Imperial College London

Department of Theoretical Physics

Integrable Systems with $\mathrm{U}_q(\mathfrak{sl}_2)$ and $\mathfrak{su}(2|2)$ symmetry

Alejandro de la Rosa Gómez

Supervised by Prof. Arkady A. Tseytlin

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Contents

1 Introduction

Symmetries are extremely important in finding exact solutions for parameters in a classical theory. Noether proved in [1] that if a physical theory is invariant under a certain group of continuous transformations, there exists a constant in time which is said to generate the symmetry. If the number of these symmetries, and hence constants, is high enough, it is possible to find exact solutions to the equations of motion. This would correspond to inserting a high enough number of constraints on a system of linearly independent differential equations until the number of equations matches the number of variables plus the number of constraints. This is known as an integrable - or solvable - system.

The quantum mechanical analog of this method is often found in 1+1 dimensional quantum field theories and statistical mechanics systems. In this case, solvability of the system is ensured by the scattering matrix of the theory or S-matrix satisfying the Yang Baxter equation. This requirement leads to a method of solving for the variables of the system known as the Bethe ansatz.

As pointed out by Dorey in [4], it is then natural to reduce the problem of solving for the variables of a quantum field theory to finding such S-matrices. Furthermore, the Yang Baxter equation implies the S-matrix of any n body process in 1+1 dimensional quantum field theory can be factorised into 2 body processes, so finding the $2 \to 2$ process S-matrix would suffice. This is simply a function which satisfies unitarity, is invariant under the symmetry of the theory, and maps different representations of such symmetry - which are interpreted physically as multi-particle states.

A topic of recent interest in the study of integrable sytems is the famous AdS/CFT correspondence. This is a conjectured duality between gauge theory and string theory in a certain space. It was proposed and motivated in 1998 by Maldacena in [19] for one particular example: $\mathcal{N} = 4$ Super Yang Mills theory (SYM) in four dimensions and type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$, where AdS_5 is the hyperbolic space in 5 dimensions. This duality provides a major advance in the search for integrable systems which arise from symmetries in these two theories. $\mathcal{N} = 4$ SYM has a symmetry group containing a variety of algebras which, if integrable, could have a corresponding solvable system in their string theory dual in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Potentially, this could be helpful in finding the form of S-matrices in a theory of gravity.

The main goal of this dissertation is to analyse the integrability of a particular sector of the full symmetry group of $\mathcal{N} = 4$ SYM by connecting it to a much simpler theory known as the Hubbard model, which was shown to be an integrable system by Shastry in [10]. This sector is the subalgebra $\mathfrak{su}(2|2)$ of $\mathfrak{su}(2|3)$. Beisert showed integrable systems may live in these sectors by computing the exact form of their corresponding factorised S-matrices in [13] and [14]. This is known as the AdS/CFT or $\mathrm{AdS}_5 \times \mathrm{S}^5$ S-matrix, whose existence was shown by Staudacher in [16] to be fully consistent with string theory in several other subalgebras of the full symmetry group of $\mathcal{N} = 4$ SYM. The connection between the $\mathfrak{su}(2|2)$ S-matrix and the Hubbard model was studied by Mitev, Staudacher and Tsuboi in [9], which has motivated the possibility of using very simple integrable models in 1+1 dimensions to study the solvability of those in a higher number of dimensions. In this dissertation, we wish to expand this study by explaining in detail the techniques and concepts involved in [9], as well as bringing together significant relatively modern discoveries on the topic of integrability in theoretical physics.

To do so, we will first introduce in detail the basic concepts of integrability, both the classical and quantum version. These will include Liouville integrability, Lax pairs and the algebraic Bethe ansatz among others. The rest of the dissertation will be divided in two main parts:

- In the first one, we will explain a procedure to find a suitable S-matrix - which we will call R-matrix to avoid confusion - for a more general version of the Hubbard model. To achieve this, elements of free fermion theory and an object known as the Tetrahedrown Zamolodchikov algebra will be used. Then, the invariance of such R-matrix under two copies of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$ will be demonstrated. In addition, we will hint the possibility of a larger symmetry group, and write a representation of the algebra $\mathrm{su}(2|2)\ltimes\mathbb{R}^2$ in terms of the operators which compose the R-matrix.

- In the second one, we will briefly introduce the basic concepts of the AdS/CFT correspondence. Then we will proceed to analyse a particular the sector the full gauge theory symmetry which is invariant under $\mathfrak{su}(2|3)$ and $\mathfrak{su}(2|2)$ and show that the R-matrix for the Hubbard model is a valid candidate for an S-matrix in such sector, thus concluding that this part of $\mathcal{N} = 4$ Super Yang Mills - and its string theory dual - is an integrable system.

2 Classical Integrability

Before studying the conditions which determine whether a quantum theory is integrable, we shall look at basic concepts of integrability in classical physics. These should seem very familiar and intuitive as it is always easier to find solutions to a classical problem if the number of symmetries is big enough. These symmetries impose constraints on the equations of motion in the form of a parameter which does not evolve with time.

2.1 The Liouville Theorem

Among the different spaces studied in theoretical physics, perhaps the most important one is the *symplectic manifold*. A manifold $\mathcal{M}$ of dimension $2n$ is called symplectic if it is equipped with a non-degenerate, closed differential 2-form ω. If we assign coordinates (p_i, q_i) to each point in $\mathcal{M}$, the choice of ω becomes obvious:

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i \qquad i = 1, ...n \tag{1}$$

The variables p_i and q_i are interpreted in classical physics as momentum and position, respectively, and the manifold $\mathcal{M}$ is the *phase space*. The fact that ω is non-degenerate is crucial - it implies there is an unique vector field X such that

$$\omega(X) = dH \tag{2}$$

where H is a function on $\mathcal{M}$ called the *Hamiltonian* with the property that $\delta H = 0$ for any arbitraty variation δ. We shall call the vector field $X \equiv X_H$ associated with it a *Hamiltonian vector field*, which has the following form:

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \tag{3}$$

If we let p_i and q_i depend on a continous paramenter t - which is physically interpreted as time - the property $\delta H = 0$ leads to the familiar Hamilton's equations of motion:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{4}$$

The evolution of any function F in the phase space is determined by the Hamiltonian vector field. To see this, consider $X_H(F)$. Using Hamilton's equations of motion, we obtain

$$X_H(F) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} = \sum_{i=1}^{n} \dot{q}_i \frac{\partial F}{\partial q_i} + \dot{p}_i \frac{\partial F}{\partial q_i} = \frac{dF}{dt} \tag{5}$$

The first sum of (5) is known as the *Poisson bracket* of H and F, and is denoted by the symbol $\{H, F\}$. So, using the fact that ω is non-degenerate, we have been able to obtain the following remarkable result for any function on the phase space:

$$\frac{dF}{dt} = \{H, F\} \tag{6}$$

This establishes a relation between the time evolution of a function F on the phase space and the Hamiltonian. In the case where $\{H, F\} = 0$, the function does not evolve, and is hence a conserved quantity of the system.

Conserved quantities are the key to solving differential equations and, consequently, physical systems. Most classical systems, such as the Lagrange or Euler top, would be impossible to solve if we did not impose conservation of energy and angular momentum. In quantum field theory, scattering amplitudes would be impossible to compute if we did not constrain the 4-momentum to be conserved at each vertex of Feynman diagrams.

Given their importance, we shall focus our introduction to classical integrability on functions that Poisson-commute with the Hamiltonian. We say dynamical system represented by a phase space $\mathcal{M}$ of dimension $2n$ is *Liouville integrable* if there exists n functions F_i on $\mathcal{M}$ such that

$$\dot{F}_i = \{H, F_i\} = 0, \quad i = 1, 2, ..., n \tag{7}$$

and the F_i's are in involution:

$$\{F_i, F_j\} = 0, \quad i = 1, 2, ..., n \tag{8}$$

The Liouville theorem states that *Liouville integrable systems can be solved by first integrals.* This provides us with a way of determining whether we can find exact solutions to a Hamiltonian dynamical system and if possible, obtain these. The key to the proof of this theorem is the existence of a canonical transformation from coordinates p_i, q_i to F_i and a function ψ_i such that the nondegenerate closed 2-form on $\mathcal{M}$ can be written as

$$\omega = \sum_i dF_i \wedge d\psi_i \tag{9}$$

which, imposing Hamilton's equations of motion on these these two variables, leads to solutions for the canonical variables p_i and q_i.

As we can see, a function H which satisfies all of the above conditions can be written as a linear combination of all the quantities F_i in involution. Therefore the Hamiltonian of the system is a linear functional of the conserved quantities on the phase space.

2.2 Lax Pairs and the Monodromy Matrix

On the phase space, sometimes one can find matrices L and M such that Hamilton's equations of motion are encoded in the following relation

$$\dot{L} = [M, L] \tag{10}$$

where $[\ \ ,\ \ ]$ is the usual matrix commutator. These matrices L and M are then said to form a *Lax pair* of the system. For example, consider the one dimensional harmonic oscillator, with Hamiltonian

$$H(p, q) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{11}$$

where ω is the frequency of oscillations. Then for matrices

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix} \tag{12}$$

one can check that equation $\dot{L} = [M, L]$ leads to $(\dot{p}, \dot{q}) = (-\omega^2 q, p)$ which are the same equations we would obtain using (5).

The importance of Lax pairs lies in that the eigenvalues of the matrix L correspond to the conserved quantities of the system. This provides on a straightforward way of finding the form of these quantities $H^{(n)}$ by computing the following trace

$$H^{(n)} = \text{Tr } L^n \tag{13}$$

In the case of the harmonic oscillator, $H^{(2)} = \text{Tr}L^2 = 2(p^2 + \omega^2 q^2)$ which is four times the Hamiltonian in (11).

Unfortunately, a procedure to determine the existence of Lax pairs in a particular physical system does not exist. There is, however, a general method to construct Lax pairs which encode the equations of motion of integrable systems, called the *Zhabarov Shabat construction.* This method treats matrices M and L as functions of a complex variable λ, called the *spectral parameter*, and studies the equivalence of poles of both sides of (10) and the diagonalizability of L around those poles. The reason diagonizability is so important is that if we can find a way to diagonalise L and M into L_d and M_d, this will result in $\dot{L_d} = 0$ since diagonal matrices commute, and we would have found a matrix which encodes all conserved quantities of the system.

For the purpose of this disseration, we will not discuss this method any further. We will simply state the result in [2] that for every $L(\lambda)$ there exist an infinite number of $M(\lambda)$'s such that (10) holds. This implies there exists a continuous curve corresponding to these choices pf $M(\lambda)$. We shall incorporate an integer label i to denote the different M's

$$\partial_{t_i} L = [M_i, L], \quad i \in \mathbb{Z} \tag{14}$$

Consider now, M_i and M_j, such that the Lax equation holds for a specific L at times labeled t_i and t_j respectively. Since $[\partial_{t_i}, \partial_{t_j}] = 0$, one obtains the relation

$$[\partial_{t_i} M_j - \partial_{t_j} M_i, L] + [M_i, [M_j, L] + [M_j, [L, M_i] = 0 \tag{15}$$

Using the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ and $[A, B] = -[B, A]$, the relation becomes

$$[\partial_{t_i} M_j - \partial_{t_j} M_i - [M_i, M_j], L] = 0 \tag{16}$$

If this holds for infinitely many M's, it leads to the following result

$$\partial_{t_i} M_j - \partial_{t_j} M_i - [M_i, M_j] = 0 \tag{17}$$

which can be recognized as the *zero curvature condition* for any pair of matrices M_i and M_j. Hence we can think of the Lax equation for this pair of matrices as L being covariantly transported along the connection (M_{t_i}, M_{t_j}).

Now suppose the equations of motion of a 1+1 dimensional field theory are encoded by the following equation

$$[\partial_x - U, \partial_t - V] = 0 \tag{18}$$

where U, V are matrices which depend on parameters of the theory. This is equivalent to the zero curvature condition for matrices U and V. For any function $T(t, x)$ that is parallely transported along the curve defined by the connection (U, V), we have the conditions

$$(\partial_x - U)T(t, x) = 0, \quad (\partial_t - V)T(t, x) = 0 \tag{19}$$

Choosing a path λ from $(0, 0)$ to (t, x), the solution of $T(t, x)$ reads

$$T(t, x) = \mathcal{P}exp[\int_\lambda (U dx - V dt)] \tag{20}$$

where $\mathcal{P}$ represents the path dependence of the integral. This is known as the *monodromy matrix* and is the equivalent of the matrix L in classical field theory in 1+1 dimensions. We can choose a particular path, in this case with a circular parametrization $\lambda : [0, 2\pi] \to \mathcal{M}$, and fix time, to obtain

$$T(t) = \mathcal{P}exp[\int_0^{2\pi} U(t,x)dx] \tag{21}$$

Just as with L, the trace of the powers of $T(t)$ encodes de conserved quantities of the classical field theory, which correspond to the eigenvalues of $T(t)$. It also obeys a similar relation to (10), which becomes apparent after taking the time derivative

$$\begin{aligned} \partial_t T(t) &= \int_0^{2\pi} dx \mathcal{P}exp^{\int_x^{2\pi} U(t,y)dy} (\partial_t U) \mathcal{P}exp^{\int_0^x U(t,y)dy} \\ &= \int_0^{2\pi} dx \mathcal{P}exp^{\int_x^{2\pi} U(t,y)dy} (\partial_x V + [V, U]) \mathcal{P}exp^{\int_0^x U(t,y)dy} \\ &= \int_0^{2\pi} dx \partial_x (\mathcal{P}exp^{\int_x^{2\pi} U(t,y)dy} (V) \mathcal{P}exp^{\int_0^x U(t,y)dy}) \\ &= [V(t,x), T(t)] \end{aligned} \tag{22}$$

3 Quantum Integrability

Now that we have reviewed at what it means to be classically integrable, it is time to immerse ourselves in the topic of quantum field theory and study what it means for a system to be quantum integrable. First, we will present a property of the 1+1 dimensional quantum field theory scattering matrix. Then, we will discuss how this property leads to the solvability of the system by introducing a familty of integrable systems called spin chains.

