# SUPERQUANTUM CORRELATIONS IN NON-LOCAL HIDDEN VARIABLE THEORIES

A study of superquantum correlations in quantum measure theory and crypto-nonlocal theory

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**Abstract**

Quantum mechanical phenomena have affected our lives in numerous ways. Their impact on modern technology ranges from everyday applications such as transistors, lasers and solar cells to ground breaking equipment such as Scanning-Tunneling electron microscopes. However, in spite all results in fundamental quantum theory research, various aspects of the theory are still not fully understood. This report investigates one of these challenges, namely the quantum correlations between particles in an entangled state. Early research into entangled states by Einstein, Podolsky and Rosen concluded that quantum mechanics theory is incomplete, and subsequently Bell showed that is must exhibit nonlocal correlations. This report reviews and compares two recent nonlocal hidden variable models that exhibit correlations that are more nonlocal than quantum mechanics. The first model by Barnett, Dowker and Rideout is based on quantum measure theory, while the model by Ghirardi and Romano is a crypto-nonlocal model. Our comparison addresses the question why PR boxes, a theoretical device that allows superquantum correlations, have not been observed in nature. We argue that the type of correlations that a hidden variable models allows, are highly dependent on the type of framework used to express the model as well as on the parameters of the variables.

# Contents

# 1 Introduction

Quantum mechanical phenomena have affected our lives in numerous ways. Their impact on modern technology ranges from everyday applications such as transistors, lasers and solar cells to ground breaking equipment such as Scanning-Tunneling electron microscopes [1]. Moreover, recent research in the field has produced significant advances in quantum information theory, which has the potential to create a whole new range of technical solutions for high-performance computing, data exchange and cryptography [2].

However, in spite all results in fundamental quantum theory research, various aspects of the theory are still not fully understood [3]. As an example, researchers continue to look for ways to unify classical theory and quantum mechanics. This report on the other hand investigates another peculiarity, namely the quantum correlations between particles in an entangled state. Entanglement is a fundamental, quantum mechanical concept associated with correlations between systems [1, 2]. An entangled state describes a system which consists of more than one object where it is impossible to treat each of these objects independently.

Early research into entangled systems resulted in the discovery of the famous EPR (Einstein, Podolsky and Rosen) paradox [4]. The authors examined a system made out of two particles that are in an entangled state [1]. Furthermore they assumed no superluminal signalling possible. The resulting mathematical model showed that by performing measurements on one particle it is possible to make predictions regarding the state of the other particle, which contradicts the Copenhagen interpretation of quantum mechanics. Since Einstein et al. believed in the validity of the principle of locality, they argued that quantum mechanics is an incomplete theory and that there must exist additional parameters that are introduced to particles during the preparation at the source. In the literature these parameters are referred to as *hidden variables* [1]. A few years later, Bohm [5, 6] presented a simpler example illustrating the entangled state described in the EPR paper which is now commonly referred to as the EPRB setup. The EPRB setup will be discussed in greater detail in section 2.1.

By considering a system in the EPRB setup, J.S. Bell derived his celebrated inequalities [7] using the same setup for the state as in [5, 6]. However, Bell showed that predictions of quantum mechanics violate these inequalities. He concluded that local hidden variable theories cannot hold. Later experimental results validated Bell's statement [1]. Until today these ground breaking inequalities set a limit to any fundamental hidden variable theory. The most familiar form of Bell inequalities, the CHSH (Clauser-Horne-Shimony-Holt) [8] inequalities, will be introduced in section 2.2. In addition to that, Tsirelson [9] derived a set of inequalities, which saturate the quantum mechanical bound. These inequalities are explained in sections 2.3 and 3.1.1. More recent research introduced a type of device called a PR box

[10], which can maximally violate Bell's inequalities. To date it is believed that a PR box does not exist in nature [11]. This report introduces two different nonlocal hidden variable models and examines their correlation function with respect to the above-mentioned inequalities.

## 1.1 Project aims and contributions

In the following an overview of the aims and contributions will be given and put into context with the structure of the report. The aim of this report is to enhance our understanding of the principles governing quantum mechanics. The overall focus lies on the investigation of the nonlocal nature of quantum theory. One of the key challenges that is addressed is the questions as to why models, which exhibit correlations that are more nonlocal than quantum correlations, are not believed to exist in nature [10, 12, 13].

In order to do so, two hidden variable models are introduced and compared. The first model was developed by Barnett et al. [11], the second one is an even more recent model introduced by Ghirardi et al. [14]. While the former model is expressed in the framework of quantum measure theory [15], the latter approaches the problem by looking at crypto-nonlocal theories [16]. Both models exhibit correlations which violate the Tsirelson bound, however, the first model [11] only examines correlations that maximally violate the Bell inequalities. In contrast to that Ghirardi et al. propose a model that allows a wide range of superquantum correlations that asymptotically approach a PR box.

The report aims to achieve two main goals. Firstly, it provides the reader with the foundations that are required to understand the two models. Secondly, it compares and contrasts the two theories. The following enumeration gives an account of the work presented.

1. Unified notation is introduced to simplify the understanding and comparison of various theories and models
2. A detailed summary of related background theory from text books and historical articles
3. Recent research results related to the two main models are reviewed
4. Where appropriate numerical and conceptual examples were added to illustrate theories better
5. Steps in the derivation of equations and proofs, which were omitted in the literature due to space constraints were added
6. Some additional proofs for conjectures are given that are commonly used in the literature without being shown explicitly
7. A discussion and analysis of the two main models is done and potential future research opportunities are highlighted

The unification of the notation is necessary since the different ideas presented in this report were originally derived in a number of different formalisms, making it harder to identify similarities and differences. Furthermore, the summary of the relevant background theory is introduced in order to remind the reader of the main concepts behind the theories and models discussed in this report. However, it is assumed that the reader is already familiar with most topics covered in this background review. Additionally, recent research results related to the two main models are reviewed. Any frameworks or formalisms required to understand the models are explained. Numerical and conceptual examples are provided to illustrate theories better and to validate some of the results quoted in the references. These examples are found in section 3.1.2 equations 30 and 31 and in the four appendices. Furthermore, additional steps were added to some of the derivations presented in this report. These steps were omitted in the literature, but added in here to help the reader to follow the derivations more easily. Examples of such additional steps can be found in equations 19, 20, 26, 28, 68 and Appendix D. Moreover, proofs for conjectures are given to explicitly show the origin of some commonly used concepts, cf. equation 32 and the proof found in section 3.2. Finally, a detailed comparison of the two main models is given. It is concluded that the range of values for the correlation function in any valid model is strongly dependent on the formalism in which it is derived. Furthermore, Ghirardi et al. [13, 14] argue that superquantum correlations exist in the intermediate level of the integration of the hidden variables. This report argues that this could be a possible reason as to why PR boxes cannot be found in nature. If hidden variable models were to reflect reality then it would be impossible to solely consider partial averages over hidden variables since it is important to consider the entire range of pairwise hidden variable values that might affect the outcome of experiments.

The report is organised as follows. In chapter 2 some essential concepts and theory related to the foundations of quantum mechanics are reviewed. This is done in historical order, starting with the introduction of EPR paradox [4] and the EPRB [5, 6] setup in section 2.1. This section is important in order to understand the concept behind Bell's theorem and inequalities [7] which are introduced in section 2.2. The inequalities are presented in their more familiar form known as the CHSH inequalities [8]. Moreover the Tsirelson inequalities [9], a weaker set of inequalities which are satisfied by quantum mechanics, are defined in section 2.3. Finally, the main concept behind a PR box [10] are explained in section 2.4. The PR box concept is portrayed as the two hidden variable models that are investigated in this report exhibit such a device.

The next chapter begins with an introduction to quantum measure theory [15] (cf. section 3.1). This formalism is needed to understand the first hidden variable model developed by Barnett et al. [11] in section 3.2. In addition to this in section 3.1.1 the Tsirelson's inequalities are derived using results from quantum measure theory [11, 12]. Furthermore, the relation between Tsirelson's inequalities and ordinary quantum mechanics [9] is illustrated in section 3.1.2, to allow a comparison between the Barnett's hidden variable model and the laws of ordinary quantum mechanics.

Chapter 4 examines Ghirardi's model of nonlocal hidden variable theory [14] in class of theories known as crypto-nonlocal theories [16]. Research into this class of theories is relatively new and therefore in section 4.1 the main principles behind these theories are explained for both 2- and N-dimensional systems [13]. The hidden variable model is described in section 4.2.

A detailed analysis of the two models is given in chapter 5. In chapter 6 the main ideas and concepts from this report are summarised and suggestions for future research are made.

# 2 Background Theory

In this chapter basic concepts and theory related to the foundations of quantum mechanics are reviewed in historical order. While it is generally assumed that the reader has come across these concepts before, some of the notation that is used in later chapters will be explained. Section 2.1, introduces the EPRB setup which later inspired the formulation of Bell's inequalities (cf. section 2.2). Subsequently Tsirelson's inequalities are stated in section 2.3 and finally PR-boxes, which violate both Bell's and Tsirelson's inequalities, and superquantum correlations are defined in section 2.4.

## 2.1 The EPRB setup

As mentioned in the introduction the concept of the EPRB setup resulted from research conducted by Einstein, Podolsky and Rosen in [4], which was then revisited by Bohm [5, 6]. They assumed the underlying quantum mechanical formalism. The general EPRB setup consists of a particle emitting source which generates a pair of spin-half particles in the singlet spin state [17, 18]. The particles are sent to two distant parties, commonly known as Alice and Bob, which reside in two spacelike separated regions, A and B . Each party then uses a Stern-Gerlach analyzer to measure the spin of the particle in a particular direction. As this entangled state has zero total spin it can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\mathbf{n}\uparrow\rangle_A |\mathbf{n}\downarrow\rangle_B - |\mathbf{n}\downarrow\rangle_A |\mathbf{n}\uparrow\rangle_B) \tag{1}$$

where $\boldsymbol{n}$ is a unit vector in three dimensional space and $|\mathbf{n}\uparrow\rangle_A$ represents particle A's spin-up in the $\boldsymbol{n}$ direction. Therefore, if Alice or Bob were to measure their particles in the $\boldsymbol{n}$ direction they would obtain $\pm\frac{\hbar}{2}$ with probability $\frac{1}{2}$ each [17, 18, 19]. Moreover, it was shown that by applying the laws of quantum mechanics that there exists a perfect anti-correlation between particles A and B. In other words, when Alice and Bob measure the state of their particle in same direction and Alice obtains $+\frac{\hbar}{2}$ she immediately knows that Bob's result is $-\frac{\hbar}{2}$ and vice versa. Hence one person can precisely determine the measurement result of the other instantaneously and without any information exchange [1]. Einstein et al. [4] believed that the physical world adheres to the principle of locality. Locality implies that particles can only be instantaneously affected by their immediate surroundings[1]. This observation led to the suggestion [4] that any outcomes of these experiments are predetermined by supplementary parameters also called *hidden variables*. In this view there must exist an unknown, more fundamental deterministic theory to which quantum mechanics is a probabilistic approximation [3]. Interestingly, in the year following the initial findings related to the EPRB setup, relatively few advances were made in the field of hidden variable theories [13]. It was only when J.S. Bell [7] published his findings that are discussed in the next section that novel hidden variable models became popular once more and nonlocal hidden variables theories such as Nelson stochastic mechanics [20] were developed.

## 2.2 Bell's theorem and CHSH inequalities

Working the same setup described in section 2.1 J.S Bell found that local deterministic hidden variable theories cannot successfully reproduce certain predictions made by quantum mechanics [7]. To show this he derived a set of inequalities from first principles, which any local hidden variable theory needs to respect. He then compared these bounds to similar bounds derived using quantum mechanical theory. The comparison showed that the bounds required by local hidden variable theories can be violated by quantum mechanics[1]. Later Bell's theorem was confirmed by experiments that showed that the theoretical inequalities for local hidden variable theories were indeed breached [21, 22]. However, while the research community generally accepts Bell's theorem, it is important to note that up to date there are still some concerns about the accuracy of the experiment, since certain problems such as the detection loophole [23], might have affected the results more than originally assumed.

