Seiberg-Witten Theory and Duality in $\mathcal{N} = 2$ Supersymmetric Gauge Theories

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Abstract

In this thesis we review Seiberg-Witten theory in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. In their seminal work, Seiberg and Witten derived the Wilsonian low-energy effective action of the $\mathcal{N} = 2$ $SU(2)$ gauge theory by encoding the problem in a two-dimensional holomorphic curve. The proposal was that the family of algebraic curves determine the prepotential, a holomorphic function which contains information about both perturbative and nonperturbative corrections to the action, and thus defining the theory. The proposal has since been generalized to other gauge groups and various matter contents. The original work found the curves by carefully studying the singularity structure and monodromies of the moduli space and then guessing the family of curves. Since the proposal, many other methods have been employed to derive the curves. In this thesis we will review the original approach taken by Seiberg and Witten in solving the low energy effective theory, which will merit extensive discussion of $\mathcal{N} = 2$ gauge theories and the structure of the moduli space. Then we will discuss a similar approach where by looking at solutions to a differential equation with the correct monodromy considerations, we find the curves using integral representations of hypergeometric functions. Finally, we will discuss how Seiberg-Witten curves may be found through the uplifting of Type IIA brane constructions to M-theory and studying the minimal surface of the resulting M5-brane.

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We can forgive a man for making a useful thing as long as he does not admire it. The only excuse for making a useless thing is that one admires it intensely.

-Oscar Wilde

Contents

1. Introduction

For the most part, much of our understanding of modern physics from quantum field theory is rooted in perturbative expansions around free fields. These expansions give reliable and accurate answers when the deviations from the free fields are small, that is, when the coupling constants are reasonably small. The predictive power of QED is a good example of this. But our knowledge of strongly coupled theories, like QCD at low energies, is fairly poor. This is why even with the many successes of the Standard Model we still have a fairly poor understanding of low energy QCD phenomena like confinement and chiral symmetry breaking. In theories with supersymmetry our understanding of the strongly coupled regime improves as nonremormalization theorems and holomorphicity put limits on the perturbative corrections to components of the theory. Thus, it follows that studying some of these properties in supersymmetric gauge theories is a worthwhile endeavor.

Experimental evidence to date has not shown any indication that supersymmetry is indeed a symmetry of nature, although many physicists, myself included, would like to believe it is for the same reason we pursue many ideas in theoretical physics, the belief that nature is governed by laws that are fundamentally elegant. But keeping in mind the many problems in particle physics that supersymmetry resolves quite nicely, it is still presently not clear what lessons can ultimately be drawn for the non-supersymmetric gauge theories by which we know nature to be governed. Yet there are many properties of supersymmetric theories that make them incredibly important playgrounds for studying the structure of gauge theories and displaying physically relevant features, for which the phenomenological and mathematical applications have already been substantial. In the work reviewed in this thesis we will see how studying properties of supersymmetric theories can help us understand some poorly understood properties of gauge theories, like the aspects of strongly coupled theories mentioned before. We also hope to illustrate how the connection to certain aspects in string theory indicates the importance of string theory as an incredibly powerful tool in studying the structure of these gauge theories. Let's hope that this motivates, to some extent, the reason this line of research is being pursued in this thesis.

Now we will briefly discuss some nice properties of supersymmetric theories in four dimensions. Nonrenormalization theorems provide limitations on how the field theory may be renormalized both perturbatively and nonperturbatively, i.e. by loop corrections and instanton corrections. These nonrenormalization properties of supersymmetric theories are often the result of the fact that some quantities or operators must have a holomorphic dependance on the fields and couplings of the theory. This holomorphic structure leads to vacuum degeneracies and allows one to use the important techniques in complex analysis. Dualities are also a common feature in supersymmetric theories. Duality between strong and weak coupling, electric and magnetic components, or short and long distances, are manifestly important properties of supersymmetric theories. Theories with a higher degree of supersymmetry are further restricted by these properties.

There are three unique types of globally supersymmetric theories in four dimensions, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (SYM), $\mathcal{N} = 2$ SYM, and $\mathcal{N} = 4$ SYM. $\mathcal{N} = 4$ gauge theories have some extremely interesting properties and dualities. The beta function of $\mathcal{N} = 4$ SYM is zero and thus it is an example of a scale invariant (conformal) field theory. It is hard to mention $\mathcal{N} = 4$ SYM without mentioning it's role in the most successful, or rather most studied, example of the AdS/CFT correspondence between $\mathcal{N} = 4$ SYM and Type IIB string theory on $AdS_5 \times S^5$. Part of the excitement of the AdS/CFT correspondence is that it relates a strongly coupled Yang-Mills theory in four dimensions to a weakly coupled theory with gravity in five dimensions. Regardless, this theory is too simple to play a role in our discussion as we are trying to study nontrivial quantum corrections and the theory has none, and is exactly solvable. $\mathcal{N} = 1$ SYM has the opposite problem, the theory is not immune from perturbative corrections as only certain objects are holomorphic functions. Another way of stating this is that there are fewer nonrenormalization theorems governing the theory. The $\mathcal{N} = 1$ chiral superpotential is a holomorphic object and is therefore protected from perturbative corrections. So one may obtain some interesting results for exact effective superpotentials. But in general, due to uncontrollable loop corrections, $\mathcal{N}=1$ gauge theories are not fully solvable. On the other hand, $\mathcal{N}=2$ theories have some intriguing properties which make them particularly interesting to study. It is not trivially solvable but nor does it suffer from higher order perturbative corrections. The low-energy limit of the theory is exactly solvable. The low energy effective action is completely determined by a holomorphic function called the prepotential $\mathcal{F}$, for which the perturbative corrections only occur at one-loop order. This is why we are particularly interested in studying $\mathcal{N}=2$ super Yang-Mills theory. For a more technical discussion of some of the details mentioned above regarding supersymmetric theories, nonrenormalization theorems etc., see [23], [24], [25].

In 1994, Seiberg and Witten proposed an exact solution for certain properties of $\mathcal{N}=2$ supersymmetric gauge theories with the gauge group $SU(2)$. The properties determined by the solution are exact at all perturbative and nonpertubative orders, and at strong coupling. In their subsequent paper, Seiberg and Witten **??** generalized their results to $\mathcal{N}=2$ theories with fundamental matter hypermuliplets. There are also generalizations to from the gauge group $SU(2)$ to $SU(N)$ and to all classical and exceptional groups, most of which are addressed in [4] and [6].

We will first review extended supersymmetry and derive the BPS bound on supermultiplets, and review some important properties of $\mathcal{N}=2$ gauge theories, including the holomorphic prepotential, global symmetries of the theory, and the low energy effective theory that we are ultimately interested in. We will be considering supersymmetric gauge theories with the gauge group $SU(2)$ without matter hypermultiplets. Then following the approach of Seiberg and Witten, we will discuss the structure of the moduli space and the existence of singularities on the moduli space of the effective theory. We will discuss the possibility of dyonic states on the moduli space and a dual description of the theory. Then in constructing the relevant monodromies we will be able to determine the exact solution from a family of elliptic curves. We will then present a similar approach of finding the exact solution using hypergeometric functions. Then we will consider a third approach of looking at $\mathcal{N}=2$ on the world volume of a brane construction and then lift to M-theory and study the curves on the resulting surface. In presenting the Seiberg-Witten solution we will encounter some fascinating phenomena that occur at strong coupling. We will also see evidence of an electric-magnetic duality in $\mathcal{N} = 2$ that occurs between regions of strong and weak coupling. Our discussion relies heavily on the review by Alvarez-Gaumé and Hassan [3] as well as on the original paper by Seiberg and Witten [1].

2. $\mathcal{N} = 2$ Supersymmetric Gauge Theories

In this chapter we will review some aspects of supersymmetric gauge theories in four dimensions which will be relevant in our discussion of Seiberg-Witten theory. Appendix A on conventions and notations describes the supersymmetry conventions we use in this thesis. We will start by discussing representations of the supersymmetry algebra with and without central charges and then find the lower bound on the mass of supermultiplets. Then we will discuss the relevant properties of $\mathcal{N} = 2$ supersymmetric theories, including the holomorphic prepotential, global symmetries of the theory, and the structure of the moduli space of vacua.

2.1 Supersymmetric Algebras

In 1971, it was shown that the possible symmetries of a quantum field theory in four dimensions need not only consist of the Poincaré group and internal symmetries, but by allowing both commuting and anticommunting generators, can admit supersymmetry as a nontrivial extension. The supersymmetric algebra without admitting any central charges is written as

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J \\ \{Q^I_\alpha, Q^J_\beta\} &= \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\alpha}J}\} = 0, \end{aligned} \tag{2.1}$$

where the indices I, J run over the supersymmetries $1, 2, \ldots, \mathcal{N}$

In 1975, it was shown that the supersymmetric algebra Eq. 2.1 admits a central extension

[7] which can be generalized to

$$
\begin{aligned}
\{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J \\
\{Q^I_\alpha, Q^J_\beta\} &= 2\sqrt{2}\ \epsilon_{\alpha\beta} Z^{IJ} \\
\{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= 2\sqrt{2}\ \epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{IJ},
\end{aligned}
\tag{2.2}
$$

where the Z's are the central charges, which are antisymmetric in their indices. When $\mathcal{N}$ is even we may skew-diagonalize the central charges so that they take the form $Z = \epsilon \otimes D$, where D is an $\mathcal{N}/2$ dimensional diagonal matrix. Then the index I will run over the ϵ indices $a = 1, 2$ and the D indices. Since we are only considering extended supersymmetric theories of $\mathcal{N} = 2$ the discussion of the D indices is not really relevant, but it can be shown that a chiral rotation allows one to consider just the $\mathcal{N} = 2$ supersymmtric extension anyway. Thus we find that the $\mathcal{N} = 2$ supersymmetric algebra takes the form

$$
\begin{aligned}
\{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J \\
\{Q^a_\alpha, Q^b_\beta\} &= 2\sqrt{2}\ \epsilon_{\alpha\beta}\epsilon^{ab} Z \\
\{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\alpha}b}\} &= 2\sqrt{2}\ \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab} Z^*.
\end{aligned}
\tag{2.3}
$$

2.1.1 BPS Bound on the SUSY Multiplets

We now define a general linear combination of of the supersymmetric generators Q^1 and Q^2 denoted by $\tilde{Q}$:

$$
\tilde{Q}_\alpha = \frac{1}{2}\left(Q^1_\alpha + w\,\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right), \tag{2.4}
$$

where w is a complex number with unit modulus. Assuming we are sitting in a rest frame, we calculate the algebra with the only nonvanishing anticommutators as

$$\begin{aligned}\{\tilde{Q}_\alpha, \tilde{Q}^\dagger_\gamma\} &= \frac{1}{4}\{Q^1_\alpha + w\,\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger, (Q^1)^\dagger_\gamma + w^*\,\epsilon_{\gamma\delta}Q^2_\delta\} \\ &= \frac{1}{2}\left(\sigma^0_{\alpha\gamma}M + w\sqrt{2}\delta_{\alpha\gamma}\epsilon_{21}Z^* + w^*\sqrt{2}\delta_{\alpha\gamma}\epsilon^{12}Z + ww^*\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\sigma^0_{\beta\delta}M\right) \\ &= \delta_{\alpha\gamma}M + \frac{\sqrt{2}}{2}\delta_{\alpha\gamma}\left(wZ^* + w^*Z\right) \\ &= \delta_{\alpha\gamma}\left(M + \sqrt{2}\,\mathrm{Re}\left(w^*Z\right)\right), \end{aligned} \tag{2.5}$$

where we have used the standard contraction of ϵ matrices. Since w has unit absolute value, the quantity w^*Z is simply a rotation of Z on the complex plane, and thus $|Z| \geq \mathrm{Re}\,(w^*Z) \leq -|Z|$. If α equals γ then the left-hand side of Eq. 2.5 must be a positive quantity as all physical states must have positive definite norm. Therefore we see that

$$M \geq \sqrt{2}|Z|. \tag{2.6}$$

This is known as the Bogomol'nyi-Prasad-Sommerfield bound, or BPS bound, and is an incredibly important result in supersymmetric quantum field theories, stating that the mass of a supersymmetric multiplet in any representation of the supersymmetric algebra, is bounded from below by the central charge of the algebra. The bound is trivially realized for massless states where the central charge is zero. For an $\mathcal{N} = 2$ multiplet where the bound Eq. 2.6 is strict has $2^4 = 16$ states. This is because there are eight supercharges which can be reformulated as four creation and four annihilation operators, where the four creation operators generate 16 states from a vacuum state which is annihilated by the annihilation operators. But when the bound Eq. 2.6 is saturated the algebra is again trivially realized and the dimension of the representation is reduced, as two of the creation operators are zero and thus the multiplet contains $2^2 = 4$ states, just as with the massless multiplets. Massive representations of the $\mathcal{N} = 2$ supersymmetric algebra where the bound is saturated are called short multiplets or BPS states, and massive representations where the bound is strict are called long mulitplets. This distinction is important because for BPS states the mass and charge is determined by supersymmetry and are thus immune from perturbative and nonperturbative corrections when considering a full quantum theory. Consequentially, BPS states at weak coupling are also valid in the strong coupling regime.

2.2 $\mathcal{N} = 2$ Super Yang-Mills Theory

In rigid theories with 8 supercharges there are two allowed massless supermultiplets, vector multiplets and hypermultiplets. The vector multiplets belong to the adjoint representation of some gauge group G, and the hypermultiplets may belong to any complete representation. All supersymmetric theories with 8 supercharges are dimensional reductions of the consistent theory in the highest number of dimensions which allows for 8 supercharges, which in this case is $d = 6$. In $d = 6$ we have the chiral $\mathcal{N} = (1, 0)$ theory where vector multiplets consist of a gauge field and a chiral Weyl spinor and the hypermultiplet consists of Weyl spinor two complex scalars. Dimensional reduction to four dimensions gives us a nonchiral $\mathcal{N} = 2$ theory in $d = 4$. Thus, in $\mathcal{N} = 2$ supersymmetric theories in four dimensions the vector multiplet is made up by a gauge field, two left-handed Weyl spinors, and one complex scalar, $(A_\mu, \psi, \lambda, \phi)$. While the hypermultiplet is made up by two left-handed Weyl spinors and two complex scalars, $(\chi, \varphi, \tilde{\varphi}, \tilde{\chi})$. Each massless mulitplet has eight on-shell degrees of freedom. The R-symmetry of the theory is $SU(2) \times U(1)$. For future reference, we note that these $\mathcal{N} = 2$ multiplets may be decomposed into $\mathcal{N} = 1$ multiplets. The two relevant types of massless multiplets in $\mathcal{N} = 1$ are the vector multiplet with a vector and a left-handed Weyl spinor, $V = (A_\mu, \lambda)$ and the chiral multiplet with a left-handed Weyl spinor and a complex scalar $\Phi = (\chi, \phi)$. The hypermultiplet is comprised of two $\mathcal{N} = 1$ chiral multiplets Φ and $\tilde{\Phi}$, while the vector multiplet is comprised of an $\mathcal{N} = 1$ vector multiplet V and chiral multiplet $\mathcal{A}$.

