(m_1,m_2) &= \frac{(1-t^{N_f})(1-t^{N_f-1})}{(1-t)(1-t^2)(1-t^{N_f/2-1})^2(1-t^{N_f/2})^2} \\ &= PE[t+t^2+2t^{N_f/2-1}+2t^{N_f/2}-t^{N_f-1}-t^{N_f}]. \end{aligned} \tag{45}$$

Again the Coloumb branch is a complete intersection, now of quaternionic dimension 2, complex dimension 4, generated by six operators under two relations.

5.2.1 $N_f = 4$

In the same way that we found a symmetry enhancement for the $U(1)$ theory with two flavors, we now find the same enhancement for $U(2)$ with four flavors. Evaluating the Hilbert series of the Coloumb branch at $N_f = 4$ we find

$$HS_{U(2),4} = PE[3t+3t^2-t^3-t^4]. \tag{46}$$

In the unrefined case we only see that we have as many generators at order one and at order two as the dimension of the adjoint of $SU(2)$. We could add a fugacity in the Hilbert series keeping track of the value of the conserved charge corresponding to the $U(1)$ symmetry of the topological conserved current and we would see that those generators have charges +1,0,-1 corresponding to the adjoint of $SU(2)$. The conserved charge is the sum of the magnetic charges, m_1+m_2. However, it is not really necessary to repeat the whole calculation with the extra fugacity, we can know what are the charges of the generators just by identifying those generators. Evaluating the formula for the conformal dimension at $N_f = 4$ we see that the only possible monopole operators of conformal dimension one are those with magnetic charges (+1,0), (-1,0),(0,+1),(0,-1). Imposing Weyl invariance we are left with two linear combinations: (1,0)+(0,1) and (-1,0)+(0,-1). Hence, we have two generators at order one that are monopole operators of conserved charge ± 1. The third generator at order one is the first Casimir operator, which has no magnetic charge, and can be written as a trace, $Tr\Phi = \phi_1 + \phi_2$, of the adjoint scalar in the vector multiplet. Dressing the two monopole generators with ϕ_1 and ϕ_2 we find the independent generators at order two with charges ± 1:

$$\{V_{1,0}\phi_2 + V_{0,1}\phi_1, V_{-1,0}\phi_2 + V_{0,-1}\phi_1, Tr\Phi^2 = \phi_1^2 + \phi_2^2\}, \tag{47}$$

the third operator being the second Casimir invariant, neutral under the conserved current.

5.3 $U(3)$ with N_f flavors

Starting with the set of positive roots

$$\Phi^+ = \{e_1 - e_2, e_1 - e_3, e, e_2 - e_3\}, \tag{48}$$

we find the conformal dimension of monopole operators

$$\Delta = \frac{N_f}{2}\left(\mid m_1 \mid + \mid m_2 \mid + \mid m_3 \mid\right) - \left(\mid m_1 - m_2 \mid + \mid m_1 - m_3 \mid + \mid m_2 - m_3 \mid\right). \tag{49}$$

The different cases of breaking of the gauge symmetry by the choice of magnetic charges are:

$\vec{m}$	Unbroken group	Degrees of Casimirs	$P(\vec{m})$
(m,m,m)	U(3)	1,2,3	$\frac{1}{1-t}\frac{1}{1-t^2}\frac{1}{1-t^3}$
(m_1,m_1,m_2)	U(2)xU(1)	1,2,1	$\frac{1}{1-t}\frac{1}{1-t^2}\frac{1}{1-t}$
(m_1,m_2,m_3)	U(1)xU(1)xU(1)	1,1,1	$\frac{1}{1-t}\frac{1}{1-t}\frac{1}{1-t}$

Table 4

Taking into account that the Weyl group is S_3, the Hilbert series is:

$$\begin{aligned} HS_{U(3),N_f} &= \sum_{m_1\geq m_2\geq m_3\in\mathbb{Z}}^{\infty} t^{\Delta} P(m_1,m_2,m_3) = \cdots = \\ &= \frac{(1-t^{N_f})(1-t^{N_f-1})(1-t^{N_f-2})}{(1-t)(1-t^2)(1-t^3)(1-t^{N_f/2-2})^2(1-t^{N_f/2-1})^2(1-t^{N_f/2})^2} \\ &= PE[t+t^2+t^3+2t^{N_f/2-2}+2t^{N_f/2-1}+2t^{N_f/2}-t^{N_f-2}-t^{N_f-1}-t^{N_f}] \end{aligned} \tag{50}$$

Once again, the Coloumb branch is a complete intersection of quaternionic dimension three, generated by the three Casimir invariants of $U(3)$ and six monopole operators (four of them dressed by classical operators), under three relations. As before, for $N_f = 6$ there is an enhancement of the topological $U(1)$ to an $SU(2)$ symmetry such that the generators transform in three adjoints of this group.