3.1 Factorisability of the Scattering Matrix

We shall begin by looking at a particular result in [4]: the $2 \to 4$ scattering amplitude f of 1+1 dimensional ϕ^4 theory, computed via

$$f = \langle out| \, S \, |in\rangle = \langle S \rangle \tag{23}$$

where S is the S-matrix and $\langle out|$ and $|in\rangle$ are representations of the 1+1 dimensional Lorentz group SO(1,1). These representations are to be intepreted physically as ingoing and outgoing multiparticle states. To proceed, we may first calculate the $3 \to 3$ process and cross one of the out and in momenta in the end, and in doing so, we will use light-cone coordinates

$$p_i = (p^+, p^-) = (p^0 + p^1, p^0 - p^1) \tag{24}$$

Let the in particles be labeled by a, b, c and the out particles d, e, f such that the momenta of each is labeled by $p_a = (ma, ma^{-1})$ in order to satisfy the mass-shell condition $p^+p^- = m^2$. The two possible diagrams for the $3 \to 3$ process in ϕ^4 theory are the following:

Figure 1: ϕ^4 theory 3-body processes

Calculating the S-matrix for both diagrams, and imposing conservation of momentum in their addition, one obtains the simple result $S_{2\to 4} = -1$. This means that in 1+1 dimensional ϕ^4 theory, the $2 \to 4$ scattering amplitude is constant. In fact, by adding an suitable ϕ^6 term to the Lagrangian, we can set this amplitude to be exactly zero.

But then we find that, for this new Lagrangian, the $2 \to 6$ amplitude turns out to be constant as well, so we can add a ϕ^8 term to cancel such amplitude. We can repeat this technique for all powers of ϕ to come to the following conclusion:

- In several two dimensional quantum field theories with interacting lagrangians, the $2 \to n$ S-matrix vanishes unless $n = 2$. Therefore, there is no particle production.

Indeed, if one starts the above procedure with a ϕ^4 term, the resulting theory is precisely described by the sinh-Gordon Lagrangian. One can generalise this argument to all quantum field theories in 1+1 dimensions by studying conserved charges. To be consistent with special relativity, all quantum field theories must be manifestly Lorentz invariant. This symmetry is generated by a conserved charge Q_s , where the index s determines the representation of SO(1,1) it transforms under, called the *spin.* This charge acts on a multiparticle state $|a_1 a_2 ... a_n\rangle$ as

$$Q_s \, |a_1 a_2 ... \, a_N\rangle = (q_{a_1}^{(s)} + q_{a_2}^{(s)} + ... \, q_{a_n}^{(s)}) \, |a_1 a_2 ... a_N\rangle \tag{25}$$

Then for a $n \to m$, scattering amplitude, with ingoing multiparticle state $|a_1 a_2 ... a_n\rangle$ and outgoing multiparticle state $|b_1 b_2 ... b_m\rangle$, this implies

$$q_{a_1}^{(s)} + q_{a_2}^{(s)} + ... \, q_{a_n}^{(s)} = q_{b_1}^{(s)} + q_{b_2}^{(s)} + ... \, q_{b_m}^{(s)} \tag{26}$$

Which, if need to be satisfied for infinitely many spins s, requires that $n = m$. Therefore,

- In any 1+1 dimensional (scalar) quantum field theory process, there is no particle production.

Hence we can study the case of three incoming particles, which can only lead to three outgoing particles. The only three possible diagrams for the $3 \to 3$ process in any 1+1D quantum field theory are given in Figure 2.

Although at first sight these might seem like different processes, they all encode the same scattering amplitude. The first and last case are mirror images of each other, and the S-matrix of the middle case can be converted into one of the other two by having the operator $e^{i\theta P_s}$ act on the S-matrix and translate the second particle. This can be done without altering the form of the S-matrix

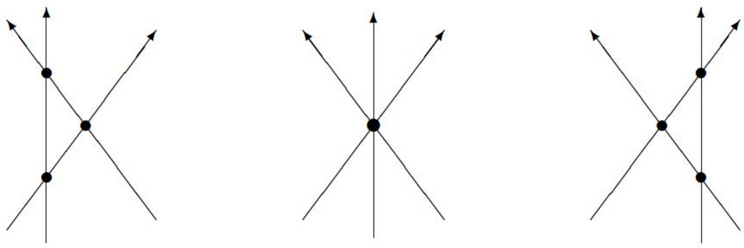

Figure 2: 1+1D QFT 3-body processes

using its invariance under SO(1,1):

$$\langle out| \, [S, e^{i\theta P_s}] \, |in\rangle = 0 \tag{27}$$

One also notices that since these diagrams are composed of 2-body collisions, they must factorise into $2 \to 2$ processes. The latter is a consequence of the number of dimensions this quantum field theory lives in. As shown in [5], if the particles all have different momenta they will cross in the one spatial dimension at different times. Due to SO(1,1) symmetry, one can think of the possibility of performing a boost on these particles so that every time there is a crossing, it happens between no more than two particles.

Factorizability of the $3 \to 3$ S-matrix can be done in two ways, represented by the first and second diagram. Denote the S-matrix of the interaction between 2 particles i and j by S_{ij}, and from left to right, label the particles by 1,2 and 3. Then

$$S_{123} = S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \tag{28}$$

This is known as the *Yang Baxter equation.* Just as for the 3-body process, a 1+1 dimensional quantum field theory is said to be *quantum integrable* if the S-matrix of any process factorises into 2-body S-matrices and, consequently, the 3-body S-matrix satisfies the Yang Baxter equation.

Furthermore, this equation defines an algebraic object called the Yang Baxter algebra. An element $\mathcal{R}$ in $A \otimes A$ belongs to the Yang Baxter algebra in $A \otimes A \otimes A$ if it satisfies the Yang Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{29}$$

where the lower indices indicate a tensor product with the identity matrix in a certain order. Explicitely, if $\mathcal{R} = a \otimes b$, then $\mathcal{R}_{12} = a \otimes b \otimes \mathbb{I}$, $\mathcal{R}_{23} = \mathbb{I} \otimes a \otimes b$ and $\mathcal{R}_{13} = a \otimes \mathbb{I} \otimes b$.

Quantum integrability, just as Liouville integrability, allows us to solve for the variables of the system. To explain how, it is convenient to focus on a family of systems called spin chains.

3.2 Spin Chains and The Algebraic Bethe Ansatz

A *spin chain* is a model of spin one-half particles distributed in a lattice which have spin-spin interaction. Although it is statistical mechanical, the spins of each particle are treated quantum mechanically, and each lattice site can be thought as a magnetic dipole.

The most famous of these models in the Heisenberg model, also known as the XXX spin chain. This model possesses the property that the coupling constant of interaction between two neighboring sites is the same in each direction, and it is therefore rotationally invariant. The Hamiltonian for such model is

$$H^{XXX} = J\sum_{j=1}^{L}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z) \tag{30}$$

where σ_j^α are spin operators pauli matrices at site j with periodic boundary conditions $\sigma_{L+1}^\alpha = \sigma_1^\alpha$ and L is the length of the spin chain. Pictorically,

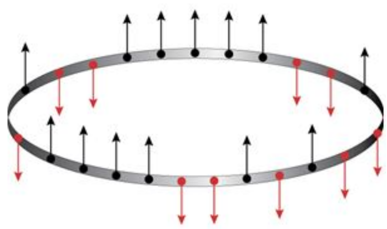

Figure 3: the XXX Spin Chain

Its basis of states is given by spins up or down. There exists an operator that exchanges the spins at sites i and j, called the permutation operator, of the form

$$P_{ij} = \frac{1}{2}(1 + \sigma_i^z\sigma_j^z + \sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+) \tag{31}$$

where $\sigma^\pm = (\sigma^x \pm \sigma^y)/2$. One can now check that the operator

$$R_{ij}(u,v) \equiv u - v + P_{ij} \tag{32}$$

where u and v are complex numbers, satisfies the Yang Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v) \tag{33}$$

which implies the Heisenberg model is quantum integrable. In fact, all spin chains are quantum integrable models, and therefore very useful in the study of integrable systems arising in theories invariant under classical lie algebras. R_{ij} is called the R-matrix, which is just another name for a scattering matrix.

The Heisenberg model is called the XXX-model because there is only one coupling constant - in other words, $J_x = J_y = J_z$. A simple variation of this model is the XXZ model, where $J_x = J_y \neq J_z$, or the XYZ model, where $J_x \neq J_y \neq J_z$. Similarly, R-matrices which satisfy the Yang Baxter equation can be found in every single one of these models.

As it will be useful in the beginning of the next section, one wishes to establish a connection between the quantum version of the monodromy matrix, the permutation operator and the R-matrix.

Let us return to the XXX spin chain, and imagine we face the task of obtaining the Hamiltonian density from the R-matrix alone. The first step is to define a curve in the complex plane such at the starting point, call it u_0, the R-matrix is just the permutation operator. The reasoning behind this is simple: the R-matrix encodes the possible processes to happen between n particles while taking into account the overall conservation of one or many quantities of the system. This set of processes defines a path in the complex plane. Since permuting the value of the conserved charge for each particle satisfies the conservation law, this should be one of the possible actions of the R-matrix on a multiparticle state. Note that, in the case of the XXX spin chain, this first step was already accomplished by simply choosing $R_{ij}(u,v) \equiv u - v + P_{ij}$, where $u_0 - v_0 = 0$.

The next step consists in rearranging (37). After some algebraic manipulation, we obtain the following equivalence

$$R_{12}(u-v)(L_{ij}(u) \otimes \mathbb{I})(\mathbb{I} \otimes L_{ij}(v)) = (\mathbb{I} \otimes L_{ij}(v))(L_{ij}(u) \otimes \mathbb{I})R_{12}(u-v) \quad (34)$$

where $L_{ij}(u) = u - \frac{i}{2} + iP_{ij}$ is called the Lax operator, which is a representation of the Yang Baxter algebra with R-matrix R_{ij}. This operator acts on representations of the Hilbert space $\mathcal{H}_j$ which are labeled by a specific spin j. If we wanted, however, to define an operator which acted the same way as L_{ij} but for any representation of the Hilbert Space of all possible spins, as done in [6], we must define the monodromy matrix in quantum mechanical systems as

$$T_a(u) = L_{La}(u)L_{L-1a}(u)...L_{1a}(u) \quad (35)$$

which acts in multi-spin states in $\mathcal{H} = \bigotimes_j \mathcal{H}_j$. Hence now, the Yang Baxter equation for $\mathcal{H}$ is

$$R_{12}(u,v)(T_a(u) \otimes \mathbb{I})(\mathbb{I} \otimes T_b(v)) = (\mathbb{I} \otimes T_b(v))(T_a(u) \otimes \mathbb{I})R_{12}(u,v) \quad (36)$$

The operator $T_a(u)$ is called the monodromy matrix because it plays the same role as the field theory monodromy matrix: it generates conserved quantities, one of them being of course, the Hamiltonian. So see this explicitely, we shall multiply equation (41) by R^{-1} and take the trace of both sides. Since the trace is invariant under similatity transformations,

$$\text{Tr }(T_a(u)\otimes I)(I\otimes T_b(v) = \text{Tr }(T_b(v)\otimes I)(I\otimes T_a(u)) \tag{37}$$

which using the identity Tr $(X\otimes Y)=$ TrX TrY, leads to

$$\text{Tr}_a\ T_a(u)\ \text{Tr}_b\ T_b(v) = \text{Tr}_b\ T_b(v)\ \text{Tr}_a\ T_a(u) \tag{38}$$

where Tr_a is the trace taken over the auxiliary space. Since all $T_a(u)$'s commute for any u, we can expand $T_a(u)$ in a power series $T^{(p)}(u) = \sum_n a_n(u-u_0)^n$ whose coefficients are such that Tr $a_n = Q_n$ commmute with each other. In other words, the monodromy matrix is a generating function of conserved quantities in involution. Each of the Q_n can be obtained by

$$Q_n = \frac{1}{n!}\frac{d^n}{du^n}\text{Tr }T^{(p)}(u)\mid_{u=u_0} \tag{39}$$

As mentioned in section 2.5, the Hamiltonian can be written as a linear combination of all conserved quantities. Making use of the monodromy matrix, this can be accomplished by the following formula:

$$H = \frac{d}{du}\ln\text{Tr}_a\ T_a(u)\mid_{u=u_0} = (\text{Tr}_a\ T(u_0))^{-1}\frac{d}{du}\text{Tr}_a\ T_a(u)\mid_{u=u_0} \tag{40}$$

For the XXX spin chain, this equality is easy to check. Note that at $u=i/2$, the R-matrix and Lax operator become the permutation operator. This implies that $T_a(u_0)$ becomes the product of all permutation operators P_{na} where n goes from N to 1. It is then possible to rearrange $T_a(u_0)$ into

$$T_a(i/2) = i^L P_{12}P_{23}...\ P_{L-1,L}P_{La} \tag{41}$$

Since $\text{Tr}_a P_{La} = \mathbb{I}_L$, the trace of (41) is simply the same but multiplied by P_{aL} on the right. Hence

$$\begin{aligned}\frac{d}{du}\ln\text{Tr}_a\ T_a(u)\mid_{u=i/2} &= (P_{12}P_{23}...\ P_{L-1,L})^{-1}\sum_n P_{12}...\ P_{n-1,n+1}...\ P_{L-1,L} \\ &= \sum_{n=1}^{L} P_{n,n+1} = \frac{H^{XXX}}{J_x} - \frac{L}{2J_x}\end{aligned} \tag{42}$$

