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Born-Infeld Action and Its Applications

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Abstract

This dissertation is a review for the Born-Infeld action. An introduction of bosonic string theory and the object D-brane are given with its light-cone quantization procedure. We then explain the method of obtaining effective action. The derivation for the Born-Infeld action as an effective theory is presented. Later we discuss the properties, supersymmetric extensions, and applications of the Born-Infeld action.

The thesis aims at an elementary and self-contained level. In the bosonic string theory part, the derivation outside the lectured courses of Quantum Fields and Fundamental Forces (QFFF) program 2012-2013 will be presented in detail. No originality is claimed.

Contents

1 Introduction

Our established understanding of nature consists with the Standard Model (SM) and General Relativity (GR). Besides their numerous success, there are still unsolved problems calling a deeper theory. For instance, the singularity in GR [1] and the cosmological constant problem [2] may require quantum gravity. Quantum gravity is also a concern from consistency. Since matters are described by the quantum theory, it is unnatural to construct a classical energy momentum tensor determining the gravity from matter.

We know from the Colella-Overhauser-Werner (COW) gravitational shift experiment [3] that the non-relativistic quantum mechanics at least partially describes the gravity. From the effective field theory point of view, we can take the Einstein-Hilbert action as a leading term. There may be ultraviolate fixing point for gravity. In this direction, it may be expected that an arbitrary-energy-scale-capable quantum gravity theory can be found inside Quantum Field Theory (QFT). It is also true that there are many alternative approaches of quantum gravity, such as causal set [4].

Nevertheless, string theory, which was discovered during studied strong interaction [5], has several attractive features: 1) it is a quantum theory automatically includes gravity; 2) no adjustable parameter; 3) it includes the gauge fields in SM ; 4) it incorporates supersymmetry.

However, the predictions of string theory are mainly at Planck scale [6], for instance the scattering amplitudes [7, p. 166]. The compactification into low energy is far from trivial [8].

The thesis will discuss the Born-Infeld action [9] which is a low-energy effective action for the electromatism of D-brane. In section 2, we introduce the object D-brane. Then by quantizing the bosonic string, the vector field A_μ is found on D-brane. We then discuss the method of obtaining effective action, namely preserving the conformal invariance at quantum level. In the following sections, the properties, supersymmetric extensions, and application of Born-Infeld action are presented. The non-abelian Born-

Infeld action [10] is outside the scope of the present thesis.

2 Born-Infeld action from string theory

2.1 Dirichlet branes

The Born-Infeld action is a low-energy effective action for the D-brane. Here we introduce the object D-brane.

Let S_{p} be a Polyakov (Brink-Di Vecchia-Howe-Deser-Zumino) action [7, p. 12]

$$S_{\mathrm{p}} = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{2.1}$$

where $\frac{1}{2\pi\alpha'}$ is a string tension. τ and σ are time and spatial variables parameterizing the string, respectively. γ_{ab} is the world-sheet metric, with $(-+)$ signature. $\gamma = \det \gamma_{ab}$. X^μ is the spacetime coordinates where the string moves (embedded). The Minkowski metric, $(-+++\cdots)$, is used for the spacetime coordinates. Einstein summation convention is assumed.

Consider a variation for X^μ in the region $\tau \in (-\infty, \infty)$ and $\sigma \in [0, l]$ [7,

p. 14]

$$
\begin{aligned}
\delta S_{\mathrm{p}} &= -\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma(-\gamma)^{1/2}\gamma^{ab}\partial_a\delta X^\mu\partial_b X_\mu \\
&= -\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma\partial_a\left[(-\gamma)^{1/2}\gamma^{ab}\delta X^\mu\partial_b X_\mu\right] \\
&+ \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma\partial_a\left[(-\gamma)^{1/2}\gamma^{ab}\partial_b X_\mu\right]\delta X^\mu \\
&= -\frac{1}{2\pi\alpha'}\int_0^l d\sigma\ (-\gamma)^{1/2}\delta X^\mu\partial^\tau X_\mu\Big|_{-\infty}^{\infty} \\
&- \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau\ (-\gamma)^{1/2}\delta X^\mu\partial^\sigma X_\mu\Big|_0^l \\
&+ \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma(-\gamma)^{1/2}\nabla^2 X_\mu\delta X^\mu.
\end{aligned}
\tag{2.2}
$$

In the second and third lines, integration by parts was performed. The last line was obtained with the help of the expression of the Laplace operator

$$
\partial_a\left[(-\gamma)^{1/2}\gamma^{ab}\partial_b X_\mu\right] = (-\gamma)^{1/2}(-\gamma)^{-1/2}\partial_a\left[(-\gamma)^{1/2}\gamma^{ab}\partial_b X_\mu\right] = (-\gamma)^{1/2}\nabla^2 X_\mu. \tag{2.3}
$$

The surface term of the fourth line in Eq. (2.2) vanishes as the usual asymptotic condition. In analogy with classical mechanics (there is no other sensible reason I have found), we require the surface term of the fifth line in Eq. (2.2) vanishes. There are three classes for the boundary conditions: open-string Neumann boundary condition,

$$
\partial^\sigma X^\mu|_{\sigma=0} = \partial^\sigma X^\mu|_{\sigma=l} = 0, \quad \forall\mu, \tag{2.4}
$$

closed-string periodic boundary condition,

$$
\begin{aligned}
&X^\mu|_{\sigma=0} = X^\mu|_{\sigma=l}\,, \partial^\sigma X^\mu|_{\sigma=0} = \partial^\sigma X^\mu|_{\sigma=l}\,, \quad \forall\mu, \\
&\gamma_{ab}|_{\sigma=0} = \gamma_{ab}|_{\sigma=l}\,,
\end{aligned}
\tag{2.5}
$$

and the Dirichlet boundary condition

$$
X^\mu|_{\sigma=0} = X^\mu|_{\sigma=l} = 0, \quad \forall\mu. \tag{2.6}
$$

Formally $(-\gamma)^{1/2}|_{\sigma=0} = (-\gamma)^{1/2}|_{\sigma=l} = 0$ could also vanish the surface term.

This option is not appearing to be physical. The zero determinant of the metric implies singularity (the author is indebted to Benedict Crampton for this explanation), which we do not expect to happen.

The Dirichlet boundary condition can be combined with open string Neumann boundary condition, such as [11, p. 51],

$$\partial^\sigma X^\mu|_{\sigma=0} = \partial^\sigma X^\mu|_{\sigma=l} = 0, \qquad \mu = 0, 1, 2, \cdots, p, \tag{2.7}$$

$$X^\mu|_{\sigma=0} = X^\mu|_{\sigma=l} = 0, \qquad \mu = p+1, \cdots, D-1, \tag{2.8}$$

where D is the number of space-time dimension. Eq. (2.7) describes open-string ending points moving in $p+1$ dimension. Conventionally this $p+1$ dimensional hypersurface is called Dp brane. Eq. (2.8) corresponds to the open-string ending points in $D-p-1$ dimension fixed on that brane. Eq. (2.6) is for D-1 brane (instanton).

The open- and closed-string boundary conditions are the only possibilities preserves the Poincaré invariance for δS_{p} [7, p. 14]. The Poincaré invariance does not hold for the Dirichlet boundary condition, since the ending points of string are fixed in the corresponding dimension. Therefore the symmetry group is reduced from $\mathrm{SO}(1, D-1)$ to $\mathrm{SO}(1, p) \times \mathrm{SO}(D-p-1)$ [11, p. 51]. It is not a problem from our experience. The Poincaré invariance is for the action, but not the solution of equation of motion. It is like putting an object in the universe. The action for the known underlying theory is Poincaré invariant, but the initial condition for creating such object is not (the author is indebted to Benedict Crampton for this explanation). It should be noticed that continues experimental efforts for testing Lorentz symmetry are carried out. So far, no violation of Lorentz symmetry was found [12].

In addition, the length of world sheet, l, does not contradict with the Poincaré invariance of the action by length contraction. The transformation is for the embedded variable [7, p. 13]

$$X^\mu(\tau, \sigma) \to X'^\mu(\tau, \sigma) = \Lambda^\mu_\nu X^\nu(\tau, \sigma) + c^\mu \tag{2.9}$$

$$\gamma_{ab}(\tau, \sigma) \to \gamma'_{ab}(\tau, \sigma) = \gamma_{ab}(\tau, \sigma) \tag{2.10}$$

not the parameters τ, σ.

2.2 Quantization for open-string on D-brane

In this section we shall quantize the bosonic string ending on brane under boundary conditions (2.7) and (2.8). The vector field and scalar field will be found after quantization.

Before proceeding quantization, there is a few redundency in the action (2.1) we would like to fix. Recall the Polyakov action (2.1) is invariant under the Poincaré (2.9)-(2.10), diffeomorphism, [7, p. 13]

$$\sigma^a(\sigma,\tau) \to \sigma'^a(\sigma',\tau') = \frac{\partial\sigma'^a}{\partial\sigma^b}\sigma^b(\sigma,\tau), \tag{2.11}$$

$$X^\mu(\tau,\sigma) \to X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma), \tag{2.12}$$

$$\gamma_{ab}(\tau,\sigma) \to \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma), \tag{2.13}$$

and Weyl transformations

$$\gamma_{ab}(\tau,\sigma) \to \gamma'_{ab}(\tau,\sigma) = \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma), \quad \forall\omega(\tau,\sigma), \tag{2.14}$$

$$X^\mu(\tau,\sigma) \to X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma). \tag{2.15}$$

The diffeomorphism is a reparametrization for (τ,σ). In Eqs. (2.11)-(2.13) σ^a schematically stands for (τ,σ). Obviously the $d\tau d\sigma(-\gamma)^{1/2} = d\sigma^1 d\sigma^2(-\det(\gamma_{ab}))^{1/2}$ term in action (2.1) is invariant under diffeomorphism [13, p. 103]. The Weyl transformations is a rescaling for the parametrical space (τ,σ). From the structure of indices, the $(-\gamma)^{1/2}\gamma^{ab} = (-\det(\gamma_{cd}))^{1/2}\gamma^{ab}$ term in action (2.1) is invariant under Weyl transformation.

The diffeomorphism and Weyl transformations are local properties, similar with the gauge invariance. In analogy with QFT, we would like to fix them before quantization. Since there are two indices in the metric and one Weyl transformation, we can set three conditions as following [7, p. 17]

$$\tau = X^+, \tag{2.16}$$

$$\partial_\sigma\gamma_{\sigma\sigma} = 0, \tag{2.17}$$

$$\det\gamma_{ab} = -1. \tag{2.18}$$

Using Eqs. (2.17) and (2.18) the inverse metric γ^{ab} becomes [7, p. 18]

$$\begin{bmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma} \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} \gamma_{\sigma\sigma} & -\gamma_{\sigma\tau} \\ -\gamma_{\tau\sigma} & \gamma_{\tau\tau} \end{bmatrix} = \begin{bmatrix} -\gamma_{\sigma\sigma} & \gamma_{\sigma\tau} \\ \gamma_{\tau\sigma} & \gamma_{\sigma\sigma}^{-1}(\gamma_{\sigma\tau}^2 - \gamma_{\sigma\sigma}\gamma_{\tau\tau} - \gamma_{\sigma\tau}^2) \end{bmatrix}$$
$$= \begin{bmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\sigma\tau}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & \gamma_{\sigma\sigma}^{-1}(\tau)(1 - \gamma_{\sigma\tau}(\tau,\sigma)^2) \end{bmatrix}. \tag{2.19}$$

