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Generalized Geometry, Parallelizability and Non Geometry

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Contents

1 Introduction

Throughout the course of their histories, Physics and Mathematics, once two disparate disciplines, have become ever more intertwined. With the advancement of our understanding of physical phenomena, the need for more sophisticated tools to play a part in their description has become evermore apparent. Once almost exclusively the prerogative of experimentalists, the last century has seen a paradigm shift where it is now mostly the theorists who blaze the trail of discovery. Mathematics, ever incapable of resisting its own utility, has been the centerpiece of this steady advance. Even some of its most abstract vistas have found a place in the physical world, which itself, has inspired, motivated and even progressed these areas which one could be forgiven for questioning, what their relation to reality is. As it appears, we must either embrace this inherent complexity, or sacrifice the understanding we would otherwise gain. However, beneath these intricacies, there seems to be an underlying trend, an almost irresistible force, towards simplicity and unity. It is only the superficial complexity of the subject that might have the casual observer believe that physics had descended into entropy. For at its heart lies its overarching purpose to describe the world we live in the most simple and elegant way. The answers to so many of the questions it poses are there for anybody with the patience to learn its language and skill in applying it, to find.

Hardly anything exemplifies this more than string theory. Originally conceived in the 1960's to describe the structure of Hadrons, it was quickly displaced by the more conservative theory of Quantum Chromodynamics. Not only did QCD conform to experimental predictions, it also didn't suffer the myriad of apparently irremediable deficiencies plaguing this early precursor of modern string theory. However, far from the final blow one would have expected this to provide, it continued to be developed for the promise of providing the answer to an even greater problem: unification of quantum theory and gravity. This unfulfilled fantasy had proved elusive to researchers working in both fields and now it seamed that string theory would be what brought them together. But, it has been said that 'truth resists simplicity' and verily, it was so. Far from providing a simple, straight forward theory, it ended up predicting extra spatial dimensions, negative mass states and the absence of fermions amongst many other things and it seemed, once again, as though the young theory would have to be consigned to the scrap heap of interesting, yet futile ideas. However, with the emergence of supersymmetry, it quickly transpired that the newly named superstring theory might stand a chance of remedying some of the old deficiencies. Further, string theory, in a wider context, is an example of a Kaluaza-Klein (KK) theory, named for the first people who attempted to unify gravity and quantum theory by proposing the existence of extra dimensions and providing mechanism for coping with them: compactification. With these tools it seems as though string theorists might be in a position to find a quantum theory of gravity.

The events of the following years found their high points in the so called first and second superstring revolutions. These were periods of particularly scintillating insight which lead to several fundamental paradigm shifts. The first important, slightly unsettling, realization was that there actually existed five, apparently disperate, strings theories, each living in 10 dimensions. These were Type IIA, Type IIB, SO(32) heterotic, $E_8 \times E_8$ heterotic and Type I. The second realization then showed these to be, in fact, the five different manifestations of an 11 dimensional theory called M-theory (the exact meaning of M is somewhat ambiguous, but most correctly is probably termed 'Membrane', though 'Magic' and 'Mystery' have also been used [1]). Providing a first insight into what a unified theory may look like, understanding M-theory is one of modern theoretical physics' great goals.

These developments conspired to bring to the forefront of our understanding of physics, something which at that time, had increasingly been encroaching on the center stage, namely symmetry. One of the most important properties of any theory, it provides a valuable tool to understanding the physics involved. It is no understatement to say that, one who understands symmetry, understands the theory it belongs to. A most poignant example may be the standard model, who's gauge symmetries are captured by the group $SU(3) \times SU(2) \times U(1)$ and are universally used when trying to understand particle physics. Therefore, one of the most important aims in the study of modern string and M-theory is to understand the symmetries involved.

The previously mentioned supersymmetry is the manifestation of this drive. By extending the well known traditional gauge symmetries, it introduces additional transformations relating the seemingly unrelated fermions and bosons. First used to remedy the lack of fermions in the earlier manifestations of string theory, thereby creating supersrtings, it has since become an indispensable part of the theory as a whole. Being convoluted with the well known Poincare group, the generators of the supersymmetries extend its algebra to the so called super Poincare algebra. Somewhat surprisingly, this alone provides an interface with gravity simply by promoting the usually global supersymmetries, to local ones. The result is called *supergravity*. What is perhaps most interesting, is that the same theory may also be realized as a low energy limit of the previously discussed superstring theories. As such, it comes in various flavours, depending on which string theory we are talking about. The most interesting manifestations are Type IIA and Type IIB $\mathcal{N} = 2$ along with Type I $\mathcal{N} = 1$ supergravity in 10 dimensions where $\mathcal{N}$ refers to the number of supersymmetry generators in powers of two. Being the easiest to understand, they are also stepping stones on the path to fully describe the supergravity belonging to M-theory in 11 dimensions. However, the transition from 10 dimensions is not at all straight forward, owing to the symmetries of this higher dimensional theory being but poorly understood. For this reason attempts at reaching a description have been limited to 7 dimensions, see for example [2][11].

A further important property of string theory, without which no discussion could be complete, are its dualities. Being distinct from the traditional notion of zero dimensional point particles, strings are objects possessing an additional one dimensional extent. This endows them with certain properties unparalleled in traditional high energy physics and, in a sense, extends their symmetries beyond those familiar to us from the study of particles. The most important of these are S- and T-duality, the first of which relates the coupling constants of different theories. Determining whether a theory can be solved perturbatively, the value of the coupling constant is of great importance. In string theory, S-duality allows one to map from large to small coupling constants allowing for a previously unsolvable problem to become solvable. T-duality, which will be the one of interest to us, arises from the compactification of the extra spacial dimensions. Its action is to map between winding modes (so called because they refer to how a string is 'wound' around the compact dimensions) and momentum modes. Interestingly, this is how Type IIA and Type IIB string theory are related, that is, they are T-dual to each other. This action is captured by the group $O(d, d)$.

The complete understanding of all symmetries and dualities brings us to the topic of generalized geometry. A comparatively recent development, it provides a mathematical framework with which to attain a coherent description of supergravity, where the apparently different symmetries consisting of the diffeomorphism, gauge invariance and T-duality are unified. Generalized geometry is the term used for the study of the generalized tangent space, most simply expressed as $E = T\mathcal{M} \oplus T^*\mathcal{M}$ which is the extension of the tangent bundle by the cotangent bundle of some manifold $\mathcal{M}$. This provides an elegant framework, the most fascinating property of which is probably that it already possess structures inherent to supergravity, namely the structure group $O(d, d)$. It is this surprising, yet gratifying fact which will serve as starting point for this report. Another important application of generalized geometry is to the study of compactification, which seeks to reduce the higher dimensional spacetimes on which string theory is formulated, to spacetimes of the form $\mathbb{R}^{1,3} \times \mathcal{M}$. Traditionally, the *internal space* $\mathcal{M}$ is chosen so as to be a *parallelizable manifold*, that is, a manifold on which one may find a frame field which is nowhere vanishing. Group manifolds fulfill this requirement and as such are of interest when studying these internal spaces. A similar concept holds with generalized geometry and parallelizability may be extended to the generalized tangent space. Only, as we will see, the internal space may then be a more general coset. The last subject we will turn to arises as a consequence of T-duality and is the somewhat imprecisely named *non geometry*. This refers to the situation when T-duality maps to spaces that lack a traditional geometric description and we will see how this may also be present in generalized geometry and how it is related to generalized parallelizability.

The layout of this report is as follows. The first section contains a brief summary of supergravity and and T-duality and gives an outline of the properties most relevant to the following content. This leads to an introduction of the basic properties of generalized geometry. We will develop the tools and formalisms necessary for the next section. Here will be discussed parallelizability in the context of the generalized tangent space and how it involves a larger class of manifolds than traditional geometry. The section will be concluded by providing two examples and explicitly constructing a Lie algebra like relation using the Courant bracket. Lastly, we will discuss briefly the concept of non geometry and by example, show the gauge dependence of the so called non geometric fluxes.

2 Supergravity and T-duality

2.1 Supergravity

Here is given a short introduction and outline of supergravity and how the need to better understand its symmetries provides some of the motivation to study generalized geometry. It arises as the low energy limit of superstring theory in which strings apparently loose their one dimensional extent and are treated as point particles. We will be particularly concerned with Type II supergravity, so called because of its origin in Type II string theory, in $d = 10$ spacetime dimensions, the basic structures of which are recalled here. The field contents of this type of supergravity is [7]

$$\{g, B, \phi, A^{(p)}, \psi^{\pm}, \lambda^{\pm}\} \tag{2.1}$$

where we have the metric g, two form gauge field B and the dilaton ψ. The p-form $A^{(p)}$ denotes the RR potentials where p being even or odd determines whether we are in Type IIA or Type IIB supergravity. There are also the fermionic fields consisting of the gravitino $\psi^{\pm}$ and dilatino $\lambda^{\pm}$. The Ramond-Ramond (RR) sector will not be of interest to us here, rather we will concentrate just on the graviton, two-form and dilaton or NeveuSchwarz (NSNS) sector. Specifically we will be interested in the symmetries of this sector and how they serve as part of the motivation for studying generalized geometry. The action for these fields may be written as

$$S = \int dx^{10} \sqrt{g} e^{-2\phi} \left(\mathcal{R}(x) + 4(\partial\phi(x))^2 - \frac{1}{12} H(x)^2 \right) \tag{2.2}$$

where $\mathcal{R}$ is the Ricci scalar and $H = dB$, analogous to the electromagnetic two form. We note that this action is invariant under two different kind of transformations. The first are diffeomorphisms acting by translations such that $x^{\mu} \to x^{\mu} + \epsilon v^{\mu}$. This gives us the variations

$$\delta g = \mathcal{L}_v g \quad \delta\phi = \mathcal{L}_v \phi \quad \delta B = \mathcal{L}_v B \tag{2.3}$$

We express the group of differomorphism on $\mathcal{M}$ as $Diff(\mathcal{M})$. The second is the gauge transformation of the B-field which gives the three form H up to addition of a closed two form. Hence, we have the gauge transformation

$$\delta B \to B + d\lambda \tag{2.4}$$

We also note, given that H is closed, B is only locally defined such that, given an open cover $\{U_i\}$ over $\mathcal{M}$, on the overlap $U_i \cap U_j$, we require the one forms Λ to provide the patching of B. So we also have

$$B_{(i)} = B_{(j)} + d\Lambda_{(ij)} \tag{2.5}$$

thus requiring, on $U_i \cap U_j$, that $d\lambda_{(i)} = d\lambda_{(j)}$. These form an Abelian group of closed two forms denoted by $\Omega^2_{cl}(\mathcal{M})$. The total symmetry group of the NSNS sector is then given by a semi direct product of the gauge and diffeomorphism groups i.e. $G_{NSNS} = Diff(\mathcal{M}) \ltimes \Omega^2(\mathcal{M})$. We can thus formulate the general variations

$$\tilde{\delta} g = \mathcal{L}_v g \quad \tilde{\delta}\phi = \mathcal{L}_v \phi \quad \tilde{\delta} B = \mathcal{L}_v B - d\lambda \tag{2.6}$$

which are parametrized by the one form $\lambda \in T^*\mathcal{M}$ and vector $v \in T\mathcal{M}$. A central feature of generalized geometry is that it unifies diffeomorphism and gauge symmetries and combines the NSNS fields into a single object. However, this report will not cover in detail how this is done and it will merely commented on in a later section. The reason for considering the matter here is that it provides an important part of the motivation for generalized geometry as a whole.