Now we proceed to demonstrate how quantum integrability leads to solvavility of the quantum system variables. Letting $f_j(u,v) = \delta_{j1} + i/(v-u)$ (as was used in [6] to derive the Yang Baxter equation) the operators R, L and T are written in matrix form:

$$R(u,v) = \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 1 & 0 \\ 0 & 1 & f_2 & 0 \\ 0 & 0 & 0 & f_1 \end{pmatrix}, \quad L_j(u) = \begin{pmatrix} u - \frac{i}{2}\sigma_j^z & -i\sigma_j^- \\ -i\sigma_j^+ & u + \frac{i}{2}\sigma_j^z \end{pmatrix}$$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{43}$$

The Yang Baxter equation implies the following relations between A, B, C, D and f_j:

$$\begin{aligned} A(u)B(v) &= f_1(u,v)B(v)A(u) - f_2(u,v)B(u)A(v) \\ D(u)B(v) &= f_1(v,u)B(v)D(u) - f_2(v,u)B(u)D(v) \\ B(u)B(v) &= B(v)B(u) \end{aligned} \tag{44}$$

Define now the vacuum state of an spin chain of length L to be the tensor product of L spin up states:

$$|0\rangle \equiv |\uparrow\rangle_L \otimes ... \otimes |\uparrow\rangle_1 \in \mathcal{H} \tag{45}$$

Since $T(u) = L_L(u)...L_2(u)L_1(u)$ acts on $\mathcal{H}$, each one of the Lax operators in the product acts on its corresponding element in the tensor product. For instance,

$$\begin{aligned} L_1(u)|\uparrow\rangle_1 &= \begin{pmatrix} (u - \frac{i}{2}\sigma_1^z)|\uparrow\rangle_1 & -i\sigma_1^-|\uparrow\rangle_1 \\ -i\sigma_1^+|\uparrow\rangle_1 & (u + \frac{i}{2}\sigma_1^z)|\uparrow\rangle_1 \end{pmatrix} \\ &= \begin{pmatrix} u - i/2 & b(u) \\ 0 & u + i/2 \end{pmatrix}|\uparrow\rangle_1 \end{aligned} \tag{46}$$

so acting with $T(u)$, we obtain the following result:

$$T(u)|0\rangle = \begin{pmatrix} (u - i/2)^L & B(u) \\ 0 & (u + i/2)^L \end{pmatrix}|0\rangle \tag{47}$$

We observe $C(u)$ satisfies $C(u)|0\rangle = 0$ for all complex numbers u, which is the same condition that the vacuum must satisfy for all annihilation operators in a quantum field theory. The *algebraic Bethe anstatz* is the assumption that these are actually annihilation operators and the $B(u)$ are creation operators, such that every state $|\vec{u}\rangle$ of the spin chain can be written as $|\vec{u}\rangle = B(u_1)...B(u_N)\,|0\rangle$, where $\vec{u} = (u_1, ..., u_N)$.

Acting on such state with $\mathrm{Tr}_a\; T_a(u)$ we should obtain an eingenvalue $\lambda(u)$. Thus, if

$$\mathrm{Tr}_a\; T_a(u)\,|\vec{u}\rangle = (A(u) + D(u))\,|\vec{u}\rangle = \lambda(u)\,|\vec{u}\rangle + K(u) \tag{48}$$

Therefore $K(u)$ must be equal to exactly zero. This leads to the following equations, known as the *algebraic Bethe equations*

$$\left(\frac{u_n - i/2}{u_n + 1/2}\right)^L = \prod_{j=1, j\neq n}^{L} \left(\frac{u_n - u_j - i}{u_n - u_j + i}\right), \quad n = 1, L \tag{49}$$

These allow for the solvavility of the parameters of the system. To see this, we give the expression of the eigenvalue $\lambda(u)$:

$$\lambda(u) = (u - i/2)^L \prod_{j=1}^{L} \left(1 + \frac{i}{u - u_j}\right) + (u + i/2)^L \prod_{j=1}^{L} \left(1 + \frac{i}{u_j - u}\right) \tag{50}$$

As it was shown earlier, the value of $\frac{d}{du}$ ln $\mathrm{Tr}_a\; T(u)$ at $u = u_0$ corresponds to the hamiltonian of the system. Consequently, conserved quantities Q are encoded by $\frac{d}{du}$ $\ln(\lambda(u))$, where taking $u = i/2$ we have

$$Q = \frac{d}{du} \ln\, \lambda(i/2) = \frac{d}{du} \ln\, i^L \prod_{j=1}^{L} \left(\frac{u_j + i/2}{u_j - i/2}\right) \tag{51}$$

which, together with (49), can be used to solve for Q explicitely.

4 The Tetrahedron Zamolodchikov Algebra

After covering the main concepts of classical and quantum integrability, we are ready to focus on the main topic of this dissertation. This section will cover the the free fermion model R-matrix and how its elements can be thought as representations of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$. We will explicitely show that the R-matrix is invariant under such group. Then we will explain how these symmetries lead to an analog of the Yang Baxter algebra for 1+2 dimensional quantum field theory.

4.1 The Quantum Group $\mathrm{U}_q(\mathfrak{sl}_2)$ and the Free Fermion Model

Let us first introduce the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$. This is simply the algebra $\mathfrak{sl}_2$ but with an additional generator and properties. The group SL(2,$\mathbb{C}$) consists of all 2×2 complex matrices whose determinant is one, and its algebra, $\mathfrak{sl}_2$, is generated by three elements: E, K and F. The four dimensional representation of $\mathfrak{sl}_2$ is given in [8] as:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{52}$$

Note that these matrices satisfy the relations $\{K, F\} = 0$, $\{K, E\} = 0$ and $[E, F] = K$. Any three elements which satisfy these relations correspond to a representation of $\mathfrak{sl}_2$ and are said to form an $\mathfrak{sl}_2$*-triple*.

The definition of $\mathrm{U}_q(\mathfrak{sl}_2)$ is the *q-deformed algebra of* $\mathfrak{sl}_2$, and it is constructed by taking the above properties and deforming them via the use of a constant factor, q, and an extra generator, K^{-1}. Summarising, $\mathrm{U}_q(\mathfrak{sl}_2)$ is generated by four elements, E, F, K and K^{-1}, satisfying:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KE &= q^2 EK \\ FK &= q^2 KF \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}} \end{aligned} \tag{53}$$

We shall now take the first step in showing that the free fermion model R-matrix possesses a symmetry directly connected to $\mathrm{U}_q(\mathfrak{sl}_2)$. By definition, this model contains at most quadratic fermionic field terms in its Hamiltonian. Just like in section 3.3, the Hamiltonian operator will be derived starting from the R-matrix.

As done in [9] we shall take a different approach than earlier and write the R-matrix in terms of operators which satisfy generic commutation or anticommutation relations depending on whether they are fermionic or bosonic. This will be useful later for connecting the R-matrix elements to a specific representation of $U_q(\mathfrak{sl}_2)$ when $q = i$.

Let $\mathbf{c}_j$ and $\mathbf{c}_j^\dagger$ be annihilation and creation fermionic operators, respectively, where j labels the point in the lattice where the fermion is located. They satisfy the (fermionic) anticommutation relations

$$\{\mathbf{c}_j, \mathbf{c}_k^\dagger\} = \delta_{jk} \tag{54}$$

where $\{A, B\} = AB + BA$ is the anticommutator and it shall not be confused with the Poisson bracket. To simplify future writing, using these fermionic operators we can construct compound ones: $\mathbf{n}_j = \mathbf{c}_j^\dagger \mathbf{c}_j$ and $\mathbf{m}_j = \mathbf{c}_j \mathbf{c}_j^\dagger$.

The free fermion R-matrix is a combination of products of these operators. Just as the XXX spin chain R-matrix depended on a complex parameter u, this one depends on a complex matrix A in SL(2,$\mathbb{C}$), such that $ad - bc = 1$. It satisfies $R_{ij}R_{ji} = \mathbb{I}$ (unitarity), maps representations of and is invariant under the Clifford algebra. These properties and a special condition fully determine its form, which is

$$R_{jk}^f(A) = -a\mathbf{n}_j\mathbf{n}_k - ib\mathbf{n}_j\mathbf{m}_k - ic\mathbf{m}_j\mathbf{n}_k + d\mathbf{m}_j\mathbf{m}_k + \mathbf{c}_j^\dagger\mathbf{c}_k + \mathbf{c}_j\mathbf{c}_k^\dagger \tag{55}$$

Without using operators in the construction, the representation of R_{12}^f as a 4×4 matrix would look like the following:

$$R_{12}^f(A) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & ib & 1 & 0 \\ 0 & 1 & ic & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{56}$$

The choice of the A variables to obey $|A| = 1$ is known as the *free fermion condition.* The free fermion model is quantum integrable, as its R-matrix satisfies the Yang Baxter equation $R_{12}^f(A)R_{13}^f(B)R_{23}^f(C) = R_{23}^f(C)R_{13}^f(B)R_{12}^f(A)R_{12}^f$. However, $B = CA$ must also be satisfied for this to hold, but we can insert this condition into the equation itself by defining a new operator R_{jk}^0 such that

$$R_{jk}^0(A_j, A_k) \equiv R_{jk}^f(A_k A_j^{-1}) \tag{57}$$

as $CA = A_3A_2^{-1}A_2A_1^{-1} = A_3A_1^{-1} = B$. Another choice of operator to satisfy $B = CA$ is $R_{jk}^1(A_j, A_k) \equiv R_{jk}^f(A_k\sigma^z A_j^{-1}\sigma^z)(\mathbf{n}_k - \mathbf{m}_k)$. These two options will become more significant in section 4.3, where they will be identified with a certain type of intertwiners in $\mathrm{U}_q(\mathfrak{sl}_2)$.

We wish to construct the Hamiltonian density operator for such spin chain. In the case of the Heisenberg model, we let the R-matrix depend on a complex parameter u_0. In the free fermion model, R^f_{jk} already depends on a complex matrix A, so we will let A be a function of a continous complex parameter u defining a curve λ in the complex plane

$$\lambda : [u_0, \infty) \to \mathrm{SL}(2,\mathbb{C})$$

such that at u_0 the R-matrix becomes the permutation operator. Now the task becomes to construct such operator. This is a map P_{jk} which sends $s_j \otimes s_k$ to $s_k \otimes s_j$, where s_i is just a spin operator. For the sake of simplicity, assume we can represent s_i in a two dimensional vector space. This would be useful in connecting P_{ij} to the matricial form of R^f_{12}. If X and Y are two dimensional vectors, then their tensor product is given by

$$X \otimes Y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_1 \end{pmatrix} \tag{58}$$

and $Y \otimes X$ is the same as above but with the middle two entries interchanged. Since $P(X \otimes Y) = Y \otimes X$, we easily obtain the form of P:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{59}$$

We can recognise this matrix as R^f_{12} when $a = d = 1, b = c = 0$, or $A(u_0) = \mathbb{I}^2$. So in terms of fermionic operators, the permutation operator generalised to sites i, j has the form

$$P^f_{jk} = -\mathbf{n}_j \mathbf{n}_k + \mathbf{m}_j \mathbf{m}_k + \mathbf{c}^\dagger_j \mathbf{c}_k + \mathbf{c}_j \mathbf{c}^\dagger_k \tag{60}$$

The monodromy matrix is constructed in terms of the R-matrix by taking the product of $R^0_{jk}(u)$'s in all possible indices k

$$T^f_j(u) = R^0_{jN}(u) R^0_{jN-1}(u) ... R^0_{j1}(u) \tag{61}$$

Here we have used the R-matrix instead of the Lax operator because they are both representations of the same algebra. This expression allows us to compute the Hamiltonian density using (40)

$$H^f = \frac{d}{du}\ln \text{Tr}_j \ T_j^f(u)) \mid_{u=u_0} \tag{62}$$

One natural choice for the curve is $a = d = \cos u$ and $b = c = \sin u$. This way, we obtain what is known as the XX model, which possesess a Hamiltonian of the form

$$H^{XX} = \sum_{i=1}^{L} \mathbf{c}_i^\dagger \mathbf{c}_{i+1} + \mathbf{c}_i \mathbf{c}_{i+1}^\dagger \tag{63}$$

In terms of the Pauli matrices σ_j^α at each lattice site j, this model is just the XXZ model but lacking a $\sigma_j^z \sigma_{j+1}^z$ spin-spin interaction term.