In the following the derivation of these inequality is show in the Clauser, Horne, Shimony and Holt (CHSH) form [8]. The notation for the measurements, outcomes and probability distribution is adapted from [11]. Assume Alice's apparatus can be set to measure in either $a$ or $a'$ direction with outcomes $i$ or $i'$ respectively. Equivalently Bob's analyser can measure in the $b$ or $b'$ direction with outcomes $j$ and $j'$ respectively. Since Alice and Bob do not necessarily measure in the same direction, outcomes might not be anti-correlated. However, Bell concluded from locality that both outcomes are determined during the preparation at the source that is to say there must exist a hidden variable[7, 19]. In contrast to the EPRB setup, the factor of $\frac{\hbar}{2}$ is dropped and instead outcomes can be either $+1$ or -1. Therefore a source can produce 16 possible outcomes represented by the strings $(ii'jj')$. Where the probability for each one of this string of outcomes is $\mathbb{P}(ii'jj')$, s.t.

$$0 \leq \mathbb{P}(ii'jj') \leq 1, \qquad \sum_{ii'jj'} \mathbb{P}(ii'jj') = 1 \tag{2}$$

However, since it is impossible for Alice and Bob to measure their particles simultaneously in both directions only the following marginal probabilities can be obtained experimentally:

$$\begin{aligned} \mathbb{P}_{ab}(ij) &= \sum_{i'j'} \mathbb{P}(ii'jj'), & \mathbb{P}_{a'b}(i'j) &= \sum_{ij'} \mathbb{P}(ii'jj') \\ \mathbb{P}_{ab'}(ij') &= \sum_{i'j} \mathbb{P}(ii'jj'), & \mathbb{P}_{a'b'}(i'j') &= \sum_{ij} \mathbb{P}(ii'jj') \end{aligned} \tag{3}$$

Assume that experimental probabilities of the EPRB setup admit a joint probability distribution. Then the average value of the product of the two outcome is given by the the correlator $X_{\alpha\beta}$ where $\alpha \in \{a, a'\}$ and $\beta \in \{b, b'\}$. Hence:

$$\begin{aligned} X_{ab} &= \sum_{ij} ij\mathbb{P}_{ab}(ij), & X_{ab'} &= \sum_{ij'} ij'\mathbb{P}_{ab'}(ij') \\ X_{a'b} &= \sum_{i'j} i'j\mathbb{P}_{ab}(i'j), & X_{a'b'} &= \sum_{i'j'} i'j'\mathbb{P}_{a'b'}(i'j') \end{aligned} \tag{4}$$

and define $Q_n$ as

$$\begin{aligned} Q_1 &= X_{ab} + X_{a'b} + X_{ab'} - X_{a'b'}, & Q_2 &= X_{ab} + X_{a'b} - X_{ab'} + X_{a'b'} \\ Q_3 &= X_{ab} - X_{a'b} + X_{ab'} + X_{a'b'}, & Q_4 &= -X_{ab} + X_{a'b} + X_{ab'} + X_{a'b'} \end{aligned} \tag{5}$$

By substituting equations 3 into the equations 4 and 5 and by considering the allowed values for the outcomes it can be shown that

$$-2 \leq Q_n \leq 2 \tag{6}$$

The family of inequalities shown in equation 6 are known as the CHSH inequalities, which are similar to the Bell's inequalities. The before-mentioned experimental violation of these boundaries occurred when some choices of measurement directions $a, a', b$ and $b'$ showed that a looser bound of is required[21, 22]. This bound turned out to be the Tsirelson bound [9] $|Q_n| \leq 2\sqrt{2}$ . Moreover, it was shown the all experimentally derived bounds respect predictions made by quantum mechanical theory[1]. As a result, Bell concluded that the assumption of locality was wrong[7]. Therefore only nonlocal hidden variable theories are possible.

### 2.3 Tsirelson's inequalities

The Tsirelson's inequalities [9] are analogous to Bell's inequalities shown in the section above. Their main difference is that unlike Bell's inequalities, there is no contradiction between Tsirelson's inequalities and the known experimental results discussed in section 2.2[11, 21]. Tsirelson's bounds agree with the empirical bounds of $|Q_n| \leq 2\sqrt{2}$. Any kind of theory that allow a looser bound than the Tsirelson bound give rise to *superquantum* correlations. The full derivation of the Tsirelson's inequalities in a quantum mechanics formalism is shown in [9] However, in 3.1.1 an alternate derivation using the quantum measure theory is shown in detail.

## 2.4 PR box

Popescu and Rohrlich [10] showed that, in theory, it is possible to have correlations that maximally violate the quantum mechanical bound given by Tsirelson's inequalities while adhering to no-signalling. To do so, one needs to consider a device that is known as a PR box [10, 24]. In general a box is a device that which given some input, for example a state of a particle, can give 2 possible outcomes. If the correlators take their maximum value of 1 or $-1$ the inequalities for the correlation function yield

$$-4 \leq Q_n \leq 4 \tag{7}$$

By using a PR box the correlation values one would expect are $-4, 4$. However, it is currently argued [10, 11, 13] that to date theories that allow the existence of a PR box do not faithfully describe the physical world. Therefore, there is still a vast amount of ongoing research that tries to explain as to why these correlations disagree with quantum mechanics and why PR boxes do not exist in nature. This is especially interesting because research in quantum information has shown [24] that if such a box were to exist, then the amount of classical information needed to be exchanged when two parties communicate would be significantly reduced.

# 3 A hidden variable model with Superquantum correlations in quantum measure theory

This chapter builds on the background chapter, which discussed general concepts related to quantum mechanics. In the following quantum measure theory [15], a more recently develop framework for quantum mechanics is introduced, as it is required to enable the reader to understand the hidden variable model [11] presented in section 3.2. The main body of work produced in this chapter is unification of different notations into a single format, which will simplify the comparison of the PR box model shown in this chapter with the one in chapter 4. Moreover, to help the reader follow the theory, proofs taken from the literature were augmented with additional steps. In addition to that the proofs shown in 3.1.2 were not taken from the literature and solved as part of this project. Quantum measure theory will be introduced in section 3.1 and Tsirelson's equations are derived using this framework [11, 12]. Furthermore, a nonlocal hidden variable model that allows the existence of a PR box is reviewed in section 3.2.

## 3.1 Quantum measure theory

Quantum measure theory (QMT) is another formalism which also describes quantum mechanics [15]. It was suggested by R. Sorkin and it is based on the *sum-over-histories* / *spacetime* approach. The motivation behind the theory is that it unifies classical mechanics and quantum dynamics, where the former is contained in the latter [25]. In Sorkin's hierarchy of measure theories states that the reality comprised of a set of histories [26]. A history is denoted by $\gamma$. For example, in QMT an electron in the double slit experiment follows only one path, but this path is not predetermined and therefore expressed in a probabilistic manner [15, 25, 27].

In measure theory one considers a sample space $\Omega$ which contains all the possible histories for a given system. The dynamics of the system is obtained from a measure $\mu$ on $\Omega$[15]. This measure is a non-negative real set function, which from a classical point of view can be interpreted as a probability function.

In [11, 15] the authors consider the following series of symmetric set-functions for a given $\mu$

$$
\begin{aligned}
I_1(X_1) \equiv &\quad \mu(X_1) \\
I_2(X_1, X_2) \equiv &\quad \mu(X_1 \sqcup X_2) - \mu(X_1) - \mu(X_2) \\
I_3(X_1, X_2, X_3) \equiv &\quad \mu(X_1 \sqcup X_2 \sqcup X_3) - \mu(X_1 \sqcup X_2) - \mu(X_2 \sqcup X_3) - \mu(X_3 \sqcup X_2) \\
&\quad + \mu(X_1) + \mu(X_2) + \mu(X_3) \\
\vdots & \\
I_n(X_1, X_2, \ldots, X_n) \equiv &\quad \mu(X_1 \sqcup X_2 \sqcup ...X_n) - \sum_n \mu(n-1\ sets) + \sum_{n(n-1)} \mu(n-2\ sets) \\
&\quad \cdots \pm \sum_n \mu(X_n)
\end{aligned} \tag{8}
$$

where the $\sqcup$ symbolizes a disjoint union and $X_n$ are disjoint subsets of $\Omega$. It can be shown that the above functions express the following recursive relationship [15]:

$$
I_{n+1}(X_0, X_1, X_2, \ldots, X_n) = I_n(X_0 \sqcup X_1, X_2, \cdots, X_n) - I_n(X_0, X_2, \cdots, X_n) - I_n(X_1, X_2, \cdots, X_n) \tag{9}
$$

This implies that each level of the hierarchy contains all lower levels. Furthermore, a theory that satisfies the sum rule $I_{k+1} = 0$ automatically satisfies all higher sum rules, $I_{k+n} = 0$ for all $n \geq 1$ [15]. Such theory is classified as a *measure theory of level $k$* [11, 25]. The lowest level of an additive set function hierarchy contains classical stochastic theories that satisfy the Kolmogorov sum rule (cf. equation10 below). It is referred to as a *level 1* measure theory.

$$
\mu(X_1 \sqcup X_2) = \mu(X_1) + \mu(X_2) \tag{10}
$$

However, in quantum mechanics there is an amplitude in form of a complex number, associated with each history. $I_2$ is closely related to the interference. The sum rule is given by

$$
\mu(X_1 \sqcup X_2) = \mu(X_1) + \mu(X_2) + interference \tag{11}
$$

This sum rule implies quantum mechanics can be described by a level 2 measure theory which is referred to as *quantum measure theory* [12, 15, 25]. The main dynamical quantity in this theory is called a *quantum measure* given by equation 12. The interference is represented by the *decoherence functional* $D(X_1; X_2)$. The quantal measure cannot always be interpreted as an experimental probabilities. For very specific cases were the interference term is zero the quantal measure is the the experimental probabilities. Otherwise, The quantal measure is the amplitude square of pairs of alternative spacetime histories[15, 11].

$$\mu(X_n) = D(X_n; X_n) \tag{12}$$

The decoherence functional for pairs of subsets of $\Omega$ satisfies the following[12]:

(i) Hermiticity: $D(X_i; X_j) = D(X_j; X_i)^*$, $\forall X_i, X_j$

(ii) Additivity: $D(X_i \sqcup X_j; X_k) = D(X_i; X_k) + D(X_j; X_k)$, $\forall X_i, X_j, X_k$where $X_i$ and $X_j$ are disjoint

(iii) Positivity: $D(X_i : X_j) \geq 0$ $\forall X_i, X_j$; where $i = j$

(iv) Normalization: $D(\cdot; \cdot) = 1$

For any ordinary unitary quantum mechanical theory the decoherence functional is strongly positive [10, 12]. This means that the $n \times n$ Hermitian matrix $M$, with elements$M_{ij} \equiv D(X_i; X_j)$ has no negative expectation values i.e. it is positive semi-definite for any finite collection of subsets $X_1, X_2, ...X_n$ of $\Omega$. It would be shown in section 3.1.1, that the Tsirelson's inequalities are derived in the QMT framework using the condition of strong positivity, as this condition allows to associate a Hilbert space with the quantum measure [12]. Therefore decoherence functional is strongly positive in quantum mechanics.

In the following the EPRB setup [4, 5, 6] is revised in the context of sample space and Quantum Measure Theory. Consider the EPRB setup, where a source emitting a pair of spin-half particles and sends them to two distant parties Alice and Bob as described in 2.1, where each party has a type of Stern-Gerlach analyzers in order to measure the spin in some direction with outcomes being either $+1$ or $-1$. As there are two analyzers in each experimental setting one obtains a set of $2 \times 2 = 4$ experimental probabilities. Therefore, each set of experimental probabilities admits a probability distributions, $\mathbb{P}_{\rho\epsilon}$ defined on a sample space $\Omega_{\rho\epsilon} = \Omega_\rho \times \Omega_\epsilon$ where $\rho$ and $\epsilon$ are the settings of Alice's and Bob's analyzers respectively and $\Omega_\rho$ and $\Omega_\epsilon$ are sample spaces associated with the experimental outcomes [12]. Note that additional sets of experimental probabilities can be obtained by varying the direction of the spin measurements. The sample space of the entire system $\hat{\Omega}$ consists of the collection of all such possible experimental probabilities and is called the *system of experimental probabilities.*

When looking at the outcome of joint measurements like in section 2.2, where the experimental setting consists of spin measurements with two possible settings for each apparatus:$a$ or $a'$ and $b$ or $b'$ for Alice's and Bob's experiments respectively, then there are a total of $2^4 = 16$ elements in the finite sample space $\hat{\Omega}$. This corresponds to the following possible probability distributions $\mathbb{P}_{\alpha\beta}$ with $\alpha \in \{a, a'\}$ and $\beta \in \{b, b'\}$. The impossibility of superluminal signalling is assumed [2, 11] and therefore Alice's marginal probability distribution must remain unaffected by Bob's choice of measurement and vice versa. This results in the following four *no-signalling* conditions

$$\sum_j \mathbb{P}_{ab}(ij) = \sum_{j'} \mathbb{P}_{ab'}(ij'), \qquad \sum_i \mathbb{P}_{ab}(ij) = \sum_{i'} \mathbb{P}_{a'b}(i'j)$$
$$\sum_j \mathbb{P}_{a'b}(i'j) = \sum_{j'} \mathbb{P}_{a'b'}(i'j'), \qquad \sum_i \mathbb{P}_{ab'}(ij') = \sum_{i'} \mathbb{P}_{a'b'}(i'j') \tag{13}$$

where$i$ and $i'$ are the outcomes of the $a$ and $a'$ measurements respectively, whereas $j$ and $j'$ represent the results of Bob's measurements. Therefor it is said that the *joint probability distribution* in this setting admits no-signalling conditions [12]. This means that the four experimental probabilities can be mapped onto a single sample space $\hat{\Omega} = \Omega_a \times \Omega_{a'} \times \Omega_b \times \Omega_{b'}$. The elements of $\hat{\Omega}$ are labeled by the sixteen 4-element bit strings $(ii\acute{}jj\acute{}) \; : \; i, i\acute{}, j, j\acute{} \in \{-1, 1\}$. In this setup there are only 8 independent probabilities, given explicitly in Appendix A, due to the no-signalling conditions and normalisation.

If there exists a strongly positive decoherence functional $D$ on $\hat{\Omega}$ which satisfy the conditions given in equation 14 it is said that the experimental probabilities admit a *strongly positive joint quantal measure (SPJQM). [12]*

$$\begin{aligned}
\sum_{i'j'k'l'} D(ii'jj'; kk'll') &= \mathbb{P}_{ab}(ij)\delta_{ik}\delta_{jl} \\
\sum_{i'jk'l} D(ii'jj'; kk'll') &= \mathbb{P}_{ab'}(ij')\delta_{ik}\delta_{j'l'} \\
\sum_{ij'kl'} D(ii'jj'; kk'll') &= \mathbb{P}_{a'b}(i'j)\delta_{i'k'}\delta_{jl} \\
\sum_{ijkl} D(ii'jj'; kk'll') &= \mathbb{P}_{a'b'}(i'j')\delta_{i'k'}\delta_{j'l'}
\end{aligned} \tag{14}$$

These conditions guarantee that the 24 off diagonal elements in the mentioned matrix $M$ will vanish. These are for example the elements in which $i$ and $j$ are different than $k$ and $l$ respectively [12].

