# Topics in Canonical Gravity

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**Abstract**

In this work we review some important topics in canonical theories of General Relativity. In particular we discuss the theory of constrained systems, the Hamiltonian formulation of GR, the difficulty in constructing gauge invariant objects for GR and recent methods involving relational observables to address this problem and finally we consider modern connection formulations of gravity.

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# 1 Introduction

In this dissertation we review a number of important topics in classical canonical gravity. The aim has been to provide a pedagogical introduction to these subjects, assuming a background in GR and QFT only. The choice of topics has been motivated by an interest in the area of quantum gravity and a desire to cover the classical background to the canonical quantization of GR, so that understanding the material here might make the jump to the quantum theory more manageable. We stress that none of the material is original, it is a review, but we have tried to present it in a logical way and where possible to include explicit computations to illustrate new ideas and to cover missing steps in the literature. We have also cited many references so that a reader can quickly identify some key papers and gain an appreciation of some of the controversies that have and do still exist in the subject.

The subject of quantum gravity has a long history and finding such a theory is one of the outstanding problems in theoretical physics today. There are many research programs working on this problem including string/ M-theory, canonical quantum gravity, causal set theory, causal dynamical triangulations (CDT) and several others, a recent review of several approaches can be found in [1].

We will be covering only the classical background to the canonical quantum gravity approach, which has received new impetus over the last two decades following a reformulation of GR into a theory of connections due to Ashtekar, which we shall discuss in section 5. However, we also discuss the earlier metric formulation of canonical gravity both because it is more familiar and because several problems, e.g. the problem of time, the difficulty in constructing gauge invariant observables and the nature of the constraints, are present in both these approaches.

One of the first attempts to quantise GR was to employ the standard methods used for classical electrodynamics, i.e. covariant perturbative field theory. This involves separating the metric into a fixed background part $\eta_{\mu\nu}$ and a fluctuating field to be quantised, see the review [2]. The resulting QFT suffers from the same infinities that occur in QED but unfortunately the methods of renormalization so successful for QED cannot be applied to GR, the theory is perturbatively non-renormalizable, and this fact was one of the original motivations for the development of supergravity and then the superstring theories.

Although GR is perturbatively non-renormalizable, there are arguments that can be made to suggest it is still worthwhile to consider a quantization of GR: i.) the divergent terms in the perturbative expansion occur at arbitrarily high energies (short distances) and assume that the background spacetime is Minkowski- this is a questionable assumption we do not even know whether we will have a smooth manifold structure at the Planck scale, and ii.) GR could be non-pertubatively renormalizable, i.e. there may exist viable approaches where one does not separate the metric into a fixed background part before quantizing - the canonical quantization methods we discuss come under this approach as does CDT, which involves a (covariant) discretization of the Einstein Hilbert action.

We should now like to discuss the topics we cover in more detail. In section 2 we provide an introduction to the theory of constrained Hamiltonian systems, this theory developed by Dirac and Bergmann provides the framework for the canonical analysis of singular Lagrangian systems, which includes GR and Yang Mills theories. This framework enables one to perform a Legendre transform and to compute the Hamilton-Dirac equations of motion, which will be subject to constraints. These constraints can be classified into first and second class, and enable one to identify where the gauge freedom, present in the Lagrangian, appears at the canonical level.

We further show why first class constraints $\phi_i$ are interpreted as the generators of point gauge transformations on phase space. This interpretation due to Dirac is not the only one, an alternative due to Bergmann is to view gauge transformations as maps that act on phase space trajectories, rather than points, and in this way maintain the idea of gauge transformations as a symmetry, i.e. a map from solutions of the equations of motions to other solutions. In this latter interpretation gauge transformations are generated by a function $G(t)$, which is a particular sum of first class constraints $\phi_i$, [10].

We then discuss some basic geometric results concerning constrained systems, one finds first class constraints generate surfaces in phase space, which represent the gauge equivalence classes of the system- and these surfaces foliate together to fill the entire constraint surface. First class constraints act twice i.) to reduce the effective phase space dimension and ii.) to reduce the effective dimension of the constraint surface by dividing it into equivalence classes. Second class constraints have no interpretation as gauge generators and they simply reduce the effective dimension of the phase space by restricting the dynamics to a surface in phase space.

In gauge theories an important issue is the identification of observable quantities, if one wants a consistent physics then such objects need to be gauge invariant, therefore in the current formalism they need to be invariant under the action of the first class constraints $\phi_i$, (i.e. they Poisson commute with these constraints). In the case of GR constructing such observables has proven very difficult and only relatively recently have approximation methods been developed, which we shall discuss in section 4.

We conclude section 2 by applying this formalism to Yang Mills theory on a Minkowski background, we perform the Legendre transform, identify the constraints, prove that the Poisson algebra of constraints is isomorphic to the Lie algebra of the gauge group and show how to recover the correct number of physical degrees of freedom.

In section 3 this formalism is applied to GR, in particular the canonical analysis of the Einstein Hilbert action is described. This is more involved than for Yang Mills, on a fixed background, because to express the Lagrangian in terms of spatial objects, evolving in time, one must assume spacetime is of the form $\Sigma \times \mathbb{R}$, where $\Sigma$ is a 3 dimensional spatial surface of arbitrary but fixed topology. One can then use a couple of geometric identities to reduce the Lagrangian into an appropriate $3+1$ form. The result, after performing the canonical analysis and reducing the phase space, is the ADM (Arnowitt Deser Misner) phase space, [13] consisting of canonical coordinates: a spatial Riemannian metric $q_{ab}$ and momentum $P^{ab}$, which is closely related to the extrinsic curvature of the embedding of $\Sigma$ in spacetime. These canonical coordinates are subject to the Hamiltonian and diffeomorphism constraints, which can be shown to be first class and therefore must be interpreted as generators of gauge transformations. The constraints are a crucial part of the story because they also form the Hamiltonian and therefore one concludes that the Hamiltonian is zero on shell and being a sum of first class constraints must generate gauge transformations.

This result seems to imply that one should interpret the dynamics of GR as the unfolding of a gauge transformation, which contradicts our observation of the dynamics of gravity, e.g. the expansion of the universe. This is the essence of the 'problem of time' in canonical gravity, [35, 36, 38]. If one assumes the Dirac interpretation of gauge transformations then this conclusion is forced upon us; a possible resolution, discussed in section 4, is to consider the construction of relational quantities, which can be shown to be gauge invariant, and then one can consider evolution with respect to additional clock fields rather than coordinate time. In certain special cases, de-parametrisable gravity matter coupling, it is possible to construct a physical Hamiltonian, which generates evolution of these relational observables. If one assumes the Bergmann interpretation then the same paradox does not apply because the gauge generator $G(t)$, although related to the Hamiltonian, has a different interpretation it does not map phase space points to points but rather trajectories to trajectories, [27].

We then discuss the Poisson algebra of the Hamiltonian and diffeomorphism constraints, the Dirac algebra, and try to relate it to the algebra of the full spacetime diffeomorphism group.

By analogy with the Yang Mills case one might expect some sort of isomorphism between them, unfortunately this is not the case. We show in detail why this is and conclude that the Poisson algebra is isomorphic to the diffeomorphism algebra only when the equations of motion hold, otherwise they are inequivalent. This lack of a representation of the diffeomorphism algebra in canonical gravity is quite puzzling and it can be understood in several ways.

Firstly, one can show that there exists a larger symmetry group of the Einstein Hilbert action, which has been called the 'induced diffeomorphism' group, [24, 26] and this consists of metric dependent coordinate transformations. It is the case that only a restriction of these coordinate transformations may be projected to the phase space from the Lagrangian velocity space and it is this projection that corresponds to the group generated by the Dirac algebra. This projected group is generally called the Bergmann-Komar group. Secondly, the difference between Yang Mills and GR regarding the relation between the gauge algebra and Poisson algebra can be understood by considering the way local symmetries of the action may (or may not) be projected to the phase space in each case, [28]. Thirdly we consider a method due to Isham and Kuchař, [29, 30] which shows that, by extending the ADM phase space to include the embedding variables with their canonical momenta, one can find a representation of the full diffeomorphism algebra in canonical gravity.

The remainder of section 3 considers the coupling of matter fields to gravity in the canonical formalism, we discuss the canonical analysis of a minimally coupled scalar field and compute the scalar field's contributions to the Hamiltonian and diffeomorphism constraints. Finally, we consider asymptotically flat spacetimes, and we show that to make the Hamiltonian finite and functionally differentiable one has to modify the Hamiltonian and diffeomorphism constraints by adding surface terms. One can compute the algebra of these modified constraints and conclude that they also satisfy the Dirac algebra (but now extended to the asymptotically flat case). By going to the constraint surface at spatial infinity, this algebra reduces to the Poincarè algebra and therefore we find a representation of the Poincarè group in the constraint algebra, [32]. The generators of this Poincarè group are in fact Dirac observables and therefore we have found ten observables for GR, i.e. that Poisson commute with the first class constraints of the theory.

In section 4 we discuss the problem of observables for GR. In particular we review recent work by Dittrich, [41, 42] which builds on earlier ideas of Rovelli [43] concerning the construction of relational observables. These observables consist of combinations of gauge variant phase space functions: partial observables and 'clock fields' which are combined into a gauge invariant object- a generalisation of the notion of the 'value of a quantity at a particular time'. These constructions are discussed firstly for the finite dimensional case and then for the field theoretic case. The general expressions for these objects can be approximated by rather complicated sums of nested Poisson brackets and we follow Dittrich by showing how in certain cases one can simplify these expressions.

Then, we consider an application of these ideas to de-parametrisable systems, [44, 48]. For such systems one can define a physical, non-zero Hamiltonian, which generates evolution of the relational observables and therefore could serve as a means to resolve the problem of time in the Dirac interpretation. In fact in this light we discuss work by Thiemann, [49] which couples a special scalar field to gravity that results in a de-parametrisable system and provides a gauge invariant way of re-formulating (modified) Friedmann Roberston Walker cosmological equations. This idea built on seminal work by Brown and Kuchař, [50] who showed that cosmological dust coupled to gravity could lead to a deparametrisable system.

Finally, in section 5 we discuss the reformulation of GR as a theory of connections originally due to Ashtekar, [51]. We describe the tetrad formalism for gravity and relate it to the familiar coordinate description. We introduce the Hilbert Palatini action, which has a Lorentz connection and tetrad as independent fields, and show that its equations of motion are equivalent to Einstein's equations. We perform a rather complicated Legendre transform on the Palatini action to analyze its Hamiltonian theory. This transform is complicated by the presence of second class constraints, which are generally solved by introducing a partial gauge fixing, and this has the effect of reducing the local symmetry from the Lorentz group to $SO(3)$. The end result is that one recovers a version of the ADM theory expressed in triad variables, and nothing is really gained because the constraints are still rather complicated expressions of the canonical variables.

We then discuss the reformulation of GR due to Ashtekar, this uses a self dual Lorentz connection and tetrad as independent fields in the action. This requires us to formulate the theory as complex GR because for Lorentzian metrics the self dual operator satisfies $\star \cdot \star = -1$ and hence self dual connections are necessarily complex. The result is that one has a complex tetrad and self dual connection, which when varied in the Ashtekar action lead to Einstein's equations for complex gravity. One then has to impose reality conditions to recover the real theory.

We perform the Legendre transform, which is easier than for the Palatini action because of the absence of second class constraints. Physically, this is because the self dual connection has three complex dimensions fewer independent degrees of freedom than the Lorentz connnection and so the second class constraints do not arise. After completing the canonical analysis one has a phase space with coordinates $\left(A_a^{IJ}, \tilde{E}_I^a\right)$, a self dual connection and densitised triad respectively but subject to three constraints: the familiar Hamiltonian and diffeomorphism constraints from metric gravity together with an additional constraint. In fact using an isomorphism between the self dual Lorentz algebra and either $SO(3)_{\mathbb{C}}$ or $SL(2,\mathbb{C})$ it is possible to translate these phase space coordinates into an $SO(3)_{\mathbb{C}}$ or $SL(2,\mathbb{C})$ connection and triad and the additional constraint is interpreted as a Gauss constraint generating an $SO(3)_{\mathbb{C}}$ or $SL(2,\mathbb{C})$ transformation of the triad.

The Ashtekar formalism has some attractive features including polynomial expressions for all the constraints in a connection theory of gravity, which makes contact with other gauge theories. Unfortunately, to date the reality conditions together with the non-compact gauge group have proven intractable in formulating the quantum theory.

So to conclude we consider the more recent real approach to connection gravity. This approach dates back to work by Barbero, [52, 53] and Immirzi, [64] who defined a 1 parameter family (labelled by $\beta$), of canonical transformations of the ADM triad variables. The resulting new phase space coordinates are a $SO(3)$ connection (the Ashtekar Barbero connection) and a densitised triad. The variable $\beta$ is known as the Immirzi parameter and $\beta = \pm i$ corresponds to the Ashtekar complex theory. If one takes $\beta$ real then one avoids the problem of reality conditions, and in addition has a compact gauge group but with a more complicated Hamiltonian constraint. However, these new variables do not have a direct four dimensional interpretation, in particular the real $SO(3)$ connection is not the pullback to a spatial slice of a four dimensional Lorentz connection, [65] as is the self dual connection.

These real variables can be derived from the Holst action, [54] but only by using a partial gauge fixing (known as the time gauge, this physically corresponds to choosing the zeroth component of the tetrad basis to be orthogonal to the embedded spacelike hypersurface) to avoid the second class constraints. The partial gauge fixing is a somewhat controversial part of the theory and there has been recent work investigating the consequences of either keeping the second class constraints and using the Dirac bracket or solving them but without using the gauge fixing. Both approaches usually come under the term covariant loop quantum gravity, [66].

We would have liked to have covered some aspects of the quantum theory, the canonical quantization of the real Ashtekar Barbero variables, but unfortunately this has not been possible due to time constraints. We therefore conclude with a discussion summarising the main results we have learnt and conclusions.

# 2 Constrained Hamiltonian Systems

In this section we consider the theory of *constrained Hamiltonian systems*, our aim is to describe the main ideas of this theory, introduce standard terminology and then, as an illustrative example, apply it to Yang Mills theory. The reason for doing this is that i.) GR is an example of a constrained Hamiltonian system and so we need to introduce this theory prior to describing the Hamiltonian formulation of GR, which we shall do in the next section and ii.) more generally all gauge theories can be viewed as examples of constrained Hamiltonian systems and so it provides a framework for understanding the dynamics of a wide class of physical theories.

The theory of constrained Hamiltonian systems was developed more than fifty years ago independently by Dirac, Bergmann and collaborators and also in earlier work by Rosenfeld. Our main references for this section have been [6, 7, 9, 12, 8] and particularly [10], which provides a very careful discussion of the meaning of gauge symmetry in the constrained Hamiltonian formalism.

## 2.1 Constrained Systems

We shall consider a system with a finite number of degrees of freedom and assume that results discussed for this case can be applied to the field theoretic case, for which we are ultimately interested. Let $q^i, i = 1, \ldots, n$ be coordinates on a configuration manifold $Q$ and $L(q, \dot{q})$ be a first order Lagrangian function with no explicit time dependence, then the Euler Lagrange equations are:

$$L_i := \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{2.1}$$

and using the chain rule one has:

$$\ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} = \frac{\partial L}{\partial q^i} + \dot{q}^j \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \ . \tag{2.2}$$

The matrix $W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is the Hessian and if it is not invertible we say that the Lagrangian is singular, (usual Hamiltonian dynamics without constraints involves only non-singular or regular Lagrangians).

We observe that if $W_{ij}$ is singular then the accelerations cannot be isolated in normal form, i.e. $\ddot{q}^j = g(q, \dot{q})$ and hence results concerning the existence and uniqueness of differential equations cannot be applied and it is possible for solutions $q^i(t)$ to contain arbitrary functions of time.

In the field theoretic case the same conclusions will follow, consider an action

$$S[\phi, \partial_\mu \phi] = \int dt d^3x \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.3}$$

then the Euler Lagrange equations are

$$\begin{aligned} L_a &= -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\right) + \frac{\partial \mathcal{L}}{\partial \phi^a} = 0 \\ \Rightarrow L_a &= -\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^a)\partial(\partial_\nu \phi^b)}\partial_\mu \partial_\nu \phi^b - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^a)(\partial \phi^b)}\partial_\mu \phi^b + \frac{\partial \mathcal{L}}{\partial \phi^a} \\ &\equiv -W^{00}_{ab}\ddot{\phi}^b + V_a = 0 \end{aligned} \tag{2.4}$$

and again if the matrix $W^{00}_{ab}$ is singular we will not be able to express the 'accelerations' in normal form and solutions may contain arbitrary functions of time.

At this point we should mention Noether's second theorem this states that a Lagrangian admits an infinite dimensional Lie group as a local symmetry, i.e. a gauge symmetry, if and only if there exist dependencies in the Euler Lagrange equations, involving linear combinations of $L_a$ and its derivatives. These dependencies are known as generalised Bianchi identities and further one can show that the presence of these identities implies that the matrix $W^{00}_{ab}$ is singular, [9]. Hence one concludes that all gauge theories are singular but the converse is not true.

A singular Lagrangian also affects the canonical formalism by way of the Legendre transform from the Lagrangian velocity phase space to phase space with coordinates $(q,\dot{q})$ and $(q,\pi)$ respectively. We recall the Legendre map $\mathcal{F}L$ is defined as:

$$
\begin{aligned}
\mathcal{F}L : TQ &\rightarrow T^{\star}Q \\
(q,\dot{q}) &\mapsto (q, \hat{\pi}_i(q,\dot{q}) := \frac{\partial L}{\partial \dot{q}^i})
\end{aligned}
\tag{2.5}
$$

where we use the standard notation for the tangent bundle $TQ$ and the co-tangent bundle $T^{\star}Q$. Using the definition of the canonical momentum $\pi_i$ one has $W_{ij} = \frac{\partial \hat{\pi}_j}{\partial \dot{q}^i}$ and for singular Hessians one concludes, by the inverse function theorem, that not all the velocities can be expressed as $\dot{q}^i = f^i(q,\pi)$, it then follows that the Legendre map is not invertible. We assume that the rank of the Hessian is constant throughout the velocity space and hence the same holds for the Legendre map. We assume that the rank of $\mathcal{F}L$ is equal to $2n-k$, so that the image $\mathcal{F}L(TQ)$ is a $2n-k$ dimensional space in the $2n$ dimensional phase space $T^{\star}Q$. This $2n-k$ dimensional space is called the *primary constraint surface* $\Gamma_p$ and can be locally characterized by the vanishing of $k$ independent phase space functions, denoted $\phi_\mu(q,\pi), \mu = 1,\dots,k$, where the functions $\phi_\mu$ are known as the *primary constraints.*

One can use the primary constraints to find a basis for the null eigenspace of the Hessian matrix. First consider the pullback of the primary constraints under the Legendre map one has:

$$
\left(\mathcal{F}L^{\star}\phi_\mu\right)(q,\dot{q}) := \phi_\mu(q,\hat{\pi}(q,\dot{q})) = 0\,\forall (q,\dot{q})
\tag{2.6}
$$

which implies

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \dot{q}^i}\phi_\mu(q,\hat{\pi}(q,\dot{q})) \\
&= \frac{\partial \hat{\pi}_j}{\partial \dot{q}^i}\frac{\partial \phi_\mu}{\partial \hat{\pi}_j} \\
&= W_{ij}\frac{\partial \phi_\mu}{\partial \pi_j}|_{\pi=\hat{\pi}}
\end{aligned}
\tag{2.7}
$$

and hence the $k$ independent vectors $\frac{\partial \phi_\mu}{\partial \hat{\pi}_j}$ must constitute a basis for the $k$ dimensional kernel of the Hessian $W_{ij}$.

In the singular case the Legendre map will be many to one and the inverse image of a point in the primary constraint surface is a $k$ dimensional space in $TQ$. In fact the inverse images of points in $\Gamma_p$ form a foliation of $k$ dimensional surfaces in $TQ$. One can show that the following vector fields $\Gamma_\mu$ on $Q$ define a basis for the set of vector fields tangent to these $k$ dimensional surfaces in $TQ$. Let

$$
\Gamma_\mu := \left(\frac{\partial \phi_\mu}{\partial \pi_j}|_{\pi=\hat{\pi}}\right)\frac{\partial}{\partial \dot{q}^j}
\tag{2.8}
$$

and consider two points on a particular 'leaf' of the foliation with coordinates[1] $(q,\dot{q})$ and $(q,\dot{q}+\delta\dot{q})$, by definition they map to the same point on the primary constraint surface, hence $\hat{\pi}_i(q,\dot{q}) =$

[1]It is sufficient to modify only the velocity variable as the configuration coordinate $q^i$ is fixed under the Legendre map.

$\hat{\pi}_i(q, \dot{q} + \delta\dot{q})$ and this implies $\delta\dot{q}^i \frac{\partial\hat{\pi}_j}{\partial\dot{q}^i} = 0$. By this latter result $\delta\dot{q}^i$ are null eigenvectors of the Hessian and by (2.7) must be expressible as a linear combination of the vectors $\frac{\partial\phi_\mu}{\partial\pi_j}|_{\pi=\hat{\pi}}$ hence the vector fields $\Gamma_\mu$ form a basis for the vector fields tangent to the leaves of the foliation in $TQ$.

Using the Legendre map it may be possible to project tensor fields from $TQ$ to $T^\star Q$, for example consider the simplest case of a function $f^L$ defined on $TQ$ and ask under what conditions one will be able to define a function $f^H$ on $T^\star Q$ such that

$$f^L = \left(\mathcal{F}L^\star f^H\right) \tag{2.9}$$

the answer is that the function $f^L$ must be constant on the leaves of the foliation induced by the Legendre map otherwise the projection will lead to a multi-valued function $f^H$. The projection by $\mathcal{F}L$ only uniquely defines $f^H$ on the primary constraint surface and so we can arbitrarily (but smoothly) extend the function $f^H$ off the constraint surface to define it on the full phase space. In fact this arbitrariness can be quantified by using using the following theorem[2].

**Theorem 2.1.** *If a smooth phase space function $g$ vanishes on the constraint surface defined by $\phi_\mu = 0$ then $g = g^\mu\phi_\mu$ for some functions $g^\mu$.*

Hence the function $f^H$ is only unique up to linear combinations of the primary constraints, which vanish under the pullback to $TQ$. The condition that $f^L$ be constant on the leaves of the foliation is

$$\Gamma_\mu f^L = 0,\ \mu = 1, \ldots, k \tag{2.10}$$

i.e. $f^L$ does not change in any of the directions tangent to the leaf of the foliation and hence must be constant.

This condition can be used to show that the Lagrangian energy function $E := \dot{q}^i \frac{\partial L}{\partial\dot{q}^i} - L(q, \dot{q})$ is a projectable function, one has:

$$\begin{aligned}
\Gamma_\mu E &= \left(\frac{\partial\phi_\mu}{\partial\pi_j}|_{\pi=\hat{\pi}}\right) \frac{\partial}{\partial\dot{q}^j}\left(\dot{q}^i\frac{\partial L}{\partial\dot{q}^i} - L(q,\dot{q})\right) \\
&= \left(\frac{\partial\phi_\mu}{\partial\pi_j}|_{\pi=\hat{\pi}}\right)\left(\frac{\partial L}{\partial\dot{q}^j} + \dot{q}^i\frac{\partial^2 L}{\partial\dot{q}^j\partial\dot{q}^i} - \frac{\partial L}{\partial\dot{q}^j}\right) \\
&= \dot{q}^i\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\left(\frac{\partial\phi_\mu}{\partial\pi_j}|_{\pi=\hat{\pi}}\right) \\
&= \dot{q}^i W_{ij}\frac{\partial\phi_\mu}{\partial\pi_j}|_{\pi=\hat{\pi}} \\
&= 0
\end{aligned} \tag{2.11}$$

where we have used the fact that $\frac{\partial\phi_\mu}{\partial\pi_j}$ are null eigenvectors of the Hessian, (2.7). Hence one can conclude there exists a function on $T^\star Q$, the canonical Hamiltonian $H_c(q, \pi)$, which satisfies $E = \mathcal{F}L^\star H_c$ but which in the singular case is not unique, i.e. $H_c + \lambda^\mu\phi_\mu$ is also a possible choice, where $\lambda^\mu(q, \dot{q}, t)$. The fact that the canonical Hamiltonian is not unique in the singular case and depends upon arbitrary functions is the important difference between the regular and singular cases and it is the fundamental reason for gauge freedom to appear in the constrained formalism, [10].

[2]For proof see Theorem 1.1 in [7].

Hence in the singular case one can make an ansatz for the Hamilton-Dirac (H-D) equations[3]:

$$\dot{q}^i = \left\{q^i, H_c + \lambda^\mu \phi_\mu\right\} = \left(\left\{q^i, H_c\right\} + \lambda^\mu \left\{q^i, \phi_\mu\right\}\right)|_{\Gamma_p} \tag{2.12}$$
$$\dot{\pi}_i = \left\{\pi_i, H_c + \lambda^\mu \phi_\mu\right\} = \left(\left\{\pi_i, H_c\right\} + \lambda^\mu \left\{\pi_i, \phi_\mu\right\}\right)|_{\Gamma_p} \tag{2.13}$$
$$0 = \phi_\mu(q, \pi),\ \mu = 1, \ldots, k \tag{2.14}$$

and in fact this turns out to be correct, in that one can show solutions of the H-D equations are equivalent to solutions of the Euler Lagrange equations. Note the second equality holds because the constraints are implemented by (2.14). The evolution for any phase space function $F$ immediately follows as:

$$\dot{F} = \left(\left\{F, H_c\right\} + \lambda^\mu \left\{F, \phi_\mu\right\}\right)|_{\Gamma_p} \tag{2.15}$$

The H-D equations split into differential (2.12), (2.13) and algebraic (2.14) parts but which are coupled in the sense that the constraints may be satisfied for particular initial conditions $q^i(0), \pi_j(0)$ but not satisfied at a later time for solutions to the differential equations (2.12) and (2.13) and in this way the constraints can place severe restrictions on the existence of solutions to the H-D equations.

## 2.2 Dirac-Bergmann algorithm & Classification of Constraints

We stress that the H-D equations above completely define the dynamics of our singular system. However, Dirac developed a way to modify these equations into a more convenient form, decoupling the algebraic component, and in the process discovering an important classification of constraints in the theory.

First note to satisfy the H-D equations the constraints must be preserved for all points $\left(q^i(t), \pi_i(t)\right)$ on a solution trajectory and this implies the constraints must be preserved under time evolution, i.e.

$$\dot{\phi}_\mu = \left(\left\{\phi_\mu, H_c\right\} + \lambda^\nu \left\{\phi_\mu, \phi_\nu\right\}\right)|_{\Gamma_p} = 0\,. \tag{2.16}$$

Let $C_{\mu\nu} := \left\{\phi_\mu, \phi_\nu\right\}$ we can consider two cases here: i.) $detC\,|_{\Gamma_p} \neq 0$ and ii.) $detC\,|_{\Gamma_p} = 0$.

In the first case we may fix the arbitrary functions $\lambda^\mu$ since we have

$$\lambda^\nu = -C^{\nu\mu} \left\{\phi_\mu, H_c\right\} \tag{2.17}$$

where $C^{\nu\mu}$ is the inverse of $C_{\mu\nu}$. The dynamics is now deterministic since we have identified all the previously arbitrary functions by requiring the preservation of constraints. For case i.) the H-D equations for an arbitrary phase space function become

$$\dot{f} = \left(\left\{f, H_c\right\} - \left\{f, \phi_\nu\right\} C^{\nu\mu} \left\{\phi_\mu, H_c\right\}\right)|_{\Gamma_p} \tag{2.18}$$

Unfortunately, case (ii.) is not so straightforward when $detC\,|_{\Gamma_p} = 0$ the function $\lambda^\mu$ cannot be fixed and one can only solve (2.16) by requiring certain relations hold between the phase space variables. Either these relations will be automatically fulfilled, e.g. they may be equivalent to existing primary constraints or one will be led to new constraints called secondary constraints $\phi_A, A = k+1, \ldots, k+M_1 \equiv J_1$. The primary and secondary constraints $\phi_A, A = 1, \ldots, J_1$ define a sub-manifold $\Gamma_1 \subset \Gamma_p$.

[3]Strictly speaking the Poisson bracket in $\{\cdot, \lambda^\mu\}\,\phi_\mu$ is not defined as $\lambda^\mu(q, \dot{q}, t)$ and we cannot invert for all the velocities in terms of the momenta to be able to view it as a proper phase space function. However, Poisson brackets of this kind will always have a coefficient equal to a constraint and because we are only interested in the the case where the constraints are implemented we can simply ignore this term as it will have a coefficient of zero.

One now has to repeat the process and check for preservation of the secondary constraints on $\Gamma_1$, and again this will either result in further secondary[4] constraints or restrictions on the $\lambda^\mu$. This process has to repeated every time new constraints are generated. The process stops after either an inconsistency is met or one has a discrete set of points remaining in phase space or, the case of interest, all constraints are preserved under evolution and one is left with a submanifold $\Gamma_C \subset \Gamma_p$ the final constraint surface defined by the constraints

$$\phi_A = 0,\ A = 1, \ldots, k + M \equiv J\ . \tag{2.19}$$

One is guaranteed that this algorithm will terminate after a finite number of steps because at each step a minimum of one new constraint is found and so after $2n - k$ steps one will have at least $2n - k$ secondary constraints which combined with the initial $k$ primary constraints define a finite number of points in the $2n$ dimensional phase space.

In the case of interest there exist functions $\lambda^\mu$ such that

$$\dot{\phi}_A = \left( \{\phi_A, H_c\} + \sum_{\nu=1}^{k} \lambda^\nu \{\phi_A, \phi_\nu\} \right) |_{\Gamma_C} = 0,\ A = 1, \ldots, J \tag{2.20}$$

the general solution for the functions $\lambda^\nu$ can be expressed as

$$\lambda^\nu = U^\nu + V^\nu \tag{2.21}$$

where $U^\nu$ is a particular solution to the inhomogeneous equation (2.20) and $V^\nu$ is a general solution to the homogenous equation

$$V^\nu \{\phi_A, \phi_\nu\} |_{\Gamma_C} = 0 \tag{2.22}$$

this general solution may be expressed as a linear combination of independent solutions, i.e. $V^\nu = v^a V_a^\nu$ where the number of independent solutions is equal to the dimension of the kernel of the rectangular $J \times k$ matrix $\tilde{C}_{A\nu} := \{\phi_A, \phi_\nu\}$ and $dim Ker \tilde{C} = k - Rank \tilde{C}$. Hence,

$$0 = V_a^\nu \{\phi_A, \phi_\nu\} |_{\Gamma_C} = \{\phi_A, \gamma_a\} |_{\Gamma_C} \tag{2.23}$$

where $\gamma_a := V_a^\nu \phi_\nu$. We have thus shown that the combinations of primary constraints $\gamma_a$ Poisson commute on the final constraint surface $\Gamma_C$ with all of the constraints of the theory and because $V_a^\nu$ form a basis of the kernel of $\tilde{C}_{A\nu}$ the $\gamma_a$ form a complete set of primary constraints with this property.

This property motivates the definition of a *first class* function as a phase space function, which Poisson commutes with all the constraints on the constraint surface $\Gamma_C$. So we can now state that the combinations of primary constraints $\gamma_a$ form a complete set of primary first class constraints. One can show that the set of first class functions is closed under the Poisson bracket. A function that is not first class will be called *second class*. So the constraints have now been divided into two sets first and second class, let the total number of first class constraints be $P$ and the total number of second class constraints be $N$.

These definitions may be used to transform the H-D equations into a form where the dynamics is decoupled from the constraints and where the number of arbitrary functions is made explicit. First we shall extend the primary first class basis $\gamma_a$ to $\gamma_I, I = 1, \ldots, P$ to form a basis set for all first class constraints and define $\chi_\alpha$ as a basis set for all second class constraints. One can immediately deduce that the Poisson bracket matrix of second class constraints $\Delta_{\alpha\beta} := \{\chi_\alpha, \chi_\beta\}$

[4]Sometimes these constraints are called tertiary but we shall stick to the convention that all non-primary constraints are secondary.

is non-singular. If this were not the case then i.) there would exist a null eigenvector $p^\alpha \Delta_{\alpha\beta} = 0$, which would imply $\{p^\alpha \chi_\alpha, \chi_\beta\}|_{\Gamma_C} = 0$ and hence the second class constraint $p^\alpha \chi_\alpha$ Poisson commutes with all the second class constraints on $\Gamma_C$ and ii.) since $p^\alpha \chi_\alpha$ commutes with the first class constraints (by their definition) it must be a first class constraint itself and we have a contradiction.

Now consider the preservation of the constraints defined by (2.20), using the general form of the solution for $\lambda^\nu$, (2.21) and (2.23), we can express it as

$$\dot{\phi}_A = \left( \{\phi_A, H_c\} + \sum_{\nu=1}^{k} \tilde{\lambda}^a \{\phi_A, \gamma_a\} + \tilde{\lambda}^{\alpha_p} \left\{\phi_A, \chi_{\alpha_p}\right\} + v^a \{\phi_A, \gamma_a\} \right) |_{\Gamma_C} = 0, \; A = 1, \ldots, J \tag{2.24}$$

where we have used our basis to write $U^\nu \phi_\nu = \tilde{\lambda}^a \gamma_a + \tilde{\lambda}^{\alpha_p} \chi_{\alpha_p}$ and where the indices $\alpha_p$ and $a$ are summed over all independent primary second class constraints and all independent primary first class constraints respectively. These equations imply for the first class constraints $\gamma_I$

$$0 = \{\gamma_I, H_c\}|_{\Gamma_C}, \; \forall I = 1, \ldots P \tag{2.25}$$

and for the second class constraints

$$0 = \left( \{\chi_\alpha, H_c\} + \tilde{\lambda}^{\alpha_p} \left\{\chi_\alpha, \chi_{\alpha_p}\right\} \right) |_{\Gamma_C}, \forall \alpha = 1, \ldots, N \, . \tag{2.26}$$

One can use the result that $\Delta_{\alpha\beta}$ is invertible to write

$$\left( \Delta^{\alpha\beta} \{\chi_\beta, H_c\} \right) |_{\Gamma_C} = -\delta^\alpha_{\alpha_p} \lambda_2^{\alpha_p} \tag{2.27}$$

and hence one can determine the multipliers for all the primary second class constraints, i.e.

$$\tilde{\lambda}^{\alpha_p} = \Delta^{\alpha_p \beta} \{\chi_\beta, H_c\}, \; \forall \alpha_p \, . \tag{2.28}$$

We can now use (2.27) to deduce that

$$\left( \Delta^{\alpha_s \beta} \{\chi_\beta, H_c\} \right) |_{\Gamma_C} = 0 \tag{2.29}$$

where the index $\alpha_s$ runs over all secondary second class constraints only.

Finally we can express the H-D equations, using (2.29), and re-defining the arbitrary $v^a \to v^a + \tilde{\lambda}^a$ to get

$$\begin{aligned} \dot{q}^i &= \left\{q^i, H_c\right\} - \left\{q^i, \chi_\alpha\right\} \Delta^{\alpha\beta} \{\chi_\beta, H_c\} + v^a \left\{q^i, \gamma_a\right\} && (2.30) \\ \dot{\pi}_i &= \{\pi_i, H_c\} - \{\pi_i, \chi_\alpha\} \Delta^{\alpha\beta} \{\chi_\beta, H_c\} + v^a \{\pi_i, \gamma_a\} && (2.31) \\ 0 &= \chi_\alpha && (2.32) \\ 0 &= \gamma_a && (2.33) \end{aligned}$$

where the summation over $\alpha, \beta$ is over all second class constraints and the summation over $a$ is over the primary first class constraints and $v^a(q, \dot{q}, t)$ are arbitrary functions left unspecified by the analysis. One can simplify the form of these equations by defining:

$$\{\cdot, H_c\}^\star := \{\cdot, H_c\} - \{\cdot, \chi_\alpha\} \Delta^{\alpha\beta} \{\chi_\beta, H_c\} \tag{2.34}$$

and

$$H_c' := H_c - \chi_\alpha \Delta^{\alpha\beta} \{\chi_\beta, H_c\} \tag{2.35}$$

where $\{\cdot,\cdot\}^\star$ is known as the Dirac bracket[5]. The object $H_c'$ is a first class function, one can see this because i.) by definition it Poisson commutes with the second class constraints on $\Gamma_C$, and ii.) the preservation of the first class constraints, in (2.25), implies $\{H_c',\gamma_a\}|_{\Gamma_C}=0$ and hence $H'$ indeed Poisson commutes with all constraints. Furthermore because of (2.27) $H_c'$ is equal to $H_c$ up to a combination of primary constraints on $\Gamma_C$ and by our earlier reasoning is a canonical Hamiltonian, i.e. it pulls back to the Lagrangian energy function.

In order to simplify the equations Dirac also introduced the following notation for phase space functions, which are equal on the constraint surface we write

$$f\approx g\Leftrightarrow(f-g)|_{\Gamma_C}=0\Leftrightarrow f-g=\lambda^A\phi_A \tag{2.36}$$

and call this a *weak equality.* The second relation follows from Theorem 2.1 and where the index $A$ is summed over all constraints defining $\Gamma_C$ and $\lambda^A$ are arbitrary. There is also a notion of *strong equality*, between phase space functions which is defined as

$$f\cong g\Leftrightarrow f\approx g\,\text{and}\,df\approx dg \tag{2.37}$$

i.e. both the functions and derivatives are equal on the constraint surface.

Hence the H-D equations can be equivalently expressed as

$$\begin{aligned}\dot{q}^i&\approx\left\{q^i,H_c\right\}^\star+v^a\left\{q^i,\gamma_a\right\} &(2.38)\\ \dot{\pi}_i&\approx\left\{\pi_i,H_c\right\}^\star+v^a\left\{\pi_i,\gamma_a\right\} &(2.39)\end{aligned}$$

or

$$\begin{aligned}\dot{q}^i&\approx\left\{q^i,H_c'\right\}+v^a\left\{q^i,\gamma_a\right\} &(2.40)\\ \dot{\pi}_i&\approx\left\{q^i,H_c'\right\}+v^a\left\{\pi_i,\gamma_a\right\} &(2.41)\end{aligned}$$

where in the latter pair we can impose $\chi_\alpha\left\{\cdot,\Delta^{\alpha\beta}\left\{\chi_\beta,H_c\right\}\right\}\approx 0$.

The final set of equations have a number of arbitrary functions $v^a$ equal to the number of independent primary first class constraints. This represents the 'core' indeterminacy that is present in our constrained system. Note that not all constrained systems have this indeterminacy, if there are no primary first class constraints then the evolution is completely deterministic and governed by (2.18). If one has a mixed first and second class system then by using the Dirac bracket one can use (2.38) and (2.39) to describe the system *as if* it were just a first class system but with a modified Poisson bracket. In fact the Dirac bracket can be interpreted as the Poisson bracket induced from $\{\cdot,\cdot\}$ on the constraint surface $\Gamma_C$. In more geometric terms it is the pullback to $\Gamma_C$ of the symplectic 2-form[6] $\omega$ defined on phase space $T^\star Q$, see [7, 9]. Alternatively in the mixed case one can avoid the use of the Dirac bracket and instead use a modified Hamiltonian $H_c'$ which is first class and canonical. Fortunately, most of the theories we shall deal with only have first class constraints, e.g. GR in the ADM formulation we shall study in the next section is of this type as are Yang Mills theories.

We may conclude that the Poisson bracket $\{\gamma_a,H_c'\}$ is a combination of first class constraints because i.) $H_c'$ is first class and hence its Poisson bracket with a first class constraint must be a first class function and ii.) the consistency of first class constraints, (2.25) implies that $H_c'$ weakly Poisson commutes with first class constraints and hence $\{\gamma_a,H_c'\}$ must be a linear combination of first class constraints only. The Dirac bracket $\{\gamma_a,H_c\}^\star$ also gives a combination of first class constraints for the same reason.

[5]The Dirac bracket satisfies all of the defining properties of a Poisson bracket.

[6]A symplectic 2 form $\omega$ is a closed and non-degenerate 2-form defined on phase space.

The main advantage of this system of equations over (2.12) to (2.14) other than the precise identification of where the arbitrariness in the dynamics is located is that there is now a decoupling between the differential and algebraic parts of the system. One is guaranteed that if the constraints are satisfied for the initial conditions $\left(q^i(0), \pi_i(0)\right)$ then they will be guaranteed to be satisfied for the later times $t > 0$ for all solutions to the differential equations (2.30) and (2.31) for arbitrary $v^a$.

## 2.3 Definitions of gauge symmetry

### 2.3.1 Dirac's point gauge transformations

The most important point about the final set of equations is that they contain a number of arbitrary functions, $v^a$. This means that the evolution of $q(t), \pi(t)$ is no longer (as compared to a regular system or one with only second class constraints) deterministic. In particular starting from fixed initial conditions it is clear that different choices of the function $v^a$ will trace out different solution trajectories in phase space. It was this observation that led Dirac to argue that in order to maintain a deterministic classical physics it must be that a physical state is no longer associated to a single point in phase space but is rather represented by an equivalence class of such points, points in this equivalence class will be by definition *gauge* equivalent, sometimes this equivalence class is called a *gauge orbit.* In this way one can associate a physical state to a single point in phase space but not vice versa.

We now identify specifically the relationship between these gauge equivalent points and the method by which all the points in a gauge orbit can be reached. There are essentially three processes that will allow us to identify gauge transformations on phase space, i.e. canonical transformations within gauge orbits.

Firstly consider a system with first and second class constraints at time $t_0$ with initial conditions $q^i(t_0)$ and $\pi_i(t_0)$ then the system will have evolved at time $t_0 + \delta t$ into the following state:

$$\begin{aligned} q^i(t_0 + \delta t) &= q^i(t_0) + \delta t\, \dot{q}^i(t_0) \\ &= q^i(t_0) + \delta t \left( \left\{q^i, H_c\right\}^\star + v^a \left\{q^i, \gamma_a\right\} \right)|_{t=t_0} \end{aligned} \tag{2.42}$$

and similarly for the momentum

$$\pi_i(t_0 + \delta t) = \pi_i(t_0) + \delta t \left( \{\pi_i, H_c\}^\star + v^a \{\pi_i, \gamma_a\} \right)|_{t=t_0} \tag{2.43}$$

if one chooses a different multiplier $v'^a$ then the new solution trajectories $q'^i(t), \pi'_i(t)$ at time $t_0 + \delta t$ will differ by the amount

$$\delta q^i := q^i(t_0 + \delta t) - q'^i(t_0 + \delta t) = \delta t \left(v^a - v'^a\right) \left\{q^i, \gamma_a\right\}|_{t=t_0} \tag{2.44}$$

$$\delta \pi_i := \pi_i(t_0 + \delta t) - \pi'_i(t_0 + \delta t) = \delta t \left(v^a - v'^a\right) \{\pi_i, \gamma_a\}|_{t=t_0}\ . \tag{2.45}$$

Dirac then argued that in order to maintain a deterministic physics from the common initial conditions the points in phase space $\left(q^i(t_0 + \delta t), \pi_i(t_0 + \delta t)\right)$ and $\left(q'^i(t_0 + \delta t), \pi'_i(t_0 + \delta t)\right)$ must represent the same physical state and hence the point transformations generated by $\left\{q^i, \gamma_a\right\}$ and $\{\pi_i, \gamma_a\}$ do not change the physical state at time $t_0 + \delta t$, and for this reason they are identified as *infinitesimal gauge transformations.* Of course one can make the same argument when dealing with the evolution of any dynamical variable and one identifies the primary first class constraints $\gamma_a$ as the generators of gauge transformations at a particular time.

Secondly consider the case where i.) two successive gauge transformations are performed with infinitesimal parameters $\epsilon^a, \eta^a$ respectively and ii.) repeated in the opposite order on the same initial state. The result of i.) on a dynamical variable $F$ will be

$$\begin{aligned} F_{\epsilon^a,\eta^a} : &= F_{\epsilon^a} + \left\{F_{\epsilon^a}, \eta^b\gamma_b\right\} + O(\eta^2) \\ &= F + \{F, \epsilon^a\gamma_a\} + \left\{F + \{F, \epsilon^a\gamma_a\}, \eta^b\gamma_b\right\} + O(\epsilon^2, \eta^2) \end{aligned}$$

Hence the difference in a dynamical variable $F$ at the end of these two sets of transformations will be

$$\begin{aligned} \delta F &= \left\{\{F, \epsilon^a\gamma_a\}, \eta^b\gamma_b\right\} - \left\{\left\{F, \eta^b\gamma_b\right\}, \epsilon^a\gamma_a\right\} + O(\epsilon^2, \eta^2) \\ &= \epsilon^a\eta^b \left\{\{\gamma_a, \gamma_b\}, F\right\} + O(\epsilon^2, \eta^2) \end{aligned} \tag{2.46}$$

where in the second line we have used the Jacobi identity.

Given that physically we expect the difference between two gauge transformations to leave the physical state unchanged we conclude that the Poisson bracket of the primary first class constraints $\{\gamma_a, \gamma_b\}$ also generates gauge transformations. Given that this Poisson bracket is i.) by definition weakly zero (and hence a linear combination of all constraints) and ii.) a first class function one can conclude that it must be a linear combination of first class constraints.

Finally we can consider the sequence of transformations: i.) gauge transform with a multiplier $\epsilon^a$ and ii.) evolve for time $\delta t$ with a multiplier $\eta^a$. For this case we shall use the H-D equations in the form (2.40) and (2.41), i.e. with the modified Hamiltonian rather than the Dirac bracket, the result of this transformation on a dynamical variable $F$ will be

$$\begin{aligned} F_{\epsilon^a;\delta t,\eta^a} &:= F_{\epsilon^a} + \delta t\left(\{F_{\epsilon^a}, H_c'\} + \eta^a\{F_{\epsilon^a}, \gamma_a\}\right) \\ &= F + \epsilon^a\{F, \gamma_a\} + \delta t\left(\{F + \epsilon^a\{F, \gamma_a\}, H_c'\} + \eta^b\{F + \epsilon^a\{F, \gamma_a\}, \gamma_b\}\right) \\ &= F + \epsilon^a\{F, \gamma_a\} + \delta t\left(\{F, H_c'\} + \eta^b\{F, \gamma_b\}\right) \\ &\quad +\delta t\left(\{\epsilon^a\{F, \gamma_a\}, H_c'\} + \eta^b\{\epsilon^a\{F, \gamma_a\}, \gamma_b\}\right) . \end{aligned} \tag{2.47}$$

If we now reverse the order of these transformations one has

$$\begin{aligned} F_{\delta t,\eta^a;\epsilon^a} : &= F_{\delta t,\eta^a} + \epsilon^a\{F_{\delta t,\eta^a}, \gamma_a\} \\ &= F + \delta t\left(\{F, H_c'\} + \eta^a\{F, \gamma_a\}\right) + \epsilon^b\{F + \delta t\left(\{F, H_c'\} + \eta^a\{F, \gamma_a\}\right), \gamma_b\} \\ &= F + \epsilon^b\{F, \gamma_b\} + \delta t\left(\{F, H_c'\} + \eta^a\{F, \gamma_a\}\right) \\ &\quad +\delta t\epsilon^b\left(\{\{F, H_c'\}, \gamma_b\} + \{\eta^a\{F, \gamma_a\}, \gamma_b\}\right) \end{aligned} \tag{2.48}$$

and hence the difference between these two sets of transformations is given by

$$\begin{aligned} \delta F &= \{\epsilon^a\{F, \gamma_a\}, H_c'\} + \eta^b\{\epsilon^a\{F, \gamma_a\}, \gamma_b\} \\ &\quad -\epsilon^a\{\{F, H_c'\}, \gamma_a\} - \epsilon^b\{\eta^a\{F, \gamma_a\}, \gamma_b\} \\ &= \{\epsilon^a\{F, \gamma_a\}, H_c'\} - \epsilon^a\{\{F, H_c'\}, \gamma_a\} \\ &\quad +\eta^b\{\epsilon^a\{F, \gamma_a\}, \gamma_b\} - \epsilon^b\{\eta^a\{F, \gamma_a\}, \gamma_b\} \\ &\approx \{\{F, \epsilon^a\gamma_a\}, H_c'\} - \{\{F, H_c'\}, \epsilon^a\gamma_a\} \\ &\quad + \left\{\{F, \epsilon^a\gamma_a\}, \eta^b\gamma_b\right\} - \left\{\{F, \eta^a\gamma_a\}, \epsilon^b\gamma_b\right\} \\ &= -\{\{\epsilon^a\gamma_a, H_c'\}, F\} - \left\{\left\{\epsilon^b\gamma_b, \eta^a\gamma_a\right\}, F\right\} \end{aligned} \tag{2.49}$$

where in the third line we used the fact that $H_c'$, and $\gamma_a$ are first class, and in the final line we used the Jacobi identity. Hence, arguing that the difference between a gauge transformation and time evolution and the reverse cannot change the physical state one concludes that the Poisson bracket $\{\epsilon^a \gamma_a, H'_c\}$ also generates gauge transformations. Recall we have already argued at the end of the last sub-section that the Poisson bracket $\{\gamma_a, H'_c\}$ is a combination of first class constraints.