Assuming familiarity with $\mathcal{N} = 1$ supersymmetric theories and local representations of $\mathcal{N} = 1$ supersymmetry, we start with the full $\mathcal{N} = 1$ supersymmetric Lagrangian in $\mathcal{N} = 1$ superspace coordinates

$$\mathcal{L} = \mathrm{Im}\,\mathrm{Tr}\left(\frac{\tau}{8\pi}\int d^2\theta\ W^\alpha W_\alpha\right) + \int d^4\theta\ \Phi^\dagger e^{-2V}\Phi + \int d^2\theta\ \mathcal{W} + \int d^2\bar{\theta}\ \bar{\mathcal{W}}, \tag{2.7}$$

where $\tau = \theta/2\pi + 4\pi i/g^2$ is a complex coupling constant, a packaging together of the gauge coupling and theta angle into one coupling constant. We now want to find the full $\mathcal{N} = 2$ Lagrangian for a pure gauge theory. The full $\mathcal{N} = 1$ Lagrangian contains the $\mathcal{N} = 1$ vector multiplet and chiral multiplet and thus has the same on-shell field content as the desired $\mathcal{N} = 2$ vector multiplet, but the Lagrangian does not have $\mathcal{N} = 2$ supersymmetry. The fermionic field content, λ and χ, must be symmetric and thus the superpotential $\mathcal{W}$ must be zero as it only couples to χ in the chiral multiplet, with the same line of logic we also fix the normalization between the gauge the fermionic kinetic terms, which amounts to rescaling the chiral multiplet by the coupling. The full Lagrangian for $\mathcal{N} = 2$ supersymmetric gauge theory is given in $\mathcal{N} = 1$ superspace coordinates and component expansion as

$$\begin{aligned}\mathcal{L} &= \operatorname{Im}\operatorname{Tr}\left(\frac{\tau}{8\pi}\int d^2\theta\; W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\;\Phi^\dagger e^{-2V}\Phi\right)\\ &= \frac{1}{g^2}\operatorname{Tr}\Bigg(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}[\phi,\phi^\dagger]^2\\ &\qquad - i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi - i\sqrt{2}[\lambda,\psi]\phi^\dagger - i\sqrt{2}[\bar{\lambda},\bar{\psi}]\phi\Bigg),\end{aligned} \tag{2.8}$$

where we have dropped the auxillary fields in the component expansion.

2.3 The Holomorphic Prepotential

Given the full $\mathcal{N} = 2$ supersymmetric Lagrangian 2.8 an incredibly important result is that it can be written in terms of a single holomorphic function $\mathcal{F}$, called the prepotential. We will sketch the derivation of this following Section 2.6 in [3]. Local representations of $\mathcal{N} = 2$ supersymmetry are written in terms of two additional coordinates $(\vartheta, \bar{\vartheta})$, on top of the $N = 1$ superspace coordinates, and thus we may write a general $\mathcal{N} = 2$ superfield as $F(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})$. The on-shell field content of the $\mathcal{N} = 2$ vector multiplet is $(\phi, \psi, \lambda, A_\mu)$ and thus we seek to find a superfield with this field content. In the same manner we define an $\mathcal{N} = 1$ chiral superfield, it is straightforward to define an $\mathcal{N} = 2$ chiral superfield Ψ as having to satisfy the constraints $\bar{\mathcal{D}}^{(\theta)}_{\dot{\alpha}}\Psi = 0$ and $\bar{\mathcal{D}}^{(\vartheta)}_{\dot{\alpha}}\Psi = 0$, where $\bar{\mathcal{D}}^{(\theta)}_{\dot{\alpha}}$ is the standard supercovariant derivative in $\mathcal{N} = 1$ superspace and $\bar{\mathcal{D}}^{(\vartheta)}_{\dot{\alpha}}$ is the supercovariant derivative over our additional fermionic coordinates

$$\bar{\mathcal{D}}^{(\theta)}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \qquad \bar{\mathcal{D}}^{(\vartheta)}_{\dot\alpha} = -\frac{\partial}{\partial\bar\vartheta^{\dot\alpha}} - i\vartheta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \tag{2.9}$$

Recall that the general form of an $\mathcal{N}=1$ chiral superfield Φ is given as a function of (y,θ), where $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, which we expand in powers of θ as

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{2.10}$$

giving us the expected field content (ϕ,ψ) and an auxiliary field F required for off-shell closure. Now for our $\mathcal{N}=2$ chiral superfield we consider the general form as a function of $(\tilde y,\theta,\vartheta)$, where $\tilde y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\vartheta\sigma^\mu\bar\vartheta$, and expand in powers of ϑ and find that

$$\Psi(\tilde y,\theta,\vartheta) = \Phi(\tilde y,\theta) + \sqrt{2}\vartheta^\alpha W_\alpha(\tilde y,\theta) + \vartheta\vartheta G(\tilde y,\theta), \tag{2.11}$$

where the zeroth-order term in ϑ is just be the $\mathcal{N}=1$ chiral superfield Eq. 2.10 as a function of $\tilde y$ and θ. The first-order term is the nonabelian gauge field strength for the $\mathcal{N}=1$ vector superfield, which when written in superspace notation

$$W_\alpha = \frac{1}{8g}\bar{\mathcal{D}}^2\left(e^{2gV}\mathcal{D}_\alpha e^{-2gV}\right), \tag{2.12}$$

where the $\mathcal{D}$'s are the standard supercovariant derivatives and V is the $N=1$ vector superfield. The subtlety arises with the third term which we will simply quote as being

$$G(\tilde y,\theta) = -\frac{1}{2}\int d^2\bar\theta\ [\Phi(\upsilon,\theta,\bar\theta)]^\dagger \exp[-2gV(\upsilon,\theta,\bar\theta)], \tag{2.13}$$

where $\upsilon = \tilde y - i\theta\sigma\bar\theta$. This constraint in the $\mathcal{N}=2$ superfield is necessary in order to eliminate certain unphysical degrees of freedom and is obtained from reality conditions and chirality considerations on the general $\mathcal{N}=2$ superfield [8].

Thus we have found from the expansion of Ψ that the field content is exactly what we would expect for a $\mathcal{N}=2$ vector multiplet, an $\mathcal{N}=1$ chiral field and an $\mathcal{N}=1$ vector super field, along with a component to eliminate unwanted degrees of freedom required for off-shell closure. We have been referring to Ψ as an $\mathcal{N}=2$ chiral superfield given the constraints we first imposed on it, but as we see it represents want we have been referring to as the $\mathcal{N}=2$ vector multiplet (which is sometimes referred to as the chiral multiplet). Thus the full general Lagrangian 2.8 for $N=2$ supersymmetry can be written in terms of Ψ as

$$\mathcal{L} = \mathrm{Im\,Tr}\left(\frac{\tau}{4\pi}\int d^2\theta d^2\vartheta \frac{1}{2}\Psi^2\right). \tag{2.14}$$

It is important to note that this Lagrangian only depends on Ψ and not $\Psi^\dagger$. So now we can construct the most general Lagrangian for $\mathcal{N}=2$ supersymmetric Yang-Mills theory with some function $\mathcal{F}(\Psi)$ of the $\mathcal{N}=2$ vector superfield,

$$\mathcal{L} = \mathrm{Im\,Tr}\left(\frac{1}{4\pi}\int d^2\theta d^2\vartheta \mathcal{F}(\Psi)\right), \tag{2.15}$$

which we can expand as

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\left(\int d^4\theta\ (\Phi^\dagger e^{2gV})^i\ \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^i} + \frac{1}{2}\int d^2\theta\ \frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^i\partial\Phi^j}W^{\alpha\,i}\,W^j_\alpha\right). \tag{2.16}$$

Thus $\mathcal{N}=2$ gauge theory can be completely expressed in terms of this holomorphic function $\mathcal{F}$ which we will call the prepotential. The exact determination of this function, and thus the solution to the low energy effective theory, is the purpose of Seiberg-Witten theory.

2.4 Global Symmetries in $\mathcal{N}=2$ Theories

In four dimensions, supersymmetric gauge theories with $\mathcal{N}$-extended supersymmetric algebras have a global symmetry corresponding to unitary transformations of the supercharges among themselves. This global symmetry is called the R-symmetry. For $\mathcal{N}=1$, a $U(1)$ R-symmetry acts on the supercharges as $Q\to e^{-i\alpha}Q$, which when considered as a transformation on the superspace coordinates, acts as $\theta\to e^{i\alpha}\theta$ and $\bar\theta\to e^{-i\alpha}\bar\theta$. For $\mathcal{N}=2$ this R-symmetry becomes $SU(2)\times U(1)_\mathcal{R}$, as we have a $U(1)$ symmetry which acts on the $\mathcal{N}=2$ superspace coordinates $\theta^i,\bar\theta_i$, just as for $\mathcal{N}=1$, which, to avoid confusion, we label as $U(1)_\mathcal{R}$. Additionally, we have an $SU(2)$ R-symmetry which rotates the index of the supercharges Q^i, in other words it acts on the two supercharges of a given chirality.

It is useful to note how fields in supermultiplets transform under these R-symmetries. The $SU(2)$ component of the R-symmetry will act on the different helicities in the multiplets, in other words the different helicity fields will fit into separate representations of $SU(2)$. In the $\mathcal{N}=2$ vector multiplet, the gauginos λ and ψ form a doublet while ϕ and A_μ are singlets under $SU(2)$. Similarly, for a hypermultiplet the scalars transform as a doublet while the spinors transform as singlets under $SU(2)$.

Under $U(1)_\mathcal{R}$ the fields in the multiplets will pick up a phase determined by their charge under $U(1)_\mathcal{R}$. Explicitly, if a field has a charge q under $U(1)_\mathcal{R}$ then when rotated by an angle α the field will pick up phase $e^{iq\alpha}$. We list the $U(1)_\mathcal{R}$ transformations of the component fields in the $\mathcal{N}=2$ vector and hyper multiplets,

$$\begin{aligned} A_\mu &\to A_\mu & \chi &\to e^{-i\alpha}\chi \\ (\psi,\lambda) &\to e^{i\alpha}(\psi,\lambda) & (\varphi,\tilde{\varphi}) &\to (\varphi,\tilde{\varphi}) \\ \phi &\to e^{2i\alpha}\phi & \tilde{\chi} &\to e^{-i\alpha}\tilde{\chi}. \end{aligned} \tag{2.17}$$

As we have stated, classically our theory has a global $SU(2)\times U(1)_\mathcal{R}$ symmetry. But once we go to the quantum theory, the $U(1)_\mathcal{R}$ symmetry is broken to a discrete subgroup by the standard chiral anomaly, which makes sense given that this global symmetry effects phase rotations of chiral fermions. A better way of seeing this is by combining the two-component Weyl spinors ψ and $\bar{\lambda}$ into the usual four-component Dirac spinor ψ_D and noting that under $U(1)_\mathcal{R}$ the spinor ψ_D transforms as $\psi_D \to e^{i\alpha\gamma_5}\psi_D$. Thus it should be clear that $U(1)_\mathcal{R}$ is a chiral symmetry and as we will see is broken by the triangle anomaly. For $SU(N)$ gauge theories we have

$$\partial_\mu J_5^\mu = -\frac{N}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2.18}$$

which implies that under a $U(1)_\mathcal{R}$ rotation by an angle α, the change in the effective action is given by

$$\delta\mathcal{L}_{\text{eff}} = -\frac{4\alpha N}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.19}$$

Since the integral $(1/32\pi^2)\int F\wedge F$ must give integer values, we can determine that in order for this chiral phase rotation to be a symmetry of the theory the angle α must take on values

$\alpha = \pi n/2N$ where $n \in \mathbb{Z}$. So the $U(1)_\mathcal{R}$ phase becomes $e^{i\alpha} = e^{2\pi i(n/4N)}$ where $n = 1, \ldots, 4N$. Thus we see that the chiral anomaly breaks $U(1)_\mathcal{R}$ to $\mathbb{Z}_{4N}$.

Note that if we are considering a theory coupled to matter, N_f fermions in the fundamental representation, as is the case when we consider $\mathcal{N} = 2$ gauge theory coupled to hypermultiplets, then the prefactor in the shift of the effective Lagrangian under $U(1)_\mathcal{R}$ becomes $\alpha(2N - N_f)/16\pi^2$. So in the presence of matter the chiral anomaly breaks $U(1)_\mathcal{R}$ to $\mathbb{Z}_{4N-2N_f}$.

We found that the anomaly breaks $U(1)_\mathcal{R} \to \mathbb{Z}_{4N}$, so naively we can conclude that the global symmetry group is $SU(2) \times \mathbb{Z}_{4N}$. But noting that the center of $SU(2)$, acting on the gauginos as $(\psi, \lambda) \to e^{i\pi}(\psi, \lambda)$, is also contained in $\mathbb{Z}_{4N}$, when $n = 2N$, we mod out a $\mathbb{Z}_2$ to cure the double counting. This leaves us with an unbroken R-symmetry group of $(SU(2) \times \mathbb{Z}_{4N})/\mathbb{Z}_2$.

The moduli space of vacua is parametrized by nonzero values of ϕ, or some function thereof, and as ϕ carries as R-charge under $U(1)_\mathcal{R}$ this further breaks the full R-symmetry. However, the functions of ϕ which parametrize the vacuum depend on the gauge group $SU(N)$ and thus change how $\mathbb{Z}_{4N}$ is further broken. For $G = SU(2)$, the vacuum is specified by nonzero values of $\mathrm{Tr}\,\phi^2$; we simply state this for now but will discuss these potential terms more extensively in the next section. Recalling from Eq. 2.17 how ϕ transforms under $U(1)_\mathcal{R}$, we see that ϕ^2 has an R-charge of 4, picking up a phase of $e^{4i\alpha} = e^{2\pi i(n/2)}$ upon rotation. This spontaneously breaks $\mathbb{Z}_8 \to \mathbb{Z}_4$ as the symmetry is now generated by values of $n = 2, 4, 6, 8$. All the leftover elements of $\mathbb{Z}_8$ which did not give rise to a vacuum symmetry we be of some relevance later when discussing the singularity structure of the moduli space. These elements act on $\phi^2 \to -\phi^2$, acting as $\mathbb{Z}_2$. Thus we have determined the global symmetry group of pure $\mathcal{N} = 2$ $SU(2)$ gauge theory to be $(SU(2) \times \mathbb{Z}_4)/\mathbb{Z}_2$.

As we mentioned, the breaking of R-symmetry by the vacuum depends on the gauge group. For $SU(3)$ the moduli space is parametrized by $\mathrm{Tr}\,\phi^2$ and $\mathrm{Tr}\,\phi^3$, which pick up phases of $e^{2\pi i(n/3)}$ and $e^{2\pi i(n/2)}$, respectively. The R-symmetry is now generated by the values $n = 6, 12$, spontaneously breaking $\mathbb{Z}_{12} \to \mathbb{Z}_2$, making the overall global symmetry structure for $\mathcal{N} = 2$ $SU(3)$ gauge theory $(SU(2) \times \mathbb{Z}_2)/\mathbb{Z}_2$. For $G = SU(N > 3)$ no subgroup of the $\mathbb{Z}_{4N}$ remains invariant under the R-symmetry breaking by the moduli space.

2.5 Low Energy Effective Action

If we are interested in the behavior of a field theory at energies lower than some cutoff Λ which is lower than the characteristic energy scale of the theory, the mass of the lightest massive field for instance, we want to find an effective low-energy theory. We do this using the Wilsonian approach of finding an effective action with the lower cutoff Λ with exactly the same low energy physics. We do this by course graining the theory, integrating out higher-momentum modes, $k > \Lambda$, to obtain the effective field theory and using it as if it were the fundamental theory. At energies below the cutoff, we will not encounter any on-shell massive states, thus we obtain the effective theory by integrating out all massive fields and all massless excitations about the cutoff. As straightforward as this might sound, it is an incredibly complicated procedure and cannot be done explicitly. The situation is made simpler as the supersymmetric actions are constrained by holomorphicity requirements in $\mathcal{N} = 2$ supersymmetry.