### 3.1.1 Tsirelson's inequalities in QMT

In this section Tsirelson's inequalities [9] will be derived using the QMT formalism. As mentioned in section 2.2 these bounds are essential to understand the limitations of hidden variable theories in quantum mechanics. As the discussion in 3.2 looks at a hidden variable theory [11] from QMT point of view it is important to show that Tsirelson's bounds also hold in this formalism.

It can be shown that Tsirelson's inequalities,$|Q_n| =\leq 2\sqrt{2}$ , holds if a joint quantal measure exists for $n = 1, ..., 4$. This bound agrees with the experimental result [12].

Proof - adapted from [12]:

Define $|ii'jj'\rangle$ as the set of vectors that span the Hilbert space $\mathcal{H}$. It is important to note that states denoted by$|ii'jj'\rangle$ do not span the entire vector space containing vectors $[ii'jj']$ as zero norm vectors are excluded. More formally, let $\mathcal{H}_1$ be the space spanned by $[ii'jj']$, the Hilbert space $\mathcal{H}$ is

$$\mathcal{H} = \frac{\mathcal{H}_1}{\mathcal{H}_0} \tag{15}$$

where $\mathcal{H}_0$ is the subspace of zero norm states. Since $D(ii\acute{}jj\acute{};kk\acute{}ll\acute{})$ is strongly positive the Hermitian inner product on $\mathcal{H}$ is given by

$$\langle ii\acute{}jj\acute{}, kk\acute{}ll\acute{}\rangle = D(ii\acute{}jj\acute{};kk\acute{}ll\acute{}) \tag{16}$$

where it is assumed that the experimental probabilities admits an SPJQM as specified by the set of equations given in 14. Furthermore define

$$\begin{aligned} |a\rangle &= \sum_{ii'jj'} i\,|ii'jj'\rangle\,, \qquad & |a'\rangle &= \sum_{ii'jj'} i'\,|ii'jj'\rangle \\ |b\rangle &= \sum_{ii'jj'} j\,|ii'jj'\rangle\,, \qquad & |b'\rangle &= \sum_{ii'jj'} j'\,|ii'jj'\rangle \end{aligned} \tag{17}$$

and

$$|a\pm\rangle = \sum_{i'jj'} |\pm 1i'jj'\rangle\,, \qquad |b\pm\rangle = \sum_{ii'j'} |ii' \pm 1j'\rangle \tag{18}$$

where$|a'\pm\rangle$ and $|b'\pm\rangle$ are defined in a similar manner. For the remainder of this report $+1$ and -1 will be denoted "+" and "−" respectively. This yields

$$\begin{aligned} \langle a+|a-\rangle = &\sum_{i'jj'k'll'} \langle +i'jj'\,|-k'll'\rangle = \sum_{i'jj'k'll'} D(+i\acute{}jj\acute{};-k\acute{}ll\acute{}) \\ = &\sum_{jl}\sum_{i'j'k'l'} D(+i\acute{}jj\acute{};-k\acute{}ll\acute{}) = 0 \end{aligned} \tag{19}$$

from equation 14 . Furthermore

$$\begin{aligned} \langle a|a\rangle = &\ \langle a+|a+\rangle + \langle a-|a-\rangle - \langle a+|a-\rangle - \langle a-|a+\rangle = \langle a+|a+\rangle + \langle a-|a-\rangle \\ = &\sum_{jl}\sum_{i'j'k'l'} D(+i\acute{}jj\acute{};+k\acute{}ll\acute{}) + \sum_{jl}\sum_{i'j'k'l'} D(-i\acute{}jj\acute{};-k\acute{}ll\acute{}) \\ = &\sum_{jl}\mathbb{P}_{ab}(+j)\delta_{jl} + \sum_{jl}\mathbb{P}_{ab}(-j)\delta_{jl} = \sum_{j}\mathbb{P}_{ab}(+j) + \mathbb{P}_{ab}(-j) \\ = &\ \mathbb{P}_{ab}(++) + \mathbb{P}_{ab}(-+) + \mathbb{P}_{ab}(+-) + \mathbb{P}_{ab}(--) = 1 \end{aligned} \tag{20}$$

The same applies to $\langle a'|a'\rangle = \langle b|b\rangle = \langle b'|b'\rangle = 1$. Additionally the correlator $X_{ab}$ can be written as

$$X_{ab} = \sum_{ij} ij \ \mathbb{P}_{ab}(ij) = \sum_{ijkl} il \ \mathbb{P}_{ab}(ij)\delta_{ik}\delta_{jl}$$

and substitute equation 14 into the above to obtain

$$\begin{aligned} X_{ab} = & \sum_{ijkl} il \sum_{i'j'k'l'} D(ii'jj';kk'll') = \sum_{ii'jj'kk'll'} il \quad D(ii'jj';kk'll') \\ = & \sum_{ii'jj'kk'll'} il \quad \langle ii\acute{}jj\acute{};kk\acute{}ll\acute{}\rangle = \langle a|b\rangle \end{aligned} \tag{21}$$

similarly

$$X_{ab'} = \langle a|b'\rangle\,, \qquad X_{a'b} = \langle a'|b\rangle\,, \qquad X_{a'b'} = \langle a'|b'\rangle \tag{22}$$

Now consider the case of the $Q_1$ correlations. By combining equations 4, 5, 21 and 22 one obtains

$$\begin{aligned} Q_1 = & \quad \langle a|a\rangle + \langle a'|b\rangle + \langle a|b'\rangle - \langle a'|b'\rangle = (\langle a| + \langle a'|)\,|b\rangle + (\langle a| - \langle a'|)\,|b'\rangle \\ \leq & \quad \|\,|a\rangle + |a'\rangle\| + \|\,|a\rangle - |a'\rangle\| \end{aligned}$$

since it was argued above that the vectors $|\alpha\rangle$and $|\beta\rangle$ are unit vectors and therefore $Q_1$ will be maximised when $|a\rangle + |a'\rangle \parallel |b\rangle$ and $|a\rangle - |a'\rangle \parallel |b'\rangle$

$$\|\,|a\rangle \pm |a'\rangle\|^2 = \langle a \pm a'|a \pm a'\rangle = 2 \pm 2Re\,\langle a|a'\rangle$$

also let $O$ be

$$\begin{aligned} O = & \quad \|\,|a\rangle + |a'\rangle\| + \|\,|a\rangle - |a'\rangle\| \\ O^2 = & \quad \|\,|a\rangle + |a'\rangle\| + \|\,|a\rangle - |a'\rangle\|^2 + 2\,\|\,|a\rangle + |a'\rangle\|\,\|\,|a\rangle - |a'\rangle\| \\ = & \quad 2 + 2Re\,\langle a|a'\rangle + 2 - 2Re\,\langle a|a'\rangle + 2\sqrt{(2 + 2Re\,\langle a|a'\rangle)(2 - 2Re\,\langle a|a'\rangle)} \\ = & \quad 4 + 2\sqrt{[4 - 4(Re\,\langle a|a'\rangle)^2]} \\ \leq & \quad 4 + 2\sqrt{4} \end{aligned}$$

therefore

$$O = Q_1 \leq 2\sqrt{2}$$

QED

Similarly it is possible to prove that

$$|Q_n| \leq 2\sqrt{2} \tag{23}$$

where $n = 1...4$. Equation 23 is known as *Tsirelson* I. Another set of inequalities known as the *Tsirelson II* inequalities state that [9, 11] if a system of probabilities such as portrayed in the EPRB setup admits a SPJQM then

$$|Q_{\text{II}_n}| \leq \pi \tag{24}$$

for $n = 1...4$. Where

$$\begin{aligned}
Q_{\text{II}_1} &= \quad |arcsinX_{ab} + arcsinX_{a'b} + arcsinX_{ab'} - arcsinX_{a'b'}| \\
Q_{\text{II}_2} &= \quad |arcsinX_{ab} + arcsinX_{a'b} - arcsinX_{ab'} + arcsinX_{a'b'}| \\
Q_{\text{II}_3} &= \quad |arcsinX_{ab} - arcsinX_{a'b} + arcsinX_{ab'} + arcsinX_{a'b'}| \\
Q_{\text{II}_4} &= \quad |-arcsinX_{ab} + arcsinX_{a'b} + arcsinX_{ab'} + arcsinX_{a'b'}|
\end{aligned} \tag{25}$$

where $-\frac{\pi}{2} \leq arcsinX_{\alpha\beta} \leq -\frac{\pi}{2}$. The inequalities in equation 24 can be proven using the QMT framework. The proof below is adapted from [11]. The proof illustrated here only considers the case for $Q_{\text{II}_1}$. However, similar steps can be applied to prove that the inequalities hold for any $Q_{\text{II}_n}$. Define

$$\begin{aligned}
\theta_1 &= \frac{\pi}{2} - arcsinX_{ab}, \qquad & \theta_2 &= \frac{\pi}{2} - arcsinX_{ab'} \\
\theta_3 &= \frac{\pi}{2} - arcsinX_{a'b}, \qquad & \theta_4 &= \frac{\pi}{2} - arcsinX_{a'b'}
\end{aligned}$$

hence

$$X_{ab} = sin(\frac{\pi}{2} - \theta_1) = cos\theta_1 \tag{26}$$

then from equations 17, 20, 21 , 22 and 26 it follows

$$\begin{aligned}
\langle a|b\rangle &= cos\theta_1, \qquad & \langle a|b'\rangle &= cos\theta_2 \\
\langle a'|b\rangle &= cos\theta_3. \qquad & \langle a'|b'\rangle &= cos\theta_4
\end{aligned} \tag{27}$$

The figure below portrays four coplanar unit vectors with angles $\theta_n$

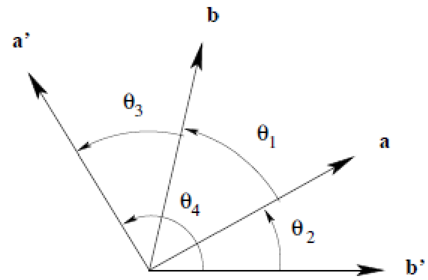


Figure 1: Coplanar unit vectors $a$,$a'$, $b$ and $b'$. From ref. [11]

If the vectors are coplanar and if $\theta_1+\theta_2+\theta_3\leq\pi$ then $\theta_1+\theta_2+\theta_3=\theta_4$. Also if the vectors are not coplanar and $\theta_1+\theta_2+\theta_3\leq\pi$ then $\theta_1+\theta_2+\theta_3\geq\theta_4$ since $\theta_4\leq\pi$. However, when $\theta_1+\theta_2+\theta_3>\pi$ then in any case $\theta_1+\theta_2+\theta_3-\theta_4\geq 0$. Thus in general one obtains $-\theta_1-\theta_2-\theta_3+\theta_4\leq 0$ which gives

$$\begin{aligned}
0 \geq & \quad -\frac{\pi}{2}+arcsinX_{ab}-\frac{\pi}{2}+arcsinX_{ab'}-\frac{\pi}{2}+arcsinX_{a'b}+\frac{\pi}{2}-arcsinX_{a'b} \\
0 \geq & \quad arcsinX_{ab}+arcsinX_{ab'}+arcsinX_{a'b}-arcsinX_{a'b}-\pi \\
\pi \geq & \quad arcsinX_{ab}+arcsinX_{ab'}+arcsinX_{a'b}-arcsinX_{a'b}
\end{aligned} \tag{28}$$

Moreover by reversing the signs of Bob's outcomes one can show in a similar manner that

$$arcsinX_{ab}+arcsinX_{ab'}+arcsinX_{a'b}-arcsinX_{a'b}\geq-\pi$$

QED

Any quantum correlations generally satisfy these inequalities which are weaker than Bell's inequalities.

#### 3.1.2 Quantum mechanics and Tsirelson's inequalities

In the following the concepts behind *ordinary quantum model for the probabilities* (OQMP) and *ordinary quantum model for the correlators* (OQMC) are explained. Moreover, the relationships between OQMP, OQMC and the Tsirelson's inequalities are stated. This is necessary for discussion of PR boxes in section 2.4.

By definition [11, 28, 29], an *ordinary quantum model for the probabilities* (OQMP) consist of a vector $|\psi\rangle$ in a Hilbert space$\mathcal{H}$ and a pair of projective decompositions of unity $\{P_a^+,P_a^-\}$, $\{P_{a'}^+,P_{a'}^-\}$,$\{P_b^+,P_b^-\}$ and $\{P_{b'}^+,P_{b'}^-\}$ such that $\left[P_\alpha^i,P_\beta^j\right]=0$. Where as before $\alpha\in\{a,a'\}$ and $\beta\in\{b,b'\}$with $i,j\in\{-1,+1\}$

and $P_\alpha^+ + P_\alpha^- = P_\beta^+ + P_\beta^- = I$ . Additionally the following statement must be true for all $\alpha$ and $\beta$, and all $i$ and $j$.

$$\mathbb{P}_{\alpha\beta}(ij) = \left\langle \psi | P_\alpha^i P_\beta^j | \psi \right\rangle \tag{29}$$

Example for measurements in OQMP:

$P_\alpha^i$ and $P_\beta^j$ are projectors. A $P_\alpha^i$measurement on $|\psi\rangle$ followed by a $P_\beta^j$ measurement on the new state has the probability

$$\mathbb{P}_{\alpha\beta}(ij) = \left\langle \psi | P_\alpha^i P_\beta^j P_\beta^j P_\alpha^i | \psi \right\rangle \tag{30}$$

this can be simplified since projectors $P_\alpha^i$ and $P_\beta^j$ satisfy $P_\beta^j P_\beta^j = P_\beta^j$ and $P_\alpha^i P_\alpha^i = P_\alpha^i$. Furthermore the commutator relation implies $P_\beta^j P_\alpha^i = P_\alpha^i P_\beta^j$. Hence

$$\mathbb{P}_{\alpha\beta}(ij) = \left\langle \psi | P_\alpha^i P_\beta^j P_\alpha^i | \psi \right\rangle = \left\langle \psi | P_\alpha^i P_\alpha^i P_\beta^j | \psi \right\rangle = \left\langle \psi | P_\alpha^i P_\beta^j | \psi \right\rangle \tag{31}$$

An *ordinary quantum model for the correlators* (OQMC) [11, 28, 29], on the other hand, contains two pairs of self-adjoint operators $S_a, S_{a'}$ and $S_b, S_{b'}$ and Hilbert space $\mathcal{H}$ with a vector $|\psi\rangle$. $[S_\alpha, S_\beta] = 0$. In an OQMC the predicted value of the correlator must be

$$X_{\alpha\beta} = \langle \psi | S_\alpha S_\beta | \psi \rangle \tag{32}$$

In the following it is formally shown that an OQMC can exists if there is a OQMP.