Introducing the light-cone coordinates as following [7, p. 16]

$$X^{\pm} = \frac{X^0 \pm X^1}{\sqrt{2}}, \tag{2.20}$$
$$X_{\pm} = \frac{X_0 \pm X_1}{\sqrt{2}}, \tag{2.21}$$

therefore

$$X_- = -X^+, X_+ = -X^-. \tag{2.22}$$

Similar relation holds for any vector.
Splitting the X^- coordinate as [7, p. 18]

$$X^-(\tau,\sigma) = x^-(\tau) + Y^-(\tau,\sigma), \tag{2.23}$$
$$x^-(\tau) = \frac{1}{l}\int_0^l d\sigma X^-(\tau,\sigma), \tag{2.24}$$
$$Y^-(\tau,\sigma) = X^-(\tau,\sigma) - x^-(\tau), \tag{2.25}$$

the Polyakov Lagrangian can be written as

$$L_{\mathrm{p}} = -\frac{1}{4\pi\alpha'}\int_0^l d\sigma \left[\gamma_{\sigma\sigma}(2\partial_\tau x^- - \partial_\tau X^i \partial_\tau X^i) - 2\gamma_{\sigma\tau}(\partial_\sigma Y^- - \partial_\tau X^i \partial_\sigma X^i) \right.$$
$$\left. + \gamma_{\sigma\sigma}^{-1}(1 - \gamma_{\tau\sigma}^2)\partial_\sigma X^i \partial_\sigma X^i\right], \quad i = 2, 3, \cdots, D-1. \tag{2.26}$$

we have used

$$\begin{aligned}
\int_0^l d\sigma \gamma^{\tau\tau} \partial_\tau X^\mu \partial_\tau X_\mu &\overset{(2.20)}{=} \int_0^l d\sigma \gamma_{\sigma\sigma} (2\partial_\tau X^+ \partial_\tau X^- - \partial_\tau X^i \partial_\tau X^i) \\
&\overset{(2.16)}{=} \int_0^l d\sigma \gamma_{\sigma\sigma} (2\partial_\tau X^- - \partial_\tau X^i \partial_\tau X^i) \\
&\overset{(2.23)}{=} \int_0^l d\sigma \gamma_{\sigma\sigma} (2\partial_\tau x^- + 2\partial_\tau Y^- - \partial_\tau X^i \partial_\tau X^i) \\
&= \int_0^l d\sigma \gamma_{\sigma\sigma} (2\partial_\tau x^- - \partial_\tau X^i \partial_\tau X^i) + 2\gamma_{\sigma\sigma}(\tau) \partial_\tau \int_0^l d\sigma Y^- \\
&= \int_0^l d\sigma \gamma_{\sigma\sigma} (2\partial_\tau x^- - \partial_\tau X^i \partial_\tau X^i) \qquad (2.27)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^l d\sigma 2\gamma^{\tau\sigma} \partial_\tau X^\mu \partial_\sigma X_\mu &\overset{(2.20)}{=} \int_0^l d\sigma 2\gamma_{\tau\sigma} \big(-\partial_\tau X^+ \partial_\sigma X^- - \partial_\sigma X^+ \partial_\tau X^- \\
&\qquad + \partial_\tau X^i \partial_\sigma X^i\big) \\
&\overset{(2.16)}{=} \int_0^l d\sigma 2\gamma_{\tau\sigma} (-\partial_\sigma X^- + \partial_\tau X^i \partial_\sigma X^i) \\
&\overset{(2.23)}{=} -\int_0^l d\sigma 2\gamma_{\tau\sigma} (\partial_\sigma Y^- - \partial_\tau X^i \partial_\sigma X^i). \qquad (2.28)
\end{aligned}$$

Concerning $\partial_\sigma Y^-$ term in the Polyakov Lagrangian (2.26), we can apply the Euler-Lagrange equation for Y^-. After integration by part, we obtain

$$\partial_\sigma \gamma_{\tau\sigma} = 0. \qquad (2.29)$$

The Neumann open-string boundary condition (2.7) can be written as

$$\gamma^{\sigma\sigma} \partial_\sigma X^\mu + \gamma^{\sigma\tau} \partial_\tau X^\mu \overset{(2.19)}{=} \gamma_{\tau\tau} \partial_\sigma X^\mu - \gamma_{\sigma\tau} \partial_\tau X^\mu = 0, \quad \sigma = 0, l; \forall \mu. \quad (2.30)$$

For $\mu = +$, Eq. (2.30) leads to

$$\gamma_{\sigma\tau}|_{\sigma=0,l} = 0. \qquad (2.31)$$

Since we can perform a Taylor expansion for $\gamma_{\sigma\tau}$ with Lagrange remainder

$$\gamma_{\sigma\tau} = \gamma_{\sigma\tau}|_{\sigma=0} + \partial_\sigma \gamma_{\sigma\tau}|_{\sigma=\sigma'}, \quad \sigma' \in (0, l), \qquad (2.32)$$

by Eqs. (2.29) and (2.31), $\gamma_{\sigma\tau} = 0$ everywhere. Therefore the Polyakov Lagrangian (2.26) is simplified to

$$L_p = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^l d\sigma\left(\gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i\partial_\sigma X^i\right). \tag{2.33}$$

The conjugated momentum for x^- and $\partial_\tau X^i$ are [7, p. 19]

$$p_- = \frac{\partial L_p}{\partial(\partial_\tau x^-)} = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma} = -p^+, \tag{2.34}$$

$$\Pi_i = \frac{\delta L}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i = \frac{p^+}{l}\partial_\tau X^i = \Pi^i. \tag{2.35}$$

The Hamiltonian for (2.33) is

$$\begin{aligned} H &= p_-\partial_\tau x^- + \int_0^l d\sigma \Pi_i \partial_\tau X^i - L \\ &= \frac{l}{4\pi\alpha' p^+}\int_0^l d\sigma\left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right). \end{aligned} \tag{2.36}$$

The Hamiltonian mechanics gives the equation of motion

$$\partial_\tau X^i = \frac{\delta H}{\delta \Pi_i} = 2\pi\alpha' c\,\Pi^i, \tag{2.37}$$

$$\partial_\tau \Pi^i = -\frac{\delta H}{\delta X_i} = \frac{c}{2\pi\alpha'}\,\partial_\sigma^2 X^i \tag{2.38}$$

$$\partial_\tau^2 X^i = c^2\partial_\sigma^2 X^i, \tag{2.39}$$

where $c = \frac{l}{2\pi\alpha' p^+}$.

The solutions for the wave equation (2.39) under boundary conditions (2.7) and (2.8) are

$$X^i(\sigma,\tau) = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{1/2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}\alpha_n^i \exp\left(-\frac{icn\pi\tau}{l}\right)\cos\left(\frac{n\pi\sigma}{l}\right), \quad i = 2,3,\cdots,p, \tag{2.40}$$

$$X^i(\sigma,\tau) = a^i + (b^i - a^i)\frac{\sigma}{l} + (2\alpha')^{1/2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}\alpha_n^i \exp\left(-\frac{icn\pi\tau}{l}\right)\sin\left(\frac{n\pi\sigma}{l}\right), \quad i = p+1,\cdots,D-1, \tag{2.41}$$

respectively. Eqs. (2.40) and (2.41) are also called mode expansions.

In Eq. (2.40) x^i and p^i are the Schrödinger operators for the center-of-mass variables $x^i(\tau)$ and $p^i(\tau)$

$$x^i = x^i(\tau)|_{\tau=0}, \qquad x^i(\tau) = \frac{1}{l}\int_0^l d\sigma X^i(\tau,\sigma), \tag{2.42}$$

$$p^i = p^i(\tau)|_{\tau=0}, \qquad p^i(\tau) = \int_0^l d\sigma \Pi^i(\tau,\sigma). \tag{2.43}$$

x^i and p^i are obtained by integrating the zero mode of Fourier expansion over the initial conditions. This can be seen from Eq. (2.42) for x^i and

$$\frac{1}{l}\int_0^l \frac{\partial X^i(\sigma,\tau)}{\partial\tau} \overset{(2.35)}{=} \frac{1}{p^+}\int_0^l \Pi^i d\sigma \overset{(2.43)}{=} \frac{p^-}{p^+} \tag{2.44}$$

for p^i.

The Hermician condition $(X^i)^\dagger = X^i$ leads to $(\alpha_n^i)^\dagger = \alpha_{-n}^i$ in both Eqs. (2.40) and (2.41).

Impose the following equal time commuting relation (only non-zero results are included) [7, p. 20]

$$[x^-, p^+] = -i, \tag{2.45}$$

$$\left[X^i(\sigma), \Pi^j(\sigma')\right] = i\delta^{ij}\delta(\sigma-\sigma'), \tag{2.46}$$

the Fourier modes for the commutators can be obtained in completely analogous as the scalar field theory

$$[x^i, p^j] = \left[\frac{1}{l}\int_0^l d\sigma \int_0^l d\sigma' [X^i(\sigma), \Pi^j(\sigma')]\right] = i\delta^{ij}. \tag{2.47}$$

By Eqs. (2.35), (2.40), and (2.41), we have the expressions of Π^i

$$\Pi^i(\tau,\sigma) = \frac{p^i}{l} + \frac{1}{(2\alpha')^{1/2} l}\sum_{n\neq 0}\alpha_n^i \exp\left(-\frac{icn\pi\tau}{l}\right)\cos\left(\frac{n\pi\sigma}{l}\right),$$
$$i = 2, 3, \cdots, p-1, \tag{2.48}$$

$$\Pi^i(\tau,\sigma) = -\frac{i}{(2\alpha')^{1/2} l}\sum_{n\neq 0}\alpha_n^i \exp\left(-\frac{icn\pi\tau}{l}\right)\sin\left(\frac{n\pi\sigma}{l}\right),$$
$$i = p, \cdots, D-1. \tag{2.49}$$

Since for $i = 2, 3, \cdots p$

$$\int_0^l X^i(\tau,\sigma)\cos\left(\frac{m\pi\sigma}{l}\right)d\sigma$$
$$= \frac{i(2\alpha')^{1/2}l}{2m}\left[\alpha_m^i \exp\left(-\frac{imc\pi\tau}{l}\right) - \alpha_{-m}^i \exp\left(\frac{imc\pi\tau}{l}\right)\right], \quad m \neq 0, \tag{2.50}$$

$$\int_0^l \Pi^i(\tau,\sigma)\cos\left(\frac{m\pi\sigma}{l}\right)d\sigma$$
$$= \frac{1}{2(2\alpha')^{1/2}}\left[\alpha_m^i \exp\left(-\frac{imc\pi\tau}{l}\right) + \alpha_{-m}^i \exp\left(\frac{imc\pi\tau}{l}\right)\right], \quad m \neq 0, \tag{2.51}$$

and for $i = p+1, \cdots, D-1$

$$\int_0^l X^i(\tau,\sigma)\sin\left(\frac{m\pi\sigma}{l}\right)d\sigma =$$
$$\frac{(2\alpha')^{1/2}l}{2m}\left[\alpha_m^i \exp\left(-\frac{imc\pi\tau}{l}\right) + \alpha_{-m}^i \exp\left(\frac{imc\pi\tau}{l}\right)\right], \quad m \neq 0, \tag{2.52}$$
$$\int_0^l \Pi^i(\tau,\sigma)\sin\left(\frac{m\pi\sigma}{l}\right)d\sigma =$$
$$\frac{-i}{2(2\alpha')^{1/2}}\left[\alpha_m^i \exp\left(-\frac{imc\pi\tau}{l}\right) - \alpha_{-m}^i \exp\left(\frac{imc\pi\tau}{l}\right)\right], \quad m \neq 0. \tag{2.53}$$

We have for $i = 2, 3, \cdots p$ and $m \neq 0$

$$\alpha_m^i = \left[\frac{m}{i(2\alpha')^{1/2}l}\int_0^l X^i(\tau,\sigma)\cos\left(\frac{m\pi\sigma}{l}\right)d\sigma\right.$$
$$\left. + \, (2\alpha')^{1/2}\int_0^l \Pi^i(\tau,\sigma)\cos\left(\frac{m\pi\sigma}{l}\right)d\sigma\right]\exp\left(\frac{imc\pi\tau}{l}\right) \tag{2.54}$$

and for $i = p+1, \cdots, D-1$ and $m \neq 0$

$$\alpha_m^i = \left[\frac{m}{(2\alpha')^{1/2}l}\int_0^l X^i(\tau,\sigma)\sin\left(\frac{m\pi\sigma}{l}\right)d\sigma\right.$$
$$\left. + \frac{(2\alpha')^{1/2}}{-i}\int_0^l \Pi^i(\tau,\sigma)\sin\left(\frac{m\pi\sigma}{l}\right)d\sigma\right]\exp\left(\frac{imc\pi\tau}{l}\right) \tag{2.55}$$

Thus for $i = 2, 3, \cdots p$

$$
\begin{aligned}
[\alpha^i_m, \alpha^j_n] = \frac{m}{il} \int_0^l \int_0^l & d\sigma d\sigma' \left[X^i(\sigma), \Pi^j(\sigma')\right] \cos\left(\frac{m\pi\sigma}{l}\right) \cos\left(\frac{n\pi\sigma'}{l}\right) \\
& \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& + \frac{n}{il} \int_0^l \int_0^l d\sigma d\sigma' \left[\Pi^i(\sigma), X^j(\sigma')\right] \cos\left(\frac{m\pi\sigma}{l}\right) \cos\left(\frac{n\pi\sigma'}{l}\right) \\
& \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& = \frac{m}{2}\delta^{ij}\left(\delta_{mn} + \delta_{m,-n}\right) \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& - \frac{n}{2}\delta^{ij}\left(\delta_{mn} + \delta_{m,-n}\right) \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& = m\delta^{ij}\delta_{m,-n}, \quad m \neq 0
\end{aligned}
\tag{2.56}
$$

and for $i = p+1, \cdots, D-1$

$$
\begin{aligned}
[\alpha^i_m, \alpha^j_n] = \frac{m}{-il} \int_0^l \int_0^l & d\sigma d\sigma' \left[X^i(\sigma), \Pi^j(\sigma')\right] \sin\left(\frac{m\pi\sigma}{l}\right) \sin\left(\frac{n\pi\sigma'}{l}\right) \\
& \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& + \frac{n}{-il} \int_0^l \int_0^l d\sigma d\sigma' \left[\Pi^i(\sigma), X^j(\sigma')\right] \sin\left(\frac{m\pi\sigma}{l}\right) \sin\left(\frac{n\pi\sigma'}{l}\right) \\
& \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& = -\frac{m}{2}\delta^{ij}\left(\delta_{mn} - \delta_{m,-n}\right) \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& + \frac{n}{2}\delta^{ij}\left(\delta_{mn} - \delta_{m,-n}\right) \exp\left(\frac{i(m+n)c\pi\tau}{l}\right) \\
& = m\delta^{ij}\delta_{m,-n}, \quad m \neq 0.
\end{aligned}
\tag{2.57}
$$