4.2 A Representation of Affine $\mathbf{U}_i(\mathfrak{sl}_2)$

Let us return our attention to the quantum group $\mathrm{U}_i(\mathfrak{sl}_2)$, where we have taken $q = i$, and find the relation of its affine extension to (55). Recall the affine extension of any algebra $\mathcal{A}$ is the vector space spanned by several copies of the generators of $\mathcal{A}$. In the case of $\mathrm{U}_i(\mathfrak{sl}_2)$ we will denote this affine extension by $\mathcal{U}_i$, which is the algebra generated by E_r, F_r, K_r and K_r^{-1}, where $r = 0, 1$. The properties (53) now become

$$\begin{aligned} K_r K_r^{-1} &= K_r^{-1} K_r = 1 \\ [K_r, K_s] &= 0 \\ \{K_r, E_s\} &= \{K_r, F_s\} = 0 \\ [E_r, F_s] &= \delta_{rs} \frac{K - K^{-1}}{2i} \end{aligned} \tag{64}$$

Furthermore, we introduce the operator H_r defined by the relation $K_r = q^{H_r}$. Imagine one would like to find a representation of $\mathcal{U}_i$. Impressively, we can use the elements of the free fermion model R-matrix to write an expression for its generators. Analysing the commutation and anticommutation relations of the generators of $\mathcal{U}_i$, we realise K_r can be interpreted as bosonic generators and E_r, F_r as fermionic generators. This gives us a hint on how to construct the representation: K_r must be a linear combination of $\mathbf{m}$ and $\mathbf{n}$ and the fermionic ones must be proportional to $\mathbf{c}$ and $\mathbf{c}^\dagger$. For the sake of simplicity, we shall ignore the lattice indices. Then it is quite easy to check that

$$\begin{array}{lll}K_0 = i^{1+u}(\mathbf{n}-\mathbf{m}) \quad , & E_0 = -\chi v^{-1}\mathbf{c}^{\dagger} & K_1 = i^{-1-u}(\mathbf{n}-\mathbf{m}), \\ E_1 = \chi w^{-1}\mathbf{c} & F_0 = \chi v\mathbf{c}, & H_0 = u-\mathbf{m}+\mathbf{n} \\ F_1 = -\chi w\mathbf{c}^{\dagger}, & H_1 = -u+\mathbf{m}-\mathbf{n} & \end{array} \tag{65}$$

satisfy the relations (64). Here u, v, w are complex numbers and χ is defined via

$$\chi^2 = \frac{i^{-u-1} - i^{u+1}}{2} \tag{66}$$

We may name this family of 2-dimensional representations $\langle u, v, w\rangle$. Now one would like to find the form of an $\mathcal{U}_i$ invariant map between tensor products of these representations, and investigate if it is related to the free fermion R-matrix. In other words, we would like to find an S-matrix for a $\mathcal{U}_i$-invariant theory and check if we can obtain R^f_{jk} from it via a series of similarity transformations. In order to do so, we must define the *coproduct* of each element in $\mathcal{U}_i$.

In algebraic terms, if $\mathcal{A}$ is a lie algebra then a coproduct Δ is a linear algebra morphism from $\mathcal{A}$ into $\mathcal{A}\otimes\mathcal{A}$ which preserves de lie bracket. It works as an extension of the lie algebra to the tensor product with itself. For a general element $g \in A$, it is usually assumed that

$$\Delta(g) = g \otimes g \tag{67}$$

We can check $[\Delta(g), \Delta(h)] = (g\otimes g)(h\otimes h) - (h\otimes h)(g\otimes g) = (gh)\otimes(gh) - (hg)\otimes(hg) = \Delta(gh) - \Delta(hg) = \Delta(gh - hg) = \Delta([g, h])$ so this map does preserve the Lie bracket. However, there are other choices for the coproduct which differ from (67). We are free to perform a so called "twist" of this case in defining the map. We see that, for example

$$\Delta(g) = g \otimes \mathbb{I} + \mathbb{I} \otimes g \tag{68}$$

does also preserve the Lie bracket. We can check this explicitely:

$$\begin{aligned}[\Delta(g), \Delta(h)] &= (g\otimes\mathbb{I} + \mathbb{I}\otimes g)(h\otimes\mathbb{I} + \mathbb{I}\otimes h) - (h\otimes\mathbb{I} + \mathbb{I}\otimes h)(g\otimes\mathbb{I} + \mathbb{I}\otimes g) \\ &= gh\otimes\mathbb{I} + \mathbb{I}\otimes gh - hg\otimes\mathbb{I} - \mathbb{I}\otimes hg \\ &= \Delta(gh) - \Delta(hg) = \Delta([g, h]).\end{aligned} \tag{69}$$

We shall define a twisted coproduct of every element in $\langle u, v, w\rangle$. To do so , one needs two extra operators: an element Z which commutes with every element in $\mathcal{U}_i$, and an operator $B = \mathbf{m} - \mathbf{n}$. The coproduct of every element in $\langle u, v, w\rangle$

is defined, for three central elements, as a regular coproduct

$$\Delta(K_r) = K_r \otimes K_r, \qquad \Delta(Z) = Z \otimes Z, \qquad \Delta(B) = B \otimes B \tag{70}$$

And for the noncentral elements (except H_r) as a twisted coproduct

$$\begin{aligned}
\Delta(H_r) &= H_r \otimes \mathbf{1} + \mathbf{1} \otimes H_r \\
\Delta(E_0) &= E_0 \otimes Z + K_0 B \otimes E_0 \\
\Delta(E_1) &= E_1 \otimes \mathbf{1} + Z K_1 B \otimes E_1 \\
\Delta(F_0) &= F_1 \otimes K_0^{-1} Z^{-1} + B \otimes F_0 \\
\Delta(F_1) &= F_1 \otimes K_1^{-1} + Z^{-1} B \otimes F_1
\end{aligned} \tag{71}$$

This choice of twist will also play an important role in section 6.3, when we connect the quantum symmetry of $\mathcal{U}_i$ and su(2|2)$\ltimes\mathbb{R}^2$.

We can use the coproduct to find operators called *intertwiners* between representations $\langle u, v, w\rangle$, which is the first step in deriving the Tetrahedron Zamolodchikov algebra and would give us a set of operators invariant under $\mathrm{U}_q(\mathfrak{sl}_2)$. An intertwiner is a map between two representations of the same algebra which is invariant under the action of the algebra itself. S-matrices are themselves a type of intertwiners. In the case we are discussing - let us call this specific intertwiner r^0_{12} - it will act in the space $\langle u_1, v_1, w_1\rangle \otimes \langle u_2, v_2, w_2\rangle$. The reason why there is no need of finding operators between tensor products of higher order than two is factorisability: if the system is integrable, the S-matrix of any process can be decomposed into the $2 \to 2$ S-matrix. This type of intertwiner must satisfy

$$r^0_{12} \Delta(X) {r^0_{12}}^{-1} = \mathrm{P}(\Delta(X)) \tag{72}$$

where X is an element of $\mathcal{U}_i$, and P is the permutation operator such that $\mathrm{P}(X \otimes Y) = Y \otimes X$. The intertwiner maps a representation $\langle u, v, w\rangle \otimes \langle u', v', w'\rangle$ to a permutation of itself. As noted in [9] the solution for equation (72) is very similar to the R-matrix R^f_{12} for the free fermion model: it contains a $(\mathbf{n}_1 + \mathbf{m}_1)(\mathbf{n}_2 + \mathbf{m}_2)$ factor with different coefficients in each compound operator combination, and the $\mathbf{c}_2^\dagger \mathbf{c}_1 + \mathbf{c}_1^\dagger \mathbf{c}_2$ factor. If we let $\eta_k = i^{-1-u_k}$, where $k = 1, 2$, it reads

$$\begin{aligned}
r^0_{12} &= (v_1 w_1 \eta_1 \eta_2 - v_2 w_2)\mathbf{n}_1 \mathbf{n}_2 + z^{-1}(v_2 w_2 \eta_1 - v_1 w_1 \eta_2)\mathbf{n}_1 \mathbf{m}_2 \\
&+ (v_2 w_2 \eta_2 - v_1 w_1 \eta_1)\mathbf{m}_1 \mathbf{n}_2 + (v_1 w_1 - v_2 w_2 \eta_1 \eta_2)\mathbf{m}_1 \mathbf{m}_2 \\
&- \sqrt{(\eta_1 + \eta_1^{-1})(\eta_2 + \eta_2^{-1})}(\mathbf{c}_2^\dagger \mathbf{c}_1 + \mathbf{c}_1 \mathbf{c}_2^\dagger)
\end{aligned} \tag{73}$$

where z is the eigenvalue of the operator Z, which we have imposed to be the same in every space $\langle u, v, w\rangle$, and physically represents the value of a conserved charge. If there exists an R-matrix which is invariant under the action of a certain algebra A, then it must be related by a similarity transformation to an intertwiner such algebra which acts on spaces of representations $\langle n\rangle \otimes \langle n'\rangle$, provided the central charges of the representations are conserved through such map. Although it is not obvious at first sight, the operator r^0_{12} satisfies the Yang Baxter equation. This is seen by studying the connection between r_{12} and R^0_{12}. Define a new operator $N_s = \mathbf{n}_s + \sqrt{\frac{w_s}{v_s}}\mathbf{n}_s$, then R^0_{12} can be written as

$$R^0_{12} = -\frac{N_1^{-1}N_2^{-1}r^0_{12}N_1N_2}{\sqrt{(\eta_1+\eta_1^{-1})(\eta_2+\eta_2^{-1})v_1w_1v_2w_2\eta_1\eta_2}} \tag{74}$$

therefore a rescaled r^0_{12} must be a representation of the Yang Baxter algebra. This expression for R^0_{12} is true provided the parameters a, b, c and d of A has been carefully adjusted as functions of v, w, η and z. This adjustment is equivalent to satisfying the conservation of central charges through maps between representations.

Since R^0_{12} can be identified with an interwiner of representations of $\mathcal{U}_i$, we come to the conclusion that it has a $\mathcal{U}_i$ symmetry, and it is therefore a suitable R-matrix for a theory invariant under such algebra. In the next section we will see the Tetrahedron Zamolodchikov algebra elements define an endomorphism of the space of operators with this type of symmetry.

4.3 The Tetrahedron Zamolodchikov Algebra

Let us now consider the elements in $\mathrm{U}_i(\mathfrak{sl}_2) \subset \mathcal{U}_i$, where we will denote the defining representation as $\langle u\rangle$ and study the possible solutions to the same intertwiner equation of the previous section, which assures the invariance of the intertwiner under the algebra,

$$r^0_{12}\Delta(X){r^0_{12}}^{-1} = \mathrm{P}(\Delta(X)) \tag{75}$$

where this time, X is an element of $\mathrm{U}_i(\mathfrak{sl}_2)$. Note it would be intuitive to label the representation with u and v instead of only u. However, since the choice of these numbers come solely from obeying the commutation relations, the representation actually depends on u and $v^{-1}v = 1$, so the second label is ignored. Due to an automorphism of the algebra, we can find two possible solutions:

- The one which was found earlier, r^0_{12}. We are working on a coproduct that was originally defined for elements in the affine extension of the algebra, so the three labels on the representation should be kept. This solution can be adjusted to $\mathrm{U}_i(\mathfrak{sl}_2)$ elements only by requiring $v_r = w_r$, so the operator belongs to the space $\langle u_1, v_1, v_1\rangle \otimes \langle u_2, v_2, v_2\rangle$.

- An operator which acts on the space $\langle u_1, v_1, v_1\rangle \otimes \langle u_2, v_2, -v_2\rangle$. The last v_2 has been multiplied by a minus sign, which comes from an automorphism between representations. If we define ψ_i to be an element such that $[\psi_i, E_j] = \delta_{ij} E_j$, $[\psi_i, F_j] = -\delta_{ij} F_j$ and $[\psi_i, H_j] = 0$, then we see that the map
$$X \mapsto e^{-i\pi\psi_1} X e^{i\pi\psi_1} \tag{76}$$
leads to the transformation $(E_0, E_1, F_0, F_1, H_i) \mapsto (E_0, -E_1, -F_0, F_1, H_i)$ which induces an automorphism from $\langle u, v, v\rangle$ to $\langle u, v, -v\rangle$.

As is shown in [9], one can perform a transformation which relates these two operators to R^0_{12} and R^1_{12}, respectively. Due to the fact that r^0_{12} and r^1_{12} possess an $\mathrm{U}_i(\mathfrak{sl}_2)$ symmetry, the operators composing the Yang Baxter equation on R^α_{jk} - such as $R^\alpha_{23} R^\beta_{13} R^\gamma_{12}$ - are also invariant under $\mathrm{U}_i(\mathfrak{sl}_2)$. However these operators do not form a linearly independent basis for the space of intertwiners in the tensor product of three representations.