Proof:

Given an OQMP one can define $S_\alpha, S_\beta$

$$\begin{aligned} S_a &= P_a^+ - P_a^- & S_{a'} &= P_{a'}^+ - P_{a'}^- \\ S_b &= P_b^+ - P_b^- & S_{b'} &= P_{b'}^+ - P_{b'}^- \end{aligned} \tag{33}$$

Since $P_\alpha^i$ and $P_\beta^j$ are projectors they are Hermitian, thus $S_a, S_{a'}$ and $S_b, S_{b'}$ are self-adjoint operators. Therefore from equation 32

$$\begin{aligned} \langle \psi | S_\alpha S_\beta | \psi \rangle = \;& = \left\langle \psi | \left(P_\alpha^+ - P_\alpha^-\right) \left(P_\beta^+ - P_\beta^-\right) | \psi \right\rangle \\ &= \left\langle \psi | \left(P_\alpha^+ P_\beta^+ - P_\alpha^+ P_\beta^- - P_\alpha^- P_\beta^+ + P_\alpha^- P_\beta^-\right) | \psi \right\rangle \\ &= \left\langle \psi | \left(P_\alpha^+ P_\beta^+\right) | \psi \right\rangle - \left\langle \psi | \left(P_\alpha^+ P_\beta^-\right) | \psi \right\rangle - \left\langle \psi | \left(P_\alpha^- P_\beta^+\right) | \psi \right\rangle + \left\langle \psi | \left(P_\alpha^- P_\beta^-\right) | \psi \right\rangle \end{aligned}$$

Hence using 29

$$\begin{aligned}\langle\psi|S_\alpha S_\beta|\psi\rangle = & \ \mathbb{P}_{\alpha\beta}(++) - \mathbb{P}_{\alpha\beta}(+-) - \mathbb{P}_{\alpha\beta}(-+) + \mathbb{P}_{\alpha\beta}(++) \\ = & \ X_{\alpha\beta}\end{aligned}$$

as by the definition of $X_{\alpha\beta}$ given by equation 4

$$X_{\alpha\beta} = \sum_{ij} ij \ \mathbb{P}_{\alpha\beta}(ij)$$

Thus if OQMP exists then an OQMC also exists. Note the converse does not hold.

QED

Finally it is important to understand that while the TsirelsonIIinequalities are necessary for both OQMP and OQMC to exist, they are a sufficient condition for the existence of an OQMC, but not for an OQMP [9, 11]. Moreover, it can be shown that a *strongly positive joint decoherence function* (SPJDF) (cf. section 3.1) exists if an OQMP exists.

### 3.2 PR box in QMT

Consider the above setup of the EPRB in the framework of QMT. If a PR box (cf. section [10]) exists it violates the TsirelsonII inequalities and therefore neither OQMP nor OQMC can exist. Given the following set of experimental probabilities, the above setup can maximally violate Tsirelson's inequalities and reach the upper bound of 4 [11]. The set of probabilities are

$$\begin{aligned}\mathbb{P}_{ab}(++) &= \mathbb{P}_{ab}(--) &= \frac{1}{2} \\ \mathbb{P}_{ab'}(++) &= \mathbb{P}_{ab'}(--) &= \frac{1}{2} \\ \mathbb{P}_{a'b}(++) &= \mathbb{P}_{a'b}(--) &= \frac{1}{2} \\ \mathbb{P}_{a'b'}(+-) &= \mathbb{P}_{a'b'}(-+) &= \frac{1}{2}\end{aligned} \tag{34}$$

where all other marginal probabilities are zero. It can now be shown that for maximum nonlocal correlations the only possible values for the marginal probabilities are $\frac{1}{2}$ and 0.

Proof:

Given $Q_1 = X_{ab} + X_{a'b} + X_{ab'} - X_{a'b'}$ where any $X_{\alpha\beta}$ is defined by the set of equations 4. In order to maximise the correlations and obtain $Q_1 = 4$, the values of $X_{\alpha\beta}$ should be $X_{ab} = X_{a'b} = X_{ab'} = 1$ and $X_{a'b'} = -1$. To have $X_{ab} = 1$ the values of $\mathbb{P}_{ab}(ij)$ for choices of $i, j$ s.t. $ij = -1$ have to be minimal, i.e $\mathbb{P}_{ab}(+-) = \mathbb{P}_{ab}(-+) = 0$, since the correlator was defined as $X_{ab} = \sum_{ij} ij\mathbb{P}_{ab}(ij)$. The same applies for

$X_{a'b}$ and $X_{ab'}$. For $X_{a'b'} = -1$, however, the marginal probabilities which yield $\mathbb{P}_{a'b'}(i'j') \neq 0$ are only needed for the anti-correlated outcomes. Combining these conditioning with the no-signalling conditions as written explicitly in Appendix A implies

$$\begin{aligned} \mathbb{P}_{ab}(++) = \quad & \mathbb{P}_{ab}(--) = \mathbb{P}_{ab'}(++) = \mathbb{P}_{ab'}(--) = \mathbb{P}_{a'b}(++) \\ = \quad & \mathbb{P}_{a'b}(--) = \mathbb{P}_{a'b'}(+-) = \mathbb{P}_{a'b'}(-+) \end{aligned} \tag{35}$$

Additionally $X_{\alpha\beta}$ can be written explicitly as

$$\begin{aligned} X_{ab} &= \mathbb{P}_{ab}(++) + \mathbb{P}_{ab}(--) &&= 1 \\ X_{a'b} &= \mathbb{P}_{a'b}(++) + \mathbb{P}_{a'b}(--) &&= 1 \\ X_{ab'} &= \mathbb{P}_{ab'}(++) + \mathbb{P}_{ab'}(--) &&= 1 \\ X_{a'b'} &= -\mathbb{P}_{a'b'}(+-) - \mathbb{P}_{a'b'}(-+) &&= -1 \end{aligned} \tag{36}$$

From 35 and 36 one obtains

$$\begin{aligned} \mathbb{P}_{ab}(++) = \quad & \mathbb{P}_{ab}(--) = \mathbb{P}_{ab'}(++) = \mathbb{P}_{ab'}(--) = \mathbb{P}_{a'b}(++) \\ = \quad & \mathbb{P}_{a'b}(--) = \mathbb{P}_{a'b'}(+-) = \mathbb{P}_{a'b'}(-+) = \frac{1}{2} \end{aligned} \tag{37}$$

with all other marginal probabilities being zero. Similarly for $Q_1 = -4$ it is required that $X_{ab} = X_{a'b} = X_{ab'} = -1$ and $X_{a'b'} = 1$. Therefore this time $\mathbb{P}_{a'b'}(i'j') = 0$ whenever the outcomes $i'$ and $j'$ are anti-correlated. Also it is required that the marginal probabilities $\mathbb{P}_{ab}(ij) = \mathbb{P}_{ab}(ij) = \mathbb{P}_{ab}(ij) = 0$ for outcomes with the same sign. Following the same procedure as for the $Q_1 = 4$ case it can be shown that for $Q_1 = -4$

$$\begin{aligned} \mathbb{P}_{ab}(+-) = \quad & \mathbb{P}_{ab}(-+) = \mathbb{P}_{ab'}(+-) = \mathbb{P}_{ab'}(-+) = \mathbb{P}_{a'b}(+-) \\ = \quad & \mathbb{P}_{a'b}(-+) = \mathbb{P}_{a'b'}(++) = \mathbb{P}_{a'b'}(--) = \frac{1}{2} \end{aligned} \tag{38}$$

The following proof is for the case of $Q_1$ and can be done in similar manner for all $Q_n$.

QED

In the EPRB setup considered above there exist 8 PR boxes corresponding to $Q_n = \pm 4$ where $n = 1 \ldots 4$. The probability distribution in equation 34 corresponds to the PR box for which the sum of correlators $Q_1 = 4$. The other 7 boxes are obtained by permuting the inputs and outputs of the two parties. For instance the probability distribution for $Q_1 = -4$ is given by equation 38.

In section 3.1.2 it was stated that a SPJDF exists if an OQMP exists. Furthermore, in [11, 12] it was shown that a PR box admits a joint quantal measure. The following decoherence functional described in equation 39 admits a joint quantal measure. Equation 39 corresponds to the PR box in equation 34 as shown in Appendix B.

$$\begin{aligned} D_{PR}(-+--;-+--) = \quad & D_{PR}(+++-;+++-) = D_{PR}(++-+;++-+) \\ = \quad & D_{PR}(---+;---+) = \frac{1}{2} \\ D_{PR}(---+;----) = \quad & D_{PR}(-+-+;+---) = D_{PR}(++-+;+++-) \\ = \quad & D_{PR}(---+;++-+) = \frac{1}{4} \\ D_{PR}(-+-+;-+--) = \quad & D_{PR}(-+-+;-+--) = D_{PR}(+++-;++--) \\ = \quad & D_{PR}(++-+;+---) = D_{PR}(---+;+---) \\ = \quad & D_{PR}(++-+;-+-+) = D_{PR}(---+;-+-+) = -\frac{1}{4} \end{aligned} \tag{39}$$

The decoherence functional is Hermitian. All remaining elements that are not specified above are equal to their Hermitian counterparts or zero. Furthermore it can be checked that this decoherence functional for the PR box in equation 34 satisfies

$$\sum_{\gamma_e \gamma_f} D_{PR}(\gamma_e;\gamma_f) \geq 0 \tag{40}$$

where $\gamma_e$ and $\gamma_f$ are specific histories represented by the 4 bit string $(ii'jj')$. This has been done by converting 39 into the matrix given below (equation 41). The labels $\gamma_e$ and $\gamma_f$ for the columns and rows were added to the matrix to portray which element refers to which history. In equation 41 the string representation of the histories was converted to binary strings, such that "+" becomes 0 and "-" becomes 1. Table 1 allocates a binary number to each history. Note that permutation of this matrix represent the

7 other possible PR boxes.

$$
\begin{array}{c|cccccccccccccccc}
 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} \\
\gamma_1 & & & & & & & & & & & & & & & & \\
\gamma_2 & & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & & & & & & & & & & & & \\
\gamma_3 & & \frac{1}{4} & \frac{1}{2} & & & & & -\frac{1}{4} & & & -\frac{1}{4} & & & & \frac{1}{4} & \\
\gamma_4 & & -\frac{1}{4} & & & & & & & & & & & & & & \\
\gamma_5 & & & & & & & & & & & & & & & & \\
\gamma_6 & & & & & & & & & & & & & & & & \\
\gamma_7 & & & & & & & & & & & & & & & & \\
\gamma_8 & & & -\frac{1}{4} & & & & & & & & \frac{1}{4} & & & & -\frac{1}{4} & \\
\gamma_9 & & & & & & & & & & & & & & & & \\
\gamma_{10} & & & & & & & & & & & & & & & & \\
\gamma_{11} & & & -\frac{1}{4} & & & & & \frac{1}{4} & & & & -\frac{1}{4} & & & -\frac{1}{4} & \\
\gamma_{12} & & & & & & & & & & & -\frac{1}{4} & \frac{1}{2} & & & & \\
\gamma_{13} & & & & & & & & & & & & & & & & \\
\gamma_{14} & & & & & & & & & & & & & & & & \\
\gamma_{15} & & & \frac{1}{4} & & & & & -\frac{1}{4} & & & -\frac{1}{4} & & & & \frac{1}{2} & \frac{1}{4} \\
\gamma_{16} & & & & & & & & & & & & & & & \frac{1}{4} &
\end{array}
\tag{41}
$$

| $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ | $\gamma_6$ | $\gamma_7$ | $\gamma_8$ | $\gamma_9$ | $\gamma_{10}$ | $\gamma_{11}$ | $\gamma_{12}$ | $\gamma_{13}$ | $\gamma_{14}$ | $\gamma_{15}$ | $\gamma_{16}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

Table 1: Allocation of binary strings to histories, where "+" was converted to 0 and "-" to 1

Technically $2^{16}$ subsets need to be considered to prove equation 40. However, it is important to note that only 8 rows and columns out of the 16 in the matrix contain non-zero elements. Hence, to show that the decoherence functional adheres to equation 40 one only has to consider the $8 \times 8$ matrix shown in equation 42.

$$
\begin{array}{c|cccccccc}
 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_8 & \gamma_{11} & \gamma_{12} & \gamma_{15} & \gamma_{16} \\
\gamma_2 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & & & & & \\
\gamma_3 & \frac{1}{4} & \frac{1}{2} & & -\frac{1}{4} & -\frac{1}{4} & & \frac{1}{4} & \\
\gamma_4 & -\frac{1}{4} & & & & & & & \\
\gamma_8 & & -\frac{1}{4} & & & \frac{1}{4} & & -\frac{1}{4} & \\
\gamma_{11} & & -\frac{1}{4} & & \frac{1}{4} & & -\frac{1}{4} & -\frac{1}{4} & \\
\gamma_{12} & & & & & -\frac{1}{4} & \frac{1}{2} & & \\
\gamma_{15} & & \frac{1}{4} & & -\frac{1}{4} & -\frac{1}{4} & & \frac{1}{2} & \frac{1}{4} \\
\gamma_{16} & & & & & & & \frac{1}{4} &
\end{array}
\tag{42}
$$

This reduction in the dimensions of the matrix might originate from the fact that there are only 8 individual probabilities in the system. The matrix elements were then used as an input for a python script [1], which checks that no n-subset violates equation 40. The script and a brief explanation of the program is provided in Appendix C.