2.2 T-duality

In this chapter is given a brief outline of an important symmetry of string theory T-duality. This duality is a phenomenon arising from the one dimensional extent of strings and has no parallel in classical quantum theories of point particles. It is purely a string theory phenomenon. In its simplest form, it arises from considering Bosonic string theory, which lives in 26 dimensions, compactified on to a circle of radius R so as to give a backgrounds that looks like $R^{25} \times S^1_R$. We then have the situation where a string may either propagate over the surface of the compact S^1 or wind around it. It turns out that these two cases are related by T-duality which maps between the two via the transformation $R \leftrightarrow \frac{1}{R}$. However, Bosonic string theory is not of much interest to us here and the reader is referred to [3] for explicit calculations. Rather we will concentrate on the generalization where the same principle is applied to string backgrounds of the form $\mathcal{M}_{d-n} \times T^n$. For this, we consider the Polyakov action in local coordinates

$$S[X] = \int_\Sigma d\tau d\sigma \sqrt{-h} h^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{2.7}$$

This is an example of what is known as a sigma model with the action describing a *worldsheet*, the two dimensional surface traced out by propagating strings, embedded in the spacetime $\mathcal{M}$ and described by the metric $g_{\mu\nu}$. The worldsheet is denoted by Σ and has coordinates τ and σ with metric h^{ab}. X gives the embedding of the world sheet such that $X : \Sigma \to \mathcal{M}$. In the rest of this section, we will assume the worldsheet metric h^{ab}, to be flat for simplicity. We now give a brief outline of the *Buscher rules.* First derived by T. Buscher in [4], they give the transformation of background fields under T-duality. The derivation given here follows [5] and [6]. To begin with, we consider the sigma model using the complex coordinates $z = \tau + i\sigma$, $\overline{z} = \tau + i\sigma$ and introducing the two form potential B encountered in the previous section

$$S[X] = \int_\Sigma d^2 z (B_{\mu\nu}(X) + g_{\mu\nu}(X)) \partial X^\mu \overline{\partial} X^\nu \tag{2.8}$$

Now suppose there is a local coordinate chart which we split to get $\{x^\mu\} = \{\theta, x^a\}$, where $a \in \{1...d-1\}$ such that B and g are independent of θ. Assume that the variation $\delta X_\mu = X_\mu + \epsilon k_\mu$ acts by translation in the θ direction giving a killing vector $k_\mu = \frac{\partial}{\partial\theta}$ which is an isometry of the metric i.e. $\mathcal{L}_k g = 0$. We further impose the condition that the isometry be periodic so that we have the isometry group $U(1)$. Hence, it is easy to see that the variation of the metric, along with the derivative terms, vanishes. For the B-field, consider $\Sigma = \partial C$ then, by Stokes' theorem

$$\int_{\partial C} X^* B = \int_C X^* H \tag{2.9}$$

We then demand that the variation $\delta H = \mathcal{L}_k H = \mathcal{L}_k dB = d\mathcal{L}_k B = 0$ vanish and thus locally get the condition $\mathcal{L}_k B = d\omega$. The gauge transformation (2.4) then gives $\delta \mathcal{L}_k B \to \mathcal{L}_k B + \mathcal{L}_k dA = d(\omega + \mathcal{L}_k A)$. With our choice of killing vector we then have $\mathcal{L}_k A_\mu = \partial_\theta A_\mu$. In order for the variation of B to vanish, we choose a gauge where $\partial_\theta A_\mu = -\omega_\mu$. Hence have that $\mathcal{L}_k B = 0$. Note we can also choose $\{x^\mu\} = \{\theta^A, x^b\}$ where $A \in \{1...N\}$ and $b \in \{N+1...d\}$ for the isometry group $U(1)^N$, provided g and B are independent of θ^A. Considering this, the action (2.8) may be written in terms of of this split basis as

$$S[X] = \int d^2z \left(Q_{AB}\partial\theta^A\overline{\partial}\theta^B + Q_{aB}\partial X^a\overline{\partial}\theta^B + Q_{Ab}\partial\theta^A\overline{\partial}X^b + Q_{ab}\partial X^a\overline{\partial}X^b\right) \tag{2.10}$$

where we have chosen $Q_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$. The T-dual case to this is obtained by *gauging the symmetry*, that is, we formulate an action where the isometry appears as a gauge symmetry. So now, a gauge variation gives $\delta X_\mu(z,\overline{z}) = X_\mu(z,\overline{z}) + \epsilon k_\mu(z,\overline{z})$. We also introduce the $U(1)^N$ gauge fields A^B and $\bar{A}^A$ and minimally couple them to θ^B by

$$\begin{aligned} \partial\theta^B &\to D\theta^B = \partial\theta^B + A^B \\ \overline{\partial}\theta^B &\to \overline{D}\theta^B = \overline{\partial}\theta^B + \overline{A}^B \end{aligned} \tag{2.11}$$

Introducing the Lagrange multiplyer $\hat{\theta}$, the action now reads

$$\begin{aligned} S[X] = \int d^2z(&Q_{AB}D\theta^A\overline{D}\theta^B + Q_{aB}\partial X^a\overline{D}\theta^B + Q_{Ab}D\theta^A\overline{\partial}X^b \\ &+Q_{ab}\partial X^a\overline{\partial}X^b + \hat{\theta}(\partial\overline{A} - \overline{\partial}A)) \end{aligned} \tag{2.12}$$

From where we get the first order action

$$\begin{aligned} S_1[X] = \int d^2z(&Q_{AB}A^A\overline{A}^B + Q_{aB}\partial X^a\overline{A}^B + Q_{Ab}A^A\overline{\partial}X^b \\ &+Q_{ab}\partial X^a\overline{\partial}X^b + \hat{\theta}(\partial\overline{A} - \overline{\partial}A)) \end{aligned} \tag{2.13}$$

To be consistent, this should reduce to (2.10) (and by extension (2.8)) and we check this by computing the equations of motion for $\hat{\theta}$ which give

$$\partial\hat{A} - \hat{\partial}A = 0 \tag{2.14}$$

Solving this by $A = \partial f$ and $\hat{A} = \hat{\partial}f$, where f is identified with θ, we indeed get (2.10). The isometry is chosen to act along a particular direction denoted by $X^\bullet$ i.e. $\theta = X^\bullet$ and $Q_{AB} = Q_{\bullet\bullet}$. Now, the next step is to calculate the equations of motion for the gauge field. Doing so, we obtain:

$$\begin{aligned} -\bar{\partial}\hat{\theta} &= g_{\bullet\bullet}\bar{A} + (g_{\bullet b} + B_{\bullet b})\bar{\partial}X^b \\ \partial\hat{\theta} &= g_{\bullet\bullet}A + (g_{a\bullet} + B_{a\bullet})\partial X^b \end{aligned} \tag{2.15}$$

Identifying $\hat{\theta} = X^0$, rearranging for A, and substituting this into (2.13) we can rearrange the result to obtain the dual action

$$S[X] = \int_V d^2z(\tilde{B}_{\mu\nu}(X) + \tilde{g}_{\mu\nu}(X))\partial X^\mu\overline{\partial}X^\nu \tag{2.16}$$

The new metric an B-field are given in terms of those in (2.8) by the relations

$$
\begin{aligned}
\tilde{g}_{\bullet\bullet} &= \frac{1}{g_{\bullet\bullet}} \\
\tilde{g}_{\bullet b} &= \frac{B_{\bullet b}}{g_{\bullet\bullet}} \\
\tilde{g}_{ab} &= g_{ab} - \frac{g_{\bullet a} g_{\bullet b} - B_{\bullet a} B_{\bullet b}}{g_{\bullet\bullet}} \\
\tilde{B}_{\bullet b} &= \frac{g_{\bullet b}}{g_{\bullet\bullet}} \\
\tilde{B}_{ab} &= B_{ab} - \frac{g_{\bullet a} B_{\bullet b} - B_{\bullet a} g_{\bullet b}}{g_{\bullet\bullet}}
\end{aligned}
\tag{2.17}
$$

These equations are known as the Buscher rules and give explicitly the T-duality transformation of a background consisting of a metric and B-field in the presence of a $U(1)$ isometry. For example, consider a circle of radius r described by the round metric $ds^2 = dr^2 + r d\phi^2$. This has the prerequisite isometry in the ϕ direction. Under T-duality we get the dual metric $ds^2 = dr^2 + \frac{1}{r} d\phi^2$. This is simply the classical T-duality from Bosonic string theory discussed earlier which inverts the radius of the circle. The final point, before moving on to the next section, is the group action of T-duality. Put simply, one may go from (2.8) to (2.16) by the action of the group $O(d,d)$, which, for the tensor $Q_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$, is denoted by

$$
Q \to Q' = (aQ+b)(cQ+d)^{-1} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d) \tag{2.18}
$$

This will prove to be of fundamental importance and serves as further motivation to study generalized geometry. We will see how this group appears as the structure group of the generalized tangent bundle and how this links to T-duality.

3 Generalized Geometry

We now turn to the subject of generalized geometry itself. Many of the concepts and structures were originally introduced by N. Hitchin [8] and further developed by M. Gualtieri [9] [10]. In laying out the basic principles, we follow [7] and [9]. Here are defined and described generalized vectors and their structure group $O(d,d)$. We then move on to defining some of the tools necessary to continue to the next topic. It should be noted that generalized geometry, when formulated to yield a description of supergravity, also requires, appart from the structure group of the generalized tangent bundle, an additional factor to account for the dilaton degrees of freedom extending the structure group to $O(d,d) \times \mathbb{R}$. However, the topic of this report does not require the inclusion of this factor and it does not have any bearing on the calculations performed. For this reason, we will be neglecting the dilaton and considering only the group $O(d,d)$

3.1 Generalized Tangent space

Consider a manifold $\mathcal{M}$ of dimension d. Replacing the traditional approach of working with the tangent bundle $T\mathcal{M}$ and cotangent bundle $T^*\mathcal{M}$ separately, generalized geometry introduces a new object: the generalized tangent bundle $E = T\mathcal{M} \oplus T^*\mathcal{M}$. If $v \in T\mathcal{M}$ and $\lambda \in T^*\mathcal{M}$, then the generalized vector $W \in \Gamma E$ has components given by

$$W = \begin{pmatrix} v \\ \lambda \end{pmatrix} \tag{3.1}$$

This space is naturally endowed with an inner product and a metric

$$\eta(W,W) = \langle W, W \rangle = i_v \lambda \tag{3.2}$$

It will be useful to introduce a notation for the components of generalized vectors. Using coordinate bases $\{\frac{\partial}{\partial x^\mu}\}$ on $TU_{(a)}$ and $\{dx^\mu\}$ on $T^*U_{(a)}$ over some patch $U_{(a)}$, we may write $W = v^\mu \frac{\partial}{\partial x^\mu} + \lambda_\mu dx^\mu$ where $\mu = 1, ..., d$. We also introduce a new index running over all components of W, i.e. $M = 1, .., 2d$, which will generally be denoted by capital Latin characters so that

$$W^M = \begin{cases} v^\mu & \text{for } M = \mu \\ \lambda_\mu & \text{for } M = \mu + d \end{cases} \tag{3.3}$$

With this in mind, we define a basis $\{E_A\}$ on E, where $A = 0, ..., 2d$ which satisfies

$$\langle E_A, E_B \rangle = \eta_{AB} \tag{3.4}$$

and is thus an orthonormal basis over E. The flat metric η, may be recast as a matrix to give

$$\eta = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{3.5}$$

Which may be diagonalized so as to give a matrix with split signature $(d, -d)$ implying its invariance under the action of the orthogonal group $O(TM \oplus T^*M) \simeq O(d,d)$

Proposition. *For $O \in O(d,d)$, η is invariant under $O(d,d)$ transformations* *such that* $\eta(OW, OW) = \eta(W, W)$

Proof. Express a general element $O \in O(d,d)$ in terms of the $d \times d$ matricies a, b, c and d as $O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then write

$$\begin{aligned} \eta(OW, OW) &= \begin{pmatrix} v & \lambda \end{pmatrix} \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} \\ &= \begin{pmatrix} v & \lambda \end{pmatrix} \begin{pmatrix} c^T a + a^T c & c^T b + a^T d \\ d^T a + b^T c & d^T b + b^T d \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} \end{aligned}$$