Eqs. (2.56) and (2.57) imply the modes of open-string and D-brane have the same role of the annihilation and creation operators, respectively [7, p. 20]

$$
\alpha^i_m \sim \sqrt{m}a, \quad \alpha^i_{-m} \sim \sqrt{m}a^\dagger, \quad m > 0, i = 2, 3, \cdots, D-1. \tag{2.58}
$$

To gain further insight into the creation and annihilation operators analogy of the operator α^i_m, we try to write the Hamiltonian (2.36) in terms of α^i_m. For $i = 2, 3, \cdots, p$

$$
\begin{aligned}
&H = \frac{l}{4\pi\alpha' p^+} 2\pi\alpha' \int_0^l d\sigma \left[\frac{p^i}{l} + \frac{1}{(2\alpha')^{1/2} l} \sum_{n\neq 0} \alpha^i_n \exp\left(-\frac{icn\pi\tau}{l} \right) \cos\left(\frac{n\pi\sigma}{l} \right) \right] \\
&\left[\frac{p^i}{l} + \frac{1}{(2\alpha')^{1/2} l} \sum_{m\neq 0} \alpha^i_m \exp\left(-\frac{icm\pi\tau}{l} \right) \cos\left(\frac{m\pi\sigma}{l} \right) \right] \\
&+ \frac{l}{4\pi\alpha' p^+} \frac{1}{2\pi\alpha'} \left[\left(\frac{-i\pi}{l} \right) (2\alpha')^{1/2} \right]^2 \int_0^l d\sigma \sum_{n\neq 0} \alpha^i_n \exp\left(-\frac{icn\pi\tau}{l} \right) \sin\left(\frac{n\pi\sigma}{l} \right) \\
&\sum_{m\neq 0} \alpha^i_m \exp\left(-\frac{icm\pi\tau}{l} \right) \sin\left(\frac{m\pi\sigma}{l} \right) \\
&= \frac{p^i p^i}{2p^+} + \frac{1}{8\alpha' p^+} \sum_{n\neq 0, m\neq 0} \alpha^i_n \alpha^i_m \exp\left(-\frac{i(n+m)c\pi\tau}{l} \right) (\delta_{n,m} + \delta_{n,-m}) \\
&- \frac{1}{8\alpha' p^+} \sum_{n\neq 0, m\neq 0} \alpha^i_n \alpha^i_m \exp\left(-\frac{i(n+m)c\pi\tau}{l} \right) (\delta_{n,m} - \delta_{n,-m}) \\
&= \frac{p^i p^i}{2p^+} + \frac{1}{4\alpha' p^+} \sum_{n\neq 0} \alpha^i_n \alpha^i_{-n} \\
&= \frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+} \left(\sum_{n>0} \alpha^i_{-n} \alpha^i_n + A \right).
\end{aligned}
\tag{2.59}
$$

For $i = p+1, \cdots, D-1$,

$$
\begin{aligned}
H &= \frac{l}{4\pi\alpha' p^+} 2\pi\alpha' \int_0^l d\sigma \left[\frac{-i}{(2\alpha')^{1/2} l} \sum_{n\neq 0} \alpha_n^i \exp\left(-\frac{icn\pi\tau}{l}\right) \sin\left(\frac{n\pi\sigma}{l}\right) \right] \\
&\left[\frac{-i}{(2\alpha')^{1/2} l} \sum_{m\neq 0} \alpha_m^i \exp\left(-\frac{icm\pi\tau}{l}\right) \sin\left(\frac{m\pi\sigma}{l}\right) \right] \\
&+ \frac{l}{4\pi\alpha' p^+} \frac{1}{2\pi\alpha'} \int_0^l d\sigma \left[(2\alpha')^{1/2} \left(\frac{\pi}{l}\right) \sum_{n\neq 0} \alpha_n^i \exp\left(-\frac{icn\pi\tau}{l}\right) \cos\left(\frac{n\pi\sigma}{l}\right) \right] \\
&\left[(2\alpha')^{1/2} \left(\frac{\pi}{l}\right) \sum_{m\neq 0} \alpha_m^i \exp\left(-\frac{icm\pi\tau}{l}\right) \cos\left(\frac{m\pi\sigma}{l}\right) \right] \\
&= -\frac{1}{8\alpha' p^+} \sum_{n\neq 0} \sum_{m\neq 0} \alpha_m^i \alpha_n^i \exp\left(\frac{-ic(n+m)\pi\tau}{l}\right) (\delta_{nm} - \delta_{n,-m}) \\
&+ \frac{1}{8\alpha' p^+} \sum_{n\neq 0} \sum_{m\neq 0} \alpha_m^i \alpha_n^i \exp\left(\frac{-ic(n+m)\pi\tau}{l}\right) (\delta_{nm} + \delta_{n,-m}) \\
&= \frac{1}{4\alpha' p^+} \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i \\
&= \frac{1}{2\alpha' p^+} \left(\sum_{n>0} \alpha_{-n}^i \alpha_n^i + A \right).
\end{aligned} \tag{2.60}
$$

Combining Eqs. (2.59) and (2.60), the Hamiltonian (2.36) can be written as

$$
H = \sum_{i=2}^{p} \frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+} \left(\sum_{i=2}^{p} \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{i=p+1}^{D-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i + A \right). \tag{2.61}
$$

The first-term in the Hamiltonian (2.61) is a kinetic energy. Since the string ending on brane does not move, there is no momentum in $p+1, \cdots, D-1$ dimension. The combination of creation and annihilation operators in the parenthesis may be interpreted as number operators. As we shall discuss later, it corresponds to the mass. The constant A is a zero-point energy which comes from (creation-annihilation) normal ordering. Therefore

$$
A = \frac{D-2}{2} \sum_{n>0} n. \tag{2.62}
$$

The present of the divergent constant, A, is unpleasant. In QFT, we met at least two kinds of infinities (here we have ignored the infrared divergence, which seems to be irrelevant). One is an infinity zero-point energy when quantizing the free scalar field theory. We may argue that only relative energy is observable and that infinity is thrown away. However, the cosmological constant problem remains [14, 15]. The constant, A, in Eqs. (2.59) - (2.61) is somehow similar to the zero-point theory in the free scalar theory. Nevertheless, string theory is a quantum gravity theory, it is not expected to simply throw infinity away.

Another kind of infinity is the ultra-violate divergence of the loop diagram. We argue that the current field theory is a low-energy theory. There is some energy scale that the present field theory is not applicable. We then regularize and renormalize the theory. For certain theory such as Quantum ChromoDynamics (QCD), the fundamental cutoff in the one-loop beta function can be taken into infinity. QCD can be defined in continuum (high-energy limit) [16, p. 79]. Therefore the low-energy-theory aspect is not necessary for QCD. We shall try to proceed similar procedure for the divergence constant, A, in the bosonic string theory.

Introducing a cutoff ε [17, p. 23]

$$\begin{aligned}
A &\to \frac{D-2}{2}\sum_{n=1}^{\infty} n e^{-\varepsilon n} \\
&= -\frac{D-2}{2}\frac{\partial}{\partial\varepsilon}\sum_{n=1}^{\infty} e^{-\varepsilon n} \\
&= -\frac{D-2}{2}\frac{\partial}{\partial\varepsilon}\frac{1}{e^{\varepsilon}-1} \\
&= -\frac{D-2}{2}\frac{\partial}{\partial\varepsilon}\frac{1}{\varepsilon\left(1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{6}+O(\varepsilon^3)\right)} \\
&= -\frac{D-2}{2}\frac{\partial}{\partial\varepsilon}\frac{1}{\varepsilon}\left(1-\frac{\varepsilon}{2}-\frac{\varepsilon^2}{6}+\frac{\varepsilon^2}{4}+O(\varepsilon^3)\right) \\
&= \frac{D-2}{2}\left(\frac{1}{\varepsilon^2}-\frac{1}{12}+O(\varepsilon)\right). \qquad (2.63)
\end{aligned}$$

To make the cutoff more physical meaningful, we make substitution $\exp(-\varepsilon n) \to$

$\exp(-\varepsilon\gamma_{\sigma\sigma}^{-1/2}|k_\sigma|), k_\sigma = n\pi/l$ [7, p. 22]. Therefore Eq. (2.63) becomes

$$A \to \frac{D-2}{2}\left(\frac{2lp^+\alpha'}{\varepsilon^2\pi} - \frac{1}{12} + O(\varepsilon)\right). \tag{2.64}$$

Expression (2.64) brings two messages: (i) the cutoff term ε^{-2} can be cancelled by a counter term proportional to $\int d^2\sigma(-\gamma)^{1/2}$ in the action; (ii) the vanishing of cutoff dependence is dictated by the Weyl invariance (2.15), since Weyl transformation is a rescaling of l. Furthermore, the vanishing of Weyl anomaly leads to zero value of the beta function [7, p. 91, p. 112]. Therefore, the low-energy-theory analogy is not necessary for this eliminating this divergence in constant, A. As a result

$$A = \frac{2-D}{24}. \tag{2.65}$$

Still, there is a question why the Weyl invariance in the classical Polyakov action (2.1) should be preserved at quantum level. There a few reasons for that: (i) it is to make theory work, by means of providing finite value; (ii) it is a kind of consistency; (iii) the Weyl invariance at quantum level could lead to the Einstein's equation of gravity [11, p. 161].

It is possible to abandon the Weyl invariance from the very beginning, by adding a term $\frac{1}{4\pi\alpha'}\int d^2\sigma\mu(-\gamma)^{1/2}$ in the classical action [11, p. 126]. This will lead to non-critical string theory. As a result, the dimension is not necessarily 26 for the bosonic string. Investigating the non-critical string theory is beyond the scope of the present thesis.

In addition, the infinities in constant, A, Eqs. (2.62) and (2.63) do not appear in the BRST quantization [7, p. 34].

After settle down the infinity, we would like to analyze the spectrum of Hamiltonian (2.36). However, we do not have an expression for mass. The expression of mass could be obtained in analogy with relativistic point particle. Consider an action for a relativistic point particle [7, p. 33]

$$S_{\text{pp}} = -m\int d\tau(-\dot{X}^\mu\dot{X}_\mu)^{1/2}, \tag{2.66}$$

where τ is a proper time and $\dot{X}$ is the τ derivative. Eq. (2.66) could be written in an equalivent form

$$S'_{\rm pp} = \frac{1}{2}\int d\tau \left(\eta^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - \eta m^2\right) \tag{2.67}$$

where $\eta(\tau) = (-\gamma_{\tau\tau}(\tau))^{1/2}$ and $\gamma_{\tau\tau}(\tau)$ is the $\tau\tau$ component of metric. The equivalence between Eqs. (2.66) and (2.67) can be seen from equation of motion for η [7, p. 74]

$$\eta^2 = -\dot{X}^{\mu}\dot{X}_{\mu}/m^2. \tag{2.68}$$

Substitute η in Eq. (2.67) will give Eq. (2.66).

Introduce the lightcone coordinate for the point particle and set $X^{+}(\tau) = \tau$, the action (2.67) can be written as [7, p. 17]

$$S'_{pp} = \frac{1}{2}\int_0^{\tau} \left(-2\eta^{-1}\dot{X}^{-} + \eta^{-1}\dot{X}^{i}\dot{X}^{i} - \eta m^2\right). \tag{2.69}$$

By the canonical momentum $p_{-} = -\eta^{-1}, p_i = \eta^{-1}\dot{X}^{i}, p^{+} = -p_{-}$, and $p^i = p_i$, we obtained the Hamiltonian for action (2.69) and an expression of mass [7, p. 17]

$$H = p_{-}\dot{X}^{-} + p_i\dot{X}^{i} - L = \frac{p^ip^i + m^2}{2p^{+}}, \tag{2.70}$$

$$m^2 = 2p^{+}H - p^ip^i. \tag{2.71}$$

Inserting Eq. (2.61) into (2.72), we have

$$m^2 = \frac{1}{\alpha'}\left(\sum_{i=2}^{p}\sum_{n>0}\alpha^{i}_{-n}\alpha^{i}_{n} + \sum_{i=p+1}^{D-1}\sum_{n>0}\alpha^{i}_{-n}\alpha^{i}_{n} + \frac{2-D}{24}\right). \tag{2.72}$$

The mass of the vacuum state, $\frac{2-D}{24}$, is the same as the open string. It is a tachyon if $D > 2$. The presence of Tachyon is similar with the maximum of the spontaneously symmetry breaking potential. Nevertheless the ground state in analogy with the symmetry breaking is unknown for bosonic string theory.