There are 16 possible combinations of such operators but the dimension of the space where they live is smaller than 16. To see this, we must calculate the dimension of the space of $\mathrm{U}_i(\mathfrak{sl}_2)$-invariant interwiners on $\langle u_1\rangle \otimes \langle u_2\rangle \otimes \langle u_3\rangle$. The representations $\langle u\rangle$ depend on the value of two functions of u, which are $f(u) = i^{-1-u}$ and $f^{-1}(u) = i^{1+u}$. This means the tensor product of two representations dependent on u_1 and u_2 depends on $(f(u) + f^{-1}(u))(f(u_2) + f^{-1}(u_2)) = f(u_1 + u_2 + 1) + f^{-1}(u_1 + u_2 + 1) + f(u_1 - u_2 - 1) + f^{-1}(u_1 - u_2 - 1)$. One notices the first two terms are the functions of $\langle u_1 + u_2 + 1\rangle$. The second two terms can be identified with $f(u_1 + u_2 - 1) + f^{-1}(u_1 + u_2 - 1)$ by realising the representation also depends on $\chi(u)^2$, which is symmetric under $u \to -u$, and so the tensor product of two $\mathrm{U}_i(\mathfrak{sl}_2)$ representations decomposes irreducibly as follows
$$\langle u_1\rangle \otimes \langle u_2\rangle = \langle u_1 + u_2 + 1\rangle \oplus \langle u_1 + u_2 - 1\rangle \tag{77}$$

The computation of the tensor product of three representations is then straightforward:

$$\begin{aligned}
\langle u_1\rangle \otimes \langle u_2\rangle \otimes \langle u_3\rangle &= (\langle u_1 + u_2 + 1\rangle \oplus \langle u_1 + u_2 - 1\rangle) \otimes \langle u_3\rangle \\
&= (\langle u_1 + u_2 + 1\rangle \otimes \langle u_3\rangle) \oplus (\langle u_1 + u_2 - 1\rangle \otimes \langle u_3\rangle) \\
&= \langle u_{123} + 2\rangle \oplus 2\,\langle u_{123}\rangle \otimes \langle u_{123} - 2\rangle \qquad (78)
\end{aligned}$$

where $u_{123} = u_1 + u_2 + u_3$. The dimension of the space of invariant intertwiners on this decomposition is obtained through Schur's Lemma: if V_k are 1-parameter dependent representations of a lie algebra $\mathfrak{h}$ and $\bigotimes_r V_r$ is decomposed irreducibly as $\bigoplus_i A_i V_i$, then

$$dim(\bigotimes_r V_r) = \sum_i A_i^2 \tag{79}$$

In this case the dimension of the tensor product happens to be 6. Therefore the operators $R_{23}^{\alpha} R_{13}^{\beta} R_{12}^{\gamma}$ must be related to each other. Such relations define the Tetrahedrown Zamolodchikov algebra elements $\mathbb{S}$: taking lightcone coordinates $R^{\pm} = \frac{1}{2}(R^0 \pm R^1)$,

$$R_{23}^{\alpha} R_{13}^{\beta} R_{12}^{\gamma} = \sum_{\alpha',\beta',\gamma'=\pm} \mathbb{S}_{\alpha'\beta'\gamma'}^{\alpha\beta\gamma}(a_k, b_k, c_k, d_k) R_{12}^{\gamma'} R_{13}^{\beta'} R_{23}^{\alpha'}, \quad k = 0, 1, 2, 3 \tag{80}$$

where a_k, b_k, c_k, d_k are the free fermion parameters. Since this shows a relation between 8 generators, and there should only be 6 linearly independent ones, there exist two linear dependence equations:

$$\sum_{\alpha,\beta,\gamma=\pm} \mathbb{H}_{\alpha\beta\gamma}^{i}(a_k, b_k, c_k, d_k) R_{12}^{\alpha} R_{13}^{\beta} R_{23}^{\gamma} = 0, \quad i = 1, 2. \tag{81}$$

which generate a set of gauge transformations. Note that adding (81) to (80) does not change (80), so the algebra is invariant under the following transformation

$$(\mathbb{S})_{\alpha'\beta'\gamma'}^{\alpha\beta\gamma} \to (\mathbb{S}')_{\alpha'\beta'\gamma'}^{\alpha\beta\gamma} = (\mathbb{S})_{\alpha'\beta'\gamma'}^{\alpha\beta\gamma} + c_1^{\alpha\beta\gamma} \mathbb{H}_{\alpha'\beta'\gamma'}^{1} + c_2^{\alpha\beta\gamma} \mathbb{H}_{\alpha'\beta'\gamma'}^{1} \tag{82}$$

and each coefficient $\mathbb{S}'$ depends on three fermion parameters only. We can label the dependence of such variables explicitely by writing $\mathbb{S'}_{rst}$ where each r, s, t can be labelered by 1,2,3 or 4 depending on which three fermion parameters each coefficient is a function of. If we consider now the product of six R-matrices in lattice order, the Tetrahedron Zamolodchikov equations can be obtained by gauging elements of the algebra suitably:

$$\mathbb{S'}_{123} \mathbb{S'}_{124} \mathbb{S'}_{134} \mathbb{S'}_{234} = \mathbb{S'}_{234} \mathbb{S'}_{134} \mathbb{S'}_{124} \mathbb{S'}_{123} \tag{83}$$

This is the analog of the Yang Baxter equation and its corresponding algebra for 1+2 dimensional physics: just as the Yang Baxter equation corresponds to the equality of two scattering matrices of a 3-body process in a two dimesional lattice, the Tetrahedron Zamolodchikov equation corresponds to this equality in a three dimensional lattice. It generates integrable three dimensional quantum field theories and statistical systems. It can also generate an infinite amount of two dimensional integrable models by interpreting the third dimension as an internal degree of freedom, every single one of these systems depending on the size of such degree.

It is convenient to perform the analog of figure 2 and represent this equation pictorically in a two dimensional lattice. To do so, let us express (77) as

$$\mathbb{R}_{12a}\mathbb{R}_{12b}\mathbb{R}_{13c}\mathbb{R}_{abc} = \mathbb{R}_{abc}\mathbb{R}_{13c}\mathbb{R}_{12b}\mathbb{R}_{12a} \tag{84}$$

where, pictorically,

Figure 4: pictorial representation of $\mathbb{R}$

are $\mathbb{R}_{12a}$ and $\mathbb{R}_{abc}$ respectively. Then (62) is just the equality of the following two lattices

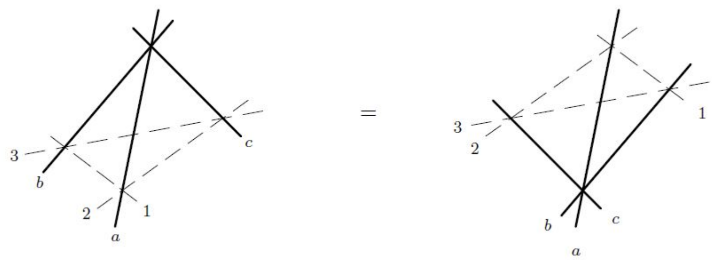

Figure 5: pictorial representation of (62)

where different shaped lines are interpreted to be apart in the third dimension. We can see that if we ignore any set of three lines that crossed at a point we obtain the Yang Baxter equation.

5 The Hubbard Model R-matrix

As in the previous section, we will begin by describing an algebraic object: centrally extended $\mathfrak{su}(2|2)$. We shall point out its relation to $\mathrm{U}_q(\mathfrak{sl}_2)$. Then we will introduce the Hubbard model and show that its R-matrix is a specific case of the solution for the Shastry ansatz. We will follow by identifying elements of the aforemenetioned R-matrix to a specific representation of centrally extended $\mathfrak{su}(2|2)$.

5.1 The $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ Lie Superalgebra

A *Lie superalgebra* is a vector space which contains both fermionic and bosonic generators, and hence does not only satisfy commutation relations but also anticommutation ones. Let a vector space generated by bosonic elements $X_r^{(b)}$ (or a regular lie algebra) be $\mathcal{A}_b$, and a space generated by fermionic elements $X_r^{(f)}$ be $\mathcal{A}_f$. A Lie superalgebra is defined as $\mathcal{A} = \mathcal{A}_b \oplus \mathcal{A}_f = \langle X_r^{(b)}, X_s^{(f)}\rangle$, where the bracket denotes $\mathcal{A}$ is spanned by both $X_r^{(b)}$ and $X_s^{(f)}$. $\mathcal{A}$ possesses an operation known as the *Superlie bracket* $[\ ,\ \}$, which is a commutator or anticommutator depending on the elements involved:

$$
\begin{aligned}
&[X_r^{(b)}, X_s^{(b)}\} \equiv [X_r^{(b)}, X_s^{(b)}] = C_{rs}^t X_t^{(b)} \\
&[X_r^{(f)}, X_s^{(f)}\} \equiv \{X_r^{(f)}, X_s^{(f)}\} = N_{rs}^k X_k^{(b)} \\
&[X_r^{(b)}, X_s^{(f)}\} \equiv [X_r^{(b)}, X_s^{(f)}] = Q_{rs}^l X_l^{(f)}
\end{aligned}
\tag{85}
$$

where C, N and Q are structure constants. Just as the elements of any algebra satisfy the Jacobi identity, an additional property of $\mathcal{A}$ is that its elements satisfy the Superjacobi identity: for any X, Y, Z in $\mathcal{A}$,

$$
[X,[Y,Z\}\} \pm [Y,[Z,X\}\} \pm [Z,[X,Y\}\} = 0 \tag{86}
$$

where the $\pm$ sign depents on whether the order of the fermionic generators has changed with respect to the first factor.

The Lie superalgebra of concern is $\mathfrak{psu}(2|2)$. Unlike matrices in the product of projective special unitary groups $\mathfrak{psu}(2)\times\mathfrak{psu}(2)$, the $2|2$ label indicates that one of the $\mathfrak{psu}(2)$ groups acts on fermionic fields and the other one acts on bosonic fields. This algebra is spanned by six bosonic generators $\mathcal{L}^{\alpha}{}_{\beta}$ and $\mathcal{R}^{a}{}_{b}$ and eight fermionic ones $\mathcal{Q}^{\alpha}{}_{a}$ and $\mathcal{S}^{a}{}_{\alpha}$. By adding three central elements $\mathcal{C}, \mathcal{P}$ and $\mathcal{K}$ we can centrally extend this Lie superalgebra to $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$. Note the group $\mathrm{PSU}(N)$ is simply $\mathrm{SU}(N)$ with a $\mathbb{Z}_N$ symmetry, hence $\mathfrak{psu}(2|2)$ is the same as $\mathfrak{su}(2|2)$ with a dimension less, and thus $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3 = \mathfrak{su}(2|2)\ltimes\mathbb{R}^2$. This is the $\mathfrak{su}(2|2)$ centrally extended Lie superalgebra.

Its nontrivial commutation relations are

$$
\begin{aligned}
&[\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{L}^{\gamma}{}_{\xi}] = \delta^{\gamma}_{\beta}\mathcal{L}^{\alpha}{}_{\xi} - \delta^{\alpha}_{\xi}\mathcal{L}^{\gamma}{}_{\beta}, && [\mathcal{R}^{a}{}_{b}, \mathcal{R}^{c}{}_{d}] = \delta^{c}_{b}\mathcal{R}^{a}{}_{d} - \delta^{a}_{d}\mathcal{R}^{c}{}_{b} \\
&[\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{Q}^{\gamma}{}_{b}] = \delta^{\gamma}_{\beta}\mathcal{Q}^{\alpha}{}_{b} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{Q}^{\gamma}{}_{b}, && [\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{S}^{a}{}_{\gamma}] = -\delta^{\alpha}_{\gamma}\mathcal{S}^{\alpha}{}_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{S}^{a}{}_{\gamma} \\
&\{\mathcal{Q}^{\alpha}{}_{a}, \mathcal{Q}^{\beta}{}_{b}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathcal{P}, && \{\mathcal{S}^{a}{}_{\alpha}, \mathcal{S}^{b}{}_{\beta}\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathcal{K} \\
&\{\mathcal{Q}^{\alpha}{}_{a}, \mathcal{S}^{b}{}_{\beta}\} = \delta^{b}_{a}\mathcal{L}^{\alpha}{}_{\beta} + \delta^{\alpha}_{\beta}\mathcal{R}^{b}{}_{a} + \delta^{b}_{a}\delta^{\beta}_{\alpha}\mathcal{C}
\end{aligned} \tag{87}
$$

As we expect, the bosonic elements generate rotations on themselves and the fermionic elements generate central charges with the exception of $\{\mathcal{Q}^{\alpha}{}_{a}, \mathcal{S}^{b}{}_{\beta}\}$, which also generates bosonic elements. As noted in [15], this centrally extended superalgebra has an $\mathfrak{sl}_2$ automorphism. The action of bosonic elements is immediately invariant under $\mathfrak{sl}_2$ since they "rotate" themselves. Grouping the central charges in one element and the fermionic generators in another as

$$
\mathcal{T}^{a}{}_{b} = \begin{pmatrix} -\mathcal{C} & \mathcal{P} \\ -\mathcal{K} & \mathcal{C} \end{pmatrix} \qquad \mathcal{J}^{a\beta c} = \begin{pmatrix} \epsilon^{ad}\mathcal{Q}^{\beta}{}_{d} \\ \epsilon^{\beta\delta}\mathcal{S}^{a}{}_{\delta} \end{pmatrix} \tag{88}
$$

The element $\mathcal{J}^{\alpha\beta c}$, being composed by fermionic generators, is immediately an $\mathfrak{sl}_2$ multiplet. Furthermore, we can infer the element $\mathcal{T}^{a}{}_{b}$ is also a multiplet of $\mathfrak{sl}_2$ by decomposing it as a linear combination of the generators in (51):

$$
\mathcal{T}^{a}{}_{b} = -\mathcal{C}K^{a}{}_{b} - \mathcal{K}F^{a}{}_{b} + \mathcal{P}E^{a}{}_{b} \tag{89}
$$

Hence we can now write the centrally extended $\mathfrak{su}(2|2)$ superalgebra in a form which explicitely manifests $\mathfrak{sl}_2$ invariance

$$
\begin{aligned}
&[\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{L}^{\gamma}{}_{\xi}] = \delta^{\gamma}_{\beta}\mathcal{L}^{\alpha}{}_{\xi} - \delta^{\alpha}_{\xi}\mathcal{L}^{\gamma}{}_{\beta}, && [\mathcal{R}^{a}{}_{b}, \mathcal{R}^{c}{}_{d}] = \delta^{c}_{b}\mathcal{R}^{a}{}_{d} - \delta^{a}_{d}\mathcal{R}^{c}{}_{b} \\
&[\mathcal{R}^{a}{}_{b}, \mathcal{J}^{c\delta e}] = \delta^{c}_{b}\mathcal{J}^{a\delta e} - \frac{1}{2}\delta^{a}_{b}\mathcal{J}^{c\delta e}, && [\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{J}^{c\delta e}] = \delta^{\delta}_{\beta}\mathcal{J}^{c\alpha e} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{J}^{c\delta e}
\end{aligned}
$$

$$
\{\mathcal{J}^{a\beta c}, \mathcal{J}^{a\beta c}\} = \epsilon^{ad}\epsilon^{\epsilon\kappa}\epsilon^{\mathfrak{cf}}\mathcal{L}^{\beta}{}_{\kappa} + \epsilon^{ak}\epsilon^{\epsilon\beta}\epsilon^{\mathfrak{cf}}\mathcal{R}^{d}{}_{k} + \epsilon^{ad}\epsilon^{\epsilon\beta}\epsilon^{\mathfrak{f}h}\mathcal{T}^{\mathfrak{c}}_{h} \tag{90}
$$