Now require $c^T a + a^T c = d^T b + b^T d = 0$ and $c^T b + a^T d = d^T a + b^T c = 1$. Thus $\eta(OW, OW) = \eta(W, W)$ □

Because of the natural $O(d,d)$ action on E, we can define the generalized structure bundle

$$K = \{(x, E_A) : x \in \mathcal{M} \text{ and } \{E_A\} \text{ is an } O(d,d) \text{ basis of } E_x\} \tag{3.6}$$

which is a principle bundle with fiber $O(d,d)$. There exists a canonical orientation on $T\mathcal{M} \oplus T^*\mathcal{M}$ [9], so that we require $O(d,d)$ to preserve this orientation. Hence we must further be able to reduce the structure group to $SO(d,d)$. Noting that the highest exterior power of E may be decomposed as

$$\wedge^{2d} T\mathcal{M} \oplus T^*\mathcal{M} = \wedge^d T\mathcal{M} \oplus \wedge^d T^*\mathcal{M} \tag{3.7}$$

so that, for $w \in \wedge^d T^*\mathcal{M}$ and $v \in \wedge^d T\mathcal{M}$ there exists a natural pairing between $\wedge^d T^*\mathcal{M}$ and $\wedge^d T\mathcal{M}$ which may be expressed as

$$(\omega, v) = \det(\omega_i(v_j)) \tag{3.8}$$

Thus, we may make the identification $\wedge^{2d} T\mathcal{M} \oplus T^*\mathcal{M} = \mathbb{R}$ so that the canonical orientation on the generalized tangent bundle is defined by a real number. We will now look in more detail at the symmetries of this space. The structure group $O(d,d)$ is generated by elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix}, \quad e^B = \begin{pmatrix} \mathbb{I} & 0 \\ B & \mathbb{I} \end{pmatrix}, \quad e^\beta = \begin{pmatrix} \mathbb{I} & \beta \\ 0 & \mathbb{I} \end{pmatrix} \tag{3.9}$$

where $A \in GL(d, \mathbb{R})$ and is the structure group of $T\mathcal{M}$ and $B \in \wedge^2 T^*\mathcal{M}$ and $\beta \in \wedge^2 T\mathcal{M}$. The additional objects are known as *B-tranforms* and *β-tranforms* and act as endomorphisms of E by $B : T\mathcal{M} \to T^*\mathcal{M}$ and $\beta : T^*\mathcal{M} \to T\mathcal{M}$. Exponentiating, one may define the mapping induced by B as

$$e^B : v + \lambda \to v + (\lambda - i_v B) \tag{3.10}$$

which acts so as to shear in the $T^*\mathcal{M}$ direction. And similarly, that of β as

$$e^\beta : v + \lambda \to (v + i_\lambda \beta) + \lambda \tag{3.11}$$

which shears in the $T\mathcal{M}$ direction. To see the role of these transformations, it will be useful to consider what happens in the overlap between two coordinate patches U_a and U_b. On $U_a \cap U_b$, the vectors and one forms must be patched in the usual way, that is

$$V_{(a)} = v_{(a)} + \lambda_{(a)} = A_{(ab)}v_{(b)} + \left(A^{-1}_{(ab)}\right)^T \lambda_{(b)} - i_{A_{(ab)}x_{(b)}} d\Lambda_{(ab)} = V_{(b)} \quad (3.12)$$

where $\Lambda_{(ab)}$ are the patching one forms. Recasting this this in matrix form gives

$$V_{(a)} = \begin{pmatrix} v_{(a)} \\ \lambda_{(a)} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ d\Lambda_{(ab)} & \mathbb{I} \end{pmatrix} \begin{pmatrix} A_{(ab)} & 0 \\ 0 & \left(A^{-1}_{(ab)}\right)^T \end{pmatrix} \begin{pmatrix} v_{(b)} \\ \lambda_{(b)} \end{pmatrix} = M_{(ab)}V_{(b)} \quad (3.13)$$

We now see the correspondence between the patching one forms and the B-transformation. It is no accident that we have chosen B to denote these transformations. As will become apparent, it does indeed correspond to the two form NSNS potential. Here it is treated as a generator of the subgroup $G_B \subset SO(d,d)$. Hence it becomes apparent that that the overall structure group G_{Geom} must be given by the semi direct product $G_{Geom} = G_B \ltimes GL(d)$ and embeds as a subgroup of $SO(d,d)$. Before moving on, it is important to note that from (3.12) we see that all vectors $v_{(a)}$ are globally equivalent up to some transformation by the structure group of the tangent bundle, whereas the one forms $\lambda_{(a)}$ are not. This follows from the additional term involving $\Lambda_{(ab)}$, which describe the fibration of $T^*\mathcal{M}$ over $T\mathcal{M}$. It is this patching that is captured by the B-transform. Hence we note that while a particular choice $v_{(a)}$ is globally equivalent up to some $GL(d,\mathbb{R})$ rotation, the one forms require additional patching in order for the generalized vector $V_{(a)}$ to be a global section of E. Taking this into account, $V \in \Gamma E$ is a global section if we define

$$V = \begin{pmatrix} v \\ \lambda - i_v B \end{pmatrix} \quad (3.14)$$

We also note, that one can make a change of basis by

$$V^M \to O^M{}_N V^N \quad V_M \to V_N (O^{-1})^N{}_M \quad (3.15)$$

where $O \in O(d,d)$. We may also define a basis on E which will generally be denoted E_A where $A \in [0, ..., 2d]$. Any choice of basis is defined up to $O(d,d)$ transformations, that is, the two bases related by $E_A = O^B{}_A \widetilde{E}_B$ are equivalent. This is a point we will return to in a later section where it will prove to be of central importance.

3.2 Dorfman derivative and Courant bracket

Now the structure of $T\mathcal{M} \oplus T^*\mathcal{M}$ has been explored, we turn our attention to another important property. That is, there exists a natural generalization of the Lie bracket.

Definition. *The Courant bracket is the skew symmetric bilinear form* $[\![\cdot,\cdot]\!] : E \times E \to E$ *s.t. for* $W = w + \lambda \in \Gamma E$ *and* $V = v + \alpha \in \Gamma E$ we have

$$[\![W,V]\!] = [w,v] + \mathcal{L}_w\alpha - \mathcal{L}_v\lambda - \frac{1}{2}d(i_w\alpha - i_v\lambda) \quad (3.16)$$

where $[\cdot,\cdot]$ is the ordinary Lie bracket. Though defined as an analogue of the Lie bracket on the generalized tangent space and having some properties in common, it does not satisfy the Jacobi identity, something that will be discussed in more detail later on.

Now we wish to investigate the symmetries of the Courant bracket. At the most basic level, one would expect to find the same symmetries as for the Lie bracket, that is we expect it to be diffeomorphism invariant. For the normal Lie bracket, under the projection $\pi : T\mathcal{M} \to \mathcal{M}$, we have the bundle automorphism (F, f) such that $F : T\mathcal{M} \to T\mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ preserve the bracket

$$F[X, Y] = [FX, FY] \tag{3.17}$$

$\forall X, Y \in T\mathcal{M}$ and $F = f_*$, where $f_* : T\mathcal{M}_p \to T\mathcal{M}_{f(p)}$ is the push-forward. Now we need to define an analogous action on $T\mathcal{M} \oplus T^*\mathcal{M}$, that is to say, a generalized bundle automorphism $(\tilde{F}, \tilde{f})$. Thus, $\tilde{F}$ needs to satisfy

$$\tilde{F}[\![W, V]\!] = [\![\tilde{F}W, \tilde{F}V]\!] \tag{3.18}$$

$\forall W, V \in \Gamma E$. Hence, we see that $F = f_* \oplus f^*$ where $f^* : T\mathcal{M} \to T\mathcal{M}$ is the pull-back. However, this isn't the full story; the Courant bracket possesses an additional symmetry inherited from the structure group of the generalized tangent bundle. Recall that we also had the action of G_B, the B-transformation.

Proposition. *For $W = w + \lambda \in \Gamma E$ and $V = v + \alpha \in \Gamma E$* we have the identity $[\![e^B W, e^B V]\!] = e^B [\![W, V]\!] + i_w i_v dB$

Proof. $[\![e^B W, e^B V]\!] = [\![w + \lambda + i_w B, v + \alpha + i_w B]\!] = [\![W, V]\!] + [w, i_v B] + [i_w B, v] = [\![W, V]\!] + \mathcal{L}_w i_v B - \mathcal{L}_v i_w B - d i_w i_v B = [\![W, V]\!] + \mathcal{L}_w i_v B - i_w d i_v B = [\![W, V]\!] + i_{[v,w]} B + i_w i_v dB = e^B [\![W, V]\!] + i_w i_v dB$ □

Hence we see that if B satisfies $dB = 0$, the Courant bracket is invariant under B-transformations. It can thus be shown that the total automorphism group of the Courant bracket [10] consists of the the Diffeomophism group on $\mathcal{M}$ and the B-transformations G_B. Thus, similarly to the structure group on E, we have the total automorphism group of the Courant bracket, $G_C = G_B \ltimes Diff(\mathcal{M})$. Before Continuing, it will be useful to note the existence of a further object related to the Courant bracket: the Dorfman bracket.

Definition. *The Dorfman bracket is the bilinear form $[\cdot, \cdot]_D : E \times E \to E$ s.t. for $W = w + \lambda \in \Gamma E$, $V = v + \alpha \in \Gamma E$ and $X = x + \omega \in \Gamma E$ we have the expression*

$$[W, V]_D = \mathcal{L}_w v + \mathcal{L}_w \alpha - d i_v \lambda \tag{3.19}$$

which also satisfies the identity

$$[X, [W, V]_D]_D = [[X, W]_D, V]_D + [W, [X, V]_D]_D \tag{3.20}$$

This bracket is indented as a generalization of the Lie derivative which however, does not satisfy the Jacobi identity but rather a modified version called the Leibniz identity as shown. To exemplify the correspondence between the Lie derivative and Dorfman bracket, we may express the latter in the form

$$[W, V]_D = \mathbb{L}_W V \tag{3.21}$$

which is referred to as the Dorfman derivative. We do not have the same relation between the Courant bracket and the Dorfman derivative as between the Lie bracket and Lie derivative. However the two are related

Proposition. *$W = w + \lambda \in \Gamma E$ and $V = v + \alpha \in \Gamma E$ we have the identity $[\![W, V]\!] = \frac{1}{2}(\mathbb{L}_W V - \mathbb{L}_V W)$*

Proof. $[\![W,V]\!] = \mathcal{L}_w v + \mathcal{L}_w \alpha - \mathcal{L}_v \lambda - \frac{1}{2} d(i_w\alpha - i_v\lambda) = \frac{1}{2}\mathcal{L}_w v - \frac{1}{2}\mathcal{L}_v w + di_w\alpha + i_w d\alpha - di_v\lambda + i_v d\lambda - \frac{1}{2} di_w\alpha + \frac{1}{2} di_v\lambda = \frac{1}{2}\mathcal{L}_w v + \frac{1}{2}(di_w\alpha + i_w d\alpha) - \frac{1}{2} di_v\lambda - \frac{1}{2}\mathcal{L}_v w - \frac{1}{2}(di_v\lambda + i_v d\lambda) + \frac{1}{2} di_w\alpha = \frac{1}{2}(\mathcal{L}_w v + \mathcal{L}_w\alpha - di_v\lambda) - \frac{1}{2}(\mathcal{L}_v w + \mathcal{L}_v\lambda - di_w\alpha) = \frac{1}{2}(\mathbb{L}_W V - \mathbb{L}_V W)$ □

The Dorfman derivative may be made to look more familiar by defining the the partial derivative operator on E

$$\partial^M = \begin{cases} \partial^\mu & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases} \tag{3.22}$$

Now we may express the Dorfamn derivative entirely in terms of generalized vectors

$$\mathbb{L}_W V^M = V^N \partial_N W^M + \left(\partial^M V^N - \partial^N V^M\right) W_N \tag{3.23}$$