SO$(1,p)$ $\times$ SO$(D-p-1)$ group has $D-1$ spin states for the massive and $D-2$ spin states for the massless vector representations. The lowest excitation in Eq. (2.72) has $D-2$ modes. Therefore it is a massless excited state. In this case, $m^2 = \frac{26-D}{24}$. The (critical) dimension is therefore 26.

The first excited state has two classes

$$\alpha_{-n}^{i}|0;k\rangle, \quad i=2,3,\cdots,p, \tag{2.73}$$

which corresponds to a massless vector representation, A_μ. Since these states are transformed as vector under SO(1,p).

$$\alpha_{-n}^{i}|0;k\rangle, \quad i=p+1,\cdots,D-1, \tag{2.74}$$

which corresponds to a massless scalar representation, X_μ. Since these states are invariant under SO(1,p).

2.3 Derivation of the Born-Infeld action from beta function

2.3.1 Conformal invariance

The Born-Infeld action is a low-energy effective action for the electromagnetism of D-brane. There are two methods of obtaining the effective action. One is analyzing the string scattering process [18]. The other is from the string moving in a background [19]. Here we follow the second approach.

We have shown there are vector field A_μ and scalar field X_μ living on D-brane. Therefore we can use the effective action of string ending on D-brane to identify the electromagnetism of D-brane. The effective action is obtained by preserving the conformal invariance at quantum level.

To define the conformal invariance, we first rewrite the Polyakov action (2.1) under (++) Euclidean metric δ_{ab} as [7, p. 32]

$$\begin{aligned} S_{\mathrm{p}} &= \frac{1}{4\pi\alpha'}\int d\sigma^1 d\sigma^2\left(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu\right) \\ &= \frac{1}{2\pi\alpha'}\int d^2 z \partial X^\mu \bar{\partial} X_\mu \end{aligned} \tag{2.75}$$

where $\partial_1 = \partial_{\sigma^1}, \partial_2 = \partial_{\sigma^2}, z = \sigma^1 + i\sigma^2,\ \bar{z} = \sigma^1 - i\sigma^2, \partial = \partial_z,\ \bar{\partial} = \partial_{\bar{z}}$, and

$d^2z = dzd\bar{z}$. We have used

$$\partial_1 = \frac{\partial}{\partial\sigma_1} = \frac{\partial}{\partial z}\frac{\partial z}{\partial\sigma_1} + \frac{\partial}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\sigma_1} = \partial + \bar{\partial} \tag{2.76}$$

$$\partial_2 = \frac{\partial}{\partial\sigma_2} = \frac{\partial}{\partial z}\frac{\partial z}{\partial\sigma_2} + \frac{\partial}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\sigma_2} = i\left(\partial - \bar{\partial}\right) \tag{2.77}$$

$$d\sigma^1 d\sigma^2 = \left|\frac{\partial(\sigma_1,\sigma_2)}{\partial(z_1,z_2)}\right| dzd\bar{z} = \left|\frac{1}{2}\frac{-1}{2i} - \frac{1}{2}\frac{1}{2i}\right| d^2z = \frac{1}{2}d^2z. \tag{2.78}$$

The conformal transformation is defined as [7, p. 44]

$$X'^\mu(z',\bar{z}') = X^\mu(z,\bar{z}), \qquad z' = f(z), \tag{2.79}$$

where $f(z)$ is any holomorphic function. The transformation (2.79) changes the distances between points.

The action (2.75) is invariant under conformal transformation (2.79) due to the cancellation of transformation between dz and ∂_z, similarly for $d\bar{z}$ and $\partial_{\bar{z}}$. Since the conformal invariance is a changing of variables from $z, \bar{z}$ to $z', \bar{z'}$ and changing of distance between points, it can be done by diffimorphism and Weyl transformation. The breaking of conformal invariance at quantum level is the same as breaking Weyl invariance, since it is not expect to break diffimorphism.

We can identify the condition which the anomoly does not happen as the effective action.

2.3.2 Open string couples to a gauge field A_μ

The starting point is an open string coupled to a gauge field A_μ [19]

$$S = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\partial^a X^\mu\partial_a X_\mu + i\int_{\partial M} d\tau A_\mu(X)\partial_\tau X^\mu. \tag{2.80}$$

Here we have defined a conformal transformation to map the open string into upper half complex plane

$$w = \sigma^1 + i\sigma^2, \tag{2.81}$$

$$z = -\exp(-iw), \tag{2.82}$$

$$\tau + i\sigma =: z, \tag{2.83}$$

where the spatial dimension $\sigma^1 \in [0,\pi]$ and the time dimension $\sigma^2 \in$

$(-\infty, +\infty)$. Therefore $\sigma \in [0, +\infty)$ and $\tau \in (-\infty, +\infty)$. The action (2.80) is obtained by the state-operator mapping technique [11, p. 164]. Here we outline the procedure of obtaining the coupling term.

Similar with the idea of GR, we generalize the Polyakov action into curved space [7, p. 108]

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \tag{2.84}$$

Expanding the metric $G_{\mu\nu}(X)$ around flat spacetime

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X), \tag{2.85}$$

the integrand of a path integral can be then written as

$$\exp(-S_\sigma) = \exp(-S_\text{p}) \left[1 - \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} g^{ab} \chi_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + O(\alpha'^{-2})\right]. \tag{2.86}$$

On the other hand, we can work out the state-operator mapping term as following: [7, p. 104]

$$V_I = \frac{g_c}{\alpha'} \int d^2\sigma g^{1/2} \left\{ \left(g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu} \right) \left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X} \right]_r + \alpha' \phi R \left[e^{ik\cdot X} \right]_r \right\}, \tag{2.87}$$

here $s_{\mu\nu}$, $a_{\mu\nu}$, and ϕ are symmetric 2-tensor, antisymmetric 2-tensor, and a scalar, respectively. g^{ab} and ϵ^{ab} are symmetric and antisymmetric tensors for the graviton and 2-form in string, respectively. $[\cdots]$ is a renormalized operator [7, p. 102]

$$[\mathcal{F}]_r = \exp\left(\frac{1}{2} \int d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\delta}{\delta X^\mu(\sigma)} \frac{\delta}{\delta X_\mu(\sigma')} \right) \mathcal{F}, \tag{2.88}$$

where

$$\Delta(\sigma, \sigma') = \frac{\alpha'}{2} \ln d^2(\sigma, \sigma'). \tag{2.89}$$

Definition (2.89) is proposed to remove singularity in operator $\mathcal{F}$. This can be seen from the following procedure. Consider a path integral for action (2.75) with a scalar field

$$\int [dX] \exp(-S) X^\nu(z', \bar{z}'). \tag{2.90}$$

The total derivative for (2.163) is zero by the usual asymptotic condition

$$
\begin{aligned}
0 &= \int [dX] \frac{\delta}{\delta X_\mu(z,\bar z)} \left[\exp(-S) X^\nu(z',\bar z')\right] \\
&\overset{(2.75)}{=} \int [dX] \exp(-S) \left[\delta^{\mu\nu}\delta^2(z-z',\bar z-\bar z') + \frac{1}{\pi\alpha'}\partial_z\partial_{\bar z}X^\mu(z,\bar z)X^\nu(z',\bar z')\right] \\
&= \delta^{\mu\nu}\left\langle \delta^2(z-z',\bar z-\bar z')\right\rangle + \frac{1}{\pi\alpha'}\partial_z\partial_{\bar z}\langle X^\mu(z,\bar z)X^\nu(z',\bar z')\rangle, \qquad (2.91)
\end{aligned}
$$

where $\langle\cdots\rangle$ is denoted to expectation value $\int[dX]\exp(-S)\cdots$ [7, p. 131]. In the second line we have used integration by part. The expectation $\langle\cdots\rangle$ can be removed from Eq. (2.91), therefore

$$\frac{1}{\pi\alpha'}\partial_z\partial_{\bar z}X^\mu(z,\bar z)X^\nu(z',\bar z') = -\eta^{\mu\nu}\delta^2(z-z',\bar z-\bar z'). \qquad (2.92)$$

Based on Eq. (2.92), we define the conformal normal ordering

$$: X^\mu(z,\bar z) : = X^\mu(z,\bar z), \qquad (2.93)$$

$$: X^\mu(z,\bar z)X^\nu(z',\bar z') : = X^\mu(z,\bar z)X^\nu(z',\bar z') + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2. \qquad (2.94)$$

The conformal normal ordering has the following properties

$$\partial_z\partial_{\bar z} : X^\mu(z,\bar z)X^\nu(z',\bar z') := 0, \qquad (2.95)$$

since

$$\partial_z\partial_{\bar z}\ln|z|^2 = 2\pi\delta^2(z,\bar z). \qquad (2.96)$$

Eq. (2.96) is a result in exericse 2.1 of Polchinski [7, p. 11]. The solution is given by Headrick [20].

Comparing Eqs. (2.86) and (2.87), we identify the coupling with graviton is exactly the same is the bending of metric

$$\chi_{\mu\nu}(X) = -4\pi g_c e^{ik\cdot X}s_{\mu\nu}. \qquad (2.97)$$

This implies a general idea, that the interactions between string and fields are produced by varying the metric. The adjustable coupling constants in QFT are determined by the dynamics of spacetime. Taken into account for all terms in Eqs. (2.86) and (2.87), we have

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d^2\sigma g^{1/2}\left[\left(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\right]. \tag{2.98}$$

Similarly, the vertex operator for opening string coupled with photon is [7, p. 107]

$$-i\frac{g_o}{(2\alpha')^{1/2}}e_\mu \int_{\partial M} ds\left[\dot{X}^\mu e^{ik\cdot X}\right]_r. \tag{2.99}$$

By a similar procedure, we obtain the coupling between open-string and gauge field A_μ in Eq. (2.80).

2.3.3 Effective action from beta function

The notations here basically follows Tong's lecture note [11, p. 184], since it is consistent with the convention of Polchinski [7, p. 12]. $d^2\sigma$ comes from a Wick rotation. In the original paper of Abouesaood et al [19], the action was written as

$$S = \frac{1}{2\pi\alpha'}\left[\frac{1}{2}\int_{M^2} d^2 z \partial^a X_\mu \partial_a X^\mu + i\int_{\partial M} d\tau A_\mu \partial_\tau X^\mu\right] \tag{2.100}$$

since $d^2 z = 2d\sigma^1 d\sigma^2$, α' in Eq. (2.100) is two times of in Eq. (2.80). It does not matter since α' is dimensional. In addition, A_μ in Eq. (2.100) is rescaled by a factor of $2\pi\alpha'$.
By splitting the scalar X^μ as

$$X^\mu(\tau,\sigma) = \bar{X}^\mu(\tau,\sigma) + \sqrt{\alpha'}Y^\mu(\tau,\sigma) \tag{2.101}$$

where $\bar{X}$ satisfies the classical equation of motion, the action (2.80) can be written as

$$\begin{aligned} S[\bar{X}+Y] = S[\bar{X}] + \frac{1}{4\pi}\int_M d^2\sigma\left(\frac{2}{\sqrt{\alpha'}}\partial^a\bar{X}^\mu\partial_a Y_\mu + \partial^a Y^\mu \partial_a Y_\mu\right) \\ + i\alpha'\int_{\partial M} d\tau\left(\frac{1}{\sqrt{\alpha'}}F_{\mu\nu}Y^\mu\partial_\tau\bar{X}^\nu + \frac{1}{2}\partial_\nu F_{\lambda\mu}Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \frac{1}{2}F_{\mu\nu}Y^\mu\partial_\tau Y^\nu \right. \\ \left. + O(Y^3)\right) \end{aligned} \tag{2.102}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here we have performed the following manipulations

$$
\begin{aligned}
S[\bar{X}+Y] &= \frac{1}{4\pi\alpha'}\int_M d^2\sigma\left[\partial^a(\bar{X}^\mu+\sqrt{\alpha'}Y^\mu)\,\partial_a(\bar{X}_\mu+\sqrt{\alpha'}Y_\mu)\right] \\
&+ i\int_{\partial M} d\tau\left[A_\mu+\sqrt{\alpha'}\partial_\nu A_\mu Y^\nu+\frac{\alpha'}{2}\partial_\nu\partial_\lambda A_\mu Y^\nu Y^\lambda+O(Y^3)\right]\partial_\tau\left[\bar{X}^\mu+\sqrt{\alpha'}Y^\mu\right] \\
&= S[\bar{X}]+\frac{1}{4\pi}\int_M d^2\sigma\left(\frac{2}{\sqrt{\alpha'}}\partial^a\bar{X}^\mu\partial_a Y_\mu+\partial^a Y^\mu\partial_a Y_\mu\right) \\
&+ i\int_{\partial M} d\tau\left[\sqrt{\alpha'}\left(\partial_\nu A_\mu\partial_\tau\bar{X}^\mu Y^\nu+A_\mu\partial_\tau Y^\mu\right)\right. \\
&+ \left.\alpha'\left(\frac{1}{2}\partial_\nu\partial_\lambda A_\mu Y^\nu Y^\lambda\partial_\tau\bar{X}^\mu+\partial_\nu A_\mu Y^\nu\partial_\tau Y^\mu\right)+O(Y^3)\right]. \qquad (2.103)
\end{aligned}
$$

The $O(Y)$ term in the last line of Eq. (2.103) can be written as

$$
\begin{aligned}
& i\int_{\partial M} d\tau\sqrt{\alpha'}\left(\partial_\nu A_\mu\partial_\tau\bar{X}^\mu Y^\nu+A_\mu\partial_\tau Y^\mu\right) \\
&= i\int_{\partial M} d\tau\sqrt{\alpha'}\left[\partial_\nu A_\mu\partial_\tau\bar{X}^\mu Y^\nu-\left(\partial_\tau A_\mu\right)Y^\mu\right] \\
&= i\int_{\partial M} d\tau\sqrt{\alpha'}\left[\partial_\nu A_\mu\partial_\tau\bar{X}^\mu Y^\nu-\left(\partial_\tau\bar{X}^\nu\partial_\nu A_\mu\right)Y^\mu\right] \\
&= i\int_{\partial M} d\tau\sqrt{\alpha'}\left[\partial_\mu A_\nu\partial_\tau\bar{X}^\nu Y^\mu-\left(\partial_\tau\bar{X}^\nu\partial_\nu A_\mu\right)Y^\mu\right] \\
&= i\alpha'\int_{\partial M} d\tau\frac{1}{\sqrt{\alpha'}}F_{\mu\nu}Y^\mu\bar{X}^\nu \qquad (2.104)
\end{aligned}
$$

where in the second, third, and fourth lines, integration by parts, substitution of variables, and renaming indices have been made, respectively.