Since Tsirelson II is violated the measure is not strongly positive. However, according to the QMT framework introduced in section 3.1the experimental probability admits a joint quantal measure [11, 15]. In fact, it was proven in [11] that for the EPRB setup, a $(2, 2, 2)$ setup with 2 parties, 2 direction measurements and 2 output, any system of experimental probability admits a joint quantal measure. It is interesting to note that this model of hidden variables fails for larger systems where more than one PR box exists [26].

In case of two PR boxes the sample space is $\hat{\Omega}_{2PR} = \Omega_{PR1} \times \Omega_{PR2}$ where $\Omega_{PR1}$ and $\Omega_{PR2}$ are the sample spaces corresponding to the two individual boxes. Hence $\hat{\Omega}_{2PR}$ consists of $2^8 = 256$ histories with $2^{256}$ possible subsets. In this case the decoherence function takes the form

$$D_{2PR}((\gamma_e; \gamma_f); (\gamma_{\overline{e}}; \gamma_{\overline{f}})) = D_{PR1}(\gamma_e; \gamma_f) D_{PR2}(\gamma_{\bar{e}}; \gamma_{\bar{f}}) \tag{43}$$

where $\gamma_e$ and $\gamma_f$ are specific histories in the sample space of the first PR box and $\gamma_{\bar{e}}$, $\gamma_{\bar{f}}$ are histories in the sample space of the second PR box. $D_{PR1}(\gamma_e; \gamma_f)$ and $D_{PR2}(\gamma_{\bar{e}}; \gamma_{\bar{f}})$ are the decoherence functions in the sample spaces $\Omega_{PR1}$ and $\Omega_{PR2}$ respectively. In this case the positivity condition is given by

$$\sum_{\gamma_e, \gamma_f, \gamma_{\overline{e}}, \gamma_{\overline{f}}} D_{2PR}((\gamma_e; \gamma_f); (\gamma_{\overline{e}}; \gamma_{\overline{f}})) \geq 0 \tag{44}$$

This condition is violated and therefore the hidden variable model discussed in this section cannot allow for a PR box to exist [26].

[1] The program was jointly written with Marcel Christoph Guenther

# 4 Hidden variable model with superquantum correlations in crypto-nonlocal theories

A crypto-nonlocal hidden variable model is introduced in this chapter [14]. In chapter 5 this model will then be compared to the hidden variable model from the previous chapter. In order to understand the idea behind crypto-nonlocal theories section 4.1 gives a detailed account of the main concepts behind crypto-nonlocal theories [37, 16]. After that a recently proposed hidden variable model is reviewed in section 4.2. This model is of particular interest as it allows superquantum correlations to occur, which violate predictions made by quantum mechanics. Generally the results of this model might enable a better insight into superquantum correlations, which naturally motivates the discussion in chapter 5. Similar to chapter 3, the main contribution in this chapter is the representation of the ideas shown in [13, 14, 16] using similar notation to one used in previous chapters, which creates a sold basis for the later comparison. Moreover, some numerical examples are provided to explain results better.

## 4.1 Crypto-nonlocal theories

*Crypto-nonlocal* theories are a broad class of non-local realistic theories, proposed by Leggett [16], that are incompatible with predictions of quantum mechanics. In [16] Leggett derived a new type of Bell's inequalities, which bound this particular class of nonlocal hidden variable theories. He further showed that those inequalities contradict predictions made by quantum mechanics. To illustrate this he described a type of experiment that is shown in figure 2

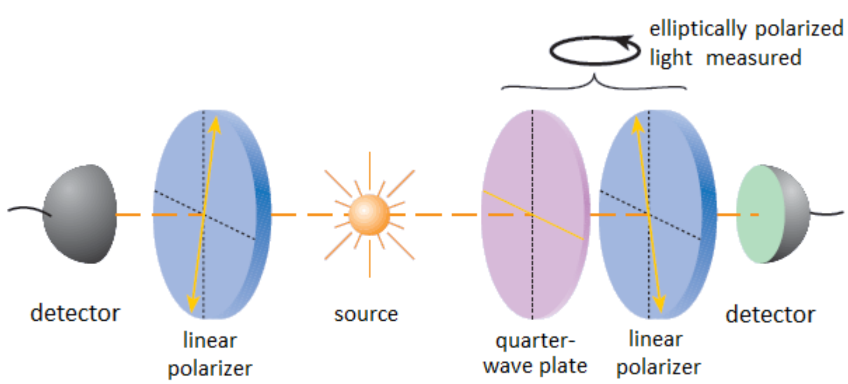


Figure 2: Experimental setup for measuring an elliptical or a linear polarization of two photons by using appropriate combinations of quarter-wave-plates and linear polarizers. From ref.[30]

### 4.1.1 Crypto-nonlocal theories for two 2-dimentional systems

In the following nonlocal realistic models, which belong to the class of Leggett's crypto-nonlocal theories, are described in [13]. However, some changes to the notation were made in order to ease the comparison between these models and the once described in previous sections.

Ghirardi et al. assume that a pair of maximally entangled photons are emitted from the source and photons have well-defined polarizations. Their state is given by equation 45

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|V_1V_2\rangle + |H_1H_2\rangle \tag{45}$$

where $|V_1\rangle$ and $|H_1\rangle$ are the vertical and horizontal states of photon 1 and $|V_2\rangle$ and $|H_2\rangle$ are polarization planes for photon 2. Alice receives a photon with polarization vector of unit length $\mathbf{u}$ and measures it in the $\alpha$ direction, while Bob receives a photon with polarization vector of unit length $\mathbf{v}$ and measures it in the $\beta$ direction. $\alpha$ and $\beta$ can be any directions. In section 3 those correspond to directions $a, a'$ for $\alpha$ and $b, b'$ for $\beta$. However, in this context $\alpha$ and $\beta$ can have arbitrary directions. The resulting measurement outcomes are either $+1$ or $-1$ due to the specific assignment of hidden variables. If locality is assumed, $A(\alpha)$ represents Alice's measurements in the $\alpha$ direction whereas $B(\beta)$ is Bob's outcome in the $\beta$ direction. In other words $i = A(a), \quad i' = A(a'), \quad j = B(b)$ and $j' = B(b')$. Yet, Leggett's models are nonlocal and the input of the first party influences the output of the second. To reflect this outputs will be noted as $A(\alpha, \beta, \mu)$ and $B(\alpha, \beta, \mu)$ where $\mu$ is a set of hidden variables corresponding to the distribution function $\rho_{\mathbf{u},\mathbf{v}}(\mu)$. Thus measurement outcomes are determined by two types of hidden variables $\mu$ and the polarization vectors ($\mathbf{u}$,$\mathbf{v}$), which is distributed according to $F(\mathbf{u},\mathbf{v})$ .

As in quantum mechanics correlators are given by equation 32. In this model, Alice's and Bob's operators are therefore represented by $\hat{A}(\alpha)$ and $\hat{B}(\beta)$ [13, 16]. Hence

$$X^{\psi_+}_{\alpha\beta} \equiv \left\langle \psi_+ | \hat{A}(\alpha)\hat{B}(\beta) | \psi_+ \right\rangle \tag{46}$$

The averages are of single photon measurements are

$$X^{\psi_+}_{\alpha} \equiv \left\langle \psi_+ | \hat{A}(\alpha) | \psi_+ \right\rangle, \qquad X^{\psi_+}_{\beta} \equiv \left\langle \psi_+ | \hat{B}(\beta) | \psi_+ \right\rangle \tag{47}$$

and the following conditions must hold

$$
\begin{aligned}
X_{\alpha}^{\psi+} &= \int d\mathbf{u}d\mathbf{v}F(\mathbf{u},\mathbf{v}) \int d\mu\rho_{\mathbf{u},\mathbf{v}}(\mu)A(\alpha,\beta,\mu) \\
X_{\beta}^{\psi+} &= \int d\mathbf{u}d\mathbf{v}F(\mathbf{u},\mathbf{v}) \int d\mu\rho_{\mathbf{u},\mathbf{v}}(\mu)B(\alpha,\beta,\mu) \\
X_{\alpha\beta}^{\psi+} &= \int d\mathbf{u}d\mathbf{v}F(\mathbf{u},\mathbf{v}) \int d\mu\rho_{\mathbf{u},\mathbf{v}}(\mu)A(\alpha,\beta,\mu)B(\alpha,\beta,\mu)
\end{aligned} \tag{48}
$$

the Leggett inequalities yield [16, 32]

$$
1-|X_{\alpha}^{\psi+} - X_{\beta}^{\psi+}| \geq X_{\alpha\beta}^{\psi+} \geq -1 + |X_{\beta}^{\psi+} + X_{\beta}^{\psi+}|
$$

Leggett uses experiments to motivate crypto-nonlocal theory. These experiments assume that two photons with indefinite polarization are emitted from a source. In this case condition $X_{\alpha\beta}^{\psi+}$ from equation 48 does not apply, though the other two conditions still do. As Leggett assumed nonlocality arises from the condition that probability of the outcome for the first party is not affected by the measured outcome of the other party [13]. This assumption is known as outcome Independence [31]. However, he claimed [13, 16] that parameter independence does not necessary hold. Parameter independence [31] is the assumption probability of the outcome of one party is not affected by the of the measurement on the other side. For crypto-nonlocal theories, the following conditions, which represent no-signalling and *washed out* nonlocality [13] must hold

$$
\begin{aligned}
\int d\mu\rho_{\mathbf{u},\mathbf{v}}(\mu)A(\alpha,\beta,\mu) &= f(\alpha,\mathbf{u}) \\
\int d\mu\rho_{\mathbf{u},\mathbf{v}}(\mu)B(\alpha,\beta,\mu) &= g(\beta,\mathbf{v})
\end{aligned} \tag{49}
$$

where $f(\alpha,\mathbf{u})$ and $g(\beta,\mathbf{v})$ are given in Leggett's proposal as follows

$$
f(\alpha,\mathbf{u}) = 2(\mathbf{u}\cdot\alpha)^2 - 1, \qquad g(\beta,\mathbf{v}) = 2(\mathbf{v}\cdot\beta)^2 - 1 \tag{50}
$$

Since nonlocality was introduced, the prediction of such this theory violated Bell's inequalities, for both linearly and elliptically polarized photons. Leggett further showed that these predictions also violate quantum mechanics. Hence, he claimed that it is insufficient solely assume nonlocality. This observation has also been validated experimentally by [33], who showed that correlations between two entangled photons violated Leggett's inequality for nonlocal realistic theories.