5.4 $SU(3)$ gauge group with N_f flavors

The Hilbert series of the Coloumb branch of the $SU(N_c)$ theories can be found by gauging away the topological $U(1)$ symmetry from the Hilbert series of the $U(N_c)$ theory. For doing this we need the refined Hilbert series with a fugacity for the conserved charge, which is the sum of magnetic charges. Let us do the case $N_c = 3$ using the Hilbert series found in the previous section.

We already mentioned that the charge under the topological $U(1)$ for the generators is 0 for the classical operators and ± 1 for the monopole operators. We can see this explicitly building all nine generators:

$$\left\{\begin{array}{l} Tr\Phi,\, Tr\Phi^2,\, Tr\Phi^3,\, V_{100}+V_{010}+V_{001},\, V_{-100}+V_{0-10}+V_{00-1}, \\ V_{100}(\phi_2+\phi_3)+V_{010}(\phi_1+\phi_3)+V_{001}(\phi_1+\phi_2), \\ V_{-100}(\phi_2+\phi_3)+V_{0-10}(\phi_1+\phi_3)+V_{00-1}(\phi_1+\phi_2) \\ V_{100}(\phi_2^2+\phi_3^2)+V_{010}(\phi_1^2+\phi_3^2)+V_{001}(\phi_1^2+\phi_2^2), \\ V_{-100}(\phi_2^2+\phi_3^2)+V_{0-10}(\phi_1^2+\phi_3^2)+V_{00-1}(\phi_1^2+\phi_2^2), \end{array}\right\} \tag{51}$$

The three Casimir invariants have charge 0 and the monopole operators, both dressed by classical operators or fundamentals, have charge either 1 or -1. We can add the fugacity with this conserved charge to the Hilbert series directly, since it does not depend on the basis chosen for the generators. We find:

$$\begin{aligned} HS_{U(3),N_f} = & \frac{(1-t^{N_f})(1-t^{N_f-1})(1-t^{N_f-2})}{(1-t)(1-t^2)(1-t^3)} \cdot \\ & \cdot \frac{1}{(1-zt^{N_f/2-2})(1-\frac{1}{z}t^{N_f/2-2})(1-zt^{N_f/2-1})(1-\frac{1}{z}t^{N_f/2-1})(1-zt^{N_f/2})(1-\frac{1}{z}t^{N_f/2})} \end{aligned} \tag{52}$$

Then we find the Hilbert series of the Coloumb branch of $SU(3)$ by gauging away the topological conserved charge using a Molien-Weyl integral and by eliminating the classical contribution of the Casimir invariant of order one:

$$\begin{aligned} HS_{SU(3),N_f} &= (1-t)\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}HS_{U(3),N_f} = (1-t)\sum_{\substack{z_i\, pole \\ |z_i|<1}} \underset{z=z_i}{Res}\left[\frac{1}{z}HS_{U(3),N_f}\right] = \\ &= (1-t)\left(\underset{z=t^{N_f/2-2}}{Res}\left[\frac{1}{z}HS_{U(3),N_f}\right] + \underset{z=t^{N_f/2-1}}{Res}\left[\frac{1}{z}HS_{U(3),N_f}\right] + \underset{z=t^{N_f/2}}{Res}\left[\frac{1}{z}HS_{U(3),N_f}\right]\right) = \\ &= \cdots = \frac{1+t^{N_f-3}+2t^{N_f-2}+t^{N_f-1}+t^{2N_f-4}}{(1-t^2)(1-t^3)(1-t^{N_f-4})(1-t^{N_f-3})} \end{aligned} \tag{53}$$

In that case the Coloumb branch is not a complete intersection. The dimension of the moduli space is the order of the pole at $t=1$, this gives four complex dimensions, equivalently two quaternionic dimensions.