The $O(Y^2)$ term in the last line of Eq. (2.103) can be written by integration by parts as

$$
\begin{aligned}
& i\alpha' \int_{\partial M} d\tau \left(\frac{1}{2} \partial_\nu \partial_\lambda A_\mu Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \partial_\nu A_\mu Y^\nu \partial_\tau Y^\mu \right) \\
& = i\alpha' \int_{\partial M} d\tau \left(\frac{1}{2} \partial_\nu \partial_\lambda A_\mu Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \frac{1}{2} \partial_\nu A_\mu Y^\nu \partial_\tau Y^\mu + \frac{1}{2} \partial_\nu A_\mu Y^\nu \partial_\tau Y^\mu \right) \\
& = i\alpha' \int_{\partial M} d\tau \left[\frac{1}{2} \partial_\nu \partial_\lambda A_\mu Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \frac{1}{2} \partial_\nu A_\mu Y^\nu \partial_\tau Y^\mu - \frac{1}{2} \partial_\tau \left(\partial_\nu A_\mu Y^\nu \right) Y^\mu \right] \\
& = i\alpha' \int_{\partial M} d\tau \left[\frac{1}{2} \partial_\nu \partial_\lambda A_\mu Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \frac{1}{2} \partial_\nu A_\mu Y^\nu \partial_\tau Y^\mu - \frac{1}{2} \partial_\nu \partial_\tau A_\mu Y^\nu Y^\mu \right. \\
& \left. - \frac{1}{2} \partial_\nu A_\mu \partial_\tau Y^\nu Y^\mu \right] \\
& = i\alpha' \int_{\partial M} d\tau \left[\frac{1}{2} \partial_\nu \partial_\lambda A_\mu Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \frac{1}{2} \partial_\nu A_\mu Y^\nu \partial_\tau Y^\mu - \frac{1}{2} \partial_\nu \partial_\lambda A_\mu Y^\nu Y^\mu \partial_\tau \bar{X}^\lambda \right. \\
& \left. - \frac{1}{2} \partial_\nu A_\mu \partial_\tau Y^\nu Y^\mu \right] \\
& = i\alpha' \int_{\partial M} d\tau \left[\frac{1}{2} \partial_\nu F_{\lambda\mu} Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu + \frac{1}{2} F_{\mu\nu} Y^\mu \partial_\tau Y^\nu \right]. \qquad (2.105)
\end{aligned}
$$

The expansion here is perturbative. We have assumed the curvature is sufficiently small [7, p. 110]. The variation of ξ gives classical equation of motion and boundary condition

$$
\Box \bar{X}^\mu = 0, \qquad (2.106)
$$

$$
\partial_\sigma \bar{X}^\mu + 2\pi\alpha' i F^{\mu\nu} \partial_\tau \bar{X}_\nu \big|_{\partial M} = 0, \qquad (2.107)
$$

where $\Box = \partial_1^2 + \partial_2^2$.

Our task is to identify the one-loop divergence and find the counter term. We expect the counter term has the form

$$
\Delta S_{\mathrm{I}}(\bar{X}) = \frac{i\alpha'}{2} \int_{\partial M} d\tau \Gamma_\mu \partial_\tau \bar{X}^\mu. \qquad (2.108)
$$

The corresponding term in (2.102) is

$$
\frac{i\alpha'}{2} \int_{\partial M} d\tau \partial_\nu F_{\lambda\mu} Y^\nu Y^\lambda \partial_\tau \bar{X}^\mu. \qquad (2.109)
$$

Thus we identify

$$\Delta S_{\mathrm{I}}(\bar{X}) = -\frac{i\alpha'}{2}\int_{\partial M} d\tau \partial_\nu F_{\lambda\mu} G^{\nu\lambda}(\tau,\tau')\partial_\tau \bar{X}^\mu \,. \tag{2.110}$$

The Green's function G satisfies

$$\frac{1}{\pi}\partial_z\partial_{\bar{z}} G(z,z') = -\delta^2(z-z',\bar{z}-\bar{z}'), \tag{2.111}$$

$$\partial_\sigma G(z,z')\big|_{\sigma=0} = 0, \tag{2.112}$$

where the delta function $\delta^2(z-z',\bar{z}-\bar{z}')$ is defined as [7, p. 33]

$$\int d^2z\delta^2(z,\bar{z}) = 1. \tag{2.113}$$

Eq. (2.111) is obtained as following [7, p. 35]. Notice a rescaling factor $\sqrt{\alpha'}$ in Eq. (2.101) on Eq. (2.91), we confirm Eq. (2.111).

Eq. (2.112) corresponds to the open-string boundary condition. Eqs. (2.111) and (2.112) can be solved in analogy as the charge imagine method [11, p. 106]. The solution is

$$G(z,z') = -\left(\ln|z-z'| + \ln|z-\bar{z}'|\right). \tag{2.114}$$

It is straightforward to verify the solution for Eq. (2.111) on the upper half plane

$$\frac{1}{\pi}\partial\bar{\partial}G = -\delta^2(z-z',\bar{z}-\bar{z}'). \tag{2.115}$$

Eq. (2.112) can be seen as following

$$\begin{aligned}
\partial_\sigma G|_{\sigma=0} &\overset{(2.83)}{=} \left(\frac{\partial}{\partial z}\frac{\partial z}{\partial\sigma} + \frac{\partial}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\sigma}\right)G \\
&\overset{(2.114)}{=} i(\partial-\bar{\partial})\left[\ln(z-z') + \ln(\bar{z}-\bar{z}') + \ln(z-\bar{z}') + \ln(\bar{z}-z')\right]\Big|_{\sigma=0} \\
&= i\left(\frac{1}{z-z'} - \frac{1}{\bar{z}-\bar{z}'} + \frac{1}{z-\bar{z}'} + \frac{1}{\bar{z}-z'}\right)\Bigg|_{\sigma=0\Rightarrow z=\bar{z}} \\
&= 0.
\end{aligned} \tag{2.116}$$

It is possible to look for counterterm based on Eqs. (2.111) and (2.112), as given in Appendix A in [19]. However, it is simpler to shift the boundary condition (2.111) with the presence of gauge field

$$\partial_\sigma G^{\mu\nu} + 2\pi\alpha' i F^\mu_\lambda \partial_\tau G^{\lambda\nu}\Big|_{\sigma=0} = 0. \tag{2.117}$$

The solution for Eq. (2.111) and (2.117) is [11, p. 186]

$$\begin{aligned} G^{\mu\nu} = -\delta^{\mu\nu}\ln|z-z'| - \frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\mu\nu}\ln(z-\bar{z}') \\ -\frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\mu\nu}\ln(\bar{z}-z'), \end{aligned} \tag{2.118}$$

where (the author is indebted to David Tong for explaining this notation)

$$\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\mu\nu} = \left(1-2\pi\alpha' F\right)^\mu_\lambda\left[\left(1+2\pi\alpha' F\right)^{-1}\right]^{\lambda\nu}. \tag{2.119}$$

The solution can be verified as following. Recall

$$\begin{aligned} \partial_\sigma &= \left(\frac{\partial}{\partial z}\frac{\partial z}{\partial \sigma} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \sigma}\right) \\ &\overset{(2.83)}{=} i\left(\partial - \bar{\partial}\right), \end{aligned} \tag{2.120}$$

$$\begin{aligned} \partial_\tau &= \left(\frac{\partial}{\partial z}\frac{\partial z}{\partial \sigma} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \sigma}\right) \\ &\overset{(2.83)}{=} \left(\partial + \bar{\partial}\right). \end{aligned} \tag{2.121}$$

Therefore

$$\begin{aligned} &\partial_\sigma G^{\mu\nu} = i\left(\partial - \bar{\partial}\right)\left[-\frac{\delta^{\mu\nu}}{2}\left(\ln(z-z') + \ln(\bar{z}-\bar{z}')\right) - \frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\mu\nu}\ln(z-\bar{z}') \right.\\ &\left.-\frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\mu\nu}\ln(\bar{z}-z')\right]\Bigg|_{\sigma=0} \\ &= i\left[-\frac{\delta^{\mu\nu}}{2}\frac{1}{z-z'} + \frac{\delta^{\mu\nu}}{2}\frac{1}{\bar{z}-\bar{z}'} - \frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\mu\nu}\frac{1}{z-\bar{z}'} \right.\\ &\left.+\frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\mu\nu}\frac{1}{\bar{z}-z'}\right]\Bigg|_{\sigma=0\Rightarrow z=\bar{z}} \\ &= i\left[-\frac{1}{2}\left(1-\frac{1+2\pi\alpha'}{1-2\pi\alpha'}\right)^{\mu\nu}\frac{1}{z-z'} + \frac{1}{2}\left(1-\frac{1-2\pi\alpha'}{1+2\pi\alpha'}\right)^{\mu\nu}\frac{1}{z-\bar{z}'}\right]\Bigg|_{\sigma=0\Rightarrow z=\bar{z}} \\ &= i\left[\frac{1}{2}\left(\frac{4\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\mu\nu}\frac{1}{z-z'} + \frac{1}{2}\left(\frac{4\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\mu\nu}\frac{1}{z-\bar{z}'}\right]\Bigg|_{\sigma=0\Rightarrow z=\bar{z}} \end{aligned} \tag{2.122}$$

and

$$
\begin{aligned}
&2\pi\alpha' i F^{\mu}_{\lambda}\partial_\tau G^{\lambda\nu} \\
&= 2\pi\alpha' i F^{\mu}_{\lambda}\left(\partial+\bar{\partial}\right)\left[-\frac{\delta^{\lambda\nu}}{2}\left(\ln(z-z')+\ln(\bar{z}-\bar{z}')\right)-\frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\lambda\nu}\ln(z-\bar{z}')\right. \\
&\left.\left.-\frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\lambda\nu}\ln(\bar{z}-z')\right]\right|_{\sigma=0} \\
&= 2\pi\alpha' i F^{\mu}_{\lambda}\left[-\frac{\delta^{\lambda\nu}}{2}\frac{1}{z-z'}-\frac{\delta^{\lambda\nu}}{2}\frac{1}{\bar{z}-\bar{z}'}-\frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\lambda\nu}\frac{1}{z-\bar{z}'}\right. \\
&\left.\left.-\frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\lambda\nu}\frac{1}{\bar{z}-z'}\right]\right|_{\sigma=0\Rightarrow z=\bar{z}} \\
&= 2\pi\alpha' i F^{\mu}_{\lambda}\left.\left[-\frac{1}{2}\left(1+\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\lambda\nu}\frac{1}{z-z'}-\frac{1}{2}\left(1+\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\lambda\nu}\frac{1}{z-\bar{z}'}\right]\right|_{\sigma=0\Rightarrow z=\bar{z}} \\
&= i\left.\left[-\frac{1}{2}\left(\frac{4\pi\alpha' F}{1-2\pi\alpha' F}\right)^{\mu\nu}\frac{1}{z-z'}-\frac{1}{2}\left(\frac{4\pi\alpha' F}{1+2\pi\alpha' F}\right)^{\mu\nu}\frac{1}{z-\bar{z}'}\right]\right|_{\sigma=0\Rightarrow z=\bar{z}}.
\end{aligned}
\tag{2.123}
$$

