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Axiomatic Topological Quantum Field Theory

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Declaration

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.

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Abstract

This dissertation provides an introduction to the ideas employed in topological quantum field theory. We illustrate how the field began by considering knot invariants of three-manifolds and demonstrating the consequences of defining such a theory axiomatically. The ideas of category theory are introduced and we show that what we are actually concerned with are symmetric monoidal functors from the category of topological cobordisms to the category of vector spaces. In two dimensions, these theories are elegantly classified through their equivalence to Frobenius algebras. Finally, we discuss possibly the simplest topological quantum field theory using a finite gauge group to define the fields.

Contents

1 Introduction

Topological Quantum Field Theoy (TQFT) has been a developing field at the interface of physics and mathematics for nearly two decades. The main attraction for mathematicians so far has been to organise previously studied invariants such as the Jones polynomials of knots in 3 dimensions or the Donaldson invariariants of 4-manifolds, but they are also applied in 2 dimensions to Riemann surface theory. For physicists, TQFTs have been a useful guide to understanding quatum theory from a category-theory perspective [3], but are mainly of interest for providing examples of background independent quantum field theories, a desirable trait for a theory of quantum gravity.

We shall provide an axiomatic definition of a TQFT in the next chapter, and remaining in this abstract language, shall provide some direct consequences of the definition. Chapter 3 specialises to the case of categories of bordisms and succinctly derives the fact a TQFT is a particular functor. Chapter 4 specialises further to the case of two dimesions, first deriving the result that classifies TQFTs and then giving an overview of the different classes. The final chapter is an introduction to what is often thought of as the most basic TQFT, giving an accessible springboard for further sophistications, in particular we show which class of TQFT this belongs to.

What remains in this section is a brief history of how the field came to be, illustrating the idea of the field quite generally.

In 1984 the mathematician Vaughan Jones discovered that for a smooth embedding of S^1 into $\mathbb{R}^3$, called a *knot* K, one can find an invariant that assigns to K a Laurent polynomial in the variable $t^{\frac{1}{2}}$ with integer coefficients. This invariant is known as the *Jones Polynomial* of K. Two knots with differing Jones polynomials cannot be smoothly deformed into each other (they are inequivalent up to isotopy of $\mathbb{R}^3$), however there does exist examples of inequivalent knots having equal Jones polynomials. At a symposium in 1988, Michael Atiyah proposed a problem for quatum field theorists: to find an intrinsically 3 dimensional definition of the Jones polynomial [1]. Edward Witten's 1989 paper [18] presented the invariant

$$\int_{\mathscr{A}} \mathcal{D}A \ \mathrm{e}^{iS[A]} W(C) \tag{1.1}$$

as precisely the Jones polynomial of the knot K in S^3 when the action used is that of Chern-Simons theory,

$$S[A] = \frac{k}{4\pi} \int_{S^3} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{1.2}$$

and

$$W(C) = \mathrm{Tr} \left(\mathcal{P}\mathrm{exp} \oint_K A \ ds \right) \tag{1.3}$$

which is the (gauge-invariant) observable known as a *Wilson loop.* In words, equation (1.3) calculates the holonomy of the connection A around K then takes the trace of this (which will be an element of SU(2)) in the fundamental representation of SU(2). Equation (1.1) takes the Feynman path integral over all equivalence classes of connections modulo gauge transformations, essentially weighting the calculation of the Wilson loop. To retrieve the polynomial form, one manipulates the integer k in (1.2) into a root of unity t in the complex plane by setting $t^{\frac{1}{2}} = \exp\left(\frac{i\pi}{k+2}\right)$.

It is worth noting that the trace in (1.2) ensures we are integrating a 3-form on S^3; in other words, a volume form, forgoing the need of a metric. Similarly, the trace in (1.3) also means there is no need to introduce a metric and as a consequence (1.1) is independent of the metric. In physics, a distinction is made between *Schwarz* (considered here) and *Witten* types of TQFTs. In the former, the correlation functions computed by the path integral are topological invariants because the path integral measure and the quantum field observables are explicitly independent of the metric. The topological invariance in the latter is more subtle: the action and the stress energy tensor are BRST exact forms so that their functional averages are zero, and therefore the topological observables form cohomological classes [17].

Jones (as with Donaldson, Floer [7] and Gromov [11]) theory deals with topological invariants, and understanding this as a quantum field theory involves constructing a theory in which all of the observables are topological invariants. The physical meaning of "topological invariance" is "general covariance." A physicist will recognise a quantum field theory defined on a manifold M without any a priori choice of metric as generally covariant. However, physicists are often indoctrinated (by General Relativity) to believe that the way to construct a quantum field theory with no a priori choice of metric is to introduce one, then integrate over all metrics[1]. The lesson from the Jones (as well as Donaldson, Floer and Gromov) theory is

[1]Physicists are therefore reminded that a generally covariant theory is not defined by one where the metric is a dynamic variable.

precisely that there are highly non-trivial quantum field theories in which general covariance is realised in other ways.

This was essentially the begining of Topological Quantum Field Theory, a surprising and elegant use of physical ideas to explain a geometric quantity. The punchline of Chapter 3 is that a Topological Quantum Field Theory is a functor from a category of cobordisms to a category of vector spaces. Graeme Segal orgininally proposed a similar idea for Conformal Field Theory in 1989 [16], and finally Michael Atiyah gave an axiomatic definition for a TQFT [2], what we provide now is a short exposition of the ideas in this dissertation which will be clarified later.

We begin with the reprisal of some well trodden physics and introduce new concepts using physical terms where possible. We introduce a quantum theory over an arbitrary manifold M without knots but retaining the Chern-Simons action, so the path integral is

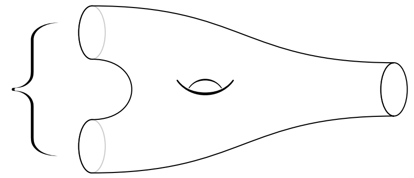
$$Z(M) = \int_{\mathscr{A}} \mathcal{D}A \exp\left(\frac{ik}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right) \tag{1.4}$$

It has been shown [18] [15] that (1.4) does give topological invariants, namely, they are proportional to the Ray-Singer analytic torsion at the stationary points[2]of the path integral.