5.5 $Sp(1) = SU(2)$ gauge group with N_f flavors

Let us now study the $SU(2)$ theory that Intriligator and Seiberg found in [1] to be mirror symmetric to a theory with gauge group $K_{SO(2N_f)} = U(1)^3 \times U(2)^{N_f-3}$ and hidden global symmetry $SO(2N_f)$ which shows explicitly in the $SU(2)$ theory as the flavor symmetry of N_f hypermultiplets. Start with the Coloumb branch. The positive root of $Sp(1)$ is

$$\Phi^+ = \{2e_1\}, \tag{54}$$

and hence the conformal dimension of magnetic monopoles:

$$\Delta = \frac{1}{2}\sum_{i=1}^{2N_f} \mid m \mid -2 \mid m \mid = (N_f - 2) \mid m \mid . \tag{55}$$

The breaking of $Sp(1)$ by the choice of magnetic charges is shown in the following table:

$\vec{m}$	Unbroken group	Degrees of Casimirs	$P(\vec{m})$
0	Sp(1)	2	$\frac{1}{1-t^2}$
m	U(1)	1	$\frac{1}{1-t}$

Table 5

Finally, restricting the summation to the Weyl chamber where $m \geq 0$, the Hilbert series is:

$$HS = \sum_{m=0}^{\infty} t^{\Delta} P(m) = \frac{1 - t^{2N_f-2}}{(1-t^2)(1-t^{N_f-2})(1-t^{N_f-1})} = PE[t^2 + t^{N_f-2} + t^{N_f-1} - t^{2N_f-2}] \tag{56}$$

In the general case of the Coloumb branch of $SU(N)$ with N_f flavors we do not get a complete intersection, as we saw in the case of $SU(3)$, but for the case $N = 2$, which coincides with the $Sp(1)$ theory, we see that it is a complete intersection.

5.5.1 $SU(2)$ with $N_f = 4$

As explained in section 2.3, the mirror of $SU(2)$ with N_f flavors can be obtained from the mirror of $U(2)$ by gauging the $U(1)$ flavor groups. For the case $N_f = 4$ we find the quiver theory:

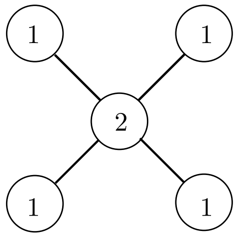


Figure 14: Mirror theory of SU(2) with 4 flavors

Denoting m_i, $i = 1, 2, 3, 4$, the magnetic charges corresponding to the $U(1)$ factors and n_1, n_2 the charges for $U(2)$, after setting $m_4 = 0$ in order to fix the translational invariance that would have the formula for the conformal dimension otherwise, we get:

$$\Delta = \frac{1}{2} \sum_{\substack{j=1,2,3 \\ i=1,2}} \mid m_j - n_i \mid - \mid n_1 - n_2 \mid + \frac{1}{2}(\mid n_1 \mid + \mid n_2 \mid) \tag{57}$$

The Hilbert series for the Coloumb branch is thus

$$HS_{K_{SO(8)}} = \frac{1}{(1-t)^3} \sum_{\substack{m_i \in \mathbb{Z} \\ i=1,2,3}} \sum_{\substack{n_i \in \mathbb{Z} \\ n_1 \leq n_2}} t^{\Delta} P_2(n_1, n_2) \tag{58}$$

with

$$P_2(n_1, n_2) = \begin{cases} a_1^2 & n_1 \neq n_2 \\ a_1 \cdot a_2 & n_1 = n_2 \end{cases} \qquad where \quad a_i = \frac{1}{1-t^i} \tag{59}$$

After some rearrangements of the summations, we find:

$$HS_{K_{SO(8)}} = \frac{1 + 18t + 65t^2 + 65t^3 + 18t^4 + t^5}{(1-t)^{10}} = 1 + 28t + 300t^2 + 1925t^3 + \dots \tag{60}$$

Notice that for order k we get as many operators as the dimension of the representation $[0, k, 0, 0]$ of D_4. We could add fugacities for each $U(1)$ magnetic charge and one fugacity for the conserved charge of the $U(2)$ factor, $n_1 + n_2$, and we would actually get the characters of these representations of D_4. Hence, the hidden symmetry $SO(8)$ is visible in the Hilbert series. Furthermore, this Coloumb branch should coincide with the Higgs branch of the original theory. In ref [6] the authors showed that the Higgs branch of certain $\mathcal{N} = 2$ theories in 4 dimensions are identified with the corresponding moduli space of instantons on $\mathbb{R}^4$. In particular, the Hilbert series for the coherent component of the one D_4 instanton moduli space is precisely

$$g^{Irr}_{D_4}(t) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{D_4} t^k. \tag{61}$$

Therefore, we have seen that the Higgs branch of $SU(2)$ with four flavors has the same Hilbert series as the Coloumb branch of its mirror, as mirror symmetry predicts.