The representations of $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ depend on the central charges only, so we can write them as $\langle\vec{C}\rangle = \langle\mathcal{C}, \mathcal{P}, \mathcal{K}\rangle$. Letting ϕ^a and ψ^α be bosonic and fermionic fields respectively, one can write the transformation rules, which must encode $\mathfrak{su}(2) \times \mathfrak{su}(2)$ covariance of the generators. Thus

$$
\mathcal{R}^{a}{}_{b}\,|\phi^a\rangle = \delta^{c}_{b}\,|\phi^a\rangle - \frac{1}{2}\delta^{a}_{b}\,|\phi^c\rangle\,, \quad \mathcal{L}^{\alpha}{}_{\beta}\,|\psi^\gamma\rangle = \delta^{\gamma}_{\beta}\,|\psi^\alpha\rangle - \frac{1}{2}\delta^{\alpha}_{\beta}\,|\psi^\gamma\rangle
$$

$$
\begin{pmatrix} \mathcal{Q}^{\alpha}{}_{a} & 0 & 0 & 0 \\ 0 & \mathcal{Q}^{\alpha}{}_{a} & 0 & 0 \\ 0 & 0 & \mathcal{S}^{a}{}_{\alpha} & 0 \\ 0 & 0 & 0 & \mathcal{S}^{a}{}_{\alpha} \end{pmatrix} \begin{pmatrix} |\phi^b\rangle \\ |\psi^\beta\rangle \\ |\phi^b\rangle \\ |\psi^\beta\rangle \end{pmatrix} = \begin{pmatrix} a\delta^b_a\,|\psi^\alpha\rangle \\ b\epsilon^{\alpha\beta}\epsilon_{ab}\,|\phi^b\rangle \\ c\epsilon^{ab}\epsilon_{\alpha\beta}\,|\psi^\beta\rangle \\ d\delta^\beta_\alpha\,|\psi^\alpha\rangle \end{pmatrix} \tag{91}
$$

where a, b, c, d determine the values of $\mathcal{C}, \mathcal{P}, \mathcal{K}$, and the closure of the algebra requires that $ad - bc = 1$.

The presence of SL(2,$\mathbb{C}$) both as the group whose algebra is deformed to obtain $\mathrm{U}_q(\mathfrak{sl}_2)$ and the automorphism of centrally extended su(2|2) suggest a connection between these two objects. Let $\mathfrak{g} = \mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ and $\mathcal{U}_i^2 \equiv \mathcal{U}_i \otimes \mathcal{U}_i$, each $\mathcal{U}_i$ corresponding to either spin up or down. We wish to convey the possibility that a subset of $\mathfrak{g}$ is isomorphic to a subset of $\mathcal{U}_i^2$, one of the possibilities of this subset being of course the space itself. To achieve this, it is necessary to connect the generators of both spaces by linear maps, or identifications.

Let us first focus on the fermionic generator $\mathcal{S}^a{}_\alpha \in \mathfrak{g}$. It satisfies the anticommutation relations $\{\mathcal{S}^a{}_\alpha, \mathcal{S}^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathcal{K}$. Note that if this anticommutator vanishes if $a = b$ or $\alpha = \beta$. In the case of $\mathcal{U}_i^2$, all anticommutation relations vanish if they involve elements with in different spin layers. Hence, if we pick $\mathcal{S}^1{}_1$ to be identified with an element with spin up, we must make sure both $\mathcal{S}^1{}_2$ and $\mathcal{S}^2{}_1$ are identified with an element with spin down.

The only nontrivial anticommutation relations are $\{\mathcal{S}^1{}_1, \mathcal{S}^2{}_2\} = -\{\mathcal{S}^1{}_2, \mathcal{S}^2{}_1\} = \mathcal{K}$. Thus, we must have one of the generators in the second anticommutator be negative. Let us now perform the identification for the generators $\mathcal{S}^a{}_\alpha$:

$$
\begin{pmatrix} \mathcal{S}^1{}_1 & \mathcal{S}^1{}_2 \\ \mathcal{S}^2{}_1 & \mathcal{S}^2{}_2 \end{pmatrix} \to \begin{pmatrix} F_{0\uparrow} & F_{0\downarrow} \\ -E_{0\downarrow}K^{-1}_{0\downarrow} & E_{0\uparrow}K^{-1}_{0\uparrow} \end{pmatrix} \tag{92}
$$

As we can see, $\{\mathcal{S}^1{}_1, \mathcal{S}^2{}_2\} = -\{\mathcal{S}^1{}_2, \mathcal{S}^2{}_1\}$ is nonzero and every other anticommutator is exactly zero because different spins anticommute. The actual value of the nontrivial anticommutators is

$$
\begin{aligned}
\{\mathcal{S}^1{}_1, \mathcal{S}^2{}_2\}_{id} &= \{F_{0\uparrow}, E_{0\uparrow}K^{-1}_{0\uparrow}\} \\
&= E_{0\uparrow}\{F_{0\uparrow}, K^{-1}_{0\uparrow}\} + [F_{0\uparrow}, E_{0\uparrow}]K^{-1}_{0\uparrow} \\
&= \frac{1}{2i}[(K^{-1}_{0\uparrow})^2 - 1]
\end{aligned} \tag{93}
$$

where the id label denotes this is not an equality but an identification. Is the result a conserved charge? We know $K_0^{-1} = \lambda(\mathbf{n} - \mathbf{m})$ so $(K_0^{-1})^2 = \lambda^2(\mathbf{n}^2 + \mathbf{m}^2)$ where the crossing elements have vanished because fermionic operators anticommute. One can now check

$$\begin{aligned}(K_0^{-1})^2 &= \eta^2(\mathbf{n}^2+\mathbf{m}^2)\\ &= \eta^2(\mathbf{c}^\dagger\mathbf{c}\mathbf{c}^\dagger\mathbf{c}+\mathbf{c}\mathbf{c}^\dagger\mathbf{c}\mathbf{c}^\dagger)\\ &= \eta^2(\mathbf{c}^\dagger(1-\mathbf{c}^\dagger\mathbf{c})\mathbf{c}+(1-\mathbf{c}^\dagger\mathbf{c})\mathbf{c}\mathbf{c}^\dagger))\\ &= \eta^2(\mathbf{c}^\dagger\mathbf{c}+\mathbf{c}\mathbf{c}^\dagger)=\eta^2. \end{aligned} \tag{94}$$

so that $\{\mathcal{S}^1{}_1, \mathcal{S}^2{}_2\} = -\{\mathcal{S}^1{}_2, \mathcal{S}^2{}_1\}$ is a constant, and hence a conserved charge.

We can also identify the elements $\mathcal{Q}^\alpha{}_a$ with operators in $\mathcal{U}_i^2$ using linear transformations. This map takes the form

$$\begin{pmatrix}\mathcal{Q}^1{}_1 & \mathcal{Q}^1{}_2\\ \mathcal{Q}^2{}_1 & \mathcal{Q}^2{}_2\end{pmatrix} \to \begin{pmatrix}F_{1\uparrow} & E_{1\downarrow}K_{1\downarrow}^{-1}\\ F_{1\downarrow} & -E_{1\uparrow}K_{1\uparrow}^{-1}\end{pmatrix} \tag{95}$$

Thus we have identified the fermionic part of $\mathfrak{g}$ with a subspace of $\mathcal{U}_i^2$. This is one of the key hints for investigating the connections between the Hubbard model, which is invariant under the latter, and a sector of a gauge theory invariant under the former.

5.2 The Shastry Ansatz

Now we proceed to introduce the 1-dimensional Hubbard model, which was shown to be exactly integrable by Shastry in [10]. As before, let $\mathbf{c}_{j,\sigma}$ and $\mathbf{c}_{j,\sigma}^\dagger$ be annihilation and creation fermionic operators respectively, and the index σ can be either spin up or down, ie $\sigma = \uparrow,\downarrow$. As in (54), define the compound operators $\mathbf{n}_{j,\sigma} = \mathbf{c}_{j,\sigma}^\dagger\mathbf{c}_{j,\sigma}$ and $\mathbf{m}_{j,\sigma} = \mathbf{c}_j\mathbf{c}_{j,\sigma}^\dagger$. The fermionic operators satisfy the usual commutation relations

$$\{\mathbf{c}_{j,\sigma}, \mathbf{c}_{k,\tau}^\dagger\} = \delta_{jk}\delta_{\sigma\tau}$$

The Hubbard model is of great importance in solid state physics. It manages to accurately describe electron conduction by assuming electrons move from atom to atom located in a lattice during such process. The Hamiltonian of the model is

$$H_h = -\sum_{j=1}^{N}\Big(\sum_{\sigma=\uparrow,\downarrow}\mathbf{c}_{j+1,\sigma}^\dagger\mathbf{c}_{j,\sigma}+\mathbf{c}_{j,\sigma}^\dagger\mathbf{c}_{j+1,\sigma}\Big)+\frac{U}{4}(\mathbf{m}_{j,\uparrow}-\mathbf{n}_{j,\uparrow})(\mathbf{m}_{j,\downarrow}-\mathbf{n}_{j,\downarrow}) \tag{96}$$

where the first part corresponds to the kinetic part of any quantum field theory hamiltonian and U is a coupling constant of interaction. Without any coupling, the Hamiltonian of this spin chain decomposes into

$$H_h^0 = -\sum_{j=1}^{N} \mathbf{c}^\dagger_{j+1,\uparrow}\mathbf{c}_{j,\uparrow} + \mathbf{c}^\dagger_{j,\uparrow}\mathbf{c}_{j+1,\uparrow} + \sum_{j=1}^{N} \mathbf{c}^\dagger_{j+1,\downarrow}\mathbf{c}_{j,\downarrow} + \mathbf{c}^\dagger_{j,\downarrow}\mathbf{c}_{j+1,\downarrow} \tag{97}$$

which is just the sum of two Hamiltonians for two free XX models. In order to compute the R-matrix for this model, it is rather intuitive to propose a linear combination of the R-matrices in section 4.3. Indeed, this is what Shastry did: proposing an Ansatz for all possible U, which was later generalized in [11]:

$$R_{jk} = (\alpha_{jk} R^0_{jk,\uparrow} + \beta_{jk} R^1_{jk,\uparrow})(\gamma_{jk} R^0_{jk,\downarrow} + \kappa_{jk} R^1_{jk,\downarrow}) \tag{98}$$

where the coefficients are complex parameters determined by the requirement that (98) obeys the Yang Baxter equation. Using light-cone coordinates and the Tetrahedron Zamolodchikov algebra, the most general expression for R_{jk} is given in [11] by

$$\begin{aligned} R_{jk} &= R^+_{jk,\uparrow} R^-_{jk,\downarrow} + R^-_{jk,\uparrow} R^+_{jk,\downarrow} \\ &+ \frac{o_k + o_j \frac{b_j c_j}{a_j d_j}}{o_j + o_k \frac{b_k c_k}{a_k d_k}} \left(\frac{a_j b_k}{b_j a_k} R^+_{jk,\uparrow} R^+_{jk,\downarrow} + \frac{d_j c_k}{c_j d_k} R^-_{jk,\uparrow} R^-_{jk,\downarrow}\right) \end{aligned} \tag{99}$$

where o_j are known as the *gluing parameters* and satisfy the equation

$$\frac{\theta_1^2 o_j}{a_j d_j} - \frac{\theta_2^2 o_k^{-1}}{b_j c_j} = -i \qquad j = 1, 2 \tag{100}$$

We can see that (99), being composed of R-matrices of the free fermion model, is invariant under $\mathcal{U}_i^2$. Hence, the full symmetry seems to be connected to $\mathfrak{g}$. To study this connection, it is convenient to write a representation using the elements in R_{jk}. The bosonic generators can be written as

$$\begin{aligned} \mathcal{R}^1{}_1 &= -\mathcal{R}^2{}_2 = \frac{1}{2}(1 - \mathbf{n}_\uparrow - \mathbf{n}_\downarrow), \qquad & \mathcal{R}^1{}_2 &= (\mathcal{R}^2{}_1)^\dagger = \mathbf{c}_\downarrow \mathbf{c}_\uparrow \\ \mathcal{L}^1{}_1 &= -\mathcal{L}^2{}_2 = \frac{1}{2}(\mathbf{n}_\uparrow - \mathbf{n}_\downarrow), & \mathcal{L}^1{}_2 &= (\mathcal{L}^2{}_1)^\dagger = \mathbf{c}^\dagger_\uparrow \mathbf{c}_\downarrow \end{aligned} \tag{101}$$

and, defining $\mathbb{P}$ to be the spin permutation operator, we can write the fermionic generators as

$$\begin{aligned} \mathcal{Q}^1{}_1 &= (a\mathbf{m}_\downarrow + b\mathbf{n}_\downarrow)\mathbf{c}^\dagger_\uparrow = \mathbb{P}\mathcal{Q}^2{}_1, \quad & \mathcal{Q}^1{}_2 &= -(b\mathbf{m}_\uparrow + a\mathbf{n}_\uparrow)\mathbf{c}_\downarrow, = -\mathbb{P}\mathcal{Q}^2{}_2 \\ \mathcal{S}^1{}_1 &= (d\mathbf{m}_\downarrow + c\mathbf{n}_\downarrow)\mathbf{c}_\uparrow = \mathbb{P}\mathcal{S}^1{}_2, & \mathcal{S}^2{}_1 &= -(c\mathbf{m}_\uparrow + d\mathbf{n}_\uparrow)\mathbf{c}^\dagger_\downarrow = -\mathbb{P}\mathcal{S}^2{}_2 \end{aligned} \tag{102}$$

Finally, the central charges may be written as

$$\mathcal{C} = \frac{ad+bc}{2}, \quad \mathcal{P} = ab, \quad \mathcal{K} = cd \tag{103}$$

where a, b, c, d are complex parameters such that $ad - bc = 1$. As mentioned earlier, this is a requirement of the closure of $\mathfrak{g}$ and these parameters determine the four dimensional representation of such algebra.