The indices are raised and lowered using the metric in (3.2). It is now straight forward to define this action on a generalized tensor of arbitrary rank. So for $P \in \Gamma E^{\otimes n}$ we have

$$\begin{aligned} \mathbb{L}_W P^{M_1\ldots M_n} = \;& V^N \partial_N P^{M_1\ldots M_n} + \left(\partial^{M_1} V^N - \partial^N V^{M_1}\right) P^{M_2\ldots M_n} + \\ & \ldots + \left(\partial^{M_n} V^N - \partial^N V^{M_n}\right) P^{M_1\ldots M_{n-1}} \end{aligned} \tag{3.24}$$

3.3 Generalized Metric

An important point needs to made regarding $E = T\mathcal{M} \oplus T^*\mathcal{M}$ which is in fact, an isomorphism rather than a definition. There is no canonical splitting of E and an appropriate isomorphism must be chosen if one wishes to describe the generalized tangent bundle in terms of vectors and one forms. This is an ambiguity that can exploited. One instance of this, which will be of interest to us here, is the existence of a principal sub-bundle of K which captures both the conventional metric g and B-field. In the context of supergravity, this brings closer the unification of the gauge and diffeomorphism invariance of the NSNS sector by defining an object that *geometrizes* both fields (though strictly speaking, the dilaton should also be included). However, we are interested in this structure for the splitting it defines which will be use later. Explicitly, we define an $O(d) \times O(d)$ principle bundle J, such that $J \subset K$. This gives us a sub-bundle $C_+ \subset E$ corresponding to one of the $O(d)$ factors where the inner product $\langle \cdot, \cdot \rangle$ is positive definite and with $dim(C_+) = dim(\mathcal{M})$. The second group then gives $C_- \subset E$ as the orthogonal complement $C_- = C_+^\perp$ with $\langle \cdot, \cdot \rangle$ negative definite, defining the splitting

$$E = C_+ \oplus C_- \tag{3.25}$$

Noting that $E^* \simeq (T\mathcal{M} \oplus T^*\mathcal{M})^* \simeq (T^*\mathcal{M} \oplus T\mathcal{M}) \simeq E$, we now define the operation $G : E = C_+ \oplus C_- \to E^* \simeq E$ where G is the generalized metric which has the form

$$G = \langle \cdot, \cdot \rangle|_{C_+} + \langle \cdot, \cdot \rangle|_{C_-} \tag{3.26}$$

To determine an explicit expression for G we can define a projection from E onto $C_+ \oplus C_-$ by

$$V^{\pm M} = \frac{1}{2}(\delta^M{}_N \pm P^M{}_N) V^N \tag{3.27}$$

where $P^M{}_S P^S{}_N = \delta^M{}_N$ and $P^M{}_S P^N{}_T \eta_{MN} = \eta_{ST}$. The operator P has a general solution of the form [12]

$$P^M{}_S = \begin{pmatrix} P^\mu{}_\nu & P_{\mu\nu} \\ P^{\mu\nu} & P_\mu{}^\nu \end{pmatrix} = \begin{pmatrix} g^{-1}B & g - Bg^{-1}B \\ g^{-1} & -Bg^{-1} \end{pmatrix} \tag{3.28}$$

Where the generalized metric may then be obtained by lowering an index and is given by

$$G_{MN} = \eta_{MS} P^S{}_N = \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{3.29}$$

where $G_{MN} = G_{NM}$ since B is antisymmetric. It will be useful to note that after the splitting $E = C_+ \oplus C_-$ it is possible to define an $O(d) \times O(d)$ invariant frame over the two sub-bundles $\{E_a^+\} \cup \{E_{\overline{a}}^-\}$. We do so defining $\{E_a^+\}$ and $\{E_{\overline{a}}^-\}$ to be orthonormal bases on C_+ and C_- respectively, with $a = 1...d$ and $\overline{a} = 1...d$. Hence they satisfy

$$\langle E_a^+, E_b^+ \rangle = \eta_{ab} \quad \langle E_{\overline{a}}^-, E_{\overline{b}}^- \rangle = -\eta_{ab} \quad \langle E_{\overline{a}}^-, E_a^+ \rangle = 0 \tag{3.30}$$

where η is the flat metric. Further taking the orthonormal frames $\{\hat{e}_a^+\}$ and $\{\hat{e}_{\overline{a}}^-\}$ of the metric g this metric may then be expressed as

$$\eta_{ab} = g(\hat{e}_a^+, \hat{e}_b^+) \quad \eta_{\overline{a}\overline{b}} = g(\hat{e}_{\overline{a}}^-, \hat{e}_{\overline{b}}^-) \tag{3.31}$$

Thus, for this choice of projection, the generalized basis, taking patching of the one forms into account, may be expressed as

$$E_a^+ = \hat{e}_a^+ + e_a^+ - i_{\hat{e}_a^+} B \quad E_{\overline{a}}^- = \hat{e}_{\overline{a}}^- - e_{\overline{a}}^- - i_{\hat{e}_{\overline{a}}^-} B \tag{3.32}$$

3.4 Generalized Connection and Torsion

Now we have a metric, it makes sense to examine some other structures known from Riemannian geometry, namely the connection and torsion, both of which have generalized counterparts. Of interest will be connections compatible with the $O(d,d)$ structure of the generalized tangent space.

Definition. *For a vector bundle W the generalized connection D is a linear operator $D : C^\infty(W) \to C^\infty(E^* \otimes W)$ which, for a generalized vector $X = X^A E_A$, may be expressed as*

$$DX = \left(dX^A + \Omega^A{}_B X^B\right) \otimes E_A \tag{3.33}$$

To understand the connection coefficients, it is helpful to to render this as

$$D_M X^A = \partial_M X^A + \Omega_M{}^A{}_B X^B \tag{3.34}$$

where $\Omega_M{}^{AB} = -\Omega_M{}^{BA}$. We note that $\Omega \in \Gamma E^* \otimes adj(K)$, where M is the general tangent space index and with the A and B indecis giving its value in the adjoint bundle representation of K. Now it remains to define the generalized torsion of D which may be done analogously to the classical torsion. Recall that conventionally the torsion $T \in \Gamma T\mathcal{M} \otimes \Lambda^2 T^*\mathcal{M}$ of a p-form $\omega \in \Gamma \Lambda^p T^*\mathcal{M}$, with respect to a metric g, is given by $(i_v T)\omega = \left(\mathcal{L}_v^\nabla - \mathcal{L}_v\right)\omega$ where ∇ is the Levi-Civita connection. Using the Dorfman derivative as the generalized counterpart to the Lie derivative, we find an analogous expression for the generalized torsion.

Definition. *For $V, W \in \Gamma E$ the generalized torsion is defined as the linear map $T : \Gamma E \to \Gamma\Lambda^2 E$ and may be expressed as*

$$T(W)V^P = (\mathbb{L}^D_W - \mathbb{L}_W)V^P \tag{3.35}$$

Here we have $\mathbb{L}^D_V$ which is simply the Dorfman derivative in terms of the generalized connection rather than ∂_M such that

$$\mathbb{L}^D_W V^M = V^N D_N W^M + \left(D^M V^N - D^N V^M\right) W_N \tag{3.36}$$

It may be noted [12] that $T \in \Gamma\Lambda^3 E$. The generalized torsion tensor T^{ABC} is thus antisymmetric on all three indices. Its components may be expressed in terms of the connection coefficients

$$T^{ABC} = \Omega^{[ABC]} \tag{3.37}$$

where the indices have been raised using η^{AB}

3.5 Lie Algebroids and Jacobi

Lastly we turn to a further property of the Courant bracket to further exemplify its relation to the Lie bracket. However, first we must introduce a new concept, the purpose of which will become clear later on.

Definition. *A Lie algebroid is the triple $(X, [\cdot,\cdot], \rho)$ where X is a vector bundle over $\mathcal{M}$, $[\cdot,\cdot]$ is the Lie bracket on $\Gamma T\mathcal{M}$ and the bundle map $\rho : X \to T\mathcal{M}$ which is called the anchor. For $x, y \in \Gamma X$ and $f \in C^\infty(\mathcal{M})$ we then have*

$$\rho([x,y]) = [\rho(x), \rho(y)]$$
$$[x, fy] = f[x,y] + \rho(x)f \cdot y$$

The simplest example of this is the tangent bundle. It is a Lie algebroid with the anchor given by the identity element and as such is the triple $(T\mathcal{M}, [\cdot,\cdot], id)$. However, in general the Courant bracket does not define a Lie algebroid as it fails to satisfy the Jacobi identity. That is, we cannot take $(T\mathcal{M} \otimes T^*\mathcal{M}, [\![\cdot,\cdot]\!], \pi)$ for the projection $\pi : T\mathcal{M} \otimes T^*\mathcal{M} \to T\mathcal{M}$. This is be captured by the *Jacobiator.*

Definition. *The Jacobiator measures the Courant bracket's failure to satisfy the Jacobi identity. For $V, W, Z \in E$ it is defined as*

$$Jac(V,W,Z) = [\![[\![V,W]\!],Z]\!] + [\![[\![W,Z]\!],V]\!] + [\![[\![Z,V]\!],W]\!] \tag{3.38}$$

A useful identity can be obtained from identifying this with the *Nijenhuis operator* [10]

Definition. *The Jacobiator may be defined in terms of the Nijenhuis operator $Nij(\cdot,\cdot,\cdot)$, as*

$$Jac(V,W,Z) = d(Nij(V,W,Z)) \tag{3.39}$$

which may in term, be expressed in terms of the Courant bracket

$$Nij(V,W,Z) = \frac{1}{3}\left(\langle[\![V,W]\!],Z\rangle + \langle[\![W,Z]\!],V\rangle + \langle[\![Z,V]\!],W\rangle\right) \tag{3.40}$$

4 Parallelizability

Now we have introduced the concept of a generalized tangent space, explored some of its properties and constructed analogous operations to the ones familiar from classical geometry, we move on to considering the implication of having such a space. The analogy we will be particularly interested in is parallelizability. This is the concept of having global frame over the entire manifold, meaning the transition functions between coordinate patches become trivial. Well known in differential geometry, their study and classification is is central to the subject of flux compatification which seeks to reduce the higher dimensional string spacetimes onto spaces of the form $\mathbb{R}^{1,3} \times G$ where G is some group manifold. Group manifolds are of interest because of the existence of a preferred frame which satisfies the property of being globally defined. The question we ask, is whether generalized geometry in any way allows for a generalized counterpart of this property. Consider the fact that such a frame will be nowhere vanishing and that we wish to find some generalized frame to satisfy the same property. Thinking of a generalized basis vector

$$E_A = \begin{pmatrix} \hat{e}_a \\ e^a \end{pmatrix} \tag{4.1}$$

where $\hat{e}_a$ and e_a are some d dimensional basis and its dual, one could imgine having a manifold where the vector components of E vanished while the one form components remained non zero. If this conspired to happen so that E would never vanish, one would have a globally defined generalized basis, thus implying that a manifold that is not conventionally parallelizable, may be generalized parallelizable, laying nigh the suspicion that the latter represent a more general class of manifolds. It is this consideration that will be of interest to us in this section

4.1 Parallelizable Manifolds and Lie Groups

Before embarking on the following discussion is will be useful to recall some basic properties of group manifolds. So we start out with the basic definition

Definition. *Let G be a lie group and X a vector field on G. For $g, h \in G$ we have the left action $L_g : G \to G$ s.t. $L_g h = gh$ which induces a diffeomorphism on $\mathcal{M}$ given by $L_{g*} \in Diff(\mathcal{M})$. Hence we define a left invariant vector field which satisfies*

$$X|_{gh} = L_{g*} X|_h \tag{4.2}$$

Here we see the link between Lie groups and geometry. Since we can use the left action to map the identity to every element of the group G, we may also use the action defined above to map $V|_e \in T_e\mathcal{M}$ to $V|_p \in T_p\mathcal{M}$ for any $p \in \mathcal{M}$. This means, we can find a nowhere vanishing vector field on a manifold provided it has vector fields which are invariant under a certain group leading to the concept of parallelizability. It should be noted that may also make an equivalent definition using right multiplication $R_g h = hg$ for $g, h \in G$.