As we already found earlier, the low energy effective action is completely determined by a holomorphic prepotential $\mathcal{F}$. In $\mathcal{N} = 1$ superspace coordinates the low energy effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left(\int d^4\theta \; \frac{\partial \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}_i}\,\bar{\mathcal{A}}^i + \int d^2\theta \; \frac{1}{2}\frac{\partial^2 \mathcal{F}(\mathcal{A})}{\partial \mathcal{A}_i \mathcal{A}_j}\,W^i_\alpha W^{j\alpha}\right). \tag{2.20}$$

As we will be considering the case where the $SU(2)$ gauge group is broken to $U(1)$ it should be clear that the low energy effective Lagrangian for the broken $U(1)$ theory is

$$\mathcal{L}^{U(1)}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left(\int d^4\theta \; \frac{\partial \mathcal{F}}{\partial \mathcal{A}}\,\bar{\mathcal{A}} + \int d^2\theta \; \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial \mathcal{A}^2}\,W_\alpha W^{\alpha}\right). \tag{2.21}$$

A quick note on notation, before we labeled $\mathcal{N} = 1$ chiral multiplets as Φ. But here and in the rest of this thesis, when discussing the chiral multiplet in the $\mathcal{N} = 2$ gauge theory, we will denote it as $\mathcal{A}$; labeling the Coulomb moduli as a should help make this intuitive. We will also try and avoid confusion by always labeling the gauge fields A_μ as indexed objects.

The kinetic term for the scalar fields is called the Kähler potential, which we see in the $\mathcal{N} = 2$ action Eq. 2.20 is the first term. Thus the Kähler potential can be written in terms of the holomorphic prepotential $\mathcal{F}$ as

$$K\left(\mathcal{A},\bar{\mathcal{A}}\right) = \mathrm{Im}\left(\bar{\mathcal{A}}_i \frac{\partial \mathcal{F}}{\partial \mathcal{A}_i}\right) \tag{2.22}$$

Classically, the prepotential is given by $\mathcal{F} = \frac{1}{2}\tau_{cl}\mathcal{A}^2$, which we saw when we derived it from the action. The full form of the perturbative part of prepotential to one-loop was first derived in [10]. We can determine the prepotential to one-loop order by requiring that the action Eq. 2.21 transforms under a $U(1)_{\mathcal{R}}$ rotation by an angle α as Eq. 2.19. In doing so one finds how the second-derivative of the prepotential transforms, taking an infinitesimally small rotation and integrating we arrive at the one-loop expression for the prepotential. Alternatively, we can find the same expression by integrating the one-loop expression for the β-function. Given the holomorphicity restrictions on $\mathcal{N} = 2$ theories, the one-loop expression is perturbatively exact. The prepotential also receives nonperturbative corrections from instanton effects. The form of the k-instanton factors was determined by Seiberg in [10] and the proportionality of the k-instanton correction to the chiral multiplet $\mathcal{A}$ was determined by Seiberg in [11]. We will simply quote the results; the full prepotential, including one-loop and instanton corrections, is written as

$$\mathcal{F} = \frac{i}{2\pi}\mathcal{A}^2 \log\frac{\mathcal{A}^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k\, \mathcal{A}^2 \left(\frac{\Lambda}{\mathcal{A}}\right)^{4k}. \tag{2.23}$$

The determination of the exact form of $\mathcal{F}$ by an alternative approach is the focus of the work by Seiberg and Witten [1] and of this chapter. The $k = 1$ instanton contribution was calculated by Seiberg in [10] and was found to be nonzero. Since the appearance of Seiberg-Witten theory there have been a number of successful methods of calculating higher k-instanton contributions. The works by Nekrasov [12] and Nekrasov and Shadchin [13] did so by determining the Seiberg-Witten prepotential by finding the instanton partition functions of the $\mathcal{N} = 2$ theories by introducing a special deformation called the Ω-background.

We are now in a position where we can outline the goal for the rest of this dissertation. We have established that determination of the prepotential amounts to knowing the low energy effective theory. We want to determine the prepotential following the work of Seiberg and Witten [1], where instead of a brute force approach we find the solution by exploiting certain insights into the theory. This is done by understanding the structure of the moduli space and calculating the Coulomb moduli a and a dual variable a_D in the dual description of the theory, as functions of u, our complex coordinate on the moduli space. The prepotential can then be found from the definition of $a_D = \partial\mathcal{F}/\partial a$. This approach will require significant discussion of properties of the moduli space, which is our next task.

3. Structure of the Moduli Space of Vacua

In the last chapter we found that the low energy effective action could be completely expressed at a single holomorphic function called the prepotential. Thus, if we are interested in finding exact solutions in $\mathcal{N} = 2$ gauge theory this amounts to exactly determining the prepotential. As we will see in this chapter, the moduli space of vacua has a highly nontrivial structure and that the intricacies of this structure can give us valuable insights into the theory and determining the prepotential.

3.1 Coordinates on the Moduli Space

As we discussed in the previous chapter, the nonzero ϕ values of the moduli space spontaneously break the R-symmetry of the supersymmetric theory. Furthermore, these vacuum expectation values also break the gauge symmetry of the $\mathcal{N} = 2$ super Yang-Mills theory. The full component expansion of the pure $\mathcal{N} = 2$ SYM Lagrangian was given in Eq. 2.8, where we see that the classical potential of the theory is

$$V(\phi) = \frac{1}{2g^2} \mathrm{Tr}\,[\phi, \phi^\dagger]^2, \tag{3.1}$$

and thus the vacuum solutions of need not require that ϕ vanish, but just that ϕ and $\phi^\dagger$ commute. The classical theory therefore has a space of vacuum configurations. For $SU(2)$, we have the generators T^1, T^2, and T^3, where $T^i = \frac{1}{2}\sigma^i$, and we can choose the vacuum to lie in the direction of the third generator T^3 as one can always make an $SU(2)$ gauge transformation to give the vacuum expectation value (vev) to the third component. For a nonzero vev, T^3 remains the unbroken generator while the other two are broken. The stability subgroup of the spontaneously broken $SU(2)$ is $U(1)$ as one of the gauge bosons remains massless while the other two become massive.

Thus moving along the flat direction $V(\phi) = 0$, up to a gauge transformation the scalar field ϕ picks up the vev $\phi = \frac{1}{2}a\sigma^3$, where a is a complex parameter spanning the moduli space. More specifically, a is the vacuum expectation value of the vector multiplet $\mathcal{A}$. Transformations from the Weyl group, rotations around the unbroken directions of $SU(2)$, can still transform $a \to -a$. Thus the gauge invariant quantity that parametrizes inequivalent vacua is $\frac{1}{2}a^2$ or $\mathrm{Tr}\,\phi^2$, which are the same as up to this point our treatment of ϕ has been classical. When we consider the full quantum theory, we want to parametrize the moduli space with the vacuum expectation value of Weyl invariants. We now define the gauge invariant quantity

$$u = \langle \mathrm{Tr}\,\phi^2 \rangle \,, \tag{3.2}$$

which in the classical limit reduces to $u = \frac{1}{2}a^2$. The complex quantity u labels inequivalent vacua and thus should be thought of as a complex coordinate on the moduli space of the $\mathcal{N} = 2$ gauge theory. Conversely, the moduli space can be thought of as a complex u-plane. We will see later that the moduli space has highly nontrivial structure; we already mentioned the global symmetries that manifest on the moduli space, but as we will see the moduli space has singularities, the behavior of which we help us solve the theory.

3.1.1 Metric on the Moduli Space

We gave the Kähler potential in the $\mathcal{N} = 2$ low energy effective action Eq. 2.20 in terms of $\mathcal{F}$ as $K = \mathrm{Im}\,(\bar{A}_i\ \partial\mathcal{F}/\partial A_i)$. It should not be too surprising that the Kähler potential is referred to as such because the moduli space of vacua for supersymmetric theories is a Kähler manifold. See Appendix B for a brief discussion of the geometry of Kähler manifolds. It is a fact that for Kähler manifolds the metric spanning the moduli space is the second derivative of the Kähler potential. As a_i span the Coulomb branch, the moduli space of vector multiplets, as a_i are the VEV's of $\mathcal{A}_i$, then the metric on the moduli space is

$$ds^2 = \frac{\partial^2 K}{\partial a_i \partial \bar{a}_j} da_i d\bar{a}_j = \frac{\partial^2}{\partial a_i \partial \bar{a}_j}\left(\mathrm{Im}\left(\frac{\partial \mathcal{F}}{\partial a_k}\bar{a}_k\right)\right) da_i d\bar{a}_j = \mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}\, da_i d\bar{a}_j. \tag{3.3}$$

When we discuss the monodromies on the moduli space we will generally be interested in the broken $U(1)$ theory with the low energy effective action Eq. 2.21, it should follow that the metric on the moduli space of the broken theory is

$$ds^2 = \mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial a^2}\, da\, d\bar{a}. \tag{3.4}$$

On the basis of $\mathcal{N}=2$ supersymmetry we can see that the metric here is the same as the full gauge coupling in the $\mathcal{N}=2$ theory. Recall that the Lagrangian for the $\mathcal{N}=1$ vector multiplet is

$$\mathcal{L} = \mathrm{Im}\,\mathrm{Tr}\left(\frac{\tau}{8\pi}\int d^2\theta\, W^\alpha W_\alpha\right). \tag{3.5}$$

Comparing this Lagrangian to the second term in the low energy effective Lagrangian 2.21 we see that

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2}, \tag{3.6}$$

and thus that the metric 3.4 can be written as $ds^2 = \mathrm{Im}\,(\tau)\, da\, d\bar{a}$.

3.2 Existence of Singularities

Before diving fully into the discussion of monodromies and how we can use the structure of the moduli space to fully determine the holomorphic prepotential, we will first try and illustrate that the moduli space has some nontrivial geometry and the existence of singularities.

We consider a large a limit corresponding to $a \gg \Lambda$, which we will call the semiclassical or weak coupling limit. We will discuss later in this section why the weakly coupled regime corresponds to taking a to be large. At large a the holomorphic prepotential $\mathcal{F}$, given exactly by one-loop perturbative and nonpertubative corrections in Eq. 2.23, becomes dominated by perturbative effects. Thus at large a we may take

$$\mathcal{F}(a) = \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2}, \tag{3.7}$$

which is perturbatively exact. In the semiclassical limit the metric on the moduli space is easily calculated. Keeping in mind the prepotential in our limit is given by Eq. 3.7, we find the metric Eq. 3.4 to be

$$ds^2 = \operatorname{Im} \frac{i}{\pi} \left(2 \log \left| \frac{a}{\Lambda} \right| + 3 \right) da\, d\bar{a} = \left(\frac{2}{\pi} \log \left| \frac{a}{\Lambda} \right| + \frac{3}{\pi} \right) da\, d\bar{a}. \tag{3.8}$$

It is a fact that the metric must be positive definite and we see that the metric given above is positive in the region of large $|a|$, where. But at smaller values of $|a|$, at approximately $\frac{1}{5}\Lambda$, the expression becomes negative. This tells us that in the weakly coupled region the expression for the metric Eq. 3.8 cannot be the correct description of the entirety of the moduli space. The fact that we mentioned considering small values of $|a|$ in a large a limit makes this argument seem less convincing. So let's consider a more general argument. Since $\mathcal{F}$ is a holomorphic function then $\operatorname{Im} \partial^2 \mathcal{F}/\partial a^2$ is a harmonic function, and harmonic functions cannot have a global minimum. So if indeed the metric is globally defined then it can not be positive definite. Since we know that the metric is positive definite and analogously the kinetic energy must be positive, means that the metric is only defined locally. This line of reasoning would imply that the moduli space has some highly nontrivial structure and contains singularities. There is not much more we can say at this point before diving into the necessary technical discussion, but we can note that if there are singularities on the moduli space that they must come in pairs. Recall in our discussion of global symmetries that since the moduli carries an R-charge, the $\mathbb{Z}_8$ is broken to $\mathbb{Z}_4$. The extraneous elements of Z_8 which did not preserve the global symmetry of the vacuum act by flipping the sign $\phi \to -\phi$, which acts as an extra $\mathbb{Z}_2$. This additional global symmetry implies that if there are singularities, that they come in mirror pairs on the moduli space.

3.2.1 Physical Interpretation of the Singularities

We have seen that the moduli space contains singularities and that understanding their structure might help us solve the theory. But first let's switch the discussion to understanding what is happening physically, as in how these singularities are arising. In general, moduli spaces may become singular at points where extraneous fields become massless. The massive multiplets correspond to nontrivial regions of the moduli space where the masses of the multiplets depend on the coordinate of the moduli space u. Thus it may be the case that for certain points on the moduli space these multiplets can become massless. In finding our low energy effective theory we integrated out all higher-momentum modes which included all on-shell massive states. At the points where the multiplets become massless, we will have integrated out the states giving rise to singularities on the moduli space. The structure of the singularity, or more relevantly the monodromy associated with the singularity, depends on the properties of the multiplet which becomes massless.

One could imagine that there may be points where the broken gauge symmetry is enhanced to the full unbroken nonabelian symmetry at some $u = \langle \mathrm{Tr}\, \phi^2 \rangle \neq 0$ where there are additional gauge fields becoming massless, and thus the scalar vev's in the vector multiplet would be contributing to the singularity. Such a phenomenon is known to occur in $\mathcal{N} = 1$ supersymmetric theories [11]. In $\mathcal{N} = 2$ theories this happens in the classical picture where at $u = 0$ the full $SU(N)$ gauge symmetry is restored. So one could image that in the quantum theory perturbative or nonperturbative corrections could shift the point where the gauge symmetry is enhanced to a nonzero u. But Seiberg and Witten [1] argued that the singularities could not arise in this way following from the behavior of the theory in the infrared.

We mentioned before that the $\mathbb{Z}_2$ symmetry of the moduli space implied that the singularities would come in pairs around $u = 0$. Now assuming that the singularities arise from an unbroken nonabelian gauge symmetry, we would expect that in the infrared the theory would be conformal. Since conformal theories do not have any dimensionful parameters, $\langle \mathrm{Tr}\, \phi^2 \rangle$ would have to be a dimension zero operator, and in a unitary theory the only operator that has a scaling dimension of zero is the identity operator. But as the identity is even under

$\mathbb{Z}_2$ and $\langle \mathrm{Tr}\, \phi^2 \rangle$ is odd, the two operators cannot mix.

Furthermore, Seiberg and Witten argued that a conformally invariant point would be fairly unlikely as such a point would have to be invariant under the full superconformal algebra, which would mean the standard chiral anomaly in the $U(1)$ R-symmetry would somehow have to disappear, along with other problems superconformal invariance would imply in the global symmetry structure of the theory.

Thus, we must conclude that the singularities are caused by multiplets with spin less than one-half, more specifically they must be massive multiplets that become massless at particular points on the moduli space. Due to $\mathcal{N} = 2$ supersymmetry, our options are fairly limited; the only particles that can exist in the theory with spin $\leq 1/2$ are hypermultiplets. This raises a few questions as not only did we start out with a theory with only vector multiplets but elementary hypermultiplets do not fit the description of what we are looking for. As no elementary particles exist, we must assume that they are composite objects, and such composite particles do exist. The only such hypermultiplets in $\mathcal{N} = 2$ theories are heavy solitonic objects which can carry both electric and magnetic charge, called dyons. These composite particles are very heavy in the weak coupling limit and thus do not play a role in the low energy effective theory. But as Seiberg and Witten [1] showed, the singularities that appear in $\mathcal{N} = 2$ theories arise from these dyonic objects becoming massless at points on the moduli space.

3.3 Monopoles and Dyons

We mentioned that the singularities on the moduli space are caused by composite objects called dyons which carry both electric and magnetic charge. The dyonic states are hypermultiplets in the $\mathcal{N} = 2$ theory and thus short multiplets, meaning they saturate the BPS bound on the masses of multiplets $M \geq \sqrt{2}|Z|$, where Z is the central charge of the $\mathcal{N} = 2$ supersymmetric algebra.