As shown in section 5.1, this algebra possesses an outer automorphism group isomorphic to SL(2,$\mathbb{C}$). As we will examine in the next section, this automorphism links the Hubbard model R-matrix, which can be decomposed in free fermion model ones, and the $AdS_5 \times S^5$ S-matrix, which must be invariant under $\mathfrak{g}$.

6 The AdS/CFT S-matrix

In this section we will first give a brief introduction to the AdS/CFT correspondence. We will then focus our attention on integrable systems which arise in $\mathcal{N} = 4$ Super Yang Mills by investigating subalgebras of the full symmetry algebra of the theory. The goal is to construct an S-matrix which is invariant under such algebras. Finally, we will show that, after a similarity transformation, a good candidate for such S-matrix is the simplest solution derived from the Shastry ansatz.

6.1 The AdS/CFT Correspondence

The Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence, also known as the gauge/gravity duality, is a conjectured connection between conformal field theories in d dimensions and quantum gravity theories in $d + 1$ dimensions. The specific one motivated by Maldacena in [19] is that type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}_5$ is equivalent to $\mathcal{N} = 4$ Super Yang Mills ($\mathcal{N} = 4$ SYM) theory in the four-dimensional conformal boundary of such space. In other words, $\mathcal{N} = 4$ SYM as a hologram of type IIB. A familiar analog of this is the theory of special relativity in 1+3 dimensions, where we can only experience its 3-dimensional spatial boundary.

The Anti-de-Sitter spacetime AdS_d, also known as hypebolic spacetime, is a d-dimensional manifold with metric signature $(-1, 1.., 1, -1)$. In other words, given coordinates $X^0, X^1, ..., X^d$ on AdS, the metric is of the form

$$ds^2 = -dX^0 + \sum_{n=1}^{d-1} dX^n - dX^d \tag{104}$$

Yand Mills theory is a gauge theory where the Lagrangian is invariant under the group $\mathrm{SU}(N)$. All fields of such a theory are represented as a linear combination of the generators t^a of the algebra $\mathfrak{su}(n)$, and thus the action of the theory is given by a trace of the non-abelian field strength $F_{\mu\nu} = F^a_{\mu\nu} t^a$:

$$\mathrm{S}_{\mathrm{YM}} = \mathrm{Tr} \int \mathrm{d}^4 x F_{\mu\nu} F^{\mu\nu} \tag{105}$$

This Lagrangian does of course admit $\mathrm{SU}(N)$-invariant functions of bosonic and fermionic fields. The field strength itself depends on the gauge field A_μ - written in terms of differential forms - as $F = \mathrm{d}A$.

Let us now construct a Yang Mills theory which is also supersymmetric. This implies that the theory is now not longer invariant under the Poincare algebra but the Superpoincare algebra. This superalgebra is composed of the bosonic elements which generate $\mathfrak{so}(1,3)$, divided in translations and rotations, and a set of $\mathcal{N}$ generators $Q^A_\alpha, \bar{Q}^A_{\dot{\alpha}}$ which anticommute with each other. Here A denotes the number of copies of generators and $\alpha = 1, 2$ is the spinor index, which corresponds to each irreducible component of a spinor ψ^α. All these generators satisfy supercommutation relations through the Superlie bracket which can be easily constructed a priori by preserving both Lorentz and spinor indices.

Each of these anticommuting generators can be identified with a *supercharge*, which is the conserved quantity that generates the supersymmetry. If we let $\mathcal{N} = 4$, the superpoincare algebra will have four copies of fermionic generators with two possible indices and a complex conjugate, or a total of 16 supercharges. If we imposed $\mathcal{N} = 4$ superpoincare and SU(N) invariance, the action of a theory with gauge field A and fields λ_a, X_j reads

$$\begin{aligned} \mathrm{S}_{\mathrm{SYM}} = \mathrm{Tr} \int & -\frac{1}{2g^2_{SYM}} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\bar{F}^{\mu\nu} - i\lambda^a \sigma^\mu D_\mu \lambda_a - D_\mu X^i D^\mu X^i \\ & + gC^{ab}_i \lambda_a [X^i, \lambda_b] + C_{iab}\lambda^a [X^i, \lambda^b] + \frac{g^2_{SYM}}{2}[X^i, X^j]^2 \end{aligned} \tag{106}$$

where $X^i, i = 1, .., 6$ are scalar fields, $\lambda^a, a = 1, .., 4$ are fermionic fields, D_μ is the covariant derivative and $\bar{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the Hodge dual of the field strength. This is $\mathcal{N} = 4$ SYM, and its fundamental element is a gauge field A^a_μ in a SU(N) gauge group.

This gauge theory is present in a particular example of a string theory. String theory is a theory of strings moving at relativistic speeds, and a candidate in the unification of the four fundamental forces of physics. It incorporates gravity to the electroweak force by increasing the number of spatial dimensions in 1+3 dimensional spacetime. In doing this, the theory proposes a description of gravity as living in all conjectured dimensions. The action of any string theory is derived through the minimization of the transvered area travelled by the string - also known as the *worldsheet*. The simplest string theory action, due to Polyakov, is given by

$$\mathrm{S} = \frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma) \tag{107}$$

where σ is the worldsheet spacelike coordinate, $g_{\mu\nu}(X)$ is the metric of the manifold the string is embedded in, $X^\mu(\sigma)$ are coordinates of position of the string, $T = (2\pi\alpha)^{-1}$ is the string tension, h^{ab} is the worldsheet metric and $h = \det(h^{ab})$. The energy spectrum of the theory is obtained by performing regular quantisation on the string coordinates. After this process , each string mode is interpreted as a different particle.

There are two types of strings: closed and open. Closed strings can move arbitrarly through space and satisfy peridiocity conditions, however open ones must have boundary conditions on their two endpoints. These can be classified in two types: Dirichlet and Neumann. The Dirichlet conditions [18] imply the string is constrained to exist in a $p+1$ dimensional hypersurface in spacetime. In [20], it was proved that in fact this hypersurface is dynamical, i.e. it can fluctuate and respond to external interactions, and it has internal degrees of freedom. It is known as a Dirichlet-brane, or D-brane. Depending on the value of p, this will be called a Dp-brane.

Imagine a string living in N coincident D-branes. In this case the string has a zero tension mode, and after quantisation one can check that the resulting particle corresponding to this mode resembles a photon. This is an SU(N) gauge field, where N is the number of D-branes. In d=4 dimensions, we have 16 supercharges or $\mathcal{N}=4$ supersymmetry, which means that the 4-dimensional worldvolume of the D3-brane will contain $\mathcal{N}=4$ SYM.

This provides a motivation to study if there is a certain string theory in a specific space which in the massless limit is equivalent to $\mathcal{N}=4$ SYM in 4 dimensions. Indeed, as suggested in [19], this theory appears to be type IIB supergravity in $\mathrm{AdS}_5\times\mathrm{S}^5$.

The AdS/CFT correspondence is widely considered on of the biggest breakthroughs of the last decades in the physical sciences. It provides both physically meaningful understanding of aspects of both theories and, most importantly, the ability to perform calculations in either theory which would otherwise be much tougher or even impossible. In the next section, we shall discuss the the search of integrable systems in a sectors of the gauge theory symmetry group.

6.2 The Gauge Theory Spin Chain

The $\mathcal{N}$=4 SYM symmetry is fixed by the group $\mathfrak{psu}(2,2|4)$, also known as the $\mathcal{N}$=4 superconformal group. In this section we may consider a subsector of $\mathcal{N}$=4 SYM by focusing on the superalgebra su(2|3) $\subset \mathfrak{psu}(2,2|4)$. In [15], Beisert constructed a set of representations of spin chain states invariant under $\mathfrak{su}(2|3)$. This was achieved with the goal of studying the integrability of structures arising from perturbative U(N) $\mathcal{N}=4$ SYM as $N\to\infty$. These representations were parametrised by a coupling constant, g, related to the gauge theory coupling constant in equation (106) by

$$g^2 = \frac{g_{SYM}^2 N}{8\pi^2} \tag{108}$$

The algebra $\mathfrak{su}(2|3)$ acts on vectors composed of 3 bosonic fields and 2 fermionic ones. Therefore, in a spin chain with a $\mathfrak{su}(2|3)$ symmetry, the spin s of each lattice site can be b_1, b_2, b_3, f_1 or f_2, where the $b_j's$ are bosonic orientations and the $f_j's$ are fermionic ones. A generic state for this spin chain is, for example

$$|\psi\rangle = |b_1 f_2 b_3 b_1 ... f_1\rangle \tag{109}$$

where each lattice site satisfies periodic boundary conditions since the spin chain is a closed object. This is equivalent to the cyclicity of the trace. Thus, the trace of a product of possible string spins, which incorporates periodicity of every site, has been replaced by a quantum mechanical state.

In [16], it is shown that the large N AdS/CFT S-matrix is equivalent to the diffractionless S-matrix of elementary excitations. Hence one shall define a vacuum state, $|0\rangle$, and then construct every other state as an excitation of $|0\rangle$. The vacuum may be assumed to be infinitely long and composed of b_1's only

$$|0\rangle = |b_1 b_1 ... b_1\rangle \tag{110}$$

Although an infinitely long spin chain is not physical, one can obtain the correct energy spectrum by considering periodically identified states. A generic state $|s_1...s_k\rangle$ is now given by a linear combination of all possible spin excitations of the vacuum of an infinitely long spin chain

$$|\psi\rangle = |s_1...s_k\rangle = \sum_{x_1<...<x_k} \psi(x_1, ..., x_k)\, |...b_1 b_1 ... s_1 ... s_n ... s_k ... b_1 b_1 ...\rangle \tag{111}$$

where the condition $x_1 < ... < x_k$ in the sum comes from the periodicity of states. Although this is a very useful way to describe the energy spectrum of the spin chain, it has one minor flaw: the number of excitations, k, is not preserved under $\mathfrak{su}(2|3)$ transformations. The largest subalgebra in which is preserved is precisely $\mathfrak{su}(2|2) \subset \mathfrak{su}(2|3)$. Hence our goal has become to study a spin chain whose states transform under $\mathfrak{su}(2|2)$ and to find its S-matrix. It is convenient to extend it by adding two central charges, in which case one obtains $\mathfrak{g} = \mathfrak{su}(2|2) \ltimes \mathbb{R}^2$. The reason behind this is, as shown in [21], that in (super)string theory this centrally extended superalgebra corresponds to the asymptotic symmetry of the the light-cone sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Therefore, by constructing a $\mathfrak{g}$-invariant spin chain, one can ensure the integrability of the aforementioned model and if possible compute string solutions.

As we noted in section 5.1, a representation of $\mathfrak{g}$ is uniquely determined by its central charges $\mathcal{C}, \mathcal{P}$ and $\mathcal{K}$. We may denote such representation $\langle \vec{C} \rangle$. Our purpose is to construct the S-matrix, a function $\mathbf{S}$ which satisfies the following properties:

- **S** maps representations of $\mathfrak{g}$ to each other via a permutation σ:

$$\mathbf{S}_\sigma : \langle \vec{C}_1 \rangle \otimes \; ... \; \otimes \langle \vec{C}_k \rangle \to \langle \vec{C}'_{\sigma(1)} \rangle \otimes \; ... \; \otimes \langle \vec{C}'_{\sigma(k)} \rangle \tag{112}$$

- **S** satisfies unitarity: $\mathbf{S}_{12}\mathbf{S}_{21} = \mathbf{S}_{21}\mathbf{S}_{12} = \mathcal{I}$

- **S** satisfies the Yang Baxter equation.

In the assumption of [16], excitations transform under $\mathfrak{g}$ as a tensor product - they do not affect each other. However, there must exist contributions from states with nearby excitations. In the exact action of the algebra on states of this spin chain, these contributions must be taken into account. A generic state $|\Psi\rangle = \Psi_1 |...s_k s'_l...\rangle$ would also include, for every pair of nearby excited particles in positions k and l,

- A contribution from the interaction between nearby excitated one-particle states. This can be represented by the state $\Psi_2 |...(ss)_{kl}...\rangle$.

- A contribution from particles with interchanged central charges. If close particles are allowed to interact, any generic state should include all permutations that conserve the overall central charge, and this is one of them. This can be represented by the state $\Psi_3 |...s''_l s'''_k...\rangle$.