Eqs. (2.122) and (2.123) cancel each other, therefore we confirmed the solution (2.119)

Since the interaction between scalar and gauge fields happen at boundary, we should look at $\sigma\to 0$ and $\sigma'\to 0$ in Eq. (2.119). The divergence happens at $z\to z'$. We denote $\ln\Lambda$ in Eq. (2.119) as the divergence term

$$
\begin{aligned}
G &\to -\ln\Lambda\left[1+\frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)+\frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)\right]^{\mu\nu} \\
&= -\ln\Lambda\left[1+\frac{(1-2\pi\alpha' F)^2+(1+2\pi\alpha' F)^2}{2(1-4\pi^2\alpha'^2F^2)}\right]^{\mu\nu} \\
&= -2\ln\Lambda\left(\frac{1}{1-4\pi^2\alpha'^2F^2}\right)^{\mu\nu}.
\end{aligned}
\tag{2.124}
$$

The counter term (2.110) then has form

$$
-2i\alpha'\ln\Lambda\int_{\partial M} d\tau\,\partial_\nu F_{\lambda\mu}\left(\frac{1}{1-4\pi^2\alpha'^2F^2}\right)^{\nu\lambda}\partial_\tau\bar{X}^{\mu}. \tag{2.125}
$$

Define beta function with respect to the cutoff scale Λ

$$\beta_\mu = \Lambda \frac{\partial}{\partial \Lambda} \Gamma_\mu = 4 \partial_\nu F_{\lambda\mu} \left(\frac{1}{1 - 4\pi^2 \alpha'^2 F^2} \right)^{\nu\lambda}. \tag{2.126}$$

The scale dependence vanishes if and only if

$$\partial_\nu F_{\lambda\mu} \left(\frac{1}{1 - 4\pi^2 \alpha'^2 F^2} \right)^{\nu\lambda} = 0. \tag{2.127}$$

Condition (2.127) can be obtained by the equation of motion of the Born-Infeld action

$$L_{\mathrm{BI}} = \sqrt{-\det(1+F)}, \tag{2.128}$$

where we have rescaled the field strength tensor F as $F \to 2\pi\alpha' F$. To show Eq. (2.128) implies (2.127), we start from Eq. (2.128)

$$\begin{aligned} \delta L_{\mathrm{BI}} &= \delta \sqrt{-\det(1+F)} \\ &= \delta \left[\det(1+F) \det((1+F)^{\mathrm{T}}) \right]^{1/4} \\ &= \delta \left[\det(1+F) \det(1-F) \right]^{1/4} \\ &= \delta \left[\det(1-F^2) \right]^{1/4}, \end{aligned} \tag{2.129}$$

where we have used the antisymmetric properties of $F_{\mu\nu}$ in transposing the matrix. Here F^2 should be understood as $F_{ab}F^{cd}$ since we can let $\det(1_{ab} + F_{ab}) = \det(\eta_{ac}(1^{cd} + F^{cd})\eta_{db}) = \det(F^{cd})$. Then by the matrix identity $\det A = e^{\mathrm{tr}\ln A}$ [21, p. 99], we have

$$
\begin{aligned}
\delta L_{\mathrm{BI}} &= \delta \exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right) \\
&= -\frac{1}{2}\exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta F_{\nu\mu} \\
&= \frac{1}{2}\exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\
&= -\partial_\mu\left[\exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\left(\frac{F}{1-F^2}\right)^{\mu\nu}\right]\delta A_\nu \\
&= -\partial_\mu\left[\exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\right]\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta A_\nu \\
&- \exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\partial_\mu\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta A_\nu \\
&= -\frac{1}{4}\exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\partial_\mu\left[\mathrm{tr}\ \ln(1-F^2)\right]\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta A_\nu \\
&- \exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\partial_\mu\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta A_\nu,
\end{aligned}
\tag{2.130}
$$

in the fourth line we have performed integration by part. To proceed, we compute

$$
\begin{aligned}
\partial_\mu\left[\mathrm{tr}\ \ln(1-F^2)\right] &= 2\left(\frac{F}{1-F^2}\right)^{\alpha\beta}\partial_\mu F_{\alpha\beta} \\
&= 2\left(\frac{F}{1-F^2}\right)^{\alpha\beta}\left(-\partial_\alpha F_{\beta\mu} - \partial_\beta F_{\mu\alpha}\right) \\
&= -4\left(\frac{F}{1-F^2}\right)^{\alpha\beta}\partial_\alpha F_{\beta\mu},
\end{aligned}
\tag{2.131}
$$

in the second line we have used the Bianchi identity. Insert Eq. (2.131) into (2.130), we obtain

$$
\begin{aligned}
\delta L_{\mathrm{BI}} = &\exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\left(\frac{F}{1-F^2}\right)^{\alpha\beta}\partial_\alpha F_{\beta\mu}\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta A_\nu \\
&- \exp\left(\frac{1}{4}\mathrm{tr}\ \ln(1-F^2)\right)\partial_\mu\left(\frac{F}{1-F^2}\right)^{\mu\nu}\delta A_\nu.
\end{aligned}
\tag{2.132}
$$

We further notice an identity [22]

$$\partial_\mu \left(\frac{F}{1-F^2}\right)^{\mu\nu} = \left(\frac{F}{1-F^2}\right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left(\frac{F}{1-F^2}\right)^{\sigma\nu}$$
$$+ \left(\frac{1}{1-F^2}\right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left(\frac{1}{1-F^2}\right)^{\sigma\nu}, \tag{2.133}$$

the proof of Eq. (2.133) can be found on web [23].
Combining Eqs. (2.132) and (2.133) we obtain

$$\delta L_{\mathrm{BI}} = -\exp\left(\frac{1}{4}\mathrm{tr}\,\ln(1-F^2)\right)\left(\frac{1}{1-F^2}\right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left(\frac{1}{1-F^2}\right)^{\sigma\nu}, \tag{2.134}$$

which gives the condition of vanishing beta function (2.127). Therefore we obtained the Born-Infeld action (2.128) from string theory.

Consequently, the Born-Infeld action is

$$S_{\mathrm{BI}} = \frac{1}{2\pi\alpha'}\int d^{p+1}x\sqrt{-\det\left(1+2\pi\alpha' F\right)}. \tag{2.135}$$

It is possible to include the scalar fluctuation of brane, the action is known as the Dirac-Born-Infeld (DBI) action [24]

$$S_{\mathrm{BI}} = \frac{1}{2\pi\alpha'}\int d^{p+1}x\sqrt{-\det\left(1+\partial_m X^s\partial_n X^s+2\pi\alpha' F_{mn}\right)}. \tag{2.136}$$

2.4 Derivation of Born-Infeld action from path integral approach

2.4.1 Path integral for string interactions

We start from the path-integral for string interactions [18]

$$\Gamma[\phi, A_\mu, B_{\mu\nu}, \cdots] = \sum_{\chi=1,0,\cdots} e^{\sigma\chi}\int [dg_{ab}][dx^\mu] e^{-I_2}\mathrm{tr}(\mathrm{P}e^{-I_1}), \tag{2.137}$$

$$I_2 = \frac{1}{4\pi\alpha'}\int_M d^2z\sqrt{g}g^{ab}\partial_a x^\mu\partial_b x_\mu, \tag{2.138}$$

$$I_1 = \int_{\partial M} dt[e\phi(x(t)) + i\dot{x}^\mu + e^{-1}\dot{x}^\mu\dot{x}^\nu B_{\mu\nu}(x(t)) + \cdots]. \tag{2.139}$$

Since the path integral is carried out over all possible configurations including the conformations of strings, the metric g_{ab} should be taken into the integration measure. χ is the Euler character. It comes from an extension of Polyakov action and Gauss-Bonnet theorem.

We hope we construct the theory from most general action includes all the Poincaré, diffeomorphism, and Weyl symmetries. The extension of Polyakov action based on two assumptions (i) local; (ii) the action is in polynomial form in derivatives. The result is

$$\chi = \frac{1}{4\pi}\int_M d\tau d\sigma (-\gamma)^{1/2} R, \tag{2.140}$$

$$S'_p = S_p - \lambda\chi \tag{2.141}$$

$$= -\int_M d\tau d\sigma (-\gamma)^{1/2}\left(\frac{1}{4\pi\alpha'}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}R\right), \tag{2.142}$$

where λ is any number. R is Ricci scalar. The extra term is invariant under Weyl transformation. Since

$$(-\gamma')^{1/2}R' = (-\gamma)^{1/2}(R - 2\nabla^2\omega) \tag{2.143}$$

and taking $\nabla\omega$ as a vector

$$(-\gamma)^{1/2}\nabla_a \nu^a = \partial_a((-\gamma)^{1/2}\nu^a), \tag{2.144}$$

we see the variation in Eq. (2.143) is a total derivative.

Eq. (2.143) can be verified by brute force computation. Recall the definions of Christoff symbol, Rienmann tensor, Ricci curvature, and Ricci scalar

$$\Gamma^i_{kl} = \frac{1}{2}\gamma^{im}\left(\partial_l\gamma_{mk} + \partial_k\gamma_{ml} - \partial_m\gamma_{kl}\right), \tag{2.145}$$

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\nu_{\mu\sigma}, \tag{2.146}$$

$$R_{ij} = R^k_{ikj}, \tag{2.147}$$

$$R = R^i_i. \tag{2.148}$$

We perform a Weyl transformation $\gamma'_{ab} = \gamma_{ab}\exp(2\omega)$ from a local inertia frame $\gamma_{\mu\nu} = \eta_{\mu\nu} + O(x^2)$, the Christoff symbol becomes

$$\begin{aligned}\Gamma'^i_{kl} &= \Gamma^i_{kl} + \partial_l\omega\gamma^{im}\gamma_{mk} + \partial_k\omega\gamma^{im}\gamma_{ml} - \partial_m\omega\gamma^{im}\gamma_{kl} \\ &= \Gamma^i_{kl} + \partial_l\omega\delta^i_k + \partial_k\omega\delta^i_l - \partial^i\omega\gamma_{kl}\end{aligned} \tag{2.149}$$

hence

$$
\begin{aligned}
R'^{\rho}_{\sigma\mu\nu} &= R^{\rho}_{\sigma\mu\nu} + (\partial_\mu\partial_\sigma\omega)\,\delta^\rho_\nu - (\partial_\mu\partial^\rho\omega)\,\eta_{\nu\sigma} - (\partial_\nu\partial_\sigma\omega)\,\delta^\rho_\mu + (\partial_\nu\partial^\rho\omega)\,\eta_{\mu\sigma} \\
&\quad + \left[(\partial_\lambda\omega)\,\delta^\rho_\mu + (\partial_\mu\omega)\,\delta^\rho_\lambda - (\partial^\rho\omega)\,\eta_{\mu\lambda}\right]\left[(\partial_\sigma\omega)\,\delta^\lambda_\nu + (\partial_\nu\omega)\,\delta^\lambda_\sigma - (\partial^\lambda\omega)\eta_{\nu\sigma}\right] \\
&\quad - \left[(\partial_\lambda\omega)\delta^\rho_\nu + (\partial_\nu\omega)\delta^\rho_\lambda - (\partial^\rho\omega)\eta_{\nu\lambda}\right]\left[(\partial_\sigma\omega)\delta^\lambda_\mu + (\partial_\mu\omega)\delta^\lambda_\sigma - (\partial^\lambda\omega)\eta_{\mu\sigma}\right]
\end{aligned}
\tag{2.150}
$$

hence

$$
\begin{aligned}
R'_{\sigma\nu} &= R'^{\rho}_{\sigma\rho\nu} \\
&= R_{\sigma\nu} + (\partial_\rho\partial_\sigma\omega)\delta^\rho_\nu - (\partial_\rho\partial^\rho\omega)\eta_{\nu\sigma} - (\partial_\nu\partial_\sigma\omega)\delta^\rho_\rho + (\partial_\nu\partial^\rho\omega)\eta_{\rho\sigma} \\
&+ \left[(\partial_\lambda\omega)\,\delta^\rho_\rho + (\partial_\rho\omega)\,\delta^\rho_\lambda - (\partial^\rho\omega)\,\eta_{\rho\lambda}\right]\left[(\partial_\sigma\omega)\,\delta^\lambda_\nu + (\partial_\nu\omega)\,\delta^\lambda_\sigma - (\partial^\lambda\omega)\eta_{\nu\sigma}\right] \\
&- \left[(\partial_\lambda\omega)\delta^\rho_\nu + (\partial_\nu\omega)\delta^\rho_\lambda - (\partial^\rho\omega)\eta_{\nu\lambda}\right]\left[(\partial_\sigma\omega)\delta^\lambda_\rho + (\partial_\rho\omega)\delta^\lambda_\sigma - (\partial^\lambda\omega)\eta_{\rho\sigma}\right]
\end{aligned}
\tag{2.151}
$$