We will simply quote some of the results regarding solitonic objects such as monopoles and dyons in the classical theory. Keep in mind that an in depth treatment of the topic requires discussion of 't Hooft-Polyakov monopoles. A good review with Seiberg-Witten theory in mind is given in [3]. We start with a discussion of the mass bound on monopoles. A monopole is an example of a soliton, a time-independent, finite energy solution to the classical field equations of some theory. The lower bound on the mass for a monopole is

$$M \geq \sqrt{2} n_m \frac{4\pi}{g^2} a, \tag{3.9}$$

where n_m is an integer call the winding number of magnetic charge. The Dirac quantization says that magentic and electric charges must be quantized as integer multiples of the respective charge. More specifically, as a monopole is a soliton we can think of n_m as the degree of the map from the sphere at spatial infinity S^2_∞ to the vacuum manifold, and from homotopy considerations n_m can be thought of as a winding number of the vacuum configuration. The above bound, which is called the Bogomol'nyi bound, and the 't Hooft-Polyakov monopole saturates the bound, $M = \sqrt{2} n_m \frac{4\pi}{g^2} a$. We note that the mass of the monopole is inversely proportional to the coupling g^2. When we are considering the low energy physics in the semiclassical limit, where the coupling is weak, the monopole becomes very massive and thus becomes irrelevant. Conversely, even though this is the bound on the classical monopole it still seems to suggest when considering the strong coupling regime the monopole becomes light and plays an important role.

The equivalent bound on the mass of a particle that carries both electric and magnetic charge is $M \geq \sqrt{2}|Z|$, where

$$Z = n_e a + n_m \tau_{cl} a, \tag{3.10}$$

where the classical complex coupling is $\tau_{cl} = \theta/2\pi + 4\pi i/g^2$ and $n_e, n_m \in \mathbb{Z}$. It is not a coincidence that this bound resembles the BPS bound on supersymmetric multiplets we found earlier. It turns out that in supersymmetric theories, solitons generate a central extension in the algebra, in calculating the anticomutation relations for the supercharges there are surface terms that are usually neglected, but in the presence of solitonic solutions these surface terms are nontrivial and give us a central charge. The end result of this discussion is that the classical solitonic solution that saturates the bound above appears in the quantum $\mathcal{N} = 2$ theory as a composite object which saturates the BPS bound.

We now turn the discussion to the full $\mathcal{N} = 2$ theory, where the bound on dyon mass is altered slightly. Suppose we start with an $\mathcal{N}=2$ theory with matter hypermultiplets, these multiplets acquire mass when the Higgs vev $a \neq 0$. The coupling between the $\mathcal{N}=1$ chiral multiplets in the $\mathcal{N}=2$ hypermultiplet, Φ and $\tilde{\Phi}$, and the $\mathcal{N}=1$ chiral multiplet $\mathcal{A}$ in the vector multiplet is fixed by $\mathcal{N}=2$ supersymmetry, so if the matter multiplet is carrying an electric charge n_e, then the coupling is fixed as $\sqrt{2}n_e\mathcal{A}\Phi\tilde{\Phi}$ as determined in [9]. From this we see that the central charge should be $Z = an_e$. Equivalently, if we considering the coupling of a magentic monopole with charge n_m to the vector multiplet we would find that $Z = n_m a_D$. Thus we determine that the exact mass spectrum for dyons is the saturation of the bound

$$M = \sqrt{2}|Z| \qquad \text{where} \qquad Z = an_e + a_D n_m, \tag{3.11}$$

and since the bound is saturated we claim that it should protected from corrections and is exact at strong coupling. We also note that there is an electric-magnetic duality as the exchange of a and a_D along with n_e and n_m leaves the spectrum invariant.

3.4 Monodromies on the Moduli Space of Vacua

We have just discussed the structure of the moduli space of the $\mathcal{N}=2$ theory. In our discussion we noted that understanding the transformations of a and a_D around the singularities on the moduli space, or monodromies, could help us determine the exact form of $\mathcal{F}$ and thus solve the theory. First we will work through a few relevant issues which will elucidate the eventual discussion.

It is claimed in the work of Seiberg and Witten [1] that the semiclassical limit where the theory becomes weakly coupled corresponds to taking the limit as a becomes large. Let's briefly discuss why this is the case. Recall we defined the energy scale a to be the Higgs scale at which the gauge symmetry in the theory breaks down to $U(1)$. Additionally, consider an energy scale Λ where the coupling becomes strong so at energies $< \Lambda$ the theory is strongly coupled, and suppose the large a limit is simply defined as the Higgs scale being greater than the strong coupling regime $a \gg \Lambda$. At energies higher than a the theory has an unbroken nonabelian gauge symmetry and as there are no matter couplings the beta function must be negative, and thus the theory is asymptotically free. At energies lower than a the theory has a broken $U(1)$ gauge symmetry, which means the beta function must either be zero or positive, or rather the coupling either does not run or the theory is free in the infrared. This means that above the Higgs scale the coupling only becomes weaker, and that below the Higgs scale only either doesn't run or becomes weaker. Thus if Λ is the scale at which the coupling becomes strong, then $a \gg \Lambda$ is the limit as the theory becomes weakly coupled.

As we did not consider a theory with matter hypermultiplets, we can assume that in the weak coupling limit there is no matter, but we cannot assume that at strong coupling the same holds true as composite particles appear, in which case the theory is that of an abelian gauge theory coupled matter.

3.4.1 The Monodromy at Infinity

In the weak coupling limit where we take $a \gg \Lambda$ the holomorphic prepotential is approximated as in Eq. 3.7, from which we find the dual Coulomb moduli to be

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \log \frac{a}{\Lambda} + \frac{ia}{\pi}. \tag{3.12}$$

In calculating the monodromy we want to make a closed loop on the complex plane at large u around the origin $u = 0$, represented by the counterclockwise contour on the complex u-plane $u \to e^{2\pi i}u$, which is equivalent to encircling the point at infinity on the Riemann sphere. We can rewrite this as $\log u \to \log u + 2\pi i$, and recalling that in the semiclassical limit $u = \frac{1}{2}a^2$. Thus for a we have that $a \to e^{i\pi}a$ or $a \to -a$, and equivalently $\log a$ gets incremented as $\log a \to \log a + i\pi$. The dual variable a_D transforms as

$$a_D \to -\frac{2ia}{\pi} \log \frac{a}{\Lambda} - \frac{ia}{\pi} + 2a = -a_D + 2a \tag{3.13}$$

and thus combining the transformations of both a and a_D into a single matrix equation we find

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}. \tag{3.14}$$

Thus the transformation around the singularity is implemented by the monodromy matrix which we will call M_∞

$$M_\infty = -T^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{3.15}$$

where T is the $SL(2,\mathbb{Z})$ translation matrix. What follows in the next subsection is a brief introduction to some relevant aspects of the group $SL(2,\mathbb{Z})$ which will play a role in our discussion of monodromies.

3.4.2 An Aside on $SL(2,\mathbb{Z})$

The group $SL(2,\mathbb{Z})$ is the special linear group of degree 2 over the integers; it is the group of 2×2 matrices with integer entries, where the matrices have unit determinant.

$$SL(2,\mathbb{Z}) = \left\{ g : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{Z} \;, \; \det g = 1 \right\}. \tag{3.16}$$

There are two matrices in $SL(2,\mathbb{Z})$ which are worth noting, the S and T matrices, which are important because one can construct any matrix in $SL(2,\mathbb{Z})$ with a finite sequence of S and T matrices and can be thought of as generators of $SL(2,\mathbb{Z})$. These matrices are

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.17}$$

The elements of $SL(2,\mathbb{Z})$ act on the complex plane by Möbius transformations:

$$z \to \frac{az+b}{cz+d}. \tag{3.18}$$

Now returning to our discussion of duality we can see the action of $SL(2,\mathbb{Z})$ on our theory. The full classical gauge coupling for $\mathcal{N}=2$ SYM is $\tau = \theta/2\pi + 4\pi i/g^2$. We see that the action of the T matrix on τ is an effective phase shift of the theta angle θ by 2π, or rather $\tau \to \tau + 1$. The action of the S matrix is a duality transformation $\tau \to -1/\tau$. We will see the importance of this S duality in the $\mathcal{N}=2$ theory. It is worth mentioning why the duality group we are considering is in fact $SL(2,\mathbb{Z})$ and not $SL(2,\mathbb{R})$, for instance. The group $SL(2,\mathbb{R})$ is generated by the same S matrix but additionally by T_b, where the top right entry of T is a real number b. Such a transformation would shift the θ angle of the theory by $2\pi b$, which would only leave the theory invariant if b were integral. We will now further explore the S duality in the $\mathcal{N}=2$ theory.

3.5 An S-Duality

Considering for a moment just the gauge field terms in the full $\mathcal{N}=2$ supersymmetric action 2.8, including the theta term we have the Lagrangian

$$\mathcal{L}_{A_\mu} = \mathrm{Im}\,\frac{\tau}{32\pi}(F + i*F)^2, \tag{3.19}$$

expanding and recalling that the coupling constant τ combines the gauge coupling and theta angle as $\tau = 4\pi i/g^2 + \theta/2\pi$ and that $(*F)^2 = -F^2$, where $*F$ is the Hodge dual of F, we can expand the Lagrangian as

$$-\mathrm{Im}\,\frac{\tau}{32\pi}(F + i*F)^2 = -\frac{1}{16\pi}\mathrm{Im}\ \tau\,(F^2 + iF*F) = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{3.20}$$

and recover the usual form of the gauge field kinetic term and theta term. Thus we start with the action

$$-\frac{1}{32\pi}\mathrm{Im}\int\tau(F + i*F)^2. \tag{3.21}$$

Usually we consider the gauge field A_μ to be our free variable, but instead we now want to regard the field strength F to be an independent variable. When we do this we must impose the usual Bianchi identity $dF=0$ as a constraint in the action, which we accomplish by introducing a Lagrange multiplier in the action. Since there are four Bianchi identities, the Lagrange multiplier field we introduce is an abelian vector field B_μ with the field strength $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The Lagrange multiplier term is now constructed by coupling the vector field to a magnetic monopole, which satisfies the constraint equation $\epsilon^{0\rho\mu\nu}\partial_\rho F_{\mu\nu} = 8\pi\delta^{(3)}$;

the Lagrange multiplier term is written as

$$
\begin{aligned}
\frac{1}{8\pi}\int B_\mu\,\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= -\frac{1}{8\pi}\int \epsilon^{\mu\nu\rho\sigma}\partial_\nu B_\mu\, F_{\rho\sigma} + \text{ total derivative} \\
&= \frac{1}{16\pi}\int \epsilon^{\mu\nu\rho\sigma}(\partial_\mu B_\nu - \partial_\nu B_\mu)\, F_{\rho\sigma} \\
&= \frac{1}{16\pi}\int \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu} F_{\rho\sigma} = \frac{1}{8\pi}\int *G\,F,
\end{aligned} \tag{3.22}
$$

where we get from the first to second lines by noting the antisymmetry in the μ and ν indices. Noting the form of the action in Eq. 3.21 we rewrite the Lagrange multiplier term as

$$
\frac{1}{8\pi}\int *G\,F = \frac{1}{16\pi}\mathrm{Im}\int (G + i*G)(F + i*F), \tag{3.23}
$$

and adding the Lagrange multiplier term to the action Eq. 3.21 and then completing the square we find

$$
\begin{aligned}
&\frac{1}{32\pi}\mathrm{Im}\int \left(-\tau(F + i*F)^2 + 2(G + i*G)(F + i*F)\right) \\
&= -\frac{1}{32\pi}\mathrm{Im}\int \left(\left(\sqrt{\tau}(F + i*F) - \frac{1}{\sqrt{\tau}}(G + i*G)\right)^2 - \frac{1}{\tau}(G + i*G)\right) \\
&= \frac{1}{32\pi}\mathrm{Im}\,\frac{1}{\tau}\int (G + i*G)
\end{aligned} \tag{3.24}
$$

where in going from the second to third lines we perform the Gaussian path integral over F which results in canceling the larger square term. Thus we have found a dual theory completely in terms of the Lagrange multiplier vector field with the exact same functional form as the original action 3.21 except the complex coupling is now $-1/\tau$.

We can repeat the same steps as above in $\mathcal{N} = 1$ superspace and find the dual form of the gauge terms in the supersymmetric action. We are interested in the transformation of the of the kinetic term for the $\mathcal{N} = 1$ vector superfield in terms of the field strength W_α, which we now will treat as an independent chiral field

$$
\frac{1}{8\pi}\mathrm{Im}\,\tau\int d^2\theta\; W^2, \tag{3.25}
$$

where here $\tau = \tau(\mathcal{A})$ is a function of chiral multiplets $\mathcal{A}$. The superfield Bianchi identity is $\mathrm{Im}\,\mathcal{D}_\alpha W^\alpha = 0$ where $\mathcal{D}_\alpha$ is the supercovariant derivative defined earlier. We introduce a Lagrange multiplier vector superfield Y with field strength Z_α. Recalling how integration by parts can be performed in superspace coordinates using the supercovariant derivative, we find the Lagrange multiplier term

$$\begin{aligned}\frac{1}{16\pi}\mathrm{Im}\int d^4\theta\; Y\,\mathcal{D}_\alpha W^\alpha &= -\frac{1}{16\pi}\mathrm{Im}\int d^4\theta\; \mathcal{D}_\alpha Y\, W^\alpha = \frac{1}{16\pi}\mathrm{Im}\int d^2\theta\; \bar{\mathcal{D}}^2(\mathcal{D}_\alpha Y\, W^\alpha) \\ &= \frac{1}{16\pi}\mathrm{Im}\int d^2\theta\; (\bar{\mathcal{D}}^2\mathcal{D}_\alpha Y)W^\alpha = -\frac{1}{4\pi}\mathrm{Im}\int d^2\theta\; Z_\alpha W^\alpha, \qquad (3.26)\end{aligned}$$

where at the beginning of the second line we used the fact that the field strength W_α is a chiral superfield $\bar{\mathcal{D}}_{\dot\alpha} W^\alpha = 0$ and the definition of the abelian field strength for $Z_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\,\mathcal{D}_\alpha Y$. Adding the term to the above action and just as before we complete the square and perform the Gaussian integral over W to find

$$\frac{1}{8\pi}\mathrm{Im}\int d^2\theta\left(\tau W^2 - 2\,Z\,W\right) = -\frac{1}{8\pi}\mathrm{Im}\,\frac{1}{\tau}\int d^2\theta\, Z^2, \qquad (3.27)$$

which is a dual action in terms of the vector superfield Y with the complex coupling $-1/\tau$.

We observed a duality in the supersymmetric action both in the Yang-Mills and superspace formalisms, which mapped a gauge field theory that couples to charges electrically to a dual gauge field theory which couples magnetically. Let us define a dual complex coupling $\tau_D = -1/\tau$. The duality is then implemented by the transformation of the coupling

$$\tau \to \tau_D = -\frac{1}{\tau}. \qquad (3.28)$$

This is exactly the action of the S matrix in $SL(2,\mathbb{Z})$. We mentioned in our discussion of $SL(2,\mathbb{Z})$ that the theory is invariant under the shift $\tau \to \tau + 1$ as such a shift corresponds to shifting the θ angle by 2π, which is implemented by the T matrix. This symmetry transformation, along with the duality transformation we found, generates the $SL(2,\mathbb{Z})$ group, which is therefore the duality group of our theory.

It is important to keep in mind that the T action is a symmetry, leaving the theory invariant, and the S action is not a symmetry but a duality transformation, giving us another description of the same theory. This duality is also the electric-magnetic duality we mentioned in our discussion of the central charge of a dyon which can be written as $Z = (n_m, n_e)\ (a_D, a)^T$. We note then that multiplying a state $(a_D, a)^T$ by the $SL(2,\mathbb{Z})$ S matrix we see that it simply exchanges the electric and magentic components.