We can conclude that gauge transformations are generated by i.) primary first class constraints $\gamma_a$, ii.) Poisson brackets of primary first class constraints $\{\gamma_a, \gamma_b\}$ and iii.) Poisson brackets between primary first class constraints and the Hamiltonian $H'$. The transformations generated by ii.) and iii.) are equivalent to linear combinations of first class constraints however they are not necessarily primary first class constraints and some (though not necessarily all) secondary first class constraints may also be generated. Motivated by this Dirac conjectured that all first class constraints generated gauge transformations. This conjecture has held true for all physically realistic theories[7] studied to date and may be proven under certain simplifying assumptions. We shall assume it throughout this work.

### 2.3.2 Gauge transformations as a map from solutions to solutions

There is another notion of gauge transformation, [10], in the canonical formalism, which is related to the idea of *symmetry* as a transformation of solutions to the equations of motion into other solutions. In the canonical formalism this involves thinking of a gauge transformation as a map from one solution trajectory in phase space to another. More precisely consider two solution trajectories $q(t)$ and $q'(t)$, they are to be considered gauge equivalent if at each time $t$ there is a gauge transformation, in the sense of Dirac, between the points of $q(t)$ and $q'(t)$.

Consider two such trajectories with the same initial conditions, which satisfy the equations of motion:

$$\begin{aligned} \dot{q}(t) &\approx \{q, H'_c\} + v^a \{q, \gamma_a\} \\ \dot{q}'(t) &\approx \{q', H'_c\} + v'^a \{q', \gamma_a\} \end{aligned}$$

then we can define the equal time variation $\Delta q(t) := q'(t) - q(t)$. The variation $\Delta q(t)$ can be generated as a canonical transformation, it is the canonical transformation obtained by starting from $q'(t)$ and evolving backward in time to $q'(t_0) = q(t_o)$ and then evolving forward in time to $q(t)$ and therefore may be written as:

$$\Delta q(t) = \{q, G(t)\} \tag{2.50}$$

where $G(t) = G(q, \pi; t)$ is the *gauge generator* and is an explicitly time dependent phase space function. We can deduce that since $G(t)$ is a vector field that maps one solution trajectory to another it must preserve the constraints of the theory and therefore be a first class function.

In [10] using the fact that $\Delta q(t)$ is an equal time variation a number of properties are derived for the gauge generator $G(t)$

$$\frac{\partial G}{\partial t} + \{G, H'_c\} \cong pfcc \tag{2.51}$$

$$\{G, \gamma_a\} \cong pfcc \tag{2.52}$$

where the notation $pfcc$ means any linear combination of primary first class constraints and we have a strong equality. These two equations mean that the gauge generator is a constant of motion $\dot{G} \cong pfcc$ which is also the case for the Dirac gauge transformations where of course all first class constraints satisfy $\dot{\gamma}_a \approx 0$. However, in this approach the gauge generator $G(t)$ satisfies a strong equality which distinguishes it from the first class constraints.

[7] A counterexample to Dirac's conjecture is given in [7] involving the Lagrangian $L = \frac{1}{2}e^y \dot{x}^2$.

The conditions on the first class function $G(t)$ can be viewed as defining a canonical generator of symmetries, that maps solutions to solutions, whether it be a global or local symmetry. $G(t)$ becomes a gauge generator when it depends upon arbitrary functions.

The following ansatz for the general form of $G(t)$ has been proposed and it can be shown that there exist solutions to the gauge generator conditions derived from it:

$$G(t) = G_0\xi(t) + G_1\dot{\xi}(t) + G_2\ddot{\xi}(t) + \cdots + G_N\xi^{(N)}(t) \tag{2.53}$$

where $\xi(t)$ are arbitrary and $G_i$ are phase space functions to be determined.

Substituting this ansatz into i.) the condition (2.52) implies

$$\{G_i, \gamma_a\} \cong pfcc \tag{2.54}$$

and ii.) the condition (2.51) implies, by considering the coefficients of $\xi^{(i)}(t)$, and noting that the relations must hold separately for all time $t$, that first

$$\begin{aligned} G_N\xi^{(N+1)}(t) &\cong pfcc \\ \Rightarrow G_N &\cong pfcc \end{aligned} \tag{2.55}$$

and subsequently that

$$\begin{aligned} \left(G_{N-1} + \{G_N, H_c'\}\right)\xi^{(N+1)}(t) &\cong pfcc \\ \Rightarrow G_{N-1} + \{G_N, H_c'\} &\cong pfcc \end{aligned} \tag{2.56}$$

and for $i = 1, \ldots, N$

$$\Rightarrow G_{i-1} + \{G_i, H_c'\} \cong pfcc \tag{2.57}$$

and finally one has

$$\begin{aligned} \{G_0, H_c'\}\,\xi(t) &\cong pfcc \\ \Rightarrow \{G_0, H_c'\} &\cong pfcc\,. \end{aligned} \tag{2.58}$$

Hence $G_N$ is strongly equal to a primary first class constraint (i.e. up to terms quadratic in the constraints), and the $G_i$ $i < N$ will correspond to secondary (if there are any) first class constraints (up to $pfcc$ terms) because i.) $\{G_N, H_c'\}$ generates the time evolution of the primary first class constraint $G_N$ which must be weakly zero, i.e. a combination of constraints and ii.) these constraints can only be first class because $H_c'$ is a first class function and the Poisson bracket of first class functions is first class. So using this approach one concludes that the gauge generator $G(t)$ is a particular combination of first class constraints.

Using the ansatz for $G(t)$ one can make contact with Dirac's notion of point gauge transformations quite easily. Consider a point gauge transformation relating two points at the time $t_0 + \delta t$, as considered earlier, generated by a first class constraint. In the current framework this can be derived from

$$\delta q(t) = \{q, G(t)\} = \sum_{i=0}^{N} \{q, G_i\}\,\xi^{(i)}(t) \tag{2.59}$$

by using the fact that the Dirac gauge transformation does not change the state at time $t_0$, we must have $\delta q(t_0) = 0$, this implies $\xi^{(i)}(t_0) = 0$. At the infinitesimally later time $t_0 + \delta t$ one has to first order

$$\xi^{(i)}(t_0 + \delta t) = \xi^{(i)}(t_0) + \delta t\xi^{(i+1)}(t_0) \tag{2.60}$$

hence one may conclude $\xi^{(i)}(t_0+\delta t)=0,\ i=0,\dots,N-1$ and $\xi^{(N)}(t_0+\delta t)=\delta t\xi^{(N+1)}(t_0)$ and where $\xi^{(N+1)}(t_0)$ is arbitrary. So we have $G(t_0+\delta t)=\delta t\xi^{(N+1)}(t_0)G_N$ and hence

$$\delta q(t_0+\delta t)=\delta t\xi^{(N+1)}(t_0)\{q,G_N\} \tag{2.61}$$

since $G_N$ is a primary first class constraint we have recovered the earlier result that primary first class constraints generate point gauge transformations.

### 2.3.3 Geometry of gauge orbits

Before applying the formalism we have discussed to Yang Mills theory, we would like to describe some of the above results in geometric terms, which help further understand the picture of the dynamics and also enable one to determine the number of physical degrees of freedom.

We recall that by *symplectic structure* we mean a closed and non-degenerate two form $\omega_{\mu\nu}$, $\mu\nu=1,\dots,2n$ defined on the $2n$ dimensional phase space, $T^\star Q$. This structure defines i.) the (Hamiltonian) vector field $X^\mu_f$ associated with a phase space function $f$ as

$$\begin{aligned}\omega(X^\mu_f,\cdot) &:= df &(2.62)\\ \Rightarrow X^\mu_f\omega_{\mu\nu} &= \partial_\nu f &(2.63)\\ \Rightarrow X^\mu_f &= -\omega^{\mu\nu}\partial_\nu f &(2.64)\end{aligned}$$

and ii.) the Poisson bracket between two phase pace functions as

$$\begin{aligned}\{f,g\} &:= \omega(X_f,X_g)\\ &= X^\mu_f\omega_{\mu\nu}X^\nu_g\\ &= -\partial_\nu f\,\omega^{\nu\gamma}\partial_\gamma g\\ &= X^\nu_g\partial_\nu f &(2.65)\end{aligned}$$

this implies that the Poisson bracket $\{f,g\}$ can be interpreted as the change in $f$ in the direction defined by the Hamiltonian vector field associated to $g$.

In order to understand the symplectic geometry of constrained Hamiltonian systems one has to understand the nature of the induced 2-form $\tilde\omega$ on the constraint surface $\Gamma_c$, this can be formally defined as

$$\tilde\omega := i^\star\omega \tag{2.66}$$

where $i$ is the inclusion map $i:\Gamma_c\to T^\star Q$. This induced form inherits the closure property from $\omega$ but it is not necessarily non-degenerate.

In the case of first class systems the Hamiltonian vector fields associated to the constraints $\gamma_a$ are tangent to the constraint surface $\Gamma_c$, one can see this as

$$X^\mu_{\gamma_b}\partial_\nu\gamma_a=\{\gamma_a,\gamma_b\}\approx 0 \tag{2.67}$$

and this implies that all constraints are weakly constant in the direction of the Hamiltonian vector fields $X^\mu_{\gamma_b}$ and hence these vectors must be tangent to the constraint surface. The following two theorems allow us to complete the picture of first class constraints as the generators of gauge transformations.

**Theorem 2.2.** *For a first class system the induced form is maximally degenerate, i.e. it has rank equal to $2n-2m$, where $2n$ is the dimension of the phase space and $m$ is the total number of (first class) constraints. Furthermore the null directions are spanned by the Hamiltonian vector fields associated to the first class constraints*[8].

[8] For a proof see Theorem 2.1 in [7].

**Theorem 2.3.** *The vector fields $X^{\mu}_{\gamma_a}$ associated to the first class system are 'surface forming', i.e. integrate to form $m$ dimensional surfaces in the constraint surface $\Gamma_c$.*

*Proof.* The proof uses Frobenius' theorem, which states that a set of vector fields are 'surface forming' if and only if the set of vectors are closed under the Lie bracket. We use the following relation between the Lie bracket and the Poisson bracket[9]

$$[X_f, X_g] = -X_{\{f,g\}} \tag{2.68}$$

and let $\{\gamma_a, \gamma_b\} = f^c_{ab}\gamma_a$ then immediately one has

$$[X_{\gamma_a}, X_{\gamma_b}] = -X_{f^c_{ab}\gamma_c} \tag{2.69}$$

and then by definition

$$\begin{aligned} -X^{\mu}_{f^c_{ab}\gamma_c} &= \omega^{\mu\nu}\partial_\nu(f^c_{ab}\gamma_c) \\ &= -f^c_{ab}X^{\mu}_{\gamma_c} + \gamma_c\omega^{\mu\nu}\partial_\nu(f^c_{ab}) \\ &\approx -f^c_{ab}X^{\mu}_{\gamma_c} \end{aligned} \tag{2.70}$$

and hence one has that the vector fields are closed, on the constraint surface, and so are surface forming. □

The surface forming property of the Hamiltonian vector fields $X_{\gamma_a}$ only applies off the constraint surface if the functions $f^c_{ab}$ are constant, i.e. only when the Poisson algebra closes with structure constants as opposed to structure functions[10].

So the final picture of first class systems is that i.) motion in phase space is restricted to the constraint surface $\Gamma_c$, ii.) that this $2n-m$ dimensional constraint surface is made up of (foliated by) $m$ dimensional surfaces, formed by the integral curves of the $m$ Hamiltonian vector fields $X^{\mu}_{\gamma_a}$ and iii.) because these vector fields generate gauge transformations we have a picture of each $m$ dimensional surface as representing a gauge orbit of physically equivalent states and by Theorem 2.2 we see that these gauge directions correspond to null directions in the induced form $\tilde{\omega}$.

As the induced form is degenerate it does not define a symplectic form and hence there is no Poisson bracket definable on this surface. In order to do so one must construct the reduced phase space by forming the quotient space of $\Gamma_c$ with the gauge orbits and thereby identifying all points on each orbit.

The picture with regard to pure second class systems, with constraints $\chi_\alpha$, is different. One can deduce that the vector fields associated to second class constraints are not tangent to the constraint surface, one has

$$\forall\alpha\, \exists\beta\; s.t.\; 0 \neq \{\chi_\alpha, \chi_\beta\} = X^{\mu}_{\chi_\alpha}\partial_\nu\chi_\beta \tag{2.71}$$

(otherwise $\chi_\alpha$ would not be second class) and hence there exists at least one one constraint which changes in the direction of each vector field $\chi_\alpha$ and therefore the vectors are not tangent to $\Gamma_c$.

**Theorem 2.4.** *For second class systems the induced two form has maximal rank and is non-degenerate. This means it defines a symplectic form, which is realised by the Dirac bracket.*[11]

[9] See for example section 2.7 in [11].

[10] We shall see in the next section that GR is a theory where the Poisson algebra only closes with structure functions.

[11] See Chapter 2 in [7].

Hence there are no null directions on the induced form mathematically co-inciding with the fact that physically we have a deterministic physics for second class systems, computed by the Dirac bracket.

In the case of a mixed first, second class system, [7], one finds that the induced form has a rank $2n-m-k$, where $m$ is the total number of constraints and $k$ is the number of first class constraints. This means that there exist $k$ null directions tangent to the constraint surface, which again define the gauge orbits.

The above results enable one to compute the number of physical degrees of freedom associated with a constrained Hamiltonian system

$$2 \times (No.\, of\, physical\, dofs) = 2n - m - k \tag{2.72}$$

so one has to subtract the total number of constraints to find the dimension of the constraint surface and then subtract only the number of first class constraints in order to find the dimension of the reduced phase space. This reduced phase space will have dimension equal to twice the number of physical degrees of freedom.

We must also mention 'observables', these are defined to be gauge invariant phase space functions, which are therefore candidates for representing real physical measurements. In the notion of Dirac these phase space functions must Poisson commute with all the first class constraints. Given the geometric analysis of this sub-section we can also see that these observables will have to constant on the gauge orbits generated by the first class constraints. Similarly, gauge invariant observables in the notion of Bergmann will have to Poisson commute with the gauge generator $G(t)$, which itself is a particular sum of first class constraints. In section 4 we shall discuss relational Dirac observables for GR and first class systems in general.

## 2.4 Yang Mills as a constrained Hamiltonian system

We apply the constrained Hamiltonian formalism to a Yang Mills field with action

$$S_{YM} = -\frac{1}{4}\int d^3x dt\, tr\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{2.73}$$

where $(x^i, t)$ are coordinates on Minkowski spacetime, and the (Lie Algebra valued) Yang Mills field strength $F_{\mu\nu}$ is defined by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] && (2.74)\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + A^a_\mu A^b_\nu f^c_{ab} T_c && (2.75)\end{aligned}$$

and the gauge (potential) field $A_\mu = A^a_\mu T_a$ where $T^a$ are basis elements in the Lie algebra of the gauge group $G$, denoted $\mathfrak{g}$ chosen orthogonal with respect to the Killing metric, and the trace $tr(T^aT^b) = \delta^{ab}$. The commutator $[T_a, T_b] = if^c_{ab}T_c$ and the structure constants $f^c_{ab}$ are totally anti-symmetric.

The configuration space $Q$ of this theory is given by the space of gauge potential configurations $A_\mu(x,t)$ at a fixed time $t$, and hence the Lagrangian field velocity space is $TQ$, coordinatised by $\left(A_\mu, \dot{A}_\mu\right)$.

This action has a local gauge symmetry given by the standard gauge transformations of the gauge potential

$$\begin{aligned} \delta A^a_\mu &= -\left(\partial_\mu\theta^a + f^a_{bc}A^b_\mu\theta^c\right) && (2.76)\\ &= -\left(\partial_\mu\theta^a - i[A_\mu,\theta]^a\right) && (2.77)\\ &=: -\left(D_\mu\theta\right)^a && (2.78)\end{aligned}$$

*Claim* 2.5. The Euler Lagrange equations derived from (2.73) are

$$D_\mu F^{\mu\nu} = 0 \tag{2.79}$$

*Proof.* We have

$$\begin{aligned}
F^a_{\mu\nu} &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu f^a_{bc}\right) \\
\Rightarrow \delta F^a_{\mu\nu} &= \partial_\mu \delta A^a_\nu - \partial_\nu \delta A^a_\mu + \delta A^b_\mu A^c_\nu f^a_{bc} + A^b_\mu \delta A^c_\nu f^a_{bc} \\
&= \partial_\mu \delta A^a_\nu + A^b_\mu \delta A^c_\nu f^a_{bc} - \left(\partial_\nu \delta A^a_\mu + A^c_\nu \delta A^b_\mu f^a_{cb}\right) \\
&= D_\mu \delta A^a_\nu - D_\nu \delta A^a_\mu \, .
\end{aligned} \tag{2.80}$$

Hence, after taking the trace in (2.73) and varying the resulting action one has

$$\begin{aligned}
\delta S_{YM} &= -\frac{1}{4}\delta \int d^3x dt \, F^a_{\mu\nu} F^{\mu\nu\, a} \\
&= -\frac{1}{2}\int d^3x dt \, \delta F^a_{\mu\nu} F^{\mu\nu\, a} \\
&= -\frac{1}{2}\int d^3x dt \, F^{\mu\nu\, a} \left(D_\mu \delta A^a_\nu - D_\nu \delta A^a_\mu\right) \\
&= -\int d^3x dt \, F^{\mu\nu\, a} D_\mu \delta A^a_\nu \\
&= -\int d^3x dt \, F^{\mu\nu\, a} \left(\partial_\mu \delta A^a_\nu + A^b_\mu \delta A^c_\nu f^a_{bc}\right) \\
&= \int d^3x dt \, \partial_\mu \left(F^{\mu\nu\, a}\right) \delta A^a_\nu + \int d^3x dt \, F^{\mu\nu\, a} A^b_\mu f^c_{ba} \delta A^c_\nu \\
&= \int d^3x dt \, D_\mu F^{\mu\nu\, a} \delta A^a_\nu
\end{aligned} \tag{2.81}$$

where we have used (2.80) for the variation in the field strength, the definition of the adjoint covariant derivative and in the sixth line done an integration by parts, dropped the boundary term and also used the total anti-symmetry of the structure constants. Given arbitrary variations in the gauge potential, the field equations are indeed given by (2.79). □

Given these field equations one can observe that only $D_\mu F^{\mu i} = 0$, $i = 1, 2, 3$ will contain second time derivatives of the gauge potential and therefore represent evolution equations. The remainder $D_\mu F^{\mu 0} = D_{\mu i} F^{i0}$ contain only $(A_\mu, \dot{A}_\mu)$ and therefore are Lagrangian constraints in $TQ$.

*Claim* 2.6. As already mentioned the invariance of the action under local gauge symmetries implies, by Noether's second theorem, a generalised Bianchi identity, which we recall is an off-shell, first order linear partial differential equation involving the Euler Lagrange equations. These identities further imply a singular Hessian matrix and so we have a singular Lagrangian system. The generalised Bianchi identities for the Yang Mills field are given by:

$$D_\mu D_\nu F^{\mu\nu} = 0 \tag{2.82}$$

*Proof.* (2.82) can be proved by first observing the anti-symmetry in $F_{\mu\nu}$ implies $D_\mu D_\nu F^{\mu\nu} = \frac{1}{2}\left[D_\mu, D_\nu\right] F^{\mu\nu}$. Then by considering an arbitrary field $\theta$, which transforms in the adjoint representation one has

$$
\begin{aligned}
2D_\mu D_\nu \theta &= [D_\mu, D_\nu]\,\theta \\
&= D_\mu \left[\partial_\nu\theta - i\left[A_\nu,\theta\right]\right] - D_\nu\left[\partial_\mu\theta - i\left[A_\mu,\theta\right]\right] \\
&= \partial_\mu\partial_\nu\theta - i\partial_\mu\left[A_\nu,\theta\right] - i\left[A_\mu,\partial_\nu\theta - i\left[A_\nu,\theta\right]\right] - \partial_\nu\partial_\mu\theta \\
&\quad + i\partial_\nu\left[A_\mu,\theta\right] + i\left[A_\nu,\partial_\mu\theta - i\left[A_\mu,\theta\right]\right] \\
&= -i\left[\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu,A_\nu\right]\right)\theta - \theta\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu,A_\nu\right]\right)\right] \\
&= -i\left(F_{\mu\nu}\theta - \theta F_{\mu\nu}\right) \\
&= -i\left[F_{\mu\nu},\theta\right] \\
&= F^a_{\mu\nu}\theta^b f^c_{ab}T_c
\end{aligned}
$$

where in the second and third lines we have used the definition of the adjoint covariant derivative, and subsequently just cancelled terms. Hence, $D_\mu D_\nu F^{\mu\nu} = \frac{1}{2}F^a_{\mu\nu}F^{\mu\nu b}f^c_{ab}T_c \equiv 0$, by symmetry, anti-symmetry in the $a, b$ indices. $\square$

We now proceed with the Hamiltonian analysis the first step is to compute the canonical momenta, this can be done by using the variation in the action from above

$$
\begin{aligned}
\delta S_{YM} &= -\int d^3x dt\, F^{\mu\nu\, a} D_\mu \delta A^a_\nu \\
&= -\int d^3x dt\, \left(F^{0\nu\, a}\partial_0\delta A^a_\nu + F^{i\nu\, a}\partial_i\delta A^a_\nu - iF^{\mu\nu\, a}\left[A_\mu,\delta A_\nu\right]^a\right) \qquad (2.83)
\end{aligned}
$$

hence

$$
\pi^\mu_b \equiv \frac{\delta S_{YM}}{\delta \dot{A}^b_\mu} = -F^{0\mu\, b}\,. \qquad (2.84)
$$

Immediately, by anti-symmetry in the spacetime indices of the field strength, (2.84) implies the primary constraints

$$
\phi_b(x) \equiv \pi^0_b(x) = 0,\ b = 1,\ldots, dim\mathfrak{g}\,. \qquad (2.85)
$$

We can also state the canonical Poisson brackets as

$$
\begin{aligned}
\left\{A^a_\mu(x), \pi^\nu_b(y)\right\} &= \delta^\nu_\mu\delta^a_b\delta^3(x,y) && (2.86) \\
\left\{A^a_\mu(x), A^b_\nu(y)\right\} &= 0 && (2.87) \\
\left\{\pi^\mu_a(x), \pi^\nu_b(y)\right\} &= 0\,. && (2.88)
\end{aligned}
$$

The next step is to write down the primary Hamiltonian, which we recall is the usual canonical Hamiltonian up to an arbitrary combination of the primary constraints. Before doing this we need to express the action in terms of the phase space variables $A^a_\mu, \pi^\mu_a$ and so we first observe that the invertible momenta may be expressed

$$
\begin{aligned}
\pi^i_a &= -F^{0i}_a \\
&= -\left(\partial^0 A^{i\,a} - \partial^i A^{0\,a} + A^{0\,b}A^{i\,c}f^a_{bc}\right) \\
&= \dot{A}^a_i - \partial_i A^a_0 - A^b_i A^c_0 f^a_{bc} \\
&= \dot{A}^a_i - D_i A^a_0\,. \qquad (2.89)
\end{aligned}
$$

Hence the canonical Hamiltonian $H_c$ may be expressed, after a Legendre transform, as

$$\begin{aligned}
H_c : \quad &= \int d^3x \pi^i_a \dot{A}^a_i - \int d^3x L(A, \pi) \\
&= \int d^3x \pi^i_a \dot{A}^a_i + \frac{1}{4}\int d^3x\, F^a_{\mu\nu} F^{\mu\nu\, a} \\
&= \int d^3x \pi^i_a \dot{A}^a_i + \frac{1}{4}\int d^3x\, F^a_{ij} F^{ij\, a} + \frac{1}{2}\int d^3x\, F^a_{0i} F^{0i\, a} \\
&= \int d^3x \left(\pi^i_a \left(\pi^i_a + D_i A^a_0\right) + \frac{1}{4} F^a_{ij} F^{ij\, a} - \frac{1}{2}\pi^i_a \pi^i_a\right) \\
&= \int d^3x \left(\frac{1}{2}\pi^i_a \pi^i_a + \frac{1}{4} F^a_{ij} F^{ij\, a} + \pi^i_a D_i A^a_0\right) \\
&= \int d^3x \left(\frac{1}{2}\pi^i_a \pi^i_a + \frac{1}{4} F^a_{ij} F^{ij\, a} - D_i(\pi^i_a) A^a_0\right) 
\end{aligned} \tag{2.90}$$

where we have used (2.89), the definition of the momenta and in the final line performed an integration by parts[12] and dropped the boundary term.

Hence, we can write down the evolution equations for any phase space function as

$$\dot{F}[A, \pi] \approx \{F, H_p\} \tag{2.91}$$

where the weak equality is defined on the primary constraint surface and the primary Hamiltonian $H_p$ is

$$H_p = H_c + \int d^3x\, v^a \phi_a \tag{2.92}$$

and the $v^a$ are arbitrary functions.

Now we can impose the preservation of the primary constraints in order to determine whether Yang Mills theory is a first, second class or mixed dynamical system. The preservation of primary constraints requires

$$\begin{aligned}
0 \quad &\approx \quad \dot{\phi}_a \approx \{\phi_a(y), H_p\} \\
&= \int d^3x - D_i(\pi^i_b)(x) \left\{\pi^0_a(y), A^b_0(x)\right\} \\
&= \int d^3x D_i(\pi^i_b)(x) \left\{A^b_0(x), \pi^0_a(y)\right\} \\
&= D_i(\pi^i_a)(y) \equiv \tilde{\phi}_a(y)
\end{aligned} \tag{2.93}$$

where in the second line we have immediately dropped all terms from $H_p$ that manifestly give zero in a Poisson bracket with the primary constraints. It is clear that these relations are not automatically fulfilled and therefore constitute secondary constraints called the generalised *Gauss constraints*, because of their similarity to the Gauss constraint in electrodynamics $\nabla.\vec{E} = 0$. We now have a new constraint surface $\Gamma_1$ in phase space, where both the primary and Gauss constraints are satisfied.

Following the Dirac-Bergmann algorithm we now have to impose preservation of the Gauss constraints on $\Gamma_1$, i.e. we require

$$\begin{aligned}
0 \quad &\approx \quad \left\{D_i(\pi^i_a)(y), H_p\right\} \\
&= \int d^3x \left\{D_i(\pi^i_a)(y), \left(\frac{1}{2}\pi^j_b \pi^j_b + \frac{1}{4} F^b_{jk} F^{jk\, b} - D_j(\pi^j_b) A^b_0 + v^a \phi_a\right)\right\} .
\end{aligned} \tag{2.94}$$

[12] We have already established in the variation of the action above that the covariant derivative behaves as if we can perform an integration by parts directly.

Computing the Poisson bracket above is more involved than for the primary constraints because of the implicit presence of the gauge field in the covariant derivative. We first observe that

$$\left\{D_i(\pi_a^i)(y), v^a\phi_a\right\} \approx 0 \tag{2.95}$$

this is because the Gauss constraint only depends upon $\pi_i$ and $A_i$, the latter through the covariant derivative, which all Poisson commute with the primary constraints $\phi_a = \pi_a^0$. So we are left with three remaining terms in $H_p$, which we compute in turn. The first is

$$\begin{aligned}
\frac{1}{2}\int d^3x \left\{D_i(\pi_a^i)(y), \pi_b^j\pi_b^j(x)\right\} &= \int d^3x\pi_b^j(x)\left\{D_i(\pi_a^i)(y), \pi_b^j(x)\right\} \\
&= \int d^3x\pi_b^j(x)\left\{(A_i^d\pi_c^i)(y)f_{dc}^a, \pi_b^j(x)\right\} \\
&= \int d^3x\pi_b^j(x)\pi_c^i)(y)f_{dc}^a\left\{A_i^d(y), \pi_b^j(x)\right\} \\
&= \pi_b^i(y)\pi_c^i(y)f_{bc}^a \\
&\equiv 0
\end{aligned} \tag{2.96}$$

where in the second line we have just kept the $A_i$ term from the covariant derivative, as only this term contributes to the Poisson bracket, and in the final line used the symmetry, anti-symmetry in the $b, c$ indices. The second term is

$$\begin{aligned}
\frac{1}{4}\int d^3x \left\{D_i(\pi_a^i)(y), F_{jk}^bF^{jk\,b}(x)\right\} &= \frac{1}{2}\int d^3xF^{jk\,b}(x)\left\{D_i(\pi_a^i)(y), F_{jk}^b(x)\right\} \\
&= \frac{1}{2}\int d^3xF^{jk\,b}(x)D_i^y\left\{\pi_a^i(y), F_{jk}^b(x)\right\} \\
&= -\frac{1}{2}\int d^3xF^{jk\,b}(x)D_i^y\left[\frac{\delta F_{jk}^b(x)}{\delta A_i^a(y)}\right] \\
&= -\frac{1}{2}\int d^3xF^{jk\,b}(x)D_i^y\left[D_j\frac{\delta A_k^b(x)}{\delta A_i^a(y)} - D_k\frac{\delta A_j^b(x)}{\delta A_i^a(y)}\right] \\
&= -\int d^3x\left(F^{ij\,a}(x)D_i^yD_j\delta^3(x,y)\right) \\
&= D_i^y\int d^3xD_jF^{ij\,a}(x)\delta^3(x,y) \\
&= D_iD_jF^{ij\,a}(y) \\
&= \frac{1}{2}F_{ij}^bF^{ij\,c}f_{bc}^a \\
&\equiv 0
\end{aligned} \tag{2.97}$$

where in the fourth line we have used the result (2.80) for the variation in $F_{jk}^b$, in the sixth line done an integration by parts and in the final two lines used the generalised Bianchi identity. The third term is

$$
\begin{aligned}
\int d^3x \left\{\tilde{\phi}_a(y), -D_j(\pi^j_b)A^b_0(x)\right\} &= -\int d^3x A^b_0(x)\left\{D_i(\pi^i_a)(y), D_j(\pi^j_b)(x)\right\} \\
&= -\int d^3x A^b_0(x)\Big[\pi^{j\,f}(x) f^b_{ef}\partial^y_i \left\{\pi^i_a(y), A^e_j(x)\right\} + \\
&\quad \pi^{i\,d} f^a_{cd}\partial^x_j \left\{A^c_i(y), \pi^j_b(x)\right\} + f^a_{cd} f^b_{ef}\left\{A^c_i\pi^{i\,d}(y), A^e_j\pi^{j\,f}(x)\right\}\Big] \\
&= \partial_i\left(A^b_0\pi^{i\,f}\right) f^b_{af} + \pi^{i\,d}\partial_i(A^b_0) f^a_{bd} \\
&\quad -\int d^3x A^b_0(x) f^a_{cd} f^b_{ef}\left\{A^c_i\pi^{i\,d}(y), A^e_j\pi^{j\,f}(x)\right\} \\
&= A^b_0\partial_i\pi^{i\,f} f^b_{af} - \int d^3x A^b_0(x) f^a_{cd} f^b_{ef}\left\{A^c_i\pi^{i\,d}(y), A^e_j\pi^{j\,f}(x)\right\} \\
&= -A^b_0\partial_i\pi^{i\,fc} f^c_{ab} - A^b_0 f^c_{ab} A^d_i \pi^{i\,e} f^c_{de} \\
&= -f^c_{ab} A^b_0 D_i(\pi^i_c) \\
&\approx 0
\end{aligned} \tag{2.98}
$$

where in the second line we have expanded out all the covariant derivatives, and kept the non-zero Poisson brackets, in the fourth line we have used the total anti-symmetry of the structure constants and in the fifth line we have stated the result of $f^a_{cd} f^b_{ef}\left\{A^c_i\pi^{i\,d}(y), A^e_j\pi^{j\,f}(x)\right\}$, which requires using the Jacobi identity for the structure constants.

So we have shown that the Gauss constraints are weakly preserved in time and therefore there are no further constraints and the Dirac-Bergmann algorithm has been completed. We can now determine whether the constraints are first or second class or of mixed type, immediately we have

$$\{\phi_a, \phi_b\} = 0 \tag{2.99}$$

$$\left\{\phi_a, \tilde{\phi}_b\right\} = 0 \tag{2.100}$$

where the first follows from the canonical Poisson brackets and the second from (2.95), that just leaves the Poisson bracket

$$\left\{\tilde{\phi}_a, \tilde{\phi}_b\right\} = f^c_{ab} D_i(\pi^i_c) = f^c_{ab}\tilde{\phi}_c \tag{2.101}$$

this follows from the computation in (2.98). Hence all constraints weakly commute with each other and we have a first class Hamiltonian system. Observe that the Poisson algebra of the Gauss constraints is isomorphic to the Lie algebra, $\mathfrak{g}$, of the Yang Mills gauge group.

So we have a phase space $T^*Q$ with coordinates $A^a_\mu, \pi^\mu_a$, a $8dim\mathfrak{g} \times \infty^3$ space and a final constraint surface $\Gamma_c$ defined by $\phi_a(x) = \tilde{\phi}_a(x) = 0$ and a set of infinitesimal gauge transformations generated by all these first class constraints, which defines a set of $2dim\mathfrak{g} \times \infty^3$ dimensional surfaces, the gauge orbits, which foliate the constraint surface, a $(8dim\mathfrak{g} - 2dim\mathfrak{g}) \times \infty^3 = 6dim\mathfrak{g} \times \infty^3$ dimensional surface. Following Dirac each point on one of the gauge orbits represents the same physical state and should be identified.

A general point gauge transformation, by Dirac's conjecture, can be generated by an arbitrary combination of the first class constraints we have

$$
\begin{aligned}
\delta A^a_\mu &= \int d^3x \left\{A^a_\mu(y), \lambda^b_1\phi_b(x) + \lambda^b_2\tilde{\phi}_b(x)\right\} \\
&= \int d^3x \lambda^b_1\left\{A^a_\mu(y), \pi^0_b(x)\right\} + \lambda^b_2 D^x_i\left\{A^a_\mu(y), \pi^i_b(x)\right\} \\
&= \lambda^a_1\delta^0_\mu - D_i(\lambda^a_2)\delta^i_\mu
\end{aligned} \tag{2.102}
$$

and hence to recover the Noether gauge symmetry, (2.76), one must choose

$$\lambda_1^a = D_0\theta^a \tag{2.103}$$
$$\lambda_2^a = -\theta^a \,. \tag{2.104}$$

The final point we wish to make concerns a reduction in the dimension of phase space, which can be achieved by effectively ignoring the conjugate coordinates $A_o^a$ and $\pi_a^0 \equiv \phi_a$. First consider the evolution of these variables

$$\begin{aligned}\dot{A}_o^a(x) &= \{A_o^a(x), H_p\} \\ &= \int d^3x\, v^b \left\{A_o^a(x), \pi_b^0\right\} \\ &= v^a(x)\end{aligned} \tag{2.105}$$

and since $\pi_a^0$ is a constraint we have $\dot{\pi}_a^0 \approx 0$. This means that i.) the evolution of $A_o^a$ is an arbitrary function and that ii.) $\pi_a^0$ is (weakly) constant. Furthermore, these variables do not affect the evolution of any other phase space variables, i.e. $A_i^a$ and $\pi_a^i$ Poisson commute with the only term in $H_p$ that depends upon them, $\int d^3xv^a\pi_a^0$. In other words one can ignore the variables $A_o^a$ (gauge fix by re-scaling it to be zero) and $\pi_a^0 \equiv \phi_a$ and just consider a reduced phase space with coordinates $A_i^a$ and $\pi_a^i$ now subject only to the Gauss constraints, $\tilde{\phi}_a$. This 'reduced theory' can be described by the following canonical action

$$\begin{aligned}S[A_i^a, \pi_a^i, N^a] &= \int d^3xdt \left(\dot{A}_i^a\pi_a^i - \left(\frac{1}{2}\pi_a^i\pi_a^i + \frac{1}{4}F_{ij}^aF^{ij\,a}\right) - N^a\tilde{\phi}_a\right) \\ &= \int d^3xdt \left(\dot{A}_i^a\pi_a^i - \frac{1}{2}\left(\pi_a^i\pi_a^i + B_i^aB^{i\,a}\right) - N^a\tilde{\phi}_a\right)\end{aligned} \tag{2.106}$$

where $N^a$ are Lagrange multipliers, whose variation results in the implementation of the Gauss constraints, and $B^{i\,a} := -\frac{1}{2}\epsilon^{ijk}F_{jk}^a$ are known as the generalised magnetic fields. This form of the canonical Yang Mills action is the one that is often presented in the literature, but one should be aware that the full phase space as constructed using constrained systems theory is the one which also contains $A_o^a$ and $\pi_a^0 \equiv \phi_a$ as coordinates. We shall see in the next section, when we discuss GR, that a similar reduction is performed in order to obtain the ADM theory.

# 3 Hamiltonian Formulation of GR

In this section we discuss the Hamiltonian formulation of GR, this is crucial preparation for the canonical quantization, which is the subject of our final section. As indicated in the previous section the Hamiltonian formulation of GR was initially developed by Dirac and Bergmann and completed in the seminal work [13] by Arnowitt, Deser and Misner (ADM). There have been several reviews and discussions of this work since and the following have been useful in the preparation of this section [18, 16, 17, 3, 14, 15, 19] and particularly [5], which has a very comprehensive discussion.

Our goal is to perform a Legendre transform on the Einstein Hilbert action and analyse the resulting Hamiltonian constrained system. We shall discuss the phase space structure, the interpretation of the constraint algebra and its relation to the diffeomorphism group, possible matter coupling and finally consider the case of asymptotically flat spacetimes.

Any canonical analysis requires a notion of time from which the canonical momenta can be defined, in GR this forces one to assume that spacetime is topologically of the form $\Sigma \times \mathbb{R}$, where $\Sigma$ is a 3 dimensional manifold of arbitrary but fixed topology and $\mathbb{R}$ is 'time'. In fact any spacetime $M$ with a well defined initial value problem will be of this form, this is a result due to Geroch, which states that any globally hyperbolic spacetime[13] will necessarily be homeomorphic to $\Sigma \times \mathbb{R}$. Not all solutions to Einstein's equations are globally hyperbolic, e.g. the maximally extended Reissner-Nordstrom solution, representing the spacetime for a spherically symmetric charged particle, has a Cauchy horizon meaning there is a region from which there exist past directed causal curves that do not pass through any candidate Cauchy surface[14]. Generally such solutions are not regarded as physically realistic and from the classical theory the restriction to globally hyperbolic spacetime required for the Hamiltonian formulation is not considered a strong limitation. However, in the quantum theory the situation is more controversial as it seems that a canonical quantization of GR will prevent topology change at the quantum level, which is considered to be a feasible option.[15].

## 3.1 3+1 analysis of Einstein Hilbert action

Initially we shall perform the Hamiltonian analysis for a spatially compact $M$ without boundary, this has the advantage that all boundary terms can be dropped when performing an integration by parts. We shall also assume a vacuum and only briefly mention the inclusion of matter for reasons of simplicity.

Our notation will be as follows: spacetime $M$ is a four dimensional Lorentzian manifold of metric signature$(- + ++)$, points of $M$ will be denoted by $X$ with local coordinates using Greek indices $X^\mu$, $\mu = 0, 1, 2, 3$. By above we can assume that $M$ is diffeomorphic to $\Sigma \times \mathbb{R}$, and we shall label points on $\Sigma$ by $\sigma$, with local coordinates using Latin indices $\sigma^a$, $a = 1, 2, 3$ and denote points $t \in \mathbb{R}$. The diffeomorphism, denoted $Y$, means that one can define a 1-parameter family

[13] A globally hyperbolic spacetime is one which has a Cauchy surface $S$, i.e. a 3-surface for which all past or future directed causal curves from all points in spacetime have to pass through. It means that "data" on $S$ must determine all future events and retrodict all past events, see [14].

[14] See pages 158-159 in [20].

[15] We shall assume this limitation in the sequel. However, in the modern form of canonical GR there is a possibility for the metric to become degenerate in the first order formalism and therefore there may be scope for topology change at the quantum level.

of embeddings of $\Sigma$ in $M$ as follows:

$$\begin{aligned} Y:\Sigma\times\mathbb{R} &\rightarrow M \\ Y_t:\Sigma &\rightarrow \Sigma_t\subset M \\ Y_t:\sigma &\mapsto X=Y_t(\sigma):=Y(\sigma,t) \end{aligned} \tag{3.1}$$

having these relations means that one can pullback covariant tensor fields on $M$ to $\Sigma$ via these embeddings. This is important because in performing the 3+1 split of the Einstein Hilbert action we shall re-write it as an integral over the manifold $\Sigma\times\mathbb{R}$.

We shall only consider a restriction of embeddings such that all $\Sigma_t$ are spacelike hypersurfaces in $M$, this means we can define a unit normal timelike vector to $\Sigma_t$, which we denote by $n^\mu$. Further, we need to define a direction of time $T^\mu$ in spacetime, a natural choice will be the pushforward of the vector $\frac{\partial}{\partial t}$, the components of which will be:

$$T^\mu(X):=\left(Y_\star(\frac{\partial}{\partial t})\right)^\mu_{|X=Y(\sigma,t)}=\frac{\partial}{\partial t}\left(X^\mu\circ Y(\sigma,t)\right)_{|X=Y(\sigma,t)}\ . \tag{3.2}$$

This vector $T^\mu$ can be decomposed into components tangential and orthogonal to the hypersurfaces $\Sigma_t$, as follows:

$$T^\mu(X)=:N(X)n^\mu(X)+N^\mu(X) \tag{3.3}$$

where the function $N$ is the lapse and the tangential vector $N^\mu$ the shift. The pushforward of the vectors $\frac{\partial}{\partial\sigma^a}$ will be $\frac{\partial}{\partial\sigma^a}\left(X^\mu\circ Y(\sigma,t)\right)_{|X=Y(\sigma,t)}=:X^\mu_{,a}$ tangent to $\Sigma_t$ by construction and hence normal to $n^\mu$.

In order for a proper foliation of $M$ by the spacelike $\Sigma_t$ one requires that the shift is everywhere positive so that i.) increasing $t$ corresponds to the future direction and ii.) to prevent different$\Sigma_t$ from intersecting.

Our starting point is the Einstein Hilbert action without boundary terms, given by[16]:

$$S_{EH}=\frac{1}{16\pi G}\int_M \mathrm{d}^4X\,\sqrt{|g|}\,{}^4R(X) \tag{3.4}$$

where ${}^4R$ is the spacetime Ricci scalar and $g$ is the determinant of the spacetime metric $g_{\mu\nu}$. This action is invariant under general coordinate transformations (or passive spacetime diffeomorphisms, ${}_PDiff(M)$) and immediately, as the passive diffeomorphisms are a subset of the local Noether symmetries one can conclude, by Noether's second theorem, that GR is a singular system and the equations of motion will have this symmetry. We therefore expect to find a constrained Hamiltonian system once we have completed the canonical analysis.

In order to perform the Legendre transform one must write this action as an integral over $\Sigma\times\mathbb{R}$ of 'time' varying three dimensional objects built from our canonical fields. This can be done by introducing the following two objects, the spatial metric $q_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$ defined as:

$$q_{\mu\nu} := g_{\mu\nu}+n_\mu n_\nu \tag{3.5}$$

$$K_{\mu\nu} := q^\alpha_\mu q^\beta_\nu\nabla_\alpha n_\beta \tag{3.6}$$

where indices are raised and lowered by $g_{\mu\nu}$. These tensors are *spatial* in that they give zero when contracted on any index by $n^\mu$, for example $q_{\mu\nu}n^\nu=n_\mu+n_\mu(n_\nu n^\nu)=0$ by timelike normality

[16] In units where c=1.

of $n^\mu$. Using this it is easy to show that $q^\alpha{}_\mu$ projects spacetime vectors onto their components tangent to $\Sigma_t$ since for orthogonal vectors $q^\alpha{}_\mu n^\mu = 0$ and for tangent vectors it is the identity:

$$\begin{aligned} q^\alpha{}_\mu N^\mu &= g^{\alpha\lambda}(g_{\lambda\mu} + n_\lambda n_\mu) N^\mu \\ &= (\delta^\alpha{}_\mu + n^\alpha n_\mu) N^\mu \\ &= N^\alpha \end{aligned}$$

where in the second line we used $N^\mu n_\mu = 0$. It follows that any spacetime tensor can be projected to its spatial version by contraction with $q^\alpha{}_\mu$ on all its free indices, i.e.

$$T_{Spatial}^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = q^{\alpha_1}{}_{\mu_1}\cdots q^{\alpha_p}{}_{\mu_p} q_{\beta_1}{}^{\nu_1}\cdots q_{\beta_q}{}^{\nu_q} T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} \tag{3.7}$$

The interpretation of $q_{\mu\nu}$ is that it is the spatial metric on the hypersurface $\Sigma_t$, it is equivalent to $g_{\mu\nu}$ on all vectors tangent to the hypersurface but degenerate on perpendicular vectors, proportional to $n^\mu$. As $\Sigma_t$ is spacelike this means that $q_{\mu\nu}$ is a Riemannian metric for all vectors in the hypersurface. The extrinisic curvature[17] can be interpreted as the curvature of $\Sigma_t$ in $M$, it is the projection onto $\Sigma_t$ of the gradient of the unit normal vector $n^\mu$. We stress that the notion of extrinsic curvature should not be confused with the intrinsic curvature of a Riemannian or Lorentzian manifold- the latter is measured by the Riemann tensor and is not dependent upon the embedding of a hypersurface into an ambient space. Some simple examples in [17] illustrate this point, e.g. a cylinder $S^1 \times \mathbb{R}$ embedded in $\mathbb{R}^3$ has zero intrinsic curvature, a cylinder is flat, but non-zero extrinsic curvature by contrast a sphere $S^2$ in $\mathbb{R}^3$ has both non-zero intrinsic and extrinsic curvature.

The extrinsic curvature is closely related to the time derivative of the spatial metric as we can now show.

*Claim* 3.1. The extrinsic curvature satisfies:

$$K_{\mu\nu} = \frac{1}{2N}\left(\mathcal{L}_T q - \mathcal{L}_N q\right)_{\mu\nu} \tag{3.8}$$

where $(\mathcal{L}_T q)_{\mu\nu}$ and $(\mathcal{L}_N q)_{\mu\nu}$ are the Lie derivatives of the spatial metric with respect to the direction of time $T^\mu$and shift vector $N^\mu$ respectively.

*Proof.* First we show that $K_{\mu\nu}$ is symmetric. It is always possible to define the hypersurface $\Sigma_t$ by an equation $g(X) = t$ and hence the normal co-vector $n_\mu$ must satisfy $n_\mu = h\nabla_\mu g$ for some function $h$, as $\nabla_\mu g$ is orthogonal to the spacelike hypersurface. If we substitute this into the definition (3.6) one has:

$$\begin{aligned} K_{\mu\nu} &= q^\alpha_\mu q^\beta_\nu \left(\nabla_\alpha h \nabla_\beta g + h\nabla_\alpha\nabla_\beta g\right) \\ &= q^\alpha_\mu q^\beta_\nu \left(h^{-1}\left(\nabla_\alpha h\right) n_\beta + h\nabla_\alpha\nabla_\beta g\right) \\ &= q^\alpha_\mu q^\beta_\nu h\nabla_\alpha\nabla_\beta g \end{aligned}$$

where in the final step we used $q^\beta_\nu n_\beta = 0$. Now anti-symmetrizing one has:

$$\begin{aligned} 2K_{[\mu\nu]} &= q^\alpha_\mu q^\beta_\nu h\nabla_{[\alpha}\nabla_{\beta]} g \\ &= 0 \end{aligned} \tag{3.9}$$

where the final step follows because the Levi-Civita connection $\nabla$ is torsion free.

[17] Another interpretation of the extrinsic curvature is that it provides a measure of the extent to which geodesics in $\Sigma_t$ are geodesics in $M$. Only if the extrinsic curvature is zero will the two geodesics coincide, [17].