Definition. *A manifold $\mathcal{M}$ is parallelizable iff the tangent bundle $T\mathcal{M}$ is trivial*

So, if the structure group is trivial, this means that if, on a d dimensional manifold, we can define an everywhere linearly independent vector field $\{X_n\}$ where $n = 1, ..., d$, it is said to be parallelizable. The vector field then forms a global frame, which we denote $\{\hat{e}_a\}$, and its dual $\{e^a\}$. These then give us the familiar relationship

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \tag{4.3}$$

along with the Maurer-Cartan equation of the dual frame

$$e^a = f_{bc}{}^a e^b \wedge e^c \tag{4.4}$$

where $f_{ab}{}^c$ are the usual Lie algebra structure constants. Thus we see that parallelizable manifolds are group manifolds. Such manifolds will admit a metric of the form

$$g = \eta_{\mu\nu} e^\mu \otimes e^\nu \tag{4.5}$$

where $\eta_{\mu\nu}$ is constant.

4.2 Generalized Parallelizable Manifolds

In the context of the previous section, we may well ask the question of whether there is any analogous structure for generalized spaces, that is, do there exist generalized parallelizable manifolds? The answer is yes and may be realized by defining an analogous expression to (4.3). We let $E_A \in \Gamma F(E)$ be a global section of the frame bundle on E which allows us to write down a similar expression using the Courant bracket

$$[\![E_A, E_B]\!] = F_{AB}{}^C E_C \tag{4.6}$$

where we assume $F_{AB}{}^C$ to be constant and take the role of structure constant. This frame will have an $O(d, d)$ metric as dfined earlier

$$\eta = \eta^{AB} E_A \otimes E_B \tag{4.7}$$

such that we have $\langle E_A, E_A \rangle = \eta_{AB}$. It is worth recalling here, that the basis E_A, and by extension the structure constant, is only defined up to $O(d, d)$ transformations and that apparently different algebras may be equivalent if the corresponding frames are equivalent. It will be useful to note that for a given frame, one may always define a generalized connection D such that $DE_A = 0$ for all values of A. This connection will generally not be torsion free, but may be expressed in terms of the generalized torsion free connection $\tilde{D}$. Hence, for some generalized vector V, we have

$$D_M V^A = \tilde{D}_M V^A + \Sigma_M{}^A{}_B V^B \tag{4.8}$$

with the new connection coefficients $\Sigma_M{}^A{}_B \in \Gamma E^* \otimes adj(K)$ similar to those in (3.34) and is constructed so as to be compatible with our frame. We use this to understand an important property of the structure constant itself. Consider an expression similar to (4.6) involving the Dorfman derivative

$$\mathbb{L}_{E_A} E_B = \hat{F}_{AB}{}^C E_C \tag{4.9}$$

As demonstrated earlier, the Courant bracket is the anti symmetrization of the Dorfman derivative giving

$$F_{AB}{}^C = \frac{1}{2}\left(\hat{F}_{AB}{}^C - \hat{F}_{BA}{}^C\right) \tag{4.10}$$

Hence ${F_{AB}}^C$ is antisymmetric in its lower indices. This, however is not all and we make a further proposition.

Proposition. *The structure constant F_{ABC} is totally antisymmetric*

Proof. Given a basis $\{E_A\}$ we define a connection D which is compatible s.t. $DE_A = 0$. Now, with the definition of the generalized torsion tensor in (3.35), we have

$$\begin{aligned} {T_A}^C{}_B E_C &= \mathbb{L}^D_{E_A} E_B - \mathbb{L}_{E_A} E_B \\ &= -\mathbb{L}_{E_A} E_B \end{aligned}$$

Recall $T^{ABC} = \Omega^{[ABC]}$. Thus, lowering indices appropriately, we get

$$T_{ABC} = -\hat{F}_{ABC}$$

showing the structure constants $\hat{F}$ and by extension F to be totally antisymmetric. □

We now take the frame $E_A \in \Gamma F(E)$ to satisfy (4.6).

Proposition. *For $E_A \in \Gamma F(E)$ and constant ${F_{AB}}^C$, we have $Jac(E_A, E_B, E_C) = 0$ and hence the algebra (4.6) is a Lie algebra $\mathfrak{h}$*

Proof. Recall that the Jacobiator may be given in terms of the Nijenhuis operator (3.39). For the latter we get

$$\begin{aligned} Nij(E_A, E_B, E_C) &= \tfrac{1}{3}\left(\langle [\![E_A, E_B]\!], E_C\rangle + \langle [\![E_B, E_C]\!], E_A\rangle + \langle [\![E_C, E_A]\!], E_B\rangle\right) \\ &= \tfrac{1}{3}\left({F_{AB}}^D\langle E_D, E_C\rangle + {F_{BC}}^D\langle E_D, E_A\rangle + {F_{CA}}^D\langle E_D, E_C\rangle\right) \\ &= \tfrac{1}{3}\left({F_{AB}}^D\eta_{DC} + {F_{BC}}^D\eta_{DA} + {F_{CA}}^D\eta_{DC}\right) \end{aligned} \tag{4.11}$$

As $Jac(E_A, E_B, E_C) = dNij(E_A, E_B, E_C)$ and the above expression is constant, the Jacobiator vanishes and (4.6) defines the Lie algebra $\mathfrak{h}$. □

Proposition. *The Lie algebra $\mathfrak{h}$ is a subgroup of $\mathfrak{o}(d, d)$ s.t. $\mathfrak{h} \subset \mathfrak{o}(d, d)$*

Proof. From [10] we get that the we may defined a derivative of the inner product in terms of the Courant bracket as

$$\pi(V)\langle W, Z\rangle = \langle [\![V, W]\!] + d\langle V, W\rangle, Z\rangle + \langle W, [\![V, Z]\!] + d\langle V, Z\rangle\rangle \tag{4.12}$$

For $W, V, Z \in \Gamma E$. Substituting the basis vectors E_A we get

$$\begin{aligned} \pi(E_A)\langle E_B, E_C\rangle &= \langle [\![E_A, E_B]\!] + d\langle E_A, E_B\rangle, E_C\rangle + \langle E_B, [\![E_A, E_C]\!] + d\langle E_A, E_C\rangle\rangle \\ &= {F_{AB}}^D\langle E_D, E_C\rangle + {F_{AC}}^D\langle E_B, E_D\rangle \\ &= {F_{AB}}^D\eta_{DC} + {F_{AC}}^D\eta_{BD} \end{aligned} \tag{4.13}$$

Notice also that the left hand side is equal to 0 given that inner product $\langle E_B, E_C\rangle$ is constant. Hence we get

$${F_{AB}}^D\eta_{DC} + {F_{AC}}^D\eta_{BD} = 0 \tag{4.14}$$

Hence the algebra $\mathfrak{h}$ defined by E_A closes. Given that $F_{AB}{}^C$ transforms in the adjoint representation of the generalized structure bundle K, owing to its definition in terms of the torsion, we have $\mathfrak{h} \subset \mathfrak{o}(d,d)$ □

We may also relate the Courant to the Lie Bracket

Definition. *We define the projection $\pi : E \to T\mathcal{M}$, such that the Courant bracket of $V, W \in E$ reduces to the Lie Bracket*

$$\pi([\![V,W]\!]) = [\pi(V), \pi(W)] \tag{4.15}$$

Though, in general this does not define a Lie algebroid unless the above propositions are satisfied. Now if we consider the projection of the generalized basis vectors, we get a set of $2d$ vectors on $T\mathcal{M}$ i.e. $\pi(E_A) = x_A$ though there may be some degeneracy. To form a basis, x_A must have at least d non vanishing components. As $\pi(\cdot)$ conserves the structure of the Lie algebra, we also have the algebra $\mathfrak{h}$ now defined over $T\mathcal{M}$ by the vector field x_A. Exponentiatiating gives us the group H on $\mathcal{M}$. However, we may expect that some vectors vanish under projection, defining a closed subalgebra of $\mathfrak{h}$. We take P to be the corresponding group and fix a point $p \in \mathcal{M}$. The algebra is given by elements of E_A with vanishing projection at a point, that is

$$\mathfrak{p} = \{E_A \in \mathfrak{h} : \pi(E_A)|_p = 0\} \tag{4.16}$$

We further note that as the Lie bracket of two vanishing vector fields also vanishes, confirming that $\mathfrak{p}$ must indeed be a sub-algebra of $\mathfrak{h}$. Thus we see that

$$\mathcal{M} = H/P \tag{4.17}$$

Where $dim(H/P) = dim(\mathcal{M}) = d$, $dim(H) = d + k$ and $dim(P) = k$ where k is some positive integer. Compare this to the earlier example of paralellizable manifolds which corresponded to groups. Now we see that for a manifold to be generalized parralellizable, the condition is relaxed so as to include cosets. Thus it appears, as though generalized paralellizability is more general than paralellizability. In the rest of this section, we show two examples of generalized paralellizable manifolds

4.3 Three Sphere

In this section, we consider the example of generalized bases on the three sphere. The easiest way of doing so, is to first consider the conventional basis of S^3 by recalling that it is the group manifold of $SU(2)$. Hence, utilizing (4.2), we may define a left invariant basis and thus see that the manifold is parallelizable. Consider

$$g = \begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ -x^3 + ix^4 & x^1 - ix^2 \end{pmatrix} \quad \text{where } g \in SU(2) \tag{4.18}$$

where the radius has been chosen to be one and hence $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1$ with $\{x^1, x^2, x^3, x^4\} \in \mathbb{R}$. We may define a left invariant basis $\{\hat{e}_a^{(L)}\}$ where $\hat{e}^{(L)} \in \Gamma T\mathcal{M}$, by $\hat{e}^{(L)} = L_{g*}X|_e$ where e denotes the identity element. By performing the push-forward, one may construct this basis explicitly as

$$\begin{aligned}
\hat{e}_1^{(L)} &= -x^2\partial_{x^1} + x^1\partial_{x^2} + x^4\partial_{x^3} - x^3\partial_{x^4} \\
&= \cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi} + \sin(\phi)\frac{\partial}{\partial\theta} - \frac{\cos(\phi)}{\sin(\theta)}\frac{\partial}{\partial\psi} \\
\hat{e}_2^{(L)} &= -x^3\partial_{x^1} - x^4\partial_{x^2} + x^1\partial_{x^3} + x^2\partial_{x^4} \\
&= -\cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi} + \cos(\phi)\frac{\partial}{\partial\theta} + \frac{\sin(\phi)}{\sin(\theta)}\frac{\partial}{\partial\psi} \\
\hat{e}_3^{(L)} &= -x^4\partial_{x^1} + x^3\partial_{x^2} - x^2\partial_{x^3} + x^1\partial_{x^4} \\
&= \frac{\partial}{\partial\phi}
\end{aligned} \tag{4.19}$$

where θ, ψ and ϕ denote the Euler angles, which in terms of the rectangular coordinates are given as

$$\begin{aligned}
x^1 &= \sin\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\psi-\phi}{2}\right) \\
x^2 &= \cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\psi+\phi}{2}\right) \\
x^3 &= \cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\psi+\phi}{2}\right) \\
x^4 &= \sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\psi-\phi}{2}\right)
\end{aligned} \tag{4.20}$$