3.5.1 Implication of the Duality

We have stated that if we are trying the find the prepotential we can do so from the relation $a_D = \partial\mathcal{F}/\partial a$ but we have yet to properly justify or make precise such a statement. Let us make the definition $h(\mathcal{A}) = \partial\mathcal{F}/\partial\mathcal{A}$ for some function h of chiral multiplets $\mathcal{A}$, the coupling is then defined as $\tau = \partial h(\mathcal{A})/\partial\mathcal{A}$. In the dual theory we have the dual variables $\mathcal{A}_D$, $\mathcal{F}_D$, and τ_D. Similarly introducing $h_D(\mathcal{A}_D) = \partial\mathcal{F}_D/\partial\mathcal{A}_D$, we see that from the S-duality above $\tau_D = \partial h_D/\partial\mathcal{A}_D = -\partial\mathcal{A}/\partial h = -1/\tau$, implies the relations

$$\mathcal{A}_D = \frac{\partial\mathcal{F}}{\partial\mathcal{A}} \qquad \mathcal{A} = -\frac{\partial\mathcal{F}_D}{\partial\mathcal{A}_D}. \tag{3.29}$$

We also note that the duality transformation on the chiral Lagrangian leaves it invariant

$$\mathrm{Im}\int d^4\theta\ \frac{\partial\mathcal{F}}{\partial\mathcal{A}}\bar{\mathcal{A}} = \mathrm{Im}\int d^4\theta\ \frac{\partial\mathcal{F}_D}{\partial\mathcal{A}_D}\bar{\mathcal{A}}_D, \tag{3.30}$$

by the relationship between dual vairables given in 3.29. We know that the Coulomb moduli of $\mathcal{A}$ are a and we will denote the moduli in the dual theory from $\mathcal{A}_D$ as a_D, we arrive the desired relationship

$$a_D = \frac{\partial\mathcal{F}}{\partial a}. \tag{3.31}$$

Therefore it stands that exactly determining the functions $a(u)$ and $a_D(u)$ we allow us to solve the theory.

Before continuing let us just note the physical implication of the duality in terms of the field content of the theory and it's dual. In finding the duality above we have elucidated a useful physical description of monopoles in our theory. We see that the magnetic monopoles and dyonic states in our theory, which are hypermultiplets in the $\mathcal{N} = 2$ gauge theory, do not couple to the $\mathcal{N} = 1$ chiral $\mathcal{A}$ or vector W multiplets in the $\mathcal{N} = 2$ theory, but instead couple to dual fields $\mathcal{A}_D$ and W_D. This makes sense in terms of an electric-magnetic duality as electrically charged objects in $\mathcal{N} = 2$ theories with matter are hypermultiplets which couple to $\mathcal{A}$ and W.

3.5.2 $U(1)$ One-loop Beta Function

To construct monodromies we will need to use the β-function for the broken $U(1)$ theory coupled to a hypermultiplet. We will use the result in calculating the monodromy around the singularities, where the monopoles become massless. This calculation follows that done in Section 3.7 of [3]. The β function for the gauge coupling at one-loop is given in [23] Eq. 3.16 as

$$\beta(g) = \mu\frac{dg}{d\mu} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}T(\mathrm{Ad}) - \frac{2}{3}T(F) - \frac{1}{3}T(S)\right), \tag{3.32}$$

where $T(\mathbf{r})$ indicates the sum of the indices of the representation $\mathbf{r}$ over the fermions $T(F)$ and complex scalars $T(S)$. Now considering a theory with Weyl fermions with charge Q_f and complex scalars with charge Q_s, the β-function for an abelian theory is given by

$$\beta(g) = \mu\frac{dg}{d\mu} = \frac{g^3}{48\pi^2}\left(2\sum_f Q_f^2 + \sum_s Q_s^2\right). \tag{3.33}$$

We note that the β-function is positive and thus the theory is free in the infrared and the coupling increases at higher energy scales. In the case we are studying the matter content of the $\mathcal{N} = 2$ hypermultiplet includes two Weyl fermions and two complex scalars all with the same charge Q. Keeping in mind that by using the anomaly that we have set the theta-angle to zero by a chiral rotation of the fermions, we have $\tau = 4\pi i/g^2$. Thus we calculate the β-function to be

$$\beta(\tau) = \mu\frac{d\tau}{d\mu} = -\frac{8\pi i}{g^3}\left(\frac{g^3}{48\pi^2}Q^2\left(2\cdot 2 + 2\cdot 1\right)\right) = -\frac{i}{\pi}Q^2. \tag{3.34}$$

We can identify the scale μ with the natural scale of the theory a. Setting $Q = 1$, we integrate to find that

$$\tau = -\frac{i}{\pi} \log\left(\frac{a}{\Lambda}\right). \tag{3.35}$$

We will use this in the calculation of the monodromies at finite u. If we are instead interested in the contribution from monopoles we simply consider the same calculation above in terms of the dual moduli a_D, which will give us

$$\tau_D = -\frac{i}{\pi} \log\left(\frac{a_D}{\Lambda}\right). \tag{3.36}$$

3.6 Monodromies at Finite u

We already found that there was a singularity at ∞, where the monodromy is described by the monodromy matrix $M_\infty = -T^{-2}$. We found that a_D can be expressed as $a_D = -\frac{i}{\pi}\sqrt{2u}\left(\log\frac{2u}{\Lambda^2} + 1\right)$ where the singularity at $u = \infty$ is a branch point of a_D, which is why refer to it as a singularity. The existence of a monodromy at $u = \infty$ implies that there are other monodromies on the moduli space. As we discussed before, the existence of a $\mathbb{Z}_2$ global symmetry on the moduli space means that they come in pairs around $u = 0$; a singularity at finite u_0 means that there is also a singularity at $-u_0$. We will now show that there must be at least three singularities to satisfy what we know already about the structure of our moduli space. The $\mathbb{Z}_2$ action of $u \to -u$ only has two fixed points, $u = 0$ and $u = \infty$. We already know there is a monodromy at $u = \infty$, so if there were only two singularities then the other would have to be $u = 0$. As monodromies are related to the contours around branch cuts, we can clearly deform the contour around $u = 0$ and observe that the monodromy at $u = 0$ is the same as the monodromy at $u = \infty$. If the singularities only exist at 0 and ∞ then the metric moduli space in the semiclassical limit is globally defined, and a is a good global coordinate. But recall from our discussion in Section 3.2 that such a coordinate cannot be globally defined as the metric would no longer be positive definite.

Opting for minimal assumptions, the next simplest choice is that of three singularities at u_0, $-u_0$, and ∞, for some finite nonzero u_0. By a contour deformation, a counterclockwise loop surrounding both finite monodromies should be equivalent to the monodromy at infinity, and thus the monodromy matrices should be related as $M_\infty = M_{u_0} M_{-u_0}$.

We briefly digress from our discussion of monodromies to address a subtlety in considering the quantum moduli space. In the classical theory, the gauge symmetry is enhanced at $u = 0$ and the additional gauge bosons become massless, where classically we have $u = \frac{1}{2}a^2$, and in the effective theory the point becomes a singularity on the moduli space. Now when considering the full quantum theory, as a becomes smaller the theory becomes strongly coupled and the same reasoning no longer applies and $a = 0$ no longer corresponds to $u = 0$. Dyons saturate the BPS bound on mass, meaning $M = \sqrt{2}|Z|$, where $Z = n_m a_D + n_e a$. So we easily see that $a = 0$ can correspond to a massless state. As the BPS bound is protected from perturbative corrections, naively a point where $a = 0$ gives rise to massless states could still exist at strong coupling. As it turns out this point does not exist in the moduli space of the quantum theory.

We already know that the singularities on the moduli space seem to arise from dyonic states becoming massless, and now our task is to calculate the associated monodromies. As we did not begin with hypermultiplets in our theory, the effective description cannot contain couplings to dyonic fields in a consistent manner. But our theory is dual to a theory with monopoles coupled to a dual vector multiplet, so we may perform the calculation theory where the equivalent low energy description is an abelian $\mathcal{N} = 2$ gauge theory coupled to matter (the monopole hypermultipet).

Considering now a point u_0 on the moduli space where a monopole becomes massless. From the BPS bound on a monopole $M = \sqrt{2}|a_D|$, we see this happens at $a_D = 0$. In the previous section we calculated the $U(1)$ β-function and integrated to find an expression for $\tau(a)$ and from duality we found that

$$\tau_D = -\frac{i}{\pi} \log \frac{a_D}{\Lambda}. \tag{3.37}$$

Since $\tau_D = -da/da_D$, we can integrate to find that

$$a = a_0 + \frac{i}{\pi} a_D \log \frac{a_D}{\Lambda} - \frac{i}{\pi} a_D, \tag{3.38}$$

where a_0 is a constant. It is worth noting that a_0 cannot have been zero, as then all electric objects would become massless at u_0 and our calculation considering only coupling to monopoles would have been incorrect. As we are interested in the region of the moduli space in the vicinity of u_0, a_D should be a good coordinate to first order, $a_D \approx c_0(u - u_0)$ near u_0, where c_0 is some complex number. Thus we find that in this region

$$a \approx a_0 + \frac{i}{\pi} c_0(u - u_0)\log(u - u_0) + \ldots, \tag{3.39}$$

where we are ignoring the terms that become small when $u \approx 0$. In calculating the monodromy we consider the counterclockwise circle around u_0: $(u - u_0) \to e^{2\pi i}(u - u_0)$. The monodromy at u_0 and monodromy matrix are then calculated as

$$\begin{aligned} a_D &\to a_D \\ a &\to a - 2a_D \end{aligned} \qquad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{3.40}$$

From the condition $M_\infty = M_{u_0} M_{-u_0}$ we find the monodromy matrix for the monodromy around the singularity at $u = -u_0$:

$$M_{-u_0} = M_{u_0}^{-1} M_\infty = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{3.41}$$

As M_{u_0} and M_{-u_0} do not commute, it might seem like there is some ambiguity in specifying an order but as we are considering the representation of the fundamental group on the moduli space for a nonabelian monodromy group, the definition of the monodromies requires a choice of a base point P. Thus under $\mathbb{Z}_2$ we could start with the base point $-P$, which would give us $M_\infty = M_{-u_0} M_{u_0}$.

We will simply quote the most general monodromy matrix for a (n_m, n_e) dyon as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}. \tag{3.42}$$

The derivation of this equation can be found in Section 3.6 of [3]. We check that the M_{u_0} monodromy corresponds to a monopole or $(1, 0)$ dyon becoming massless. We also see that the M_{-u_0} monodromy corresponds to a $(1,-1)$ dyon becoming massless. As the singularity at infinity does not correspond to a massless hypermultiplet, M_∞ will not be of the same form.

Considering the choice of base point at $-P$, we find the monodromy matrix for the singularity at $-u_0$ to be $M_{-u_0} = M_\infty M_{u_0}^{-1}$, which corresponds to a $(1,1)$ dyon. Thus the global $\mathbb{Z}_2$ symmetry on the moduli space exchanges the $(1,-1)$ dyon with the $(1,1)$ dyon. In fact upon further manipulation we find that the electric winding number is less fundamental. Considering the contour where we take the clockwise loop around infinity, around the u_0 monodromy, then counterclockwise around infinity, we find the monodromy $M_\infty M_{u_0} M_\infty^{-1}$ corresponding to a $(1,2)$ dyon becoming massless, and the reverse action gives us a monodromy where a $(1,-2)$ dyon becomes massless.

The more elegant way of seeing what is happening is by noting that all monodromy matrices as given by the general form Eq. 3.42 belong to a subgroup $\Gamma(2)$ of $SL(2,\mathbb{Z})$, comprised of matrices congruent to the identity mod 2. Thus at points on the moduli space where dyons become massless, the electric winding number is defined modulo 2, taking even values at the u_0 singularity and odd values at the $-u_0$ singularity. This is equivalent a conjugation of the fundamental group by M_∞^n.

Before moving on to the solution of the model there is one more subtlety we would like to address. We eliminated the possibility of there being two singularities and then moved ahead under the minimal assumption that there were three singularities. Let us comment briefly on why the assumption is likely correct the correct one. Assume that in addition to the singularity at infinity there are k singularities on the moduli space at points $u_1, u_2, \ldots, u_k$, ignoring the fact that they come in pairs under $\mathbb{Z}_2$. The contour deformation around all the finite singularities reproduces the expected monodromy relationship: $M_\infty = M_{u_1} M_{u_2} \ldots M_{u_k}$. Each monodromy corresponds to a (n_m, n_e) dyon becoming massless, with the condition that both winding numbers are integral. There is confidence that there do not exist solutions for values $k > 2$ that satisfy this condition.

4. Seiberg-Witten Solutions in $\mathcal{N} = 2$ Gauge Theories

We have extensively discussed properties of $\mathcal{N} = 2$ supersymmetric gauge theories and the structure of the associated moduli space of vacua. We have come to understand the singularity structure and the relevant monodromies on the moduli space and in doing so are now in a position to solve the low energy effective theory. Following [1] [3], we will first present the original solution by Seiberg and Witten [1] of finding a family of elliptic curves with the right monodromy considerations and using them to determine $a(u)$ and $a_D(u)$ as functions of the moduli space. Then we will discuss a similar approach following [5] [6], where by justifying $a(u)$ and $a_D(u)$ as solutions to a differential equation with the correct monodromy considerations, we find the explicit expressions of the quantities using hypergeometric functions. We will then try to plot the metric on the moduli space of vacua in the interest of a visual representation of the singularity structure. Finally, by considering the theory on the world-volume of a brane construction, we will again find the Seiberg-Witten curves by uplifting the brane configuration to M-theory and looking at the nontrivial cycles.

4.1 Exact Solution from Elliptic Curves

We are now in a position to determine the holomorphic prepotential by using the monodromies we have calculated to determine $a(u)$ and $a_D(u)$, and thus solve the model. As we are now confident that there are only three singularities on the moduli space we will define the points u_0 and $-u_0$ as $u = 1$ and $u = -1$. So we have found in the last section that the moduli space has singularities at $u = \infty, 1, -1$, where the monodromies we calculated are

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \qquad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

In their paper [1] Seiberg and Witten found a family of elliptic curves

We know that the metric must be positive definite and that the monodromy matrices belong to the subgroup $\Gamma(2)$ of $SL(2,\mathbb{Z})$ of matrices congruent to the identity. We consider then the quotient space $\mathbb{H}/\Gamma(2)$ where $\mathbb{H}$ is the upper half plane. Keeping in mind the singularities at $u = -1, 1, \infty$, we start by defining an elliptic curve E_u at each point u on the moduli space

$$y^2 = (x-1)(x+1)(x-u) \tag{4.2}$$

where $(x, y) \in \mathbb{C}$. The elliptic curve E_u is a Riemann surface of genus one, or more visually, a torus with two nontrivial cycles. In Appendix B.3 we review some basic facts about the geometry of elliptic curves relevant to our discussion. The space $\mathbb{H}/\Gamma(2)$ naturally parametrizes this family of curves where the singularities appear as cusped curves, generated by the quotient $\Gamma(2)$. Just as in our discussion in the appendix, we have four branch points, three at the poles $x = -1, 1, u$ and one at ∞, joined by two cuts. These cuts allow the function y to become single-valued on the complex plane. With a stereographic projection we take two sheets as two Riemann spheres, each with two cuts. Joining the spheres the two curves C_1 and C_2 become the nontrivial cycles c_1 and c_2 on the resulting torus. These curves are the same we found in Appendix B.3 with $r_1 = -1$, $r_2 = 1$ and $r_3 = u$.