However, the path integral formulation of quantum field theory was not initially intended to calculate bare partition functions as in (1.4); there should be dynamics involved, so that one can express the probability amplitude for one field configuration to evolve into another. To this end, we heuristically consider a field theory which associates a set of fields (connec-

[2]The stationary points are called "flat connections," and are identified as those gauge fields for which the curvature vanishes, $F_{ij} = 0$.

tions here) to a "space" which we call a boundary. The boundary can be thought of as[3] a manifold of fixed dimension $d = 2$ appearing as a boundary of a "spacetime" which we think of as $d + 1$ dimensional manifolds. The boundary of a spacetime can be divided into two sets, the "incoming" boundary and the "outgoing" boundary, then the field theory gives a way to start with the fields on the incoming boundary and propagate them across the spacetime to give some fields on the outgoing boundary:



To a space Σ we associate the space of fields living on it, which here is the space of connections $\mathcal{A}(\Sigma)$. A physical state Ψ corresponds to a functional on this space of fields. This is the Scrödinger picture of quantum mechanics, if $A \in \mathcal{A}(\Sigma)$ then $\Psi(A)$ represents the probability that the state Ψ will be found in the field configuration A. The dynamics of the theory cause the states to evolve with time $\Psi(A) \to \Psi(A, t)$. The natural basis for a space of states are delta functionals $\hat{A}$ satisfying $\left\langle \hat{A} | \hat{A}' \right\rangle = \delta(A - A')$. The amplitude for a system in the state $\hat{A}_1$ (that is, expanded in the $\hat{A}$ basis) on the space Σ_1 to propogate to a state $\hat{A}_2$ on Σ_2 is then

$$\left\langle \hat{A}_2 | U | \hat{A}_1 \right\rangle = \int_{A_1}^{A_2} \mathcal{D}\mathcal{A} \; \mathrm{e}^{iS[A]} \tag{1.5}$$

We have then constructed U, the *time evolution operator* associated to the spacetime M which brought about this propogation of fields. One can see now that specifying quantum field theory just amounts to giving the rules for constructing Hilbert spaces $\mathcal{A}(\Sigma)$ and the rules for calculating the

[3]This is a working definition which will ofcourse be generalised later.

time evolution operator $U(M) : \Sigma_1 \to \Sigma_2$. What we have described in the previous sentance is a *functor* from the category of bordisms to the category of Hilbert spaces, all of which shall be explained in what follows.

2 Axioms

We essentially follow Atiyah's definition [2]. It is useful to first recall that homology theory has been given an axiomatic definition [6] which has proven to be extremely useful. The motivation for the axioms of homology theory is geometric and there are many geometric constructions for homology (de Rham, simplicial, singular) which are important for applications. However the purely formal properties are best studied independently of any geometric realisation. The same applies to TQFT. In fact, producing a rigorous geometric or analtic construction may not yet be possible in all cases, hence having axioms provides a framework to aim for.

A homology theory can be described as a functor F from the category of topological spaces to the category of Λ-modules, where Λ is some fixed ground ring that is commutative and has a unit element, such as $\Lambda = \mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$. Some key axioms this functor then satisfies are

(i) *homotopy invariance*, expressed using "cylinders" $X \times I$

(ii) *additivity*[1] on disjoint sums given by $F(X_1 \sqcup X_2) = F(X_1) \oplus F(X_2)$

A TQFT also assigns a vector space functorially to each topological space but differs by the following

(a) a TQFT will be defined for a *manifold* of a fixed dimension

[1] This implies that for the empty set, $F(\varnothing) = 0$.

(b) axiom (i) is strengthened by replacing cylinders with *cobordisms*

(c) axiom (ii) is replaced by a *multiplicative* axiom for disjoint sums[2]

Physically, (b) is related to the need for relativistic invariance. (c) is indictive of a characteristic property of quantum field theories that they take disjoint unions of spaces to tensor products of state modules. For the tensor product to make sense here, the sets we end up with must have an algebraic structure (be a module over a ring), in contrast to “classical” theories which take unions to cartesian products, needing no extra structure.

Now the axioms. A TQFT in dimension d defined over a ground ring Λ consists of the following data:

(A) A finitely generated Λ-module $Z(\Sigma)$ associated to each oriented smooth closed smooth $(d-1)$-manifold Σ

(B) An element $Z_M \in Z(\partial M)$ associated to each oriented smooth d dimensional manifold (with boundary) M

These data are subject to the following axioms:

functoriality Z is functorial with respect to orientation preserving diffeomorphisms of Σ and M

involutarity $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ is the dual module

multiplicativity for disjoint unions, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

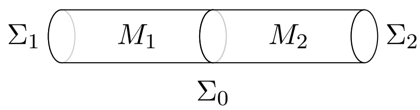
The functoriality axiom means that an orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ induces the isomorphism $Z(f) : Z(\Sigma) \to Z(\Sigma')$. Also

[2]The empty set then has value Λ rather than 0.

if f extends to an orientation preserving diffeomorphism $M \to M'$, with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then $Z(f)$ takes Z_M to $Z_{M'}$.

The involutarity axiom becomes most clear when Λ is a *field*, as $Z(\Sigma)$ and $Z(\Sigma)^*$ are then just dual vector spaces, and we shall ussually adopt the case best suited for physics, $\Lambda = \mathbb{C}$.