Furthermore it was shown that in the two qubit system case, crypto-nonlocal theories compatible with quantum mechanics must satisfy

$$
f(\alpha,\mathbf{u}) = g(\beta,\mathbf{v}) = 0 \tag{51}
$$

because local observables average over $\psi$in quantum mechanics

$$X_{\alpha}^{\psi_{+}}=X_{\beta}^{\psi_{+}}=0 \tag{52}$$

#### 4.1.2 Crypto-nonlocal theories of a two $N$-dimensional systems

In [13, 14] Ghirardi et al. considered a more general case of the model introduced in section 4.1.1, in which the state $\psi$ is a maximally entangled state of a two $N$-level systems and $N \geq 2$. A maximally entangled state is given by the Schmidt decomposition [2]:

$$|\psi\rangle=\frac{1}{\sqrt{N}}\sum_{j}^{N}|W_j\rangle\,|V_j\rangle \tag{53}$$

where $\{|V_j\rangle\,;j\}$and $\{|W_j\rangle\,;j\}$ are orthonormal bases

Under these circumstances [13] $\lambda$ is the set of hidden variables $\lambda=(\mu,\tau)$ with distribution function $\rho(\lambda)\equiv\rho(\mu,\tau)$. Note, $\mu$ is also referred to as *deep level* hidden variable. Moreover, the distribution function$\rho(\lambda)$ can be written as

$$\rho(\lambda)=\rho(\mu,\tau)=\rho(\mu|\tau)\rho(\tau) \tag{54}$$

This requires the following more general set conditions (analogous to equation 48)

$$\begin{aligned}
X_{\alpha}^{\psi} &=\int d\lambda\rho(\lambda)A_{\psi}(\alpha,\beta,\lambda) &&=\int d\tau\rho(\tau)f_{\psi}(\alpha,\tau)\\
X_{\beta}^{\psi} &=\int d\lambda\rho(\lambda)B_{\psi}(\alpha,\beta,\lambda) &&=\int d\tau\rho(\tau)g_{\psi}(\beta,\tau)\\
X_{\alpha\beta}^{\psi} &=\int d\lambda\rho(\lambda)A_{\psi}(\alpha,\beta,\lambda)B_{\psi}(\alpha,\beta,\lambda) &&=\int d\tau\rho(\tau)X_{\alpha\beta}^{\psi,\tau}
\end{aligned} \tag{55}$$

where $f_{\psi}(\alpha,\tau)$ and $g_{\psi}(\beta,\tau)$ are intermediate averages over $\mu$ (equation 57), $A_{\psi}(\alpha,\beta,\lambda)$ and $B_{\psi}(\alpha,\beta,\lambda)$ are the outcomes (similarly to section 4.1.1) and

$$X_{\alpha\beta}^{\psi,\tau}=\int d\mu\rho(\mu|\tau)A_{\psi}(\alpha,\beta,\mu,\tau)B_{\psi}(\alpha,\beta,\mu,\tau) \tag{56}$$

the intermediate averages over $\mu$. These averages

$$\begin{aligned}
f_{\psi}(\alpha,\tau) &=\int d\mu\rho(\mu|\tau)A_{\psi}(\alpha,\beta,\mu,\tau)\\
g_{\psi}(\beta,\tau) &=\int d\mu\rho(\mu|\tau)B_{\psi}(\alpha,\beta,\mu,\tau)
\end{aligned} \tag{57}$$

are equivalent to the no-signaling conditions and result in washed out nonlocality. In addition to this

Ghirardi et al. proved in [13] that for systems with an arbitrarily large number of dimensions, a crypto-nonlocal theory is equivalent to quantum mechanics if it satisfies

$$f_\psi(\alpha, \tau) = g_\psi(\beta, \tau) = 0 \tag{58}$$

Equation 58 is a generalization of equation 51. Moreover they showed that when equation 58 applies one has to have

$$f_\psi(\alpha, \tau) = X^\psi_\alpha, \qquad g_\psi(\beta, \tau) = X^\psi_\beta \tag{59}$$

Previous research on crypto-nonlocal theories [33, 34, 35] argued that due to these conditions crypto-nonlocal theories cannot assign local values which quantum mechanics cannot. Hence, the authors in [33, 34, 35] considered that as a proof that these theories are simply a local version of quantum mechanics and thus not of particular interest.

## 4.2 PR box in crypto-nonlocal theories

In the following a crypto-nonlocal hidden variables model is introduced that can give rise to both quantum and superquantum correlation. In this model the framework is the same as the one introduced in section 4.1.2[13]. Alice can measure in direction $\alpha$ and her outcome is denoted by $A_\psi(\alpha, \beta, \lambda)$, where $\lambda$ is the hidden variable in the model. Similarly, Bob can measures in the $\beta$ direction with outcome $B_\psi(\alpha, \beta, \lambda)$. $A_\psi(\alpha, \beta, \lambda)$ and $B_\psi(\alpha, \beta, \lambda)$ can take values $\pm 1$. The correlators are given by equation 55. As in section 4.1.2 $\lambda$ is specified by two variables $\mu$ and $\tau$ corresponding to lower and upper level hidden variables respectively. In Ghirardi's model [14] $\lambda$ is a unit vector in $\mathbb{R}^3$ with orthogonal reference frame (x,y,z). $\lambda$ is uniformly distributed over a unit sphere and $\mu$ and $\tau$ are the polar angles specifying this sphere. $\mu$ and $\tau$ are defined such that $\mu \in [0, 2\pi)$ and $\tau \in [0, \pi)$. The relation between $\mu$,$\tau$ and the conventional polar angles $\theta$and $\phi$ is portrayed in equation 60:

$$\begin{array}{llllll} \mu = & \theta & \tau & = & \phi & \text{for} \quad y \geq 0 \\ \mu = & 2\pi - \theta & \tau & = & \phi - \pi & \text{for} \quad y < 0 \end{array} \tag{60}$$

The unit vectors of the direction $\alpha$ and $\beta$ are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively and $\omega$ is the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where $0 \leq \omega \leq 0$. Vectors $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ lie in the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-plane and form an angle $\hat{\omega}$. By definition $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ must be each others image when mirrored with respect to $\omega$'s bisector. Much like in Bell's model the relationship between$\omega$ and $\hat{\omega}$ in the case of this model is given by

$$\hat{\omega} = \pi \sin^2(\frac{\omega}{2}) \tag{61}$$

where

$$\begin{aligned} \hat{\omega} \leq \quad & \omega \qquad \text{for} \quad \omega \quad \leq \frac{\pi}{2} \\ \hat{\omega} > \quad & \omega \qquad \text{for} \quad \omega \quad > \pi \end{aligned}$$

The measurement operators are $\hat{A}(\alpha) = \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}$ and $\hat{B}(\beta) = \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ represents the three Pauli matrices. The outcomes of the measurements are uniquely determined by the values assigned to hidden variable $\lambda = (\mu, \tau)$ where

$$\begin{aligned} A_\psi(\alpha, \beta, \mu, \tau) \quad &= \text{sgn}(\hat{\boldsymbol{\alpha}} \cdot \lambda) \\ B_\psi(\alpha, \beta, \mu, \tau) \quad &= -\,\text{sgn}(\hat{\boldsymbol{\beta}} \cdot \lambda) \end{aligned} \tag{62}$$

Hence the values $\pm 1$ of the observables are related to opposite hemispheres of the unit sphere of $\lambda$. Moreover, integration over a maximal circle is equivalent to integration over the deeper level hidden variable $\mu$ resulting in equation 58. Hence from equations 55 and 58 Ghirardi and Raffaele [14] showed that this crypto-nonlocal model gives similar predictions as quantum mechanics when integrating over all hidden variables.

In order to show that this model exhibits superquantum correlations, integration over only the deeper level hidden variable, equation 63 needs to be evaluated. This equation represents the averages over $\mu$ of the correlators and it is analogous to equation 56

$$X_{\alpha\beta}^{\psi,\tau} = \frac{1}{4}\int_0^{2\pi} d\mu \, |\sin(\mu)| \, A_\psi(\alpha, \beta, \mu, \tau) B_\psi(\alpha, \beta, \mu, \tau) \tag{63}$$

The evaluation of the above integration is done for a model [14] with four measurement in the (x,z)-plane $\alpha \in \{\mathbf{a}, \mathbf{a}'\}$ and $\beta \in \{\mathbf{b}, \mathbf{b}'\}$ corresponding to

$$\begin{aligned} \mathbf{a} \quad &= (\sin(\eta), 0, \cos(\eta))\,, \qquad & \mathbf{a}' = \quad & (\text{-}\sin(\eta), 0, \cos(3\eta)) \\ \mathbf{b} \quad &= (\text{-}\sin(\eta), 0, \cos(\eta))\,, \qquad & \mathbf{b}' = \quad & (\sin(3\eta), 0, \cos(3\eta)) \end{aligned} \tag{64}$$

where $\eta \in [\mathbf{0}, \frac{\pi}{\mathbf{4}})$. Ghirardi and Raffaele defined the following parameters $\chi_n$ for $n = 1, ..., 4$

$$\chi_n = \chi_n(\eta, \tau) = \frac{\cos(\tau)}{\sqrt{\cos^2(\tau) + \cot^2(\frac{\kappa_n(\eta)}{2})}} \tag{65}$$

where $\kappa_n(\eta)$ are simply

$$\begin{aligned} \kappa_1(\eta) &= \pi\sin^2(\eta), & \kappa_2(\eta) &= \pi\sin^2(3\eta) \\ \kappa_3(\eta) &= 4\eta + \pi\sin^2(\eta), & \kappa_4(\eta) &= 4\eta - \pi\sin^2(\eta) \end{aligned} \tag{66}$$

The joint correlation $X^{\psi,\tau}_{\mathbf{ab}}$ can be computed in the following way. Using the relation given in equation 67 (cf. Appendix D for derivation).

$$\mathrm{sgn}(\hat{\boldsymbol{a}} \cdot \lambda)\mathrm{sgn}(\hat{\boldsymbol{b}} \cdot \lambda) = \mathrm{sgn}(\hat{\boldsymbol{a}} \cdot \lambda)(\hat{\boldsymbol{b}} \cdot \lambda) = \mathrm{sgn}\left(\chi_1^2 - \cos^2(\mu)\right) \tag{67}$$

and hence

$$\begin{aligned} X^{\psi,\tau}_{\mathbf{ab}} &= \frac{1}{2}\int_0^\pi d\mu \sin(\mu)\mathrm{sgn}\left(\chi_1^2 - \cos^2(\mu)\right) \\ &= \frac{1}{2}\int_1^{-1} - dy\mathrm{sgn}\left(\chi_1^2 - y^2\right) \\ &= \frac{1}{2}\int_{-1}^{1} dy\mathrm{sgn}\left(|\chi_1|^2 - y^2\right) \\ &= 2|\chi_1| - 1 \end{aligned} \tag{68}$$

The derivation of the other correlations are rather difficult but they are given in [14] as

$$\begin{aligned} X^{\psi,\tau}_{\mathbf{ab}} &= 2|\chi_1| - 1, & X^{\psi,\tau}_{\mathbf{a'b'}} &= 2|\chi_2| - 1 \\ X^{\psi,\tau}_{\mathbf{a'b}} &= |\chi_3 - \chi_4| - 1, & X^{\psi,\tau}_{\mathbf{ab'}} &= |\chi_3 - \chi_4| - 1 \end{aligned} \tag{69}$$

when $0 \le \eta \le \tilde{\eta}$.

$$\begin{aligned} X^{\psi,\tau}_{\mathbf{ab}} &= 2|\chi_1| - 1, & X^{\psi,\tau}_{\mathbf{a'b'}} &= 2|\chi_2| - 1 \\ X^{\psi,\tau}_{\mathbf{a'b}} &= 1 - |\chi_3 + \chi_4|, & X^{\psi,\tau}_{\mathbf{ab'}} &= 1 - |\chi_3 - \chi_4| \end{aligned} \tag{70}$$

when $\tilde{\eta} \le \eta \le \frac{\pi}{4}$. Where $\tilde{\eta}$ is the solution of $4\eta + \pi\sin^2\eta = \pi$. Analogously to equation 5 the following correlations can be obtained

$$\begin{aligned} Q_1^{\psi,\tau} &= X^{\psi,\tau}_{\mathbf{ab}} + X^{\psi,\tau}_{\mathbf{a'b}} + X^{\psi,\tau}_{\mathbf{ab'}} - X^{\psi,\tau}_{\mathbf{a'b'}} \\ &= 2|\chi_1| + 1 + |\chi_3 - \chi_4| - 1 + |\chi_3 - \chi_4| - 1 - 2|\chi_2| \\ &= 2\left(|\chi_1| - |\chi_2| + |\chi_3 - \chi_4| - 1\right) \end{aligned} \tag{71}$$

for $0 \leq \eta \leq \tilde{\eta}$

$$\begin{aligned} Q_1^{\psi,\tau} &= X_{\mathbf{ab}}^{\psi,\tau} + X_{\mathbf{a'b}}^{\psi,\tau} + X_{\mathbf{ab'}}^{\psi,\tau} - X_{\mathbf{a'b'}}^{\psi,\tau} \\ &= 2\left|\chi_1\right| + 1 + 1 - \left|\chi_3 + \chi_4\right| + 1 - \left|\chi_3 - \chi_4\right| - 1 - 2\left|\chi_2\right| \\ &= 2\left(\left|\chi_1\right| - \left|\chi_2\right| - \left|\chi_3 + \chi_4\right| + 1\right) \end{aligned} \tag{72}$$

for $\tilde{\eta} \leq \eta \leq \frac{\pi}{4}$. The values for $\tau$ and $\eta$ that allow a PR box are in the neighbourhood of $\tau = \frac{\pi}{2}$ and $\eta = \frac{\pi}{6}$ for the lower bound and $\tau = (\frac{\pi}{2})$ and $\eta = \tilde{\eta}$ for the upper bound. Hence $-4 \leq Q_{1\psi,\tau} \leq 4$. It is important to note that $Q_1^{\psi,\tau}$ reaches this bound asymptotically. This is because $\cos(\frac{\pi}{2}) = 0$ and therefore $\chi_n = 0$. Hence $Q_1^{\psi,\tau} = \pm 2$. But in the neighbourhood of the above angles the bound is reached asymptotically. An explicit example for this is given in Appendix D.