Using the symmetry of $K_{\mu\nu}$ allows us to express (3.6) as:

$$\begin{aligned}
2K_{\mu\nu} &= q^{\alpha}_{\mu} q^{\beta}_{\nu} 2\nabla_{(\alpha} n_{\beta)} \\
&= q^{\alpha}_{\mu} q^{\beta}_{\nu} \left(\mathcal{L}_n g\right)_{\alpha\beta} \\
&= q^{\alpha}_{\mu} q^{\beta}_{\nu} \left(\mathcal{L}_n q - \mathcal{L}_n \left(n \otimes n\right)\right)_{\alpha\beta} \\
&= q^{\alpha}_{\mu} q^{\beta}_{\nu} \left(\mathcal{L}_n q - n \otimes \mathcal{L}_n n - \mathcal{L}_n n \otimes n\right)_{\alpha\beta} \\
&= q^{\alpha}_{\mu} q^{\beta}_{\nu} \left(\mathcal{L}_n q\right)_{\alpha\beta} \\
&= (\delta^{\alpha}{}_{\mu} + n^{\alpha} n_{\mu})(\delta^{\beta}{}_{\nu} + n^{\beta} n_{\nu}) \left(\mathcal{L}_n q\right)_{\alpha\beta} \\
&= \left(\mathcal{L}_n q\right)_{\mu\nu} \, . 
\end{aligned} \tag{3.10}$$

The second line follows immediately from the definition of the Lie derivative of the spacetime metric and metric compatibility of $\nabla$, and the fifth line follows because $\mathcal{L}_n n = 0$. The final line follows by taking the Lie derivative of $0 = n^{\alpha} q_{\alpha\beta}$, then using both that the Lie derivative commutes with contractions and Leibniz to see that $0 = n^{\alpha} \left(\mathcal{L}_n q\right)_{\alpha\beta} + q_{\alpha\beta} \left(\mathcal{L}_n n\right)^{\alpha} = n^{\alpha} \left(\mathcal{L}_n q\right)_{\alpha\beta}$. The claim follows by replacing the unit normal in (3.10) by (3.3) and simplifying as:

$$\begin{aligned}
K_{\mu\nu} &= \frac{1}{2} \left(\mathcal{L}_{N^{-1}(T-N)} q\right)_{\mu\nu} \\
&= \frac{1}{2N} \left(\mathcal{L}_{(T-N)} q\right)_{\mu\nu} + \frac{1}{2} \left(q_{\mu\lambda} \nabla_{\nu}(N^{-1}) + q_{\lambda\nu} \nabla_{\mu}(N^{-1})\right) (T-N)^{\lambda} \\
&= \frac{1}{2N} \left(\mathcal{L}_T q - \mathcal{L}_N q\right)_{\mu\nu}
\end{aligned}$$

the second line follows from the definitions of the Lie derivatives and the final line follows from the fact that $(T-N)^{\lambda}$ is orthogonal to the spacelike hypersurface $\Sigma_t$ and so contracts with the spatial metric to give zero. Hence the extrinsic curvature is closely related to the 'time derivative' of the spatial metric, where the 'time derivative' is the Lie derivative with respect to $T^{\mu}$. □

The final step we need in order to perform the Legendre transform is to express the spacetime Ricci scalar and metric determinant in terms of quantities defined on the spatial slice $\Sigma_t$, they will be expressions involving both the spatial metric and the extrinsic curvature. This can be done by using geometric identities, which we now discuss.

*Claim* 3.2. The hypersurface $\Sigma_t$ can be viewed as a manifold with a Riemannian metric $q_{\mu\nu}$, hence there exists a unique torsion free, (spatial) metric compatible connection, which we denote $D$. It can be defined through its relation with the Levi-Civita connection as:

$$D_{\varrho} T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} := q^{\alpha_1}{}_{\mu_1} \cdots q^{\alpha_p}{}_{\mu_p} q_{\beta_1}{}^{\nu_1} \cdots q_{\beta_q}{}^{\nu_q} q_{\varrho}{}^{\sigma} \nabla_{\sigma} T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \, . \tag{3.11}$$

*Proof.* This follows since $D$ is linear, torsion free and satisfies Leibniz by the definition. Furthermore, it is compatible with the spatial metric since:

$$\begin{aligned}
D_{\varrho} q_{\alpha\beta} : &= q_{\alpha}{}^{\nu} q_{\beta}{}^{\sigma} q_{\varrho}{}^{\mu} \nabla_{\mu} q_{\nu\sigma} \\
&= q_{\alpha}{}^{\nu} q_{\beta}{}^{\sigma} q_{\varrho}{}^{\mu} \nabla_{\mu} \left(g_{\nu\sigma} + n_{\nu} n_{\sigma}\right) \\
&= 0
\end{aligned}$$

where the last line follows since $\nabla$ is metric compatible and by Leibniz we shall have a contraction of the form $q^{\sigma}{}_{\beta} n_{\sigma} = 0$ in the second term. □

The spatial connection $D$ allows us to define the intrinsic curvature of $\Sigma_t$ using three dimensional Riemann and Ricci tensors, which we denote as ${}^3R_{\mu\nu\sigma}{}^{\varrho}$ and ${}^3R_{\mu\nu}$ respectively. We can use (3.11) to express ${}^3R_{\mu\nu\sigma\varrho}$ in terms of ${}^4R_{\mu\nu\sigma\varrho}$ and the extrinsic curvature by the Gauss equation. Then further contractions and manipulation can be used to derive the Codacci equation.

*Claim* 3.3. Using the relation between the spatial and spacetime curvatures and extrinsic curvature one can prove i.) the Gauss equation:

$$ {}^3R_{\mu\nu\sigma\varrho} = q_\alpha{}^\nu q_\beta{}^\sigma q_\varrho{}^\mu q_\alpha{}^{\nu 3}R_{\mu\nu\sigma\varrho} - 2K_{\varrho[\mu}K_{\nu]\sigma} \tag{3.12} $$

and ii.) the Codacci equation:

$$ {}^4R = {}^3R + \left(K_{\mu\nu}K^{\mu\nu} - K^2\right) - 2\nabla_\mu\left(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu\right) \tag{3.13} $$

where $K := g^{\mu\nu}K_{\mu\nu}$.

Using the Codacci equation we can now substitute for ${}^4R$ in the Einstein-Hilbert action, replacing it with spatial quantities only. Notice the third term in (3.13) is a total derivative and therefore we shall drop it due to the assumption of no boundary. The remaining term to consider is the covariant volume element $\mathrm{d}^4X\,\sqrt{|g|}$.

Before we consider this we shall use the embeddings $Y_t$ to pull back the above spatial equations to the abstract manifold $\Sigma$, this will enable us to write $S_{EH}$ as an integral over $\Sigma\times\mathbb{R}$. The key definitions which allow us to do this are:

$$ \begin{aligned} q_{ab}(\sigma,t) &:= X^\mu_{,a}X^\nu_{,b}q_{\mu\nu}(X)_{|X=Y_t(\sigma)} = X^\mu_{,a}X^\nu_{,b}g_{\mu\nu} && (3.14)\\ K_{ab}(\sigma,t) &:= X^\mu_{,a}X^\nu_{,b}K_{\mu\nu}(X)_{|X=Y_t(\sigma)} = X^\mu_{,a}X^\nu_{,b}\nabla_\mu n_\nu && (3.15) \end{aligned} $$

where in both lines we used that $n_\mu X^\mu_{,a} = 0$ and the definitions (3.5) and (3.6). All covariant tensors can be pulled back in this way. In general one cannot pullback contravariant tensors but we can use the induced metric to raise indices in $\Sigma$ so that we can define shift vectors $N^a(\sigma,t) := q^{ab}\left(X^\mu_{,b}g_{\mu\nu}N^\nu\right)$ on $\Sigma$ and similarly for other vectors. Finally one can show that the curvature scalar ${}^3R(\sigma,t)$ computed through $q_{ab}$ equals the one on $\Sigma_t$ as used in the Gauss and Codacci equations. Given this we can now express both (3.8) and the right hand side of (3.13) (dropping boundary term) as

$$ \begin{aligned} K_{ab}(\sigma,t) &= \frac{1}{2N}\left(\dot q_{ab} - \left(\mathcal{L}_{\vec N}q\right)_{ab}\right)(\sigma,t) && (3.16)\\ {}^4R &= \left({}^3R + \left(K_{ab}K^{ab} - K^2\right)\right)(\sigma,t) && (3.17) \end{aligned} $$

where $K := q^{ab}K_{ab}$ and the Lie derivative is with respect to $N^a(t,\sigma)$ and we have used the definition of the time derivative $X^\mu_{,a}X^\nu_{,b}\left(\mathcal{L}_T q\right)_{\mu\nu} = \mathcal{L}_T\left(q_{\mu\nu}X^\mu_{,a}X^\nu_{,b}\right) =: \dot q_{ab}$.

The remaining obstacle to writing $S_{EH}$ in 3+1 form is expressing the volume element in the adapted frame- we can do this by using the diffeomorphism $Y$ to pullback the (scalar) line element $ds^2$ to $\Sigma\times\mathbb{R}$ and thereby both read off the metric components and compute the determinant that we need. So by definition on $\Sigma\times\mathbb{R}$ one has:

$$ ds^2 = \left(Y^\star g\right)_{tt}dtdt + \left(Y^\star g\right)_{ta}dtd\sigma^a + \left(Y^\star g\right)_{at}d\sigma^a dt + \left(Y^\star g\right)_{ab}d\sigma^a d\sigma^b $$

and then by re-expressing $ds^2 = g_{\mu\nu} dX^\mu dX^\nu$ we get:

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dX^\mu dX^\nu \\
&= g_{\mu\nu} \left( X^\mu_{,t} dt + X^\mu_{,a} d\sigma^a \right) \left( X^\nu_{,t} dt + X^\nu_{,b} d\sigma^b \right) \\
&= g_{\mu\nu} \left( \left( N n^\mu + N^\mu \right) dt + X^\mu_{,a} d\sigma^a \right) \left( \left( N n^\nu + N^\nu \right) dt + X^\nu_{,b} d\sigma^b \right) \\
&= g_{\mu\nu} \left( N n^\mu dt + X^\mu_{,a} \left( d\sigma^a + N^a dt \right) \right) \left( N n^\nu dt + X^\nu_{,b} \left( d\sigma^b + N^b dt \right) \right) \\
&= g_{\mu\nu} \Big[ \left( N^2 n^\mu n_\nu + N n^\mu X^\nu_{,b} N^b + X^\mu_{,a} N^a N n^\nu + X^\mu_{,a} N^a X^\nu_{,b} N^b \right) dt dt \\
&\quad + \left( N n^\mu X^\nu_{,b} + X^\mu_{,a} X^\nu_{,b} N^a \right) dt d\sigma^b + \left( N n^\nu X^\mu_{,a} + X^\mu_{,a} X^\nu_{,b} N^b \right) d\sigma^a dt + X^\mu_{,a} X^\nu_{,b} d\sigma^a d\sigma^b \Big] \\
&= \left( -N^2 + q_{ab} N^a N^b \right) dt dt + q_{ab} N^a dt d\sigma^b + q_{ab} N^b d\sigma^a dt + q_{ab} d\sigma^a d\sigma^b \qquad (3.18)
\end{aligned}
$$

where in the third line we use $X^\mu_{,t} = T^\mu$ and (3.3), in the fourth line $N^\mu = X^\mu_{,a} N^a$. and in the last two lines the fact that $X^\nu_{,b}$ is orthogonal to $n^\nu$. We can now immediately read off the components of $Y^\star g$: $(Y^\star g)_{tt} = -N^2 + q_{ab} N^a N^b$, $(Y^\star g)_{ta} = q_{ab} N^a$ and $(Y^\star g)_{ab} = q_{ab}$, one can then compute $det\,(Y^\star g) = -N^2 det(q_{ab}) =: -N^2 q$.

We can now write $S_{EH}$ in the 3+1 form, by noting that integration is invariant under diffeomorphisms using the new volume form $\sqrt{|det(Y^\star g)|} dt d^3\sigma = N\sqrt{q}$. Hence we can write:

$$S_{3+1} = \frac{1}{16\pi G} \int_{\mathbb{R}} dt \int_\Sigma \mathrm{d}^3\sigma \, N \sqrt{q} \left( {}^3R + K_{ab} K^{ab} - K^2 \right) \qquad (3.19)$$

## 3.2 Hamiltonian Analysis

We can now define the conjugate momenta and then perform the Legendre transform to obtain the Hamiltonian. The action $S_{3+1}$ is a functional of three configuration variables: the induced metric, the lapse and the shift vector, where the in the above action $K_{ab}$ is to be expressed as (3.16). Immediately we can write down the conjugate momenta to these variables:

$$
\begin{aligned}
P^{ab}(t,\sigma) &:= \frac{\delta S}{\delta \dot{q}_{ab}(t,\sigma)} = \frac{\sqrt{q}}{16\pi G} \left( K^{ab} - q^{ab} K \right)(t,\sigma) \qquad (3.20) \\
\Pi(t,\sigma) &:= \frac{\delta S}{\delta \dot{N}(t,\sigma)} = 0 \qquad (3.21) \\
\Pi_a(t,\sigma) &:= \frac{\delta S}{\delta \dot{N}^a(t,\sigma)} = 0 \qquad (3.22)
\end{aligned}
$$

We can see that the momenta $\Pi(t,\sigma)$ and $\Pi_a(t,\sigma)$ are both zero and this means we have constraints because we cannot express $\dot{N}(t,\sigma)$ and $\dot{N}^a(t,\sigma)$ as functions of their momenta. By contrast $\dot{q}_{ab}(t,\sigma)$ can be so inverted since contracting $P^{ab}$ with the induced metric one has:

$$P = \frac{\sqrt{q}}{16\pi G} (K - 3K) = -\sqrt{q} \frac{K}{8\pi G} \qquad (3.23)$$

and then first substituting back into (3.20) and then (3.16) we have:

$$
\begin{aligned}
P^{ab} &= \frac{\sqrt{q}}{16\pi G} \left( K^{ab} + \frac{8\pi G}{\sqrt{q}} q^{ab} P \right) \\
\Rightarrow K^{ab} &= \frac{16\pi G}{\sqrt{q}} P^{ab} - \frac{8\pi G}{\sqrt{q}} q^{ab} P = \frac{8\pi G}{\sqrt{q}} \left( 2P^{ab} - q^{ab} P \right) \qquad (3.24) \\
\Rightarrow \dot{q}_{ab} &= \frac{16 N \pi G}{\sqrt{q}} \left( 2P_{ab} - q_{ab} P \right) + \left( \mathcal{L}_{\vec{N}} q \right)_{ab} . \qquad (3.25)
\end{aligned}
$$

So we just have the primary constraints (3.21) and (3.22).

Following the Dirac algorithm for expressing the primary Hamiltonian one has:

$$
\begin{aligned}
H_{Prim} : &= \int_\Sigma \mathrm{d}^3\sigma \left(\dot{q}_{ab}P^{ab} - L_{3+1} + \lambda\Pi + \lambda^a\Pi_a\right) \\
&= \int_\Sigma \mathrm{d}^3\sigma \Big[\left(\frac{16N\pi G}{\sqrt{q}}\left(2P_{ab} - q_{ab}P\right) + \left(\mathcal{L}_N q\right)_{ab}\right) P^{ab} \\
&\quad -\frac{N\sqrt{q}}{16\pi G}\left({}^3R + 128\pi G^2\left(2P_{ab}P^{ab} - P^2\right)\right) + \lambda\Pi + \lambda^a\Pi_a\Big] \\
&= \int_\Sigma \mathrm{d}^3\sigma \Big[\frac{16N\pi G}{\sqrt{q}}\left(P_{ab}P^{ab} - \frac{1}{2}P^2\right) + \left(\mathcal{L}_N q\right)_{ab} P^{ab} - \frac{N\sqrt{q}}{16\pi G}\,{}^3R + \lambda\Pi + \lambda^a\Pi_a\Big] \\
&= \int_\Sigma \mathrm{d}^3\sigma \Big[\frac{16N\pi G}{\sqrt{q}}\left(P_{ab}P^{ab} - \frac{1}{2}P^2\right) + 2\left(D_a N_b\right) P^{ab} - \frac{N\sqrt{q}}{16\pi G}\,{}^3R + \lambda\Pi + \lambda^a\Pi_a\Big] \\
&= \int_\Sigma \mathrm{d}^3\sigma \Big[N\left(\frac{16\pi G}{\sqrt{q}}\left(P_{ab}P^{ab} - \frac{1}{2}P^2\right) - \frac{\sqrt{q}}{16\pi G}\,{}^3R\right) - 2N_b D_a P^{ab} + \lambda\Pi + \lambda^a\Pi_a\Big] \\
&\quad + \int_\Sigma \mathrm{d}^3\sigma 2D_a\left(N_b P^{ab}\right) \\
&= \int_\Sigma \mathrm{d}^3\sigma \Big[N\left(\frac{16\pi G}{\sqrt{q}}\left(P_{ab}P^{ab} - \frac{1}{2}P^2\right) - \frac{\sqrt{q}}{16\pi G}\,{}^3R\right) - N^a 2q_{ac}D_b P^{bc} + \lambda\Pi + \lambda^a\Pi_a\Big] \\
&= \int_\Sigma \mathrm{d}^3\sigma \Big[N\left(\frac{16\pi G}{\sqrt{q}}\left(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}\right) P^{ab}P^{cd} - \frac{\sqrt{q}}{16\pi G}\,{}^3R\right) - N^a 2q_{ac}D_b P^{bc} \\
&\quad + \lambda\Pi + \lambda^a\Pi_a\Big]
\end{aligned}
\tag{3.26}
$$

where $\lambda(t,\sigma)$ and $\lambda^a(t,\sigma)$ are arbitrary functions. The second line follows from using the 3+1 action but with extrinsic curvature replaced by momenta using (3.24) and replacing $\dot{q}_{ab}$ with (3.25). The fourth line follows from the definition of the Lie derivative and metric compatibility of the spatial connection $D$. We have then done an integration by parts and subsequently dropped the term $\int_\Sigma \mathrm{d}^3\sigma 2D_a\left(N_b P^{ab}\right)$ because the covariant divergence of a vector density of weight $+1$ is equal to its ordinary divergence, i.e. $\int_\Sigma \mathrm{d}^3\sigma 2D_a\left(N_b P^{ab}\right) = \int_\Sigma \mathrm{d}^3\sigma 2\partial_a\left(N_b P^{ab}\right)$ and hence it is a total derivative, and can be dropped by our boundary assumptions. Note we have chosen to re-arrange the coefficient of lapse in the final step so that the presence of canonical variables is made explicit.

We now have to ensure the consistency of the constraints, i.e. that they are preserved by evolution generated by $H_{Prim}$. This is easy to do, since using the canonical Poisson brackets:

$$\left\{q_{ab}(\sigma), P^{cd}(\sigma')\right\} = \delta^c_{(a}\delta_{b)}^{\;d}\delta^3(\sigma,\sigma') \tag{3.27}$$

$$\left\{N(\sigma), \Pi(\sigma')\right\} = \delta(\sigma,\sigma') \tag{3.28}$$

$$\left\{N^a(\sigma), \Pi_b(\sigma')\right\} = \delta^a_b\delta^3(\sigma,\sigma') \tag{3.29}$$

(and with all other Poisson brackets zero), we have:

$$\dot{\Pi} = \left\{\Pi, H_{Prim}\right\} = \frac{16\pi G}{\sqrt{q}}\left(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}\right)P^{ab}P^{cd} - \frac{\sqrt{q}}{16\pi G}\,{}^3R =: H(\sigma; q, P] \tag{3.30}$$

$$\dot{\Pi}_a = \left\{\Pi_a, H_{Prim}\right\} = -2q_{ac}D_b P^{bc} =: H_a(\sigma; q, P]\,. \tag{3.31}$$

As the primary constraints are not conserved by above, we have to impose the secondary constraints $H(\sigma; q, P] = 0$ and $H_a(\sigma; q, P] = 0$, for all spatial points $\sigma$. The constraints should be viewed as separate phase space functionals, defined for each point $\sigma$. (It happens that they are *local functionals* of the canonical fields, i.e. functionals that return a function of the canonical fields at the chosen point $\sigma$). The notation $H(\sigma; q, P]$ and $H_a(\sigma; q, P]$ indicates that the constraints are functions of the spatial points $\sigma$ and functionals of the canonical fields $\left(q_{ab}, P^{ab}\right)$, see [35]. $H$ is known as the Hamiltonian constraint and $H_a$ the spatial diffeomorphism constraint. One should now check that these secondary constraints are preserved, i.e. compute both $\{H, H_{Prim}\}$ and $\{H_a, H_{Prim}\}$, a non-trivial calculation shows that these Poisson brackets generate combinations of the secondary constraints and so we do not have tertiary constraints.

We can express the Hamiltonian constraint as:

$$H(\sigma; q, P] = 16\pi G G_{abcd} P^{ab} P^{cd} - \frac{\sqrt{q}}{16\pi G}\, {}^3R \tag{3.32}$$

where we have[18]:

$$G_{abcd}(\sigma, q] = \frac{1}{2\sqrt{q}} \left(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}\right) . \tag{3.33}$$

$G_{abcd}[q]$ is known as the (inverse) De Witt metric, it can be interpreted as a metric on the space of contravariant Riemannian metrics. The De Witt metric $G^{abcd}$ defines an interval, [19], between two infinitesimally separated Riemannian metrics: $q_{ab}$ and $q_{ab} + \delta q_{ab}$ as:

$$(\delta q_{ab}, \delta q_{ab}) := \int_\Sigma d^3\sigma G^{abcd}[q_{ef}]\delta q_{ab}(\sigma)\delta q_{cd}(\sigma) . \tag{3.34}$$

Presently our phase space consists of the following canonical coordinates:

$$\left(q_{ab}, N, N^a; P^{ab}, \Pi, \Pi_a\right)$$

subject to the constraints defined above. However, it is possible to reduce this by considering i.) that:

$$\begin{aligned} \dot{N} &= \{N, H_{Prim}\} = \lambda \\ \dot{N}^a &= \{N, H_{Prim}\} = \lambda^a \end{aligned}$$

i.e. the evolution of the lapse and shift are completely arbitrary, ii.) that $\Pi, \Pi_a$do not evolve- they are constraints and iii.) the evolution of $q_{ab}$ and $P^{ab}$ is equivalently described by the Hamiltonian:

$$H_{ADM} := \int_\Sigma \mathrm{d}^3\sigma \left[NH + N^a H_a\right] \tag{3.35}$$

in this sense the $N, N^a$ are non-dynamical variables. This is not surprising since $N, N^a$ serve to parameterise the spacelike embeddings $Y_t$ and foliation of $M$ but subject to defining a proper foliation they can be arbitrary. One can also implement this from an action $S_{ADM}$ by using Lagrange multipliers to implement the Hamiltonian and diffeomorphism constraints, i.e.

$$S_{ADM}[q_{ab}, P^{ab}, N, N^a] = \int_{\mathbb{R}} dt \int_\Sigma \mathrm{d}^3\sigma \left(P^{ab}\dot{q}_{ab} - NH - N^a H_a\right) . \tag{3.36}$$

Hence we define a new constrained Hamiltonian system, with phase space $\Gamma_{ADM}$, canonical coordinates $\left(q_{ab}(\sigma); P^{ab}(\sigma)\right)$, and an evolution generated by $H_{ADM}$, which is a linear combination

[18] Just symmetrise on the (cd) indices from (3.30) to show this.

of the constraints $H$ and $H_a$. In this theory the lapse and shift are viewed as fixed phase space independent functions on the spatial manifold. The Hamilton equations:

$$\dot{q}_{ab} = \{q_{ab}, H_{ADM}\} \quad (3.37)$$
$$\dot{P}^{ab} = \left\{P^{ab}, H_{ADM}\right\} \quad (3.38)$$

subject to the Hamiltonian and diffeomorphism constraints are equivalent to the Einstein field equations $G_{\mu\nu} = 0$. (Since one is considering the GR as an evolution of a spatial metric the term *geometrodynamics* is often used to describe this framework). In particular, if $g_{\mu\nu}$ satisfies the Einstein equation and $Y_t$ is a 1-parameter family of spacelike embeddings of $\Sigma$ with associated lapse and shift then the 1 parameter family of induced metrics $q_{ab}(t,\sigma)$ and momenta $P^{ab}(t,\sigma)$ (defined through (3.20) and (3.16)) will satisfy (3.37), (3.38) and the Hamiltonian and diffeomorphism constraints. Conversely, if there is a spacelike foliation $Y_t$ of a Lorentzian manifold $M$ such that the $q_{ab}$and $P^{ab}$ satisfy the above Hamilton equations and constraints then the reconstructed metric $g_{\mu\nu}$is a solution to the Einstein field equation, [35]. To be clear the solutions to Hamilton's equations subject to the constraints allows us to reconstruct the spacetime metric $g_{\mu\nu}$ because we can define its components in the adapted coordinates defined by (3.18). In fact one can further show that if the constraints (3.30) and (3.31) are satisfied on every spacelike hypersurface in the spacetime $M$ then the metric $g_{\mu\nu}$ satisfies the Einstein equations, [35].

One can also show[19] that i.) the Hamilton equations above are equivalent to the full spatial projection of the four dimensional Einstein equations, i.e. $q^{\alpha}{}_{\mu} q^{\beta}{}_{\nu} \, {}^4R_{\alpha\beta} = 0$ and ii.) the Hamiltonian and diffeomorphism constraints are respectively equivalent to the full and partial projection of the Einstein equations orthogonal to the hypersurface $\Sigma_t$, i.e.

$$H = 0 \Leftrightarrow n^{\alpha} n^{\beta} \, {}^4R_{\alpha\beta} = 0 \quad (3.39)$$
$$H_a = 0 \Leftrightarrow n^{\alpha} \, {}^4R_{\alpha\beta} = 0 \, . \quad (3.40)$$

As these constraints form a closed algebra, we have a first class Hamiltonian system. In the Dirac picture of gauge transformations, section 2.3.1, one has to view the GR Hamiltonian as a gauge generator since it is simply a sum of first class constraints. This has the consequence that one should interpret 'time' evolution in GR as a gauge transformation and also means that Dirac observables, defined to Poisson commute with all the constraints, will therefore be constant in coordinate time. This latter issue is known as the 'frozen formalism' and the general issues surrounding how to interpret evolution in GR, reconcile it with our observations and extract gauge invariant information form the important problem of time in canonical gravity. We shall discuss these issues in section 4 in the context of *relational* Dirac observables for GR and first class constrained systems in general.

In the Bergmann picture of gauge transformations, section 2.3.2, one can construct a gauge generator $G(t)$, (3.80) and [26], which is a particular sum of the above constraints on the full phase space with coordinates $(q_{ab}, N, N^a; P^{ab}, P, P_a)$. This generator by definition maps between the gauge equivalent phase space trajectories. However, although it is numerically related to the Hamiltonian, it has a different interpretation- the Hamiltonian by contrast acts upon phase space points and maps one point to another point representing the dynamical system at a later time. Hence, in this view, one does not think of the dynamics generated by the Hamiltonian as a gauge transformation. This proposal due to [27] is rather recent but seems to provide a viable way to avoid some of the issues of the problem of time. However, one still has the problem of determining gauge invariant observables and therefore the discussion in section 4 is of interest to both interpretations of gauge transformations.

[19] See for example chapter 5 in [17].

Finally, we can count the number of physical degrees of freedom in the gravitational field. We have a phase space $\Gamma_{ADM}$, which has a dimension of $12 \times \infty^3$ since the coordinates are symmetric and defined on a 3 dimensional manifold, we have $4 \times \infty^3$ first class constraints, this means the physical phase space[20] is $(12 - 2 \times 4) \times \infty^3 = 4 \times \infty^3$dimensional and hence there are $2 \times \infty^3$ degrees of freedom in GR. This result agrees with the linear field analysis, which shows that the gravitational wave propagating on a fixed background spacetime has two degrees of freedom (or polarizations). We remark that such a counting argument always holds up to a finite number of degrees of freedom.

## 3.3 Constraint Algebra analysis

We now state and discuss the Poisson algebra formed by $H$ and $H_a$. It is convenient to consider the Poisson brackets between smeared constraints[21], defined as:

$$H(N) := \int_\Sigma d^3\sigma N(\sigma) H(\sigma) \tag{3.41}$$

$$\vec{H}(\vec{N}) := \int_\Sigma d^3\sigma N^a(\sigma) H_a(\sigma) \tag{3.42}$$

where $N$ and $\vec{N}$ are arbitrary scalar and vector fields on $\Sigma$. The first class constraint algebra for GR, often called the Dirac, [6] or *hypersurface deformation algebra* [21] is given by:

$$\left\{\vec{H}(\vec{N}_1), \vec{H}(\vec{N}_2)\right\} = \vec{H}(\mathcal{L}_{\vec{N}_1}\vec{N}_2) \tag{3.43}$$

$$\left\{\vec{H}(\vec{N}_1), H(N)\right\} = H(\mathcal{L}_{\vec{N}_1}N) \tag{3.44}$$

$$\{H(N_1), H(N_2)\} = \vec{H}(\vec{N}) \tag{3.45}$$

where $N^a := q^{ab}(N_1 N_{2\,,b} - N_2 N_{1\,,b})$.

The first observation is that elements $\vec{H}\left[\vec{N}\right]$ for arbitrary $\vec{N}$ form a sub-algebra, but they do not form an ideal because there is not closure under the Poisson bracket with elements $H\left[N\right]$. This sub-algebra has a straightforward interpretation, recall that diffeomorphisms on $\Sigma$ are generated by spatial vector fields, $\vec{N}$, which form a Lie algebra with the commutator as Lie bracket[22], then given $\mathcal{L}_{\vec{N}_1}\vec{N}_2 = \left[\vec{N}_1, \vec{N}_2\right]$ we can see that the map $\vec{N}_1 \to \vec{H}\left[\vec{N}_1\right]$ is a homomorphism from the Lie algebra of Diff($\Sigma$)to this sub-algebra.

We can confirm that $\vec{H}\left[\vec{N}\right]$ is the generator of infinitesimal (spatial) diffeomorphisms on phase space functionals by first computing its action on the canonical fields. First observe that:

$$\begin{aligned}
\vec{H}(\vec{N}) &= -2\int_\Sigma d^3\sigma N^a(\sigma) q_{ac} D_b P^{bc} \\
&= 2\int_\Sigma d^3\sigma q_{ac}\left(D_b N^a\right) P^{bc} \\
&= \int_\Sigma d^3\sigma P^{ab}\left(\mathcal{L}_{\vec{N}} q\right)_{ab}
\end{aligned} \tag{3.46}$$

[20] Recall for first class constraints one subtracts twice- first to get to the constraint surface and second to factor out the gauge symmetry.

[21] $H(N)$ and $\vec{H}(\vec{N})$ are functionals of $N, N^a$ and so more properly should be expressed as $H[N]$ and $\vec{H}[\vec{N}]$ however the usual notation in the literature is as above and so we shall follow it here.

[22] Strictly speaking the Lie Bracket for the Diffeomorphism group is in fact the negative of the commutator, [35], so the map $\vec{N}_1 \to \vec{H}\left[\vec{N}_1\right]$ is an anti-homomorphism.

where we have done an integration by parts and used the symmetry of $P^{ab}$. Then we have for the action on the metric:

$$\begin{aligned}\left\{q_{ab}(\sigma), \vec{H}(\vec{N})\right\} &= \int_\Sigma d^3\sigma' \left(\mathcal{L}_{\vec{N}} q\right)_{cd}(\sigma') \left\{q_{ab}(\sigma), P^{cd}(\sigma')\right\} \\ &= \left(\mathcal{L}_{\vec{N}} q\right)_{ab}(\sigma) . \end{aligned} \tag{3.47}$$

Hence $\vec{H}(\vec{N})$ generates infinitesimal (spatial) diffeomorphism of the metric. Similarly, for the momentum we can use the Leibniz property of the Lie derivative to express (3.46) as:

$$\vec{H}(\vec{N}) = -\int_\Sigma d^3\sigma q_{ab} \left(\mathcal{L}_{\vec{N}} P\right)^{ab} + \int_\Sigma d^3\sigma \mathcal{L}_{\vec{N}}(P) \tag{3.48}$$

where $P := q_{ab}P^{ab}$ is a scalar density of weight $+1$ and one can show the second term is a total derivative by using the definition of the Lie derivative of $P$[23]:

$$\begin{aligned}\mathcal{L}_{\vec{N}}(P) &= \sqrt{q}\mathcal{L}_{\vec{N}}\left(\frac{P}{\sqrt{q}}\right) + PD_aN^a \\ &= \sqrt{q}N^aD_a\left(\frac{P}{\sqrt{q}}\right) + PD_aN^a \\ &= D_a\left(N^aP\right) = \partial_a\left(N^aP\right) \end{aligned} \tag{3.49}$$

where we used the result that $D_aq = 0$. Hence we can drop the second term in (3.48) and compute the action of $\vec{H}(\vec{N})$ on the momentum as:

$$\begin{aligned}\left\{P^{ab}(\sigma), \vec{H}(\vec{N})\right\} &= -\int_\Sigma d^3\sigma' \left(\mathcal{L}_{\vec{N}} P\right)^{cd} \left\{P^{ab}(\sigma), q_{cd}(\sigma')\right\} \\ &= \left(\mathcal{L}_{\vec{N}} P\right)^{ab}(\sigma) . \end{aligned} \tag{3.50}$$

Hence $\vec{H}[\vec{N}]$ generates spatial diffeomorphisms of the canonical momentum. Finally, we show $\vec{H}[\vec{N}]$ generates spatial diffeomorphisms on arbitrary functionals of the canonical variables, let $f = f[q,p]$ then we can compute:

$$\begin{aligned}\left\{f, \vec{H}(\vec{N})\right\} &= \int d^3\sigma' \frac{\delta f}{\delta q_{ab}(\sigma')}\frac{\delta \vec{H}[\vec{N}]}{\delta P^{ab}(\sigma')} - \frac{\delta f}{\delta P^{ab}(\sigma')}\frac{\delta \vec{H}[\vec{N}]}{\delta q_{ab}(\sigma')} \\ &= \int d^3\sigma' \frac{\delta f}{\delta q_{ab}(\sigma')}\left\{q_{ab}(\sigma'), \vec{H}[\vec{N}]\right\} + \frac{\delta f}{\delta P^{ab}(\sigma')}\left\{P^{ab}(\sigma'), \vec{H}[\vec{N}]\right\} \\ &= \int d^3\sigma' \frac{\delta f}{\delta q_{ab}(\sigma')}\left(\mathcal{L}_{\vec{N}} q\right)_{ab}(\sigma') + \frac{\delta f}{\delta P^{ab}(\sigma')}\left(\mathcal{L}_{\vec{N}} P\right)^{ab}(\sigma') \\ &= \mathcal{L}_{\vec{N}} f \end{aligned} \tag{3.51}$$

and $\vec{H}(\vec{N})$ is the generator of spatial diffeomorphisms.

These results allows us to derive, [22], (3.43) and (3.44), let

$$f[M] := \int_\Sigma d^3\sigma M^{a\cdots b}{}_{c\cdots d}(\sigma) f^{c\cdots d}{}_{a\cdots b}(q,p)$$

[23] See for example [16] for Lie derivatives of tensor densities.

where $M^{a\cdots b}{}_{c\cdots d}$ is an arbitrary tensor on $\Sigma$, independent of the canonical fields and $f^{c\cdots d}{}_{a\cdots b}(q,p)$ is a tensor density of weight $+1$, which is a function of the canonical fields. Then we can compute the Poisson bracket with $\vec{H}[\vec{N}]$ as follows:

$$
\begin{aligned}
\left\{\vec{H}(\vec{N}), f[M]\right\} &= \int d^3\sigma' \frac{\delta \vec{H}[\vec{N}]}{\delta q_{ab}(\sigma')}\frac{\delta f[M]}{\delta P^{ab}(\sigma')} - \frac{\delta \vec{H}[\vec{N}]}{\delta P^{ab}(\sigma')}\frac{\delta f[M]}{\delta q_{ab}(\sigma')} \\
&= \int d^3\sigma' \left(-\left(\mathcal{L}_{\vec{N}}P\right)^{ab}\right)\frac{\delta f[M]}{\delta P^{ab}(\sigma')} - \left(\mathcal{L}_{\vec{N}}q\right)_{ab}\frac{\delta f[M]}{\delta q_{ab}(\sigma')} \\
&= -\int d^3\sigma' \Big[\left(\mathcal{L}_{\vec{N}}P\right)^{ab}\int d^3\sigma M^{e\cdots f}{}_{c\cdots d}(\sigma)\frac{\delta f^{c\cdots d}{}_{e\cdots f}(q,p)}{\delta P^{ab}(\sigma')} \\
&\quad + \left(\mathcal{L}_{\vec{N}}q\right)_{ab}\int d^3\sigma M^{e\cdots f}{}_{c\cdots d}(\sigma)\frac{\delta f^{c\cdots d}{}_{e\cdots f}(q,p)}{\delta q_{ab}(\sigma')}\Big] \\
&= -\int d^3\sigma' M^{e\cdots f}{}_{c\cdots d}\left[\left(\mathcal{L}_{\vec{N}}P\right)^{ab}\frac{\partial f^{c\cdots d}{}_{e\cdots f}(q,p)}{\partial P^{ab}} + \left(\mathcal{L}_{\vec{N}}q\right)_{ab}\frac{\partial f^{c\cdots d}{}_{e\cdots f}(q,p)}{\partial q_{ab}}\right] \\
&= -\int d^3\sigma' M^{e\cdots f}{}_{c\cdots d}\mathcal{L}_{\vec{N}} f^{c\cdots d}{}_{e\cdots f} \\
&= \int d^3\sigma' \left(\mathcal{L}_{\vec{N}} M^{e\cdots f}{}_{c\cdots d}\right) f^{c\cdots d}{}_{e\cdots f}
\end{aligned} \tag{3.52}
$$

where on the fourth line we have used the functional derivative chain rule to remove the $\int d^3\sigma$ integration and on the final line used an integration by parts and dropped the boundary term.

Immediately by i.) replacing $M^{a\cdots b}{}_{c\cdots d}$ by $M^a$ and $f^{c\cdots d}{}_{a\cdots b}(q,p)$ by $H_a$ (3.52) is equivalent to (3.43) and ii.) using a scalar $M$ and scalar density $H$ (3.52) is equivalent to (3.44).

Having shown that the Lie algebra of spatial diffeomorphisms is represented in the Dirac algebra it is natural to ask whether the full spacetime diffeomorphism Lie algebra can also be mapped to the Dirac algebra in this way. This is not the case because the Dirac algebra is not a Lie algebra, we can see this in (3.45) as $\{H(N_1), H(N_2)\}$ closes with a 'phase space' dependent coefficient (the vector $N^a$involves the inverse 3-metric $q^{ab}$) of $H_a$. (Usually it is stated that the Dirac algebra involves structure functions rather than structure constants). Hence, although the original symmetry of the theory was the spacetime diffeomorphism group[24], in the canonical analysis we have found a genuinely different algebra.

We now compute the action of the Hamiltonian constraint on the phase space variables in order to determine precisely where this 'failure' to represent the diffeomorphism comes from.

[24]See section 3.5 for further discussion of this point.

Consider first the Poisson bracket of the smeared Hamiltonian constraint with the 3-metric:

$$\begin{aligned}
\{q_{ab}(\sigma), H(N)\} &= 16\pi G \int d^3\sigma' \frac{N}{\sqrt{q}} \left( q_{ce}q_{df} - \frac{1}{2}q_{cd}q_{ef} \right) \left\{ q_{ab}(\sigma), P^{cd}P^{ef} \right\} \\
&= 16\pi G \int d^3\sigma' \frac{N}{\sqrt{q}} \left( q_{ce}q_{df} - \frac{1}{2}q_{cd}q_{ef} \right) \left[ P^{cd} \left\{ q_{ab}(\sigma), P^{ef} \right\} + (c \leftrightarrow d,\, e \leftrightarrow f) \right] \\
&= 16\pi G \int d^3\sigma' \frac{N}{\sqrt{q}} \left( q_{ce}q_{df} - \frac{1}{2}q_{cd}q_{ef} \right) \left[ P^{cd} \delta^e_{(a}\delta^f_{b)} + P^{ef}\delta^c_{(a}\delta^d_{b)} \right] \delta^3(\sigma, \sigma') \\
&= 16\pi G \int d^3\sigma' \frac{2N}{\sqrt{q}} \left( q_{c(a}q_{b)d} - \frac{1}{2}q_{cd}q_{ab} \right) P^{cd}\delta^3(\sigma, \sigma') \\
&= 32\pi G \frac{N}{\sqrt{q}} \left( P_{ab} - \frac{q_{ab}P}{2} \right) \\
&= 2NK_{ab} \\
&= \dot{q}_{ab} - \left( \mathcal{L}_{\vec{N}} q \right)_{ab} \\
&= \mathcal{L}_{N\vec{n}} q \,. \qquad (3.53)
\end{aligned}$$

In the first line we dropped the ${}^3R$ from the Hamiltonian constraint, as it is independent of the momentum and also Poisson commuted the factor involving the 3-metrics. In the sixth line we used (3.20) and (3.23) to substitute for the momentum. In the penultimate line we used (3.16) and then one can compute the components of the spacetime vector $Nn^\mu$ in the adapted embedded coordinates to reach the final line, also see (3.3).

The end result is that we have shown the Hamiltonian constraint generates diffeomorphisms of the 3-metric in the direction orthogonal to the hypersurface. Given that we have shown above that the Dirac algebra is not isomorphic to the spacetime diffeomorphism algebra and yet have shown just such an action on the canonical variables ($\vec{H}(\vec{N})$ on both canonical variables and $H(N)$ on the 3-metric) it must mean that there is a problem with the one remaining Poisson bracket $\left\{ P^{ab}, H(N) \right\}$.

A non-trivial calculation gives the following result:

$$\begin{aligned}
\left\{ P^{ab}, H(N) \right\} &= -2\frac{N}{\sqrt{q}} \left[ P^{ac}P_c^b - \frac{P^{ab}P}{2} \right] + \frac{q^{ab}NH}{2} + N\sqrt{q} \left( q^{ab}\,{}^3R - {}^3R^{ab} \right) \\
&\quad + \sqrt{q} \left[ -q^{ab}D_cD^cN + D^aD^bN \right] \,. \qquad (3.54)
\end{aligned}$$

The reason this calculation is more involved is that i.) we need to perform a variation of $\sqrt{q}$ and its inverse, this can be done using the standard result:

$$\delta q = q q^{ab} \delta q_{ab} \qquad (3.55)$$

and ii.) we cannot drop the ${}^3R$ term and therefore need to compute the functional derivative $\frac{\delta^3 R}{\delta q_{ab}(\sigma)}$ , this can be done by using two identities:

$$\begin{aligned}
q^{ab}\delta^3 R_{ab} &= q^{ab} \left[ -D_a \delta\Gamma^c_{cb} + D_c \delta\Gamma^c_{ab} \right] \qquad (3.56) \\
\delta\Gamma^a_{bc} &= q^{ad} \left[ D_c\delta q_{bd} + D_b \delta q_{cd} + D_d \delta q_{bc} \right] \,. \qquad (3.57)
\end{aligned}$$

The former is the 'Palatini identity' whose 4-dimensional equivalent is used in the standard variation of the Einstein Hilbert action. The second identity can be derived by using the definition of $\Gamma^a_{bc}$ in terms of the 3-metric and then performing a variation, if one then uses normal coordinates the coefficient of $\delta q^{ad}$ will be zero because $\Gamma^a_{bc} = 0$ in these coordinates and similarly in the second term partial derivatives can be replaced by covariant ones. This leaves a tensor equation, which must hold in all frames and hence the result follows.

The challenge is now to interpret (3.54) in terms of an infinitesimal diffeomorphism orthogonal to the spacelike hypersurface just as for the metric variable. In other words what is the relationship between the right hand side of (3.54) and the Lie derivative $\mathcal{L}_{N\vec{n}}P^{\mu\nu}$ this will allow us to see precisely how the spacetime diffeomorphism group is not represented in the action $H\left[N\right]$ on the momentum. A non-trivial calculation[25] shows that:

$$\{P^{\mu\nu}, H(N)\} = \frac{q^{\mu\nu}NH}{2} - N\sqrt{q}\left[q^{\mu\varrho}q^{\nu\sigma} - q^{\mu\nu}q^{\varrho\sigma}\right]^4 R_{\rho\sigma} + \mathcal{L}_{N\vec{n}}P^{\mu\nu}\,. \tag{3.58}$$

This means that $H(N)$ can be interpreted as generating infinitesimal diffeomorphisms on the momentum but only when $H = 0$ and ${}^4R_{\mu\nu} = 0$, i.e the Einstein vacuum equations hold (however the latter is sufficient since ${}^4R_{\mu\nu} = 0 \Rightarrow H = 0$) . Hence we come to the important conclusion that the gauge transformations generated by the constraints can be interpreted as infinitesimal spacetime diffeomorphisms but only when the equations of motion hold ('on-shell'). We shall return to this topic in section 3.5, where we consider from a broader perspective the invariance groups of general relativity and their relationships. However, first we consider the addition of matter degrees of freedom in the canonical formalism.

## 3.4 Matter coupling in the canonical formalism

So far we have only considered vacuum general relativity, it is possible to consider matter degrees of freedom in the canonical formalism. We shall then have matter contributions to the Hamiltonian and diffeomorphism constraints, i.e.

$$H^{Tot} = H + H^{Matter} \tag{3.59}$$
$$H_a^{Tot} = H_a + H_a^{Matter} \tag{3.60}$$

The situation simplifies greatly when non-derivative coupling is considered, i.e. when the matter Lagrangian couples only to functions of the spacetime metric and not derivatives of it. In this case $H^M$ and $H_a^M$ will depend upon the matter canonical variables and the spatial metric only. Unfortunately, if one considers derivative coupling then it is possible for the $H^M$ constraint to depend upon the gravitational momentum $P^{ab}$and the matter momentum $\pi$, [19]. We shall only consider non-derivative coupling in what follows.

Under these conditions the algebra of the total Hamiltonian constraints separates into matter

[25] See section 1.3 [5] or equivalently [22].

and gravity contributions as we can see:

$$\begin{aligned}
\left\{H^{Tot}(N), H^{Tot}(M)\right\} &= \{H(N), H(M)\} + \left\{H(N), H^M(M)\right\} + \left\{H^M(N), H(M)\right\} \\
&\quad + \left\{H^M(N), H^M(M)\right\} \\
&= \{H(N), H(M)\} + \left\{H^M(N), H^M(M)\right\} + \\
&\quad + \int_\Sigma \int_\Sigma d^3\sigma d^3\sigma' N(\sigma) M(\sigma') \left\{H(\sigma), H^M(\sigma')\right\} \\
&\quad + \int_\Sigma \int_\Sigma d^3\sigma d^3\sigma' N(\sigma) M(\sigma') \left\{H^M(\sigma), H(\sigma')\right\} \\
&= \{H(N), H(M)\} + \left\{H^M(N), H^M(M)\right\} + \\
&\quad \int_\Sigma \int_\Sigma d^3\sigma d^3\sigma' \left(N(\sigma) M(\sigma') - N(\sigma) M(\sigma')\right) \delta^3(\sigma, \sigma') g(q, p) \\
&= \{H(N), H(M)\} + \left\{H^M(N), H^M(M)\right\} \\
&= \vec{H}(\vec{N}_2) + \left\{H^M(N), H^M(M)\right\} \qquad (3.61)
\end{aligned}$$

where in the third line we have used the fact that i.) $H$ does not depend upon derivatives of the canonical variables and ii.) by the assumption of non-derivative coupling $H^M$ will also not depend upon such derivatives and hence cross term Poisson brackets are ultra-local because no integration by parts need to be performed. In the final line we have used the Dirac algebra for the Hamiltonian constraints.

All forms of bosonic degrees of freedom can be described using the canonical formalism developed for $(q_{ab}, P^{ab})$ including scalar, Maxwell and Yang Mills fields. Unfortunately, fermionic degrees of freedom cannot be described using these variables, one has to use the triad basis for this[26]. The reason that fermionic degrees of freedom cannot be described by $(q_{ab}, P^{ab})$ is the impossibility of defining spinors for the general linear group. By impossibility we mean that there do not exist (finite dimensional) double valued representations of the general linear group[27], which would be required to define a spinor.

We shall briefly discuss a minimally coupled scalar field with potential and derive the corresponding Hamiltonian and diffeomorphism constraints. The spacetime action for this theory is:

$$S[\Phi] = -\frac{1}{2} \int_M d^4X \sqrt{g} \left(g^{\mu\nu}(X) \partial_\mu \Phi(X) \partial_\nu \Phi(X) + V(\Phi)\right) \qquad (3.62)$$

where $V(\Phi)$ is a self-interaction term.