The multiplicativity axiom is very strong. If we decompose a boundary as $\partial M = \Sigma_1 \sqcup \Sigma_0^*$, then $Z_M \in Z(\Sigma_0)^* \otimes Z(\Sigma_1) = \mathrm{Hom}(Z(\Sigma_0), Z(\Sigma_1))$. Hence we can view any *cobordism* M from Σ_0 to Σ_1 as inducing a linear transformation $Z_M : Z(\Sigma_0) \to Z(\Sigma_1)$. The mulipicativity axiom asserts that this is *transitive* when composing bordisms. That is, if $\partial M_1 = \Sigma_1 \sqcup \Sigma_0$, $\partial M_2 = \Sigma_2 \sqcup \Sigma_0^*$ and we construct a manifold[3] $M = M_1 \sqcup_{\Sigma_0} M_2$:



Then we require $Z_M = \langle Z(M_1), Z(M_2) \rangle$ to be the composition where $< \ , \ >$ gives the natural pairing $Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2)$.

A further application of the multiplicativity axiom shows that when considering $\Sigma = \varnothing$ the empty $(d-1)$-dimensional manifold, a projection of the vector space $Z(\Sigma)$ is idempotent and therefore we can set $Z(\Sigma)$ as zero or isomorphic to the ground field Λ. If now we have a *closed* d-manifold M so that $\partial M = \varnothing$, then $Z_M \in Z(\varnothing) = \Lambda$ is a constant element of the ground field. Hence we see the theory produces numerical *invariants* of closed d-manifolds, which play an important part in the theory.

For the cylinder cobordism $\Sigma \times I$, an immediate implication of the axioms is that $Z_{\Sigma \times I} : Z(\Sigma) \to Z(\Sigma)$ must be independent of the length of the interval I. This is known as *homotopy invariance.* Furthermore, it is clear

[3]We depict this manifold with M_1 and M_2 as cylinders in the diagram, though they may not necessarily be.

that the element $Z_{\Sigma\times I} \in \mathrm{End}(Z(\Sigma))$ is an idempotent, or more generally, the identity on the subspace of $Z(\Sigma)$ spanned by all the elements Z_M such that $\partial M = \Sigma$. We lose little generality in assuming then that the cylinder is the identity operation. In fact, this is a characteristic property of a "topological" field theory: If $Z_{\Sigma\times I} = \mathrm{id}$, then Z_X can only be non-trivial if X is non-trivial. This contrasts to the physically interesting theories, which require a Riemannian metric and other data on X, that can give highly non-trivial Z_X on cylinders $Z_{\Sigma\times I}$.

If we form the product manifold $\Sigma \times S^1$ by identifying opposite ends of the cylinder, then our axioms imply that

$$Z(\Sigma \times S^1) = \mathrm{Tr}(\mathrm{Id}) = \dim(\Sigma) \tag{2.1}$$

More generally, if $f : \Sigma \times \Sigma$ is an orientation preserving diffeomorphism, and we identify opposite ends of $\Sigma \times I$ by f, then this given a manifold Σ_f and our axioms imply $Z(\Sigma_f) = \mathrm{Tr}(Z(f))$ where $Z(f)$ is the induced automorphism of Z(Σ).

To reconcile with the example given in the first chapter: Σ indicates the physical space at an instant in time; the extra dimension in $\Sigma \times I$ is "imaginary" time; $Z(\Sigma)$ is the Hilbert space of the quantum theory; the vector Z_M (where $\partial M = \Sigma$) in the Hilbert space is to be thought of as the *vacuum state* defined by M; the number Z_M for a closed manifold M is the *vacuum expectation value*, also called the partition function.

2.1 Recent Developments

Although we do not consider these facts further in this dissertation, it has been realised that such a formalization as provided here fails to capture some important aspects of the main examples of TQFTs. For example, Reshetikhin, Turaev and others found a setting where the ideas of Witten could be made mathematically rigorous by interpreting everything in terms of the category of representations of a quantum group [14], which turned out to be a braided monoidal category. It was found that this description and the Atiyahl monoidal functor description were somehow different sides of the same coin, and the problem was to find a formalism that described both well. Another problem was that the formalisation of a TQFT as a monoidal functor only captured a small subset of the gluing laws (see for example [13]) which actually held in practice. The action in a quantum field theory is usually of local nature, which suggests that the theory should be natural with respect to all possible gluing laws of all codimensions, not just gluing two $(d-1)$-manifolds along an d-dimensional cobordism. These considerations gave rise to the notion of an extended TQFT as one which behaves well with respect to these extra gluing laws [4] [8] [19].

3 Categories and Cobordisms

We briefly give the necessary definitions from category theory as we wish to interpret the axiomatic definiton of a TQFT. We find that when we apply the axioms to bordisms, they define a TQFT as a functor.

Definition 1 (Category)**.** *A* category $\mathcal{C}$ *consists of a class of objects* $\mathrm{ob}(\mathcal{C})$*, a class of maps called* morphisms $\hom(\mathcal{C})$ *and for every* a*,* b*,* $c \in \mathrm{ob}(\mathcal{C})$ *a binary operation* $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ *called the* composition*. These are then subject to two axioms, associativity of morphism compositions and the existence of a morphism* $\mathrm{id}_x : x \to x$ *for every object* x*.*

As promised, we wish to explain a category of interest, **nCob**, the category of n-dimensional cobordisms. Elements of ob(**nCob**) are $(n-1)$-dimensional closed[1] oriented manifolds Σ. Elements of hom(**nCob**) are *cobordisms*, compact oriented n-manifolds $M : \Sigma \to \Sigma'$. If $M : \Sigma \to \Sigma'$ and $N : \Sigma' \to \Sigma''$ are two cobordisms, then the composition axiom defines the morphism $N \circ M : \Sigma \to \Sigma''$ which can be thought of as "gluing" N on the end of M.

The other category of interest to us is $\mathbf{Vect}_{\mathbb{K}}$ where objects are vector spaces over a field $\mathbb{K}$ and $\mathbb{K}$-linear maps are the morphisms.