The values that of $Q_1^{\psi,\tau}$ for different combinations of $\eta$ and $\tau$ is given in figure 3

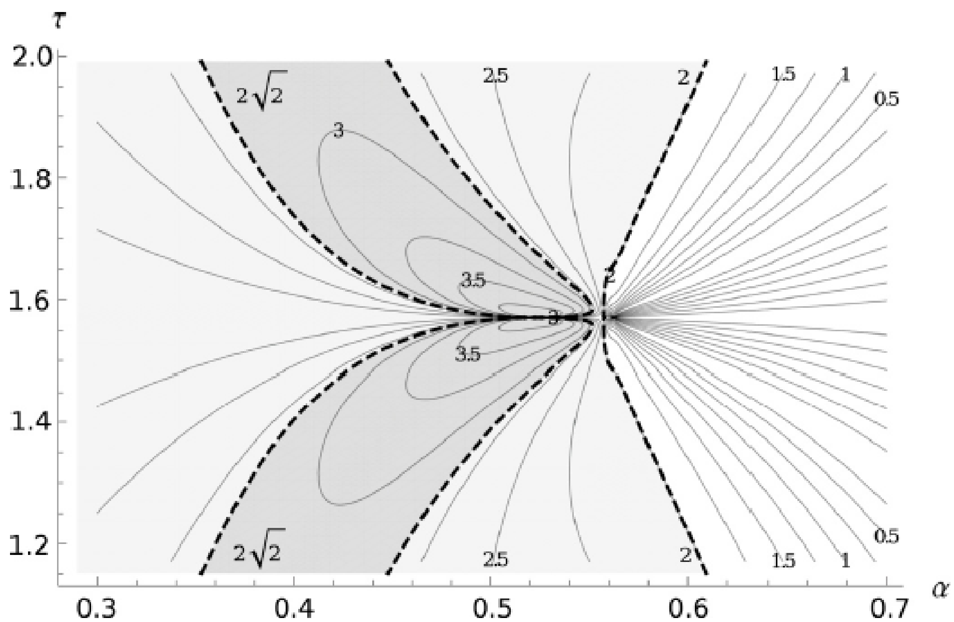


Figure 3: Contour plot of the function $\left|Q_1^{\psi,\tau}\right|$ in the space of $\eta$ and $\tau$. Three regions can be observed. The dark grey region corresponds to $\left|Q_1^{\psi,\tau}\right| \geq 2\sqrt{2}$ where Tsirelson's bound is violated and superquantum correlation are observed. The area in light grey represents quantum nonlocality which respect the Tsirelson bound but violate Bell's inequalities $2 \leq \left|Q_1^{\psi,\tau}\right| \leq 2\sqrt{2}$. Finally the white area is the one for which local hidden variables theories hold. From ref.[14]

Finally, by averaging $\left|Q_1^{\psi,\tau}\right|$over the higher level hidden variable $\tau$, correlations corresponding to quantum mechanics are reproduced. This model therefore encapsulates all possible forms of hidden variables theories for the (2,2,2) setup. In addition it reaches asymptotically correlations that exhibits a PR box. Hence, the model allows a comparison between crypto-nonlocal theories, which were possibly prematurely discarded, and quantum mechanics[14].

# 5 Discussion

In this chapter the two hidden variable models introduces in sections 3.2, 4.2 are analysed and compared. The first model by Barnett et al. [11] was derived in the mathematical QMT framework, whereas the second model by Ghirardi et al. [14] is derived using crypto-nonlocal theory.

The correlations that are described in Barnett's model maximally violate Tsirelson's inequalities. On the other hand, in Ghirardi's model the maximum bound is only reached asymptotically and produces correlations for all known bounds, i.e. local, quantum mechanical and superquantum bounds. The exact type of correlations depends on the intervals over which the hidden variables are integrated. According to Ghirardi et al. [14], local correlations arise since nonlocality is washed out when imposing the conditions in equation 57.

In section 3.2it was mentioned that the Barnett's model does not faithfully describe larger sample spaces that contain more than one PR box[26]. This is because the positivity condition given by equation 44 does not hold. However, Ghirardi's model does not cover the case where more than one box is assumed. Therefore, future research into this model should investigate this scenario and see if the revised model is still consistent with original one and still produces the same type of correlations.

Barnett's model features a joint probability which allows the existence of a PR box distribution which admits a joint quantal measure [12]. This joint quantal measure is not strongly positive as was stated in [11], cf. section 3.2. Hence the correlation function in their model violates the Tsirelson inequalities and also contradicts OQMP. Hence they concluded that the question of whether a PR Box can exist in nature is related to the mathematical condition of strong positivity [11]. In contrast to Barnett's model which was formulated in the QMT framework, the crypto-nonlocal theories used by Ghirardi et al. to describe their model, belong to the class of nonlocal hidden variable theories which describe physical experiments [16]. As crypto-nonlocal theories are equivalent to quantum mechanics only when the condition given by equation 58 holds [13], Ghirardi et al. [14] argue that their model gives some sort of physical interpretation to the relationship between superquantum correlations and quantum mechanics. However, we argue that the last statement does not necessarily imply that the superquantum correlations exist. Ghirardi's model allows superquantum correlations only at intermediate level of crypto-nonlocal theories[14, 13], i.e. only when integrating over the deeper level hidden variables. Yet, in order to reproduce the quantum mechanical bound one has to integrate over both hidden variables. Therefore, we conclude that in this model the question whether a PR Box exists or not, is related to the integration over the different levels of hidden variables. Hence as far as we can see, it is vital to average over the entire joint hidden variable space as one should not ignore possible effects on the final outcome.

# 6 Conclusion & Further work

In the introduction section it was mentioned that one of the main challenges in quantum mechanics research is its lack of unification with classical theories. An elegant formalism that partially solves this problem is QMT [15]. As it has been highlighted in chapter 3, the significance of this framework is that classical deterministic theories are contained within the path integral formulation of quantum mechanics. This is because the classical deterministic interpretation of our world can be expressed by level 1 sum rules in the hierarchy of measure theory, whereas the dynamics of quantum mechanics are contained in level 2 hierarchies. Moreover, it was shown that lower levels of the hierarchy are contained within higher levels.

The derivation of the Tsirelson inequalities in this framework was shown in [11, 12], cf. section 3.1.1. The authors derived two bounds, the Tsirelson I inequality (cf. equation 23) and the stronger Tsirelson II bound (cf. equation 24). Moreover, for an OQMP to exist Tsirelson II must hold. Since Tsirelson's inequalities were derived assuming the existence of SPJQM, this implies that the existence of SPJQM is also a necessary condition for the existence of OQMP [12]. Whether the converse holds remains an open question and is subject to future research.

The concepts behind OQMP and SPJQM were introduced in order to investigate Barnett's nonlocal hidden variable model [11] in the QMT framework (c.f section 3.2). The authors concluded that the question of whether a PR Box can exist in nature is related to the condition of strong positivity shown in equation 40. The joint probability distribution given in the model allows the existence of a PR box which admits a joint quantal measure. However, the joint quantal measure is not strongly positive since the probability distribution of a PR box maximally violate Tsirelson bound. Hence, in [11] the authors argue that the reason why OQMP does not allow superquantum correlations could be due to the positive nature of the decoherence function. Furthermore it was stated that the positivity condition does not hold when two PR boxes are combined.

One of the interesting findings of this report with regard to Barnett's model is that the reduction of the matrix representation for the decoherence function in equation 41 can be reduced to the $8 \times 8$ matrix shown in equation 42. This reduction makes it much easier to determine if the matrix is strongly positive, as it makes the calculations, usually done computationally, quicker and less expensive. The $8 \times 8$ matrix might naturally relate to the fact that there are only 8 independent marginal probabilities (cf. Appendix A).

In general, research into quantum mechanics using the QMT framework might help to answer open questions about the true nature of quantum mechanics. In particular it is an important research goal to develop theories that can satisfy level 3 sum rules in the QMT hierarchy. If those were to exist it would be of further interest to see whether they allow the existence of a PR box. Even if it was possible to show the existence of a level 3 hierarchy in some hidden variable model, such systems may still not exist in nature. On the other hand, if level 3 QMT were shown not to exist, one would need to investigate as to why the description of nature stops at level 2 in QMT hierarchy[11].

In this report the ideas of Barnett et al. were compared to those by Ghirardi et al. [14]. In their model, Ghirardi et al. aim to show the existence of correlations that are stronger than quantum correlations. Rather than using QMT, their work is conducted using a class of theories known as crypto-nonlocal theories, cf. section 4.1. These theories consist of two levels of hidden variables and quantum correlations are obtained by averaging over both. However, a special condition was imposed, cf. equation 57. As a consequence when averaging locally over the deeper level hidden variable nonlocality is washed out. However, Leggett [16] previously showed that this type of models conflict with quantum mechanics, which was later shown experimentally. Therefore in order for Ghirardi's model to agree with quantum mechanics, equations 57 and 58 must be satisfied. They further show that averaging the correlations over the deeper hidden variable results in superquantum correlations, while averaging over all hidden variables results in the quantum mechanical correlations. Hence, superquantum correlations exist at the intermediate level of crypto-nonlocal theories.

Previous research on crypto-nonlocal theories argued [33, 34, 35] that due to these conditions given in equations 57 and 58, crypto-nonlocal theories cannot assign local values which quantum mechanics cannot. Hence, the authors in [33, 34, 35] argued that these theories are simply a local version of quantum mechanics and thus not of particular interest, whereas Ghirardi et al. challenge this statement with their model. Despite the fact that Ghirardi's hidden variable model violates quantum mechanics, when integrating over all hidden variables, one obtains correlations that reproduce the Tsirelson bound. Hence, Ghirardi et al. claim that their model is a useful means for comparing superquantum correlations with the Tsirelson bound and Leggett's model [14, 13].

Our comparison between Barnett's and Ghirardi's models highlight that the type of correlations that a hidden variable models allows, are highly dependent on the type of framework used to express the model as well as on the parameters of the variables. Finally, we believe that if nature is accurately described by hidden variable models, Ghirardi's model provides a strong indication as to why superquantum correlations cannot exist in nature, since all levels of hidden variables must be considered when calculating the outcomes of experiments.

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# Appendix A

The no-signalling conditions are given in equation 13 and revised here

$$\sum_j \mathbb{P}_{ab}(ij) = \sum_{j'} \mathbb{P}_{ab'}(ij'), \qquad \sum_i \mathbb{P}_{ab}(ij) = \sum_{i'} \mathbb{P}_{a'b}(i'j)$$
$$\sum_j \mathbb{P}_{a'b}(i'j) = \sum_{j'} \mathbb{P}_{a'b'}(i'j'), \qquad \sum_i \mathbb{P}_{ab'}(ij') = \sum_{i'} \mathbb{P}_{a'b'}(i'j')$$

These can be explicitly written as

$$\begin{aligned}
\mathbb{P}_{ab}(++) + \mathbb{P}_{ab}(+-) &= \mathbb{P}_{ab'}(++) + \mathbb{P}_{ab'}(+-) \\
\mathbb{P}_{ab}(-+) + \mathbb{P}_{ab}(--) &= \mathbb{P}_{ab'}(-+) + \mathbb{P}_{ab'}(--) \\
\mathbb{P}_{ab}(++) + \mathbb{P}_{ab}(-+) &= \mathbb{P}_{a'b}(++) + \mathbb{P}_{a'b}(-+) \\
\mathbb{P}_{ab}(+-) + \mathbb{P}_{ab}(--) &= \mathbb{P}_{a'b}(+-) + \mathbb{P}_{a'b}(--) \\
\mathbb{P}_{a'b}(++) + \mathbb{P}_{a'b}(+-) &= \mathbb{P}_{a'b'}(++) + \mathbb{P}_{a'b'}(+-) \\
\mathbb{P}_{a'b}(-+) + \mathbb{P}_{a'b}(--) &= \mathbb{P}_{a'b'}(-+) + \mathbb{P}_{a'b'}(--) \\
\mathbb{P}_{ab'}(++) + \mathbb{P}_{ab'}(-+) &= \mathbb{P}_{a'b'}(++) + \mathbb{P}_{a'b'}(-+) \\
\mathbb{P}_{ab'}(+-) + \mathbb{P}_{ab'}(--) &= \mathbb{P}_{a'b'}(+-) + \mathbb{P}_{a'b'}(--)
\end{aligned} \tag{73}$$

It was stated in section 3.1 that only eight independent probabilities exist for the (2,2,2) setup. An explicit example for a set of eight independent probabilities is given below. This set is not unique. The probabilities were found using the above no-signalling conditions combined with the normalisation conditions portrayed in equation 74

$$\begin{aligned}
\sum_{ij} \mathbb{P}_{ab}(ij) &= 1 \\
\sum_{i'j} \mathbb{P}_{a'b}(i'j) &= 1 \\
\sum_{ij'} \mathbb{P}_{ab'}(ij') &= 1 \\
\sum_{i'j'} \mathbb{P}_{a'b'}(ij) &= 1
\end{aligned} \tag{74}$$

which can be written as

$$
\begin{aligned}
\mathbb{P}_{ab}(--) &= 1-\mathbb{P}_{ab}(++)-\mathbb{P}_{ab}(+-)-\mathbb{P}_{ab}(-+)\\
\mathbb{P}_{a'b}(--) &= 1-\mathbb{P}_{a'b}(++)-\mathbb{P}_{a'b}(+-)-\mathbb{P}_{a'b}(-+)\\
\mathbb{P}_{ab'}(--) &= 1-\mathbb{P}_{ab'}(++)-\mathbb{P}_{ab'}(+-)-\mathbb{P}_{ab'}(-+)\\
\mathbb{P}_{a'b'}(--) &= 1-\mathbb{P}_{a'b'}(++)-\mathbb{P}_{a'b'}(+-)-\mathbb{P}_{a'b'}(-+)
\end{aligned}
\tag{75}
$$

By combining equation 13 with equation 75 one arrives to the following set of probabilities

$$
\begin{aligned}
\mathbb{P}_{ab}(--) &= 1-\mathbb{P}_{ab'}(++)-\mathbb{P}_{ab'}(+-)-\mathbb{P}_{ab}(-+)\\
\mathbb{P}_{a'b}(--) &= 1+\mathbb{P}_{ab}(+-)-\mathbb{P}_{ab'}(++)-\mathbb{P}_{ab'}(+-)\\
&\quad -\mathbb{P}_{ab}(-+)-\mathbb{P}_{a'b}(+-)\\
\mathbb{P}_{ab'}(--) &= 1-\mathbb{P}_{ab'}(++)-\mathbb{P}_{ab'}(+-)-\mathbb{P}_{ab'}(-+)\\
\mathbb{P}_{a'b'}(--) &= 1-\mathbb{P}_{a'b'}(+-)-\mathbb{P}_{ab'}(++)-\mathbb{P}_{ab'}(-+)\\
\mathbb{P}_{a'b}(++) &= -\mathbb{P}_{a'b}(+-)+\mathbb{P}_{a'b'}(++)+\mathbb{P}_{a'b'}(+-)\\
\mathbb{P}_{ab}(++) &= -\mathbb{P}_{ab}(+-)+\mathbb{P}_{ab'}(++)+\mathbb{P}_{ab'}(+-)\\
\mathbb{P}_{a'b'}(-+) &= -\mathbb{P}_{a'b'}(++)+\mathbb{P}_{ab'}(++)+\mathbb{P}_{ab'}(-+)\\
\mathbb{P}_{a'b}(-+) &= -\mathbb{P}_{ab}(+-)+\mathbb{P}_{ab'}(++)+\mathbb{P}_{ab'}(+-)\\
&\quad +\mathbb{P}_{ab}(-+)+\mathbb{P}_{a'b}(+-)-\mathbb{P}_{a'b'}(++)\\
&\quad -\mathbb{P}_{a'b'}(+-)
\end{aligned}
\tag{76}
$$

# Appendix B

In this appendix it will be shown that the decoherence functional described by 39 corresponds to the PR box in equation 34. When verifying this statement it is sufficient to calculate the values of only four experimental probabilities. These probabilities corresponds to the four different combinations of the measurements directions. The values of the other 12 experimental probabilities can then be inferred as shown in section 3.2.