This frame satisfies the Lie algebra relation (4.3) with ${f_{12}}^3 = {f_{31}}^2 = {f_{23}}^1 = 1$. We may similarly compute the Maurer-Cartan one forms which are obtained by considering the pull-back $e = L_g^*\tilde{X}|_e$ and are given by

$$\begin{aligned}
e^{1(L)} &= \sin(\phi)d\theta - \cos(\phi)\sin(\theta)d\psi \\
e^{2(L)} &= \cos(\phi)d\theta + \sin(\phi)\sin(\theta)d\psi \\
e^{3(L)} &= d\phi + \cos(\theta)d\psi
\end{aligned} \tag{4.21}$$

which similarly, satisfy (4.4). These allow us to define a top form on S^3 which we identify with the H-flux as $H = ke^1 \wedge e^2 \wedge e^3 = k\sin(\theta)d\phi \wedge d\theta \wedge d\psi$. The B-field, defined as $H = dB$, then given by $B = k\cos(\theta)d\phi \wedge d\psi$ up to gauge transformations. Now that we have an $SU(2)$ basis over S^3, we may construct the generalized basis vectors. Following the expression (3.14) we may define

$$E_A^{(L)} = (\hat{e}_1^{(L)}, \hat{e}_2^{(L)}, \hat{e}_3^{(L)}, e^{1(L)} + i_{\hat{e}_1^{(L)}}B + e^{2(L)} + i_{\hat{e}_2^{(L)}}B, e^{3(L)} + i_{\hat{e}_3^{(L)}}B) \tag{4.22}$$

This is then split so as to separate the vector and one form parts so that we get the split frame

$$\begin{aligned}
E_a^{(L)} &= (\hat{e}_1^{(L)}, \hat{e}_2^{(L)}, \hat{e}_3^{(L)}) \\
E^{\overline{a}(L)} &= (e^{1(L)} - i_{\hat{e}_1^{(L)}}B, e^{2(L)} - i_{\hat{e}_2}^{(L)}B, e^{3(L)} - i_{\hat{e}_3}^{(L)})
\end{aligned} \tag{4.23}$$

with $a, \overline{a} \in \{1...3\}$. We compute the algebra of this frame as being

$$\begin{aligned}
&[\![E_a^{(L)}, E_b^{(L)}]\!] = {f_{ab}}^c E_c \\
&[\![E_a^{(L)}, E^{\overline{b}(L)}]\!] = -{f_{a\overline{c}}}^{\overline{b}} E^{\overline{c}} \\
&[\![E^{\overline{a}(L)}, E^{\overline{b}(L)}]\!] = 0
\end{aligned} \tag{4.24}$$

Where ${f_{12}}^3 = \frac{1}{2}$. It is interesting to note that the form of this algebra is reminiscent of one well known in physics. Recall that the Poincare algebra has generators corresponding to boosts J and translation P for which we have

commutations relationships of the form $[P,P] \sim 0$, $[P,J] \sim P$ and $[J,J] \sim J$ similar to those above.
Next, define a similar basis using right multiplication. The procedure for doing so is exactly analogous to that for left invariant vector fields. We denote the right invariant basis by $\{\hat{e}_a^{(R)}\}$ such that

$$\begin{aligned} \hat{e}_1^{(R)} &= -x^2\partial_{x^1} + x^1\partial_{x^2} - x^4\partial_{x^3} + x^3\partial_{x^4} \\ &= -\cot(\theta)\cos(\psi)\frac{\partial}{\partial\psi} - \sin(\psi)\frac{\partial}{\partial\theta} + \frac{\cos(\psi)}{\sin(\theta)}\frac{\partial}{\partial\phi} \\ \hat{e}_2^{(R)} &= -x^3\partial_{x^1} + x^4\partial_{x^2} + x^1\partial_{x^3} - x^2\partial_{x^4} \\ &= -\cot(\theta)\sin(\psi)\frac{\partial}{\partial\psi} + \cos(\psi)\frac{\partial}{\partial\theta} + \frac{\sin(\psi)}{\sin(\theta)}\frac{\partial}{\partial\phi} \\ \hat{e}_3^{(R)} &= -x^4\partial_{x^1} - x^3\partial_{x^2} + x^2\partial_{x^3} + x^1\partial_{x^4} \\ &= \frac{\partial}{\partial\psi} \end{aligned} \tag{4.25}$$

And also the Maurer-Cartan forms

$$\begin{aligned} e^{1(R)} &= -\sin(\psi)d\theta + \cos(\psi)\sin(\theta)d\phi \\ e^{2(R)} &= \cos(\psi)d\theta + \sin(\psi)\sin(\theta)d\phi \\ e^{3(R)} &= d\psi + \cos(\theta)d\phi \end{aligned} \tag{4.26}$$

The form of H doesn't change. We now define the generalized basis similar to before

$$\begin{aligned} E_a^{(R)} &= (\hat{e}_1^{(R)}, \hat{e}_2^{(R)}, \hat{e}_3^{(R)}) \\ E^{\overline{a}(R)} &= (e^{1(R)} - i_{\hat{e}_1^{(R)}}B, e^{2(R)} - i_{\hat{e}_2}^{(R)}B, e^{3(R)} - i_{\hat{e}_3}^{(R)}) \end{aligned} \tag{4.27}$$

Which now give the algebra

$$\begin{aligned} &[\![E_a^{(R)}, E_b^{(R)}]\!] = -f_{ab}{}^{c}E_c \\ &[\![E_a^{(R)}, E^{\overline{b}(R)}]\!] = f_{a\overline{c}}{}^{\overline{b}}E^{\overline{c}} \\ &[\![E^{\overline{a}(R)}, E^{\overline{b}(R)}]\!] = 0 \end{aligned} \tag{4.28}$$

As may be expected from the classical example of left and right invariant vector fields on S^3. Generalized geometry allows us to realized both algebras simultaneously on S^3 which, aside from being the group manifolds of $SU(2)$, is also the coset $\frac{SO(4)}{SO(3)}$. We then have the isomorphism $SO(4) \simeq SU(2) \times SU(2)$ which we may realized as both the left invariant and right invariant vector fields. In splitting the generalized baisis we will choose the $O(d) \times O(d)$ invariant, orthogonal subspace $C_+ \oplus C_-$ so that we split the frame into E_a^+ and $E_{\overline{a}}^-$ as in (3.30). Hence we get

$$\begin{aligned} E_a^- &= \hat{e}_a^{(L)} + e^{a(L)} - i_{\hat{e}_a^{(L)}}B \\ E_{\overline{a}}^+ &= \hat{e}_{\overline{a}}^{(R)} + e^{\overline{a}(R)} - i_{\hat{e}_{\overline{a}}^{(R)}}B \end{aligned} \tag{4.29}$$

It is worth noting that H has the same form in both frames. We now obtain the algebra

$$\begin{aligned} [\![E_a^-, E_b^-]\!] &= \hat{f}_{ab}{}^{c}E_c^+ \\ [\![E_a^+, E_{\overline{b}}^-]\!] &= 0 \\ [\![E_{\overline{a}}^+, E_{\overline{b}}^+]\!] &= -\hat{f}_{ab}{}^{c}E_c^- \end{aligned} \tag{4.30}$$

And we thus see that the two bases, while not commuting with themselves, commute with each other giving us an $SU(2) \times SU(2)$ structure on S^3. This provides a good example of how the isomorphism taking the generalized tangent bundle into some sub-bundle is not canonical and that by considering different splitting we may realized different structures.

4.4 Three Torus

In this next section, we explore another example of a generalized paralellizable manifold: the three torus. As in the previous section we will first examine the straight forward case of a simple T^3 with non zero H-flux and see how this leads to an algebra of the form (4.6). The next step will be to perform a T-duality transformation by application of the Buscher rules (2.17) and see whether the resulting manifold yields a similar structure.

Now, recall that a torus may be represented as a product manifold such that it can be represented as $T^3 = S^1 \times S^1 \times S^1$. This gives us a simple coordinate basis $\{\partial_1, \partial_2, \partial_3\}$ on the tangent space where we use the convention $\partial_a \equiv \frac{\partial}{\partial x^a}$. The dual basis is then $\{dx^1, dx^2, dx^3\}$. It is easy to see that we have the Lie algebra relation $[\partial_a, \partial_b] = 0$. One may already construct a generalized vector as

$$E_A = (\partial_1, \partial_2, \partial_3, dx^1, dx^2, dx^3) \tag{4.31}$$

Substituting this into the Courant bracket we get the algebra

$$\begin{aligned} [\![E_a, E_b]\!] &= 0 \\ [\![E_a, E^b]\!] &= 0 \\ [\![E^a, E^b]\!] &= 0 \end{aligned} \tag{4.32}$$

However, this is not very interesting and recall that in order to define a global section of E, we must take into account the B-transformations as in (3.14). To this end we consider the following: we identify H with the volume form on the 3-torus such that $H = \frac{k}{2\pi}\mathrm{Vol}(T^3) = \frac{k}{2\pi} dx^1 \wedge dx^2 \wedge dx^3$ for some $k \in \mathbb{R}$. We then choose $B = \frac{k}{4\pi} x^1 dx^2 \wedge dx^3$, though we are free to make any another choice. We let $x^1 \in [0, 2\pi]$ so the normalization has been chosen accordingly. This now allows us to define a global basis on E with patching given by the B-field.

$$E_A = (\partial_1, \partial_2 + \frac{kx^1}{2\pi} dx^3, \partial_3 - \frac{kx^1}{2\pi} dx^2, dx^1, dx^2, dx^3) \tag{4.33}$$

This gives us the split generalized basis with

$$\begin{aligned} E_a &= (\partial_1, \partial_2 + \tfrac{kx}{2\pi}^1 dx^3, \partial_3 - \tfrac{kx}{2\pi}^1 dx^2) \\ E^a &= (dx^1, dx^2, dx^3) \end{aligned} \tag{4.34}$$

The Counrant bracket then gives the algebra

$$\begin{aligned} [\![E_a, E_b]\!] &= -H_{abc} E^c \\ [\![E_a, E^b]\!] &= 0 \\ [\![E^a, E^b]\!] &= 0 \end{aligned} \tag{4.35}$$

Where $H_{123} = \frac{k}{2\pi}$ in addition to all other permutations. We see that, with the introduction of the B-field, the constant F_{ABC} has taken the form of the H flux. We now take this torus with non zero B-field and the transformation given by the Buscher rules in (2.17). In order to do so, we write down the metric of the torus and the B-field explicitly as

$$B_{12} = kx^1, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.36}$$

Applying the Buscher rules in the x^3 direction then gives

$$B_{\mu\nu} = 0, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + (\frac{kx^1}{2\pi})^2 & \frac{kx^1}{2\pi} \\ 0 & \frac{kx^1}{2\pi} & 1 \end{pmatrix} \tag{4.37}$$

This is a so called *twisted torus* owing to the dependence on the S^1 coordinate x^1, which causes the two torus to twist as one moves around the circle. Another way to think of it is a fibration of T^2 over the base, S^1. From this metric, we may construct again the generalized basis vectors which is most readily done considering the generalized metric in (3.29). In the absence of the B-field this reduces to the simple form

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \tag{4.38}$$

giving the basis as

$$E_A = (\partial_1, \partial_2 - \frac{kx^1}{2\pi}\partial_3, \partial_3, dx^1, dx^2, dx^3 - \frac{kx^1}{2\pi}dx^2) \tag{4.39}$$

which may be split

$$\begin{aligned} E_a &= (\partial_1, \partial_2 - \tfrac{kx^1}{2\pi}\partial_3, \partial_3) \\ E^a &= (dx^1, dx^2, dx^3 - \tfrac{kx^1}{2\pi}dx^2) \end{aligned} \tag{4.40}$$

From this, we may now compute its algebra as

$$\begin{aligned} &[\![E_a, E_b]\!] = f_{ab}{}^c E_c \\ &[\![E_a, E^b]\!] = -f_{ac}{}^b E^c \\ &[\![E^a, E^b]\!] = 0 \end{aligned} \tag{4.41}$$

So the first T-duality transformation has changed the flux H to f. The reason for choosing this relabeling will become clear in the next chapter. At first glance, this result bears no apparent correspondence with the previous results. However, recall that equivalent frames are related by $O(d, d)$ transformations. So we may also act by an element of $O \in O(3, 3)$ on (4.33). Explicitly, if we take

$$O \in \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.42}$$

(4.33) becomes

$$E_A = (\partial_1, \partial_2 - kx^1 dx^3, dx^3, dx^1, dx^2, \partial_3 + kx^1 dx^2) \tag{4.43}$$

Which also gives us the same algebra as in (4.41). Relabeling the above such that $dx^3 \leftrightarrow \partial_3$ gives us back the exact form of (4.39) and see that this is what the action of the T-duality essentially boils down to. Hence we must conclude that the algebras (4.41) and (4.35) are equivalent. This is valid, not only on a local level but also on a global as the transformation O is globally defined.