Definition 2 (Monoidal Category)**.** *A (*strict*)*[2] monoidal category *is a*

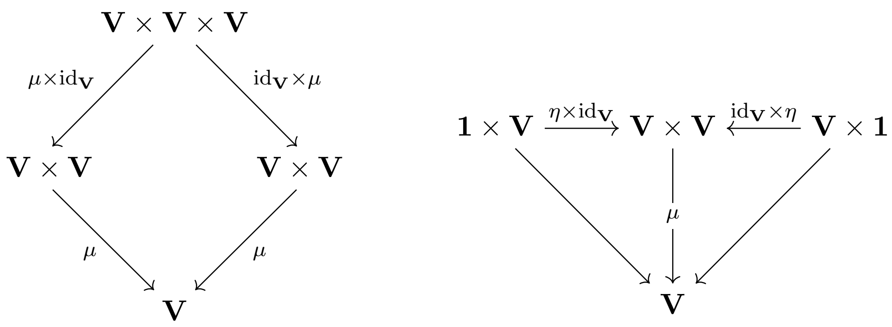
[1] By which we mean compact and without boundary

[2] The weaker alternative, which is mathematically and philosophically more correct, has

category **V** *equipped with two* functors

$$\mu : \mathbf{V} \times \mathbf{V} \to \mathbf{V}, \qquad\qquad \eta : \mathbf{1} \to \mathbf{V}$$

such that the following diagrams (called the "associativity" and "neutral element" axioms, respectively) commute



where $\mathbf{1}$ *is the neutral object,* $\mathrm{id}_{\mathbf{V}}$ *is the identity functor* $\mathbf{V} \to \mathbf{V}$ *and the diagonal functors without labels are projections.*

The commutative diagrams show why this category has been named as such: if **V** were a set and μ and η were functions, we would have defined a *monoid.* The mathematical name for this upgrade is *vertical categorification.* It is important to note that, as functors, μ and η operate on both objects and morphisms:

$$\begin{array}{rcl} \mathbf{V} \times \mathbf{V} & \xrightarrow{\mu} & \mathbf{V} \\ (\mathrm{X,Y}) & \longmapsto & X \Box Y \\ (\mathrm{f,g}) & \longmapsto & f \Box g \end{array}$$

where we have used $\Box$ as an infix for μ, it will be some kind of binary composition, but dependent on the specific category. So for a pair of objects

axioms which only hold up to coherent isomorphisms. By coherent we mean that these isomorphisms have diagrams of their own which must commute for the whole category to be consistent. Mac Lane has proven [12] that all monoidal categories are equivalent to strict ones, and we do not need to consider them for our purposes.

X, Y, a new object $X \Box Y$ is associated and to each pair of morphisms $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ a new morphism $X \Box Y \xrightarrow{f \Box g} X' \Box Y'$.

A monoidal category is specified by the triplet $(\mathbf{V}, \Box, I)$ where I is the object which is the image of η. Some examples include

Definition 3 (Symmetric Monoidal Category)**.** *A monoidal category* $(\mathbf{V}, \Box, I)$ *is called* symmetric *if for each pair of objects* X*,* Y *there is a* twist map

$$\tau_{X,Y} : X \Box Y \to Y \Box X$$

subject to three axioms:

(i) naturality*: for every arrow in* $\mathbf{V} \times \mathbf{V}$*, therefore every pair of arrows* $f : X \to X'$ *and* $g : Y \to Y'$ *the following diagram commutes:*

$$\begin{array}{ccc} X \Box Y & \xrightarrow{\tau_{X,Y}} & Y \Box X \\ \Big\downarrow{\scriptstyle f \Box g} & & \Big\downarrow{\scriptstyle g \Box f} \\ X' \Box Y' & \xrightarrow{\tau_{X',Y'}} & Y' \Box X' \end{array}$$

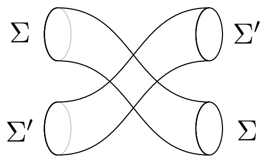
(ii) symmetry*: for every three objects* X*,* Y*,* Z*, the two following diagrams commute:*

$$\begin{array}{ccccc} X \Box Y \Box Z & & \xrightarrow{\tau_{X,Y \Box Z}} & & Y \Box Z \Box X \\ & {\scriptstyle \tau_{X,Y} \Box \mathrm{id}_Z} \searrow & & \nearrow {\scriptstyle id_Y \Box \tau_{X,Z}} & \\ & & Y \Box X \Box Z & & \end{array} \qquad \begin{array}{ccccc} X \Box Y \Box Z & & \xrightarrow{\tau_{X \Box Y,Z}} & & Z \Box X \Box Y \\ & {\scriptstyle \mathrm{id}_Z \Box \tau_{Y,Z}} \searrow & & \nearrow {\scriptstyle \tau_{X,Z} \Box \mathrm{id}_Y} & \\ & & X \Box Z \Box Y & & \end{array}$$

which says that twists compose like permutations.

(iii) identity*: we have* $\tau_{X,Y} \tau_{Y,X} = \mathrm{id}_{X \Box Y}$*.*

Again, to give a symmetric monoidal category is to specify the quadruplet $(\mathbf{V}, \Box, I, \tau)$. Relevant examples are the category of n-cobordisms, $(\mathbf{nCob}, \sqcup, \varnothing, T)$, where the twist cobordism is $T_{\Sigma,\Sigma'} : \Sigma \sqcup \Sigma' \to \Sigma' \sqcup \Sigma$, or diagramatically,



Also, the category of vector spaces, $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$, has a canonical symmetry for pairs of vector spaces, $\sigma : V \otimes W \to W \otimes V$.