5.5.2 $SU(2)$ with $N_f = 5$

When increasing the number of flavors for the $SU(2)$ theory the quiver of the mirror theory grows adding $U(2)$ factors, and the computation of the Hilbert series of the Coloumb branch of the mirror becomes quite complicated to do analytically. In the case of five flavors the quiver is

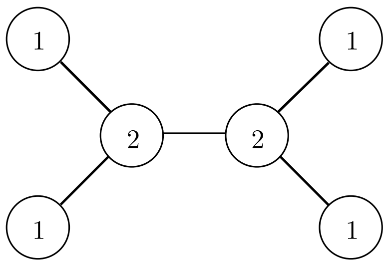


Figure 15: Mirror theory of SU(2) with 5 flavors

and the conformal dimension of monopole operators:

$$\Delta = \frac{1}{2} \sum_{\substack{i=1,2 \\ j=1,2}} \mid m_i - n_j \mid + \frac{1}{2} \sum_{\substack{i=3,4 \\ j=1,2}} \mid m_i - l_j \mid + \frac{1}{2} \sum_{\substack{i=1,2 \\ j=1,2}} \mid n_i - l_j \mid - \mid n_1 - n_2 \mid - \mid l_1 - l_2 \mid \tag{62}$$

where now l_1, l_2 stand for the magnetic charges of the second $U(2)$ factor. The Hilbert series:

$$HS_{K_{SO(10)}} = \frac{1}{(1-t)^3} \sum_{\substack{m_1,m_2,m_3\in\mathbb{Z} \\ n_1\leq n_2\in\mathbb{Z} \\ l_1\leq l_2\in\mathbb{Z}}} t^{\Delta} P_2(n_1,n_2) P_2(l_1,l_2) \tag{63}$$

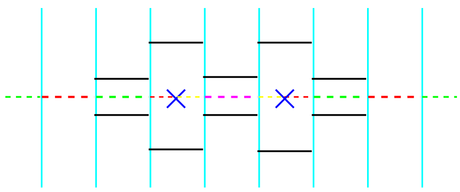
This case could still be done exactly analytically, but it does not worth the time. Calculating order by order we find

$$HS_{K_{SO(10)}} = 1 + 45t + 770t^2 + 7644t^3 + \dots \tag{64}$$

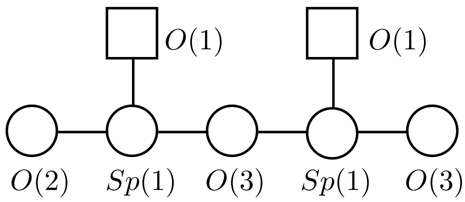
The coefficient of the order k term corresponds to the dimension of the representation $[0,k,0,0,0]$ of $SO(10)$, thus the correspondence with the Higgs branch of the original theory is confirmed, as well as the $SO(10)$ hidden symmetry.

5.5.3 $Sp(1)$ with $N_f = 4$ flavors

Consider now the mirror theory of $Sp(1)$ with 4 flavors found from the brane picture of $Sp(1)$. This theory is depicted in the following figure:



(a) Mirror of Sp(1) with 4 flavors



We find a different mirror than that of $SU(2)$, there is no uniqueness for the mirror of a theory. Although this quiver theory is very different from the $K_{SO(8)}$ theory, both Higgs branches and Coloumb branches must coincide. It would be interesting calculating the Hilbert series for both branches of this theory and confirming that statement.