We have seen the first demonstration of the fact that the fluxes depend on the choice of frame, something that will be further explored in the following section. Before moving on, we notice two things in particular. Firstly, the choice to name the structure constant in (4.35) to the same as the H-flux is no accident. The two do indeed correspond and the structure constant F encodes information of the flux. Indeed, we may regard the constant f as a flux, though perhaps it is best imagined as measuring the twisting of the torus fibers (in (4.37) one torus coordinate is shifted as a function of the coordinate on the base leading to a twisting of the torus).

Secondly, we see that the flux f is related to H by a T-duality transformation, which is also encoded by the $O(d, d)$ transformations of the generalized space. In fact, the fluxes are equivalent and, on the level of physics, indistinguishable as they can be deformed into each other globally. Hence this fact is also captured by F which leads to question of whether we may see in it, the emergence of more structure. This question leads us on to the next chapter.

5 Non Geometry

Here we investigate the emergence of non geometry from T-duality and how this may be linked to generalized geometry. Conventionally, one would consider a manifold which possesses an atlas containing several coordinate charts with which one may describe the manifold. In the overlap between two such charts there is a transition function in order to switch from one to another and so obtain a global covering of the manifold. In our case, non geometry refers to the absence of the possibility to obtain such a description. How so becomes clear in the following pages. One may well ask how such backgrounds come to be. The answer lies in string theory. As discussed in a previous section, the action of T-duality relates theories in different backgrounds by mapping between high and low momentum modes into each other. This is not an inherently geometric transformation but rather a physical one, which also has geometric consequences. We will see the effect of repeated T-duality transformations and how this produces backgrounds which elude a classical geometric description.

These backgrounds are characterized by the appearance of certain non geometric fluxes, usually denoted Q and R, which are T-dual to the, what we will now term, geometric fluxes H and f already encountered. Within this section we will see how these fluxes emerge from the algebra of the generalized frame by following the work done on this topic in [13].

In this context, we explore another example of generalized paralellizable manifolds. This time however, we will go further and consider bases that do not conform to the standard expression (3.14). In doing so, we will see the emergence of additional structure constants, which are in fact the additional fluxes. These, as it will turn out, are what constitute the constant F in (4.6). We consider the simplest general case which differentiates itself from the previous examples in that it also contains a β-transformation as defined in (3.11).

5.1 β-transformation

Let $\{E_A\}$ be an orthonormal basis on E which is then split to give

$$E_A = \begin{cases} E_a \text{ for } A = a \\ E^a \text{ for } A = a + d \end{cases} \tag{5.1}$$

where $a \in [1, ..., d]$. We further let these span $C_+ \oplus C_-$ as defined in (3.25). We have chosen a slightly different convention to before so as to not clutter the notation. The frame is now restricted by choosing $e^+ = e^-$. What this does is essentially to fix the action both $O(d)$ groups to be equal and not to let them vary independently. This is in contrast to the S^3 case where both groups were able to act on both the left and right bases independently. Thus, the $O(d, d)$ and the $O(d) \times O(d)$ metrics now take the form

$$\eta = E^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E \quad G = E^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} E \tag{5.2}$$

Using the fact that we have fixed the $O(d)$ actions, E may be expressed as

$$E = \begin{pmatrix} e & 0 \\ -\hat{e}^T B & \hat{e}^T \end{pmatrix} \tag{5.3}$$

We will also consider the β-transformation as defined in (3.11). Having been absent from the discussion so far, one may well ask what is its effect. In answering this we not that it is possible to define a metric analogous to (3.29) by defining the β-transformed frame $\{\tilde{E}_A\}$. We then have

$$\eta = \tilde{E}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{E} \quad \tilde{G} = \tilde{E}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{E} \tag{5.4}$$

Where we may express $\tilde{E}$ by

$$\tilde{E} = \begin{pmatrix} e & e\beta \\ 0 & \hat{e}^T \end{pmatrix} \tag{5.5}$$

In this frame, we may rewrite the generalized metric (3.29) in terms of β as

$$\tilde{G}_{MN} = \begin{pmatrix} \tilde{g} & \tilde{g}\beta \\ \beta\tilde{g} & \tilde{g}^{-1} - \beta\tilde{g}\beta \end{pmatrix} \tag{5.6}$$

where we have relations by $\tilde{g} = (g+B)g^{-1}(g-B)$ and $\beta = -(g+B)^{-1}g(g-B)^{-1}$. The β-transformations are problematic in that they do not generally provide consistent patching conditions as demanded by (3.12) and hence usually do not result in globally defined frames. We will exploit this to provide an example of how non geometric fluxes emerge and demonstrate a link with T-duality. It is possible for a β-transformed frame to give a consistent description of the entire manifold, but only when there exists a globally defined transformation between the frame (5.5) and (5.3). This will be explored later on.

We now consider first the frame under the B-transform so that we have

$$E^a = e^a \quad E_a = \hat{e}_a - i_{\hat{e}_a} B \tag{5.7}$$

Proposition. *Given the above transformation, we have the identities*

$$\begin{aligned} &(1) \quad [\![E_a, E_b]\!] = f_{ab}{}^{c} E_c - H_{abc} E^c \\ &(2) \quad [\![E_a, E^b]\!] = -f_{ac}{}^{b} E^c \\ &(3) \quad [\![E^a, E^b]\!] = 0 \end{aligned} \tag{5.8}$$

where $f_{ab}{}^{c}$ are the structure constants as identified in (4.3), $H_{abc} = i_{\hat{e}_{[a}} dB_{bc]} + f_{[ab}{}^{d} B_{c]d}$ and $B = B_{ab} e^a \wedge e^b$

Proof. Using the expression for the Courant bracket as given in (3.16), the above may be shown as follows

$$\begin{aligned} (1) \quad [\![E_a, E_b]\!] &= [\hat{e}_a, \hat{e}_b] + \mathcal{L}_{\hat{e}_a} i_{\hat{e}_b} B - \mathcal{L}_{\hat{e}_b} i_{\hat{e}_a} B - \tfrac{1}{2} d\,(i_{\hat{e}_a} i_{\hat{e}_b} B - i_{\hat{e}_b} i_{\hat{e}_b} B) \\ &= f_{ab}{}^{c} \hat{e}_c + d i_{\hat{e}_a} i_{\hat{e}_b} B + i_{\hat{e}_a} d i_{\hat{e}_b} B - i_{\hat{e}_b} i_{\hat{e}_a} B - i_{\hat{e}_b} d i_{\hat{e}_a} B \\ &\quad + \tfrac{1}{2} d\,(i_{\hat{e}_a} i_{\hat{e}_b} B - i_{\hat{e}_b} i_{\hat{e}_b} B) \\ &= f_{ab}{}^{c} \hat{e}_c + \mathcal{L}_{\hat{e}_a} i_{\hat{e}_b} B - i_{\hat{e}_b} d i_{\hat{e}_a} B \\ &= f_{ab}{}^{c} \hat{e}_c + \mathcal{L}_{\hat{e}_a} (B_{bd} e^d) - i_{\hat{e}_b} d(B_{ad} e^d) \\ &= f_{ab}{}^{c} \hat{e}_c + i_{\hat{e}_a} dB_{bd} e^d + B_{bd} d i_{\hat{e}_a} e^d + B_{bd} i_{\hat{e}_a} de^d - i_{\hat{e}_b} dB_{ad} e^d \\ &\quad + dB_{ad} i_{\hat{e}_b} e^d \\ &= f_{ab}{}^{c} \hat{e}_c + (i_{\hat{e}_{[a}} dB_{bc]} + f_{[ab}{}^{d} B_{c]d}) e^c \\ &= f_{ab}{}^{c} \hat{E}_c + H_{abc} E^c \end{aligned}$$

$$(2) \quad [\![E_a, E^b]\!] = \mathcal{L}_{\hat{e}_a} e^b - \tfrac{1}{2} d i_{\hat{e}_a} e^b$$

$$
\begin{aligned}
&= i_{\hat{e}_a} de^b \\
&= -\tfrac{1}{2} i_{\hat{e}_a} \left(f^b{}_{cd} e^c \wedge e^d \right) \\
&= -f^b{}_{ac} e^c \\
&= -f^b{}_{ac} E^c
\end{aligned}
$$

(3) $[\![E^a, E^b]\!] = 0$ trivially □

We may also consider the β-transformation. While a part of the $O(d,d)$ structure group of the generalized tangent space, it is not however, an automorphism of the Courant bracket. To see the effect of the β-transform, we consider its action on the generalized basis vector such that (5.7) now becomes

$$E^a = e^a + i_{e^a}\beta \quad E_a = \hat{e}_a \tag{5.9}$$

Proposition. *Given the above transformation, we have the identities*

(1) $[\![E_a, E_b]\!] = f_{ab}{}^c E_c$

(2) $[\![E_a, E^b]\!] = -f_{ac}\ E^c + Q^{bc}{}_a E_c$ (5.10)

(3) $[\![E^a, E^b]\!] = Q^{ab}{}_c E^c + R^{abc} E_c$

where $Q^{ab}{}_c = \beta^{ad} i_{\hat{e}_d} d\beta^{bc} - \beta^{bd} i_{\hat{e}_d} d\beta^{ac} + \beta^{ad}\beta^{be} f_{de}{}^c$, $R^{abc} = i_{\hat{e}_c} d\beta^{ab} + \beta^{ad} f_{dc}{}^b + \beta^{bc} f_{ec}{}^a$ *and* $f_{ab}{}^c$ *is defined as above.*

Proof. Using the courant bracket similar to the previous proof, the following may be shown

(1) $[\![E_a, E_b]\!]$
$$
\begin{aligned}
&= [\hat{e}_a, \hat{e}_b] \\
&= f_{ab}{}^c \hat{e}_c \\
&= f_{ab}{}^c E_c
\end{aligned}
$$

(2) $[\![E_a, E^b]\!]$
$$
\begin{aligned}
&= \mathcal{L}_{\hat{e}_a} e^b + \mathcal{L}_{\hat{e}_a} i_{e^b}\beta \\
&= -f_{ac}{}^b e^c + \mathcal{L}_{\hat{e}_a} \beta^{bd} (\delta^b{}_c \hat{e}_d - \delta^b{}_d \hat{e}_c) \\
&= -f_{ac}{}^b e^c + i_{\hat{e}_a} d\beta^{bc} \hat{e}_c + \beta^{ac} [\hat{e}_a, \hat{e}_d] - \beta^{bc} [\hat{e}_a, \hat{e}_c] \\
&= -f_{ac}{}^b e^c + i_{\hat{e}_a} d\beta^{bc} \hat{e}_c + \beta^{ac} f_{ad}{}^c \hat{e}_c - \beta^{bc} f_{ac}{}^d \hat{e}_d \\
&= -f_{ac}\ E^c + Q^{bc}{}_a E_c
\end{aligned}
$$