Definition 4 (Symmetric Monoidal Functor)**.** *Given two symmetric monoidal categories* $(\mathbf{V}, \Box, I, \tau)$ *and* $(\mathbf{V}', \Box', I', \tau')$ *a monoidal functor between them* $F : V \to V'$ *is required to preserve the symmetry property if it is to earn the title* symmetric*, hence for every pair of objects we have* $F\tau_{X,Y} = \tau'_{FX,FY}$*.*

Definition 5 (Monoidal Natural Transformation)**.** *Given two monoidal functors* $G, F : \mathbf{V} \to \mathbf{V}'$*, a* monoidal natural transformation $\alpha : F \Rightarrow G$ *satisfies, for every pair of objects* X*,* $Y \in \mathbf{V}$*,*

$$\alpha_{X\Box Y} = \alpha_X \Box' \alpha_Y$$

and also $\alpha_I = \mathrm{id}_{I'}$*.*

The latter equation makes sense because $FI = GI = I'$. The first equation is becuase the following diagram should commute:

$$\begin{array}{ccc} F(X\Box Y) & \xrightarrow{\alpha_{X\Box Y}} & G(X\Box Y) \\ \| & & \| \\ FX\Box' FY & \xrightarrow[\alpha_X \Box' \alpha_Y]{} & GX\Box' GY \end{array}$$

An important category for our purposes is the category of symmetric monoidal functors, defined as follows. Given two symmetric monoidal categories $(\mathbf{V}, \Box, I, \tau)$ and $(\mathbf{V}', \Box', I', \tau')$ there is a category $\mathbf{SymMonCat}(\mathbf{V}, \mathbf{V}')$ whose objects are the symmetric monoidal functors from $\mathbf{V}$ to $\mathbf{V}'$ and whose morphisms are monoidal natural transformations.

Definition 6 (Linear Representation)**.** *A* linear representation *of a symmetric monoidal category* $(\mathbf{V}, \Box, I, \tau)$ *is a symmetric monoidal functor* $(\mathbf{V}, \Box, I, \tau) \to (\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)$

So the set of all linear representations of $\mathbf{V}$ are the objects of a category denoted $\mathbf{Rep}_{\mathbb{K}}(\mathbf{V}) = \mathbf{SymMonCat}(\mathbf{V}, \mathbf{Vect}_{\mathbb{K}})$.

We can now see that an n-dimensional[3] topological quantum field theory is a symmetric monoidal functor from $(\mathbf{nCob}, \sqcup, \varnothing, T)$ to $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)$. Recalling Atiyah's definition of a TQFT, it begins by associating a vector space to each n-manifolds. In other words, we have a map from $\mathbf{nCob}$ (the category of disjoint unions of $(n-1)$-manifolds and n-manifolds between them) to $\mathbf{Vect}_{\mathbb{K}}$. The functoriality axiom says that this map is actually a functor, it respects the identity (the cylinder) and composition. The multiplicativity axiom then says that this functor is monoidal as it takes disjoint unions to direct products and the empty manifold to the ground field. Finally the cobordism which switches two manifolds around should be sent to the linear map which swaps the factors in the tensor product of their corresponding vector spaces, that is, the twist operation is also

[3] In Chapter 2 we declared a TQFT in d dimensions, whereas in this chapter we have used n for the category of cobordisms. These dimensions have been labelled differently so far because we wanted to be general; there was no reason, until now, to assume $d = n$.

preserved by the functor, hence it is symmetric. We have established then, that

$$\mathbf{nTQFT}_{\mathbb{K}} = \mathbf{Rep}_{\mathbb{K}}(\mathbf{nCob}) = \mathbf{SymMonCat}(\mathbf{nCob}, \mathbf{Vect}_{\mathbb{K}})$$

It is worth noting now that the original axioms make no mention of cobordisms, but as we have shown, using oriented cobordisms for the d-manifolds between vector spaces satisfies all the axioms. We have therefore deferred the following section until now as it is most intuitive when seen in the language of cobordisms, though need not necessarily be stated in these terms.

3.1 Non-Degenerate Pairings

As a direct consequence of the axioms we determine two further results. If we take a cylinder $\Sigma \times I$, map a closed manifold Σ to one end and the dual Σ^* to the other then we have a cylinder with two incoming boundaries, , which we call the *pair*. This is then a cobordism with two incoming boundaries and an empty out-boundary, in other words it is the map : $\Sigma \sqcup \Sigma \to \varnothing$. The axioms state that we are to take the image of Σ^* under the TQFT as the dual vector space to the image of Σ, so $Z(\Sigma^*) = Z(\Sigma)^* =: V^*$. Hence we have a linear map given by the pair cobordism which we denote $\beta : V \otimes V \to \mathbb{K}$. Similarly, we can define a *copair* by a cylinder with two outgoing boundaries, . We denote the linear map for this by $\gamma : \mathbb{K} \to V \otimes V$. This implies we have an isomorphism $V \cong V^*$ and thus V must be finite-dimensional[4]. What we have then shown is that the pairing β is non-degenerate, that is, acting on $V = V \otimes \mathbb{K}$ with

[4]This is because the dual of an infinite dimensional vector space is of strictly higher dimension than the original, hence there could not be an isomorphism between them.

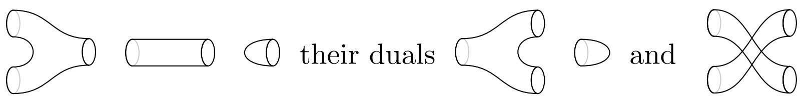
the composite $\beta \otimes \mathrm{id}_V \circ \mathrm{id}_V \otimes \gamma$ produces $\mathbb{K} \otimes V = V$, and hence acts as the identity map of V.

4 Classification

Since connected $2d$-manifolds are completely classified by their genus and number of boundary circles, it is perhaps not completely a shot in the dark to hope that TQFTs in 2 dimensions can be completely classified: this will be the goal then of this chapter.