As for vacuum gravity we can perform a 3+1 analysis of this action - this involves i.) pulling back the scalar field to $\Sigma \times \mathbb{R}$ under our family of embeddings $Y_t$, where we define $\phi(\sigma, t) := \Phi(Y_t(\sigma))$, ii.) using the pullback of the volume form $d^4X \sqrt{g} = N\sqrt{q} dt d^3\sigma$ and iii.) computing the components of the inverse spacetime metric in the adapted frame. We computed the metric in the adapted frame, see above (3.19), and this implies that the inverse metric components are:

$$g^{tt} = -N^{-2}, g^{at} = \frac{N^a}{N^2}, g^{ab} = q^{ab} - \frac{N^a N^b}{N^2} \qquad (3.63)$$

[26] We shall consider triad (and tetrad) bases in section 5 when we cover the Ashtekar variables.

[27] See [23] for a proof that there are no finite dimensional representations of the universal cover of the special linear group.

and we can then compute on $\Sigma \times \mathbb{R}$ the pullback of the scalar Lagrangian as:

$$\begin{aligned} g^{\mu\nu}\partial_\mu\phi(\sigma,t)\partial_\mu\phi(\sigma,t) &= -N^{-2}\dot{\phi}^2 + 2\frac{N^a}{N^2}\dot{\phi}\phi_{,a} + \left(q^{ab} - \frac{N^a N^b}{N^2}\right)\phi_{,a}\phi_{,b} \\ &= -N^{-2}\left(\dot{\phi} - N^a\phi_{,a}\right)^2 + q^{ab}\phi_{,a}\phi_{,b}\,. \end{aligned} \tag{3.64}$$

Combining these results one can write the action in 3+1 form as:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}} dt \int_\Sigma d^3\sigma N\sqrt{q}\left(N^{-2}\left(\dot{\phi} - N^a\phi_{,a}\right)^2 - q^{ab}\phi_{,a}\phi_{,b} - V(\phi)\right) \tag{3.65}$$

and hence compute the canonical momentum $\pi$ as:

$$\pi(\sigma,t) := \frac{\delta S[\phi]}{\delta\dot{\phi}(\sigma,t)} = \frac{\sqrt{q}}{N}\left(\dot{\phi} - N^a\phi_{,a}\right)\,. \tag{3.66}$$

Clearly, this is not a constraint as one can invert the momentum in terms of the velocity using (3.66).

Hence, one can perform the Legendre transform to compute the Hamiltonian $\mathrm{H}^\phi$as:

$$\begin{aligned} \mathrm{H}^\phi : &= \int_\Sigma d^3\sigma\left(\pi\dot{\phi} - L(\phi,\pi)\right) \\ &= \int_\Sigma d^3\sigma\left(\pi\dot{\phi} - \frac{1}{2}\frac{N}{\sqrt{q}}\pi^2 + \frac{1}{2}N\sqrt{q}q^{ab}\phi_{,a}\phi_{,b} + \frac{N}{2}\sqrt{q}V(\phi)\right) \\ &= \int_\Sigma d^3\sigma\left(\frac{1}{2}\pi\frac{N}{\sqrt{q}}\pi + \pi N^a\phi_{,a} + \frac{1}{2}N\sqrt{q}q^{ab}\phi_{,a}\phi_{,b} + \frac{N}{2}\sqrt{q}V(\phi)\right) \\ &= \int_\Sigma d^3\sigma\left[\frac{N}{2}\left(\frac{\pi^2}{\sqrt{q}} + \sqrt{q}\left(q^{ab}\phi_{,a}\phi_{,b} + V(\phi)\right)\right) + N^a\pi\phi_{,a}\right] \end{aligned} \tag{3.67}$$

this Hamiltonian combines with (3.35) to form the total Hamiltonian of the gravity scalar system. Since in the gravity sector we still have the same primary constraints(3.21) and (3.22) one must conclude that the coefficients of lapse and shift are secondary constraints as before and this implies we can read off the scalar field contributions to these constraints as:

$$H^\phi(\sigma) = \frac{1}{2}\left[\frac{\pi^2}{\sqrt{q}} + \sqrt{q}\left(q^{ab}\phi_{,a}\phi_{,b} + V(\phi)\right)\right](\sigma) \tag{3.68}$$

$$H^\phi_a(\sigma) = \pi\phi_{,a}(\sigma) \tag{3.69}$$

where $H^\phi(\sigma)$, $H^\phi_a(\sigma)$ are the scalar field contributions to the Hamiltonian and diffeomorphism constraints respectively. Note that $H^\phi_a$ does not depend upon the metric, this results holds for arbitrary matter fields, [21].

As $H^\phi_a(\sigma)$ does not depend upon the canonical gravitational variables all cross terms drop out when computing: $\left\{\vec{H}^{Tot}(\vec{N}), \vec{H}^{Tot}(\vec{M})\right\}$ and one has:

$$\begin{aligned} \left\{\vec{H}^{Tot}(\vec{N}), \vec{H}^{Tot}(\vec{M})\right\} &= \left\{\vec{H}(\vec{N}), \vec{H}(\vec{M})\right\} + \left\{\vec{H}^\phi(\vec{N}), \vec{H}^\phi(\vec{M})\right\} \\ &= \vec{H}(\mathcal{L}_{\vec{N}}\vec{M}) + \left\{\vec{H}^\phi(\vec{N}), \vec{H}^\phi(\vec{M})\right\} \end{aligned} \tag{3.70}$$

where in the final line we have used the Dirac algebra.

Since we have a first class system this means that the final line above must be a combination of the total constraints, using this one can conclude that:

$$\left\{\vec{H}^{\phi}(\vec{N}), \vec{H}^{\phi}(\vec{M})\right\} = \vec{H}^{\phi}(\mathcal{L}_{\vec{N}}\vec{M})\,. \tag{3.71}$$

The same conclusion can be made for the $\left\{H^{\phi}, H^{\phi}\right\}$Poisson bracket of the scalar field from (3.61), i.e.

$$\left\{H^{\phi}(N), H^{\phi}(M)\right\} = \int_{\Sigma} d^3\sigma q^{ab}\left(NM_{,b} - MN_{,b}\right) H^{\phi}_a(\sigma)\,. \tag{3.72}$$

Finally, one can consider the $\left\{H^{Tot}, H^{Tot}_a\right\}$ Poisson bracket:

$$\begin{aligned}
\left\{H^{Tot}(N), \vec{H}^{Tot}(\vec{M})\right\} &= \left\{H(N), \vec{H}(\vec{M})\right\} + \left\{H^{\phi}(N), \vec{H}(\vec{M})\right\} + \left\{H^{\phi}(N), \vec{H}^{\phi}(\vec{M})\right\} \\
&= -H(\mathcal{L}_{\vec{M}}N) + \int_{\Sigma} d^3\sigma N(\sigma)M^a(\sigma')\left\{H^{\phi}(\sigma), -2q_{ac}D_bP^{bc}(\sigma')\right\} \\
&\quad + \left\{H^{\phi}(N), \vec{H}^{\phi}(\vec{M})\right\} \\
&= -H(\mathcal{L}_{\vec{M}}N) + \left\{H^{\phi}(N), \vec{H}^{\phi}(\vec{M})\right\} + \\
&\quad \int_{\Sigma} d^3\sigma \int_{\Sigma} d^3\sigma' N(\sigma)M^a(\sigma') - 2q_{ac}D_b\left\{H^{\phi}(\sigma), P^{bc}(\sigma')\right\} \\
&= -H(\mathcal{L}_{\vec{M}}N) + \left\{H^{\phi}(N), \vec{H}^{\phi}(\vec{M})\right\} \\
&\quad -2\int_{\Sigma} d^3\sigma \int_{\Sigma} d^3\sigma' N(\sigma)M^a(\sigma')q_{ac}D_b\left[\frac{\delta H^{\phi}(\sigma)}{\delta q_{bc}(\sigma')}\right]
\end{aligned} \tag{3.73}$$

where in the first line we used the fact that $\left\{H(N), \vec{H}^{\phi}(\vec{M})\right\} = 0$ as $H^{\phi}_a$ is independent of the gravitational variables, in the second line we used the Dirac algebra, and in the third line used the fact that $H^{\phi}$ does not depend upon $P^{ab}$, which allows the metric and spatial covariant derivative to be taken outside the Poisson bracket. Again in order to have a first class system the right hand side of (3.73) must be a total constraint and hence we conclude that:

$$\left\{H^{\phi}(N), \vec{H}^{\phi}(\vec{M})\right\} = -H^{\phi}(\mathcal{L}_{\vec{M}}N) + 2\int_{\Sigma} d^3\sigma \int_{\Sigma} d^3\sigma' N(\sigma)M^a(\sigma')q_{ac}D_b\left[\frac{\delta H^{\phi}(\sigma)}{\delta q_{bc}(\sigma')}\right] \tag{3.74}$$

we can view (3.71), (3.72), and (3.74) as constraints on the types of matter coupling that will lead to a consistent system in the constrained Hamiltonian formalism.

We shall use the scalar field canonical analysis again when we consider i.) the parametrised scalar field in section 3.5 and ii.) the deparametrisation of GR, when coupled to a scalar field without potential, in section 4.4.1.

## 3.5 Symmetries, diffeomorphisms & the Dirac algebra

The fact that the constraint algebra in GR is not isomorphic to the Lie algebra of diffeomorphisms leads to several important questions: i.) what precisely are the symmetries of GR in the covariant and canonical formalisms and what are their relationships, ii.) why is there a difference between GR and Yang Mills in the way the Lie algebra is represented by the constraints (recall we showed the (gauge) Lie algebra and the constraint algebra for Yang Mills theory were isomorphic in (2.101)) and finally iii.) can one, by some means, obtain a representation of the spacetime diffeomorphism group or algebra in the GR phase space. We shall try to explore these questions in this section.

### 3.5.1 Symmetries in GR

We should now like to take a broader perspective and consider the various symmetry groups that characterise GR. This will enable us to understand more fully the relationship between the symmetry groups associated with the covariant and canonical formalisms. The seminal paper on this topic is [24], but we have found recent discussions in [25, 26, 27] to be helpful also.

The Einstein Hilbert action (3.4), is invariant under general coordinate transformations or passive spacetime diffeomorphisms, ${}_PDiff(M)$ and as previously stated this will imply a singular Lagrangian and constrained Hamiltonian system, which we have now analysed. Any symmetry of the action will automatically be reflected in the equations of motion, i.e. in the dynamics. However, ${}_PDiff(M)$ is not the largest dynamical symmetry group of the Einstein equations or Einstein Hilbert action, in fact there exists a larger group, called Q, [24, 25] or the 'induced diffeomorphism group' in [26]. This group is characterised by allowing general coordinate transformations to depend not only on coordinates but the metric field as well. In particular, one can describe an element $g$ of this induced diffeomorphism gauge group of gravity as a map

$$\begin{aligned} g : Lor(M) &\rightarrow Lor(M) \\ g : g_{\alpha\beta} &\rightarrow g[g_{\alpha\beta}] \equiv d[g_{\alpha\beta}](g_{\alpha\beta}) \end{aligned} \tag{3.75}$$

where $Lor(M)$ is the space of Lorentzian metrics on spacetime and the map $d$ is defined as

$$\begin{aligned} d : Lor(M) &\rightarrow Diff(M) \\ d : g_{\alpha\beta} &\mapsto d[g_{\alpha\beta}]\,. \end{aligned} \tag{3.76}$$

In other words the action of the element $g$ is to push forward the spacetime metric by the particular diffeomorphism $d[g_{\alpha\beta}]$. Clearly, this group contains the action of the standard passive diffeomorphisms as a sub-group but what is more interesting is that this group can be directly related to the symmetry group we have obtained in the Hamiltonian analysis.

We briefly discuss the results of [26], which provide a relation between this larger group and the symmetries we have found in canonical gravity. The starting point is the analysis discussed in section 2.3.2 concerning the generators of gauge transformations in the sense of Bergmann, i.e. as maps from trajectories to trajectories in phase space. In order to translate the Lagrangian symmetries to canonical ones a restriction on the form of the general coordinate transformations has to be made, this ensures that the symmetry may be *projected* from the Lagrangian configuration velocity space to phase space. This restriction is of the form

$$X^\mu \rightarrow X^\mu - \epsilon^\mu(X,\, g_{\alpha\beta}) \tag{3.77}$$

where

$$\epsilon^\mu = \delta^\mu_a \xi^a + n^\mu \xi^0 \tag{3.78}$$

and where $\xi^0$ and $\xi^a$ are arbitrary functions of the spacetime coordinates $X^\mu$ and spatial metric. The form of the gauge generator is postulated to be

$$G(t) = \int_\Sigma d^3\sigma \left(\xi^\mu(t) G^0_\mu + \dot{\xi}^\mu(t) G^1_\mu\right) \tag{3.79}$$

and then using the results of section 2.3.2 on the form of the $G^i$ it is possible to express them in terms of the known constraints of canonical gravity. The result is that

$$G(t) = \int_\Sigma d^3\sigma \left(\xi^\mu \left(H_\mu + N^\rho P_\nu C^\nu_{\mu\rho}\right) + \dot{\xi}^\mu P_\mu\right) \tag{3.80}$$

where $H_\mu \to (H, H_a)$, $N^\rho \to (N, N^a)$, $P_\mu$ are the canonical momenta to $N^\mu$, (which get dropped in the ADM phase space), and $\{H_\mu, H_\rho\} = C^\nu_{\mu\rho} H_\nu$.

This generator $G(t)$ is the canonical generator of gauge transformations in the full GR phase space, (i.e. not the ADM space) but the original phase space and it maps solution trajectories to other solution trajectories. In this picture the canonical symmetry group, generated by $G(t)$, can be viewed as the group obtained from the projectable sub-group of the induced diffeomorphism group. This is the relation between the Lagrangian and Hamiltonian symmetries of GR. This projectable group is often called the *Bergmann-Komar group*, see also the helpful dialogue in the appendix of [27].

### 3.5.2 Projectability of Noether symmetries to phase space

The difference between Yang Mills theories and GR in the way the gauge Lie algebra is represented by the Poisson algebra of the constraints has been investigated in [28]. In their approach a projection map $\pi : \mathscr{F} \to \Gamma$ is used to construct a phase space $\Gamma$ from the space of (spacetime) field configurations $\mathscr{F}$ for an arbitrary Lagrangian field theory and it is shown that $\Gamma$ coincides with the usual canonical phase space for both Yang Mills and GR. A notion of *local symmetry* in $\mathscr{F}$ is then defined as a pair $(\delta\phi^a, \alpha^\mu)$, (where $\delta\phi^a|_{\phi^a}$ is a tangent vector on $\mathscr{F}$ , at the field configuration point $\phi^a$, and where $\alpha^\mu$ is a spacetime vector density), such that the variation of the Lagrangian density $\delta\mathscr{L}$ with respect to this infinitesimal change in the field $\delta\phi^a$ is the total derivative $\nabla_\mu \alpha^\mu$. One is then able to use the map $\pi$ to pushforward local symmetries from $\mathscr{F}$ to $\Gamma$, however because the projection map is in general many to one the projected symmetry may not be well defined. It is a result of [28] that the differences in the way the gauge symmetry is represented by the constraint algebra in Yang Mills and GR can be precisely captured by the *projectability* of the local symmetries to the phase space in each case.

### 3.5.3 Finding a representation of *L*Diff($M$) in canonical gravity

Having said this it is possible to find a representation of $L$Diff$(M)$ in canonical gravity, however in order to do so one must extend the phase space of the theory, [29, 30]. First, [29], describes a method to extend the phase space $(\phi(\sigma), \pi(\sigma))$ of a scalar field theory on a fixed spacetime background by adding the "embedding variables" and their conjugate momenta and then defines a homomorphism from $L$Diff$(M)$ into the Poisson algebra of this extended phase space. In [30] this method is applied, modulo certain technicalities, to canonical gravity. The main motivations for this work were i.) to see whether the spacetime covariance, manifestly broken in the canonical picture, could be recovered (rather as in canonical QFT where the Poincarè group can be recovered despite the breaking of Lorentz covariance in choosing a time) and ii.) the quantum theory of gravity- there exist group theoretic methods to find operators on a Hilbert space but these generally require the observables to be generators of a Lie algebra. We discuss the method by which this representation can be found.

By "embedding variables" we mean embedding maps $Z : \Sigma \to M$, the space of all such embeddings will be denoted $Emb(\Sigma, M)$, and this space inherits a differential structure from $C^\infty(\Sigma, M)$. We shall also use the set of spacelike embeddings $Emb_g(\Sigma, M)$ (with respect to a metric $g$) and it also inherits a differential structure and is an infinite dimensional manifold. One can define vectors, co-vectors in this space where for example a vector $v_Z \in T_Z(Emb_g(\Sigma, M))$ is defined to be a smooth map $v_Z$ from $\Sigma$ into the tangent vectors on $M$ where $v_Z(\sigma) \in T_{Z(\sigma)}M$ . Co-vectors can be defined as appropriate duals of these objects. The tangent space of $Emb_g(\Sigma, M)$ at the "point" $Z \in Emb_g(\Sigma, M)$ will be denoted $T_Z Emb_g(\Sigma, M)$ and the co-tangent space $T^\star_Z Emb_g(\Sigma, M)$.

One can define a coordinate basis $\delta/\delta X^\alpha$ in $T_{Y_t}(Emb_g(\Sigma, M))$ and coordinate co-vector basis $\delta X^\alpha$ in $T^\star_{Y_t}(Emb_g(\Sigma, M))$ by their action on $\sigma \in \Sigma$ and relation to their corresponding basis vectors, co-vectors in $T_pM$ and $T_p^\star M$ as:

$$\frac{\delta}{\delta X^\alpha}(\sigma) \quad := \quad \frac{\partial}{\partial X^\alpha}|_{X=Y_t(\sigma)} \tag{3.81}$$
$$\delta X^\alpha(\sigma) \quad := \quad dX^\alpha|_{X=Y_t(\sigma)} \tag{3.82}$$

The constructions we have used in the canonical analysis above can be defined as tangent vectors, co-vectors in $T_Z Emb_g(\Sigma, M)$ or $T_Z^\star Emb_g(\Sigma, M)$ respectively, e.g. the time $T^\mu$ can be thought of as an element $v_{Y_t}$defined such that $v_{Y_t}(\sigma) = T^\mu(Y_t(\sigma))$ and the unit normal $n_\mu$ can be defined as $n \in T^\star_{Y_t} Emb_g(\Sigma, M)$ such that $n(\sigma) = n_\mu(Y_t(\mu))$. Indeed one can decompose any vector $v_{Y_t}$ in $T_{Y_t} Emb_g(\Sigma, M)$ into a sum such that their action on $\sigma \in \Sigma$ is to map to vectors normal and tangential to the hypersurface $Y_t(\Sigma)$. We have:

$$v^\alpha_{Y_t}(\sigma) = N(\sigma) g^{\alpha\beta}(Y_t(\sigma)) n_\beta(\sigma) + N^a(\sigma) \frac{\partial}{\partial \sigma^a}(X^\alpha \circ Y_t(\sigma)) \tag{3.83}$$

where $N$ is the pullback of the lapse function, $n_\beta \in T^\star_{Y_t} Emb_g(\Sigma, M)$ maps to the unit normal, and the second term can be interpreted as $v_{Y_t}[N^a] \in T_{Y_t}(Emb_g(\Sigma, M))$ , i.e. $v_{Y_t}[N^a](\sigma)$ is the pushforward of the spatial vector $N^a$ at the spacetime point $Y_t(\sigma)$. One can repeat the entire ADM analysis using this formalism and reproduce the Dirac algebra as above.

In this analysis the reason the Dirac algebra acquires structure functions is as follows. First there is an action of Diff$(M)$ on the space of all embeddings $Emb(\Sigma, M)$ as follows:

$$\begin{aligned} Emb(\Sigma, M) \times \text{Diff}(M) &\rightarrow Emb(\Sigma, M) \\ (Z, \phi) &\mapsto \phi_\circ Z \end{aligned} \tag{3.84}$$

where $\phi \in$ Diff$(M)$ and one can see that this is a representation of Diff$(M)$. If one only focuses on $\phi$ sufficiently close to the identity then a spacelike hypersurface $Z(\Sigma)$ will be mapped to another spacelike hypersurface. Secondly, if one considers a vector field $V$ on $M$ generating a 1-parameter family of spacetime diffeomorphisms $\phi^V_t$ then the vector $v_Z \in T_Z Emb(\Sigma, M)$ associated to the spacetime vector $V$ , i.e. $v_Z(\sigma) = V(Z(\sigma))$, will be an element of $T_Z Emb_g(\Sigma, M)$. Hence, the vector fields on $Emb_g(\Sigma, M)$ do form a representation space for $L$Diff$(M)$. The problem in the canonical analysis is that one has to decompose spacetime vectors into their normal and tangential components and it is this that introduces a reference to the background spacetime metric, i.e. one has to use the metric to define the notion of an orthogonal vector to $\Sigma_t$. In effect one replaces the coordinate basis $\delta/\delta X^\alpha$ and uses instead the decomposition (a non-coordinate basis) $n^\alpha \delta/\delta X^\alpha$, and $\frac{\partial}{\partial \sigma^a}(X^\alpha \circ Y_t(\sigma))\delta/\delta X^\alpha$ the fact these basis vectors do not commute is the reason for $q^{ab}$ in (3.45), [29].

**Parametrised Scalar Field Theory**

The resolution to this problem is, as mentioned above, to extend the phase space of the theory. The method developed by Isham and Kuchař is first applied to the parametrisation of a minimally coupled scalar field with fixed spacetime metric $g_{\mu\nu}$.

The action we consider is:

$$S[\Phi] = \int_M d^4X - \frac{1}{2}\sqrt{g}\left(U(\Phi) g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + V(\Phi)\right) \tag{3.85}$$

where $U > 0$ and $V$ are functions (self interactions) of the scalar field. As before one can perform a 3+1 analysis on this action and pull back all quantities to $\Sigma \times \mathbb{R}$, where we define $\phi(\sigma, t)$ as the pullback of $\Phi$ under our fixed family of embeddings $Y_t$. One finds:

$$S[\phi] = \int_{\mathbb{R}} dt \int_{\Sigma} d^3\sigma \frac{1}{2} N\sqrt{q} \left( \frac{U}{N^2} \left( -\dot{\phi} + N^a \phi_{,a} \right)^2 - U q^{ab} \phi_{,a}\phi_{,b} - V \right) \tag{3.86}$$

and hence the conjugate momentum $\pi$ is:

$$\pi(\sigma, t) := \frac{\delta S[\phi]}{\delta \phi(\sigma, t)} = -\sqrt{q} \frac{U}{N} \left( -\dot{\phi} + N^a \phi_{,a} \right) . \tag{3.87}$$

After performing a Legendre transform the resulting Hamiltonian is:

$$H[N, N^a] = \int_{\Sigma} d^3\sigma \left( N H^{\phi} + N^a H^{\phi}_a \right) \tag{3.88}$$

where $H^{\phi}$and $H^{\phi}_a$ are given by:

$$\begin{aligned} H^{\phi} &= \frac{1}{2} \left( U^{-1} \frac{\pi^2}{\sqrt{q}} + U \sqrt{q} q^{ab} \phi_{,a}\phi_{,b} + \sqrt{q} V \right) && (3.89) \\ H^{\phi}_a &= \pi \phi_{,a} \, . && (3.90) \end{aligned}$$

We stress these functions are *not* constraints in the current theory, because the spacetime metric is fixed we do not generate Hamiltonian and diffeomorphism constraints. In fact $H^{\phi}$ can be interpreted as the energy density of the scalar field and $H^{\phi}_a$ as the momentum density.

There is an implicit dependence on the embedding in all of the above quantities, e.g.

$$\begin{aligned} q_{ab}(\sigma) &= g_{\alpha\beta}(Y_t(\sigma)) \frac{\partial}{\partial \sigma^a} (X^{\alpha} \circ Y_t(\sigma)) \frac{\partial}{\partial \sigma^b} (X^{\alpha} \circ Y_t(\sigma)) \\ \phi(\sigma, t) &= \Phi \circ Y_t(\sigma) \end{aligned}$$

and so on, hence more properly one should write $\phi[Y_t, \sigma)$, $q_{ab}[Y_t, \sigma)$, $N[Y_t, \sigma)$ and $N^a[Y_t, \sigma)$. Of course in the current situation this is academic as our family of embeddings are fixed. However we now wish to *parametrise* the system, i.e. promote the fixed embeddings $Y_t$ to be additional dynamical variables. This is the field theoretic generalization of the parametrization[28] of the Newtonian particle. (Given that the original spacetime action (3.85) is manifestly independent of the foliation and embedding, when one does perform a variation of the extended action with respect to the embeddings the field equations derived must be valid because of the original Hamiltonian equations).

**Extending the phase space to include the embedding variables**

By adding the embeddings as additional configuration variables one is enlarging the configuration space of the theory from $C^{\infty}(\Sigma, \mathbb{R})$, the space of infinitely differentiable scalar fields, to include $Emb_g(\Sigma, M)$ the space of all spacelike embeddings. The phase space will similarly be enlarged from the co-tangent bundle of $C^{\infty}(\Sigma, \mathbb{R})$ ($T^{\star}C^{\infty}(\Sigma, \mathbb{R})$ the space coordinatised by $(\phi, \pi)\, \phi \in C^{\infty}(\Sigma, \mathbb{R}), \pi \in T^{\star}_{\phi}C^{\infty}(\Sigma, \mathbb{R})$) to include $T^{\star}Emb_g(\Sigma, M)$. This new phase space will have coordinates $(\phi, \pi, X^{\alpha}, P_{\alpha})$ where $X^{\alpha}(\sigma) := X^{\alpha} \circ Y(\sigma)$ and $P_{\alpha}(\sigma) dX^{\alpha} := P(\sigma)$ for $P \in T^{\star}_{Y_t} Emb_g(\Sigma, M)$. Now $\phi, \pi$ are to be regarded as functions on $\Sigma \times \mathbb{R}$ independent of the embedding variables and the explicit dependence on the embedding is to be found in $q_{ab}[Y, \sigma, t)$, $N[Y, \sigma, t)$ and $N^a[Y, \sigma, t)$.

[28]See [3, 36] for discussions of parametrised field theories.

We define a new action equivalent to the canonical action

$$S_{Can} = \int_{\mathbb{R}} dt \int_{\Sigma} d^3\sigma \left(\pi\dot{\phi} - NH^{\phi} - N^a H^{\phi}_a\right)$$

but which is also a functional of the embedding variables[29] $Y(\sigma, t)$ as well as the scalar field, it is:

$$S[Y, \phi, \pi] = \int_{\mathbb{R}} dt \int_{\Sigma} d^3\sigma \left(\pi\dot{\phi} - H^{\phi}_{\alpha}\dot{X}^{\alpha}\right) \tag{3.91}$$

where $\dot{X}^{\alpha} := \frac{\partial}{\partial t}(X^{\alpha} \circ Y(\sigma, t)) = Nn^{\alpha}[Y, \sigma] + N^a (X^{\alpha} \circ Y(\sigma, t))_{,a}$ and $H^{\phi}_{\alpha}$ is defined by:

$$H^{\phi}_{\alpha} := -n_{\alpha}[Y, \sigma]H^{\phi}[Y, \phi, \pi; \sigma) + H^{\phi}_a[\phi, \pi; \sigma)X^a_{\alpha}[Y, \sigma) \,. \tag{3.92}$$

The object $X^a_{\alpha}[Y, \sigma)$ is defined as:

$$X^a_{\alpha}[Y, \sigma) := g_{\alpha\beta}(Y(t, \sigma))q^{ab}(\sigma)\frac{\partial}{\partial \sigma^b}(X^{\beta} \circ Y(\sigma, t)) \tag{3.93}$$

and we have explicitly included all dependence on the embedding variables.

This action does the job because

$$\begin{aligned} H^{\phi}_{\alpha}\dot{X}^{\alpha} &= \left(-n_{\alpha}[Y, \sigma)H^{\phi}[Y, \phi, \pi; \sigma) + H^{\phi}_a[\phi, \pi; \sigma)X^a_{\alpha}[Y, \sigma)\right)\left(Nn^{\alpha}[Y, \sigma) + N^b (X^{\alpha} \circ Y(\sigma, t))_{,b}\right) \\ &= NH^{\phi}[Y, \phi, \pi; \sigma) + N^a H^{\phi}_a[\phi, \pi; \sigma) \end{aligned} \tag{3.94}$$

using $n^{\alpha}n_{\alpha} = -1$, $n^{\alpha}(X^{\alpha} \circ Y(\sigma, t))_{,b} = 0$ and the chain rule $X^a_{\alpha}[Y, \sigma)(X^{\alpha} \circ Y(\sigma, t))_{,b} = \delta^b_a$ and also note that the momentum density $H^{\phi}_a[\phi, \pi; \sigma)$ is not a functional of the embedding.

Now one can compute the momentum conjugate $P_{\alpha}$ to the embedding[30] as:

$$P_{\alpha} := \frac{\delta S[Y, \phi, \pi]}{\delta \dot{X}^{\alpha}} = -H^{\phi}_{\alpha} \tag{3.95}$$

but observe that $H^{\phi}_{\alpha}$ is independent of $\dot{X}^{\alpha}$ by (3.92) and hence (3.95) cannot be inverted for the velocity $\dot{X}^{\alpha}$ and hence is a constraint so we have:

$$H_{\alpha} := P_{\alpha} + H^{\phi}_{\alpha} = 0 \tag{3.96}$$

as a constraint of the parametrised theory. We can implement this constraint in the action (3.91) by means of a Lagrange multiplier $\lambda^{\alpha}(\sigma, t)$ as follows:

$$S[Y, P, \phi, \pi; \lambda^{\alpha}] = \int_{\mathbb{R}} dt \int_{\Sigma} d^3\sigma \left(P_{\alpha}\dot{X}^{\alpha} + \pi\dot{\phi} - \lambda^{\alpha}H_{\alpha}\right) \,. \tag{3.97}$$

The equations of motion for this action are:

$$0 = \frac{\delta S[Y, P, \phi, \pi; \lambda^{\alpha}]}{\delta P_{\alpha}} \Rightarrow \dot{X}^{\alpha} - \lambda^{\alpha} = 0 \tag{3.98}$$

$$0 = \frac{\delta S[Y, P, \phi, \pi; \lambda^{\alpha}]}{\delta \lambda^{\alpha}} \Rightarrow H_{\alpha} = 0 \tag{3.99}$$

[29] Following [29] we are using $Y(\sigma, t)$, which is strictly the diffeomorphism to $M$ rather than the embedding $Y_t(\sigma)$.

[30] We mean $\delta\dot{X}^{\alpha} := X^{\alpha} \circ \delta\dot{Y}$ here and similarly $\delta X^{\alpha} := X^{\alpha} \circ \delta Y$.

the variation with respect to the embedding $X^\alpha$ ensures that the constraint (3.96) is preserved in time. Finally variation with respect to $\phi, \pi$ just reproduces the original Hamilton equations of motion, i.e. with respect to the Hamiltonian in (3.88).

The next step is to decompose the constraint $H_\alpha$, a co-tangent vector on $Emb_g(\Sigma, M)$, into its tangential and normal components (because the spacetime metric is not dynamic here these components are fixed functionals of the embedding), we have:

$$H_\alpha(\sigma) = -H(\sigma)n_\alpha + H_a(\sigma)X^a_\alpha(\sigma) \tag{3.100}$$

where $H(\sigma) := n^\alpha H_\alpha$ and $H_a := X^\alpha_a H_\alpha$ using this decomposition the action can be written as:

$$S = \int_{\mathbb{R}} dt \int_\Sigma d^3\sigma \left( P_\alpha \dot{X}^\alpha + \pi\dot{\phi} - \lambda H - \lambda^a H_a \right) . \tag{3.101}$$

Note variation with respect to $P^\alpha$ ensures that $\lambda, \lambda^a$ can be interpreted as lapse and shift, see (3.98). The quantities $H$ and $H_a$ are the Hamiltonian and diffeomorphism constraints of the parametrised theory.

There are two non-trivial results, [29], concerning the Poisson brackets of this theory: i.) $\{H_\alpha(\sigma), H_\beta(\sigma')\} = 0$ and ii.) $H(\sigma)$ and $H_a(\sigma)$ satisfy the Dirac algebra. We stress that although the 3-metric is not a canonical variable itself here- the spacetime metric is not dynamic- it is a functional of the embedding and therefore in the parametrised theory is a phase space dependent object. This means the appearance of $q^{ab}[X, \sigma)$ in the Dirac algebra has to be interpreted as a structure function, as before for canonical gravity, and hence we do not have a Lie algebra and that seems to imply a loss of the diffeomorphism group. However, the difference is that now the embeddings are dynamical variables and these results do allow the construction of a homomorphism from $L\text{Diff}(M)$ to the Poisson algebra of the parametrised theory.

**Homomorphism from $L$Diff$(M)$ to the Poisson algebra of the parametrised scalar theory**

Our goal is to find a homomorphism from $L\text{Diff}(M)$ to the Poisson algebra of the parametrised theory. The phase space of this theory can be defined as $T^\star Emb_g(\Sigma, M) \times T^\star C^\infty(\Sigma, M)$ subject to the constraints defined in (3.96). In [29] this homomorphism is defined in stages, which we now describe.

Firstly, in order to deal with the sign difference between the Lie bracket of elements of $LDiff(M)$ and the commutator an anti-homomorphism is defined between $L\text{Diff}(M)$ and the algebra of vector fields with the commutator as product.

Secondly, we recall the group representation of $\text{Diff}(M)$ on the space of embeddings in (3.84). This implies there is an induced representation on the algebra, so a vector field $V$ generating a diffeomorphism $\phi$ can be mapped to a vector field $v[V]$ on $Emb(\Sigma, M)$ such that it is a Lie algebra homomorphism. The image $v[V]$ can be defined as:

$$\begin{aligned} v[V] : Emb(\Sigma, M) \times \Sigma &\rightarrow TM \\ v[V] : (Y, \sigma) &\mapsto v[V](Y, \sigma) = V^\alpha(Y(\sigma))\frac{\partial}{\partial X^\alpha}|_{X=Y(\sigma)} \end{aligned} \tag{3.102}$$

and one has the homomorphism $[v[U], v[V]] = v[[U, V]]$. The problem is that in the canonical theory one is not interested in $Emb(\Sigma, M)$ but spacelike embeddings $Emb_g(\Sigma, M)$. This latter space is not invariant under the action of the diffeomorphism group because as mentioned previously there always exists a diffeomorphism which can map a spacelike surface to a non-spacelike one. However, if one restricts to diffeomorphisms sufficiently close to the identity then the spacelike surface can be preserved, this is because $Emb_g(\Sigma, M)$ is an open sub-manifold of $Emb(\Sigma, M)$. The result is that the representation of $LDiff(M)$ by vector fields on $Emb(\Sigma, M)$ can be restricted to $Emb_g(\Sigma, M)$. The price to be paid is that these vector fields are not complete in $TEmb_g(\Sigma, M)$ in other words they define integral curves that will not remain in $Emb_g(\Sigma, M)$.

Thirdly, we use a general result that there exists an anti-homomorphism from the space of vector fields on a configuration manifold $Q$ with coordinates $q^a$ to the Poisson algebra of phase space functions on $T^\star Q$ . The map in question can be defined on a vector field $v = v^a \frac{\partial}{\partial q^a}$ on $Q$ by mapping it to the phase space function $p(v)$ as follows:

$$v \mapsto p(v) = v^a(q) p_a \tag{3.103}$$

where $p_a$ is the momentum conjugate to $q^a$. Further, the rate of change of a function $g$ along the integral curves generated by the vector $v$ can now be represented as a Poisson bracket,

$$\begin{aligned} \dot{g}(q) &= (vg)(q) \\ &= \{g, p(v)\} \ . \end{aligned} \tag{3.104}$$

This map is an anti-homomorphism, i.e. one can show:

$$p([u, v]) = -\{p(u), p(v)\} \tag{3.105}$$

and we can apply this result to the parametrised field theory where $Q \to Emb_g(\Sigma, M)$. This means the map[31]:

$$\begin{aligned} v[U] \mapsto \langle P, v[U] \rangle [X^\alpha, P_\alpha] &:= \int_\Sigma d^3\sigma v[U](Y, \sigma) P(\sigma) \\ &= \int_\Sigma d^3\sigma U^\alpha(Y(\sigma)) P_\alpha(\sigma) \\ &=: P(U)[X^\alpha, P_\alpha] \end{aligned} \tag{3.106}$$

where $v[U]$ is a vector field on $Emb_g(\Sigma, M)$ and $P$ is co-vector field on $Emb_g(\Sigma, M)$ is an anti-homomorphism, i.e. $\{P(U), P(V)\} = -P([U, V])$. So combining these three steps one has a homomorphism from LDiff$(M)$ to the Poisson algebra of functions on $T^\star Emb_g(\Sigma, M)$, i.e. $C^\infty(T^\star Emb_g(\Sigma, M), \mathbb{R})$.

As in (3.104) we can generate the evolution of functions $g[X]$ on $Emb_g(\Sigma, M)$ by computing the Poisson bracket with respect to $P(U)$, i.e.

$$\dot{g}[X] = \{g, P(U)\} \tag{3.107}$$

by "evolution" here we mean the change in $g[X]$ along the flow of the vector field $v[U]$.

The final step is to extend this homomorphism to the Poisson algebra of functions on the full phase space $T^\star Emb_g(\Sigma, M) \times T^\star C^\infty(\Sigma, M)$. One can do this using the map $P(U)$ in (3.106) by noting that it has a trivial action on the scalar field variables. However, one can modify the homomorphism so that it actually generates the evolution of all the dynamical variables, the map then becomes the Hamiltonian $H(U)$ defined as:

$$\begin{aligned} H(U)[X^\alpha, P_\alpha, \phi, \pi] : &= \int_\Sigma d^3\sigma U^\alpha(Y(\sigma)) H_\alpha(\sigma) \\ &= P(U) + \int_\Sigma d^3\sigma U^\alpha(Y(\sigma)) H^\phi_\alpha(\sigma) \\ &=: P(U)[X, P] + H^\phi(U)[X, \phi, \pi] \end{aligned} \tag{3.108}$$

[31]The first line below defines the appropriate notion of dual hinted at earlier in the discussion of vectors and co-vectors on $Emb_g(\Sigma, M)$.

when $H(U)$ acts on $g[X]$ it reproduces the result in (3.107) because $H^{\phi}(U)$ Poisson commutes with $g[X]$ as it is independent of the momentum. However, when the Hamiltonian acts upon the scalar field variables $\phi, \pi$ the $P(U)$ term Poisson commutes so that one is left with $H^{\phi}(U)$ and hence reproduces the Hamiltonian in (3.92).

One can confirm that $H(U)$ is a homomorphism by using the fact that i.) the $H_{\alpha}(\sigma)$ Poisson commute, see below (3.101) and ii.) $H^{\phi}_{\alpha}$ Poisson commutes with the embeddings $X^{\alpha}$. We have:

$$
\begin{aligned}
\{H(U), H(V)\} &= \int_{\Sigma} d^3\sigma \int_{\Sigma} d^3\sigma' \left\{U^{\alpha}(Y(\sigma))H_{\alpha}(\sigma), V^{\beta}(Y(\sigma'))H_{\beta}(\sigma')\right\} \\
&= \int_{\Sigma} d^3\sigma \int_{\Sigma} d^3\sigma' \Big[\left\{U^{\alpha}(Y(\sigma)), P_{\beta}(\sigma')\right\} H_{\alpha}(\sigma)V^{\beta}(Y(\sigma')) \\
&\quad + \left\{P_{\alpha}(\sigma), V^{\beta}(Y(\sigma')\right\} U^{\alpha}(Y(\sigma))H_{\beta}(\sigma')\Big] \\
&= \int_{\Sigma} d^3\sigma \int_{\Sigma} d^3\sigma' \left[U^{\alpha}_{,\beta}H_{\alpha}(\sigma)V^{\beta}(Y(\sigma')) - V^{\beta}_{,\alpha}U^{\alpha}(Y(\sigma))H_{\beta}(\sigma')\right] \delta^3(\sigma, \sigma') \\
&= -\int_{\Sigma} d^3\sigma H_{\beta}(\sigma) \left[V^{\beta}_{,\alpha}U^{\alpha} - V^{\alpha}U^{\beta}_{,\alpha}\right] \\
&= -\int_{\Sigma} d^3\sigma H_{\beta}(\sigma) \left[U, V\right]^{\beta} \\
&= -H([U, V])
\end{aligned}
\tag{3.109}
$$

where we used the simple result:

$$
\begin{aligned}
\left\{U^{\alpha}(Y(\sigma)), P_{\beta}(\sigma')\right\} &= \frac{\partial U^{\alpha}}{\partial X^{\gamma}} \left\{X^{\gamma}(\sigma), P_{\beta}(\sigma')\right\} \\
&= \frac{\partial U^{\alpha}}{\partial X^{\gamma}} \delta^{\gamma}_{\beta}\delta^3(\sigma, \sigma') \\
&= U^{\alpha}_{,\beta}\delta^3(\sigma, \sigma') \, .
\end{aligned}
$$

Hence we have found a homomorphism from $L\mathrm{Diff}(M)$ to the Poisson algebra of phase space functions of the parametrised scalar field theory.

### Applying the method to canonical gravity

In [30] the Isham Kuchař method is used to obtain a representation of $L\mathrm{Diff}(M)$ in the Poisson algebra of an extended canonical gravity phase space. There is one key subtlety, which we briefly discuss before stating the main result.

First in GR, by contrast to the parametrised field theory, the lapse and shift are not definite functionals of the embedding this is because the spacetime metric is dynamic and the notion of orthogonal and tangential vectors cannot be defined until the metric is also known- it is not enough to have the embedding only. In the GR case we wish to make the lapse and shift definite functionals of the embedding and gravitational variables $\left(q_{ab}, P^{ab}\right)$. However in order to do so one must implement a *spacetime gauge fixing* on the metric $g_{\mu\nu}$ (which removes the diffeomorphism redundancy) this is done by imposing an additional structure to which one fixes four projections of the spacetime metric. This has the effect of preventing arbitrary variations in the lapse and shift and suspends the Hamiltonian and diffeomorphism constraints, which will need to be recovered at the end of the process.

The additional conditions imposed are defined with respect a reference foliation $Y^{Ref} : \Sigma \times \mathbb{R} \to M$, its inverse enables space and time to be defined by $Y^{-1}_{Ref}(X) = \sigma(X) \times \tau(X) \in \Sigma \times \mathbb{R}$. In [30] the following class of metrics and embeddings are used i.) all globally hyperbolic spacetime metrics $g \in GHPRiem(M)$ and ii.) all embeddings[32] $X \in Emb(\Sigma, M)$; the latter because in GR we cannot specify an external metric, with which to define the notion of spacelike as we could with the scalar field theory. Finally we denote by $Emb_{Y^{Ref}}(\Sigma, M)$ the space of embeddings which may be continuously deformed into one of the embeddings $Y_t^{Ref}$. The extended configuration space for GR will then be taken as $Emb_{Y^{Ref}}(\Sigma, M) \times Riem\Sigma$, where $Riem\Sigma$, the space of Riemannian metrics on $\Sigma$, is the original configuration space for canonical GR. We remark that for any pair $(X, q_{ab})$ there exists a spacetime metric $g$ such that $q = X^\star g$, in fact there exist infinitely many.

The four conditions imposed on $g$ are as follows:

$$g^{-1}(d\tau, d\tau) = -1 \tag{3.110}$$

$$g^{-1}(d\tau, \sigma^\star(\cdot)) = 0 \tag{3.111}$$

the sub-set of globally hyperbolic metrics which satisfy these conditions will be called Gaussian with respect to the diffeomorphism $Y^{Ref}$ , $g_{Gauss} \in GaussPRiem_{Y^{Ref}}M$. One could regard the above conditions as fixing the spacetime gauge if every globally hyperbolic metric $g$ could be mapped to a Gaussian metric $g_{Gauss}$ by a diffeomorphism $\phi$, i.e. $g_{Gauss} = \phi^\star g$.[33] Unfortunately, this only holds in some finite time of a particular slice $Y_t$.

These conditions enable one to express, [30], the spacetime metric $g_{\alpha\beta}(X(\sigma))$ as a linear function of $q_{ab}(\sigma)$ and functional of the embedding $X$, i.e. $g = g[X, q_{ab}(\sigma))$. It follows that because the covectors $n_\mu$and $X^a_\mu$ are dependent on the spacetime metric (see (3.93) for the latter) they may also be expressed as functionals of the embedding and functions of the induced metric $q_{ab}$. So the original ADM action $S_{ADM}[q_{ab}, P^{ab}, N, N^a]$, (3.36), can now be written as:

$$S'_{ADM}[q_{ab}, P^{ab}; X] = \int_{\mathbb{R}} dt \int_\Sigma d^3\sigma \left( P^{ab}\dot{q}_{ab} - H^{GR}_\alpha \dot{X}^\alpha \right) \tag{3.112}$$

which is the analogue of (3.91) for the parametrised scalar field and where we have:

$$\dot{X}^\alpha = N[X, q)n^\alpha[X, q_{ab}(\sigma)) + N^a[X, q)(X^\alpha_{,a}) \tag{3.113}$$

and

$$H^{GR}_\alpha := -Hn_\alpha + H_a X^a_\alpha\,. \tag{3.114}$$

Now the lapse and shift can no longer be be varied whilst keeping our additional structure (the reference embedding and gauge conditions) fixed hence, in contrast to $S_{ADM}$, $S'_{ADM}$ is no longer a functional of lapse and shift. As mentioned above this means one does not recover the Hamiltonian and diffeomorphism constraints and they have been lifted in this formalism. Again as with the scalar field we can compute the momentum conjugate to the embedding to obtain:

$$P_\alpha := \frac{\delta S[X, q_{ab}, P^{ab}]}{\delta \dot{X}^\alpha} = -H^{GR}_\alpha\,. \tag{3.115}$$

As with the scalar field case, this momentum cannot be inverted in terms of the velocity and hence is a constraint. We have:

$$\Pi_\alpha := P_\alpha - Hn_\alpha + H_a X^a_\alpha = 0 \tag{3.116}$$

and can implement this constraint by using Lagrange multipliers $N, N^a$ to obtain:

$$S[q_{ab}, P^{ab}; X, P, N, N^a] = \int_{\mathbb{R}} dt \int_\Sigma d^3\sigma \left( P^\alpha \dot{X}_\alpha + P^{ab}\dot{q}_{ab} - N\Pi - N^a \Pi_a \right) \tag{3.117}$$

[32]We shall use $X$ for an embedding in this sub-section to avoid confusion with the reference foliation $Y$.

[33]Strictly one should require a unique such diffeomorphism in order to provide a perfect gauge fixing.

where $\Pi := n^\alpha \Pi_\alpha = n^\alpha[X,q]P_\alpha + H$ and $\Pi_a := X^\alpha_{,a}\Pi_\alpha = X^\alpha_{,a}P_\alpha + H_a$.

Variation with respect to $N, N^a$ recovers the constraints $\Pi_\alpha = 0$, variation with respect to $P_\alpha$ ensures that the multipliers can be interpreted as lapse and shift[34]. Finally, we can compute the evolution of the gravitational variables$\left(q_{ab}, P^{ab}\right)$ using Hamilton's equations:

$$\begin{aligned}
\dot{q}_{ab}(\sigma) &= \left\{q_{ab}(\sigma), \int_\Sigma d^3\sigma' N\Pi + N^a\Pi_a\right\} \\
&= \left\{q_{ab}(\sigma), \int_\Sigma d^3\sigma' N\left(n^\alpha[X,q]P_\alpha + H\right) + N^a\left(X^\alpha_{,a}P_\alpha + H_a\right)\right\} \\
&= \left\{q_{ab}(\sigma), \int_\Sigma d^3\sigma' NH + N^a H\right\} \qquad (3.118)
\end{aligned}$$

where the final line follows because only the expressions $H, H_a$ contain the gravitational momentum $P^{ab}$ and hence can give non-zero Poisson brackets with the 3-metric. Of course (3.118) just reproduces the standard ADM Hamilton equation for the 3-metric. We now do the same for the momentum:

$$\begin{aligned}
\dot{P}^{ab}(\sigma) &= \left\{P^{ab}(\sigma), \int_\Sigma d^3\sigma' N\Pi + N^a\Pi_a\right\} \\
&= \left\{P^{ab}(\sigma), \int_\Sigma d^3\sigma' N\left(n^\alpha[X,q]P_\alpha + H\right) + N^a\left(X^\alpha_{,a}P_\alpha + H_a\right)\right\} \\
&= \left\{P^{ab}(\sigma), \int_\Sigma d^3\sigma' NH + N^a H_a\right\} + \int_\Sigma d^3\sigma' NP_\alpha\left\{P^{ab}(\sigma), n^\alpha[X,q]\right\} \qquad (3.119)
\end{aligned}$$

the final line follows because the terms $X^\alpha_{,a}P_\alpha$ are both independent of the 3-metric (notice $X^a_\alpha[X,q]$ depends on the metric because we have to lower a spacetime index). In this case the first term above reproduces the ADM Hamilton equation for the momentum but we also have a second term. This is a result of the fact that $n^\alpha$ depends on the 3-metric and hence it does not Poisson commute with $P^{ab}$. Observe that this second term can be (weakly) dropped when the original Hamiltonian and diffeomorphism constraints are recovered since the coefficient $P_\alpha \simeq 0$ in that case. Hence we can conclude that the action (3.117) is equivalent to the original ADM action when we impose the Hamiltonian and diffeomorphism constraints. As in the parametrised scalar a key result, [30], now follows:

$$\{\Pi_\alpha(\sigma), \Pi_\beta(\sigma')\} = 0\,. \qquad (3.120)$$

This enables us to use the same steps as above to obtain a representation of $L\mathrm{Diff}(M)$ in the Poisson algebra of the extended phase space. The following map is the homomorphism:

$$V \mapsto \Pi(V)[X^\alpha, P_\alpha, q_{ab}, P^{ab}] := \int_\Sigma d^3\sigma V^\alpha(X(\sigma))\Pi_\alpha(\sigma) \qquad (3.121)$$

where $V$ is a vector in $L\mathrm{Diff}(M)$.