hence

$$
\begin{aligned}
R' &= R'_{\sigma\nu}\eta^{\sigma\nu} \\
&= R + (\partial_\rho\partial_\sigma\omega)\eta^{\sigma\rho} - (\partial_\rho\partial^\rho\omega)2 - 2(\partial_\nu\partial_\sigma\omega)\eta^{\sigma\nu} + (\partial_\nu\partial^\rho\omega)\delta^\nu_\rho \\
&+ 2(\partial_\lambda\omega)(\partial_\sigma\omega)\delta^\lambda_\sigma\eta^{\sigma\nu} + 2(\partial_\lambda\omega)(\partial_\nu\omega)\delta^\lambda_\sigma\eta^{\sigma\nu} - 2(\partial_\lambda\omega)(\partial^\lambda\omega)\eta_{\nu\sigma}\eta^{\sigma\nu} \\
&+ (\partial_\rho\omega)(\partial_\sigma\omega)\delta^\rho_\lambda\delta^\lambda_\nu\eta^{\sigma\nu} + (\partial_\rho\omega)(\partial_\nu\omega)\delta^\rho_\lambda\delta^\lambda_\sigma\eta^{\sigma\nu} - (\partial_\rho\omega)(\partial^\lambda\omega)\delta^\rho_\lambda\eta^{\sigma\nu}\eta_{\nu\sigma} \\
&- (\partial^\rho\omega)(\partial_\sigma\omega)\eta_{\rho\lambda}\delta^\lambda_\nu\eta^{\sigma\nu} - (\partial^\rho\omega)(\partial_\nu\omega)\eta_{\rho\lambda}\delta^\lambda_\sigma\eta^{\sigma\nu} + (\partial^\rho\omega)(\partial^\lambda\omega)\eta_{\rho\lambda}\eta_{\nu\sigma}\eta^{\sigma\nu} \\
&- (\partial_\lambda\omega)(\partial_\sigma\omega)\delta^\rho_\nu\delta^\lambda_\rho\eta^{\sigma\nu} - (\partial_\lambda\omega)(\partial_\rho\omega)\delta^\rho_\nu\delta^\lambda_\sigma\eta^{\sigma\nu} + (\partial_\lambda\omega)(\partial^\lambda\omega)\delta^\rho_\nu\eta_{\rho\sigma}\eta^{\sigma\nu} \\
&- (\partial_\nu\omega)(\partial_\sigma\omega)\delta^\rho_\lambda\delta^\lambda_\rho\eta^{\sigma\lambda} - (\partial_\nu\omega)(\partial_\rho\omega)\delta^\rho_\lambda\delta^\lambda_\sigma\sigma^{\rho\nu} + (\partial_\nu\omega)(\partial^\lambda\omega)\delta^\rho_\lambda\eta_{\rho\sigma}\eta^{\sigma\nu} \\
&- (\partial^\rho\omega)(\partial_\sigma\omega)\eta_{\nu\lambda}\delta^\lambda_\rho\eta^{\sigma\nu} - (\partial^\rho\omega)(\partial_\rho\omega)\eta_{\nu\lambda}\delta^\lambda_\sigma\eta^{\sigma\nu} + (\partial^\rho\omega)(\partial^\lambda\omega)\eta_{\nu\lambda}\eta_{\rho\sigma}\eta^{\sigma\nu} \\
&= R - 2\partial_\rho\partial^\rho\omega \\
&+ 2(\partial_\lambda\omega)(\partial^\lambda\omega) + 2(\partial_\lambda\omega)(\partial^\lambda\omega) - 4(\partial_\lambda\omega)(\partial^\lambda\omega) \\
&+ (\partial_\lambda\omega)(\partial^\lambda\omega) + (\partial_\lambda\omega)(\partial^\lambda\omega) - 2(\partial_\lambda\omega)(\partial^\lambda\omega) \\
&- (\partial_\lambda\omega)(\partial^\lambda\omega) - (\partial_\lambda\omega)(\partial^\lambda\omega) + 2(\partial_\lambda\omega)(\partial^\lambda\omega) \\
&- (\partial_\lambda\omega)(\partial^\lambda\omega) - (\partial_\lambda\omega)(\partial^\lambda\omega) + 2(\partial_\lambda\omega)(\partial^\lambda\omega) \\
&- 2(\partial_\lambda\omega)(\partial^\lambda\omega) - (\partial_\lambda\omega)(\partial^\lambda\omega) + (\partial_\lambda\omega)(\partial^\lambda\omega) \\
&+ 2(\partial_\lambda\omega)(\partial^\lambda\omega) + (\partial_\lambda\omega)(\partial^\lambda\omega) - (\partial_\lambda\omega)(\partial^\lambda\omega) \\
&= R - 2\partial_\rho\partial^\rho\omega.
\end{aligned}
\tag{2.152}
$$

The last two lines in Eq. (2.151) do not contribute to the Ricci scalar as we saw in the explicitly computation. It may be intuitively viewed as coming from exchanging indices μ and ν in the definition (2.146) with different signs. Therefore their contractions in the Ricci scalar cancel.

Other forms of action than Eq. (2.142) seem to be impossible, since the diffeomorphism fixed the form $d\sigma d\tau(-\gamma)^{1/2}$. We can only add scalars. Higher power Ricci scalar is not possible, since it breaks the Weyl invariance. Nevertheless proving Ricci scalar is the only possibility is beyond the author's knowledge.

For the open string with boundary, an additional term can be added in the action [7, p. 30]

$$\chi = \frac{1}{4\pi}\int_M d\tau d\sigma(-\gamma)^{1/2}R + \frac{1}{2\pi}\int_{\partial M} dsk, \tag{2.153}$$

where ds is the proper time and $k = \pm t^a n_b \nabla_a t^b$. Here t^a is the unit tagent vector to the boundary. n^b is the unit vector orthogonal to t^a.

Combing Eqs. (2.142) and (2.153), we write the general path integral as

$$\int [dXdg]\exp(-S), \tag{2.154}$$

$$S = S_X + \lambda\chi, \tag{2.155}$$

$$S_X = \frac{1}{4\pi\alpha'}\int_M d^2\sigma g^{1/2} g^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{2.156}$$

$$\chi = \frac{1}{4\pi}\int_M d^2\sigma g^{1/2}R + \frac{1}{2\pi}\int_{\partial M} dsk. \tag{2.157}$$

The Gauss-Bonnet theorem [25, p. 81] can convert Eq. (2.157) into $e^{-\lambda\chi}$ where χ here is the Euler character. The difference in the minus sign here and the plus sign with σ in Eq. (2.137) is conventional.

2.4.2 General expressions for present computation

Similar with the treatment in the β function approach, we split the coordinate $x^\mu = y^\mu + \xi^\mu$ in Eq. (2.137). Here ξ is the non-constant part. Thus

$$\int [dx^\mu] e^{-I_1 - I_2} = \int d^D y \int [d\xi^\mu]\exp\left\{-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\hat{g}}(\partial\xi^\mu)^2 - i\int dt\dot{\xi}^\mu A_\mu(y+\xi)\right\}. \tag{2.158}$$

Integrating the Gaussian integrals, Eq. (2.158) becomes

$$\int [d\xi^\mu] e^{-I_2 - I_1[\xi|_{\partial M}]} \sim \int [d\eta^\mu] e^{-1/2\eta G^{-1}\eta - I_1[\sqrt{2\pi\alpha'}\eta]}, \tag{2.159}$$

where a change of variables was made, that η is simply connected component on the boundary, such that $1 = \Pi_A \int [d\eta_A^\mu \delta(\xi^\mu|_{C_A} - \eta_A)]$. Expanding A_μ in Eq. (2.158) and integrating by parts, we have

$$\int dt \dot{\xi}^\mu A_\mu(y+\xi) = \frac{1}{2} F_{\nu\mu}(y) \int dt \dot{\xi}^\mu \xi^\nu + O(\xi^3). \tag{2.160}$$

The path integral is then proportional to a Gaussian one

$$\int [d\eta^\mu] \exp\left(-\frac{1}{2}\eta G^{-1}\eta + \frac{i}{2}\bar{F}_{\mu\nu} \int dt \dot{\eta}^\mu \eta^\nu \right), \tag{2.161}$$

where $\bar{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu}$. The field strength tensor is antisymmetric, therefore it cannot be diagonalized in general. Nevertheless it can be rotated into a block diagonal form

$$\bar{F}_{\mu\nu} = \begin{pmatrix} 0 & \bar{f}_1 & & & & \\ -\bar{f}_1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & \bar{f}_n \\ & & & & -\bar{f}_n & 0 \end{pmatrix}, \tag{2.162}$$

where $n = D/2$ and $\bar{f}_n = 2\pi\alpha' f_n$.

Hence the path integral becomes

$$\Gamma(F) = \sum_\chi e^{\sigma\chi} \int d\mu(\lambda) Z(0) \bar{Z}(F), \tag{2.163}$$

$$Z(0) = \int [dx^\mu] e^{-I_2}, \tag{2.164}$$

$$\bar{Z}(F) = \Pi_{k=1}^{D/2} \int [d\tilde{\eta}] \exp\left(-\frac{1}{2}\tilde{\eta}\Delta_k\tilde{\eta} \right) = \Pi_{k=1}^{D/2} (\det \Delta_k)^{-1/2}, \tag{2.165}$$

$$\Delta_k = 1 + \bar{f}_k^2 \ddot{G} \cdot G. \tag{2.166}$$

2.4.3 Tree level approximation

The perturbative expansion is controlled by the Euler character χ. $\chi = 1$ correspond to the tree-level approximation. We map the world sheet into a unit disc by [26, p. 85]

$$z = re^{i\theta}, \tag{2.167}$$

where $\theta \in [0, 2\pi)$ and $r \in [0, 1]$. The Neumann boundary problem becomes

$$\Delta G(z, z') = \partial_z \partial_{\bar{z}} G(z, z') = \delta(z - z'), \tag{2.168}$$

$$\left.\frac{\partial G(z, z')}{\partial r}\right|_{r=1} = 0. \tag{2.169}$$

The solution

$$G(z, z') = \frac{1}{2\pi} \ln |z - z'||z - \bar{z}'^{-1}| \tag{2.170}$$

could be obtained by the charge image method. By Eq. (2.167) we have [26, p. 134]

$$\begin{aligned} G &= \frac{1}{4\pi}\left[\ln(e^{i\theta} - e^{i\theta'}) + \ln(e^{i\theta} - e^{i\theta'}) + \text{c.c.}\right] \\ &= \frac{1}{2\pi}\left[\ln(e^{i\theta} - e^{i\theta'}) + \ln(e^{-i\theta} - e^{-i\theta'})\right] \\ &= \frac{1}{2\pi}\left[\ln(1 - e^{i(\theta'-\theta)}) + \ln(1 - e^{-i(\theta'-\theta)})\right] \\ &= \frac{1}{2\pi}\left[-\sum_{n=1}^{\infty} \frac{e^{-in(\theta-\theta')}}{n} - \sum_{n=1}^{\infty} \frac{e^{in(\theta-\theta')}}{n}\right] \\ &= -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos(n(\theta - \theta'))}{n}. \end{aligned} \tag{2.171}$$

here c.c. denote to complex conjugation.

We can then find the inverse Green function

$$G^{-1} = -\frac{1}{\pi}\sum_{n=1}^{\infty} n \cos(n(\theta - \theta')) = \ddot{G} \tag{2.172}$$

Therefore Eq. (2.166) becomes

$$\Delta_k = (1 + \bar{f}_k^2)\bar{\delta}(\xi). \tag{2.173}$$

The expression of inverse Green function (2.172) can be verified by

$$
\begin{aligned}
&\int_0^{2\pi} d\theta' G(\theta,\theta') G^{-1}(\theta',\theta'') \\
&= \frac{1}{\pi^2}\int_0^{2\pi}\sum_{n,m=1}^{\infty}\frac{m}{n}\left(\cos n\theta\cos n\theta' + \sin n\theta \sin m\theta'\right)\left(\cos m\theta'\cos m\theta'' + \sin\theta' \sin m\theta''\right) d\theta' \\
&= \frac{1}{2\pi^2}\sum_{n,m=1}^{\infty}\int_0^{2\pi}\frac{m}{n}\left(\cos n\theta \cos(n-m)\theta' \cos m\theta'' + \sin n\theta \cos(n-m)\theta' \sin m\theta''\right) \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(n(\theta-\theta'')) \\
&= \delta(\theta-\theta'') - \frac{1}{2\pi} \\
&\equiv \bar{\delta}(\theta-\theta''),
\end{aligned}
\tag{2.174}
$$

in the fifth line we have used the completeness relation in the Fourier series.