4.1 Generators of 2Cob

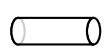
We have established in Chapter 3 that a 2-dimensional TQFT is a representation of **2Cob**, and this is a category we fully control. The category of 2-cobordisms has objects $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, ...\}$ which are disjoint unions of circles, and the *generating* morphisms are as follows:

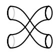


To be explicit, by "generating" set we mean that every morphism in **2Cob** is obtainable by some composition of the above set of generators. We want to specify a symmetric monoidal functor $Z : \mathbf{2Cob} \to \mathbf{Vect}_{\mathbb{K}}$, so we specify the vector space $V := Z(\mathbf{1})$ and a linear map (to follow) for each generator. Because we require a monoidal functor, we must have $V^{\otimes \mathbf{n}} = Z(\mathbf{n})$. Because we also require this functor to be symmetric, the twist maps must be preserved. As promised, we declare the following:

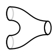
$$Z : \mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\mathbb{K}}$$

$$\mathbf{1} \longmapsto V$$

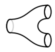
 $\longmapsto [\mathrm{id}_V : V \to V]$

 $\longmapsto [\sigma : V^{\otimes 2} \to V^{\otimes 2}]$

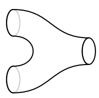
 $\longmapsto [\eta : \mathbb{K} \to V]$

 $\longmapsto [\mu : V^{\otimes 2} \to V]$

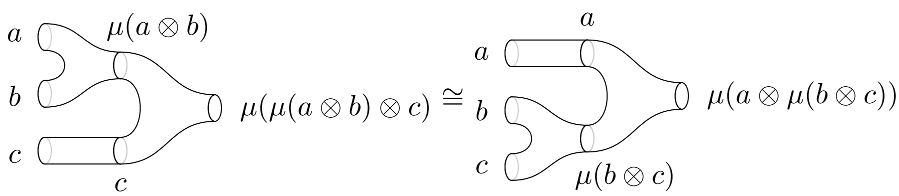
 $\longmapsto [\epsilon : V \to \mathbb{K}]$

 $\longmapsto [\delta : V \to V^{\otimes 2}]$

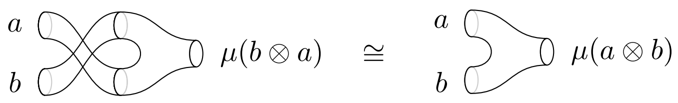
Consider now the "pair of pants" generator. We show that this acts as a "multiplication" for $Z(\mathbf{1}) = V$. If we denote elements of the "incoming" vector spaces as a and b then the element (of the outgoing vector space) that we are left with is given by $\mu(a \otimes b)$:



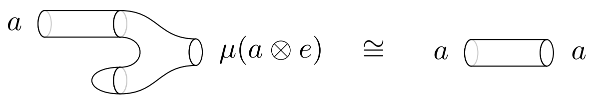
Therefore consider the following homeomorphism:



These two cobordisms must be topologically equivalent (they are both the 2-sphere with 4 "holes" in it) and therefore we attain **associativity** for our multiplication: $(ab)c = a(bc)$, writing $\mu(a \otimes b)$ as ab. Furthermore, the following equivalence:



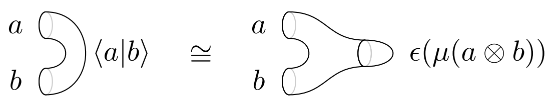
shows that our multiplication is **commutative**, as we have $ab = ba$. Another equivalence (where we label the element of outgoing vector space of η by e)



shows that our multiplication is **unital** as we have shown $ae = a$ and it is clear that $ea = a$ also, hence the map η acts as a unit. We denote the inner product of elements defined by the pairing given in Section (3.1)as

$$\begin{aligned} \beta : V^{\otimes 2} &\longrightarrow \mathbb{K} \\ a \otimes b &\longmapsto \langle a|b\rangle \end{aligned}$$

then we can establish the following equivalence



Which shows $\epsilon(ab) = \langle a|b\rangle$. In other words a **non-degenerate bilinear form** ϵ where the corresponding pairing is given by $\langle\ |\ \rangle$. The following well-timed definitions are now relevant.

Definition 7 (Frobenius Algebra)**.** *A finite dimensional, unital, associative algebra A defined over a field $\mathbb{K}$ is said to be a* Frobenius algebra *if A is equipped with a non-degenerate bilinear form $\epsilon : A \times A \to \mathbb{K}$. The bilinear form is known as the* Frobenius form *of the algebra.*

Definition 8 (Commutative Frobenius Algebra)**.** *A Frobenius algebra is* commutative *if the action of its associated monoid is commutative.*

Clearly, given a 2 dimensional TQFT Z, its image vector space $V = Z(\mathbf{1})$ is equivalent to a commutative Frobenius algebra. Conversely, if we start with a commutative Frobenius algebra (A, ϵ) we can construct a TQFT Z by using the above linear maps given for the generators as definitions. Thus we have proved our main theorem of this section, the equivalence

$$\mathbf{nTQFT}_{\mathbb{K}} \simeq \mathbf{cFA}_{\mathbb{K}}$$

where $\mathbf{cFA}_{\mathbb{K}}$ is the category of commutative Frobenius algebras in which the morphisms are algebra homomorphisms ϕ that preserve the Frobenius form, $\epsilon = \phi\epsilon'$.

4.2 Examples of Frobenius Algebras

- **Matrix algebras.** A is the ring of all $N \times N$ matrices with complex entries, $\mathrm{Mat}_N(\mathbb{C})$ and the trace ϵ is the actual trace operation Tr. To see that this is non-degenerate, if we have $\mathrm{Tr}(XY) = 0\ \forall Y$, then we require $X = 0$. Taking the linear basis of $\mathrm{Mat}_N(\mathbb{C})$ as the matrices E_{ij} with only one non-zero entry, $e_{ij} = 1$, clearly the number of such basis-matrices is $N^2 = \dim(\mathrm{Mat}_N(\mathbb{C}))$. The dual basis element is then E_{ji} and we have $\mathrm{Tr}(E_{ij}E_{ji}) = 1$. This is a symmetric Frobenius algebra as the trace operation is cyclical.