The final step is to re-introduce the original constraints in order to ensure that this canonical system reconstructs a spacetime metric, which satisfies Einstein's equations. This is achieved by varying the gauge conditions $\tau(X)$ and $\sigma(X)$. An arbitrary variation in these conditions together with variations in the embedding and the 3-metric is equivalent to an arbitrary variation in the spacetime metric [30] and hence an arbitrary variation in lapse and shift. This ensures that the

[34] As pre-empted in the notation $N, N^a$.

original Hamiltonian and diffeomorphism constraints can be recovered but without affecting the homomorphism defined above because we always maintain the Gaussian conditions, we just vary the functions used to define it. Hence a representation of $L\text{Diff}(M)$ has been found in an extended canonical gravity phase space.

We have shown that there exists a representation of $L\text{Diff}(M)$ in canonical gravity. However, we would like to mention several points regarding this work. Firstly the representation involves vector fields on a manifold $Emb_g(\Sigma, M)$, which are not complete as the representation does not extend to $\text{Diff}(M)$. Secondly, the gauge fixing used cannot be defined globally on the phase space so the results derived here have to be considered as having local validity in phase space only. Finally, Gaussian conditions were used to gauge fix the metric but this is not mandatory other choices are possible, see [31].

### 3.6 Asymptotically flat case

In this section we conclude by considering what happens when we extend the analysis to asymptotically flat spacetimes, which are non-compact with boundary and which have appropriate fall off conditions on the spacetime metric motivated by the spacetime surrounding a compact object, e.g. a star or black hole alone in the universe. We have used [5, 32] as the principal references for this section.

From the covariant perspective the case of spacetime boundaries (for spatially compact or non-compact spacetime) requires one to add boundary terms to the Einstein Hilbert action in order that the resulting action be functionally differentiable. One should then go through the 3+1 decomposition again and carefully consider all the surface terms that were previously dropped. An important point is that after the 3+1 analysis is performed the additional boundary terms do not contain time derivatives of the 3- metric and therefore the conjugate momentum is unchanged. The difference is that the Hamiltonian ends up with additional surface terms, which are non-zero on the constraint surface, defined by the usual Hamiltonian and diffeomorphism constraints. Rather than do this we shall simply consider whether the functionals $H(N)$ and $\vec{H}(\vec{N})$ need to be modified in asymptotically flat spacetimes. A review of boundary terms in GR can be found in [15] and a careful discussion in light of potentially different asymptotic spacetimes from the covariant view and which demonstrates equivalence to the method we are adopting here is given in [33].

We shall define asymptotic flatness by the following fall off conditions on the ADM variables:

$$q_{ab}(\sigma) = \delta_{ab} + \frac{f_{ab}(t, \frac{\sigma}{r})}{r} + h_{ab}(\sigma) \tag{3.122}$$

$$P^{ab}(\sigma) = \frac{F^{ab}(t, \frac{\sigma}{r})}{r^2} + g^{ab}(\sigma) \tag{3.123}$$

where $r = \sqrt{\sigma^a \sigma^b \delta_{ab}}$ and $h_{ab}$ and $g^{ab}$ are remainders that fall off as $r^{-1-\epsilon}$ and $r^{-2-\epsilon}$ for $\epsilon > 0$ and we also assume that the derivatives satisfy $q_{ab,c} \sim O(r^{-2})$ and $P^{ab}_{,c} \sim O(r^{-3})$. We impose opposite parity conditions on $f_{ab}$ and $F^{ab}$, even and odd respectively.

We have shown that when the equations of motion hold $H(N)$ and $\vec{H}(\vec{N})$ generate spacetime diffeomorphisms. However, in the spatially non-compact case this interpretation only holds if $N$ and $N^a$ fall off sufficiently fast as otherwise the spatial integrals defining $H(N)$ and $\vec{H}(\vec{N})$ may diverge. In the context of asymptotically flat spacetime we wish to consider decay behaviour for the smearing functions, which would correspond to asymptotic Poincarè transformations (translations, rotations and boosts), and this motivates the following:

$$
\begin{aligned}
N(t,\sigma) &= b_0 + \beta_a \sigma^a + S(t,\sigma) && (3.124)\\
N^a(t,\sigma) &= b^a + \omega_{ab}\sigma^b + S^a(t,\sigma) && (3.125)
\end{aligned}
$$

where $b_0, b^a$ are translations, $\beta^a$ boosts and the antisymmetric $\omega_{ab}$ rotations. The functions $S$ and $S^a$ are known as *supertranslations* and can be interpreted as angle dependent translations, they are defined to be odd $O(1)$ functions on the asymptotic sphere (and also include the higher powers of $r^{-1}$). In fact these group of transformations (viewed as coordinate transformations on $\{\sigma^a\}$) preserve the asymptotic flatness as defined above, [17].

The problem is that for these non-trivial fall off conditions the quantities $H(N)$ and $\vec{H}(\vec{N})$ either diverge or are not functionally differentiable. For $\vec{H}(\vec{N})$ with $N^a \sim O(1)$ one has the integrand $N^a q_{ac} D_b P^{bc} \sim O(r^{-3})$ and given the volume element is $\sim O(r^2)$ one has a logarithmic divergence. Similarly, for $H(N)$ one has a problem with the $\sqrt{q}\,{}^3RN$ term in the integrand since ${}^3R \sim {}^3R_{abcd} \sim \partial\Gamma \sim O(r^{-3})$ where we use that $q_{ab} \sim O(1)$ and $\sqrt{q} \sim O(1)$ and this implies $\sqrt{q}\,{}^3RN \sim O(r^{-3})$ and hence again we have a logarithmic divergence for $N \sim O(1)$.

The point is that by adding an appropriate boundary term one can render both the resulting functional and its variation well defined. One has for $\vec{H}(\vec{N})$

$$
\begin{aligned}
\vec{H}(\vec{N}) &= -2\int_\Sigma d^3\sigma N^a q_{ac} D_b P^{bc}\\
&= 2\int_\Sigma d^3\sigma D_b(N^a q_{ac})P^{bc} - 2\int_\Sigma d^3\sigma D_b(N^a q_{ac} P^{bc})\\
&= \int_\Sigma d^3\sigma \left(\mathcal{L}_{\vec{N}} q\right)_{bc} P^{bc} - 2\int_\Sigma d^3\sigma D_b(N^a q_{ac} P^{bc})\\
&= \int_\Sigma d^3\sigma \left(\mathcal{L}_{\vec{N}} q\right)_{bc} P^{bc} - 2\int_{\partial\Sigma} d^2 S_b(N^a q_{ac} P^{bc}) && (3.126)
\end{aligned}
$$

where in the second line we have done an integration by parts, in the third line we have used the definition of $\mathcal{L}_{\vec{N}} q$ and in the final line converted the volume integral of the divergence to one over the boundary of $\Sigma$, and $d^2 S_b$ is the volume element on the asymptotic sphere $S^2$. The variation can now be computed as:

$$
\begin{aligned}
\delta\vec{H}(\vec{N}) &= \delta\int_\Sigma d^3\sigma \left(\mathcal{L}_{\vec{N}} q\right)_{bc} P^{bc} - 2\delta\int_\Sigma d^3\sigma D_b(N^a q_{ac} P^{bc})\\
&= \int_\Sigma d^3\sigma \left(\mathcal{L}_{\vec{N}} q\right)_{bc} \delta P^{bc} + \int_\Sigma d^3\sigma \frac{\delta}{\delta q_{cd}(\sigma)}\left[\int_\Sigma d^3\sigma' \left(\mathcal{L}_{\vec{N}} q\right)_{ab} P^{ab}\right]\delta q_{cd}(\sigma)\\
&\quad -2\delta\int_{\partial\Sigma} d^2 S_b(N^a q_{ac} P^{bc})\\
&= \int_\Sigma d^3\sigma \left(\delta P^{ab}\mathcal{L}_{\vec{N}} q_{ab} - \delta q_{ab}\mathcal{L}_{\vec{N}} P^{ab}\right) - 2\delta\int_{\partial\Sigma} d^2 S_b(N^a q_{ac} P^{bc}) && (3.127)
\end{aligned}
$$

where in the first line we have used (3.126), and in the third line we have used the result that the functional derivative in the second term is $\mathcal{L}_{\vec{N}} P^{ab}$. This latter result is easy to see if one is permitted to drop a boundary term after an integration by parts however we cannot do this here as $\Sigma$ has a boundary. However, one can compute:

$$\begin{aligned}
\frac{\delta}{\delta q_{ab}(\sigma)}\left[\int_{\Sigma} d^3\sigma' \left(\mathcal{L}_{\vec{N}} q\right)_{cd} P^{cd}\right] &= \frac{\delta}{\delta q_{ab}(\sigma)}\left[\int_{\Sigma} d^3\sigma' P^{ed} 2 q_{dc} D_e N^c\right] \\
&= 2P^{c(a} D_c N^{b)} - D_c(P^{ab}N^c) \\
&= 2P^{c(a} D_c N^{b)} - N^c D_c(P^{ab}) - P^{ab} D_c N^c \\
&= -\left(\sqrt{q}\left(N^c D_c(\frac{P^{ab}}{\sqrt{q}}) - \frac{2}{\sqrt{q}} P^{c(a} D_c N^{b)}\right) + P^{ab} D_c N^c\right) \\
&=: -\mathcal{L}_{\vec{N}} P^{ab}
\end{aligned}$$

where the non-trivial step is the second line which involves using the identity (3.57), see [16], in the penultimate line we have used $D_a q = 0$ and in the final line we have just used the definition of the Lie derivative of a rank 2 contravariant tensor of density weight +1.

One can show that i.) all the integrals over $\Sigma$ in (3.126) and (3.127) are finite for the decay behaviour on $N^a$ above for both $\vec{H}(\vec{N})$ and $\delta\vec{H}(\vec{N})$ and ii.) both boundary integral terms diverge for the decay behaviour above. This motivates the definition of a new generator $\vec{J}(\vec{N})$

$$\vec{J}(\vec{N}) := \vec{H}(\vec{N}) + \vec{P}(\vec{N}) \tag{3.128}$$

where $\vec{P}(\vec{N}) := 2\int_{\partial\Sigma} d^2 S_b N^a q_{ac} P^{bc} = 2\int_{\Sigma} d^3\sigma D_b(N^a q_{ac} P^{bc})$ and whose variation $\delta\vec{J}(\vec{N})$ is the first two terms in (3.127). Using (3.126) we can see that

$$\vec{J}(\vec{N}) = \int_{\Sigma} d^3\sigma \left(\mathcal{L}_{\vec{N}} q\right)_{ab} P^{ab} \tag{3.129}$$

which is finite for the fall-off conditions on $N^a$. Finally if we let $N^a = b^a$, i.e. an asymptotic spatial translation, one has:

$$\vec{P}(\vec{N}) = b^a 2\int_{\partial\Sigma} d^2 S_b N^a q_{ac} P^{bc} =: b^a P_a^{ADM} \tag{3.130}$$

where $P_a^{ADM}$ is known as the ADM momentum.

One can repeat the above argument for $H(N)$, i.e. consider it for the above fall off conditions on $N$, also its variation and again add an appropriate boundary term to $H(N)$ to define a new functional, which is both finite and functionally differentiable. The result is that one defines a new generator $J(N)$ as:

$$J(N) = H(N) + E(N) \tag{3.131}$$

and where the appropriate boundary term $E(N)$ is given by

$$\begin{aligned}
E(N) :&= \int_{\partial\Sigma} \sqrt{q} q^{cd} q^{ef} \left[dS_d \left(q_{ef} - \delta_{ef}\right) D_c N - dS_c \left(q_{df} - \delta_{df}\right) D_e N\right] \\
&+ \int_{\partial\Sigma} \sqrt{q} q^{cd} N \left(-dS_c \Gamma^e_{ed} + dS_e \Gamma^e_{cd}\right) .
\end{aligned} \tag{3.132}$$

This expression greatly simplifies if $N$ consists of just a translation, i.e. $N = b_0$ in this case one has:

$$\begin{aligned}
E(N) &= \int_{\partial\Sigma} \sqrt{q} q^{cd} b_0 \left(-dS_c \Gamma^e_{ed} + dS_e \Gamma^e_{cd}\right) \\
&= \int_{\partial\Sigma} \delta^{cd} \delta^{ef} b_0 dS_c \left(q_{ed,f} - q_{ef,d}\right) \\
&=: 16\pi G b_o E_{ADM}
\end{aligned} \tag{3.133}$$

where in the second line we used the asymptotic properties of the spatial metric, i.e. $q^{cd} \to \delta^{cd}$, $\sqrt{q} \to 1$ and the definition of the connection components in terms of the metric. The quantity $E_{ADM}$ is known as the ADM mass or energy, when computed on for example the Schwarzschild metric $E_{ADM} = m$, where $m$ is the mass term in the Schwarzschild line element. It has been proven that $E_{ADM} \geq 0$ and that $E_{ADM} = 0$ if and only if the spacetime is Minkowski. Note that although $E(N)$ is well defined for translations it diverges for both boosts and rotations, one can see this from the first two terms in (3.132), in the integrand $D_c N \sim O(1)$, $(q_{ef} - \delta_{ef}) \sim O(r^{-1})$, $dS_d \sim O(r^2)$ and one can ignore the prefactor $\sqrt{q} q^{cd} q^{ef}$ as it is $O(1)$ even and hence one has an $O(r)$ integrand to be computed on the asymptotic sphere in the limit $r \to \infty$, which diverges.

The result is that one now has two new generators $J(N)$ and $\vec{J}(\vec{N})$ which are well defined for the above asymptotic behaviour on $N, N^a$ leading to asymptotic (generalised, including supertranslations) Poincarè transformations. The question then is how one should interpret the transformations generated by $J(N)$ and $\vec{J}(\vec{N})$, i.e. do they generate gauge transformations in the same sense as $H(N)$ and $\vec{H}(\vec{N})$ for the spatially compact case. In the case of supertranslations one can see that $J(N) = H(N)$ and $\vec{J}(\vec{N}) = \vec{H}(\vec{N})$ this is because for $E(S)$ and $\vec{P}(\vec{S})$ the integrand is even with respect to the odd measure $dS_a$ and hence the boundary terms vanish identically. Hence for supertranslations $J(S)$ and $\vec{J}(\vec{S})$ should be viewed as generating gauge transformations. However, for proper asymptotic Poincarè transformations $J(N)$ and $\vec{J}(\vec{N})$ need to considered as independent functionals and the interpretation of their action is not clear.

This question can be addressed after computing the algebra of $J(N)$ and $\vec{J}(\vec{N})$. This algebra is calculated in [5] and it is the Dirac algebra, which we state in compact form as:

$$
\begin{aligned}
\left\{ J(N_1, \vec{N}_1), J(N_2, \vec{N}_2) \right\} &= J\left( \mathcal{L}_{\vec{N}_2} N_1 - \mathcal{L}_{\vec{N}_1} N_2, \mathcal{L}_{\vec{N}_2} \vec{N}_1 - \vec{N}_{12}(q) \right) \\
&=: J(N_3, \vec{N}_3)
\end{aligned}
\tag{3.134}
$$

where $J(N_1, \vec{N}_1) := J(N_1) + \vec{J}(\vec{N}_1)$ and $\vec{N}_{12}(q) = q^{ab}\left(N_1 \partial_b N_2 - N_2 \partial_b N_1\right)$. The difference now is that we have realised the Dirac algebra for non-trivial asymptotic behaviour of lapse and shift, which using $H(N, \vec{N})$ would have led to ill-defined functionals.

First we consider the case where both $N_1, \vec{N}_1$ and $N_2, \vec{N}_2$ are supertranslations in this case one finds that $N_3, \vec{N}_3$ are also supertranslations. This means that the algebra closes for supertranslations. Secondly, if one of $N_1, \vec{N}_1$ and $N_2, \vec{N}_2$ is a supertranslation and the other non-trivial then one has that $N_3, \vec{N}_3$ are supertranslations. In other words the action of a supertranslation on a non-trivial Poincarè generator is to turn it into a supertranslation generator. Since the supertranslation generators equal $H(N, \vec{N})$, which vanish on the constraint surface, one has the important result that the non-trivial generators weakly Poisson commute with the gravitational constraints. This means that $J(N_1, \vec{N}_1)$ for Poincarè transformations are by definition Dirac observables for gravity. Finally, if both $N_1, \vec{N}_1$ and $N_2, \vec{N}_2$ are non-trivial then $N_3, \vec{N}_3$ are non-trivial.

On the constraint surface all dependence upon the supertranslations drops out and hence one can define $J(N_1, \vec{N}_1)$ by using the ten parameters $b_o, b^a, \omega_{ab}$, and further asymptotically we can replace $q^{ab}$ in $\vec{N}_{12}$ by $\delta^{ab}$which removes the presence of structure functions when we consider only the non-trivial asymptotic behaviour. Then we can define:

$$
J(N_1, \vec{N}_1) =: b_o P^0 + \beta^a B_a + b^a P_a + \phi^a J_a \tag{3.135}
$$

where $\omega_{ab} = \epsilon_{abc}\phi^c$. By then using the notation: $-M^{a0} = M^{0a} := B_a$, $M^{ab} := \epsilon^{acb} J_c$ one has:

$$
J(N_1, \vec{N}_1) = b_\alpha P^\alpha + \frac{1}{2}\omega_{\alpha\beta} M^{\alpha\beta} \tag{3.136}
$$

and finally using the algebra above one can conclude that $P^\alpha, M^{\alpha\beta}$ satisfy the Poincarè algebra,

$$\begin{aligned} \left\{P^\alpha, P^\beta\right\} &= 0 \\ \left\{M^{\alpha\beta}, P^\mu\right\} &= \eta^{\mu\alpha}P^\beta - \eta^{\mu\beta}P^\alpha \\ \left\{M^{\alpha\beta}, M^{\mu\nu}\right\} &= -\left(\eta^{\mu\alpha}M^{\beta\nu} - \eta^{\mu\beta}M^{\alpha\nu} - \eta^{\nu\alpha}M^{\beta\mu} + \eta^{\nu\beta}M^{\alpha\mu}\right) \end{aligned} \tag{3.137}$$

hence we have managed to recover the Poincarè group on the constraint surface in the algebra of the $J(N_1, \vec{N}_1)$.

The conclusion is that because i.) $J(N_1, \vec{N}_1)$ for Poincarè transformations are non-zero on the constraint surface, and ii.) they are Dirac observables, they represent physical observations of an asymptotic observer, (e.g. $P^0 = E_{ADM}$ is the ADM mass of the space) and we should therefore not consider them as generators of gauge transformations but rather as generating real observable transformations, e.g. under a boost at infinity the ADM mass transforms non-trivially as the zeroth component of a four vector. The fact that $P^0, P^a, J^a$ define Dirac observables to be interpreted as energy, momentum and angular momentum is an important result, in fact they are the only non-trivial globally defined Dirac observables on the canonical phase space and exist only in the asymptotically flat case.

In the case of compact $\Sigma$ without boundary it is a result, [34], that there are no *local* Dirac observables, by local we mean that there are no observables which can be expressed as spatial integrals of densities over $\Sigma$ involving only a finite number of derivatives of the canonical variables $q_{ab}, P^{ab}$. Certainly the ten Poincarè charges do not capture the dynamics of the gravitational field and it is an important problem as to how one can define further observables for GR. In the next section we shall consider one approach to this problem by discussing relational observables for both first class systems and then GR in particular.

# 4 Dirac Observables in GR

We have seen that the Hamiltonian is totally constrained in GR and therefore that time evolution in the Dirac picture appears to be the unfolding of a gauge transformation. This is in stark contrast with our perception of the dynamics of gravity, for example the expansion of the universe. Furthermore observables, defined by Dirac to be invariant under the first class constraints of the theory, will be frozen as this invariance implies that $\frac{dO}{dt} = \{O, H(N, N^a)\} = 0$ for any such observable $O$. Hence it appears that nothing changes in canonical gravity! This is the 'frozen formalism' and is in essence the famous problem of time in canonical gravity. Notice that this occurs already at the classical level. Important reviews of this topic can be found in [35] and [36] and more recently in [37]. This problem has also been considered from a philosophical viewpoint in [38] and [39].

In this section we shall describe one particular approach to addressing this conceptual problem. This involves describing observables not with respect to co-ordinates but rather with respect to other dynamical fields, for example matter fields. More precisely consider a first class Hamiltonian system with one constraint and two gauge variant functions $f$ and $g$ the goal is to compute the value of $f$ when $g$ takes a particular value say $g_0$. One can show that this construction is indeed gauge invariant[35] and assuming that $g$ takes values in the continuum of $\mathbb{R}$ one will have constructed a 1-parameter family of gauge invariant functions and it is this family that will represent our perception of evolution. Since each element of the family is gauge invariant it will by definition be constant in co-ordinate time and for this reason this construction is called the 'Evolving Constants' approach. The phase space function $g$ is acting as a "physical clock" and it is with respect to this generalised notion of clock that evolution is understood. Unfortunately, this approach is not directly applicable to GR as for gravity we have an infinite number of constraints and in this section we shall describe how it may be generalised to accommodate gravity. (The short answer is that there will be a physical clock for every one of the constraints of GR).

The construction of gauge invariant observables in GR is important because they will be required in canonical quantization. In quantization these observables will become operators and their Poisson algebra will need to be represented on the physical Hilbert space. This representation will be required[36] regardless of whether one performs a reduced phase space quantization or a Dirac quantization .[37]

As mentioned in the previous section, below (3.40), the Bergmann interpretation of gauge transformation avoids one identifying time evolution with a gauge transformation, as in this picture gauge transformations act on phase space trajectories and not points. However, of course there is still a notion of Bergmann gauge invariant observable and this will have to Poisson commute with the gauge generator, which is still a combination of first class constraints, (3.80).

We have chosen to focus on this particular approach as i.) there has been a significant amount of research conducted over the last two decades since [35] and [36], and ii.) the idea of co-incidental observables removes dependence on co-ordinates, which in any generally covariant theory cannot be considered as invariant.

[35] We shall do so in this section.

[36] See introduction in [40].

[37] In a 'reduced phase space' quantization one solves the constraints at a classical level, by constructing the quotient of the phase space with the gauge orbits, and then quantizes the resulting coset space to obtain the physical Hilbert space. In 'Dirac quantization' one quantizes the entire classical phase space and then reduces to the physical Hilbert space by selecting only those state vectors which are elements of the kernel of the operator versions of the constraints. (Strictly speaking this procedure is implemented in a generalised sense for canonical gravity using tools from functional analysis, [5]). In general these procedures do not yield equivalent quantum theories.

## 4.1 Relational Observables for finite dimensional constrained systems

In order to introduce the idea of relational observables we first consider finite dimensional first class constrained systems and then subsequently generalise to field theories. Our discussion is primarily based upon [41] and [42].

We assume a Hamiltonian system with a finite number of constraints $C_j,\ j = 1,\ldots,n$ , a $2p$ dimensional phase space $\Gamma$ ( with points $x \in \Gamma$), and canonical co-ordinates $(q_a,\ p_a,\ a = 1,\ldots,p)$ such that $\{q_a, p_b\} = \delta_{ab}$. We can define the flow $\alpha^t_C(x)$ (generated by a phase space function $C$) of $x \in \Gamma$ as the curve $c : \mathbb{R} \ni t \to \alpha^t_C\,(x) \in \Gamma$, that has tangent vector equal to the Hamiltonian vector field $\chi_C$[38]. Based on this flow one can define an action or flow on phase space functions by

$$\alpha^t_C(f)(x) := f(\alpha^t_C(x))\,. \tag{4.1}$$

This action can be calculated by the series

$$\alpha^t_C(f)(x) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \{C, f\}_m\,(x) \tag{4.2}$$

where the iterated Poisson bracket is defined by $\{C, f\}_0 := f$ and $\{C, f\}_{m+1} = \{C, \{C, f\}_m\}$.

In this formalism a gauge transformation is a map from $\Gamma$ to itself, which can be expressed as a composition of the above flows, where the generating function is a constraint. As discussed in section 1 the set of all gauge transformations of a point $x \in \Gamma$ forms the gauge orbit of $x$, denoted $G_x$ and this, in a first class system (at least on the constraint surface), is guaranteed to form an n-dimensional surface.

Given the above definitions we can now introduce the concept of partial and complete observables first used by Rovelli, see [43] for further discussion and examples. A partial observable is a physical quantity that can be measured by an observer but that cannot be uniquely predicted by theory. A complete observable by contrast is a physical quantity whose value can be predicted by the classical theory. A partial observable is therefore any phase space function, whereas a complete observable is gauge invariant and therefore must Poisson commute with the Hamiltonian constraints.

### Systems with a single constraint

In a system with one constraint $C(x)$ it is possible to use two partial observables: $f(x)$ and a "physical clock" $T(x)$ to construct a 1 parameter family, labelled by $\tau$ , of complete observables denoted $F_{[f,T]}(\tau, x)$. This complete observable is defined to be:

$$F_{[f,T]}(\tau, x) := \alpha^t_C(f)(x)_{|\alpha^t_C(T)(x)=\tau} \tag{4.3}$$

that is $F_{[f,T]}(\tau, x)$ is the value of $f(\alpha^t_C(x))$ when $\alpha^t_C(T)(x) = T(\alpha^t_C(x)) = \tau$. Intuitively $F_{[f,T]}(\tau, x)$ is the value of the partial observable $f$ at a specified point on the gauge orbit $G_x$, determined by the value of the "clock" $T$. To actually compute this complete observable one must solve $\alpha^t_C(T)(x) = T(\alpha^t_C(x)) = \tau$ for t (let the solution be $t = t_0(\tau, x)$) and then substitute it into the expression for the flow of $f$ to get $\alpha^{t_o(\tau,x)}_C(f)(x)$. In order for this to be well defined the clock must provide a good parametrization of the gauge orbit.

*Claim* 4.1. The complete observable $F_{[f,T]}(\tau, x)$ is gauge invariant with respect to the single constraint $C(x)$.

[38] Recall for any phase space function $f$ $\chi_C$is defined by $\chi_C(f) := \{C, f\}$.

*Proof.* Since any gauge transformation can be generated by the flow $\alpha_C^t(x)$ we need to show that $\alpha_C^\beta(F_{[f,T]}(\tau,x)) = F_{[f,T]}(\tau,x)$ , for any parameter $\beta$. We have

$$\begin{aligned}
\alpha_C^\beta(F_{[f,T]}(\tau,x)): &= F_{[f,T]}(\tau,\alpha_C^\beta(x)) = \alpha_C^t(f)(\alpha_C^\beta(x))_{|\alpha_C^t(T)(\alpha_C^\beta(x))=\tau} \\
&= \alpha_C^\beta \circ \alpha_C^t(f)(x)_{|\alpha_C^\beta \circ \alpha_C^t(T)(x)=\tau} \\
&= \alpha_C^{\beta+t}(f)(x)_{|\alpha_C^{\beta+t}(T)(x)=\tau} \\
&= F_{[f,T]}(\tau,x)
\end{aligned} \tag{4.4}$$

where in the penultimate line we used the fact that composition of flows are additive in the evolution parameter. □

**Systems with several constraints**

The extension to n constraints $C_j$, $j = 1,\ldots,n$ essentially relies on finding a good parametrization of the gauge orbit. Such a good parametrization can be viewed as a gauge fixing, where $T_j(x) = \tau_j$ defines a surface that cuts every gauge orbit precisely once[39].

In such a system the orbit will be n-dimensional and therefore n "clocks" $T_j$, $j = 1,\ldots,n$ will be required. We can then define a complete observable associated to a partial observable $f(x)$ and the n clocks in an obvious extension of the single constraint case (4.3):

$$F_{[f,T_1,\ldots,T_n]}(\tau_1,\ldots,\tau_n,x) := \alpha_{\beta_j C_j}(f)(x)_{|\alpha_{\beta_j C_j}(T_i)(x)=\tau_i} \tag{4.5}$$

note that $\alpha_{\beta_j C_j}(f)(x) := \alpha^{t=1}_{\beta_j C_j}(f)(x)$. Again to compute the complete observable one has to solve the system of equations $\alpha_{\beta_j C_j}(T_i)(x) = \tau_i$ for the evolution parameters$\beta_i = \beta_i(\tau_j,x)$ and substitute the solution for $\beta_i$ into $\alpha_{\beta_j C_j}(f)(x)$.[40]

*Claim* 4.2. The complete observable $F_{[f,T_1,\ldots,T_n]}(\tau_1,\ldots,\tau_n,x)$ is gauge invariant, i.e. it is invariant under the flow generated by the constraints $C_j$.

*Proof.* Under an arbitrary flow of the complete observable one has:

$$\begin{aligned}
\alpha_{\beta_j C_j}\left(F_{[f,T_k]}(\tau_m,x)\right) &= F_{[f,T_k]}\left(\tau_m,\alpha_{\epsilon_j C_j}(x)\right) \\
&= \alpha_{\beta_j C_j}(f)\left(\alpha_{\epsilon_l C_l}(x)\right)_{|\alpha_{\beta_j C_j}(T_i)\left(\alpha_{\epsilon_l C_l}(x)\right)=\tau_i} \\
&= \alpha_{\epsilon_j C_j} \circ \alpha_{\beta_j C_j}(f)(x)_{|\alpha_{\epsilon_j C_j} \circ \alpha_{\beta_j C_j}(T_i)(x)=\tau_i} \\
&= F_{[f,T_k]}(\tau_m,x)
\end{aligned} \tag{4.6}$$

where the last line follows because a composition of flows stays on the gauge orbit $G_x$ and any gauge transformation may always be written as a single flow, i.e. $\alpha_{\epsilon_j C_j} \circ \alpha_{\beta_j C_j}(f)(x) = \alpha_{\gamma_j C_j}(f)(x)$ for some $\gamma_k$. □

[39] Such a gauge fixing will in general not exist. In non-Abelian gauge theories there is a topological obstruction (Gribov ambiguity) that prevents such a gauge fixing. For this reason the approximation schemes discussed in (4.2.1) represent possible ways forward because they only rely upon the local properties of the physical clocks $T_i$. This is a similar problem to that encountered in the Fadeev-Popov method of dealing with gauge theories in the path integral formalism. Again a gauge fixing is required that will not exist in general but can be employed in a perturbative analysis.

[40] An important point here is that the solution parameters $\beta_i = \beta_i(\tau_j,x)$ become phase space dependent. However, one should not treat them as such inside Poisson brackets when evaluating $\alpha_{\beta_j C_j}(f)(x)$, i.e. first take the $\beta_i$ outside the iterated Poisson brackets and then substitute the solutions $\beta_i = \beta_i(\tau_j,x)$. This is illustrated in a number of examples in [41].

## 4.2 Important results concerning complete observables

A number of useful results are derived in [41] regarding complete observables, which we now discuss.

### 4.2.1 Approximation scheme for complete observables

In general computing complete observables is difficult because finding analytic expressions for the $\beta_i = \beta_i(\tau_j, x)$ will not be possible. A formal power series expression for the complete observables has been derived in [41] and it is this we now discuss.

First one can write a Taylor series in n variables about the point $\tau_i = T_i(y)$ for $F_{[f,T_i]}(\tau_i, , x)$:

$$F_{[f,T_i]}(\tau_i, x) = \sum_{k_1,\ldots,k_n}^{\infty} \frac{1}{k_1!\ldots k_n!} \frac{\partial^{k_1\ldots k_n}}{\partial^{k_1}\tau_1\ldots\partial^{k_n}\tau_n} F_{[f,T_i]}(T_i(y), x)\,(\tau_1 - T_1(y))^{k_1}\ldots(\tau_n - T_n(y))^{k_n} \tag{4.7}$$

where $y \in G_x$. Second the partial derivative of any complete observable is also a complete observable - this is intuitive because $F_{[f,T_i]}(\tau_i, x)$ is a complete observable for any values of the $\tau_i$. More precisely one has:

$$\frac{\partial^{k_1\ldots k_n}}{\partial^{k_1}\tau_1\ldots\partial^{k_n}\tau_n} F_{[f,T_i]}(T_i(y), x) = F_{[g_{k_1\ldots k_n},T_i]}(T_i(y), x) = g_{k_1\ldots k_n}(y) \tag{4.8}$$

where $g_{k_1\ldots k_n}(y)$ is the operator given by

$$g_{k_1\ldots k_n}(y) = (S_1)^{k_1} \circ \ldots \circ (S_n)^{k_n}. \tag{4.9}$$

The $S_i$ are operators, which act on an arbitrary phase space function space $h$ as:

$$S_i(h) = A^{-1}_{il}\{C_l, h\} \tag{4.10}$$

and $A^{-1}_{il}$ is the inverse matrix element of $A_{ij} := \{C_i, T_j\}$.

Substituting (4.8) into (4.7) one has:

$$F_{[f,T_i]}(\tau_i, x) = \sum_{k_1,\ldots,k_n}^{\infty} \frac{1}{k_1!\ldots k_n!} g_{k_1\ldots k_n}(y)\,(\tau_1 - T_1(y))^{k_1}\ldots(\tau_n - T_n(y))^{k_n} \tag{4.11}$$

as a formal solution for the complete observable.

A (weakly) related expression for $F_{[f,T_i]}(\tau_i, x)$ can also be derived, which makes use of an equivalent set of constraints $\tilde{C}_i$,$i = 1, \ldots, n$. (By equivalent we mean that the $\tilde{C}_i$define the same geometric constraint surface as the original $C_i$ constraints). The $\tilde{C}_i$are defined by:

$$\tilde{C}_i := A^{-1}_{ij} C_j\ . \tag{4.12}$$

The constraints $\tilde{C}_i$are said to be *Weakly Abelian* because they generate flows which commute on the constraint surface, i.e.

$$\left\{\tilde{C}_i, \left\{\tilde{C}_j, f\right\}\right\} \simeq \left\{\tilde{C}_j, \left\{\tilde{C}_i, f\right\}\right\} \tag{4.13}$$

for arbitrary $h$ and by the Jacobi identity this is equivalent to $\left\{\tilde{C}_i, \tilde{C}_j\right\} \sim O(C^2)$.

The reason the $\tilde{C}_i$ are used is that the flow of the "physical clocks" $T_i$ with respect to these constraints is (weakly) linear in the evolution parameters $\beta_i$, i.e.

$$\alpha_{\beta_j C_j}\left(T_k\right)(x) \simeq T_k(x) + \delta_{kj}\beta_j \tag{4.14}$$

and hence one can easily solve the system of equations $\alpha_{\beta_j C_j}\left(T_k\right)(x) \simeq \tau_k(x)$ by setting $\beta_i = \tau_i - T_i(x)$. One can show that in terms of these weakly abelian constraints the formal solution (4.11) for $F_{[f,T_i]}(\tau_i, x)$ can be weakly written as:

$$F_{[f,T_i]}(\tau_i, x) \simeq \sum_{k_1,\ldots,k_n}^{\infty} \frac{1}{k_1!\ldots k_n!}\tilde{S}_1^{k_1} \circ \ldots \circ \tilde{S}_n^{k_n}(f)(x)\,(\tau_1 - T_1(x))^{k_1}\ldots(\tau_n - T_n(x))^{k_n} \;\;, \tag{4.15}$$

where $\tilde{S}_i(h) := \left\{\tilde{C}_i, h\right\}$ for arbitrary $h$.

The advantage of the approximation schemes outlined here (4.15) is that they define a complete observable in a *local* way, i.e. possibly only for certain points in phase space, depending upon the convergence properties of the series and the (local) parametrization of the gauge orbit $G_x$ by the clock variables $T_i$. Recall that the original definition of the complete observable (4.5) required a good parametrization or gauge fixing, which in general may not exist.

### 4.2.2 Poisson Algebra of Complete Observables

An important goal is to understand the Poisson algebra of the set of complete observables - as indicated at the beginning of this section knowledge of this algebra will be required in the quantum theory, when the Poisson algebra of gauge invariant functions should be promoted to the commutator algebra of the associated operators on the physical Hilbert space.

Assuming that one has a good gauge fixing, then the complete observable $F_{[f,T_i]}(\tau_i, x)$ satisfies:

$$F_{[f,T_i]}(\tau_i, x)_{|T_i(x)=\tau_i} \simeq f(x)_{|T_i(x)=\tau_i} \tag{4.16}$$

restricted to the 'gauge fixed surface' (intersection of the constraint hypersurface and the gauge fixing surface $T_i(x) = \tau_i$). From this perspective the complete observable $F_{[f,T_i]}(\tau_i, y)$ is the (weakly) unique gauge invariant extension of the gauge restricted phase space function $f_{|T_j=\tau_j}$.

As stated in section (2) the Poisson bracket induced on the gauge fixed surface is the Dirac Bracket. It is stated in [7][41] and proven in [41] that the Poisson bracket between two gauge invariant functions or complete observables $F_{[f]}$ and $F_{[g]}$ is given weakly by the complete observable associated to the Dirac bracket $\{f, g\}^{\star}$of the partial observables $f$ and $g$, i.e.:

$$\left\{F_{[f,T_i]}(\tau_i, \cdot), F_{[g,T_i]}(\tau_i, \cdot)\right\}(x) \simeq F_{[\{f,g\}^{\star},T_i]}(\tau_i, x)\,. \tag{4.17}$$

This result (4.17) and the following one regarding addition and multiplication:

$$F_{[fg+h,T_i]}(\tau_i, x) = F_{[f,T_i]}(\tau_i, x)\cdot F_{[g,T_i]}(\tau_i, x) + F_{[h,T_i]}(\tau_i, x) \tag{4.18}$$

derived in [41]enable one to conclude that the map $\mathbf{F}$, which returns the complete observable associated to a gauge restricted partial observable is a Poisson algebra homomorphism. We have:

$$\mathbf{F}_{[T_i]}(\tau_i) : (\mathcal{C}^{\infty}(\Gamma)/\mathcal{I}(\Gamma), \{\cdot,\cdot\}^{\star} \;\;\rightarrow\;\; (\mathcal{D}(\Gamma)/\mathcal{I}(\Gamma), \{\cdot,\cdot\}) \tag{4.19}$$

$$f \;\;\mapsto\;\; F_{[f,T_i]}(\tau_i, \cdot) \tag{4.20}$$

[41]See section 1.5.4.

where $\mathcal{C}^\infty(\Gamma)$ is the space of infinitely differentiable phase space functions, $\mathcal{D}(\Gamma)$ is the space of gauge invariant phase space functions and $\mathcal{I}(\Gamma)$ is the space of smooth functions vanishing on the constraint surface. The reason the quotient is taken is that physically (in the classical theory) one cannot distinguish between phase space functions, which are equal on the constraint surface. We also know that any element of $\mathcal{I}(\Gamma)$ can be expressed as $u^i C_i$ where the $u^i$ are a set of arbitrary phase space functions. This means any two functions $g$ and $h$ which share this property must be related as $g = h + u^i C_i$, and should be identified. Since the product of an arbitrary function $g$ with an element of $\mathcal{I}(\Gamma)$ vanishes on the constraint surface, $\mathcal{I}(\Gamma)$ forms an ideal in the algebra of phase space functions.

### 4.2.3 Partially complete observables

The final result from [41] we wish to discuss, before moving to GR, is one concerning the simplification of the construction of complete observables. The question asked is whether it is possible to construct a complete observable in stages, i.e. suppose one has found a partial observable $f$ which is invariant with respect to a sub-algebra $\mathfrak{C}_1$of the constraints $\mathfrak{C}_1 \subset \mathfrak{C}$ can one then compute the complete observable associated to $f$ but only with respect to the remaining constraints $\mathfrak{C}_2 = \mathfrak{C} - \mathfrak{C}_1$and be sure that this observable is still complete with respect to $\mathfrak{C}_1$? The question is not trivial because there are two possible problems which need to be considered.

Firstly, if the sub-algebra $\mathfrak{C}_1$ is not an ideal, then it is possible that $\alpha_{C_k}(f)$ is not invariant under $\mathfrak{C}_1$ for every $C_k \in \mathfrak{C}_2$. We can see this just by considering $\{C_i, \alpha_{C_k}(f)\}$ to first order for $C_i \in \mathfrak{C}_1$ this is $\{C_i, \{C_k, f\}\}$and by the Jacobi identify only if $\{f, \{C_i, C_k\}\} = 0$ will $\{C_i, \{C_k, f\}\} = 0$, assuming $f$ is $\mathfrak{C}_1$ invariant ($\{C_i, f\} = 0$). So we need that $\{C_i, C_k\} \in \mathfrak{C}_1$, i.e. $\mathfrak{C}_1$is an ideal.

Secondly, the set $\mathfrak{C}_2$ may not form a sub-algebra and so Poisson brackets between constraints in $\mathfrak{C}_2$ may involve constraints in $\mathfrak{C}_1$ . This has the effect that the Hamiltonian vector fields associated to constraints in $\mathfrak{C}_2$ may not integrate to form a dim$\mathfrak{C}_2$ hypersurface, i.e the gauge orbits of $\mathfrak{C}_2$.

It is a result of [41] that the there is a positive answer to the above question provided certain clock variables can be found.

*Claim* 4.3. For a first class constraint system with n constraints $\mathfrak{C} = \{C_j\}$, $j = 1, \ldots, n$ , a sub-algebra $\mathfrak{C}_1 = \{C_{m+1}, \ldots, C_n\}$, an observable $f$ invariant under $\mathfrak{C}_1$ and clock variables $T_j$, $j = 1, \ldots, n$ for which the first m clocks are also $\mathfrak{C}_1$ invariant then the complete observable associated to $f$ and computed with respect to the remaining constraints $\mathfrak{C}_2 = \{C_1, \ldots, C_m\}$ is invariant under all the constraints $\mathfrak{C}$.

## 4.3 Field theories & GR

We now wish to generalise the above arguments to field theories, which have an infinite number of degrees of freedom. The phase space $\Gamma$ will be an infinite dimensional manifold[42] and the canonical fields will be defined on the 3-dim spatial manifold $\Sigma$, with points denoted by $\sigma$. Let us assume that canonical coordinates are given by the field configurations $(\phi_a(\sigma), \pi_a(\sigma))$. We shall have an infinite number of constraints labelled by the points $\sigma$ of$\Sigma$ and by some finite index $(\alpha = 1, \ldots, n)$, i.e. $C_\alpha(\sigma) = 0$. There will be an infinite number of clock variables $T_\alpha(\sigma)$ required in order to paramaterise the gauge orbits and hence define our complete observables.

In the field theoretic case phase space functions are really functionals[43] and so strictly our partial observables $f(\sigma)$ and clock variables $T_\alpha(\sigma)$ should be viewed as maps which act upon a

[42]We shall not discuss the functional analytic properties required to properly defined such infinite dimensional manifolds.

[43]We shall use these terms interchangeably for the infinite dimensional case.

phase space point or $\Sigma$ field configuration $(\phi_a(\sigma), \pi_a(\sigma))$ as:

$$\begin{aligned} T_\alpha(\sigma) : \Gamma &\rightarrow \mathbb{R} \\ T_\alpha(\sigma) : [\phi_a(\sigma'), \pi_a(\sigma')] &\mapsto T_\alpha(\sigma)\,[\phi_a(\sigma'), \pi_a(\sigma')] \end{aligned} \tag{4.21}$$

we shall generally only deal with local functionals that return the values of canonical fields or simple functions thereof at the particular point $\sigma$, e.g. $T_\alpha(\sigma)\,[\phi_a(\sigma'), \pi_a(\sigma')] = \phi_a(\sigma)$.

Other field specific notation follows in the usual way, e.g. the gauge flow of a phase space function $\psi(\sigma)$ can be written as:

$$\alpha_{C[\Lambda]}\,(\psi(\sigma)) := \sum_{n=0}^{\infty} \frac{1}{n!} \left\{\psi(\sigma), C\,[\Lambda]\right\}_n \tag{4.22}$$

where $C\,[\Lambda] = \int_\Sigma \Lambda^J(\sigma) C_J(\sigma) \mathrm{d}^3\sigma$ and $\Lambda^J(\sigma)$ is a phase space independent smearing function for the constraints $C_J(\sigma)$. The complete observable will be denoted $F_{[f,T]}(\tau, x)$ and defined to be the value of the phase space functional $f(\sigma)$ at the point $y$ in the gauge orbit $G_x$, where the clock variables satisfy $T^J(\sigma)(y) = \tau^J(\sigma)$. In order to find the point $y$ one must solve the flow of the clock variables, i.e. find the functions $\beta^J(\sigma)$ which weakly satisfy:

$$\alpha_{C[\Lambda]}(T^J(\sigma))_{\Lambda \rightarrow \beta(x)}(x) \simeq \tau^J(\sigma) \tag{4.23}$$

for all $J$ and $\sigma$. The important point here, as with the finite dimensional case, is that the $\beta$ will become phase space dependent and the same caveat regarding footnote (43) applies. Finally, given (4.23) one can weakly express the complete observable as:

$$F_{[f,T]}(\tau, x) \simeq \left[\alpha_{C[\Lambda]}(f(\sigma))_{\Lambda \rightarrow \beta(x)}\right](x) \tag{4.24}$$

As in the finite dimensional case the solution of the $\beta$ can be greatly simplified by the use of the weakly abelian constraints. In the infinite dimensional case they become:

$$\tilde{C}_I(\sigma) := \int_\Sigma \mathrm{d}^3\sigma' C_J(\sigma') (A^{-1})^J{}_I(\sigma, \sigma') \tag{4.25}$$

where $(A^{-1})^J{}_I(\sigma, \sigma')$ is the inverse (integral kernel) of the infinite dimensional matrix $A^I{}_J(\sigma, \sigma') := \left\{T^I(\sigma), C_J(\sigma')\right\}$. These constraints have the important property that their Poisson brackets are ultra-local[44] with the clock variables and hence the flow generated is easy to solve c.f. (4.14), i.e.:

$$\left\{T^I(\sigma), \tilde{C}_J(\sigma')\right\} \simeq \delta^I_J \delta(\sigma, \sigma') \tag{4.26}$$

and

$$\alpha_{\widetilde{C}[\Lambda]}(T^J(\sigma)) \simeq T^J(\sigma) + \Lambda^J(\sigma), \tag{4.27}$$

with solution $\Lambda^J(\sigma) = \tau^J(\sigma) - T^J(\sigma)$.

One can substitute this solution for $\Lambda^J(\sigma)$ back into the flow for $f$ and hence find the following expression for the complete observable:

$$\begin{aligned} F_{[f,T]}(\tau, x) \simeq\; & \sum_{s=0}^{\infty} \frac{1}{s!} \int_\Sigma \mathrm{d}^3\sigma_1 \cdots \mathrm{d}^3\sigma_s \left\{\cdots \left\{f, \tilde{C}_{J_1}(\sigma_1)\right\}, \cdots, \tilde{C}_{J_s}(\sigma_s)\right\}(x) \times \\ & \left(\tau^{J_1}(\sigma_1) - T^{J_1}(\sigma_1)\right) \cdots \left(\tau^{J_s}(\sigma_s) - T^{J_s}(\sigma_s)\right)(x) \end{aligned} \tag{4.28}$$

where there is an implicit summation on all pairs of upper and lower indices labelled $J_r$ from 1 to $n$.

[44] Ultra-local just means that the Poisson brackets are proportional to the delta function.

### 4.3.1 Reducing the number of constraints (I)

The main results of [42] are that i.) under certain conditions the infinite number of constraints present in field theories including GR may be replaced by a finite number and under still further conditions by a single constraint and ii.) that the canonical complete observables for GR can be precisely related to spacetime GR observables. We shall now discuss these in turn.