We now want to define the functions a_D and a as integrals of a one-form λ, which we will soon define, integrated over the two nontrivial one-cycles

$$a = \oint_{\gamma_1} \lambda \qquad a_D = \oint_{\gamma_2} \lambda. \tag{4.3}$$

λ can be thought of as a meromorphic one-form on the family of elliptic curves. For the one-forms on E_u we make the choice of basis

$$\lambda_1 = \frac{dx}{y} \qquad \lambda_2 = \frac{xdx}{y}. \tag{4.4}$$

As argued by Seiberg and Witten there are considerations to make the choice of λ less arbitrary, including the positive definitness of the metric $\operatorname{Im}\tau > 0$. Considering the derivatives

$$\frac{da_D}{du} = \oint_{\gamma_1} \frac{d\lambda}{du} \qquad \frac{da_D}{du} = \oint_{\gamma_2} \frac{d\lambda}{du}. \tag{4.5}$$

for the moment we take that

$$\frac{d\lambda}{du} = f(u)\frac{dx}{y}. \tag{4.6}$$

The basis element λ_1 we chose on the torus is normalized over the one-cycles of the torus as

$$b_i = \oint_{\gamma_i} \lambda_1 \tag{4.7}$$

as it is the unique holomorphic differential on the family of elliptic curves. It is an algebraic result that the torus has a parameter $\tau_u = b_1/b_2$ which obeys $\operatorname{Im}(\tau_u) > 0$. Returning to the choice of 4.6, this implies that

$$\tau = \frac{da_D/du}{da/du} = \frac{\oint_{\gamma_1} f(u)\lambda_1}{\oint_{\gamma_2} f(u)\lambda_1} = \frac{b_1}{b_2} = \tau_u \tag{4.8}$$

Thus as $\operatorname{Im}\tau_u > 0$ is a condition on the torus it also holds that $\operatorname{Im}\tau > 0$, which we need for the metric to be positive definite. Thus the choice 4.6 is valid, integrating this and fixing the integration with the asymptotic behavior of $f(u)$ as argued in [1], we find the explicit expression for the one-form

$$\lambda = \frac{\sqrt{2}}{2\pi}\left(\frac{xdx}{y} - \frac{udx}{y}\right) = \frac{\sqrt{2}}{2\pi}dx\frac{(x-u)}{\sqrt{(x^2-1)(x-u)}} = \frac{\sqrt{2}}{2\pi}dx\sqrt{\frac{x-u}{x^2-1}} \tag{4.9}$$

Recall how we defined $a(u)$ and $a_D(u)$ as integrals over the two one-cycles 4.3. We can contract the first one-cycle c_1, which looped the cut from -1 to 1, to lie along the cut. This means we just need to integrate λ along the cut, doing so we find the expression for $a(u)$

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx\sqrt{\frac{x-u}{x^2-1}}. \tag{4.10}$$

For $a_D(u)$ we contract c_2 to lie along the cut from 1 to u. Integrating λ along the cut, we find the expression for $a_D(u)$ to be

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u} dx\sqrt{\frac{x-u}{x^2-1}}. \tag{4.11}$$

We have found expressions for $a(u)$ and $a_D(u)$, now let's check that the asymptotic behavior at the monodromies is correct.

4.1.1 Monodromy at $u = \infty$

First we consider the monodromy at infinity where we are in the large u limit, calculating $a(u)$ we recover the expected semiclassical relation

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx\sqrt{\frac{x-u}{x^2-1}} \approx \frac{\sqrt{2u}}{\pi}\int_{-1}^{1} dx\sqrt{\frac{1}{x^2-1}} = \sqrt{2u} \tag{4.12}$$

which tells us $u = \frac{1}{2}a^2$ as expected. Now let's calculate $a_D(u)$ at large u

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u} dx\sqrt{\frac{x-u}{x^2-1}} = \frac{\sqrt{2u}}{\pi}\int_{1/u}^{1} dy\sqrt{\frac{y-1}{y^2-1/u^2}} \tag{4.13}$$

where we have made the substitution $x = uy$. The integral develops a logarithmic divergence in the limit as $u \to \infty$ near the origin, the at lower limit of the integral approaches zero, the numerator contributes an i, while the denominator becomes z, giving us the logarithmic divergence

$$a_D(u) \approx \frac{i\sqrt{2u}}{\pi}\log u. \tag{4.14}$$

We see that our expressions for a and a_D transform under the monodromy transformation $u \to e^{2\pi i}u$, enclosing the singularity at infinity, as: $a \to -a$ and $a_D \to -a_D + 2a$ which reproduces the expected monodromy transformation 3.14.

4.1.2 Monodromy at $u = 1$

Now we calculate $a(u)$ at $u = 1$ to be

$$a(u=1) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx\sqrt{\frac{x-1}{x^2-1}} = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx}{\sqrt{x+1}} = \frac{4}{\pi} \tag{4.15}$$

but since we wanted to find the behavior in the region around the singularity we find the first order term for $a(u \to 1)$ by taking the derivative with respect to u and take the limit as $u \to 1$

$$\frac{da}{du} = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx\sqrt{\frac{x-u}{x^2-1}} = -\frac{\sqrt{2}}{2\pi}\int_{-1}^{1} dx\frac{1}{\sqrt{(x^2-1)(x-u)}} \tag{4.16}$$

We see that the integral becomes logarithmically divergent near in the upper bound of the integral as in the limit $u \to 1$ the denominator has a $(x-1)$ factor, pulling out a $\sqrt{2}$ factor we have

$$\begin{aligned}\frac{da}{du} &= -\frac{1}{2\pi}\int dx\frac{1}{\sqrt{(x-1)(x-u)}}\Bigg|_{x=1} \\ &= -\frac{1}{2\pi}\log\left(2x-1-u+2\sqrt{x-1}\sqrt{x-u}\right)\Big|_{x=1} \\ &= -\frac{1}{2\pi}\log(1-u) = -\frac{1}{2\pi}\log(1-u)\end{aligned} \tag{4.17}$$

Integrating using the zeroth order term we found before as the constant of integration, we find

$$a = \frac{4}{\pi} - \frac{1}{2\pi}(u-1)\log(u-1) + \dots \tag{4.18}$$

Now we calculate a_D near $u = 1$, in the limit as $u \to 1$ we have a divergence in the denominator at the lower bound of the integral

$$a_D = \frac{\sqrt{2u}}{\pi}\int_{1/u}^{1} dy\frac{\sqrt{y-1}}{\sqrt{y^2-1/u^2}} \approx \frac{\sqrt{u}}{\pi}\int_{1/u}^{1} dy\sqrt{\frac{y-1}{y-1/u}} \approx \frac{1}{\pi}\int_{1/u}^{1} dy\sqrt{\frac{y-1}{y-1/u}} \tag{4.19}$$

where we have made valid approximations in the limit $u \to 1$. Evaluating we find

$$a_D = \frac{i}{2}\left(1-\frac{1}{u}\right) \approx \frac{i}{2}(u-1). \tag{4.20}$$

Considering the loop $u \to e^{2\pi i}u$ around the singularity at $u = 1$ we find the transformations $a_D \to a_D$ and $a \to a - 2a_D$, which reproduces the expected monodromy transformations 3.40.

It suffices to just check the monodromies at $u = 1$ and $u = \infty$ as the singularity at $u = -1$ is related to the singularity at $u = 1$ by the global $\mathbb{Z}_2$ symmetry.

4.2 Solutions via Hypergeometric Functions

In their original work Seiberg and Witten [1] solve the model by finding a set of elliptic curves with the correct monodromy structure which correctly determines the two desired functions $a(u)$ and $a_D(u)$. Here we briefly present an alternative approach which allows us to find explicit forms for $a(u)$ and $a_D(u)$ using the integral representation of hypergeometric functions.

If we already know the explicit form of $a(u)$ and $a_D(u)$ then we can just take a top down approach and match the solutions to hypergeometric functions. But for the sake of completeness we will take the bottom up approach find $a(u)$ and $a_D(u)$ as solutions to a second-order differential equation, which turn out to be hypergeometric functions. We follow the procedure outlined in [3], [5], and [6].

Monodromies typically arise in solutions to differential equations with periodic solutions or boundary conditions. For example a time independent Schrödinger equation with a periodic potential: $V(x) = V(x+2\pi)$. The set of solutions at x must be the same as the solutions at $x + 2\pi$. If there are two independent solutions, $\psi_1(x)$ and $\psi_2(x)$ then the solution when the system is shifted to $x + 2\pi$ is some linear combination of $\psi_1(x)$ and $\psi_2(x)$. Thus we can construct the monodromy transformation: $(\psi_1(x + 2\pi), \psi_2(x + 2\pi))^T = M\ (\psi_1(x), \psi_2(x))^T$, where M is a monodromy matrix.

The equivalent situation for differential equations on the complex plane is having meromorphic solutions as here the equivalent of periodicity is single-valuedness. So now we consider an equation with the same functional form to the Schrödinger equation

$$\left(-\frac{d^2}{dz^2} + V(z)\right)\psi(z) = 0 \tag{4.21}$$

where now $V(z)$ is some meromorphic function with finite poles z_i and one at ∞. Here the equivalent to a periodic shift is an analytic continuation around the pole z_i, giving us the monodromy transformation M_i. More explicitly, we see that for the two independent solutions $\psi_1(z)$ and $\psi_2(z)$, we have

$$\begin{pmatrix} \psi_1(z + e^{2\pi i}(z - z_i)) \\ \psi_2(z + e^{2\pi i}(z - z_i)) \end{pmatrix} \to M_i \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix}. \tag{4.22}$$

Now we turn to the moduli space of our $\mathcal{N} = 2$ gauge theory. In our theory there are three singularities at $u = -1, 1$ and ∞ on the moduli space with two functions $a(u)$ and $a_D(u)$. This exactly fits the system we discussed above. Thus we take the two functions $a(u)$ and $a_D(u)$ to be the solutions to the second-order differential equation on the complex u-plane. With the singularities in mind, we find the general form of the potential knowing that the singularities correspond to (up to) second-order poles

$$V(u) = -\frac{1}{4}\left(\frac{1-\lambda_1^2}{(u+1)^2} + \frac{1-\lambda_2^2}{(u-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(u+1)(u-1)}\right) \tag{4.23}$$

where we have used the fact that $M_1 M_{-1} = M_\infty$. Naively, one would assume that first order poles at $u = 1$ or $u = -1$ could appear but these correspond to third order poles at ∞. Taking the limit $u \to \infty$, we find that

$$V(u) \sim -\frac{1-\lambda_3^2}{4u^2} \tag{4.24}$$

We know from our discussion above that asymptotically the solutions behave like $\sim \sqrt{u}$ and $\sim \sqrt{u}\log u$, plugging these into the differential equation and working backwards, i.e. taking the second derivative then dividing by the solution we see that for both solutions we want $V \sim -\frac{1}{4u^2}$, and since a nonzero λ_3 would change the asymptotics we take $\lambda_3 = 0$. Taking the limit $u \to 1$ we find that

$$V(u) \sim -\frac{1}{4}\left(\frac{1-\lambda_2^2}{(u-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2}{2(u-1)}\right) \tag{4.25}$$

Recalling our discussion before we know the solutions near $u = 1$ behave as $\sim (u-1)$ and $\sim$

const $+(u-1)\log(u-1)$, thus it follows that we'd only want the second term in the potential to contribute and we take $\lambda_2 = 1$. The same result applies for the singularity at $u = -1$ where we take $\lambda_1 = 1$. We are left with the potential

$$V(u) = \frac{1}{(u+1)(u-1)}. \tag{4.26}$$

The solution of the differential equation we started with has known solutions [28], given that $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$ we have the two linearly independent solutions

$$\psi_{1,2}(u) = f_{1,2}\left(\frac{u+1}{2}\right) \tag{4.27}$$

The function f satisfies Euler's hypergeometric differential equation:

$$u(1-u)\frac{d^2f}{du^2} + (c - (a+b+1)u)\frac{df}{du} - abf = 0 \tag{4.28}$$

where $a = \frac{1}{2}(1 - \lambda_1 - \lambda_1 + \lambda_3)$, $b = \frac{1}{2}(1 - \lambda_1 - \lambda_1 - \lambda_3)$, and $c = 1 - \lambda_1$. Therefore we find that $a = -1/2$, $b = -1/2$, and $c = 0$. Given that we are interested in two linearly independent solutions in the neighborhood of 1 and ∞, we choose two of Kummer's 24 solutions to the hypergeometric equation that fit the criteria. From [28] we choose the basis for the hypergeometric equation as

$$\begin{aligned} \psi_1(u) &= f_{1(\infty)}(u) = u^{-a}F(a, a-c+1; a-b+1; u^{-1}) \\ \psi_2(u) &= f_{2(1)}(u) = (1-u)^{c-a-b}F(c-b, c-a; c-a-b+1; 1-u) \end{aligned}$$

where $F(\alpha, \beta; \gamma; z)$ are the hypergeometric functions. We now propose that

$$\begin{aligned} a(u) &= 2\psi_1(u) = \sqrt{2(u+1)}\ F\left(-1/2, 1/2; 1; 2/(u+1)\right) \\ a_D(u) &= i\psi_2(u) = \frac{i}{2}(1-u)\ F(1/2, 1/2; 2; (1-u)/2) \end{aligned}$$

The hypergeometric functions can be expressed in an integral representation as

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dx\ x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha} \tag{4.29}$$

where we have defined $F(\alpha,\beta;\gamma;z)$ normalized with the Beta-function $B(\beta,\gamma-\beta)$. We find the integral expressions for $a(u)$ and $a_D(u)$

$$\begin{aligned} a(u) &= \sqrt{2(u+1)}\frac{\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1 dx\ \frac{1}{\sqrt{x(1-x)}}\left(1-x\frac{2}{u+1}\right)^{\frac{1}{2}} \\ a_D(u) &= \frac{i}{2}(1-u)\frac{\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}\int_0^1 dx\ \sqrt{\frac{1-x}{x}}\left(1-x\frac{1-u}{2}\right)^{-\frac{1}{2}} \end{aligned}$$

simplifying and changing variables to avoid confusion we have

$$a(u)=\frac{\sqrt{2}}{\pi}\sqrt{u+1}\int_0^1 dt\ \frac{1}{\sqrt{t(1-t)}}\left(1-\frac{2}{u+1}t\right)^{\frac{1}{2}} \tag{4.30}$$

$$a_D(u)=\frac{i}{\pi}(1-u)\int_0^1 dt\ \sqrt{\frac{1-t}{t}}\left(1-\frac{1-u}{2}t\right)^{-\frac{1}{2}}. \tag{4.31}$$

We now want to recover the expressions for $a(u)$ and $a_D(u)$ we found in our derivation using elliptic curves 4.10 and 4.11. In the above expression for $a(u)$ 4.30, we make the substitution $t=\frac{1}{2}(x+1)$ and find

$$a(u)=\frac{\sqrt{2}}{\pi}\sqrt{u+1}\int_{-1}^1 \frac{dx}{2}\ \frac{2}{\sqrt{1-x^2}}\sqrt{\frac{u-x}{u+1}}=\frac{\sqrt{2}}{\pi}\int_{-1}^1 dx\ \sqrt{\frac{u-x}{1-x^2}}. \tag{4.32}$$

In the expression above for $a_D(u)$ 4.31, we make the substitution $t=(x-1)/(u-1)$

$$a_D(u)=\frac{i}{\pi}(1-u)\int_1^u \frac{dx}{u-1}\ \sqrt{\frac{u-x}{x-1}}\sqrt{\frac{2}{x+1}}=\frac{\sqrt{2}}{\pi}\int_1^u dx\ \sqrt{\frac{x-u}{x^2-1}}. \tag{4.33}$$

Using the integral representation of the hypergeometric function we have found the explicit form of $a(u)$ and $a_D(u)$

$$a(u)=\frac{\sqrt{2}}{\pi}\int_{-1}^1 dx\sqrt{\frac{x-u}{x^2-1}}\qquad a_D(u)=\frac{\sqrt{2}}{\pi}\int_1^u dx\sqrt{\frac{x-u}{x^2-1}} \tag{4.34}$$

which is in exact agreement with 4.10 and 4.11.