- **Finite group algebras.** If G is a finite group then we take A as the *group algebra* $\mathbb{C}[G]$ with basis $\{g \in G\}$ and multiplication given by multiplication in the group. The Frobenius form should be $\epsilon(g) = \delta_g^e$ where e is the identity element in G. The pairing corresponding to this form is non-degenerate as $\epsilon : g \otimes h \mapsto \epsilon(gh) = 1$ iff $gh = e$

which implies $h = g^{-1}$. This is clearly symmetric so long as the group multiplication is commutative. A vector $v = \sum_g v_g g \in \mathbb{C}[G]$ can be thought of as a function $v : G \to \mathbb{C}$, that is $v \in C(G)$. Now in terms of $C(G)$, the product of vectors is the convolution,

$$(v * u)(g) = \sum_h v_h u_{gh^{-1}} \tag{4.1}$$

and the trace is $\epsilon(v) = \frac{1}{|G|} v(e)$, where the customary factor ensures the total volume of the group is unity. The inner product is then

$$\epsilon(v * u) = \frac{1}{|G|} \sum_g v_g u_{g^{-1}} \tag{4.2}$$

The Peter-Weyl theorem tells us that this example of a Frobenius algebra is just a particular case of the former example, since $C(G)$ can be decomposed

$$C(G) = \bigoplus_{\rho \in \hat{G}} \mathrm{End}(V_\rho) \tag{4.3}$$

where $\hat{G}$ is the finite set of irreducible representations ρ of G. Note that the regular representation of an algebra V is a homomorphism $V \to \mathrm{End}(V)$.

- **Class Functions on a group.** The center[1] of the algebra of functions on a group $A := Z(C(G))$ is a commutative Frobenius algebra which can be understood as the space of *class functions*, that is, functions on the conjugacy classes of G:

$$Z(C(G)) = C_{class}(G) = \left\{ f : G \to \mathbb{C} | f(h^{-1}gh) = f(g) \right\} \tag{4.4}$$

[1]Given by $Z(G) = \{z \in G | \forall g \in G, zg = gz\}$.

For each irreducible representation ρ of G, the character $\chi_\rho = \mathrm{Tr}(\rho)$ is a class function. The characters χ_ρ form a basis of the space of class functions.

- **Cohomology rings.** For a compact, oriented manifold X, the de Rahm cohomology $H^*(X) = \oplus_{i=0}^n H^i(X)$ forms an algebra under the wedge product. This ring is graded in the sense that for $\alpha \in H^p(X)$ and $\beta \in H^q(X)$ we have $\alpha \wedge \beta \in H^{p+q}$. The trace ϵ is integration over X with respect to a chosen volume form. The corresponding bilinear form $(\cdot, \cdot) : H^*(X) \otimes H^*(X) \to \mathbb{R}$ is non-degenerate because of Poincaré duality[2], hence $H^*(X)$ is a Frobenius algebra over $\mathbb{R}$. If $\{e_{p,i} \in H^p(X)\}$ is a basis for $H^*(X)$ then the dual basis is given by the Poincaré duals $\{e^i_{d-p} \in H_{d-p}(X)\}$. Note that this algebra is not commutative but graded-commutative, that is $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$. The simplest example of these is the cohomology of complex projector space $\mathbb{CP}^{n-1}$. The ring is the truncated polynomial ring $\mathbb{C}[z] \, / \, z^n$ and the inner product is formed by picking out the coefficient of z^{n-1} in the product. A more general Frobenius algebra of this type arises in the next example.

- **Landau-Ginzburg models.** Let W be a polynomial[3] in n variables x_i, and suppose the zero locus $\mathcal{Z}(W) = \{x \in \mathbb{C}^n | W(x) = 0\} \subset \mathbb{C}^n$ has an isolated singularity at the origin, $0 \in \mathbb{C}^n$. Set $W_i := \frac{\partial W}{\partial x_i}$ and define the ideal $I = (W_1, \ldots, W_n) \subset \mathbb{C}[x_1, \ldots, x_n]$. The chiral ring moded by this ideal, $\mathbb{C}[x_1, \ldots, x_n] \, / \, I$, then has a canonical non-degenerate form defined on it. Precisely, we define ϵ as the residue

[2] Recall this states that the ith cohomology group is isomorphic to the $(d-i)$th homology group, $H^i(X) \cong H_{d-i}(X)$, where $\dim(X) = d$.

[3] This is often used as the superpotential for a set of chiral $N = 2$ superfields.

obtained by integrating along the singularity around a real n-ball, so if $B = \{x | W_i(x) = \rho\}$ for some small $\rho > 0$ then (writing $x_1, \ldots, x_n$ as just x)

$$\begin{aligned} \epsilon_W : \mathbb{C}[x_1, \ldots, x_n] \,/\, I &\longrightarrow \mathbb{C} \\ g &\longmapsto \int_B \frac{g(x)\ dx_1 \wedge \cdots \wedge dx_n}{W_1(x) \cdots W_n(x)} \end{aligned}$$

Local duality (see [10]) then states that the corresponding bilinear pairing is non-degenerate.

5 Dijkgraaf-Witten theory

Often when studying quantum field theory, a"toy model", usually ϕ^4 theory, is examined in detail to understand formalities as well as to give a simple foundation for more sophisticated developments. We therefore agree with Freed and Quinn [9] that a similarly illustrative theory would prove instructive. As suggested, we present a TQFT with a finite gauge group, which has come to be named, for its progenators in [5], Dijkgraaf-Witten theory.