By further using Fourier expansion

$$\eta(\theta) = \sum_{m=1}^{\infty}\frac{1}{\sqrt{\pi}}\left(a_m \cos m\theta + b_m \sin m\theta\right), \tag{2.175}$$

the following results can be obtained

$$\int [d\eta] e^{-c\eta^2} \sim \int \Pi_{m=1}^{\infty} da_m db_m e^{-c(a_m^2+b_m^2)} \sim \Pi_{m=1}^{\infty} c^{-1} = e^{-\zeta(0)\ln c} = c^{1/2} \tag{2.176}$$

$$\bar{Z}(F) = \Pi_{k=1}^{D/2}\Pi_{m=1}^{\infty}(1+\bar{f}_k^2)^{-1} = \Pi_{k=1}^{D/2}(1+\bar{f}_k^2)^{1/2}. \tag{2.177}$$

Eq. (2.176) employs the zeta function regularization. Namely define the value of Riemann zeta function

$$\xi(s) = \sum_{n=1}^{\infty} n^{-s} \tag{2.178}$$

through continuation.

The physical meaning of zeta function regularization may be understood in analogy with Eq. (2.62). In Eq. (2.62) we can perform an alternative approach

$$A = \frac{D-2}{2}\zeta(-1) = -\frac{D-2}{24}, \tag{2.179}$$

since the removing infinity treatment in quantizing Bosonic string has to physical effect due to vanishing beta function, we can use the zeta function regularization as well.

The path integral (2.163) becomes

$$\Gamma(F) = Z_0 \frac{1}{\alpha'^{D/2} g_0^2} \int d^D y \left[\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})\right]^{1/2}, \tag{2.180}$$

which provides the Born-Infeld action. In the next section, we shall supplement the supersymmetry from path integral approach.

3 Properties of the $D = 4$ Born-Infeld action

The Born-Infeld action in four dimension can be written as

$$L_{\mathrm{BI}} = \sqrt{-\det_4 \left(\eta_{\mu\nu} + F_{\mu\nu}\right)} - 1. \tag{3.1}$$

here $\eta_{mn} = - + ++$.

The factor -1 in the end of Eq. (3.1) is conventional, in order to reproduce the Maxwell action for small F

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3.2}$$

Explicitly computing the determinant in terms of $\mathbf{E}$ and $\mathbf{B}$ for Eq. (3.1) gives

$$-\det_4 \left(\eta_{\mu\nu} + F_{\mu\nu}\right) = 1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16} \left(F_{\mu\nu} F^{*\mu\nu}\right)^2, \tag{3.3}$$

where $F^{*\mu\nu} = \varepsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$ is the dual tensor. Therefore for small field strength, the Born-Infeld action (3.1) reduces to the Maxwell action (3.2). Eq. (3.3) also shows the Lorentz invariance of the Born-Infeld action.

In the Euclidean signature there is an inequality

$$L_{\mathrm{BI}} = \sqrt{\left(1 + \frac{1}{4} F_{\mu\nu} F^{*\mu\nu}\right)^2 + \frac{1}{4} \left(F_{\mu\nu} - F^*_{\mu\nu}\right)^2} - 1 \geq \frac{1}{4} F_{\mu\nu} F^{*\mu\nu}, \tag{3.4}$$

which implies the minimum value of action is at $F_{\mu\nu} = F^*_{\mu\nu}$ (self-dual)

The Born-Infeld action is invariant under duality transformation $F \to F^*$.

4 Supersymmetric $D = 4$ Born-Infeld action

The bosonic string theory is generally regarded as an unrealistic theory, since it does not describe fermion. The incorporation of the supersymmetry is the superstring theory. Nevertheless, the bosonic string string usually remains in the lecture course of string theory. Many techniques in bosonic string theory could apply to superstring theory as well. Here we first review the properties of supersymmetry and then introduce on the extension of $D = 4$ Born-Infeld theory with supersymmetry.

4.1 Brief review of Supersymmetry

Supersymmetry is a symmetry between boson and fermion [27]

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle, \qquad Q|\text{Fermion}\rangle = |\text{Boson}\rangle. \tag{4.1}$$

It was explored by Golfand in the late 1960s for solving weak interaction [28]. The supersymmetric quantum field theory has much less divergence than the ordinary QFT [27]. This feature makes it very attractive. Although supersymmetry is not the only possibilities to reduce the divergence in QFT, discrete spacetime can also avoid the divergences [29].

The supersymmetry algebra is [30, p. 3]

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B, \qquad A, B = 1, \cdots, N, \tag{4.2}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0, \tag{4.3}$$

$$[P_m, Q^A_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}A}] = 0, \tag{4.4}$$

$$[P_m, P_n] = 0. \tag{4.5}$$

here $[\cdots, \cdots]$ and $\{\cdots, \cdots\}$ denote to commutator and anticommutator, respectively. The reason for the anticommutator in Eqs. (4.2) and (4.3) is highly remarkable. It is restricted by the Coleman-Mandula theorem [31].

The Coleman-Mandula theorem is a result of all the possible symmetries of the S-matrix in QFT. Based on the assumptions of Poincaré invariance, finiteness of particle number, elastic analyticity, existence of scattering, and a technical assumption, the theorem concludes that the symmetry group G is locally isomorphic to a direct product of a compact internal symmetry group and the space-time Poincaré group (external symmetry).

Since Eq. (4.2) mixes the internal and external symmetries and we are not dealing with conformal field theory at this moment, the anticommutator is used. In conformal field theory, in/out state cannot be defined. There is no S-matrix.

Furthermore, the Haag-Łopuszański-Sohnius theorem limits that the only possible spin change by supertransformation is 1/2 [32]. The 3/2 and higher spin transitions are excluded. Therefore, the logic restriction fixed the form Eqs. (4.2) and (4.3). This feature somehow strength the hope of supersymmetry.

For massless particle, the four momentum $p^m = (E, 0, 0, E)$. Eq. (4.2) can be written as [30, p. 12]

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = \delta^A_B \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.6}$$

Therefore we can introduce the creation and annihilation operators

$$a^A_\alpha = \frac{1}{\sqrt{4E}} Q^A_\alpha \tag{4.7}$$

to construct states with supersymmetry partners.

4.2 Born-Infeld action from superstring theory

The superstring analogy of the path integral approach is to add extra terms [33, 9]

$$S[\psi, A] = \frac{i}{4\pi\alpha'} \int d^2z \left(\bar{\psi}\partial_z\bar{\psi} + \psi\partial_{\bar{z}}\psi\right) - \frac{i}{2}\int_0^{2\pi} d\theta \psi^\mu F_{\mu\nu}\psi^\nu \bigg|_{r=1} \tag{4.8}$$

in the action. The tree-level path integral becomes

$$S = \int \mathcal{D}x\mathcal{D}\psi e^{-I}, \tag{4.9}$$

$$I = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_z X^\mu \partial_{\bar{z}} X^\mu + \bar{\psi}\partial_z\bar{\psi} + \psi\partial_{\bar{z}}\psi\right) + i\int d\theta\left[\dot{x}^\mu A_\mu(x)\right.$$
$$\left. -\frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x)\right], \tag{4.10}$$

$$\partial_z = \partial_1 + i\partial_2, \quad \partial_{\bar{z}} = \partial_1 - i\partial_2. \tag{4.11}$$

It can be shown that the new action is invariant under supersymmetry transformation [26, p. 90]. By a similar procedure as the bosonic string path integral, the effective action is the same as the bosonic case.

Beyond the leading term, there are derivative corrections to the Born-Infeld action. For open bosonic theory, the result is [34, 35]

$$L = \sqrt{-\det(\eta_{mn} + F_{mn})} - \frac{1}{48\pi}\left(F_{kl}F_{kl}\partial_a F_{mn}\partial_a F_{mn} + 8F_{kl}F_{lm}\partial_a F_{mn}\partial_a F_{nk}\right.$$
$$\left. -4F_{la}F_{lb}\partial_a F_{mn}\partial_b F_{mn}\right) + O(\partial^2 F^6). \tag{4.12}$$

For open superstring theory, the result is [34]

$$L = \sqrt{-\det(\eta_{mn} + F_{mn})} - \frac{1}{96}\left(\partial_a\partial_b F_{mn}\partial_a\partial_b F_{nl}F_{lr}F_{rm}\right.$$
$$+\frac{1}{2}\partial_a\partial_b F_{mn}F_{nl}\partial_a\partial_b F_{lr}F_{rm} - \frac{1}{4}\partial_a\partial_b F_{mn}F_{mn}\partial_a\partial_b F_{lr}F_{lr}$$
$$\left. -\frac{1}{8}\partial_a\partial_b F_{mn}\partial_a\partial_b F_{mn}F_{lr}F_{lr}\right) + O(\partial^4 F^6). \tag{4.13}$$

4.3 Supersymmetric extensions of $D = 4$ Born-Infeld action with $N = 1$, $N = 2$, and $N = 4$

In this section, the notations follow the review of Tseytlin [9]. Based on the Born-Infeld action from superstring theory, Cecotti and Ferrara constructed

$N = 1$ supersymmetric Born-Infeld action [36, 37]

$$S = \frac{1}{2}\int d^4x \left(\int d^2\theta W^\alpha W_\alpha + h.c.\right) + \int d^4x \int d^2\bar{\theta} B(K, \bar{K}) W^\alpha W_\alpha \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}, \tag{4.14}$$

$$B \equiv \frac{1}{1 - \frac{1}{2}(K + \bar{K}) + \sqrt{1 - (K + \bar{K}) + \frac{1}{4}(K - \bar{K})^2}}, \tag{4.15}$$

$$K \equiv D^2(W^\alpha W_\alpha), \quad \bar{K} = \bar{D}^2(\bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}), \tag{4.16}$$

$$D^2 \equiv -\frac{1}{4}D^\alpha D_\alpha, \quad \bar{D}^2 \equiv -\frac{1}{4}\bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}, \tag{4.17}$$

$$D_\alpha \equiv \partial_\alpha + i\sigma^m_{\alpha\dot{\alpha}}\partial_m, \quad \bar{D}_{\dot{\alpha}} \equiv -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}}\partial_m, \tag{4.18}$$

where θ is the anticommuting variables in the superspace. The $N = 1$ supersymmetric Born-Infeld action is interpreted as a supersymmetry breaking from $N = 2$ actions. The corresponding $N = 2$ action is not unique [37, 9].

One $N = 2$ supersymmetric Born-Infeld action was constructed by Ketov [38, 9]

$$S = \frac{1}{2}\int d^4x \left[\left(\int d^4\theta \mathcal{W}^2 + \text{h.c.}\right) + \frac{1}{4}\int d^4\theta d^4\bar{\theta} \mathcal{B}(\mathcal{K}, \bar{\mathcal{K}}) \mathcal{W}^2 \bar{\mathcal{W}}^2\right], \tag{4.19}$$

$$\mathcal{B} = \frac{1}{1 - \frac{1}{2}(\mathcal{K} + \bar{\mathcal{K}}) + \sqrt{1 - (\mathcal{K} + \bar{\mathcal{K}}) + \frac{1}{4}(\mathcal{K} - \bar{\mathcal{K}})^2}}, \tag{4.20}$$

$$\mathcal{K} = \frac{1}{2}D^4 \mathcal{W}^2, \quad \bar{\mathcal{K}} = \frac{1}{2}\bar{D}^4 \bar{\mathcal{W}}^2. \tag{4.21}$$

Similarly with the $N = 1$ action (4.14), Eq. (4.19) was interpreted as symmetry breaking from $N = 4$ action [39].

The $N = 4$ supersymmetric Born-Infeld action has been constructed by several authors [40, 41, 42, 43, 44], although the expressions have not been in form of W_α and Φ_a. For a D3 brane the bosonic part is the DBI action

$$S = \int d^4x \sqrt{-\det(\eta_{mn} + \partial_m X^s \partial_n X^s + F_{mn})}, \quad s = 1, 2, \cdots, 6, \tag{4.22}$$

where

$$\varphi^a = X^a + iX^{a+3}, \quad \partial_m X^s \partial_n X^s = \partial_{(m}\varphi^a \partial_{n)}\bar{\varphi}^a. \tag{4.23}$$

5 Application to cosmology

Inflation is a proposal to solve the homogeneity and magnetic monopole problems in cosmology. In principle, the homogeneity in the region without causal contact in the universe and the absence of magnetic monopole could be explained by a very special initial condition of big bang. Inflation provides a more natural explanation. More important, the inflation predicts the cosmological background spectrum. The detailed prediction depends on particular models of inflation [45].

The DBI action has been used in describing inflation in cosmology [46, 47]

$$S = \int \frac{1}{2} M_p^2 \sqrt{-g} \mathcal{R} + \mathcal{L}_{\text{eff}} + \cdots , \tag{5.1}$$

$$\mathcal{L}_{\text{eff}} = -\frac{1}{g_s} \sqrt{-g} \left(f(\phi)^{-1} \sqrt{1 + f(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} + V(\phi) \right) , \tag{5.2}$$

where M_p is the planck mass. $f(\phi)$ is an Anti-de Sitter (AdS)-type warp factor. $V(\phi)$ is a scalar potential from Ramond-Ramond flux and compactification. The action (5.1) describes D3 brane couples with gradational field. One singificant feature of this model is, it predicts large non-Gaussianities on the CMB.

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