5.6 $Sp(2)$ gauge group with N_f flavors

Starting with the set of positive roots

$$\Phi^+ = \{2e_1, 2e_2, e_1 - e_2, e_1 + e_2\} \tag{65}$$

we find the following formula for the conformal dimension of monopole operators:

$$\begin{aligned}\Delta &= \frac{1}{2}\sum_{i=1}^{2N_f}(\mid m_1 \mid + \mid m_2 \mid) - (\mid 2m_1 \mid + \mid 2m_2 \mid + \mid m_1 - m_2 \mid + \mid m_1 + m_2 \mid) \\ &= (N_f - 2)(\mid m_1 \mid + \mid m_2 \mid) - \mid m_1 - m_2 \mid - \mid m_1 + m_2 \mid\end{aligned} \tag{66}$$

The breaking of the gauge group by the set of magnetic charges:

$\vec{m}$	Unbroken group	Degrees of Casimirs	$P(\vec{m})$
$(0,0)$	Sp(2)	2,4	$\frac{1}{1-t^2}\frac{1}{1-t^4}$
$(m,0)$	U(1)xSp(1)	1,2	$\frac{1}{1-t}\frac{1}{1-t^2}$
(m,m)	U(2)	1,2	$\frac{1}{1-t}\frac{1}{1-t^2}$
(m_1,m_2)	U(1)xU(1))	1,1	$\frac{1}{1-t}\frac{1}{1-t}$

Table 6

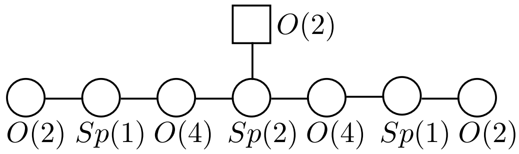
And finally, the Hilbert series of the Coloumb branch is

$$\begin{aligned}HS_{Sp(2),5} &= \sum_{m_1 \geq m_2 \geq 0}^{\infty} t^{\Delta} P(m_1, m_2) \\ &= \frac{(1-t^{2N_f-2})(1-t^{2N_f-4})}{(1-t^2)(1-t^4)(1-t^{N_f-4})(1-t^{N_f-3})(1-t^{N_f-2})(1-t^{N_f-1})} \\ &= PE[t^2 + t^4 + t^{N_f-4} + t^{N_f-3} + t^{N_f-2} + t^{N_f-1} - t^{2N_f-4} - t^{2N_f-2}]\end{aligned} \tag{67}$$

As for $Sp(1)$, the Coloumb branch is a complete intersection, as it will be for any $Sp(k)$ theory.

5.6.1 $N_f = 5$

The brane picture of this theory and its mirror is depicted in figure 11. The quiver for the mirror theory is



Calculating the Hilbert series for either the Higgs branch or the Coloumb branch of this theory can be quite complicated. Let us check that the dimensions of each branch are in correct correspondence with the dimensions of the branches of the original theory. For the original theory, the dimensions of the Coloumb branch and Higgs branch are $d_V = 2$ and $d_H = 10$, respectively. For the mirror:

$$\begin{aligned} d_V &= 1+1+2+2+2+1+1 = 10 \\ d_H &= \frac{1}{2}\left(2\times 2+2\times 4+4\times 4+4\times 2+4\times 4+4\times 2+2\times 2\right) \\ &-(1+3+6+10+6+3+1) = 32-30 = 2 \end{aligned} \tag{68}$$

5.6.2 $N_f = 6$

One last example of $Sp(2)$ theory, now with six flavors. The quiver of the mirror theory is

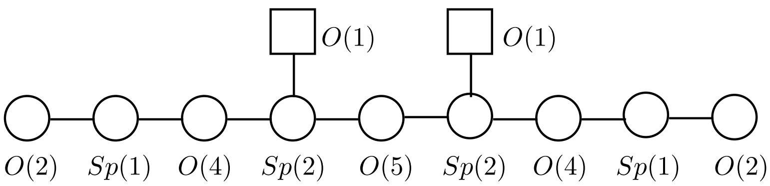


Figure 16: Mirror theory of Sp(2) with 6 flavors

The dimensions of the Higgs and Coloumb branches of the original theory are $d_H = 14$ and $d_V = 2$, respectively. For the mirror:

$$\begin{aligned} d_V &= 1+1+2+2+2+2+2+1+1 = 14 \\ d_H &= \frac{1}{2}\left(2\times 2+2\times 4+4\times 4+4\times 1+4\times 5+5\times 4+4\times 1+4\times 4+4\times 2+2\times 2\right) \\ &-(1+3+6+10+10+10+6+3+1) = 52-50 = 2 \end{aligned} \tag{69}$$