#### General Arguments

We shall now see that the expression for the complete observable in (4.28) can be simplified provided that both the clock variables $T_I(\sigma)$ and the partial observable $f : \Gamma \ni x \mapsto f(x) =: \chi(\sigma^\star) \in \mathbb{R}$, have weakly ultra-local Poisson brackets with the constraints $C_I(\sigma)$. This implies weakly ultra-local Poisson brackets with the weakly abelian constraints as well. Hence we may write the first order term in the power series (4.28) as:

$$\begin{aligned}\int d^3\sigma \left\{\chi(\sigma^\star), \tilde{C}_J(\sigma)\right\} \left(\tau^J(\sigma) - T^J(\sigma)\right) &\simeq \int d^3\sigma \left\{\chi(\sigma^\star), \tilde{C}_J\,[1]\right\} \delta(\sigma^\star, \sigma) \left(\tau^J(\sigma) - T^J(\sigma)\right) \\ &\simeq \left\{\chi(\sigma^\star), \tilde{C}_J\,[1]\right\} \left(\tau^J(\sigma^\star) - T^J(\sigma^\star)\right) \end{aligned} \tag{4.29}$$

where $\tilde{C}_J\,[1] := \int d^3\sigma \Lambda_J \tilde{C}_J(\sigma)$ and $\Lambda_J(\sigma^\star) = 1$. The first (weak) equality can be shown by expanding both sides and using the weak ultra-locality of the Poisson brackets together with the definition of $\tilde{C}_J\,[1]$. The important point is that now the first order term in (4.28) depends upon $\tau^J(\sigma^\star)$ and not the fields $\tau^J(\sigma)$. This result holds see [42] for the higher order terms and means that all integrals in (4.28) disappear - as they are integrated against a delta function and we are left with:

$$\begin{aligned} F_{[\chi(\sigma^\star),T]}(\tau, x) \simeq\ & \sum_{s=0}^{\infty} \frac{1}{s!} \left\{\cdots \left\{\chi(\sigma^\star), \tilde{C}_{J_1}\,[1]\right\}, \cdots, \tilde{C}_{J_s}\,[1]\right\} \times \\ & \left(\tau^{J_1}(\sigma^\star) - T^{J_1}(\sigma^\star)\right) \cdots \left(\tau^{J_s}(\sigma^\star) - T^{J_s}(\sigma^\star)\right)(x) \end{aligned} \tag{4.30}$$

This expression means that the complete observable $F_{[\chi(\sigma^\star),T]}(\tau, x)$ can be computed with respect to a finite number of constraints $\tilde{C}_J\,[1]$ and specified only with respect to a finite number of parameters $\tau^J(\sigma^\star)$ for the clock variables to take.

#### Application to GR

We now aim to apply these ideas to GR with the aim of reducing the difficulty of computing complete observables by reducing the number of constraints. First however we make some remarks concerning spacetime observables[45] in GR.

GR is a generally covariant theory[46] and a consequence of this is that determinism is only maintained provided that field equation solutions related by active diffeomorphisms are identified. In other words GR can be viewed as a gauge theory with the four dimensional diffeomorphism gauge group. This is essentially the conclusion of the famous hole argument also discussed in [45] and [4]. In this context any four dimensional tensor, which depends upon co-ordinates cannot be viewed as gauge invariant because for example the Ricci scalar $R(X)$ at a spacetime point $X$ would need to be constant throughout spacetime to be invariant under spacetime diffeomorphisms,

[45] Up until this point we have only considered canonical observables for both finite and infinite constrained Hamiltonian systems.

[46] An excellent discussion of general covariance can be found in the historical review [45].

[38]. Therefore one needs to be more careful with the definition of spacetime observables. This problem has been solved see [4], [46] and [47] using, as in the canonical case, a notion of relational or co-incidental observables.

*Claim* 4.4. A spacetime diffeomorphism invariant observable can be defined by using four spacetime scalar fields $\phi^I(X)$ where $I = 0, \dots 3$ a further scalar field $\psi(X)$, and the set of parameters $\tau^I$ . In particular, the value of $\psi$ at the spacetime point where $\phi^I = \tau^I$ is indeed spacetime diffeomorphism invariant.

There is a similarity between the spacetime observable defined above and the complete canonical observables we have been discussing - both are specified by using other fields (clock variables) to specify "when" a partial observable should be evaluated. However, in the canonical case these observables are defined on phase space and in the spacetime case the clock conditions on $\phi^I$ define a spacetime point. Furthermore, in the spacetime case only four parameters $\tau^I$ are required to specify the "when" in contrast with the (general) canonical case where the fields $\tau^I(\sigma)$ are needed. One of the results of [42] is that using the ideas above of reducing the number of constraints to the $\tilde{C}_J\,[1]$ one can then relate the canonical complete observable to the spacetime observable defined in claim 4.4.

In order to apply the result (4.30) to GR we need to ensure that both the clock variables and the partial observable $f$ have ultra-local Poisson brackets with the constraints, i.e. the spatial diffeomorphism and Hamiltonian constraints $H_a(\sigma)$ and $H(\sigma)$ discussed in section 3. In fact this requirement of ultra-locality is precisely satisfied by those fields defined on $\mathbb{R}\times\Sigma$ , which may be mapped to spacetime scalars on $M$. Recall that in the canonical analysis of GR one has a 1-parameter family of embeddings of $\Sigma$ onto $M$, denoted $X_t$, this family of embeddings enables any field $\phi(t, \sigma)$ (built from the canonical fields) to be mapped to a spacetime field $\tilde{\phi}(X) := \phi(t, \sigma)_{|X=X_t(\sigma)}$. The necessary and sufficient conditions under which $\tilde{\phi}(X)$ is a spacetime scalar are provided by the reconstruction theorems of Kuchař. The conditions are simply that $\phi(\sigma)$ satisfy the following:

$$\left\{\phi(\sigma), \int_\Sigma d^3\sigma' N(\sigma') H(\sigma')\right\} \simeq 0 \tag{4.31}$$

$$\left\{\phi(\sigma), \int_\Sigma d^3\sigma' N^a(\sigma') H_a(\sigma')\right\} = N^a(\sigma)\frac{\partial\phi}{\partial\sigma^a} \tag{4.32}$$

The first condition is to hold for all lapse functions $N$ with $N(\sigma) = 0$. This means that the Poisson brackets with the Hamiltonian constraint are ultra-local. The second condition is equivalent to the statement that $\phi(\sigma)$ is a scalar field on $\Sigma$. It also means that the Poisson brackets of $\phi(\sigma)$ are ultra-local with the diffeomorphism constraint. So indeed those canonical fields $\phi(\sigma)$ that can be reconstructed as spacetime scalars may be used as clock variables and partial observables to simplify the expression for a complete observable as in (4.30).

Let us use clock variables $T^I(\sigma)$ where $I = 0, \dots, 3$ , a partial observable $f : \Gamma \ni x \mapsto f(x) =: \chi(\sigma^\star) \in \mathbb{R}$, where both the clocks $T^I(\sigma)$ and $\chi(\sigma^\star)$ are reconstructable as spacetime scalars and the clock parameters $\tau^I(\sigma)$. Given these objects one can immediately re-write (4.30) for GR specifically as:

$$\begin{aligned} F_{[\chi(\sigma^\star),T]}(\tau, x) \simeq{} & \sum_{s=0}^{\infty}\frac{1}{s!}\left\{\cdots\left\{\chi(\sigma^\star), \tilde{C}_{J_1}[1]\right\}, \cdots, \tilde{C}_{J_s}[1]\right\} \times \\ & \left(\tau^{J_1}(\sigma^\star) - T^{J_1}(\sigma^\star)\right)\cdots\left(\tau^{J_s}(\sigma^\star) - T^{J_s}(\sigma^\star)\right)(x) \end{aligned} \tag{4.33}$$

where now we use $\tilde{C}_J\,[1] := \int d^3\sigma \tilde{C}_J(\sigma)$. The $\tilde{C}_J(\sigma)$ are the weakly abelian constraints for GR and determined from the original constraints: $C_0(\sigma) := H(\sigma)$ and $C_a(\sigma) := H_a(\sigma)$.

Using the definition of the matrix $A^L{}_K(\sigma, \sigma')$ and ultra-locality of the Poisson brackets one has:

$$\begin{aligned} A^L{}_K(\sigma,\sigma') : &= \left\{T^L(\sigma), C_K(\sigma')\right\} \\ &= B^L{}_K(\sigma)\delta(\sigma,\sigma') \end{aligned} \tag{4.34}$$

One can substitute this expression for $A^L{}_K(\sigma, \sigma')$ into the defining relation for its integral kernel and solve for the inverse to get:

$$\begin{aligned} \int d^3\sigma' B^L{}_K(\sigma)\delta(\sigma,\sigma') \left(A^{-1}\right)^K{}_M(\sigma',\sigma'') &= \delta^L_M \delta(\sigma,\sigma'') \\ \Rightarrow \left(A^{-1}\right)^K{}_M(\sigma,\sigma'') &= \left(B^{-1}\right)^K{}_M(\sigma)\delta(\sigma,\sigma'') \end{aligned} \tag{4.35}$$

Now substituting this solution into the defining relation for the weakly abelian constraints, (4.25), one has:

$$\begin{aligned} \tilde{C}_I(\sigma) &= \int_\Sigma \mathrm{d}^3\sigma' C_J(\sigma')(B^{-1})^J{}_I(\sigma)\delta(\sigma,\sigma') \\ &= C_J(\sigma)(B^{-1})^J{}_I(\sigma)\,. \end{aligned} \tag{4.36}$$

Hence the weakly abelian constraints are just linear combinations of the original ones.

The expression (4.33) depends upon the four constraints $\tilde{C}_J$ [1], rather than the infinite number of original Hamiltonian and diffeomorphism constraints. Furthermore, (4.33) only depends upon the clock parameters $\tau^J(\sigma^\star)$, i.e. four parameters. This makes the connection with the spacetime observable defined in claim 4.4 which also only depended upon four parameters.

### 4.3.2 Reducing the number of constraints (II)

In this section we extend the theory developed above in order to reduce the number of constraints from four to one. The approach relies on combining the ideas of the previous section together with the fact that complete observables can be computed in stages, as discussed in 4.2.3. In the context of GR one can compute complete observables by i.) choosing clock variables and a partial observable $f$ to be invariant under the sub-algebra of diffeomorphism constraints and ii.) computing the complete observable associated to $f$ but only with respect to the Hamiltonian constraint. One is then guaranteed (by the result of section 4.2.3) that this complete observable will be invariant under all the constraints of GR. In the construction of the (final) complete observable one will need as many clock variables as there are Hamiltonian constraints, i.e. one per point $\sigma$ of $\Sigma$.

In order to simplify this task the following strategy in [42] is employed instead of using the Hamiltonian constraints an equivalent set of constraints are used, which are also invariant under the diffeomorphism constraints. This means that when computing the final complete observable one is working only with diffeomorphism invariant objects both clocks, partial observable and constraints.

*Claim* 4.5. [47]The complete observable $D_{[H(\sigma),T^A]}(\tau^A, x)$, associated to the Hamiltonian constraint and computed with respect to the sub-algebra of diffeomorphism constraints[48] $H_a(\sigma)$ form a new

[47]Note we have written this claim in the context of GR. In fact it can be proved for first class systems in general, [42].

[48]Following [42] we use the notation $D_{[H(\sigma),T]}(\tau, x)$ for the complete observable computed only with respect to the sub-algebra of constraints rather than $F$ which is reserved for complete observables with respect to all constraints.

set of constraints, which are equivalent to the original Hamiltonian constraints. Implicit in the definition are clock fields $T^I(\sigma)$, $I = 1, \ldots, 3$ and parameter fields $\tau^I(\sigma)$. Hence the constraint surface defined by both sets of constraints ($H_a(\sigma)$ and $D_{[H(\sigma),T^A]}(\tau^A, x)$) is equivalent to the original GR constraint surface.

In order to use the results of 4.3.1 we need to work with spatial tensors on $\Sigma$ that have ultra-local Poisson brackets with the constraints. We therefore assume that our clock variables $T^A(\sigma)$, $A = 1, \ldots, 3$ satisfy the conditions (4.31) and (4.32). If we choose a partial observable $f : \Gamma \ni x \mapsto f(x) =: \chi(\sigma^\star) \in \mathbb{R}$ where $\chi(\sigma)$ is a spatial scalar field then the complete observable $D_{[\chi(\sigma^\star),T]}(\tau, x)$ will be invariant under the diffeomorphism constraints by definition, and only depend upon three constraints and three parameters through the arguments in (4.3.1)[49].

The argument made for $\chi(\sigma)$ will apply to any spatial scalar and it is possible to make any spatial tensor on $\Sigma$ into a spatial scalar by expressing that tensor in the "clock frame", e.g. the inverse canonical 3-metric on $\Sigma$, $q^{ab}(\sigma)$ can be expressed in the clock variable coordinates as:

$$q^{AB}(\sigma) = \frac{\partial T^A}{\partial \sigma^a} \frac{\partial T^B}{\partial \sigma^b} q^{ab}(\sigma) \tag{4.37}$$

The components $q^{AB}(\sigma)$ are spatial scalars because under a (passive) spatial diffeomorphism the contravariant nature of $q^{ab}(\sigma)$ cancels with the covariant nature of the Jacobian $T^A{}_{,a}$- the $A, B$ indices are by definition spatial scalars and so do not transform. Similarly we may use the inverse Jacobian $S^a{}_A(\sigma)$ to express the canonical 3-metric $q_{ab}(\sigma)$ as:

$$q_{AB}(\sigma) = S^a{}_A S^b{}_B q_{ab}(\sigma) \tag{4.38}$$

where each component of $q_{AB}(\sigma)$ is a spatial scalar. Finally, one can also convert spatial tensor densities to spatial scalars by multiplication with the appropriate power of the determinant of the Jacobian as well as factors of the Jacobian matrix or inverse. The canonical conjugate momentum $p^{ab}(\sigma)$ is a tensor density of weight 1 and can therefore be de-densitised by multiplication with a factor of the Jacobian of the inverse $s := \det(S^a{}_A)$ and therefore one has:

$$p^{AB}(\sigma) = s T^A{}_{,a} \, T^B{}_{,b} \, p^{ab}(\sigma) \tag{4.39}$$

where the components of $P^{AB}$ are all spatial scalars.

We can now easily convert the original Hamiltonian constraint, $H(\sigma)$, into a spatial scalar by multiplication with the inverse Jacobian $s$. Hence we may compute the partially complete observable $D_{[sH(\sigma),T]}(\tau, x)$, which satisfies $D_{[sH(\sigma),T^A]}(\tau^A, x) = D_{[s,T^A]}(\tau^A, x) \times D_{[H(\sigma),T^A]}(\tau^A, x)$ by the Poisson algebra homomorphism result (4.19). Hence the condition $D_{[sH(\sigma),T]}(\tau, x) = 0$ is equivalent to $D_{[H(\sigma),T]}(\tau, x) = 0$ provided

$$\begin{aligned} 0 &\neq D_{[sH(\sigma),T]}(\tau, x) \\ \Leftrightarrow 0 &\neq s \end{aligned} \tag{4.40}$$

We assume this condition on $s$ and note it amounts to assuming that the fields $T^A$ form a good local coordinate system on $\Sigma$. This means by claim 4.5 and (4.40) that we may use the partially invariant Hamiltonian constraints $D_{[sH(\sigma),T^A]}(\tau^A, x)$ instead of $H(\sigma)$ in computing the final complete observable. The advantage of using $D_{[sH(\sigma),T^A]}(\tau^A, x)$ is that because $sH(\sigma)$ is a spatial scalar by applying the results of section 4.3.1 the formal series for $D_{[sH(\sigma),T^A]}(\tau^A, x)$ will only depend upon a finite number of constraints and parameters.

[49]We only require that $\chi(\sigma)$ is a spatial scalar, i.e. satisfies (4.32) because we are only computing the complete observable with respect to the diffeomorphism constraints and not the full set of GR constraints. If the latter we would indeed require that $\chi(\sigma)$ be reconstructable as a spacetime scalar and also satisfy (4.31).

Hence the goal is to now compute the complete observable $F_{\left[D_{[\chi,T^A]}(\tau^A,\cdot),T^0\right]}(\tau^0,\cdot)$ with respect to the constraints $D_{[sH(\sigma),T^A]}(\tau^A,x)$ and where we have introduced $T^0$ as the clock variable for the new Hamiltonian constraint with the gauge fixing $T^0=\tau^0$. This observable will be a complete observable for GR. The formal series for $F_{\left[D_{[\chi,T^A]}(\tau^A,\cdot),T^0\right]}(\tau^0,\cdot)$ can be greatly simplified if both $D_{[\chi,T^A]}(\tau^A,\cdot)$ and $T^0$ have ultra-local Poisson brackets with the Hamiltonian constraints because only one constraint and one parameter will then be required. This condition on $D_{[\chi,T^A]}(\tau^A,\cdot)$ can be shown to follow from the assumption that $\chi(\sigma)$ is reconstructable as a spacetime scalar.

*Claim* 4.6. If the spatial field $\chi(\sigma)$ is reconstructable as a spacetime scalar then the partially complete observable $D_{[\chi(\sigma),T^A]}(\tau^A,\cdot)$ will have ultra-local Poisson brackets with respect to the partially invariant Hamiltonian constraints $D_{[sH(\sigma),T^A]}(\tau^A,\cdot)$, that is:

$$\left\{D_{[\chi(\sigma^\star),T^A]}(\tau^A,\cdot),D_{[sH(\sigma^{\star\star}),T^A]}(\tau^A,\cdot)\right\}\sim\delta\left(\tau^A(\sigma^\star),\tau^A(\sigma^{\star\star})\right) \tag{4.41}$$

where we stress that because both $sH(\sigma)$ and $\chi(\sigma)$ are spatial scalars the partially complete observables inside the above Poisson bracket only depend upon the parameters $\tau^A(\sigma^\star)$ and $\tau^A(\sigma^{\star\star})$ respectively.

*Proof.* By the result (4.17) we know that the Poisson bracket in (4.41) is given by the complete observable associated to the Dirac bracket of the corresponding partial observables. This means we need to compute:

$$D_{[\{\chi(\sigma^\star),sH(\sigma^{\star\star})\}^D,T^A]}(\tau^A,\cdot)$$

where $\{\cdot,\cdot\}^D$is the Dirac bracket.

We can (weakly) write the Dirac bracket as $\{\chi(\sigma^\star),sH(\sigma^{\star\star})\}^D\simeq s\left(\sigma^{\star\star}\right)\{\chi(\sigma^\star),H(\sigma^{\star\star})\}^D$ where the Dirac bracket is defined by:

$$\begin{aligned}\{\chi(\sigma^\star),H(\sigma^{\star\star})\}^D &= \{\chi(\sigma^\star),H(\sigma^{\star\star})\}-\int_\Sigma d^3\sigma\left\{\chi(\sigma^\star),S^b{}_AH_b(\sigma)\right\}\left\{T^A(\sigma),H(\sigma^{\star\star})\right\}\\ &\quad+\int_\Sigma d^3\sigma\left\{\chi(\sigma^\star),T^A(\sigma)\right\}\left\{S^b{}_AH_b(\sigma),H(\sigma^{\star\star})\right\}\end{aligned} \tag{4.42}$$

Now the final term on the right hand side is weakly zero by the first class nature of GR. The first term on the right hand side is proportional to a delta function by the assumption that $\chi(\sigma)$ is reconstructable as a spacetime scalar and hence:

$$\{\chi(\sigma^\star),H(\sigma^{\star\star})\}=\{\chi(\sigma^\star),H[1]\}\,\delta(\sigma^\star,\sigma^{\star\star})\,.$$

The second term can be simplified as follows:

$$\begin{aligned}\int_\Sigma d^3\sigma\left\{\chi(\sigma^\star),S^b{}_AH_b(\sigma)\right\}\left\{T^A(\sigma),H(\sigma^{\star\star}\right\} &\simeq \int_\Sigma d^3\sigma S^b{}_A(\sigma)\left\{\chi(\sigma^\star),H_b(\sigma)\right\}\left\{T^A(\sigma),H(\sigma^{\star\star})\right\}\\ &= \int_\Sigma d^3\sigma S^b{}_A(\sigma)\chi_{,b}\delta(\sigma^\star,\sigma)\left\{T^A(\sigma),H(\sigma^{\star\star})\right\}\\ &= S^b{}_A\chi_{,b}(\sigma^\star)\left\{T^A(\sigma^\star),H(\sigma^{\star\star})\right\}\\ &\simeq S^b{}_A\chi_{,b}(\sigma^\star)\left\{T^A(\sigma^\star),H[1]\right\}\delta(\sigma^\star,\sigma^{\star\star})\end{aligned}$$

where in the second line we used the fact that $\chi(\sigma)$ is a spatial scalar and in the final step used the fact that $T^A$also have ultra-local Poisson brackets with the Hamiltonian constraint, as they are reconstructable as spacetime scalars. Putting the previous results together we can (weakly) write for the Dirac bracket:

$$\{\chi(\sigma^\star), sH(\sigma^{\star\star})\}^D \simeq s\left(\sigma^{\star\star}\right)\delta(\sigma^\star,\sigma^{\star\star})\left(\{\chi(\sigma^\star), H[1]\} - S^b{}_A\chi_{,b}(\sigma^\star)\left\{T^A(\sigma^\star), H[1]\right\}\right). \quad (4.43)$$

Now both terms on the right hand side are spatial scalars, which means that the partially invariant observables associated to them, i.e. the $D_{[\cdot]}$ complete observable will simplify and only depend upon the three parameters $\tau^A(\sigma^\star)$ and the three constraints $H_a[1]$. Hence the Poisson bracket in (4.41) can be expressed as:

$$\begin{aligned}
\{\chi(Y^\star), sH(Y^{\star\star})\} \simeq\ & D_{\left[s(\sigma^{\star\star})\delta(\sigma^\star,\sigma^{\star\star})\left(\{\chi(\sigma^\star),H[1]\}-S^b{}_A\chi_{,b}(\sigma^\star)\{T^A(\sigma^\star),H[1]\}\right),T^A\right]}(\tau^A,\cdot) \\
=\ & \left(D_{\left[\{\chi(\sigma^\star),H[1]\}-S^b{}_A\chi_{,b}(\sigma^\star)\{T^A(\sigma^\star),H[1]\},T^A\right]}(\tau^A,\cdot)\right)\times \\
& \left[\det\tau^C_{,c}(\sigma^{\star\star})\right]^{-1}\delta(\sigma^\star,\sigma^{\star\star}) \\
=\ & \left(D_{\left[\{\chi(\sigma^\star),H[1]\}-S^b{}_A\chi_{,b}(\sigma^\star)\{T^A(\sigma^\star),H[1]\},T^A\right]}(\tau^A,\cdot)\right)\times \\
& \delta\left(\tau^A(\sigma^\star),\tau^A(\sigma^{\star\star})\right) \\
\sim\ & \delta\left(Y^\star,Y^{\star\star}\right)
\end{aligned} \quad (4.44)$$

where we used the abbreviations $\chi(Y^\star)$ for $D_{[\chi(\sigma^\star),T^A]}(\tau^A,\cdot)$ and $sH(Y^{\star\star})$ for $D_{[sH(\sigma^{\star\star}),T^A]}(\tau^A,\cdot)$ and where $Y$ represents the gauge fixing conditions $\tau^A$ at the point $\sigma$. The first line follows from the result[50] that if phase space functions $f$ and $g$ are weakly equal then their corresponding complete observables will also be weakly equal, i.e. $F_{[f,T]}(\tau,x) \simeq F_{[g,T]}(\tau,x)$. The second line follows from (4.19) and since $D_{[T^B(\sigma),T^A]}(\tau^A,\cdot) = \tau^B(\sigma)$ and since the determinant is an algebraic function involving additions and multiplications we will have det $\left(D_{[T^B_{,c}(\sigma),T^A]}(\tau^A,\cdot)\right) = D_{[\det\left(T^B_{,c}(\sigma)\right),T^A]}(\tau^A,\cdot)$. Further as det $\left(T^B_{,c}(\sigma)\right)$ is invertible by assumption we will have

$$\left[det\left(\tau^B_{,c}(\sigma)\right)\right]^{-1} = D_{[s(\sigma),T^A]}(\tau^A,\cdot)\,.$$

The final line follows for the expression for the change in variables of the delta function. Hence we have proved the claim that the Poisson bracket of the partially invariant observables is proportional to the delta function as in (4.41). □

This means that by using partially invariant observables $\chi(Y)$, $T^0(Y)$ (computed from $T^0(\sigma)$ a spacetime reconstructable scalar), and an adapted gauge fixing $\tau^0(Y)$ one can define the following GR gauge invariant observable:

$$F_{[\chi(Y^\star),T^0(Y)]}(\tau^0(Y),\cdot) = \sum_{m=0}^{\infty}\frac{1}{m!}\left\{\chi(Y^\star),\tilde{H}[1]\right\}_m\left(\tau^0(Y^\star) - T^0(Y^\star)\right)^m \quad (4.45)$$

where we have used the weakly abelian constraints $\tilde{H}(Y)$ derived from the partially invariant Hamiltonian constraints $sH(Y)$ and defined in the usual way $\tilde{H}(Y) := B^{-1}(Y)H(Y)$ for $\left\{T^0(Y), H(Y^\star)\right\} = B(Y)\delta(Y,Y^\star)$ and $\tilde{H}[1] := \int_{\tau^A(\Sigma)} H(Y)d^3\tau$.

The advantage of this series is that compared to the general expression for a complete observable, (4.15), it only depends upon 1 constraint $\tilde{H}[1]$ and 1 parameter $\tau^0(Y^\star)$.

[50] See section 8 [41].

## 4.4 Deparametrisation of constrained systems

The final topic we wish to discuss are deparametrisable systems, essentially a sub-set of systems for which the construction of complete observables greatly simplifies together with other nice properties. However, we stress deparametrisation relies upon mathematical assumptions, not satisfied by pure GR or GR coupled with standard model matter. Thus one can take the (positive) view regarding GR that it is a study of 'toy models' that might lead to some insight regarding true phenomenological gravity or (more speculatively) that it leads to the hypothesis of new matter fields. From our perspective we shall first describe the simplifications that occur in deparametrisable systems and then discuss an example from the literature.

Recall in section 2 we discussed the example of the (parametrised) non-relativistic particle where 'parametrisation' involved adding, to the usual canonical variables $\left(q^i, p_i\right)$, the pair $(t, p_t)$. The subsequent analysis revealed that there was a Hamiltonian constraint $H_s = p_t + H$, where $H = H(q,p)$, i.e. independent of the additional configuration variable $t$. In fact these properties for the parametrised particle capture the main points of a general deparametrisable system, which we now discuss.[51]

We consider a constraint system subject to finite first class constraints $C_I$. Such a system *deparametrises* if we can split the phase space into two sets of canonical pairs $(q^a, p_a)$ and $\left(T^I, \Pi_I\right)$ where the constraints can be solved for the momenta $\Pi_I$, such that

$$C_I \quad = \quad 0 \Leftrightarrow c_I = \Pi_I + h_I \tag{4.46}$$

and the phase space function $h_I = h_I(q^a, p_a)$ is independent of the $T^I$. It follows that i.) the constraints $c_I$ Poisson commute and ii.) the functions $h_I$ are gauge invariant.

These properties mean that one can apply the framework of relational observables (using the $T^I$ as clock variables) to express the gauge invariant extension of $f$ with the constraints $c_I$- note their Poisson commutativity makes redundant the need to compute weakly abelian constraints. Therefore using the definition of the complete observable (4.5) and the fact that $\left\{c_I, T^J\right\} = \delta_I^J$ one has for the complete observable:

$$F_{[f,T^J]}(\tau^J, \cdot) = \left(\sum_{m=0}^{\infty} \frac{1}{m!}\left\{\beta^I c_I, f\right\}_m\right)_{|\beta^J = (\tau^J - T^J)} \tag{4.47}$$

where the substitution $\beta^J = \left(\tau^J - T^J\right)$ occurs after all the iterated Poisson brackets have been computed, see footnote 43.

The important result is that one can use a linear combination of the $h_I$ to define a *physical Hamiltonian*, which is non-vanishing and which generates evolution of the complete observables.

*Claim* 4.7. Physical evolution of the complete observables can be generated by the functions $h_I$, i.e.

$$\frac{\partial F_{[f,T^J]}(\tau^J, \cdot)}{\partial \tau^K} = \left\{h_K, F_{[f,T^J]}(\tau^J, \cdot)\right\} \tag{4.48}$$

provided $f$ only depends upon $(q^a, p_a)$.

*Proof.* First expanding the complete observable one has:

$$\begin{aligned} F_{[f,T^J]}(\tau^J, \cdot) \quad = \quad & f + \left(\tau^I - T^I\right)\{c_I, f\} + \frac{1}{2!}\left(\tau^I - T^I\right)\left(\tau^J - T^J\right)\{c_I, \{c_J, f\}\} \\ & + \frac{1}{3!}\left(\tau^I - T^I\right)\left(\tau^J - T^J\right)\left(\tau^K - T^K\right)\{c_I, \{c_J, \{c_K, f\}\}\} \\ & + \cdots + \frac{1}{n!}\left(\tau^{I_1} - T^{I_1}\right) \times \cdots \times \left(\tau^{I_n} - T^{I_n}\right)\{c_{I_1}, \cdots, \{c_{I_n}, f\}\cdots\} + \cdots \end{aligned}$$

[51] A good discussion of parametrised systems can be found in [3].

Taking the derivative with respect to $\tau^L$ implies:

$$
\begin{aligned}
\frac{\partial F_{[f,T^J]}}{\partial \tau^L} &= \{c_L, f\} + \frac{1}{2!}\left[\left(\tau^J - T^J\right)\{c_L,\{c_J,f\}\} + \left(\tau^J - T^J\right)\{c_I,\{c_L,f\}\}\right] \\
&+\frac{1}{3!}\left[\left(\tau^J - T^J\right)\left(\tau^K - T^K\right)\{c_L,\{c_J,\{c_K,f\}\}\} + \left(\tau^I - T^I\right)\left(\tau^K - T^K\right)\{c_I,\{c_L,\{c_K,f\}\}\}\right. \\
&\left.+\left(\tau^I - T^I\right)\left(\tau^J - T^J\right)\{c_I,\{c_J,\{c_L,f\}\}\}\right] + \cdots + \\
&\frac{1}{n!}\left[\left(\tau^{I_2} - T^{I_2}\right)\times\cdots\times\left(\tau^{I_n} - T^{I_n}\right)\{c_L,\{c_{I_2},\cdots,\{c_{I_n},f\}\cdots\}\}\right. \\
&+\left(\tau^{I_1} - T^{I_1}\right)\times\left(\tau^{I_3} - T^{I_3}\right)\times\cdots\times\left(\tau^{I_n} - T^{I_n}\right)\{c_{I_1},\{c_L,\{c_{I_3}\cdots,\{c_{I_n},f\}\cdots\}\}\} \\
&\left.+\cdots+\left(\tau^{I_1} - T^{I_1}\right)\times\cdots\times\left(\tau^{I_{n-1}} - T^{I_{n-1}}\right)\left\{c_{I_1},\cdots,\left\{c_{I_{n-1}},\{c_L,f\}\right\}\cdots\right\}\right] + \cdots
\end{aligned}
$$

Now one can re-order the iterated Poisson brackets so that the action of $c_L$ is last because the $c_I$ form an abelian algebra, hence the $n$ terms contributing at each order are equal and will cancel with the $\frac{1}{n!}$ to give $\frac{1}{(n-1)!}$. Secondly one can observe that the action of $c_L$ can be replaced by $h_L$ because the iterated Poisson brackets $\{c_{I_1},\{c_{I_2},\cdots,\{c_{I_n},f\}\cdots\}\}$ cannot be functions of $T^I$ as both $f$ and $c_I$ are independent of $T^I$ and hence $\{\Pi_L,\{c_{I_1},\{c_{I_2},\cdots,\{c_{I_n},f\}\cdots\}\}\} = 0$. This gives:

$$
\begin{aligned}
\frac{\partial F}{\partial \tau^L} &= \{h_L, f\} + \left(\tau^I - T^I\right)\{h_L,\{c_I,f\}\} + \frac{1}{2!}\left(\tau^I - T^I\right)\left(\tau^J - T^J\right)\{h_L,\{c_I,\{c_J,f\}\}\} \\
&+\cdots+\frac{1}{(n-1)!}\left(\tau^{I_1} - T^{I_1}\right)\times\cdots\times\left(\tau^{I_{n-1}} - T^{I_{n-1}}\right)\left\{h_L,\left\{c_{I_1},\cdots,\left\{c_{I_{n-1}},f\right\}\cdots\right\}\right\} + \cdots \\
&= \{h_L, f + \left(\tau^I - T^I\right)\{c_I,f\} + \frac{1}{2!}\left(\tau^I - T^I\right)\left(\tau^J - T^J\right)\{c_I,\{c_J,f\}\} + \cdots \\
&+\frac{1}{(n-1)!}\left(\tau^{I_1} - T^{I_1}\right)\times\cdots\times\left(\tau^{I_{n-1}} - T^{I_{n-1}}\right)\left\{c_{I_1},\cdots,\left\{c_{I_{n-1}},f\right\}\cdots\right\} + \cdots\} \\
&= \left\{h_L, F_{[f,T^I]}(\tau^I,\cdot)\right\}
\end{aligned}
$$

where the penultimate line follows because $\left\{T^I, h_J\right\} = 0$. □

So we have shown that in the deparametrisable case there is a physical Hamiltonian, which is gauge invariant, non-zero and which generates time evolution of observables $F_{[f(q,p),T]}$. As this Hamiltonian is gauge invariant it can be viewed as constant in the evolution parameters $\tau^I$ and therefore we have a conservative system.

As we shall concentrate only upon partial observables $f = f(q^a, p_a)$ for which the associated complete observable has a time evolution generated by a physical Hamiltonian, we note that in this case the expression, (4.47), for the complete observable simplifies to the following:

$$
F_{[f,T^J]}(\tau^J,\cdot) = \left(\sum_{m=0}^{\infty}\frac{1}{m!}\left\{\beta^I h_I, f\right\}_m\right)_{|\beta^J = (\tau^J - T^J)} \tag{4.49}
$$

the reason is that, as above, $\{\Pi_I, f(q,p)\} = 0$ and all iterated Poisson brackets of $\beta^I h_I$ and $f$ are independent of $T^I$ because both $h_J$ and $f$ are independent of $T^I$.

We now discuss gravity matter coupling, which results in a deparametrisation of GR. An important historical work in this subject is [50] in which a pressure less dust is shown to deparametrise GR. This work did not use the terminology of partial and complete observables but is essentially equivalent. Later work in [48] and [49] applied and extended this deparametrisation for both dust and a minimally coupled scalar field without potential. As stressed earlier both these matter types are to be viewed as special in the sense that they are non-standard model matter. For reasons of simplicity we shall concentrate only upon the scalar model.

### 4.4.1 Deparametrisation of GR using matter fields

We follow [49] to the extent that the overall scalar field deparametrisation of GR can be seen as a special case of the relational framework and summarise the results that have been obtained. The purpose of [49] was to describe a gauge invariant cosmological model, which could avoid the general interpretational problem already mentioned regarding the problem of time. The FRW equations, which describe the evolution of the observed universe to a good accuracy, do not involve gauge invariant objects and their evolution is described with respect to the Hamiltonian constraint, which is the generator of a gauge transformation. In [49] an alternative set of equations are derived, which involve only gauge invariant objects and evolution with respect to a physical Hamiltonian. The model relies on the presence of a special scalar or phantom field, which serves as the clock variable, and further requires that this field has negative energy density to avoid singular behaviour[52]. It is argued that this is still acceptable because the field is not observable[53] (pure gauge) and the sufficient presence of ordinary matter with a positive energy density will stabilise the theory. The results of this analysis do not yield simply a gauge invariant version of the FRW equations but rather an evolution that differs significantly from FRW in the late universe, or for large values of the scale factor. In general it seems one should expect deparametrisation models to describe different physics from their gauge variant counterparts. In this way one could have falsifiable predictions to rule out particular deparametrisable models.

It is a non-trivial fact that scalar fields involving only first order derivatives can lead to a deparametrisation of GR. Consider the following gravity matter system including a class of scalar (phantom) fields, described by the action:

$$\begin{aligned} S_{Tot} &= S_{EH} + S_M + S_{Ph} & (4.50) \\ S_{Ph} &= \int_M d^4X \sqrt{g} L \left( -\frac{1}{2} g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} \right) & (4.51) \end{aligned}$$

where $L$ is an arbitrary function and $S_M$ refers to the action of all matter fields other than the phantom scalar.

By performing a canonical analysis and assuming that $\Sigma \times \mathbb{R}$ is diffeomorphic to $M$ as in section 3 one can pullback all spacetime fields to $\Sigma$ using the one parameter family of embeddings $Y_t(\sigma) = X$ . The result is that the argument of $L$ can be expressed as $I := -g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} = (\nabla_n \phi)^2 - q^{ab}\phi_{,a}\phi_{,b}$ where $\phi(\sigma,t) = (Y_t^\star \Phi)(\sigma)$ and $n$ is the unit normal $n = N^{-1}\left(T - N^a Y_{t,a}\right)$ to the spacelike hypersurface $\Sigma_t := Y_t(\Sigma)$ and $N$ and $N^a$ are the lapse and shift respectively - this means that $\nabla_n \phi = N^{-1}\left(\dot{\phi} - N^a \phi_{,a}\right)$. It follows that the phantom action can be written:

$$S_{Ph} = \int_{\mathbb{R}} dt \int_\Sigma d^3\sigma N \sqrt{q} L \left( \frac{1}{2} \left( N^{-2} \left( \dot{\phi} - N^a \phi_{,a} \right)^2 - q^{ab}\phi_{,a}\phi_{,b} \right) \right) \tag{4.52}$$

and hence the (phantom) canonical momentum $\pi(\sigma,t) = \sqrt{q}\left(\nabla_n \phi\right) L'(I/2)$ . Squaring this expression for $\pi$ implies $K := \left(\frac{\pi}{\sqrt{q}}\right)^2 = \left(L'(I/2)\right)^2 (I + V)$ where $V := q^{ab}\phi_{,a}\phi_{,b}$. If we assume this latter expression can be solved for $I$ as $I = J(K,V)$ then the defining equation for the momentum can be written:

$$\frac{\pi}{\sqrt{q}} \left(L'(J/2)\right)^{-1} = \nabla_n \phi \tag{4.53}$$

[52] The universe reaches infinite size in a finite amount of time.

[53] All we mean here is that the scalar field $\Phi$ is used to provide a clock to define complete observables with respect to all other partial observables, which do not depend on $\Phi$ or its conjugate momentum.

this enables the Legendre transform to be completed and we have:

$$\begin{aligned} H_{Ph} &= \int_\Sigma d^3\sigma \left(\dot\phi\pi - N\sqrt{q}L\left(J/2\right)\right) \\ &= \int_\Sigma d^3\sigma \left(\left(N\nabla_n\phi + N^a\phi_{,a}\right)\pi - N\sqrt{q}L(J/2)\right) \\ &= \int_\Sigma d^3\sigma \left(N\left(\frac{\pi^2}{\sqrt{q}}\left(L'(J/2)\right)^{-1}\right) + N^a\phi_{,a}\pi - N\sqrt{q}L(J/2)\right) \\ &= \int_\Sigma d^3\sigma \left(N\sqrt{q}\left(p^2\left(L'(J/2)\right)^{-1} - L(J/2)\right) + N^a\phi_{,a}\pi\right) \end{aligned} \tag{4.54}$$

where $p := \frac{\pi}{\sqrt{q}}$. Now we have a constrained system because the total action of the gravity matter system does not depend upon the velocities $\dot N$ and $\dot N^a$and this implies primary constraints $p_N = p_{N^a} = 0$. By requiring consistency of these primary constraints one will be led to secondary constraints, which define the total Hamiltonian and diffeomorphism constraints as in the pure GR case discussed in section 3. We can read off the phantom field contributions to this totally constrained Hamiltonian from (4.54) as follows:

$$H_a^{Ph} = \pi\phi_{,a} \tag{4.55}$$

$$H^{Ph} = \sqrt{q}\left(p^2\left(L'(J/2)\right)^{-1} - L(J/2)\right) \tag{4.56}$$

where $H_a^{Ph}$ is the scalar field contribution to the diffeomorphism constraint and $H^{Ph}$ is the scalar field contribution to the Hamiltonian constraint.

Deparametrisation follows from the initial observation that the diffeomorphism constraint $H_a^{Tot} = 0$ can be used to remove dependence on the phantom field from the Hamiltonian constraint $H^{Tot} = 0$ as follows:

$$\begin{aligned} V &= q^{ab}\phi_{,a}\phi_{,b} \\ &= q^{ab}\frac{H_a^{Ph}H_b^{Ph}}{\pi^2} \\ &= q^{ab}\frac{\left(H_a^{EH} + H_a^M\right)\left(H_b^{EH} + H_b^M\right)}{\pi^2} \\ &= K^{-1}\frac{q^{ab}\left(H_a^{EH} + H_a^M\right)\left(H_b^{EH} + H_b^M\right)}{q^2} \\ &=: K^{-1}h \end{aligned} \tag{4.57}$$

where in the third line we have imposed the diffeomorphism constraint. The phantom field only appears in the Hamiltonian constraint (4.56) through $V$ in the function $J = J(K, V)$, but by (4.57) we can express $V = K^{-1}d$ as a function independent of the phantom field $\phi$. This mechanism works because the initial action only depended upon the derivative of the phantom field, this is why no non-trivial potentials are allowed in the class of scalar models we are considering.

One can substitute this expression for $V$ back into (4.56) and therefore obtain an equivalent total Hamiltonian constraint:

$$\bar H^{Tot} = H^{EH} + H^M + \sqrt{q}\left(K\left(L'(\bar J/2)\right)^{-1} - L(\bar J/2)\right) \tag{4.58}$$

where $\bar J := J(K, V = K^{-1}h)$. Finally, if one solves $\bar H^{Tot} = 0$ for $K$ one will have $K = G(H^{EH}, H^M, h)$ and using the definition $K := \frac{\pi^2}{q}$ one is led to a second equivalent Hamiltonian constraint:

$$\tilde H_{i,j}^{Tot} := (-1)^i\pi + (-1)^j\sqrt{q}\sqrt{G} \tag{4.59}$$

where $i, j = 0, 1$ enumerate the positive and negative square roots. In order to define the entire Hamiltonian constraint surface one should take all sign combinations for the square roots as indicated above. However, in [49] a choice was made to focus only upon the positive roots $i = j = 0$ because this led to a physical Hamiltonian with the desired weak gravitational limit.

We have now found a Hamiltonian constraint which is in the deparametrised form[54], (4.46) since $\sqrt{q}\sqrt{G}$ is independent of the phantom field. It follows from the general theory that these Hamiltonian constraints will Poisson commute with each other- recall this is not the case for the original Dirac algebra- and hence also $\sqrt{q}\sqrt{G}$ will commute. A formal proof of this Poisson commutativity is given in [49]. The family of functions which Poisson commute in this way has been classified and can be shown to be solutions of a PDE. The solution space of this PDE is in fact greater than the set of $\sqrt{q}\sqrt{G}$ that may be derived from a covariant Lagrangian of the form (4.51).

We can now construct complete observables (with respect to all the constraints of GR) of the form (4.49) for our deparametrised GR. Let $\tilde{H} := \sqrt{q}\sqrt{G}$ then:

$$O_{[f,\phi]}(\tau,\cdot) := \left( \sum_{m=0}^{\infty} \frac{1}{m!} \left\{ \tilde{H}(\beta), f \right\}_m \right)_{|\beta=(\tau-\phi)} \tag{4.60}$$

where $\tilde{H}(\beta) := \int_\Sigma d^3\sigma \beta(\sigma)\tilde{H}(\sigma)$ and $f$ is both independent of $(\phi, \pi)$ and spatially diffeomorphism invariant, is invariant under the diffeomorphism and Hamiltonian constraints $H_a^{Tot}$ and $\tilde{H}_{i,j}^{Tot}$ respectively. Note we have used a different notation here as opposed to $F_{[f,T^J]}(\tau^J, \cdot)$ because of one important difference we must have $\tau$ a constant rather than a spatial field $\tau(\sigma)$. This latter condition ensures that $O_{[f,T^J]}(\tau, \cdot)$ is spatially diffeomorphism invariant. This condition was not needed in the earlier discussion of complete observables for GR because when we reduced the complete observable to one parameter in (4.45), the clock variables were spatially diffeomorphism invariant. In the case here the clock variables $\phi$ are spatial scalar fields and therefore are not invariant under the spatial diffeomorphism constraints but rather transform as in (4.32). The proofs of both the diffeomorphism and Hamiltonian invariance of (4.60) can be found in [49].

The next property is that the deparametrisation of GR provides a physical Hamiltonian, see claim 4.7. In fact H defined as:

$$\mathrm{H} := \int_\Sigma d^3\sigma \tilde{H}(\sigma) \tag{4.61}$$

is a physical Hamiltonian that is gauge invariant and that generates evolution of the complete observables $O_{[f,T^J]}(\tau, \cdot)$ as follows:

$$\frac{d}{d\tau} O_{[f,\phi]}(\tau, \cdot) = \left\{ \mathrm{H}, O_{[f,\phi]}(\tau, \cdot) \right\}. \tag{4.62}$$

The proof involves the same manipulations as used in claim 4.7, see [49].

We shall not discuss the detailed selection criteria and subsequent derivation of the Lagrangian function for this model. We simply note that this selection involved using: i.) existence of algebraic solutions for $I = J(K, V)$, ii.) spatial homogeneity for the phantom field $\phi$, motivated by having a synchronised clock on $\Sigma$ and iii.) a weak gravitational limit close to the standard model, which is equivalent to $\tilde{H} \approx \left| H^{GR} + H^M \right|$ in the small $\alpha$ limit. These criteria lead to the following expression for $L$:

$$L = -\beta + \alpha\sqrt{g}\sqrt{(1 + g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu})} \tag{4.63}$$

[54] This construction has only deparametrised the Hamiltonian constraints while the diffeomorphism constraints have been left in their original form. If one wanted to deparametrise the diffeomorphism constraints as well a further three scalar fields would be required. In fact the dust gravity coupling does indeed fully deparametrise GR, [48].

where $\alpha, \beta$ are constants of dimensions cm$^{-2}$ and $\alpha > 0$. One can then run through the deparametrisation mechanism as described above for this particular Lagrangian and in turn define complete observables and a physical Hamiltonian.

The usual FRW equation for our gravity, standard matter and phantom field system would be derived by computing $\dot{a} = \left\{H^{Tot}, a\right\}$ where $a$ is the scale factor and $H^{Tot} = H^{GR} + H^M + H^{Ph}$ the original total Hamiltonian constraint. By contrast the physical gauge invariant evolution equations obtained through this deparametrisation process would be computed by

$$\frac{d}{d\tau} O_{[a,\phi]}(\tau, \cdot) = \left\{\mathrm{H}, O_{[a,\phi]}(\tau, \cdot)\right\} \tag{4.64}$$

where H is the physical Hamiltonian derived from the specific Lagrangian model in (4.63). Note the FRW equations without the phantom matter, $\dot{a} = \left\{H^{GR} + H^M, a\right\}$ are approximated by evolution with respect to the physical Hamiltonian precisely because $\tilde{H} \approx \left|H^{GR} + H^M\right|$ in the small $\alpha$ limit. However, this relationship is never exact and so there will always be a discrepancy between the two models and in [49] this is significant for large scale factors.

The final point to mention is that the negative energy density requirement for the phantom field has to be compensated by the remaining matter so that the total energy density is positive and this requires either positive dust and cosmological constant contributions or a k-essence field, [49].

## 4.5 Discussion

In this section we have discussed a number of topics concerning relational Dirac observables for GR. The key issue that one is trying to understand is that the canonical analysis of GR implies evolution with respect to coordinate time is a gauge transformation and yet i.) gauge variant equations evolving with respect to coordinate time provide excellent agreement with experiment, e.g. the FRW equations for the expansion of the universe and ii.) we do not view such time evolution as a gauge transformation we appear to observe real physical change and iii.) in the gauge invariant view one is forced to conclude complete observables do not change in coordinate time- the frozen formalism. All these issues come broadly under the problem of time in quantum gravity.