4.2.1 An Explicit Form of the Metric

What we would like to do now is find an explicit form of the metric and plot moduli space, which we are able to do in terms of hypergeometric functions. To make things less messy, we make some definitions in terms of hypergeometric functions, it is common to express the complete elliptic integrals as

$$K(\omega) = \frac{\pi}{2} F(1/2, 1/2; 1; \omega^2) \qquad E(\omega) = \frac{\pi}{2} F(-1/2, 1/2; 1; \omega^2) \tag{4.35}$$

Recalling that

$$\begin{aligned} a(u) &= \sqrt{2(u+1)}\ F\left(-1/2, 1/2; 1; 2/(u+1)\right) \\ a_D(u) &= \frac{i}{2}(1-u)\ F(1/2, 1/2; 2; (1-u)/2) \end{aligned}$$

it should be clear that we can express $a(u) = \frac{4}{\pi k} E(\omega)$ when $\omega^2 = 2/(1+u)$. We know we can find the metric from the coupling τ, which can be expressed as

$$\tau = \frac{\partial a_D}{\partial a} = \frac{da_D}{du} \bigg/ \frac{da}{du} \tag{4.36}$$

using Mathematica to compute the derivatives of the hypergeometric functions in terms of the complete elliptic integrals we find that

$$\frac{da}{du} = \frac{\sqrt{2}}{\pi\sqrt{1+u}}\ K\left(\sqrt{\frac{2}{1+u}}\right) \tag{4.37}$$

$$\frac{da_D}{du} = \frac{i}{\pi}\ K\left(\sqrt{\frac{1-u}{2}}\right) \tag{4.38}$$

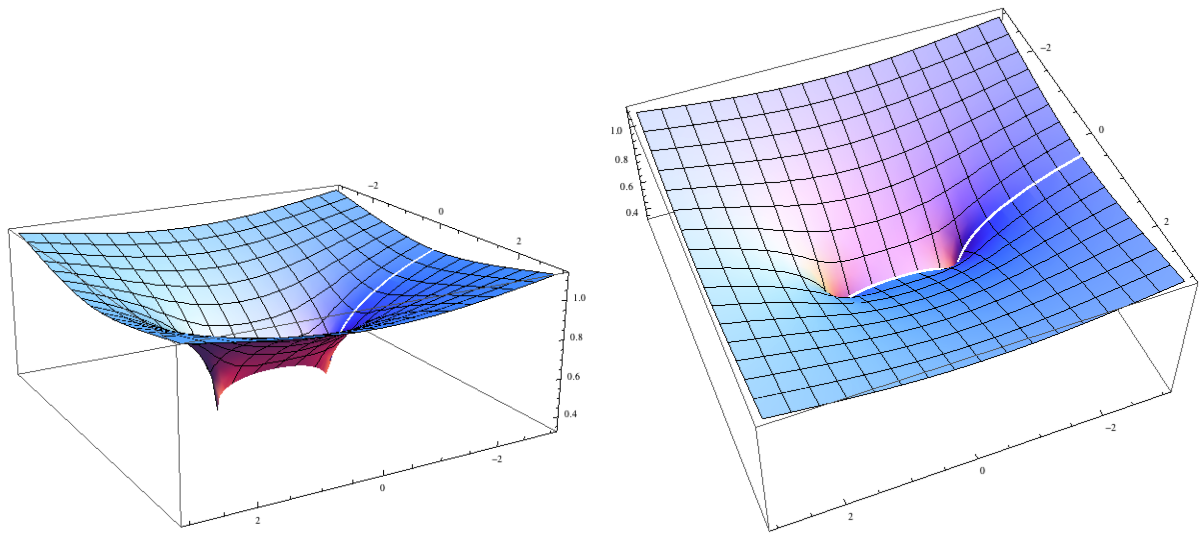

Figure 4-1: Plot of the metric on the moduli space, defined as $\operatorname{Im}\tau$ and given in terms of complete elliptic functions in 4.40, we observe the expected singularity structure.

Noting that $K(iv/\omega) = \omega K(v)$, letting $\omega^2 = 2/(1+u)$ and $v^2 = 1 - \omega^2$, meaning that $-v^2/\omega^2 = (1-u)/2$, we can rewrite

$$\frac{da_D}{du} = \frac{i}{\pi}\, K\left(\frac{iv}{\omega}\right) = \frac{i\omega}{\pi}\, K\left(v\right) = \frac{i}{\pi}\sqrt{\frac{2}{1+u}}\, K\left(v\right) \tag{4.39}$$

we find that

$$\tau = \frac{iK(v)}{K(\omega)} \tag{4.40}$$

Recalling that the metric on the moduli space is $\operatorname{Im}\tau$ we plot our results in Figure 4-1. We can see that in addition to seeing singularities at $u = 1$ and $u = -1$, we can see evidence of the singularity at $u = \infty$ because such a singularity is a branch point of $a_D(u)$.

We have found the functions $a(u)$ and $a_D(u)$ in agreement with the results using the original approach of determining a family of elliptic curves. It may seem as though in this approach the use of the monodromies was more subtle, at least compared to the previous approach. But recall it was the consideration of the monodromies that implied the two functions were solutions to the second-order differential equation we used. The singularity structure was used in determining the functional form of the potential. Furthermore consideration of the behavior around $u = 1$ and $u \to \infty$ allowed us to correctly determine which Kummer solutions to the hypergeometric differential equation, and thus find the correct hypergeometric functions which correctly gave us the functions $a(u)$ and $a_D(u)$. As $a(u)$ and $a_D(u)$ completely determine the holomorphic prepotential, which in turn completely determines the low energy effective theory, we have successfully solved our model using hypergeometric functions.

4.3 Seiberg-Witten Curves from M-theory Lifts of Brane Constructions

4.3.1 Branes in Type II and M-theory

First we will quickly review some basic facts about branes in Type II string theories and in M-theory. The fields that make up the $\mathcal{N} = 1$ supergravity multiplet in 11-dimensional M-theory are the graviton, the gravitino, and the 3-form gauge field $(g, \psi_\mu^\alpha, C^{(3)})$. Just as in electromagnetism, in string theory and M-theory there are electric and magnetic objects which are charged under gauge fields. The conserved electric and magnetic charges of the 3-form field are $dF^{(4)} = \delta^{(5)} Q_M$ and $d * F^{(4)} = \delta^{(8)} Q_E$, which gives us a codimension 8 electric object in 11d, called the M2-brane, and a codimension 5 magnetic object, called the M5-brane. The fields in the $\mathcal{N} = (1,1)$ supergravity multiplet in 10-dimensional Type IIA string theory an be found from dimensional reduction of M-theory. The bosonic fields are the graviton, the dilaton, a 1-form, a 2-form, and a 3-form field; $(g, \phi, C^{(1)}, B^{(2)}, C^{(3)})$. For completeness the fermionic fields in Type IIA are two gravitini, a left-handed Majorana-Weyl (MW) gravitino and a right-handed MW gravitino, and two spinors, a left MW spinor and right MW spinor. As this is a $\mathcal{N} = (1,1)$ theory, and is nonchiral, the left and right MW gravitini transform as one Majorana gravitino, and the left and right spinors transform as one Majorana spinor. As in M-theory, the gauge fields will source electric and magnetic objects. In the same manner as above, the branes sourced by the 1-form $C^{(1)}$ and 3-form $C^{(3)}$ are the D0, D2, D4, and D6 branes. The Kalb-Ramond 2-form $B^{(2)}$ magnetically sources the NS5-brane and electrically sources the fundamental string F1. The dilaton ϕ is noncompact and is not gauge invariant and does not source charges. From T-duality we can also include the D8-brane in the brane content of Type IIA. In the $\mathcal{N} = (2,0)$ supergravity multiplet of Type IIB, the gauge fields $B^{(2)}, C^{(0)}, C^{(2)}, C^{(4)+}$ give us the brane content of the theory to be: F1, NS5, D(-1), D1, D3, D5, D7. Here the scalar $C^{(0)}$, usually called the axion, is compact and does source any branes.

This was a rather terse review of branes in Type II string theories and M-theory. For a more in depth discussion and better rooted derivations of the facts above please see the textbook [26] or the course notes from Prof. Hanany's String Theory course [27].

4.3.2 Field Theories on the Worldvolume of Branes

It is possible to find and study many interesting aspects of both gauge theories and string theories by investigating the quantum field theories that appear on the world-volume of branes.

Consider a D4-brane ending on an NS5-brane. Now suppose we have $n+1$ NS5-branes labeled by $0, 1, \dots, n$, and there are k_α D4-branes, where the index α indicates that the D4-brane is stretched between the $(\alpha-1)$-th and α-th NS5-brane. Then the resulting low energy effective theory on the D4-brane world volume is an $\mathcal{N} = 2$ SYM theory with gauge group $G = \prod SU(k_\alpha)$ and hypermultiplets in the bifundamental representation $(k_1, \bar{k}_2) \oplus \ldots \oplus (k_{n-1}, \bar{k}_n)$.

An important distinction to make is that the gauge group of the brane construction is in fact $SU(k_\alpha)$ and not $U(k_\alpha)$ as there is an overall $U(1)$ factor. This is due to the center of mass motion of the D4-branes on the NS5-branes is logarithmically divergent.

We can also add a D4-branes and b D4-branes on to each end of the brane construction which give rise to a hypermultiplets in the fundamental of $SU(k_1)$ and b hypermultiplets in the fundamental of $SU(k_n)$. We can also add c D6-branes between two of the NS5-branes, say $\alpha - 1$ and α, giving us d hypermultiplets in the fundamental of $SU(k_\alpha)$. Let's see how certain theories arise in these brane constructions. If we have an NS5-brane with k D4-branes ending on it, where the direction along which the D4-branes run tangent to the NS5-brane, is compactified on a circle, the resulting theory on the world volume is an $\mathcal{N} = 4$ $U(k)$ theory. Supposedly, if we fix one of these D4 branes giving it a vev, then the theory becomes a $SU(k)$ theory. Adding mass to the adjoint hypermultiplet gives us an $\mathcal{N} = 2$ theory (don't really understand how this happens, breaks half the SUSY supposedly). $\mathcal{N}=2$ QCD can be obtained by adding infinite D4-branes on either side of the NS5-branes, or by adding D6-branes between them. This is the flavor brane setup referred to in the literature.

We will now just focus on the $\mathcal{N}=2$ $SU(N)$ setup of D4-branes between NS5-branes and the $\mathcal{N}=2$ QCD setup obtained by adding D6-branes. Considering M-theory on $\mathbb{R}^{9+1}\times S^1$, i.e. compactifying x^{10} on a circle, we get Type IIA. As we would expect, the brane content of Type IIA is just the dimensional reduction of the branes or transverse modes in M-theory. The D4-brane is the M5-brane wrapped over the compactified dimension x^{10}, and the NS5-brane is the unwrapped M5-brane, transverse to x^{10}. D6-branes and D0-branes are sourced by the 1-form field in Type IIA, which comes from the dimensional reduction of the metric in M-theory. Thus the D6-brane and D0-brane are lifted to the KK-monopole and momentum mode, respectively, in 11 dimensions.

We see now that lifting the D4/NS5 brane construction to M-theory will give us a single M5-brane with highly nontrivial geometry. To preserve the $\mathcal{N}=2$ supersymmetry the world volume of the M5-brane is of the form $\mathbb{R}^4\times\Sigma$, where Σ is a holomorphic curve. This curve should be embedded in $\mathbb{R}^3\times S^1$. To see which directions the holomorphic curve is let's look at the spatial configuration of the D4/NS5 brane setup:

NS5:	1	2	3	4	5	-	-	-	-
D4:	1	2	3	-	-	6	-	-	-

In this configuration, in the world volume of the M5-brane the holomorphic curve should be embedded in $\mathbb{R}^3\times S^1$, comprised of directions x^4, x^5, x^6 and x^{10}. As a side note, in the $\mathcal{N}=4$ brane construction, where the x^6 direction is compactified, then the holomorphic curve is embedded in $\mathbb{R}^2\times T^2$.

Our goal is to understand the low energy effective theory living on the world volume $\mathbb{R}^4\times\Sigma$ of the M5-brane by determining the gauge couplings from the holomorphic curve Σ.

4.3.3 The Holomorphic Curve as a Seiberg-Witten Curve

What we need to do in order to study the low energy effective theory on the world volume of the M5-brane, $\mathbb{R}^4\times\Sigma$, and just as in Seiberg-Witten theory we want to find the gauge couplings from the curve on Σ. The coupling of the scalars can then be determined from the gauge couplings from supersymmetry. The gauge fields come from the reduction of the self-dual 2-form potential along Σ on the world volume. The self-dual field strength can be decomposed as such $G = (1 + *)F \wedge \omega$ where F is the usual gauge field strength and ω is a harmonic 1-form on Σ. If Σ has a genus of g then there are g $U(1)$ gauge fields. The gauge couplings are then found from the reduction of the kinetic term

$$\int_{\mathbb{R}^4 \times \Sigma} |T|^2 \tag{4.41}$$

What we see is that the gauge couplings are the period matrix of Σ, or rather the intersection form of ω. An intersection form is defined on an oriented compact 4-manifold $\mathcal{M}$, and is a symmetric bilinear form $\alpha \in \Omega^2(\mathcal{M})$. When the manifold is smooth, which for us is true, then the intersection form may be expressed as the integral

$$Q(a, b) = \int_{\mathcal{M}} \alpha \wedge \beta \tag{4.42}$$

where α and β are 2-forms. So since the gauge couplings are the period matrix of Σ, then Σ is our Seiberg-Witten curve. There should probably be some explanation of why we may write this down as an algebraic curve, related to the fact that although Σ is generically noncompact, we may compactify the space by adding a point at infinity.

4.3.4 Finding Solutions of the Theory

Solutions for a large class of $\mathcal{N} = 2$ theories, where by solutions we mean that we write down the Seiberg-Witten curve that depends on the moduli of the theory, so that we may compute the gauge couplings of the theory from the periods of the curve. We start with the brane construction for $\mathcal{N} = 2$ $SU(n)$ pure gauge theory, which is n D4-branes suspended between two NS5-branes. We will write the relevant coordinates x^4, x^5, x^6, and x^{10}, as the complex coordinates $v = x^4 + ix^5$ and $t = e^{-s} = \exp(-(x^6 + ix^{10})/R)$, since the M-theory circle x^{10} has a periodicity of $2\pi R$. In a general brane construction the degree of the equation defining the Seiberg-Witten curve in t is the number of NS5-branes and the degree in v is the number of D4-branes. Thus, it should be clear that $f(t,v)$ should be quadratic in t, and so should have the form

$$f(t,v) = A(v)t^2 + B(v)t + C(v) = 0. \qquad (4.43)$$

The limit of $t \to 0$ corresponds to one NS5-brane, which we will call the left one, and the limit of $t \to \infty$ corresponds to the other NS5-brane, the right one. As the D4-branes end on the NS5-branes and the $\mathcal{N} = 2$ theory lives on the world volume of the D4-branes, then $f(t,v) = 0$ should have no solution for finite v at $t = 0$ and $t = \infty$. So we must set $A(v)$ and $C(v)$ to be constants and without loss of generality we may set $A = C = 1$. Solving the above equation and shifting the variables we find that

$$t^2 = (u_1 v^n + u_2 v^{n-1} + \ldots + u_n)^2 - 1 \qquad (4.44)$$

this is precisely the $SU(n)$ Seiberg-Witten curve with the scale $\Lambda = 1$.