5.7 $Sp(3)$ gauge group with N_f flavors

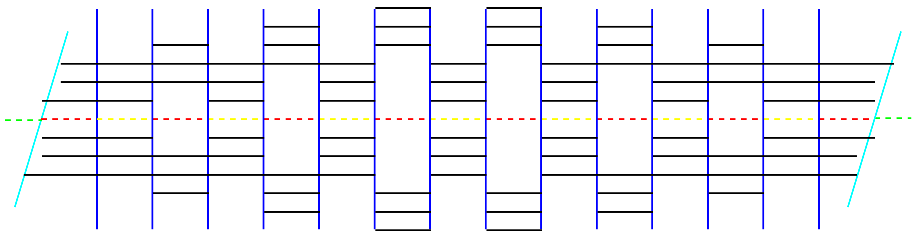
As a last result, we give the Hilbert series for the Coloumb branch of $Sp(3)$ with N_f flavors:

$$HS_{Sp(3),7} =$$
$$= PE[t^2 + t^4 + t^6 + t^{N_f-6} + t^{N_f-5} + t^{N_f-4} + t^{N_f-3} + t^{N_f-2} + t^{N_f-1} - t^{2N_f-6} - t^{2N_f-4} - t^{2N_f-2}] \tag{70}$$

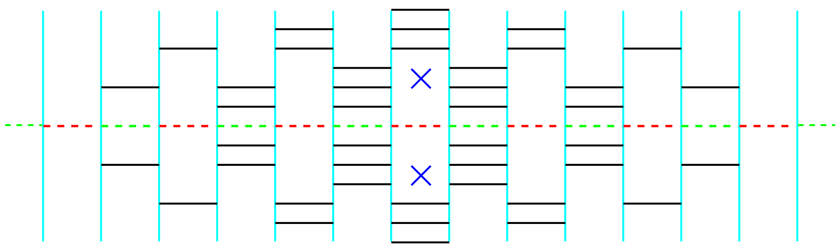
Taking into account the form of the Hilbert series for $Sp(1)$ and $Sp(2)$, it is clear how the general case $Sp(k)$ is going to be.

5.7.1 $N_f = 7$

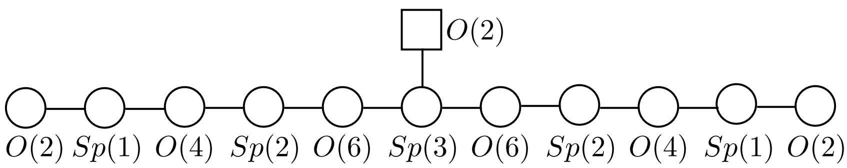
And one more example of brane configuration, its mirror and the corresponding quiver.



(a) Higgs branch of Sp(3) with 7 flavors



(b) Mirror of Sp(3) with 7 flavors



6 Conclusions

Mirror symmetry has been confirmed for U(1) (with 3 hypermultiplets) and SU(2) (with 4 and 5 hypermultiplets) , the theories considered by Intriligator and Seiberg in [1], comparing the Hilbert series of the Higgs branch of each theory with the Hilbert series of the Coloumb branch of the corresponding mirror theory. For the $SU(2)$ theory, an enhanced global symmetry has been observed for the cases with four and five flavors, giving $SO(8)$ and $SO(10)$, respectively, which generalizes to an $SO(2N_f)$ symmetry for N_f number of flavors, and corresponds to the flavor symmetry of the original theory. The Hilbert series of the Coloumb branches of U(1), U(2) and U(3) theories with any number of flavors have been computed and an enhancement of the $U(1)$ topological symmetry to an $SU(2)$ has been observed for the cases with 2, 4 and 6 flavors, respectively. For Sp(1), Sp(2) and Sp(3) the Hilbert series of the Coloumb branch has also been computed, showing that the Coloumb branch of these theories is a complete intersection, which has also been observed to be true for $U(N)$ theories. The computation of the $SU(3)$ theory has shown that in this case, as well as for the G_2 theory, the Coloumb branch is not a complete intersection.

The $SO(N)$ theories have not been computed, as well as the F_4, E_6, E_7 and E_8. These theories, and all the ones considered in this work, are treated in a very recent paper [4].

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