The relational framework attempts to address these issues by defining complete observables with respect to non-gravitational dynamic fields and recovering a notion of evolution with respect to the values these fields or ‘clock variables’ can take. Such observables have been shown to be gauge invariant for first class constrained systems. These complete observables are in general quite complicated, the perturbative definitions involve multiple iterated Poisson brackets, the possible inversion of (in the field theoretic case) infinite dimensional matrices and multiple-fold integrals, see (4.28). For this reason we have discussed several methods to reduce the complexity of these expressions: i.) reducing the effective number of constraints and parameters by using partial observables and clock variables which have ultra-local Poisson brackets with the constraints, ii.) taking advantage of the result that one can compute complete observables in stages and iii.) using deparametrisation to find a physical Hamiltonian from which evolution equations for complete observables can be computed. Further it is not clear what the convergence properties of these expressions are in general and presumably they would depend upon the choice of clock variables and should need to be examined on a case by case basis.

Observables in the reduced constraint form for GR, (4.45) and the phantom deparametrisation case (4.60) have a similar level of complexity involving just one type of iterated Poisson bracket. In the deparametrised case one has the advantage of a physical conservative Hamiltonian and no requirement to abelianise the constraints as this is automatic after deparametrisation. In the reduced constraint case one needs to work with diffeomorphism invariant partial observables and clocks, which will be quite complicated expressions of the canonical fields.

It is an important result that deparametrisation can provide a method to obtain physical evolution in GR that can also approximate the gauge evolution with respect to the Hamiltonian constraint, e.g. the FRW equations, [49]. It is an interesting speculation[55] that something similar might happen with GR coupled to any matter, i.e. gauge variant equations are always reduced gauge invariant expressions, i.e the coordinates which one measures are really 'physical coordinates' depending upon matter degrees of freedom but that the observer does not describe this relation with respect to other fields and so is left with gauge variant expressions. However, to the author's knowledge for GR coupled with standard model matter it is not known if or how one could show this, as GR is not deparametrisable in this case.

The discussion in this section has been classical in nature, it is an open research problem how to construct gauge invariant operators on the physical Hilbert space in canonical gravity. The main problems are that the physical Hilbert space is not well understood due to the difficulty in solving the Hamiltonian constraint and that there will be operator ordering ambiguities. In LQG geometrical (area and volume) operators have been constructed and their spectra have been found to be discrete. Unfortunately, these operators are not gauge invariant in that they do not commute with the Hamiltonian constraint and so it is unclear whether these kinematical results will hold at the physical level.

[55] See section 2.2 in [44].

# 5 Connection Formalism

In this final section we discuss a reformulation of GR, originally due to Ashtekar, [51], in the mid 1980s, which has formed the basis of modern attempts to canonically quantize general relativity, resulting in the theory known as *Loop Quantum Gravity* (LQG). The essence of the approach is to consider GR as a theory of connections (just like in Yang Mills theory) rather than metrics. In particular a SU(2) connection and metric (tetrad) are chosen as conjugate dynamical variables (configuration and momentum respectively), whose combined equations of motion are equivalent[56] to those derived from the Einstein Hilbert action. By describing GR in this way it can be viewed as a Yang Mills theory subject to additional constraints[57].

The idea to describe GR in terms of a dynamic connection is not new and dates back to the Palatini action, where a Lorentz connection and tetrad are treated as independent dynamical variables. However the Palatini approach ultimately does not affect the Hamiltonian theory and one recovers the ADM phase space after performing the 3+1 analysis, and solving additional second class constraints.

The modification due to Ashtekar rests on the important result that i.) one can express a complex version of the Palatini action as a sum of two terms involving only Lorentz self dual and anti-self dual connections and ii.) that either one of these terms contains the theory of complex GR within it. Thus one can recover complex GR by using only half of the information in the original complex Palatini action. After performing a canonical analysis one finds that the configuration variable is a SO(3) connection one form and the conjugate momentum is a complex densitised triad of weight +1.

This self dual approach has some important consequences i.) it describes complex GR so that reality conditions must be imposed to recover real GR at the end, ii.) the Hamiltonian theory involves an additional first class constraint, an SO(3) version of the Gauss constraint found in Yang Mills theory and iii.) all the constraints involve only polynomial expressions in the canonical variables. This latter point was considered the main advantage of the Ashtekar self dual formalism because it offered potential benefits in the quantum theory, (regarding operator ordering and regularization issues), e.g. we recall by contrast that the expressions for the Hamiltonian and diffeomorphism constraints in the ADM formalism involve the inverse determinant $\frac{1}{\sqrt{q}}$ and the inverse metric[58] respectively. Unfortunately, this early promise has not materialised because although the constraints are easier to handle the imposition of reality conditions in the quantum theory has proven intractable.

For this reason a generalised Ashtekar formalism has been developed in the mid 1990s, involving a modified connection, [52, 53], (the Ashtekar Barbero Immirzi connection) and tetrad, which is chosen to describe a real GR from the outset. One can show that the new phase space variables, (a SU(2) connection[59] and densitised triad as configuration and momentum respectively) are the

[56] Strictly speaking this formalism extends metric GR because there is no requirement for the tetrad to be invertible. Hence, although every solution of Einstein GR is a solution in the connection formalism, the converse is not true.

[57] We stress that this difference is, from the point of view of the dynamics, huge. The fact that the phase spaces for GR and a SU(2) Yang Mills fields are the same is a kinematical relation only the dynamics is of course quite different. Indeed an extreme example of this is the relation between a certain topological field theory called 'BF theory' which has the same phase space as GR. In fact GR can be viewed as BF theory with additional constraints. The fact that the additional constraints can turn a theory with a finite number of degrees of freedom into one with an infinite number of degrees of freedom (GR) is a result of the new constraints not commuting with the original BF constraints and therefore in effect turning what were gauge equivalent phase space points into physically distinct states.

[58] The inverse metric appears in the covariant derivative.

[59] The connection could be interpreted as SO(3) because of the Lie algebra isomorphism between SU(2) and SO(3). However, in order to describe fermionic degrees of freedom one is required to interpret it as an SU(2)

result of i.) a canonical transformation of the ADM variables subject to the imposition of a Gauss constraint and ii.) a 3+1 analysis of the Holst action, [54], for which, however, a partial gauge fixing is required in order to complete the canonical analysis. The result is a first class system, like the self dual formalism, but which is chosen real and for which the constraints are no longer of polynomial form.

One could, at this point, question what has been achieved over the ADM variables. However work has shown that progress (at the kinematical level) in the quantum theory can be made by using techniques imported from Yang Mills theories (relating to Wilson loops), and which may be applied regardless of the fact that the Hamiltonian constraint still contains a factor of $\frac{1}{\sqrt{q}}$.

Our main references for this section have been for the Palatini and self dual formalisms the introduction in [55] and detailed reviews in [22, 56, 57, 58]. The modern approach using the generalised Ashtekar Barbero connection is described in detail in [5], [16] and in recent LQG reviews including [59, 60].

## 5.1 Tetrad formalism

Before we write down the Palatini action we need to introduce the notion of a tetrad (or vielbein or frame field) and some related connection and curvature structures. The tetrad most basically can be viewed as a set of orthonormal, with respect to the metric $g_{\mu\nu}$, basis vectors at each point in spacetime $M$. They can be physically understood as describing the frame of reference of an inertial observer, who in a sufficiently small region recovers, by the equivalence principle, special relativity.

More formally one can define, [55], a tetrad as a vector bundle isomorphism $e$ between a local trivialization and the tangent bundle of spacetime, i.e.

$$\begin{aligned} e : M \times \mathbb{R}^4 &\rightarrow TM && (5.1)\\ e : (X, v) &\mapsto e(X, v)^\mu \partial_\mu && (5.2)\end{aligned}$$

where $\mathbb{R}^4$ represents Minkowski spacetime, and $\partial_\mu$ is a coordinate basis in $TM$. One has then trivialised the tangent bundle into a vector bundle, which has a copy of Minkowski space at each point of $M$. This approach makes contact with the mathematical formalism of classical gauge and matter fields, which are described by principal and associated vector fibre bundles respectively, see [61] for an introduction. In this way one views the copy of Minkowski spacetime as an "internal space" in the same way that one views either the gauge group $G$ or its representation space as an internal space in Yang Mills / matter theory.

If one chooses a set of orthonormal vectors, $\xi_I$, $I = 0, 1, 2, 3$ in $\mathbb{R}^4$ then one defines

$$e(X, \xi_I) = e(X)^\mu_I \partial_\mu \qquad (5.3)$$

as usual we shall use Greek letters to denote spacetime indices and capital Latin letters $I, J, K$ to denote the internal Minkowski space indices. All spacetime indices can be raised or lowered only with the spacetime metric $g_{\mu\nu}$. As $\mathbb{R}^4$ is Minkowski spacetime it comes with the standard metric $\eta_{IJ} = diag(-1, +1, +1, +1)$, and similarly one can raise or lower Minkowski (internal space) indices only with $\eta_{IJ}$.

We shall assume that the tetrad map is such that the two tetrads $e_I, e_J$ are orthonormal with respect to the spacetime metric and then it follows

$$\begin{aligned} \eta_{IJ} &= g(e_I, e_J) \\ &= e^\mu_I e^\nu_J g_{\mu\nu} && (5.4)\end{aligned}$$

connection.

and because the map $e$ can be inverted, one obtains

$$g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ} \tag{5.5}$$

where the object $e_\mu^I$ is known as a co-tetrad field.

It is clear from (5.5) that knowledge of the tetrad field implies one can determine the spacetime metric. However, the converse is not true there are an infinity of frame fields satisfying (5.5), all related to each other by local Lorentz transformations, i.e.

$$\begin{aligned} e_\mu^I(X)e_\nu^J(X)\eta_{IJ} &= e_\mu^I(X)e_\nu^J(X)\Lambda^{I'}_I(X)\Lambda^{J'}_J(X)\eta_{I'J'} \\ &= e_\mu^{I'}(X)e_\nu^{J'}(X)\eta_{I'J'} \end{aligned} \tag{5.6}$$

using the invariance of the Minkowski metric under Lorentz transformations. Thus the local Lorentz transformations are to be interpreted as gauge in this formalism. Of course this can be seen from the number of independent components in (5.5), the spacetime metric has 10 such components whereas the tetrad has 16 components, the difference 6 corresponds to the dimension of the Lorentz group SO(1,3).

The other structure we need is the connection on the local trivialization, $M \times \mathbb{R}^4$, as with any vector bundle one cannot define differentiation without this additional structure, and in general the connection will be a spacetime 1-form $A_\mu$, with values in the general linear group $GL(4, \mathbb{R})$. We may define the action of this connection on Minkowski indices by

$$D_\mu v^I = \partial_\mu v^I + A_{\mu\ J}^{\ I} v^J \tag{5.7}$$

and extend it to Minkowski tensors by linearity, the Leibniz rule and commutation with contractions, as is done with the spacetime connection $\nabla_\mu$.

We also require that this connection be compatible with the Minkowski metric $\eta_{IJ}$, this implies

$$\begin{aligned} 0 &= D_\mu \eta_{IJ} \\ &= \partial_\mu \eta_{IJ} - A_{\mu\ I}^{\ K}\eta_{KJ} - A_{\mu\ J}^{\ K}\eta_{IK} \\ &= -2A_{\mu(IJ)} \end{aligned} \tag{5.8}$$

and hence the connection $A_{\mu IJ}$ is anti-symmetric, in its internal indices, and for this reason is known as a Lorentz connection and takes values in the Lie algebra $\mathfrak{so}(1,3)$.

The isomorphism $e$ allows one to define a spacetime connection $\tilde{\nabla}$, induced from the Lorentz connection, by

$$\tilde{\nabla}_U V := e(D_U(e^{-1}(V))) \tag{5.9}$$

where $U$, $V$ are arbitrary spacetime vectors and in components this implies

$$\begin{aligned} \tilde{\Gamma}^\alpha_{\mu\nu}\partial_\alpha &:= \tilde{\nabla}_\mu \partial_\nu \\ &= e\left((\partial_\mu e_\nu^I + A_{\mu\ J}^{\ I} e_\nu^J)\xi_I\right) \\ &= (\partial_\mu e_\nu^I + A_{\mu\ J}^{\ I} e_\nu^J)e_I^\alpha \partial_\alpha \end{aligned} \tag{5.10}$$

and hence

$$\tilde{\Gamma}^\alpha_{\mu\nu} = (\partial_\mu e_\nu^I + A_{\mu\ J}^{\ I} e_\nu^J)e_I^\alpha \tag{5.11}$$

we have placed a tilde on the spacetime connection induced in this way to distinguish it from the Levi Civita connection.

We should like to consider two properties of this induced spacetime connection i.) its action on the spacetime metric and ii.) its torsion. First consider its action on the spacetime metric $g_{\mu\nu}$

$$
\begin{aligned}
\tilde{\nabla}_\mu g_{\nu\sigma} &= \partial_\mu g_{\nu\sigma} - \tilde{\Gamma}^\gamma_{\mu\nu} g_{\gamma\sigma} - \tilde{\Gamma}^\gamma_{\mu\sigma} g_{\nu\gamma} \\
&= \partial_\mu g_{\nu\sigma} - g_{\gamma\sigma} e^\gamma_I (\partial_\mu e^I_\nu + A_\mu{}^I{}_J e^J_\nu) - g_{\nu\gamma} e^\gamma_I (\partial_\mu e^I_\sigma + A_\mu{}^I{}_J e^J_\sigma) \\
&= \partial_\mu (e^I_\nu e^J_\sigma \eta_{IJ}) - g_{\gamma\sigma} e^\gamma_I \partial_\mu e^I_\nu - g_{\nu\gamma} e^\gamma_I \partial_\mu e^I_\sigma \\
&\quad - g_{\gamma\sigma} e^\gamma_I A_\mu{}^I{}_J e^J_\nu - g_{\nu\gamma} e^\gamma_I A_\mu{}^I{}_J e^J_\sigma \\
&= -e^I_\sigma e^J_\nu A_{\mu IJ} - e^J_\sigma e^I_\nu A_{\mu IJ} \\
&= 0
\end{aligned}
\tag{5.12}
$$

where in the second line we have used (5.11), in the third line we have substituted for the spacetime metric using (5.5), in the fourth line cancelled the terms involving derivatives using Leibniz and in the final line used the anti-symmetry of the Lorentz connection. Hence we have shown that the induced spacetime connection is compatible with the spacetime metric.

Second the torsion of the induced connection, we have

$$
\begin{aligned}
\tilde{\Gamma}^\alpha_{\mu\nu} - \tilde{\Gamma}^\alpha_{\nu\mu} &= 2(\partial_{[\mu} e^I_{\nu]} + A_{[\mu}{}^I{}_{|J|} e^J_{\nu]}) e^\alpha_I \\
&= T^\alpha_{\mu\nu}
\end{aligned}
\tag{5.13}
$$

where $T^\alpha_{\mu\nu}$ is the torsion tensor[60], and one can then express this using differential forms as

$$
\begin{aligned}
T^I &:= e^I_\rho T^\rho_{\mu\nu} dx^\mu \wedge dx^\nu \\
&= 2 e^I_\rho e^\rho_J \left( \partial_{[\mu} e^J_{\nu]} + A_{[\mu}{}^J{}_{|K|} e^K_{\nu]} \right) dx^\mu \wedge dx^\nu \\
&= 2(\partial_{[\mu} e^I_{\nu]} + A_{[\mu}{}^I{}_{|K|} e^K_{\nu]}) dx^\mu \wedge dx^\nu \\
&=: d_D e^I
\end{aligned}
\tag{5.14}
$$

where $T^I$ is the torsion two form, $d_D$ is the exterior covariant derivative, and (5.14) is the first Cartan structure equation. Note that the induced connection is not necessarily torsion free, in fact we shall see that in the Palatini action the torsion free condition $T^I = 0$ is one of the Euler Lagrange equations derived from it.

The final result we need before discussing the Palatini action is the relation between the curvature on the Minkowski vector bundle and the spacetime curvature it defines via the induced connection. First the internal curvature two form $F^{IJ}_{\mu\nu}$ is defined by

$$
2D_{[\mu} D_{\nu]} v_I := F_{\mu\nu I}{}^J v_J \tag{5.15}
$$

where $v_I$ is an arbitrary Lorentz co-vector and $F^{IJ}_{\mu\nu}$ can be expressed, in terms of the Lorentz connection coefficients, as

$$
\begin{aligned}
F^{IJ}_{\mu\nu} &= \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + [A_\mu, A_\nu]^{IJ} \\
&= \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + A_\mu{}^I{}_K A^{KJ}_\nu - A_\nu{}^I{}_K A^{KJ}_\mu \, .
\end{aligned}
\tag{5.16}
$$

Again we may express this relation using differential forms as

$$
F^{IJ} = dA^{IJ} + A^I{}_K \wedge A^{KJ} \, . \tag{5.17}
$$

[60] We are using a coordinate spacetime basis, labelled by the Greek indices, so we do not need the additional term in the definition of the torsion tensor involving the commutator of the basis vectors, which is automatically zero in this case.

Second recall that the induced spacetime Riemann curvature is defined by

$$2\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}v_\sigma := \tilde{R}^{\rho}_{\mu\nu\sigma}\, v_\rho \tag{5.18}$$

for an arbitrary spacetime co-vector $v_\sigma$.

By using the abstract definition for the induced connection in (5.9) one can deduce

$$\begin{aligned}
\tilde{\nabla}_U\tilde{\nabla}_V W &= \tilde{\nabla}_U\left(e(D_V(e^{-1}(W)))\right) \\
&= e\left(D_U\left(e^{-1}\circ e\left(D_V\left(e^{-1}(W)\right)\right)\right)\right) \\
&= e\left(D_U D_V\left(e^{-1}(W)\right)\right)
\end{aligned} \tag{5.19}$$

where $U$, $V$, $W$ are arbitrary spacetime vectors and then anti-symmetrising we get

$$\tilde{R}(U,V)W = e\left(F(U,V)(e^{-1}(W))\right) \tag{5.20}$$

and hence one can read off the components of this equation when acting on basis vectors to get

$$\begin{aligned}
\tilde{R}^{\rho}_{\mu\nu\sigma}\partial_\rho &= e\left(F_{\mu\nu I}{}^{J}\xi_J e^I_\sigma\right) \\
&= F_{\mu\nu I}{}^{J} e^I_\sigma e^\rho_J \partial_\rho
\end{aligned} \tag{5.21}$$

where we have used $F(\partial_\mu,\partial_\nu)(e^{-1}\partial_\sigma) = F_{\mu\nu}(e^I_\sigma\xi_I) = e^I_\sigma F_{\mu\nu I}{}^{J}\xi_J$, and hence

$$\tilde{R}^{\rho}_{\mu\nu\sigma} = F_{\mu\nu I}{}^{J} e^I_\sigma e^\rho_J\,. \tag{5.22}$$

In short one can transfer between the internal and spacetime curvature by appropriate contractions of indices with the tetrad and co-tetrad. Using (5.22) one can express the spacetime curvature Ricci tensor and scalar in terms of contractions of the internal curvature and tetrad as

$$\begin{aligned}
\tilde{R}^{\rho}_{\mu\nu\sigma} &= e^I_\sigma e^\rho_J F_{\mu\nu I}{}^{J} \\
\Rightarrow \tilde{R}_{\mu\sigma} &= e^I_\sigma e^\rho_J F_{\mu\rho I}{}^{J} && (5.23) \\
\Rightarrow \tilde{R} &= g^{\sigma\mu} e^I_\sigma e^\rho_J F_{\mu\rho I}{}^{J} \\
&= e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}\,. && (5.24)
\end{aligned}$$

## 5.2 Hilbert Palatini Action

We can now write down the Hilbert-Palatini action, which is just the Einstein Hilbert action, (3.4), but with the Lorentz connection, defining the curvature, and the tetrad defining the metric now considered as independent variables. Recall by the previous section we know that the induced connection, though metric compatible, is not the Levi Civita connection because it is not torsion free.

One has from (5.5) that

$$\sqrt{|g|} = |e| \tag{5.25}$$

where $e := det(e^I_\mu)$. The Palatini action is defined as

$$\begin{aligned}
S_P[e,A] &= \frac{1}{16\pi G}\int_M d^4X\,|e|\,\tilde{R}(e,A) \\
&\phantom{=} \frac{1}{16\pi G}\int_M d^4X\,|e|\,e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}(A)
\end{aligned} \tag{5.26}$$

where the action is a functional of both the tetrad and the Lorentz connection $A^{IJ}_\mu$.

We now compute the equations of motion for this action. First consider the variation with respect to the tetrad, one has the standard result for the variation of the determinant

$$\begin{aligned}
\delta|e| &= |e|e_I^\mu \delta e_\mu^I \\
&= -|e|e_\mu^I \delta e_I^\mu
\end{aligned} \tag{5.27}$$

and hence

$$\begin{aligned}
\delta S_P &= \frac{1}{16\pi G}\int_M d^4X \left(\delta|e|e_I^\mu e_J^\nu F_{\mu\nu}^{IJ} + \delta e_I^\mu |e| e_J^\nu F_{\mu\nu}^{IJ} + \delta e_J^\nu |e| e_I^\mu F_{\mu\nu}^{IJ}\right) \\
&= \frac{1}{8\pi G}\int_M d^4X\, |e|\delta e_I^\mu \left(-\frac{1}{2}e_\mu^I e_K^\sigma e_J^\nu F_{\sigma\nu}^{KJ} + \delta_K^I e^{K\sigma}\tilde{R}_{\mu\sigma}\right) \qquad (5.28)\\
&= \frac{1}{8\pi G}\int_M d^4X\, |e|\delta e_I^\mu \left(-\frac{1}{2}e_J^\nu \eta^{IJ} g_{\mu\nu}\tilde{R} + \eta^{IJ}\eta_{JK}e^{K\sigma}\tilde{R}_{\mu\sigma}\right) \\
&= \frac{1}{8\pi G}\int_M d^4X\, |e|\delta e_I^\mu \left(-\frac{1}{2}e_J^\nu \eta^{IJ} g_{\mu\nu}\tilde{R} + \tilde{R}_{\mu\nu}\right) \\
&= \frac{1}{8\pi G}\int_M d^4X\, |e|\delta e_I^\mu \eta^{IJ} e_J^\nu \left(\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}\right) \qquad (5.29)
\end{aligned}$$

this implies the equation of motion

$$|e|\eta^{IJ}e_J^\nu \left(\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}\right) = 0 \tag{5.30}$$

which of course would be Einstein's equations (assuming the tetrad is non-degenerate, which we require in the variation) if our induced connection were torsion free.

The other equation of motion comes from varying the connection $A$. In order to compute this variation we need the result of the variation of the curvature $F_{\mu\nu}^{IJ}$, we have from (5.16)

$$\begin{aligned}
\delta F_{\mu\nu}^{IJ} &= \partial_\mu \delta A_\nu^{IJ} - \partial_\nu \delta A_\mu^{IJ} + \delta A_{\mu\,K}^{I} A_\nu^{KJ} + A_{\mu\,K}^{I}\delta A_\nu^{KJ} - \delta A_{\nu\,K}^{I} A_\mu^{KJ} - A_{\nu\,K}^{I}\delta A_\mu^{KJ} \\
&= \partial_\mu \delta A_\nu^{IJ} + A_{\mu\,K}^{I}\delta A_\nu^{KJ} + A_{\mu\,K}^{J}\delta A_\nu^{IK} - \partial_\nu \delta A_\mu^{IJ} - A_{\nu\,K}^{I}\delta A_\mu^{KJ} - A_{\nu\,K}^{J}\delta A_\mu^{IK} \\
&= D_\mu \delta A_\nu^{IJ} - D_\nu \delta A_\mu^{IJ} \\
&= 2D_{[\mu}\delta A_{\nu]}^{IJ} \qquad (5.31)\\
&= (d_D \delta A)^{IJ} \qquad (5.32)
\end{aligned}$$

where in the second line we just used the anti-symmetry of the Lorentz connection and then in the penultimate line the definition of the covariant derivative.

Hence the initial variation in the Palatini action with respect to the connection is given by

$$\begin{aligned}
\delta S_P &= \frac{1}{16\pi G}\int_M d^4X\, |e|\, e_I^\mu e_J^\nu \delta F_{\mu\nu}^{IJ} \\
&= \frac{1}{16\pi G}\int_M d^4X\, 2|e|\, e_I^\mu e_J^\nu D_{[\mu}\delta A_{\nu]}^{IJ} \\
&= \frac{1}{16\pi G}\int_M d^4X\, 2|e|\, e_I^{[\mu} e_J^{\nu]} D_\mu \delta A_\nu^{IJ}\,.
\end{aligned} \tag{5.33}$$

At this point, prior to performing an integration by parts, it is convenient to use the following identity, [58],

$$2|e|\, e_I^{[\mu} e_J^{\nu]} = \epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\rho^K e_\sigma^L \tag{5.34}$$

where to be clear $\epsilon_{IJKL}$ is the $SO(1,3)$ $\epsilon$ symbol which is totally anti-symmetric and for which we define $\epsilon_{0123} = 1$, its indices can be raised by the Minkowski metric only and we observe that this implies $\epsilon^{0123} = -1$. The objects $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ are both totally anti-symmetric and satisfy $\epsilon^{0123} = \epsilon_{0123} = 1$ in every coordinate system, and are tensor densities of weight $+1$ and $-1$ respectively.

and this implies

$$\begin{aligned}
\delta S_P &= \frac{1}{16\pi G}\int_M d^4X\, \epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma D_\mu \delta A^{IJ}_\nu \\
&= -\frac{1}{16\pi G}\int_M d^4X\, D_\mu\left(\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma\right)\delta A^{IJ}_\nu \\
&\quad +\frac{1}{8\pi G}\int_M d^4X\, \partial_\mu\left(|e|\, e^{[\mu}_I e^{\nu]}_J \delta A^{IJ}_\nu\right) \\
&= -\frac{1}{16\pi G}\int_M d^4X\, D_\mu\left(\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma\right)\delta A^{IJ}_\nu
\end{aligned} \tag{5.35}$$

where we have dropped the boundary term. Hence the equation of motion derived from the variation with respect to the connection is

$$\begin{aligned}
0 &= D_\mu\left(\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma\right) \\
0 &= \epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}D_\mu\left(e^K_\rho e^L_\sigma\right) \\
\Rightarrow 0 &= \epsilon_{IJKL}e^K_\rho \epsilon^{\mu\nu\rho\sigma}D_\mu e^L_\sigma \\
\Rightarrow 0 &= D_{[\mu}e^L_{\sigma]} \\
\Leftrightarrow 0 &= T^L_{\mu\sigma}
\end{aligned} \tag{5.36}$$

where in the second line we used the anti-symmetry of the Lorentz connection, which ensures the covariant constancy of $\epsilon_{IJKL}$ and in the final line used the definition of the torsion two form, given in (5.14). The torsion free condition now implies the induced spacetime connection is the unique Levi Civita connection and hence that (5.30) is now equivalent to Einstein's equations in vacuum. The Palatini formalism is often called first order because the equations of motion only involve first order derivatives of the dynamical variables in contrast to the Einstein Hilbert action where e.g. the Ricci tensor involves second order derivatives of the metric.

Before we discuss the self dual formalism we should like to make some remarks concerning the Hilbert Palatini formalism, described here.

First, note that the identity (5.34) allows us to re-write the Palatini action as an integral of a four form

$$\begin{aligned}
S_P[e,A] &= \frac{1}{32\pi G}\int_M d^4X\, \epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma F^{IJ}_{\mu\nu}(A) \\
&= \frac{1}{32\pi G}\int_M \epsilon_{IJKL}\left(\epsilon^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma F^{IJ}_{\mu\nu}d^4X\right) \\
&= \frac{1}{32\pi G}\int_M \epsilon_{IJKL}e^K\wedge e^L\wedge F^{IJ}
\end{aligned} \tag{5.37}$$

where we have suppressed the spacetime indices and used the fact that the co-tetrads, and curvature are one and two forms respectively.

This differential form version of the Palatini action, is the one usually cited in modern treatments, [4] and in this formalism one only ever has to work with the co-tetrad $e^K_\mu$ and it is this field that is identified as the *gravitational field.* The Euler Lagrange equations of this action can be immediately computed, the variation with respect to the co-tetrad is

$$
\begin{aligned}
\delta S_P &= \frac{1}{32\pi G}\int_M \epsilon_{IJKL}\left(\delta e^K\wedge e^L\wedge F^{IJ} + e^K\wedge\delta e^L\wedge F^{IJ}\right)\\
&= \frac{1}{32\pi G}\int_M \epsilon_{IJKL}\left(\delta e^K\wedge e^L\wedge F^{IJ} - \delta e^L\wedge e^K\wedge F^{IJ}\right)\\
&= \frac{1}{32\pi G}\int_M \epsilon_{IJKL}\delta e^K\wedge e^L\wedge F^{IJ} + \epsilon_{IJKL}\delta e^K\wedge e^L\wedge F^{IJ}\\
&= \frac{1}{16\pi G}\int_M \delta e^K\wedge\left(\epsilon_{IJKL}e^L\wedge F^{IJ}\right) \qquad (5.38)
\end{aligned}
$$

and the variation with respect to the connection is unchanged and hence one derives the equations of motion

$$
\begin{aligned}
\epsilon_{LIJK}e^L\wedge F^{IJ} &= 0 \qquad (5.39)\\
T^I &= 0\,. \qquad (5.40)
\end{aligned}
$$

Of course the first equation, assuming the co-tetrad is non-degenerate, is equivalent to Einstein's equations subject to the torsion free condition. However, the difference with these equations is that because the tetrad is never explicitly used we may also consider $e^L = 0$, as a solution to these equations of motion. But note that this solution corresponds to a degenerate spacetime metric by (5.5) and hence the solution space to (5.39) and (5.40) is strictly greater than the solution space to the metric version of Einstein's equations, as derived from the Einstein Hilbert action. (Of course degenerate spacetime metrics leave the Einstein Hilbert action and equations of motion undefined, as one cannot even define the Levi Civita connection components without the inverse of the metric. The fact that degenerate metrics are allowed in this formalism has been used to suggest that topology change is feasible in the (pure gravity sector of the) Palatini formalism.

The second point we wish to make regards matter coupling in the Palatini formalism, all matter types may be coupled to this action including fermionic matter. Indeed as we mentioned earlier in section 3.4 only the tetrad formalism may be used to describe fermionic degrees of freedom. However, one can simply re-write the Einstein Hilbert action directly in terms of a tetrad basis but where the connection is fixed and non-dynamical such that it induces the Levi Civita connection and in this case one can describe all matter degrees of freedom, i.e one has

$$
S_{EH}[e] = \frac{1}{16\pi G}\int_M d^4X\,|e|\,e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}(A(e)) \qquad (5.41)
$$

where $A(e)$ is such that $\tilde{\Gamma}^\alpha_{\mu\nu}$ is the Levi Civita connection.

However, in the Palatini formalism the connection *is* dynamical and this leads to a non-equivalence in the dynamics for fermions coupled to gravity. This non-equivalence appears because in order to write down a covariant derivative for fermions one must use the Lorentz connection and then one has, in this formalism, a fermionic 'standard model' action term of the form, [4, 5]

$$
S_{Fermions}[e, A^{IJ}_\mu, \phi, A^A_\mu, \psi] = \int_M d^4X\,|e|\left(\bar{\psi}\gamma^I e^\mu_I\left(\partial_\mu\psi + A^J_{\mu\,K}L^K_J\psi + A^A_\mu L_A\psi\right) + Y(\phi,\psi,\bar{\psi}\right) + c.c \qquad (5.42)
$$

where $\phi$ is a scalar field, $A^A_\mu$ is a Yang Mills field with gauge group Lie algebra index $A$, $\psi$ is a Dirac spinor, $\gamma^I$ are the Gamma matrices, $L^K_J$, $L_A$ are representation matrices of the Lorentz and Yang Mills gauge group $G$, which act upon the representation space that $\psi$ is an element of and finally $Y$ is a polynomial interaction term, which will include the mass term for the fermion field after symmetry breaking. Note we have suppressed all spinor indices. The term $\partial_\mu \psi + A_{\mu\ K}^{J} L_J^K \psi + A_\mu^A L_A \psi$ can be viewed as a generalised covariant derivative acting upon the group $SO(1,3) \times G$.

Now when one performs a variation with respect to the Lorentz connection in this term there will be a non-zero contribution $\frac{\delta S_F}{\delta A^{IJ}}$ which contributes to the torsion $T^I$ and hence one finds that the spacetime connection 'on shell' is no longer Levi Civita but will have a non-zero torsion. Hence in the presence of fermions the second order and first order theories are inequivalent.

Finally we consider the Legendre transform of these tetrad formulations of gravity. The Hamiltonian formulation of the tetrad version of the Einstein Hilbert action, (5.41), is derived in detail in [58] and the result is a first class Hamiltonian system. The configuration variable is a triad $e_{ai}(\sigma)$ where $a$ is the spatial index on $\Sigma$ and $i = 1, 2, 3$ is an SO(3) index, where, in analogy with the tetrad, the triad is an orthonormal frame which satisfies

$$q_{ab} = e_a^i e_b^j \delta_{ij} \tag{5.43}$$

where $\delta_{ij}$ is the Euclidean metric on $\mathbb{R}^3$. The conjugate momentum is denoted $\pi^{ai}(\sigma)$ and is related to the extrinsic curvature of the spatial hypersurface. The constraints consist of three types the Hamiltonian and diffeomorphism constraints, as for the metric ADM case, (just now expressed in terms of the new variables), and an additional constraint, $G^i(\sigma)$ which is an SO(3) Gauss constraint, which reflects the freedom to rotate the triads arbitrarily in $\mathbb{R}^3$ without changing the induced spatial metric. Note that in order to *reduce* the gauge symmetry from SO(1,3) to SO(3) a *gauge fixing* has been employed. This involves fixing an internal timelike vector, $n^I$, which is used in the $3 + 1$ decomposition of the tetrad. One can show the equivalence of this constrained Hamiltonian system to the ADM phase space by constructing the quotient space with respect to the Gauss constraint. This follows because the equivalence class of triads is equivalent to the spatial metric $q_{ab}(\sigma)$ and the momentum can be directly related to the ADM momentum $P^{ab}(\sigma)$, after taking into account the Gauss constraint, [22].

The Legendre transform for the Hilbert Palatini action, [22, 56, 58], is more involved because the Hamiltonian system contains additional second class constraints, not present in the analysis of (5.41). One has to solve these constraints[61] and, in the process use a partial gauge fixing, in order to recover a first class system and one then finds that the Hamiltonian theory is equivalent to the tetrad formulation of Einstein Hilbert just discussed. We shall briefly discuss this Legendre transform so that one can see where the second class constraints appear and the form of the final phase space.

We begin by performing a 3+1 analysis of the Palatini action, (5.26), using the usual decomposition of the unit normal vector $n^\mu$ into the lapse and shift, $n^\mu = N^{-1}(T^\mu - N^\mu)$ one can define a spatial projection of the tetrad by

$$\begin{aligned} E_I^\mu &:= e_I^\nu \left(g_\nu^\mu + n^\mu n_\nu\right) && (5.44) \\ &= e_I^\mu + n^\mu n_I && (5.45) \end{aligned}$$

where $n_I := e_I^\mu n_\mu$ is by construction a timelike internal co-vector of unit length. Immediately one can deduce that

$$\begin{aligned} E_I^\mu n_\mu &= e_I^\mu n_\mu + n^\mu n_\mu n_I \\ &= n_I - n_I \\ &= 0 && (5.46) \end{aligned}$$

[61] In principle one does not need to do this, one could keep the second class constraints and define the dynamical system by using the Dirac bracket.

and

$$\begin{aligned}
E^\mu_I n^I &= (e^\mu_I + n^\mu n_I)\, n^I \\
&= e^\mu_I e^I_\nu n^\nu + n^\mu n_I e^{\nu I} n_\nu \\
&= n^\mu + n^\mu n^\nu n_\nu \\
&= n^\mu - n^\mu \\
&= 0
\end{aligned} \tag{5.47}$$

and hence we view $E^\mu_I$ as a triad.

Using this one can express the Palatini action, (5.26) as

$$\begin{aligned}
S &= \frac{1}{16\pi G}\int_M d^4X\, N\sqrt{q}\, (E^\mu_I - n^\mu n_I)\, (E^\nu_J - n^\nu n_J)\, F^{IJ}_{\mu\nu}(A) \\
&= \frac{1}{16\pi G}\int_M d^4X\, N\sqrt{q}\left[E^\mu_I E^\nu_J F^{IJ}_{\mu\nu} - E^\mu_I n_J N^{-1}\,(T^\nu - N^\nu)\, F^{IJ}_{\mu\nu}\right. \\
&\quad \left. -n_I E^\nu_J N^{-1}\,(T^\mu - N^\mu)\, F^{IJ}_{\mu\nu}\right] \\
&= \frac{1}{16\pi G}\int_M d^4X\, N\sqrt{q}\left[E^\mu_I E^\nu_J F^{IJ}_{\mu\nu} - \left(E^\mu_I n_J N^{-1} T^\nu - n_I E^\nu_J N^{-1} T^\mu\right) F^{IJ}_{\mu\nu}\right. \\
&\quad \left. + \left(E^\mu_I n_J N^{-1} N^\nu + n_I E^\nu_J N^{-1} N^\mu\right) F^{IJ}_{\mu\nu}\right] \\
&= \frac{1}{16\pi G}\int_M d^4X\, N\sqrt{q}\left[E^\mu_I E^\nu_J F^{IJ}_{\mu\nu} - 2N^{-1} n_I T^\mu E^\nu_J F^{IJ}_{\mu\nu} + \right. \\
&\quad \left. 2N^{-1} n_I N^\mu E^\nu_J F^{IJ}_{\mu\nu}\right]
\end{aligned} \tag{5.48}$$

where in the second line we used the anti-symmetry of the curvature in either Lorentz or spacetime indices to show $n^\mu n_I n^\nu n_J F^{IJ}_{\mu\nu} \equiv 0$.

By comparison with the ADM formulation we expect that the coefficient of $N$ will form the Hamiltonian constraint, and the coefficient of $N^\mu$ the diffeomorphism constraint. The second term with the time vector contracting an index of the curvature will contribute the $\int p\dot{q}$ term of the 3+1 action and hence will allow us to identify the conjugate momentum variable. In particular the second term will be[62]

$$
\begin{aligned}
&= \frac{1}{8\pi G}\int_M d^4X\,\sqrt{q}\,n_I E^\nu_J T^\mu\left(\partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + A^{IK}_\mu A_{\nu K}{}^J - A^{IK}_\nu A_{\mu K}{}^J\right)\\
&= \frac{1}{8\pi G}\int_M d^4X\,\sqrt{q}\,n_I E^\nu_J\left(T^\mu\partial_\mu A^{IJ}_\nu - A^J_{\nu\,K}T^\mu A^{IK}_\mu - A^I_{\nu\,K}T^\mu A^{K\,J}_\mu\right)\\
&\quad+\frac{1}{8\pi G}\int_M d^4X\,\left[\partial_\nu\left(\sqrt{q}\,n_I E^\nu_J T^\mu\right)A^{IJ}_\mu - \partial_\nu\left(\sqrt{q}\,n_I E^\nu_J T^\mu A^{IJ}_\mu\right)\right]\\
&= \frac{1}{8\pi G}\int_M d^4X\,\sqrt{q}\,n_I E^\nu_J\left(T^\mu\partial_\mu A^{IJ}_\nu - A^J_{\nu\,K}T^\mu A^{IK}_\mu - A^I_{\nu\,K}T^\mu A^{K\,J}_\mu\right)\\
&\quad+\frac{1}{8\pi G}\int_M d^4X\,\left[\sqrt{q}\,n_I E^\nu_J\partial_\nu\left(T^\mu\right)A^{IJ}_\mu - \partial_\nu\left(A^{IJ}_\mu T^\mu\right)\sqrt{q}\,n_I E^\nu_J\right]\\
&= \frac{1}{8\pi G}\int_M d^4X\,\sqrt{q}\,n_I E^\nu_J\left[T^\mu\partial_\mu A^{IJ}_\nu + A^{IJ}_\mu\partial_\nu\left(T^\mu\right)\right]\\
&\quad-\frac{1}{8\pi G}\int_M d^4X\,\sqrt{q}\,n_I E^\nu_J\left[\partial_\nu\left(A^{IJ}_\mu T^\mu\right) + A^J_{\nu\,K}T^\mu A^{IK}_\mu + A^I_{\nu\,K}T^\mu A^{K\,J}_\mu\right]\\
&= \frac{1}{16\pi G}\int_M d^4X\,\sqrt{q}\,2n_I E^\nu_J\left[\mathcal{L}_{\vec{T}}A^{IJ}_\nu - D_\nu(A_\mu T^\mu)^{IJ}\right]
\end{aligned}
\tag{5.49}
$$

where we have used the anti-symmetry of the Lorentz connection, Leibniz rule on partial derivatives and the definitions of i.) the Lie derivative of $A_\nu$ with respect to the time vector (the internal indices act as scalars here) and ii.) the (Lorentz) covariant derivative of $A_\mu T^\mu$.

Now first observe that for the first and final terms in (5.48) the curvature $F^{IJ}_{\mu\nu}$ is always contracted with a spatial object (either $E^\mu_I$ or $N^\mu$) on every index, this means we can replace $F^{IJ}_{\mu\nu}$ with its spatial projection. Secondly, in (5.49) the Lie derivative term is also contracted by a spatial object and one can hence show that $A^{IJ}_\mu$ may be replaced by its spatial projection, [56]. Finally, the covariant derivative in (5.49) is contracted with $E^\nu_J$ and hence that may also be viewed as the action of the spatial covariant derivative. To make this manifest we label the only remaining spacetime connection term (in the covariant derivative) with the superscript 4 all other connection, curvature and derivative terms are to be now interpreted as spatial objects.

Hence substituting (5.49) back into (5.48) and pulling the integral back to $\Sigma\times\mathbb{R}$ one has

$$
S = \frac{1}{16\pi G}\int dtd^3\sigma\,E\left[NE^a_I E^b_J F^{IJ}_{ab} - 2n_{[I}E^a_{J]}\left(\dot{A}^{IJ}_a - D_a({}^4A_\nu T^\nu)^{IJ}\right) + 2N^a n_I E^b_J F^{IJ}_{ab}\right]. \tag{5.50}
$$

where the $a, b$ indices are spatial indices on $\Sigma$, $E \equiv \sqrt{q}$ and $\dot{A}^{IJ}_a$ is the pullback of the Lie derivative to $\Sigma$.

Following [56] this action can be simplified to

$$
S[A_a, E^a, N, N^a, {}^4A_T] = \int dtd^3\sigma\left[\tilde{\alpha}^a_{IJ}\dot{A}^{IJ}_a - \left(-D_a\left(\tilde{\alpha}^a_{IJ}\right){}^4A^{IJ}_T - 2\underset{\sim}{N}\tilde{\alpha}^{aK}_I\tilde{\alpha}^b_{KJ}F^{IJ}_{ab} + N^a\tilde{\alpha}^b_{IJ}F^{IJ}_{ab}\right)\right] \tag{5.51}
$$

where $\underset{\sim}{N} \equiv E^{-1}N$, $\tilde{E}^a_I \equiv EE^a_I$ is a densitised triad, ${}^4A^{IJ}_T \equiv {}^4A^{IJ}_\nu T^\nu$and $\tilde{\alpha}^a_{IJ} \equiv 8\pi G\tilde{E}^a_{[I}n_{J]}$[63] and we have performed an integration by parts and dropped the boundary term. The third term in

[62] We follow the method in [56] but include all the missing steps.

[63] Using an over tilde to indicate an object of density weight +1 and an under tilde for a density of weight −1 is quite common in the literature.

(5.51) follows because working backwards one has

$$\begin{aligned}
\underset{\sim}{N}\tilde{\alpha}_I^{aK}\tilde{\alpha}_{KJ}^b F_{ab}^{IJ} &= \underset{\sim}{N}\tilde{\alpha}_{IK}^a \eta^{KL}\tilde{\alpha}_{LJ}^b F_{ab}^{IJ} \\
&= \underset{\sim}{N}\tilde{E}_{[I}^a n_{K]}\eta^{KL}\tilde{E}_{[L}^b n_{J]}F_{ab}^{IJ} \\
&= \frac{\underset{\sim}{N}}{4}\left(\tilde{E}_I^a n_K - \tilde{E}_K^a n_I\right)\eta^{KL}\left(\tilde{E}_L^b n_J - \tilde{E}_J^b n_L\right)F_{ab}^{IJ} \\
&= \frac{\underset{\sim}{N}}{4}\left(\tilde{E}_I^a n^L \tilde{E}_L^b n_J - \tilde{E}_I^a n_K \eta^{KL}\tilde{E}_J^b n_L - n_I n_J \tilde{E}_K^a \tilde{E}^{bK} + n^K \tilde{E}_K^a n_I \tilde{E}_J^b\right)F_{ab}^{IJ} \\
&= -\frac{\underset{\sim}{N}}{4}\left(n_K n^K \tilde{E}_I^a \tilde{E}_J^b\right)F_{ab}^{IJ} \\
&= -\frac{\underset{\sim}{N}}{4}e_K^\mu n_\mu e^{\nu K} n_\nu \tilde{E}_I^a \tilde{E}_J^b F_{ab}^{IJ} \\
&= -\frac{\underset{\sim}{N}}{4}n_\mu n^\mu \tilde{E}_I^a \tilde{E}_J^b F_{ab}^{IJ} \\
&= \frac{\underset{\sim}{N}}{4}\tilde{E}_I^a \tilde{E}_J^b F_{ab}^{IJ} \\
&= \frac{EN}{4}E_I^a E_J^b F_{ab}^{IJ}
\end{aligned} \tag{5.52}$$

where in the fifth line we use (5.47) and $n_I n_J F_{ab}^{IJ} \equiv 0$ (by symmetry, anti-symmetry in the internal indices), in the sixth line the definition of $n_I$ and in the seventh line the timelike normality of $n^\mu$.

We observe that the action in (5.51) is in standard canonical form $S = \int(p\dot{q} - H)$ and therefore we can read off the conjugate momenta to the configuration variables $A_a, N, N^a, {}^4A_T$ as

$$\tilde{\alpha}_{IJ}^a \equiv 8\pi G \tilde{E}_{[I}^a n_{J]} \tag{5.53}$$

$$P_N = 0 \tag{5.54}$$

$$P_a = 0 \tag{5.55}$$

$$P_{IJ} = 0 \tag{5.56}$$

respectively. Clearly the last three momenta being identically zero correspond to primary constraints. However, in addition (5.53) is a constraint because $\tilde{\alpha}_{IJ}^a$ has 18 independent components whereas $\tilde{E}_{[I}^a n_{J]}$ has 12 such components, (there are 3 independent components in the timelike vector $n_I$ and (5.47) removes 3 degrees of freedom from the the densitised triad leaving 9 independent components). Hence one expects (5.53) to be equivalent to six constraints, in fact see [56] one has equivalence with the following two constraints

$$\phi^{ab} \equiv \epsilon^{IJKL}\tilde{\alpha}_{IJ}^a \tilde{\alpha}_{KL}^b \approx 0 \tag{5.57}$$

$$\tilde{\alpha}_{IJ}^a \tilde{\alpha}_I^{bJ} > 0\,. \tag{5.58}$$

The former constraint consists of (the expected) six independent relations and the latter constraint is an inequality, which does not reduce the dimension of the phase space, it is non-holonomic and for this reason does not get treated together with the other constraints in the usual way.

Hence we can now write down the primary Hamiltonian, which generates the time evolution for this Hamiltonian system, it is given by

$$H_p = H_c + \int_\Sigma d^3\sigma \left(\lambda P_N + \lambda^a P_{N^a} + \lambda^{IJ} P_{IJ} + \lambda_{ab}\phi^{ab}\right) \tag{5.59}$$

where $\lambda$, $\lambda^a$, $\lambda^{IJ}$, $\lambda_{ab}$ are arbitrary and $H_c$ is the canonical Hamiltonian

$$H_c = \int_\Sigma d^3\sigma \left( N^a \tilde{\alpha}^b_{IJ} F^{IJ}_{ab} - 2\underset{\sim}{N} \tilde{\alpha}^{aK}_I \tilde{\alpha}^b_{KJ} F^{IJ}_{ab} - D_a \left(\tilde{\alpha}^a_{IJ}\right) {}^4A^{IJ}_T \right) . \tag{5.60}$$

Following the Dirac-Bergmann algorithm one should now impose preservation of the above constraints under evolution generated by $H_p$ to check for secondary constraints. Immediately we see that preservation of $P_N$, $P_a$, $P_{IJ}$ implies the coefficients of the corresponding configuration variables in the Hamiltonian will be secondary constraints (just as we saw for the metric variables in section 3.2), i.e.

$$\tilde{\alpha}^b_{IJ} F^{IJ}_{ab} \approx 0 \tag{5.61}$$
$$\tilde{\alpha}^{aK}_I \tilde{\alpha}^b_{KJ} F^{IJ}_{ab} \approx 0 \tag{5.62}$$
$$G_{IJ} \equiv D_a \left(\tilde{\alpha}^a\right)_{IJ} \approx 0 . \tag{5.63}$$

Finally, one should check the evolution of $\phi^{ab}$ a non-trivial result, [56], is that this leads to six independent secondary constraints[64]

$$\chi^{ab} \equiv \epsilon^{IJKL} \tilde{\alpha}^{cM}_I \tilde{\alpha}^{(a}_{MJ} D_c \left(\tilde{\alpha}^{b)}\right)_{KL} \approx 0 . \tag{5.64}$$

One should now check for the consistency of these secondary constraints but fortunately there are no further secondary (tertiary) constraints, [56]. Hence we have completed the Dirac-Bergmann algorithm all that remains is to classify the constraints we have found into first and second class. The result, [56], is that all constraints are first class except for $\phi^{ab}$ and $\chi^{ab}$. We observe i.) that all the constraints are polynomial in the dynamical variables and ii.) that (5.63) has the form of a Gauss constraint and given that out internal symmetry is $SO(1,3)$ we expect the $G_{IJ}$ to be generators of Lorentz transformations. Indeed this can be confirmed, if one computes the Poisson algebra of the smeared $G_{IJ}$ then one finds it is isomorphic to the Lorentz algebra.