Ψ_1 and Ψ_3 are transformed independently, and Ψ_2 must be adjusted so that it yields the contributions to the boundaries of the asymptotic regions. The completion of states can be performed by the operator **S**, which permutes the momenta labels of different particle states in representations of $\mathfrak{g}$. Hence

$$\mathbf{S}_{kl} |...s_k s'_l...\rangle = |...s''_l s'''_k\rangle \tag{113}$$

so that a generic state can be written as

$$|\Psi\rangle = \Psi_1(1 + \mathbf{S}_{kl}) |...s_k s'_l...\rangle + |...(ss)_{kl}...\rangle \tag{114}$$

So the requirement to ensure the consistency of Staudacher's assumption is that the state $|\Psi\rangle$ transforms adequately under $\mathfrak{g}$, which implies the operator **S** must commute with the generators of the algebra. As we will see, the R-matrix from the Shastry ansatz satisfies this property, and is therefore a suitable candidate for the S-matrix of this spin chain.

6.3 Full Symmetry of the Hubbard Model R-matrix

The goal of this section is to show that the solution to the Shastry ansatz is invariant under $\mathfrak{g}$. To do this, the representation in section 5.2 constructed via elements of the R-matrix itself will be used. Since we suppresed the lattice indices of operators $\mathbf{c}_\sigma$ and its adjoint partner, we intend to restore them via labeling each generator $g(\mathbf{c}_\sigma) \in \mathfrak{g}$ with an index j.

An operator Ω is invariant under a certain Lie algebra if $g^{-1}\Omega g = \Omega$ for every generator g, or equivalently, $[\Omega, g] = 0$. But $\mathfrak{g}$ is not only generated by bosonic elements but also fermionic ones. Thus, we can extend this condition to a superalgebra by imposing invariance under fermionic generators as well. So Ω is invariant under a fermionic generator g_f of $\mathfrak{g}$ if $g_f^{-1}\Omega g_f = -\Omega$, or equivalently, $\{\Omega, g_f\} = 0$.

Let us focus on the invariance of R_{jk} under the even subspace of $\mathfrak{g}$. In the representation of $\mathfrak{g}$ we constructed using the elements of R_{jk} each generator depended on one lattice site only. Thus, we need to show that

$$[R_{jk}, g_j + g_k] = 0, \quad g \in \mathfrak{g}, \quad j, k = 1, 2, 3. \tag{115}$$

On can observe R_{jk} depends on the operators $\mathbf{n}_{\sigma,j}$ and $\mathbf{m}_{\sigma,j}$ where the spin of the products of $\mathbf{c}_{\sigma,j}$ and $\mathbf{c}^\dagger_{\sigma,j}$ are the same. It is then easy to check then, since $[\mathbf{n}_j, \mathbf{m}_k] = [\mathbf{n}_j, \mathbf{n}_k] = [\mathbf{m}_j, \mathbf{m}_k] = 0$, that

$$\begin{aligned} &[R_{jk}, (\mathcal{L}^\alpha{}_\beta)_j + (\mathcal{L}^\alpha{}_\beta)_k] = 0 \\ &[R_{jk}, (\mathcal{R}^1{}_1)_j + (\mathcal{R}^2{}_2)_k] = 0 \end{aligned} \tag{116}$$

Showing the invariance under the generators $\mathcal{R}^1{}_2$ and $\mathcal{R}^2{}_1$ requires a little more work. One must look back at the relation between the interwiner r^0_{12} and R^0_{12} in (59). This relation was established to be true for carefully adjusted parameters v_i, w_i and η_i in terms of the SL(2,$\mathbb{C}$) matrix elements a_k, b_k, c_k and d_k. Specifically, as in [9], we obtain the following identifications:

$$\begin{aligned} a_k &= \frac{1}{\sqrt{v_k w_k (1 - \frac{1}{\lambda_k^2})}}, & b_k &= \frac{1}{iz\lambda_k}\sqrt{\frac{v_k w_k}{(1 - \frac{1}{\lambda_k^2})}} \\ c_k &= \frac{iz}{\lambda_k^2}\frac{1}{\sqrt{v_k w_k (1 - \frac{1}{\lambda_k^2})}}, & d_k &= \sqrt{\frac{v_k w_k}{1 - \lambda_k^2}}) \end{aligned} \tag{117}$$

One notices that $a_k b_k = -\frac{1}{z^2} c_k d_k$. This indicates that the value z is necessary to cover all possible SL(2,$\mathbb{C}$) variables, which is the main reason to use a central element Z to twist the coproduct. If one chooses $z \in \{-i, i\}$, which equates to $a_k b_k = c_k d_k$, we can perform two similarity transformations on R_{jk} to obtain a new R-matrix $\bar{R}_{jk}$ which satisfies

$$
\begin{aligned}
&[\bar{R}_{jk}, (\mathcal{R}^1{}_2)_j - (\mathcal{R}^1{}_2)_k] = 0 \\
&[\bar{R}_j, (\mathcal{R}^2{}_1)_j - (\mathcal{R}^2{}_1)_k] = 0 \\
&[\bar{R}_{j,}, (\mathcal{R}^a{}_b)_j + (\mathcal{R}^a{}_b)_k] = 0
\end{aligned} \tag{118}
$$

Now we proceed to show the invariance of $\bar{R}_{jk}$ under fermionic generators. First we must define the following two matrices:

$$
\mathfrak{M} = e^{-i\frac{\pi}{4}} \begin{pmatrix} \sqrt{\frac{\theta_o^3}{\theta_2}} \frac{c_j o_j}{t_j} \frac{1}{a_j d_j} & \sqrt{\frac{\theta_2^3}{\theta_1}} \frac{t_j}{c_j o_j} \\ -\frac{\sqrt{\theta_1 \theta_2}}{b_j t_j} & -\frac{t_j \sqrt{\theta_1 \theta_2}}{c_j} \end{pmatrix}, \quad \mathfrak{N} = \begin{pmatrix} \frac{c_j}{a_j} & 0 \\ 0 & \frac{a_j}{c_j} \end{pmatrix} \tag{119}
$$

The matrix $\mathfrak{N}$ is clearly in SL(2,$\mathbb{C}$). For $\mathfrak{M}$, we can see that $\det(\mathfrak{M}) = 1$ by the gluing conditions. Therefore both of these matrices belong to SL(2,$\mathbb{C}$), and they can determine a representation of $\mathfrak{g}$ subject to the outer automorphism of the algebra $\mathfrak{sl}_2$. Hence one can form other representations of $\mathfrak{g}$ by multiplying these two matrices in any desired combination. Define now matrices $\mathfrak{D}_i$ and $\mathfrak{D}'_i \in$ SL(2,$\mathbb{C}$), $i = 1, 2$, as follows:

$$
\begin{pmatrix} \mathfrak{D}_1 & \mathfrak{D}_2 \\ \mathfrak{D}'_1 & \mathfrak{D}'_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{N}_2 \mathfrak{M}_1 & \mathfrak{M}_2 \\ \mathfrak{M}_1 & \mathfrak{N}_1 \mathfrak{M}_2 \end{pmatrix} \tag{120}
$$

With this choice of matrices, and $\mathfrak{G}_i$ being any fermionic generator of $\mathfrak{g}$, one obtains the following result:

$$
\bar{R}_{12}[\mathfrak{G}_1(\mathfrak{D}_1) + \mathfrak{G}_2(\mathfrak{D}_2)]\bar{R}_{12}^{-1} = [\mathfrak{G}_1(\mathfrak{D}'_2) + \mathfrak{G}_2(\mathfrak{D}'_1)] \tag{121}
$$

The matrices $\mathfrak{D}_1$ and $\mathfrak{D}_2$ determine the representation D_{12} of the fermionic part of $\mathfrak{g}$, so the operator $\bar{R}$ happens to be an intertwiner acting on the space $D_{12} \otimes D_{1'2'}$, which implies this R-matrix is a well defined map between representations of the fermionic part of $\mathfrak{g}$ which is also invariant under the such sector of the algebra. One can also see this choice of representation and R-matrix is in agreement with the conservation of the central charges in a map between representations of $\mathfrak{g}$. We have the relation $ad - bc = 1$, which using (94), imposes a condition on the central charges

$$
\langle \mathcal{C} \rangle^2 = \mathcal{C}^2 - \mathcal{P}\mathcal{K} = \frac{1}{4} \tag{122}
$$

Hence in a process where $\langle C_1 \rangle \otimes \langle C_2 \rangle \to \langle C'_2 \rangle \otimes \langle C'_1 \rangle$ we have the following conservation laws:

- $\langle \mathcal{C}_1 \rangle^2 + \langle \mathcal{C}_2 \rangle^2 = \langle \mathcal{C}'_2 \rangle^2 + \langle \mathcal{C}'_1 \rangle^2$

- $\mathcal{C}_1 + \mathcal{C}_2 = \mathcal{C}'_1 + \mathcal{C}'_2$
- $\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}'_1 + \mathcal{P}'_2$ and $\mathcal{K}_1 + \mathcal{K}_2 = \mathcal{K}'_1 + \mathcal{K}'_2$

A solution of this process starts with the imposition that $\mathcal{C}_i$ is mapped to $\mathcal{C}'_i$. Intuitively, one can think of $\mathcal{C}$ as the energy of the system and this would be equivalent to working in the particle rest frame. Then, by the first conservation law, we obtain $\mathcal{P}_1\mathcal{K}_1 + \mathcal{P}_2\mathcal{K}_2 = \mathcal{P}'_1\mathcal{K}'_1 + \mathcal{P}'_2\mathcal{K}'_2$. If we now impose $\mathcal{P}'_i\mathcal{K}'_i = \mathcal{P}_i\mathcal{K}_i$, we obtain a solution of the transformation of the central charges under this map:

$$\mathcal{C}'_i = \mathcal{C}_i, \quad \mathcal{P}'_i = \mathcal{K}_i \frac{\mathcal{P}_1 + \mathcal{P}_2}{\mathcal{K}_1 + \mathcal{K}_2}, \quad \mathcal{K}'_i = \mathcal{P}_i \frac{\mathcal{K}_1 + \mathcal{K}_2}{\mathcal{P}_1 + \mathcal{P}_2} \tag{123}$$

This should be true for any valid $\mathfrak{g}$-invariant R-matrix. We can work out if these conditions are compatible with the defined matrices. The matrices $\mathfrak{D}_i$ and $\mathfrak{D}'_i$ provide a representation of $\mathfrak{g}$ where, for example,

$$\mathfrak{D}_1 = \begin{pmatrix} \mathfrak{a}_1 & \mathfrak{b}_1 \\ \mathfrak{c}_1 & \mathfrak{d}_1 \end{pmatrix} = e^{-i\frac{\pi}{4}} \begin{pmatrix} \frac{c_2}{a_2}\sqrt{\frac{\theta_1^3}{\theta_2}}\frac{c_1 o_1}{t_1}\frac{1}{a_1 d_1} & \frac{c_2}{a_2}\sqrt{\frac{\theta_2^3}{\theta_1}}\frac{t_1}{c_1 o_1} \\ -\frac{a_2}{c_2}\frac{\sqrt{\theta_1\theta_2}}{b_1 t_1} & -\frac{a_2}{c_2}\frac{t_1\sqrt{\theta_1\theta_2}}{c_1} \end{pmatrix} \tag{124}$$

Using equation (94), this leads to a solution for every central charge which is compatible with the form of matrices $\mathfrak{D}_\mathfrak{i}$ and $\mathfrak{D}'_i$. Hence $\bar{R}_{jk}$ is an intertwiner acting on tensor products of representations of $\mathfrak{g}$, and it is therefore a suitable S-matrix for a $\mathfrak{g}$-invariant theory. In terms of variables used in AdS/CFT, a, b, c and d in the representation of $\mathfrak{g}$ given in section 5 is written as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sqrt{g} \begin{pmatrix} \eta & \frac{\zeta}{\eta}(1 - x^+/x^-) \\ i\frac{\eta}{\zeta x^-} & \frac{x^+}{i\eta}(1 - x^-/x^+) \end{pmatrix} \tag{125}$$

where the condition $A \in \mathrm{SL}(2,\mathbb{C})$ relates variables x^+, x^- and g.

One can't help but notice the similarity between equations (122) for $\mathfrak{g}$ and (75) for $\mathrm{U}_i(\mathfrak{sl}_2)$. Indeed, this similarity is not a coincidence and is connected to the identification between the fermionic part of $\mathfrak{g}$ and a subspace of $\mathcal{U}_i^2$ found in section 5.1. Through a series of linear transformations, call these x and y, one can actually merge these two equations into one:

$$\bar{R}_{12}(x \circ \Delta(X))\bar{R}_{12}^{-1} = y(P_{12})(x \circ \mathrm{P}(\Delta(X)))y(P_{12}) \tag{126}$$

where $X \in \mathrm{U}_i(\mathfrak{sl}_2) \oplus \mathrm{U}_i(\mathfrak{sl}_2)$ and one can find the exact form x and y in [9].

7 Summary and Further Work

In this dissertation, we have shown in detail that a very simple physical model in two dimensions is nontrivially related to $\mathcal{N} = 4$ SYM, a theory which in the past few decades has had a significant role in the attempt of explaining reality due to the AdS/CFT correspondence. This relation becomes apparent if one tries to analyse the full symmetry of the scattering matrix of the model is invariant under. One finds this symmetry is closely related to a subalgebra of $\mathfrak{psu}(2,2|4)$ which, in the string dual of $\mathcal{N} = 4$ SYM, corresponds to the off-shell symmetry of the light-cone sigma model.

This discovery should motivate for further study of well-known integrable models to understand structures in more complicated theories. The task of performing calculations in perturbative $\mathcal{N} = 4$ SYM has become easier with the gauge/gravity duality, however, recent work in integrability does hint for a correspondence between integrable models which may rely on tools to perform calculations and obtain exact solutions.

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