4.4 Physical Phenomena at Strong Coupling

There are a number of interesting physical phenomena that have appeared in our treatment of $\mathcal{N} = 2$ gauge theories which we would like to address more directly. We saw that monopoles and dyonic states appeared in the strong coupling region of the moduli space. For singularities to occur on the moduli space of the low energy effective theory there need to be multiplets becoming massless, and we saw that on the moduli space we studied the two finite singularities occurred when a $(1,0)$ monopole and a $(1,-1)$ dyon became massless. We want to continue this discussion and address the condensation of monopoles and confinement, phenomena which we can observe in the theory.

In the semiclassical limit of our effective theory, the massless spectrum in $\mathcal{N} = 2$ gauge theories only consists of the $\mathcal{N} = 2$ vector multiplet, which has a broken $U(1)$ gauge symmetry. In the classical supersymmetric theory, the $\mathcal{N} = 2$ supersymmetry can be broken to $\mathcal{N} = 1$ supersymmetry with the addition of a superpotential $\mathcal{W} = m \mathrm{Tr}\, \Phi^2$. Recall our construction of the $\mathcal{N} = 2$ Lagrangian for gauge fields, where we looked at the full $\mathcal{N} = 1$ Lagrangian and enhanced to $\mathcal{N} = 2$ supersymmetry by requiring the $\mathcal{W}$ superpotential be zero. The addition of this term gives a mass to the chiral multiplet Φ. Integrating out massive states, the low energy effective theory becomes an $\mathcal{N}=1$ theory of Abelian gauge fields only.

In our low energy effective theory, we add a mass term to the chiral multiplet $\mathcal{A}$, $m\mathrm{Tr}\,\mathcal{A}^2$. The operator is itself a chiral superfield which we will call U. Clearly, the moduli of the operator is $\mathrm{Tr}\,\phi^2$, which we have defined as $u = \langle \mathrm{Tr}\,\phi^2 \rangle$. For small enough m is should be justifiable to add a mass term mU to the effective Lagrangian, which gives us massive chiral multiplets. If we also want massive vector multiplets, we either need points on the moduli space where the gauge symmetry is enhanced to the non Abelian symmetry, which can occur with additional gauge fields, or some coupling to a charged field that allows for an additional Higgs mechanism. As we already discussed the problems with gauge enhancements on the moduli space, we choose to consider that there might be a Higgs mechanism through a coupling to dyonic states. If this were to be the case the dyonic states would need to be incredibly light. We know from our discussion of duality that the monopole is an $\mathcal{N}=2$ hypermultiplet, which can be broken into two $\mathcal{N}=1$ chiral multiplets Φ and $\bar{\Phi}$. Just as electrically coupled chiral multiplets couple to vectors, the monopole couples to the dual vector multiplet $\mathcal{A}_D$. The superpotential then becomes

$$\mathcal{W} = \sqrt{2}\,\mathcal{A}_D \Phi \bar{\Phi} + mU \tag{4.45}$$

where in this dual description $U = m\mathrm{Tr}\,\mathcal{A}_D^2$. We now minimize the superpotential to find vacuum solutions. Clearly $\Phi = \bar{\Phi} = m = 0$ is a solution corresponding to the moduli space of the $\mathcal{N}=2$ theory. But as presented in [1], if $m \neq 0$ then

$$\sqrt{2}\Phi\bar{\Phi} + m\frac{du}{da_D} = 0 \qquad a_D\Phi = a_D\bar{\Phi} = 0 \tag{4.46}$$

is a solution Φ and $\bar{\Phi} \neq 0$. As the monopoles are charged and couple to the gauge field we find a magnetic equivalent of the Higgs mechanism. The nonzero vacuum solution for the massless magnetic monopoles implies that they will condense, and monopole condensation gives rise to confinement of electrically charged fields.

5. Conclusion

Our original goal was to find the prepotential that fully determined the $\mathcal{N} = 2$ supersymmetric theory, even at strong coupling, and in doing we hoped to shed light on some interesting phenomena that occur. We know that strong coupling phenomena occur in the real world, and within the realm of Yang-Mills theories most of them occur in QCD at low energies. This is because unlike QED, QCD is a strongly coupled theory, where the low energy behavior exhibits strong coupling. As a result phenomena occur, for example the confinement of quarks or chiral symmetry breaking, which are not well understood. Lattice QCD strives to obtain results by calculating QCD at points on a lattice and extrapolate information. But if our goal is to understand something fundamental about gauge theories, specifically the structure of Yang-Mills theories, then we look to studying theories like $\mathcal{N} = 2$ in the hopes of understanding strong coupling.

We sought to fully solve the low energy effective theory by determining the holomorphic prepotential, which in turn we did by solving for the quantities $a(u)$ and $a_D(u)$ as functions of the moduli space coordinate u. We succeeded in doing so following the original approach of Seiberg and Witten by determining a family of elliptic curves, and by a similar approach using hypergeometric functions, which allowed us to find the exact forms $a(u)$ and $a_D(u)$. We also discussed Seiberg-Witten curves on lifted brane constructions. We caught a glimpse of some incredible phenomena that occur at strong coupling. We were only able to determine the prepotential at strong coupling by the existence of weakly coupled monopoles and dyonic states. We also observed the condensation of magnetic monopoles and the confinement of electric charge.

A. Notation and Conventions

A.1 Notation

Greek indices in the middle of the alphabet, i.e. $\mu, \nu, \ldots$, will run over the 4-dimensional spacetime coordinates $0, 1, 2, 3$. Greek indices at the beginning of the alphabet, i.e. $\alpha, \beta, \ldots$, will run over spinor indices, and as we are working with supersymmetric theories these indices will often appear in both dotted and undotted form. Lowercase latin indices $i, j, \ldots$ will run over spatial coordinates $1, 2, 3$. Capital latin indices $I, J, \ldots$, will denote supersymmetric indices $1, 2, \ldots, \mathcal{N}$, but as we will almost exclusively be working with $\mathcal{N} = 2$ theories, $I, J, \ldots = 1, 2$.

A.2 Spinor Conventions

We use the 'particle physicist' metric

$$\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \tag{A.1}$$

defined so the contracted momenta gives a positive square mass $p^\mu p_\mu = m^2$.

The invariant $SL(2, \mathbb{C})$ tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$, and their inverses $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$, are defined as

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.2}$$

and can be used to raise and lower spinor indices. The generalized sigma matrices are defined as the four vectors

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = (\mathbb{I}, \sigma^i) \qquad \bar{\sigma}^{\mu\,\alpha\dot{\alpha}} = (\mathbb{I}, -\sigma^i), \tag{A.3}$$

where the spatial components σ^i are the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

We have the tensors $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ as antisymmetrized products of σ^μ matrices

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta \tag{A.5}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{A.6}$$

We typically write spinors with indices contracted as

$$\chi\psi = \chi^\alpha\psi_\alpha = -\chi_\alpha\psi^\alpha \qquad \bar{\chi}\bar{\psi} = \bar{\chi}^\alpha\bar{\psi}_\alpha = -\bar{\chi}_\alpha\bar{\psi}^\alpha. \tag{A.7}$$

B. Geometry and Topology

There were a number of parts of this thesis where concepts in geometry and topology were used which might merit further discussion. For the benefit of the reader, and even more so, for the benefit of the author, what follows is a discussion of a few of these topics.

B.1 Kähler Manifolds

In this section we will briefly review some aspects of cohomology on complex manifolds and then define Kähler manifolds. Given the relevance of Kähler spaces in Seiberg-Witten theory, it should be helpful to provide some brief definitions, but by no means is this discussion meant to embody sufficient mathematical depth. For further reference see [20], [21], or [22].

On an m-dimensional manifold $\mathcal{M}$, the exterior derivative acts on forms as $d:\ \Omega^p(\mathcal{M}) \rightarrow \Omega^{p+1}(\mathcal{M})$ and is nilpotent $d^2 = 0$. Due to this property it is possible for us to define cohomology as p-forms will define a chain complex, where the image of one $\Omega^p(\mathcal{M}) \rightarrow \Omega^{p+1}$ is in the kernal of the next $\Omega^{p+1}(\mathcal{M}) \rightarrow \Omega^{p+2}$. Recalling that a p-form $\omega \in \Omega^p(\mathcal{M})$ is closed if $d\omega = 0$ and is exact if $\omega = d\alpha$ where $\alpha \in \Omega^{p-1}(\mathcal{M})$, we define the p-th de Rham cohomology as the quotient space

$$H^p(\mathcal{M}) \equiv \frac{\text{space of closed } p\text{−forms on } \mathcal{M}}{\text{space of exact } p\text{−forms on } \mathcal{M}}, \tag{B.1}$$

where the p-th betti number is $b^p = \dim H^p(\mathcal{M})$.

A manifold $\mathcal{M}$ of even dimension $d = 2n$, is complex is we cam parametrize it with n complex coordinates z^i, where the transition between patches is smooth and defined by holomorphic transition functions. Formally, the manifold is a Hausdorff topological space with open sets J, and there exist an atlas of charts $\{(U_i, \psi_i)\}$ where $U_i \in J$, such that

$\cup_i U_i = \mathcal{M}$ and the maps ψ_i are homeomorphisms from U_i into open subsets of $\mathbb{C}^n$. If $U_i \cap U_j \neq 0$ then the transition functions $\phi_{ij} = \psi_j \circ \psi_i^{-1}$ from $\mathbb{C}^n \to \mathbb{C}^n$ are holomorphic functions.

On such a space it is natural to define (p,q)-forms: $\omega_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \in \Omega^{(p,q)}(\mathcal{M})$ and define an external derivative $d = \partial + \bar{\partial}$ where $\partial = dz^i \frac{\partial}{\partial z^i}$ and $\bar{\partial} = d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}$. These derivatives act on (p,q)-forms as $\partial :\ \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial} :\ \Omega^{p,q} \to \Omega^{p,q+1}$, and are both nilpotent $\partial^2 = \bar{\partial}^2 = 0$. It is then natural to define *Dolbeault cohomology* on the complex manifold $\mathcal{M}$ to be the quotient space

$$H^{p,q}(\mathcal{M}) \equiv \frac{\text{space of } \bar{\partial} \text{ closed } (p,q)\text{−forms on } \mathcal{M}}{\text{space of } \bar{\partial} \text{ exact } (p,q)\text{−forms on } \mathcal{M}}, \tag{B.2}$$

and we define the hodge numbers of $\mathcal{M}$ to be $h^{p,q}(\mathcal{M}) \equiv \dim H^{p,q}(\mathcal{M})$.

Kähler manifolds are complex manifolds equipped with a hermitean metric $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Thus we may define a two-form called the Kähler form $K = i g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$ which is closed under $dK = 0$. For Kähler manifolds the two dolbeault cohomologies are equivalent $H^{p,q}_{\partial} = H^{p,q}_{\bar{\partial}}$, the insight being that the two exterior derivatives commute both with each other and with the Laplacian. The hodge numbers are related as: $h^{p,q} = h^{q,p}$ from complex conjugation, $h^{p,q} = h^{n-p,n-q}$ from the hodge dual. Since the Kähler form is a closed two form we can define a Kähler class for $H^{1,1}$ and expand the form as $K = \sum_i x^i \omega_i$ for a basis ω_i where $i = 1, \ldots, h^{1,1}$. This means that the coordinates x^i are good coordinates on the moduli space of the Kähler manifold.

B.2 Homotopy

While homology groups assigns structure to cycles that are not boundaries and cohomology groups assigns structure to closed forms mod exact forms, in dealing with homotopy groups we are concerned with the continuous deformation of maps. Let X and Y be *standard* topological spaces and let $\mathcal{F}$ be a set of continuous maps from the space X to Y. We may define the equivalence relation in $\mathcal{F}$ as such; consider two maps f and g, $\forall\ f, g \in \mathcal{F}$, if the image $f(X)$ can be continuously deformed to $g(X)$ in the space Y, then $f \sim g$ and we say these maps are homotopic. As homotopy is an equivalence relation, the equivalence class it defines is called the homotopy class.

What follows is a more intuitive explanation of homotopy. Consider two spaces, one is a disc and the other is a disc with a hole in the center. Note that a 1-cycle can be deformed differently in each of these spaces. A loop in the disc can be continuously shrunk to a point, while on the second space the loop cannot due to the presence of the hole. Thus any loop on the disc is homotopic to a point. While there is only one homotopy class defined on the disc, on the punctured disc the homotopy classes are defined by an integer number of times the loop is wound round the hole, $n \in \mathbb{Z}$, where if $n < 0$ the loop winds clockwise and if $n > 0$ the loop winds anticlockwise, and $n = 0$ if the loop does not wind round the hole. We now consider the group structure of $\mathbb{Z}$, as $\mathbb{Z}$ is an additive group with the group operation $+$, the element $m + n$ can easily be interpreted as a loop encircling the hole first m times then n times. The group structure of homotopy classes is called the fundamental group, where the group structure of the first homotopy class is the fundamental group and deals with the deformation of loops. Higher homotopy classes are concerned with the deformation of an n-dimensional analog of a loop, where the group structure of higher homotopy groups are an n-dimensional extension of the fundamental group. For a more in depth discussion of homotopy and the fundamental group please see [20] [21] [22].

B.3 The Geometry of Elliptic Curves

We will present a brief discussion of the geometry of elliptic curves relevant to our discussion of Seiberg-Witten theory. The discussion here is largely taken from the first part of the first chapter on Algebraic Functions and Riemann surfaces in [29]. Consider a function $y(z)$ on the complex z-plane, where $y(z)$ is defined as

$$y^2 = a(z - r_1)(z - r_2)(z - r_3), \tag{B.3}$$

where $a, r_1, r_2, r_3 \in \mathbb{C}$ and $r_1 \neq r_2 \neq r_3$. We can see now that the function y is a double valued function. This is made clear by considering $y = \sqrt{a(z - r_1)(z - r_2)(z - r_3)}$. If we take the function y and deform around an odd number of poles on the complex plane the function will undergo a sign change. To make the function $y(z)$ single-valued on the complex

z-plane we make cuts on the plane. If we make a cut from r_1 to r_2 and another cut from r_3 to ∞ the function y becomes single-valued as there is now no way of deforming around an odd number of poles. Therefore no deformation around the complex plane will cause y to change sign. Let us elaborate on this point; with these two cuts in mind it should be clear that there is no way of circling any one pole without intersecting a cut. We can circle both r_1 and r_2, including the cut between them, but this does not result in a change of sign. Additionally, because of the cut from r_3 to ∞ we cannot circle all three poles.

The goal of this discussion is to better understand the nontrivial cycles that appear on the elliptic curves we deal with in our Seiberg-Witten analysis. Elliptic curves are genus one Riemann surfaces, or, in other words, a torus. Now we want to consider two copies of this complex plane with two cuts, which are referred to as sheets. One the first sheet we define C_1, a curve that goes around the r_1-r_2 cut, and on the second sheet we define C_2 to circle to poles at r_2 and r_3. C_2 is the a forbidden curve on the sheet in that it intersects both cuts. The next step is to map the sheets to Riemann spheres by the standard stereographic projection, and in doing so we map the point at ∞ to either the north or south pole on the Riemann sphere, depending on the projection. The point is that the infinite cut becomes finite on the sphere. Now we have two Riemann spheres each with two cuts in them.

Now we want to open up the cuts and extend the cuts outwards. Imagine now we have two spheres each with what looks like two short hoses extending outwards, where the open ends of the hoses correspond to the cuts we made. We now want to glue the hoses extending from each sphere together. This amounts to identifying each of the poles on the Riemann sphere to it's counterpart on the other Riemann sphere. Deforming the two spheres again we now end up with a torus. The illegal curve C_2 we made on the second sheet now extends round on the surface of the torus. The curves we started with C_1 and C_2 now correspond to the two nontrivial cycles on the torus c_1 and c_2.

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