5.1 Preliminaries

The extra data needed for this TQFT is a finite group G. Hence we can define a principle G-bundle over a manifold M as a manifold P. These are covering maps $P \to M$ with a free action of G such that $P/G = M$. G-bundles on M are classified. A map of G-bundles $\phi : P' \to P$ is a smooth map which commutes with the G-action. If the induced map $\bar{\phi} : M' \to M$ is the identity map then ϕ is called a morphism. Since any morphism must have an inverse, if there does exist a morphism $\phi : P' \to P$, we say that P' is equivalent to P. To specify a G-bundle on M, up to isomorphism, is to specify a map $M \to^{BG} /_{\text{homotopy}}$ where BG is the classifying space of G, defined as the space with homotopy groups $\pi_i(BG) \cong \begin{cases} G & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$. To a closed, connected n-manifold M Atiyah tells us to assign a number, and the idea of Dijkgraaf-Witten theory is that this number should count the G-bundles on the manifold. So we set

$$Z(M) = \frac{\#\text{ of homomorphisms } \pi_1(M) \to G}{|G|} \tag{5.1}$$

Note, because M is compact we are assured that the numerator is finitely generated. If we now generalise to where M is not connected then what we should define is

$$Z(M) = \#\text{ of } G\text{-bundles on } M\text{, counted with mass} \tag{5.2}$$

. If we now fix a point $x \in M$ and a set X with a free and transitive G-action. We focus only on G-bundles $P \to M$ where $P_x = X$. Also, fix an element $p_0 \in X$. Parallel transport around a loop σ based at x will send $p_0 \to p_0 \cdot g$. These holonomies determine a homomorphism $\phi_P : \pi_1(M,x) \to G$. This homomorphism then determines the bundle which we write as P_ϕ. If we had chosen a different reference point $p_0' = p_0 \cdot h$ the holonomy would have been conjugated $g \to h^{-1}gh$. Thus there is a right action of G on $\mathrm{Hom}(\pi_1(M,X),G)$, defined by conjugating the image of ϕ, $(\phi \cdot g)(\sigma) = g^{-1}\phi(\sigma)g$, and P_{ϕ_1} is isomorphic to P_{ϕ_2} when $\phi_2 = \phi_1 \cdot g$ for some g. Then

$$\overline{\mathcal{C}_M} = \mathrm{Hom}(\pi_1(M,x),G)/G \tag{5.3}$$

where $\overline{\mathcal{C}_M}$ denotes the set of equivalence classes of G-bundles over M. However we want to count the G-bundles with mass, so we will "remember" the isomorphisms and say that

$$\#\text{ of } G\text{-bundles on } M\text{, counted with mass } = \sum_{P \to M} \frac{1}{|\mathrm{Aut}P|} \tag{5.4}$$

If a G-bundle P is given by a homomorphsm $\alpha \in \mathrm{Hom}(\pi_1(M), G)$, then $\mathrm{Aut}P$ is the subgoup of G consisting of elements which centralise $\mathrm{Im}(\alpha)$. So we can write

$$\begin{aligned} \sum_{P\to M} \frac{1}{|\mathrm{Aut}P|} &= \sum_{\alpha/G} \frac{1}{|\text{centraliser of } \alpha|} \\ &= \sum_{\alpha} \frac{1}{|\text{centraliser of } \alpha| \times |{}^{\alpha}/_{G}|} \\ &= \sum_{\alpha} \frac{1}{|G|} \end{aligned} \tag{5.5}$$

(where we have written $|{}^{\alpha}/_{G}|$ for the number of homomorphisms that are conjugated to α). Hence we recover equation (5.1), where we have shown the "counting" of G-bundles it carries out.

Now we must define what this theory assigns to a $(n-1)$-manifold Σ. We want to associate a complex vector space, so we set

$$Z(\Sigma) = \{\text{locally constant functions on } \mathrm{Map}(\Sigma, BG)\} \tag{5.6}$$

We can think of $\mathrm{Map}(\Sigma, BG)$, the space of maps from Σ in to BG, as a topological space and because of equation (5.3), it can be shown that the connected components of the space $\mathrm{Map}(\Sigma, BG)$ are in $1:1$ correspondence with elements of $\overline{\mathcal{C}_M}$. So $\mathrm{Map}(\Sigma, BG)$ can be thought of as a space of G-bundles on Σ, it has one connected component for each G-bundle and is again the form of a classifying space. It is homotopy equivalent to

$$\bigsqcup_{\overline{\mathcal{C}_\Sigma}} B\mathrm{Aut}(P_\Sigma)$$

where we have extended our notation $\overline{\mathcal{C}_\Sigma}$ to mean the equivalence classes of G-bundles $P_\Sigma \to \Sigma$. Note that we always have $\mathrm{Aut}(P_\Sigma) \subseteq G$. What we are interested in are the locally constant functions on the space $\text{Map}(\Sigma, BG)$, that is, functions that are constant on each connected component. To describe such a function, we just assign a number to every G-bundle. Hence $Z(\Sigma)$ is a complex vector space with dim $= |\overline{\mathcal{C}_\Sigma}|$, that is, counted without considering mass.

5.2 n=2

Previously we have shown that a 2-dimensional TQFT should be the same as a commutative Frobenius algebra, so let us find which Frobenius algebra our Dijkgraaf-Witten theory corresponds to. Consider then $Z(\mathbf{1}) = Z(S^1)$. By definition we have

$$Z(S^1) = \{\text{locally constant } \mathbb{C}\text{-valued functions on } \text{Map}(S^1, BG)\} \quad (5.7)$$

in other words, we want functions on $\overline{\mathcal{C}_{S^1}}$. So fix a point $x \in S^1$. Then the fundamental group $\pi_1(S^1, x) = \mathbb{Z}$ so that to give a homomorphism $\alpha : \pi_1(S^1, x) \to G$ is to give an arbitrary $g \in G$. Conjugation of ϕ corresponds to conjugation of g so that by (5.3),

$$Z(S^1) = C_{class}(G) \quad (5.8)$$

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