(3) $[\![E^a, E^b]\!]$
$$
\begin{aligned}
&= [i_{\hat{e}_a}\beta, i_{\hat{e}_b}\beta] + \mathcal{L}_{i_{\hat{e}_a}\beta} e^b - \mathcal{L}_{i_{\hat{e}_b}\beta} e^a - \tfrac{1}{2} d \left(i_{i_{\hat{e}_a}\beta} e^b - i_{i_{\hat{e}_b}\beta} e^a \right) \\
&= \mathcal{L}_{i_{\hat{e}_a}\beta} i_{\hat{e}_a\beta} + i_{i_{\hat{e}_a}\beta} de^b - i_{i_{\hat{e}_a}\beta} de^a - \tfrac{1}{2} d i_{i_{\hat{e}_a}\beta} e^b - \tfrac{1}{2} d i_{i_{\hat{e}_b}\beta} e^a \\
&= \mathcal{L}_{\beta^{ad}\hat{e}_d} (\beta^{be}) \hat{e}_e + \beta^{be} \mathcal{L}_{\beta^{ad}\hat{e}_d} \hat{e}_e + \beta^{ad} i_{\hat{e}_d} de^b - \beta^{be} i_{\hat{e}_e} de^a + i_{\hat{e}_c} d\beta^{ab} \\
&= \left(\beta^{ad} i_{\hat{e}_d} d\beta^{bc} - \beta^{bd} i_{\hat{e}_d} d\beta^{ac} + \beta^{ad}\beta^{be} f_{de}{}^c \right) \hat{e}^c \\
&\quad + \left(i_{\hat{e}_c} d\beta^{ab} + \beta^{ad} f_{dc}{}^b + \beta^{bc} f_{ec}{}^a \right) e^c \\
&= Q^{ab}{}_c E^c + R^{abc} E_c
\end{aligned}
$$

□

Hence we see the emergence of the fluxes H, f, Q and R from the action of $O(d,d)$. Observe that the fluxes are dependent on the choice of frame E_A which are related to equivalent frames by an $O(d) \times O(d)$ rotation. Hence equivalent frames give different fluxes appearing as decomposition of the structure constant F and lead to the question if we can also, as in the case of f, get rid of Q and R just by changing the basis. The answer to this is yes, but in some cases only locally. As mentioned, it is always possible to make an $O(d) \times O(d)$ rotation of the form $E \to E' = TE$ such that the β-transformation is mapped to a B-transformation [13]. So the frame (5.5) may be transformed into (5.3). More explicitly, if we have O_- acting on C_- and O_+ acting on C_+, then the transformation $T \in O(d) \times O(d)$ is of the form

$$T = \begin{pmatrix} O_+ + O_- & O_+ - O_- \\ O_+ - O_- & O_+ + O_- \end{pmatrix} \tag{5.11}$$

To see the action of this explicitly, we split the base on $C_\pm$ so as to separate the base and the fibers with the corresponding frames $\{e_B\}$ and $\{e_F\}$ with the duals $\{\hat{e}_B\}$ and $\{\hat{e}_F\}$ recalling that T-duality, to which we wish to relate this, acts on the fibers. Thus the frame (5.5) becomes

$$\begin{pmatrix} e_B & & & \\ & e_F & & e_F\beta \\ & & \hat{e}_B^T & \\ & & & \hat{e}_F^T \end{pmatrix} \tag{5.12}$$

Which may be put [13] into the standard form (5.3) now expressed as

$$\begin{pmatrix} e_B & & & \\ & e_F & & \\ & & \hat{e}_B^T & \\ & -\hat{e}_F^T\beta & & \hat{e}_F^T \end{pmatrix} \tag{5.13}$$

if the transformation in (5.11) is given by

$$O_+ = \mathbb{I} \quad O_- = \begin{pmatrix} 1 & \\ & (\hat{e}_F^T + e_F\beta)(\hat{e}_F^T - e_F\beta)^{-1} \end{pmatrix} \tag{5.14}$$

However, in order for (5.12) to give a consistent basis, we require this transformation to be globally defined. That is, $O_\pm$ must be single valued. Otherwise, while one may locally deform Q into H, it is not possible globally.

In the next section, we will see an example of this and how the same structure is obtained by T-duality. It should be stressed that while we can see non geometric structure emerging form β-transformations, it is more of an accident and should not necessarily be seen as an inherent consequence of generalized geometry itself. The transformations are merely a method of producing different frames that can be related by the $O(d,d)$ or $O(d)\times O(d)$ symmetry groups. Non geometry is properly a consequence of T-duality and occurs here almost incidentally. The reason for introducing the concept, is that in the following example of the three torus, there is a link between the frame given by the β-transformation and the one remaining T-duality.

It is also worth noting that, while some spaces give rise to non geometric fluxes and that the spaces themselves elude a geometric description by not allowing consistent patching, this is only in a conventional geometric sense. It is in fact possible to perform patching, only it must be by T-duality transformations. So while they may be badly defined in traditional geometry, in string theory they do not present a problem. This is interesting as it seems to suggest that strings require an extension of traditional geometry, where we see manifolds as collections of points, since this doesn't seem to capture the structure necessary to describe backgrounds found in string theory. Rather, we need to include structures like T-duality which, after all, arise from a fundamental object that is not point like, but rather has one dimensional extent. It should also be noted that for this reason, the non geometric manifolds arising here cannot be described by supergravity, which treats strings as point like objects. Nonetheless we can utilize it to illustrate the fact that these backgrounds exist and that are non geometric.

5.2 Non Geometry: Torus

We now exemplify the previously laid out arguments by returning to the three torus. Having already examined the case of a torus with H-flux and the corresponding T-dual twisted torus, one may continue this analysis and look at the link between T-duality and non gemeorty.
Firstly, we consider a β-transformation of (4.31). In analogy to the B- transformation, we choose $\beta = \frac{k}{4\pi}x^1\partial_2 \wedge \partial_3$, giving

$$\{\partial_1, \partial_2, \partial_3, dx^1, dx^2 + \frac{kx^1}{2\pi}\partial_3, dx^3 - \frac{kx^1}{2\pi}\partial_2\} \tag{5.15}$$

In this basis we have the algebra

$$\begin{aligned} [\![E_a, E_b]\!] &= 0 \\ [\![E_a, E^b]\!] &= -Q^{bc}{}_a E_c \\ [\![E^a, E^b]\!] &= Q^{ab}{}_c E^c \end{aligned} \tag{5.16}$$

Where $Q^{23}{}_1 = \frac{k}{2\pi}$. As expected we see the emergence of the non geometric charge Q. To see how this corresponds to the result obtained by T-duality consider the following. We have one remaining isometry in the x^2 direction and hence one further possibility of a T-duality transformation. Taking the metric (4.37) and applying the Buscher rules once again gives

$$B_{23} = \frac{kx^1}{1+(kx^1)^2}, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1+(kx^1)^2} & 0 \\ 0 & 0 & \frac{1}{1+(kx^1)^2} \end{pmatrix} \tag{5.17}$$

Which, after some manipulation, allows us to write down the generalized basis

$$\{\partial_1, \partial_2, \partial_3, dx^1, dx^2 + \frac{kx^1}{2\pi}\partial_3, dx^3 - \frac{kx^1}{2\pi}\partial_2\} \tag{5.18}$$

This corresponds exactly with (5.15) and gives the same algebra. So we see that the second T-duality gives us the same basis as the β-transformation thus giving rise to the Q-flux. We are now left with the outstanding question of whether the two algebras (4.35) and (5.16) are related. Having linked T-duality to the $O(d,d)$ transformations inherent in generalized vectors, it has already been shown that the fluxes f and H are equivalent. We have already seen that there is a general $O(d) \times O(d)$ rotation which will take us from (5.18) to (4.33). Substituting appropriately, (5.12) yields

$$O_+ = \mathbb{I} \quad O_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta^{-2}(1-(\frac{kx^1}{2\pi})^2) & 2\Delta^{-2}\frac{kx^1}{2\pi} \\ 0 & 2\Delta^{-2}\frac{kx^1}{2\pi} & \Delta^{-2}(1-(\frac{kx^1}{2\pi})^2) \end{pmatrix} \tag{5.19}$$

where $\Delta = \sqrt{1+\frac{kx^1}{2\pi}}$. However, the coordinate x^1 is that of the base, S^1, and thus periodic. So we see that the transformation given above holds only locally as O_- is not single valued. Hence the flux Q may locally be "gauged" away but not globally. In examining the base (5.18), we also see that it does not conform to the definition of generalized vectors in (3.14) as it is not patched appropriately to be a global section of E.

While it is possible to switch from the β to a B-transformation, the absence of single valued rotation represents a global obstruction to this. We are thus lead to conclude that there is no consistent patching condition in the space arising from the second T-duality and that, while locally it may have the structure of a manifold, globally it does not.

This leaves us with the question of what happened to the R flux. We have seen that the definition of the B-field has left us two isometries along which we can use the Buscher rules to go to a T-dual configuration. However, there remains one more direction in which one may perform a further transformation, though the absence of an isometry may lead one to believe that there is not T-duality. While it is certainly not possible to use the Buscher rules, it has been argued [14] [15] that it should in principle be possible to perform a further T-duality leading to the, here absent, R-flux. Although it should be noted the the quantity itself is not well understood. While background associated with Q had at least a local description, that of R is thought to elude even this, being *non associative*. This, however is all the attention we shall give it here.

6 Conclusion

The past few pages have laid out and demonstrated the subject of generalized geometry. We observed its formulation, motivated by the symmetries of supergravity and T-duality and showed how it may be applied to better understand these. However, our discussion was limited to Type II supergravity in order construct generalized geometry in its simplest manifestation. Had we been concerned with a general case, we might have asked how these principles were to be applied to M-theory. Indeed, such studies have been made [2] [11]. Rather than having an $O(d,d)$ structure, one considers the exceptional symmetry group $E_{(d)d}$ and the study of the aptly named *exceptional generalized geometry.* Rather than the generalized tangent bundle splitting into $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$, one instead considers a space with a local isomorphism $E \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M} \oplus \wedge^5 T^*\mathcal{M} \oplus (T^*\mathcal{M} \otimes \wedge^7 T^*\mathcal{M})$. However, this theory is plagued by an significant shortcoming; it is incapable of describing 11 dimensional supergravity. Owing to the absence of understanding for exceptional groups with rank greater than E_8, M-theory is as elusive here as anywhere else. In fact, we do not even have such a theory based on E_8 and can only use this description for $d \leq 7$.

Had the comparison of the generalized objects been continued, we would also have encountered generalized curvature and the counterparts of the Riemann and Ricci tensors [7]. These would eventually result in a reformulation of the supergravity action (2.2) where the local diffeomorphism and gauge invariance had been replaced by an $O(d,d)$ invariant action unifying the entire NSNS sector into a single geometric object, the generalized Ricci scalar

$$S = \int |vol_G| \, \mathcal{R} \tag{6.1}$$

better concluding the motivation provided in the introductory section on supergravity.

In further exploring the analogy between traditional and generalized structures, we found generalized parallelizable manifolds to constitute a larger class than the conventional group manifolds. By the possession a preferred frame, they proved themselves have structure beyond the $O(d,d)$ structure group. The Courant bracket provided a method for computing the structure of these spaces. A consequence, is that they are of interest as compactification manifolds, providing a more general space on which to explore the consequences of dimensionally reduced theories. The examples chosen to illustrate this were special for already being group manifolds, though nonetheless served well to demonstrate the principles involved, including how to realize additional structure in the case of S^3. Here, we found the first example of frame dependence and how a change of basis could result in equivalent algebras.
An outstanding question in the study of generalized parralellizable manifolds, is that of classification. Unlike the internal spaces consisting of group manifolds, the cosets arising from this new method lack a consistent description. This is an outstanding topic of research and still in need of development.

Lastly, we saw a link between generalized geometry and T-duality. By considering the β-transformation, we saw that we could obtain the same frame as the T-duality transformation, leading into a curious subject area: non geometry. Characterized by the Q and R fluxes and eluding a traditional geometrical description, we saw its emergence from the symmetries inherent to string theory. As a subject, it is not well understood but has been studied in the context of β-transformations [16][17]. Generalized geometry, as a theory describing supergravity, will never be able to fully capture non geometry. It does however, provide a first and interesting insight into this comparatively new area of research. Ultimately the goal in studying it to understand string theory in a background independent way. But this is still a long way from being realized.

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