As argued in section 3.2 we can now reduce the phase space to one only coordinatised by $\left(A^{IJ}_a, \tilde{\alpha}^a_{IJ}\right)$ because i.) the evolution of $N, N^a, {}^4A_T$ will be arbitrary, ii.) the evolution of their conjugate momenta will be fixed as they are constraints and iii.) the evolution of $\left(A^{IJ}_a, \tilde{\alpha}^a_{IJ}\right)$ will not depend upon the primary constraint terms present in the primary Hamiltonian involving these conjugate momenta.

Hence we have a phase space $\left(A^{IJ}_a, \tilde{\alpha}^a_{IJ}\right)$ of $36 \times \infty^3$ dimensions with the symplectic structure

$$\left\{A^{IJ}_a(\sigma), \tilde{\alpha}^b_{KL}(\sigma')\right\} = \delta^b_a \delta^I_{[K} \delta^J_{L]} \delta^3(\sigma, \sigma') \tag{5.65}$$

and now subject to ten first class constraints (5.61), (5.62) and (5.63) and twelve second class constraints (5.57) and (5.64). A simple counting confirms we have the correct number of physical degrees of freedom for general relativity, using (2.72) one has $\frac{1}{2} \times (36 - 22 - 10) \times \infty^3 = 2 \times \infty^3$ degrees of freedom as required.

At this point one could either live with the second class constraints and use Dirac brackets or, as is the usual method, solve them in order to recover a first class system. The original motivation of finding polynomial constraints for GR has been realised in the Palatini formalism but the at the price of second class constraints, which force one to use the Dirac bracket and therefore lead to further operator ordering ambiguities in the quantum theory. We shall not discuss the method of solution of these second class constraints it is described in detail in [22, 56, 58] but the result is that one both solves the constraints and in the process partially solves the first class constraint

[64] This constraint comes from the Poisson bracket $\left\{\phi^{cd}, \tilde{\alpha}^{aK}_I \tilde{\alpha}^b_{KJ} F^{IJ}_{ab}\right\}$.

(5.63)[65]. The point is that solving the constraint $\chi^{ab}$ is most easily done by gauge fixing the timelike internal vector $n^I$ however in order to ensure this fixing is preserved one must ensure that $n^I G_{IJ} \approx 0$, it is this requirement that reduces the internal gauge symmetry from the Lorentz group to $SO(3)$.

The result is that after fixing $n_I$ there will be 9 remaining degrees of freedom in the momentum $\tilde{\alpha}^a_{IJ}$, in other words it is completely determined by knowledge of the triad. Secondly one finds that the Lorentz connection $A_a^{IJ}$ can be split into a component dependent upon the triad (the part that induces the spacetime Levi Civita connection) and ultimately an additional field $K_a^I$, which satisfies $K_a^I n_I = 0$. Hence we may use now use the coordinates $\left(\tilde{E}^a_I, K_a^I\right)$ for our phase space[66] . The fact that both the triad and $K_a^I$ annihilate the gauge fixed internal vector means that we can replace the Lorentz internal index $I$ with an index $i = 1, 2, 3$ representing an $SO(3)$ index.

So we have a phase space with coordinates $\left(\tilde{E}^a_i, K_a^i\right)$ of dimension $18 \times \infty^3$ with symplectic structure

$$\left\{\tilde{E}^a_i(\sigma), K_b^j(\sigma')\right\} = \delta^a_b \delta^j_i \delta^3(\sigma, \sigma') \tag{5.66}$$

now subject to seven first class constraints, which gives us 2 physical degrees of freedom per spatial point. The first class constraints (5.61) and (5.62) and the remaining freedom (after gauge fixing $n^I$) in (5.63), expressed in these new coordinates, are

$$\frac{1}{\sqrt{q}}\left(\tilde{E}^b_i \tilde{E}^a_j - \tilde{E}^a_i \tilde{E}^b_j\right) K_a^i K_b^j - \sqrt{q}\, {}^3R \approx 0 \tag{5.67}$$

$$D_{[b}\left(K^i_{a]} \tilde{E}^a_i\right) \approx 0 \tag{5.68}$$

$$G'_i \equiv \epsilon_{ijk} K_a^i \tilde{E}^a_i \approx 0 \tag{5.69}$$

where $D_b$ is now the unique Lorentz connection, which induces the Levi Civita connection on the spatial manifold $\Sigma$, and ${}^3R$ is the spatial Ricci scalar derived from this connection. The first two constraints are the Hamiltonian and diffeomorphism constraints of metric gravity in the new coordinates and the final constraint is an $SO(3)$ Gauss constraint. Again the interpretation of $G'_i$ as an $SO(3)$ generator can be confirmed by computing the Poisson algebra of these smeared constraints using the fundamental Poisson bracket (5.66).

This final system is precisely the Hamiltonian system of ADM gravity in tetrad formalism discussed earlier and so we have seen that the Hamiltonian analysis of the Palatini action ultimately just recovers the ADM phase space. We also note that i.) the constraints above have lost their polynomial form present in (5.61), (5.62) and (5.63) and ii.) after solving the second class constraints the interpretation of gravity as a theory of connections has been lost.

## 5.3 Ashtekar self-dual Action

In this section we describe the self dual complex formalism due to Ashtekar, [51] and also earlier work of Sen. This action finally realised the aim of simplifying the constraints of GR, whilst also maintaining their first class character. We shall discuss this action and the Hamiltonian theory obtained from its Legendre transform. This theory is a complex theory of GR and so at the end one has to impose reality conditions and it is these conditions that to date have proven intractable

[65]The partial gauge fixing is convenient but not compulsory see [58] for a solution to the second class constraints without breaking the Lorentz symmetry.

[66]We observe that the triad variable is now a configuration coordinate rather than a momentum one, as was the case before the second class constraints were solved. According to [22] this switch is required because the triad must remain invertible and we presume that this is not guaranteed were it to remain a momentum variable.

in the quantum theory. It is important to note that a complex form of the Palatini action does not itself offer any advantages one proceeds through the canonical analysis as for the real case and after solving the second class constraints recovers a complex version of ADM tetrad gravity, [56].

As we are dealing with a complex version of GR we shall generalise the tetrad to a map, [55] between the complexified Minkowski space and the complexified tangent bundle $\mathbb{C}TM$, i.e.

$$e : M \times \mathbb{C}^4 \to \mathbb{C}TM \tag{5.70}$$

where $\mathbb{C}TM$ is the vector bundle whose "internal space" at each point $p \in M$ consists of complex combinations of tangent vectors and $\mathbb{C}^4$ is the complexified Minkowski spacetime but where the Minkowski metric is kept real. This means that the tetrad components will be complex in this formalism and hence so will the spacetime metric.

The crucial difference in the Ashtekar formalism is the use of a self dual Lorentz connection as the configuration variable of the theory rather than the Lorentz connection used in the Palatini action. In order to introduce these variables we first define the the dual operator on the internal Lorentz indices, this is an internal version of the Hodge dual operator which maps $p$ forms to $n-p$ forms given a metric. In particular because we deal with anti-symmetric Lorentz connections in Minkowski space one can define an internal dual from the space of Lorentz connections to itself, in the same way that the Hodge dual maps 2 forms to 2 forms in four dimensions. The internal dual map $\star$ on two index objects is defined by

$$(\star A)^{IJ}_\mu = \frac{1}{2}\epsilon^{IJ}{}_{KL}A^{KL}_\mu \tag{5.71}$$

where the Minkowski metric is used to raise the internal indices and $\epsilon_{IJKL}$ is the totally anti-symmetric symbol.

On Lorentzian manifolds the square of this dual operator is given by $\star \cdot \star = -1$, since

$$\begin{aligned}
(\star \cdot \star A)^{IJ}_\mu &= \frac{1}{2}\epsilon^{IJ}{}_{KL}(\star A)^{KL}_\mu \\
&= \frac{1}{4}\epsilon^{IJ}{}_{KL}\epsilon^{KL}{}_{MN}A^{MN}_\mu \\
&= \frac{1}{4}\epsilon^{IJKL}\epsilon_{KLMN}A^{MN}_\mu \\
&= \frac{1}{4}\left(-4\delta^{[I}_M\delta^{J]}_N\right)A^{MN}_\mu \\
&= -A^{IJ}_\mu
\end{aligned} \tag{5.72}$$

where we have used the $\epsilon - \delta$ identity see appendix in [58] the $-$ sign ultimately follows from the fact that $\eta$ is a Lorentzian metric.

For this reason we are forced to use complex connections as our self dual variables and we say a Lorentz connection is self dual if and only if

$$(\star A)^{IJ}_\mu = iA^{IJ}_\mu \tag{5.73}$$

and anti-self dual if and only if

$$(\star A)^{IJ}_\mu = -iA^{IJ}_\mu\,. \tag{5.74}$$

This then allows us to decompose any connection into its self and anti-self dual components as

$$A^{IJ}_\mu = {}^+A^{IJ}_\mu +^- A^{IJ}_\mu \tag{5.75}$$

where

$$
{}^{+}A_{\mu}^{IJ} := \frac{1}{2}\left(A_{\mu}^{IJ} - i(\star A)_{\mu}^{IJ}\right) \tag{5.76}
$$

and

$$
{}^{-}A_{\mu}^{IJ} := \frac{1}{2}\left(A_{\mu}^{IJ} + i(\star A)_{\mu}^{IJ}\right) \; . \tag{5.77}
$$

It follows that ${}^{+}A_{\mu}^{IJ}$ and ${}^{-}A_{\mu}^{IJ}$ are self and anti-self dual respectively, since immediately one has

$$
\begin{aligned}
\star\left({}^{+}A_{\mu}^{IJ}\right) &= \frac{1}{2}\left(\star A_{\mu}^{IJ} - \star i(\star A)_{\mu}^{IJ}\right) \\
&= \frac{1}{2}\left(\star A_{\mu}^{IJ} + i A_{\mu}^{IJ}\right) \\
&= i^{+}A_{\mu}^{IJ}
\end{aligned} \tag{5.78}
$$

and similarly for ${}^{-}A_{\mu}^{IJ}$.

The following are important results that we shall use in the analysis of the self dual action, [58]

$$
\begin{aligned}
{}^{+}A^{IJ}\,{}^{-}B_{IJ} &= 0 && (5.79)\\
\Rightarrow {}^{+}A^{IJ}\,B_{IJ} &= {}^{+}A^{IJ}\,{}^{+}B_{IJ} && (5.80)\\
[A,B]^{IJ} &= \left[{}^{+}A,{}^{+}B\right]^{IJ} + \left[{}^{-}A,{}^{-}B\right]^{IJ} && (5.81)
\end{aligned}
$$

where $A$ and $B$ are arbitrary Lorentz connections, which may be proven by using the definitions above and $\epsilon - \delta$ identities. The former is an orthogonality relation between self dual and anti-self dual connections and the latter together with the fact that the Lie bracket of self dual connections is self dual (and similarly for the anti-self dual case) implies that the complexified Lorentz algebra, within which our complex connections take values, decomposes into a direct sum of self dual and anti-self dual Lie sub-algebras [55], [57] and [62] i.e.

$$
SO(1,3)_{\mathbb{C}} \simeq^{+} SO(1,3)_{\mathbb{C}} \oplus^{-} SO(1,3)_{\mathbb{C}} \; . \tag{5.82}
$$

In fact one can show, [55], that the self dual and anti-self dual sub-algebra are isomorphic to the Lie algebras $SL(2,\mathbb{C})$ which as we recall has 3 complex dimensions.

We can now state the self dual action it is given by

$$
S_{SD}[e,{}^{+}A] = \frac{1}{16\pi G}\int_{M} d^4X\,|e|\,e_I^{\mu}e_J^{\nu}F_{\mu\nu}^{IJ}({}^{+}A) \tag{5.83}
$$

which is the complex Palatini action but with the Lorentz connection replaced by the self dual connection ${}^{+}A$. One can show that the curvature of the self dual connection is self dual and similarly for the anti-self dual case and hence one can write the complex Palatini action as the sum of its self dual and anti-self dual components

$$
S_{Pal}[e,A] = S_{SD}[e,{}^{+}A] + S_{ASD}[e,{}^{-}A] \tag{5.84}
$$

the remarkable result is that one need only consider the self dual (or equivalently anti-self dual) action separately in order to recover the equations of motion of complex GR.

The equations of motion of (5.83) can be computed as follows, first variation with respect to the self dual connection reproduces the analogue of the Palatini variation in (5.33)

$$
\delta S = \frac{1}{16\pi G}\int_{M} d^4X\,2|e|\,e_I^{[\mu}e_J^{\nu]}\mathcal{D}_{\mu}\delta\,{}^{+}A_{\nu}^{IJ} \tag{5.85}
$$

where $\mathcal{D}_\mu$ is the covariant derivative on internal indices defined with respect to the self dual connection. Now because of the orthogonality relation above, the variation $\delta\, {}^+A_\nu^{IJ}$ requires us to keep only the self dual part of $e_I^{[\mu}e_J^{\nu]}$ and so after an integration by parts one has

$$\mathcal{D}_\mu\left(|e|\,{}^+\left(e_I^{[\mu}e_J^{\nu]}\right)\right) = 0\,. \tag{5.86}$$

It is shown in [22] that this equation implies ${}^+A_\nu^{IJ}$ is the self dual part of the unique Lorentz connection that induces the Levi Civita connection on spacetime, c.f. (5.11).

The variation with respect to the tetrad yields, see (5.28)

$$\begin{aligned}\delta S_{SD} &= \frac{1}{8\pi G}\int_M d^4X\,|e|\delta e_I^\mu\left(\delta_K^I e^{K\sigma}\tilde{R}_{\mu\sigma}({}^+A) - \frac{1}{2}e_\mu^I e_K^\sigma e_J^\nu F_{\sigma\nu}^{KJ}({}^+A)\right)\\ &= \frac{1}{8\pi G}\int_M d^4X\,|e|\delta e_I^\mu\left(\delta_K^I\delta_M^K e_N^\rho F_{\mu\rho}^{MN} - \frac{1}{2}e_\mu^I e_K^\sigma e_J^\nu F_{\sigma\nu}^{KJ}\right)\\ &= \frac{1}{8\pi G}\int_M d^4X\,|e|\delta e_I^\mu\left(e_J^\rho F_{\mu\rho}^{IJ} - \frac{1}{2}e_\mu^I e_K^\sigma e_J^\nu F_{\sigma\nu}^{KJ}\right)\end{aligned} \tag{5.87}$$

where curvatures are computed with respect to the self dual connection. Hence the second equation of motion is

$$e_J^\nu F_{\mu\nu}^{IJ} - \frac{1}{2}e_\mu^I e_K^\sigma e_J^\nu F_{\sigma\nu}^{KJ} = 0 \tag{5.88}$$

but now we may use the fact, determined from the first equation of motion, that $F_{\sigma\nu}^{KJ}$ is the self dual component of the internal curvature that induces the Riemann tensor on spacetime, we denote this latter curvature $R_{\mu\nu}^{IJ}$ , which, by (5.22) satisfies

$$R_{\mu\nu\sigma}^\rho = R_{\mu\nu I}{}^J e_\sigma^I e_J^\rho \tag{5.89}$$

and hence one has

$$\begin{aligned}F_{\mu\nu}^{IJ} &= \frac{1}{2}\left(R_{\mu\nu}^{IJ} - i(\star R)_{\mu\nu}^{IJ}\right)\\ &= \frac{1}{2}\left(R_{\mu\nu}^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{KL}R_{\mu\nu}^{KL}\right)\,.\end{aligned} \tag{5.90}$$

We can now substitute this expression for $F_{\mu\nu}^{IJ}$ into (5.88) to obtain

$$\begin{aligned}e_J^\nu\left(R_{\mu\nu}^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{KL}R_{\mu\nu}^{KL}\right) &= \frac{1}{2}e_\mu^I e_K^\sigma e_J^\nu\left(R_{\sigma\nu}^{KJ} - \frac{i}{2}\epsilon^{KJ}{}_{MN}R_{\sigma\nu}^{MN}\right)\\ \Rightarrow e_{\gamma I}e_J^\nu\left(R_{\mu\nu}^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{KL}R_{\mu\nu}^{KL}\right) &= \frac{1}{2}e_{\gamma I}e_\mu^I e_K^\sigma e_J^\nu\left(R_{\sigma\nu}^{KJ} - \frac{i}{2}\epsilon^{KJ}{}_{MN}R_{\sigma\nu}^{MN}\right)\\ \Rightarrow e_{\gamma I}e_J^\nu R_{\mu\nu}^{IJ} - \frac{1}{2}g_{\gamma\mu}e_K^\sigma e_J^\nu R_{\sigma\nu}^{KJ} &= \frac{i}{2}e_{\gamma I}e_J^\nu\epsilon^{IJ}{}_{KL}R_{\mu\nu}^{KL} - \frac{i}{4}g_{\gamma\mu}e_K^\sigma e_J^\nu\epsilon^{KJ}{}_{MN}R_{\sigma\nu}^{MN}\end{aligned} \tag{5.91}$$

and we claim i.) the left hand side of (5.91) is equal to the Einstein tensor $G_{\gamma\mu}$ and ii.) each term on the right hand side is identically zero and hence we have recovered Einstein's equations in the self dual formalism. We now prove these claims, firstly one has

$$\begin{aligned}e_{\gamma I}e_J^\nu R_{\mu\nu}^{IJ} - \frac{1}{2}g_{\gamma\mu}e_K^\sigma e_J^\nu R_{\sigma\nu}^{KJ} &= e_\gamma^I e_J^\nu R_{\mu\nu\alpha}^\beta e_I^\alpha e_\beta^J - \frac{1}{2}g_{\gamma\mu}e^{\sigma K}e_J^\nu R_{\sigma\nu\alpha}^\beta e_K^\alpha e_\beta^J\\ &= g_\gamma^\alpha g_\beta^\nu R_{\mu\nu\alpha}^\beta - \frac{1}{2}g_{\gamma\mu}g^{\sigma\alpha}g_\beta^\nu R_{\sigma\nu\alpha}^\beta\\ &= R_{\gamma\mu} - \frac{1}{2}g_{\gamma\mu}R\end{aligned} \tag{5.92}$$

and secondly for the first term on the right hand side of (5.91) one has

$$
\begin{aligned}
\frac{i}{2} e_{\gamma I} e^\nu_J \epsilon^{IJ}{}_{KL} R^{KL}_{\mu\nu} &= \frac{i}{2} g_{\gamma\lambda} e^\lambda_I e^\nu_J e^\alpha_K e^\beta_L \epsilon^{IJKL} R_{\beta\mu\nu\alpha} \\
&= -\frac{i}{2} g_{\gamma\lambda} \epsilon^{\lambda\beta\alpha\nu} R_{\mu[\beta\alpha\nu]} \\
&\equiv 0
\end{aligned} \tag{5.93}
$$

where in the second line we have used the relation between the (inverse) volume elements of $\eta_{IJ}$ and $g_{\mu\nu}$ and in the final line the algebraic Bianchi identity of the Riemann tensor. The second term on the right hand side of (5.91) is zero for the same reason and hence we have confirmed that the self dual action leads to Einstein's equations.

We now perform the Legendre transform to determine the Hamiltonian and the constraints of the self dual action. As with the Palatini action one must first project the tetrad onto the spatial surface $\Sigma_t$ to define $E^\mu_I$ and then, following similar manipulations to those used for the Palatini action, one has, [56]

$$
\begin{aligned}
S_{SD}[e, {}^+A] &= \frac{1}{16\pi G} \int_M d^4X \, NE \, (E^\mu_I - n^\mu n_I)(E^\nu_J - n^\nu n_J) F^{IJ}_{\mu\nu}({}^+A) \\
&= \frac{1}{16\pi G} \int_M d^4X \left( \underset{\sim}{N} \tilde{E}^\mu_I \tilde{E}^\nu_J F^{IJ}_{\mu\nu} - 2Nn^\mu n_I \tilde{E}^\nu_J F^{IJ}_{\mu\nu} \right) \\
&= \frac{1}{16\pi G} \int_M d^4X \left( \underset{\sim}{N} \tilde{E}^\mu_I \tilde{E}^\nu_J F^{IJ}_{\mu\nu} + iNn^\mu \tilde{E}^\nu_J n_I \epsilon^{IJ}{}_{KL} F^{KL}_{\mu\nu} \right) \\
&= \frac{1}{16\pi G} \int_M d^4X \left( \underset{\sim}{N} \tilde{E}^\mu_I \tilde{E}^\nu_J F^{IJ}_{\mu\nu} - iT^\mu \tilde{E}^\nu_J \epsilon^J{}_{KL} F^{KL}_{\mu\nu} + iN^\mu \tilde{E}^\nu_J \epsilon^J{}_{KL} F^{KL}_{\mu\nu} \right) \\
&= \frac{1}{16\pi G} \int_M d^4X \left( -i\tilde{E}^\nu_J \epsilon^J{}_{KL} \left( \mathcal{L}_{\vec{T}} \left({}^+A^{KL}_\nu\right) - \mathcal{D}_\nu ({}^+A_\mu T^\mu)^{KL} \right) + \right. \\
&\quad \left. + iN^\mu \tilde{E}^\nu_J \epsilon^J{}_{KL} F^{KL}_{\mu\nu} + \underset{\sim}{N} \tilde{E}^\mu_I \tilde{E}^\nu_J F^{IJ}_{\mu\nu} \right)
\end{aligned} \tag{5.94}
$$

where in the second line we dropped the term quadratic in $n_I n_J$ as it is identically zero due to symmetry and anti-symmetry in $I$, $J$, in the third line we used the fact that $F^{IJ}_{\mu\nu}$ is self dual, in the fourth line we used the decomposition of the unit normal $n^\mu$ into lapse and shift, and the definition $\epsilon^{JKL} = n_I \epsilon^{IJKL}$ and in the final line we used the identity $T^\mu F^{KL}_{\mu\nu} = \mathcal{L}_{\vec{T}} A^{KL}_\nu - \mathcal{D}_\nu (A_\mu T^\mu)^{KL}$, which was derived in the context of the Palatini Legendre transform in (5.49). As argued below (5.49) we can see again that all spacetime indices in (5.94) are contracted with spatial triads or the shift vector and hence may be interpreted as spatial tensors.

Hence one can express the self dual action in $3+1$ form as

$$
\begin{aligned}
S_{SD}[E, {}^+A_a, {}^4A_T, N^a, N] \;=\; & \frac{1}{16\pi G} \int dt d^3\sigma \left( -i\tilde{E}^a_J \epsilon^J{}_{KL} {}^+\dot{A}^{KL}_a - i\mathcal{D}_a \left( \tilde{E}^a_J \epsilon^J{}_{KL} \right) ({}^+A_\mu T^\mu)^{KL} \right. \\
& \left. + iN^a \tilde{E}^b_J \epsilon^J{}_{KL} F^{KL}_{ab} + \underset{\sim}{N} \tilde{E}^a_I \tilde{E}^b_J F^{IJ}_{ab} \right)
\end{aligned} \tag{5.95}
$$

where we have done an integration by parts and dropped the boundary term. Immediately we can compute the conjugate momenta to the configuration variables ${}^+A^{IJ}_a$, ${}^4A_T$, $N^a$ and $N$ as

$$
16\pi G P^a_{IJ} = {}^+\left( -i\tilde{E}^a_K \epsilon^K{}_{IJ} \right) \tag{5.96}
$$

$$
P_{IJ} = 0 \tag{5.97}
$$

$$
P_a = 0 \tag{5.98}
$$

$$
P_N = 0 \tag{5.99}
$$

and we observe that, as for the Palatini action, there are primary constraints associated with the momenta conjugate to ${}^4A_T$, $N^a$ and $N$. The fact that $P^a_{IJ}$ is the self dual part of $-i\tilde{E}^a_K\epsilon^K{}_{IJ}$ follows because its coefficient $\dot{A}^{KL}_a$ is self dual and so it projects onto the self dual space of whatever it multiplies. We can explicitly compute the form of the momentum as follows

$$\begin{aligned}
16\pi G P^a_{IJ} &= {}^+\left(-i\tilde{E}^a_K\epsilon^K{}_{IJ}\right) \\
&= -\frac{i}{2}\tilde{E}^a_K\epsilon^K{}_{IJ} - \frac{1}{4}\tilde{E}^a_L\epsilon^{LMN}\epsilon_{IJMN} \\
&= -\frac{1}{4}\tilde{E}^a_L\epsilon^{PLMN}\epsilon_{MNIJ}n_P - \frac{i}{2}\tilde{E}^a_K\epsilon^K{}_{IJ} \\
&= -\frac{1}{4}\tilde{E}^a_L\left(-4\delta^{[P}_I\delta^{L]}_J\right)n_P - \frac{i}{2}\tilde{E}^a_K\epsilon^K{}_{IJ} \\
&= \tilde{E}^a_I n_J - \frac{i}{2}\tilde{E}^a_K\epsilon^K{}_{IJ}\,.
\end{aligned} \tag{5.100}$$

Since the self dual algebra has three complex dimensions $P^a_{IJ}$ has $3\times 3 = 9$ complex independent components but recall that $\tilde{E}^a_K$ has 9 independent complex components also, due to the condition $\tilde{E}^a_K n^K = 0$, and hence there are no constraints in (5.96) both sides have the same number of degrees of freedom. This is an important difference to the Palatini action where we uncovered a further six primary constraints in (5.53).

One can now perform the Legendre transform, we express (5.95) in terms of the momentum variable to get, [56]

$$S_{SD} = \int dtd^3\sigma\,\left(P^a_{KL}{}^+\dot{A}^{KL}_a + \mathcal{D}_a\left(P^a_{KL}\right)\left({}^+A_\mu T^\mu\right)^{KL} - N^a P^b_{KL}F^{KL}_{ab} + \underset{\sim}{N} P^a_{IK}P^{bK}{}_J F^{IJ}_{ab}\right) \tag{5.101}$$

this action is now in canonical form and we can read off the canonical Hamiltonian $H_c$ as

$$H_c = \int d^3\sigma\left(N^a P^b_{KL}F^{KL}_{ab} - \mathcal{D}_a\left(P^a_{KL}\right)\left(A_\mu T^\mu\right)^{KL} - \underset{\sim}{N} P^a_{IK}P^{bK}{}_J F^{IJ}_{ab}\right) \tag{5.102}$$

and the primary Hamiltonian for our system will be given by

$$H_p = H_c + \int d^3\sigma\left(\lambda^{IJ}P_{IJ} + \lambda^a P_a + \lambda P_N\right)\,. \tag{5.103}$$

We shall have secondary constraints, given by the coefficients of the non-dynamical configuration variables, required by the preservation of the primary constraints under evolution generated by $H_p$, i.e.

$$\mathcal{D}_a\left(P^a_{KL}\right) \approx 0 \tag{5.104}$$
$$P^b_{KL}F^{KL}_{ab} \approx 0 \tag{5.105}$$
$$P^a_{IK}P^{bK}{}_J F^{IJ}_{ab} \approx 0 \tag{5.106}$$

we observe these constraints are polynomial in the self dual connection and its conjugate momentum. Fortunately, the above primary and secondary constraints do not require tertiary constraints and furthermore they form a first class system, [56].

At this point we can reduce the phase space to only the dynamical variables ${}^+A^{IJ}_a$ and its momentum $P^a_{KL}$ subject to only the constraints (5.104), (5.105) and (5.106). We can count the degrees of freedom of this system to ensure it is consistent with complex GR, the phase space is $2\times 3\times 3 = 18$ complex dimensions per spatial point, and subject to $3+3+1 = 7$ complex first class constraints per spatial point. Hence the number of physical degrees of freedom of this theory are $\frac{1}{2}(18 - 2 \times 7) = 2$ complex dimensions per spatial point, just as we would expect for complex GR.

The final point we wish to make is that there exists an isomorphism between the self dual Lie algebra and the complexified $SO(3)_{\mathbb{C}}$ algebra and that this fact may be used to express all of our dynamical variables, and constraints in terms of $SO(3)_{\mathbb{C}}$ valued objects. The explicit isomorphism depends upon a choice of unit timelike internal vector $n^I$ see [22]. The final picture is that the phase space coordinates will be configuration variable $A^i_a$, a $SO(3)_{\mathbb{C}}$ valued connection, and momentum variable $\tilde{E}^a_i$, a complex densitised triad, with non-trivial Poisson bracket

$$\left\{A^i_a(\sigma), \tilde{E}^b_j(\sigma')\right\} = \delta^b_a \delta^i_j \delta^3(\sigma, \sigma') \,. \tag{5.107}$$

The above first class constraints can be equivalently expressed using these new coordinates as, [22]

$$D_a \tilde{E}^a_i \approx 0 \tag{5.108}$$
$$\tilde{E}^b_i F^i_{ab} \approx 0 \tag{5.109}$$
$$\epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{abk} \approx 0 \tag{5.110}$$

where (5.108) is to be interpreted as an $SO(3)$ Gauss constraint, (5.105) as the spatial diffeomorphism constraint and (5.110) the Hamiltonian constraint. We observe that the Hamiltonian constraint has naturally come out as a scalar of density weight two. These constraints are polynomial in the new variables and unlike the Palatini action there were no second class constraints found in the analysis. This simiplification was the original motivation of the Ashtekar variables, but unfortunately the reality conditions, which we still have to impose in order to regain real GR, have proven intractable to date in the quantum theory, [5]. An additional / related problem is that the gauge group for the Ashtekar formalism is non-compact ($SO(3)_{\mathbb{C}}$ and $SL(2, \mathbb{C})$) are non-compact and this has caused difficulties in the quantum theory because most mathematical tools available from Yang Mills theory apply only to compact gauge groups, e.g. the Haar measure is only available for compact gauge groups.

## 5.4 Epilogue

In this final section we should like to make some brief remarks concerning the modern formulation of connection gravity. In particular, [53] introduced a real $SO(3)$ connection (now known as the Ashtekar Barbero connection), as a configuration variable for GR that therefore avoided the problem of reality conditions present in the self dual formalism and see also [64]. The key issue is that the Hamiltonian constraint is now a more complicated expression of the canonical variables.

The real variables were obtained by a canonical transformation of the original ADM triad formulation of gravity, which we recall has canonical variables $\left(\tilde{E}^a_i, K^i_a\right)$ where $\tilde{E}^a_i$ is a densitised triad and $K^i_a$ can be related to the extrinsic curvature $K_{ab}$, which satisfy canonical Poisson brackets. One can define a canonical transformation on these variables given by

$$\tilde{E}^a_i = \tilde{E}^a_i \tag{5.111}$$
$$A^i_a = \Gamma^i_a + \beta K^i_a \tag{5.112}$$

where $\Gamma^i_a$ is the $SO(3)$ connection that induces the Levi Civita connection on the spatial manifold $\Sigma$ and $\beta$ is an arbitrary complex parameter, now known as the Immirzi parameter. Since $\Gamma^i_a = \Gamma^i_a(E)$ the only non-trivial Poisson bracket to confirm the pair $A^i_a, \tilde{E}^a_i$ are canonical is to check that $\left\{A^i_a(\sigma), A^j_b(\sigma')\right\} = 0$, and one finds the two non-zero brackets $\left\{\Gamma^i_a(\sigma), \beta K^j_b(\sigma')\right\}$ and $\left\{\beta K^i_a(\sigma), \Gamma^j_b(\sigma'),\right\}$ cancel out. In terms of these new variables the constraints of the ADM triad theory may be expressed as

$$
\begin{aligned}
G_i &\equiv D_a \tilde{E}^a_i \approx 0 && (5.113)\\
H_a &\equiv F^i_{ab}\tilde{E}^b_i \approx 0 && (5.114)\\
\sqrt{q}H &\equiv \epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F_{abk} - \frac{2}{\beta^2}\left(1+\beta^2\right)\tilde{E}^a_{[i}\tilde{E}^b_{j]}\left(A^i_a - \Gamma^i_a\right)\left(A^j_b - \Gamma^j_b\right) \approx 0 && (5.115)
\end{aligned}
$$

where $F^i_{ab}$ is the curvature of the connection $A^i_a$ and $q{=}\left|det(\tilde{E}^a_i)\right|$. These constraints are almost identical to the final Ashtekar constraints (5.108), (5.109) and (5.110) except for the final Hamiltonian constraint, which is considerably more complicated and which is a density weight one scalar. One sees immediately that the choice $\beta = \pm i$ simplifies the Hamiltonian constraint and recovers the Ashtekar phase space variables discussed above, (in the $SO(3)$ rather than self dual version and up to a re-scaling of the Hamiltonian constraint by $\sqrt{q}$.

So from this viewpoint we have a 1 parameter family of canonical transformations of the ADM triad variables and for which the choice $\beta = \pm i$ recovers the complex Ashtekar formalism. However, we could choose real $\beta$ and be left with a phase space description of gravity, which has the advantage of i.) being real and compact (avoiding the reality conditions in the quantum theory), ii.) maintaining a connection formulation of GR which has enabled mathematical methods developed for Yang Mills theories to be imported into the quantization of gravity, e.g. the use of loop variables but at the price of complicating the Hamiltonian constraint. This choice of real $\beta$ is the current preferred option for the canonical quantization of gravity. However, it is not without several issues, which we shall discuss shortly.

The real formulation of GR just discussed has been obtained through a canonical transformation of the triad ADM phase space. It would interesting to know whether this phase space may be obtained directly from an action principle and the answer to this question is yes. In [54] it was shown by Holst that the phase space variables $A^i_a$, $\tilde{E}^a_i$ may be derived from the following action

$$
\begin{aligned}
S_{Holst}[e,A] &= \frac{1}{2}\int ee^\mu_I e^\nu_J \left(F^{IJ}_{\mu\nu}(A) - \frac{1}{\beta}\left(\star F\right)^{IJ}_{\mu\nu}(A)\right) && (5.116)\\
&= \frac{1}{2}\int ee^\mu_I e^\nu_J \left(F^{IJ}_{\mu\nu}(A) - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL}F^{KL}_{\mu\nu}\right) && (5.117)
\end{aligned}
$$

where the action is a functional of the tetrad $e^\mu_I$ and the real Lorentz connection $A^{IJ}_\mu$, and now we also require the Immirzi parameter $\beta \neq 0$. The dual operator $\star$ is the internal Hodge operator defined in (5.71). First one can see that the first term in this action is just the original Palatini action, which we have already shown, under variation in the connection and tetrad, leads to Einstein's equations. The second term does not provide any further dynamical information because it leads to a term that vanishes by a symmetry of the Riemann tensor. We can see this by first considering a variation of (5.116) with respect to the connection we have

$$\begin{aligned}
\delta S &= \int e e^\mu_I e^\nu_J \left( D_{[\mu}\delta A^{IJ}_{\nu]} - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL} D_{[\mu}\delta A^{KL}_{\nu]} \right) \\
&= \int e e^\mu_I e^\nu_J \left( \delta^I_{[K}\delta^J_{L]} - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL} \right) D_{[\mu}\delta A^{KL}_{\nu]} \\
&= \int e e^{[\mu}_I e^{\nu]}_J \left( \delta^I_{[K}\delta^J_{L]} - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL} \right) D_\mu \delta A^{KL}_\nu \\
&= -\int D_\mu \left( e e^{[\mu}_I e^{\nu]}_J \right) \left( \delta^I_{[K}\delta^J_{L]} - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL} \right) \delta A^{KL}_\nu \qquad (5.118)
\end{aligned}$$

after an integration by parts and where we have used the covariant constancy of the $\epsilon_{IJKL}$ with respect to the Lorentz connection. This implies the equation of motion

$$D_\mu \left( e e^{[\mu}_I e^{\nu]}_J \right) = 0 \qquad (5.119)$$

which is equivalent to the vanishing of the torsion of the induced connection from $A^{IJ}_\mu$, see (5.33) and the discussion below. Hence we can deduce that the curvature

$$F^{IJ}_{\mu\nu} = R^{IJ}_{\mu\nu} \qquad (5.120)$$

but this implies that the second term in the Holst action is identically zero since

$$\begin{aligned}
-\frac{1}{2\beta} e e^\mu_I e^\nu_J \epsilon^{IJ}{}_{KL} F^{KL}_{\mu\nu} &= -\frac{1}{2\beta} e e^\mu_I e^\nu_J \epsilon^{IJ}{}_{KL} R^{KL}_{\mu\nu} \\
&= -\frac{1}{2\beta} e e^\mu_I e^\nu_J e^\alpha_K e^\beta_L \epsilon^{IJKL} R_{\beta\mu\nu\alpha} \\
&= -\frac{1}{2\beta} e \epsilon^{\beta\mu\nu\alpha} R_{\beta\mu\nu\alpha} \\
&\equiv 0 \qquad (5.121)
\end{aligned}$$

by the algebraic Bianchi identity for the Riemann tensor. The remaining non-zero term is just the Palatini action and hence we recover Einstein's equations of motion from the Holst action.

One can then perform a canonical analysis on this action and, as for the Palatini action, it is complicated by the presence of second class constraints. In [54] a partial gauge fixing is chosen prior to the computation of the constraints, and this has the effect of removing the second class constraints. The gauge fixing involves setting the tetrad component $e^\mu_0 = n^\mu$, and physically it corresponds to performing local Lorentz transformations so that every time $t$, the zeroth component of the local frame is orthogonal to the spacelike hypersurface $\Sigma_t$, sometimes this is called a temporal gauge fixing or time gauge. It has the effect of reducing the local symmetry from the Lorentz group to $SO(3)$. By choosing this gauge it is then possible to show that the phase space deduced from the Holst action is coordinatised by the Ashtekar Barbero connection and the densitised triad with the above constraints.

There are several issues however with the formalism as explained. First the introduction of the Immirzi parameter, which is used to describe a 1 parameter family of canonical transformations at the classical level appears to lead to inequivalent quantum theories. More precisely one can show in the quantum theory, LQG, that the spectra of geometrical operators for area and volume depend upon the Immirzi parameter, and also a computation for black hole entropy using LQG requires a particular value of the Immirzi parameter to agree with the semi-classical arguments of

Hawking. As argued at the end of section 4 these quantum operators are not Dirac observables, i.e. not gauge invariant, and so one must be careful to conclude that inequivalent spectra at the kinematical level are necessarily physically inequivalent. But it does appear to provide some evidence that arbitrary canonical transformations can lead to inequivalent quantum theories.

A second issue raised in [65] is that the Ashtekar Barbero connection is not a spacetime gauge theory of gravity, by this we mean the Ashtekar Barbero connection is not the pullback of a spacetime connection to a spatial slice as *is* the case for the complex Ashtekar connection. This can be proved by showing that the holonomy of the Ashtekar Barbero connection on a spatial loop is not independent of the embedding of the spatial surface in the spacetime except for $\beta = \pm i$ and this implies it cannot be the pullback of a spacetime connection. Recall also that we do not reduce the symmetry group in the complex Ashtekar formalism the spatial connection $A^i_a$ still takes values in $SO(3)_{\mathbb{C}}$, or equivalently $SL(2,\mathbb{C})$, by contrast the real connection is not a Lorentz connection it is an $SU(2)$ or $SO(3)$ connection. The concern expressed in [65] is how could one arrive at a gauge theory of gravity with compact gauge group, when the spacetime view is that the group appears to be non-compact (e.g. the Lorentz or Poincarè groups). The answer is that the Legendre transform of the Holst action, in particular the partial gauge fixing, breaks the Lorentz symmetry of the theory and we lose the spacetime gauge interpretation. Of course there is a spacetime interpretation of sorts in that one can view the Ashtekar Barbero variables as canonical transformations of the ADM variables, which do have a four dimensional interpretation. However, one has lost a direct geometric interpretation of the real connection that one did have with the original complex Ashtekar connection.

Concerns with regard to some of these issues: the lack of a spacetime interpretation of the connection, the Immirzi parameter ambiguity, and the fact that the compact gauge group is the reason for the discrete geometric operator spectra found in LQG have led to approaches which avoid using the time gauge. One such approach keeps the second class constraints derived from the Holst action, and uses the Dirac bracket and therefore no loss of Lorentz symmetry is encountered, and one then tries to perform a Dirac quantisation. The main results of the analysis, [66] are that the canonical variables are a Lorentz connection and conjugate triad, and that the Immirzi parameter disappears because the area spectra then become continuous at the kinematical level. The difficulties are a non-compact gauge group[67], and a non-commutative connection (with respect to the Dirac bracket) which have to be dealt with in the quantum theory. Another approach involves solving the second class constraints, without the time gauge, and this analysis is completed in [67]. One can then show that the resulting phase space has two pairs of canonical variables $(A, E)$ the Ashtekar-Barbero variables and an additional pair of $\mathbb{R}^3$ valued canonical fields $(\chi, \zeta)$. It is stated in [69] that one can reconstruct a Lorentz connection from the fields $A$, $\chi$ and that this resulting connection is commutative. Introductions to all these approaches, which are often labelled 'covariant LQG' can be found in [66, 68, 69].

[67] We agree that this is a mathematical difficulty but if the gauge group of gravity is non-compact then this is a problem that has to be dealt with if the theory is to be viable.

# 6 Discussion

We have covered quite a broad range of topics and so we wish to conclude with a summary and conclusions, describing the main points we have discovered.

Firstly the mathematical formalism of constrained Hamiltonian system has been discussed, this is the framework required for any canonical analysis of a singular Lagrangian and it is the tool that enables us to understand the Hamiltonian formulation of GR. An important concept is the classification of constraints, into first and second class, which provides the key to understanding gauge theory in canonical form. In particular we have found two possible interpretations of gauge transformations in the Hamiltonian theory. The first, due to Dirac, identifies first class constraints as the generators of gauge transformations that map points in phase space to other points in phase space. The second, due to Bergmann, identifies, a gauge generator $G(t)$, which is a combination of first class constraints, as a map from one phase space trajectory, (or solution to the equations of motion) to another, where at each time $t$ the map from a point in one trajectory to the corresponding point is a gauge transformation in the sense of Dirac.

We then proved a number of properties relating to the geometric picture of the transformations generated by first class constraints. In particular they are 'surface forming', i.e. first class constraints generate surfaces that fill the constraint surface, where each surface is to be identified as a gauge equivalence class or gauge orbit.

The constrained Hamiltonian formalism has been applied to Yang Mills theory (on a Minkowski background) and we proved the following: i.) Yang Mills theory is a first class system, ii.) the Poisson algebra of the constraints is isomorphic to the Lie algebra of the gauge group and iii.) one can compute the physical number of degrees of freedom by subtracting the 'redundancy' present in the constraints.

This analysis has been extended to GR, where we showed that the Einstein Hilbert action can be expressed in $3+1$ form and, after following the Dirac-Bergmann algorithm, found the phase space of GR together with its first class constraints, (the Hamiltonian and diffeomorphism constraints). This phase space consists of a spatial Riemannian metric $q_{ab}$ and its conjugate momentum $P^{ab}$, (related to the extrinsic curvature of the spatial surface embedded in the spacetime).

These constraints are of crucial importance they encode the canonical symmetry of GR within their Poisson algebra (Dirac algebra) and also have to be implemented as constraints in the Dirac quantization. We have stated this algebra and tried to understand it from the point of view of reflecting the four dimensional spacetime diffeomorpism group in the canonical formalism. In fact this conclusion is only possible when the equations of motion hold. In other words it is an important result that the Dirac algebra of GR is not directly related to the diffeomorphism algebra.

This result seemed quite surprising and therefore we have tried to understand it from a couple of different perspectives. The first is that in fact there exists a larger symmetry group of the Einstein Hilbert action (than the passive diffeomorphism group, which is a sub-group) which consists of metric dependent coordinate transformations, this has been called the 'induced diffeomorphism' group. It has been shown that the Dirac algebra is the projectable part of this induced diffeomorphism group, and this projectable component is called the Bergmann-Komar group. So in fact although the diffeomorphism group and the Dirac algebra are not themselves related they are both related to this larger symmetry group. Our second perspective was to show that by extending the phase space of GR, by adding the embedding variables and their conjugate momenta, it is possible to find a representation of the diffeomorphism algebra in canonical gravity.

We also covered asymptotically flat spacetimes where it was found necessary to modify the Hamiltonian and diffeomorphism constraints with additional boundary terms. These terms ensured that the Hamiltonian remained finite and functionally differentiable and enabled us to consider non-trivial gauge transformations at infinity. By doing so and then computing the Poisson algebra of these modified constraints one can find a representation of the Poincare group at spatial infinity. This means that we have the proper notion of an asymptotically flat spacetime because we have recovered its symmetry group from the first class constraints of the theory. We also observed that the ten Poincare charges are Dirac observables.

The next topic considered was gauge invariant observables for GR, in the sense of Dirac, though the analysis also applies to Bergmann observables. The construction of gauge invariant observables is important because in the quantum theory their Poisson algebra will have to be represented in the physical Hilbert space and also from a purely classical view it is interesting to understand what an observable actually is in GR. We have reviewed recent work in this area, which has shown that there exist gauge invariant relational observables for any first class system including GR. These observables require additional non-gravitational 'clock fields' to be used in the following way one computes the value of a particular gauge variant observable $f$ when the clock fields have certain values. It is in this sense these observables are relational. The approximation schemes used to construct these observables result in rather complicated expressions of nested Poisson brackets and therefore methods to reduce the complexity have been considered. Firstly reducing the effective number of constraints and parameters by using partial observables and clock variables, which have ultra-local Poisson brackets with the constraints, secondly taking advantage of the result that one can compute complete observables in stages and finally using deparametrisation to find a physical Hamiltonian from which evolution equations for the relational observables can be computed. The deparametrisation method offers several advantages in that in addition to simplifying the observables it offers a physical Hamiltonian with which evolution can be generated.

Finally, we considered several connection formulations of gravity, these have been initially motivated by a desire to reduce the complexity of the Hamiltonian constraint in the hope that this would make the quantum theory easier to construct. We first showed that the Hilbert Palatini action led to the Einstein equations of motion but that after a complicated Legendre transform, and the solving of second class constraints, one essentially recovers the ADM theory in triad form. We then considered the complex self dual action, which has been the start of the modern formulation of canonical gravity. After performing the Legendre transform we showed that the resulting theory is a first class, connection theory of gravity, which has polynomial constraints in the dynamical variables. Unfortunately, the theory also presents reality conditions and a non-compact gauge group $SL(2,\mathbb{C})$ which have both proven intractable in the quantum theory.

These difficulties motivated the real Ashtekar Barbero (AB) connection, which can be defined through i.) a 1-parameter family of canonical transformations of the ADM triad variables leading to a $SO(3)$ or $SU(2)$ AB connection and densitised triad as canonical variables and ii.) the phase space coordinates derived from the Holst action. This theory has the advantage of involving real variables and a compact gauge group, but where the Hamiltonian constraint is considerably more complicated. However, we found that in the real case we do not get a spacetime gauge theory of gravity because the AB connection is not the pullback of a spacetime Lorentz connection, whereas the self dual connection is. Furthermore we seem to have introduced a 1-parameter family of inequivalent quantum theories (at least at the kinematical level) by performing this canonical transformation. This so called Immirzi parameter ambiguity is an open area of research.

These final issues have motivated further work on covariant loop quantum gravity, where one either tries to work with the Dirac bracket to avoid the time gauge or solves the second class constraints but without gauge fixing.

Unfortunately, due to time constraints we were not able to explore the properties of the covariant LQG approaches, and this together with a discussion of the construction of the kinematical

Hilbert space of LQG and definition of geometric operators, alluded to in the text, would be very interesting topics for future work.

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