The decoherence functional admits a joint quantal measure[12] and equation 14 holds.

$$\sum_{ijkl} D(ii'jj';kk'll') = \mathbb{P}_{a'b'}(i'j')\delta_{i'k'}\delta_{j'l'}$$

Hence

$$\begin{aligned}
\mathbb{P}_{a'b'}(+-) = & \sum_{ijkl} D(i+j-;k+l-) = \sum_{jkl} D(++j-;k+l-) + \sum_{jkl} D(-+j-;k+l-) \\
= & \sum_{kl} D(+++-;k+l-) + \sum_{kl} D(-++-;k+l-) + \sum_{kl} D(++--;k+l-) \\
& + \sum_{kl} D(-+--;k+l-) \\
= & \sum_{l} D(+++-;++l-) + \sum_{l} D(-++-;++l-) + \sum_{l} D(++--;++l-) \\
& + \sum_{l} D(-+--;++l-) + \sum_{l} D(+++-;-+l-) + \sum_{l} D(-++-;-+l-) \\
& + \sum_{l} D(++--;-+l-) + \sum_{l} D(-+--;-+l-) \\
= & D(+++-;+++-) + D(-++-;+++-) + D(++--;+++-) \\
& + D(-+--;+++-) + D(+++-;-++-) + D(-++-;-++-) \\
& + D(++--;-++-) + D(-+--;-++-) + D(-+--;-+--) \\
& + D(+++-;++--) + D(-++-;++--) + D(++--;++--) \\
& + D(-+--;++--) + D(+++-;-+--) + D(-++-;-+--) \\
& + D(++--;-+--)
\end{aligned} \tag{77}$$

Using the values given in equation 39

$$\begin{aligned}\mathbb{P}_{a'b'}(+-) &= \frac{1}{2} + 0 - \frac{1}{4} + 0 + 0 + 0 + 0 + 0 + \frac{1}{2} \\ &= -\frac{1}{4} + 0 + 0 + 0 + 0 + 0 + 0 \\ &= \frac{1}{2}\end{aligned} \tag{78}$$

Similarly it can be shown that

$$\begin{aligned}\mathbb{P}_{ab}(++) &= \frac{1}{2} \\ \mathbb{P}_{ab'}(++) &= \frac{1}{2} \\ \mathbb{P}_{a'b}(++) &= \frac{1}{2}\end{aligned} \tag{79}$$

Therefore the equation 39 corresponds to the PR box given by equation 34.

## Appendix C

In this appendix a python script, which checks that no n-subset violates equation 40, is listed. The program was jointly written with Marcel Christoph Guenther. The program requires the user to specify the dimensions of the relevant matrix and all the non zero tuples, where P2 contains all tuples with values equal to $\frac{1}{2}$, P4 with $\frac{1}{4}$ and N4 with $-\frac{1}{4}$. The matrix $\gamma_{ef}$ is Hermitian, hence it is sufficient to only input one tuple from each Hermitian conjugate pair, i.e. only tuples in the form $(e, f)$ are needed and all $(f, e)$ are automatically added. An example input for the matrix used in equation 41 is shown at the top of the script. Once the input arguments have been processed, the program adds all the Hermitian conjugate counterparts (cf. *splitTuples*). The program then determines all indices $e, f$ appearing in non-zero tuples (cf. *findIndicies*). This is done in order to speed up the process as only $N = 8$ rows and columns out of the 16 in the matrix contain non-zero elements. Subsequently all possible subspaces of size $n \in 3 \ldots N$ are explored. For each possible n-tuple the sum of all matrix values for all possible 2-tuples inside the n-tuple is checked against equation 40(cf. *trySubSetSize*). If it is violated the program halts with an error message, otherwise is continues until all possibilities have been checked.

### The script:

```
# -*- coding: utf-8 -*-
#!/usr/bin/python
# e.g usage: python solve.py -d 16 --p2 "(2,2) (3,3) (12,12) (15,15)"
# --p4 "(2,3) (3,15) (8,11) (15,16)" --n4 "(2,4) (3,8) (3,11) (8,15)
#                                                   (11,15) (11,12)"
# -v 1 from optparse import OptionParser import sys
#*******************************************************************
# Option parsing
parser = OptionParser()
parser.add_option("-v", "--verbose", default=0, type="int",
                  dest="verbose", help="Print all subset test
                  messages to stdout");
parser.add_option("-d", "--dimension", dest="dim", default=0,
                  type="int", metavar="DIM", help="Dimension=DIM
                  of square matrix (index starting at 1)");
parser.add_option("-p", "--p2", dest="p2", type="string",
```

```
                    metavar="TUPLE", help="Set \"(TUPLE)+\"
                    all tuples (X,Y)=1/2, (Y,X) will be added
                                    automatically if omitted");
parser.add_option("-q", "--p4", dest="p4", type="string",
                    metavar="TUPLE", help="Set \"(TUPLE)+\"
                    all tuples (X,Y)=1/4, (Y,X) will be added
                    automatically if omitted");
parser.add_option("-n", "--n4", dest="n4", type="string",
                    metavar="TUPLE", help="Set \"(TUPLE)+\"
                    all tuples (X,Y)=-1/4, (Y,X) will be added
                    automatically if omitted");

(options, args) = parser.parse_args();
dim = options.dim;
print ("Matrix Dimension : " + str(dim));
# Create the tuples
def splitTuples(s):
         tuples = str(s).split(' ')
         out = []
         for x in tuples:
                 a,b = x.strip('()').split(',')
                 if (int(a) > dim or int(b) > dim or
             int(a) < 1 or int(b) < 1):
                         print("Error: Elements must be between
                 1-" + str(dim) + ": ("+a+","+b+")");
                         sys.exit(0)
                 if (not (int(a),int(b)) in out):
             out.append((int(a),int(b)))
                 if (not (int(b),int(a)) in out):
             out.append((int(b),int(a)))
         return out

p2 = splitTuples(options.p2);
```

```
p4 = splitTuples(options.p4);
n4 = splitTuples(options.n4);
print("Positive 1/2 tuples : " + str(p2));
print("Positive 1/4 tuples : " + str(p4));
print("Negative 1/4 tuples : " + str(n4));

# Find indices
indices = []
def findIndicies(tuples,indices):
        for t in tuples:
                if (not t[0] in indices):
                        indices.append(t[0]);
                if (not t[1] in indices):
                        indices.append(t[1]);
        findIndicies(p2,indices);
        findIndicies(p4,indices);
        findIndicies(n4,indices) indices = sorted(indices);
        print("All indices appearing in non-zero
          tuples: " + str(indices));

# Calc sum of a single n-set
def sumUp(theComb,tuples,val):
        tempSum2 = 0.0;
        for t in tuples:
                if (t[0] in theComb and t[1] in theComb):
                        tempSum2 += val;
        return tempSum2;

# Check all possible subsets combinations of indices
def trySubSetSize(sss,comb,indices):
        if (sss == 0):
                # Translate to combination
                theComb = []
```

```
            for i in range(0,len(comb)):
                theComb.append(indices[comb[i]]);
                theSum = sumUp(theComb,p2,0.5)
                theSum += sumUp(theComb,p4,0.25)
                theSum += sumUp(theComb,n4,-0.25)
                if (options.verbose==1):
                    print("Verbose: The subset " + str(theComb)
              + " creates a sum of " + str(theSum));
                if (theSum < 0):
                    print("Error: The subset " + str(theComb)
              + " creates a sum of " + str(theSum) + " < 0");
                    sys.exit(0);
                return
            lastInd = -1;
            if (len(comb) > 0):
                lastInd = comb[len(comb)-1];
                for i in range(lastInd,len(indices)-1):
                    comb.append(i+1);
                    trySubSetSize(sss-1,comb,indices);
                    comb.pop(len(comb)-1);
# Try all n-subsets
for sss in range(2,len(indices)+1):
    trySubSetSize(sss,[],indices);
    print("All " + str(sss) + "-sets have sum >= 0");
```

# Appendix D

In the following appendix a derivation for equation 67 in section 4.2 is given. In addition, In order to check that the model gives the results claimed in section 4.2 several numerical examples checking the values for the correlation function $\left|Q_1^{\psi,\tau}\right|$ are given.

## Derivation of equation 67

Equation 67 states that

$$\mathrm{sgn}(\hat{\boldsymbol{a}} \cdot \lambda)\mathrm{sgn}(\hat{\boldsymbol{b}} \cdot \lambda) = \mathrm{sgn}\left(\chi_1^2 - \cos^2(\mu)\right)$$

where

$$\chi_n = \chi_n(\eta, \tau) = \frac{\cos(\tau)}{\sqrt{\cos^2(\tau) + \cot^2(\frac{\kappa_n(\eta)}{2})}}$$

and

$$\kappa_1(\eta) = \pi\sin^2(\eta), \qquad \kappa_2(\eta) = \pi\sin^2(3\eta)$$
$$\kappa_3(\eta) = 4\eta + \pi\sin^2(\eta), \qquad \kappa_4(\eta) = 4\eta - \pi\sin^2(\eta)$$

Derivation:

$$\mathbf{a} = (\sin(\eta), 0, \cos(\eta)), \qquad \mathbf{b} = (\text{-}\sin(\eta), 0, \cos(\eta))$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\omega$ such that

$$\omega = 2\eta \tag{80}$$

Since $\hat{\boldsymbol{a}}$ and $\hat{\mathbf{b}}$ must be each others image when mirrored with respect to $\omega$'s bisector they can be defined as

$$\hat{\mathbf{a}} = (\sin(\hat{\eta}), 0, \cos(\hat{\eta})), \qquad \mathbf{b} = (\text{-}\sin(\hat{\eta}), 0, \cos(\hat{\eta})) \tag{81}$$

The angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is $\hat{\omega} = 2\hat{\eta}$. The relationship between $\hat{\omega}$ and $\omega$ is give by equation $\left(\hat{\omega} = \pi\sin^2(\frac{\omega}{2})\right)$ Hence

$$\hat{\eta} = \frac{\pi\sin^2(\eta)}{2} \tag{82}$$

Therefore

$$\begin{aligned}
\text{sgn}(\hat{\boldsymbol{a}}\cdot\lambda)\text{sgn}(\hat{\boldsymbol{b}}\cdot\lambda) = &\ \text{sgn}(\hat{\boldsymbol{a}}\cdot\lambda)(\hat{\boldsymbol{b}}\cdot\lambda) \\
= &\ \text{sgn}(\hat{\boldsymbol{a}_x}\cdot\lambda_x + \hat{\boldsymbol{a}_z}\cdot\lambda_z)(\hat{\boldsymbol{b}_x}\cdot\lambda_x + \hat{\boldsymbol{b}_z}\cdot\lambda_z) \\
= &\ \text{sgn}\left(\hat{\boldsymbol{a}_x}\cdot\hat{\boldsymbol{b}_x}\cdot\lambda_x^2 + \hat{\boldsymbol{a}_z}\cdot\hat{\boldsymbol{b}_z}\cdot\lambda_z^2 + \lambda_x\cdot\lambda_z(\hat{\boldsymbol{a}_x}\cdot\hat{\boldsymbol{b}_z} + \hat{\boldsymbol{a}_z}\hat{\boldsymbol{b}_x})\right)
\end{aligned} \tag{83}$$

But from equation 81

$$\hat{\boldsymbol{a}_x}\cdot\hat{\boldsymbol{b}_z} = -\hat{\boldsymbol{a}_z}\hat{\boldsymbol{b}_x} \tag{84}$$

$\lambda$ is uniformly distributed over a unit sphere and $\mu$ and $\tau$ are the polar angles specifying this sphere. Hence

$$\lambda_x = \text{sin}(\mu)\text{cos}(\tau), \qquad \lambda_y = \text{sin}(\mu)\text{sin}(\tau) \qquad \lambda_x = \text{cos}(\mu) \tag{85}$$

Equations 83, 84 and 85 yield

$$\begin{aligned}
\text{sgn}(\hat{\boldsymbol{a}}\cdot\lambda)\text{sgn}(\hat{\boldsymbol{b}}\cdot\lambda) = &\ \text{sgn}(\hat{\boldsymbol{a}_x}\cdot\hat{\boldsymbol{b}_x}\cdot\lambda_x^2 + \hat{\boldsymbol{a}_z}\cdot\hat{\boldsymbol{b}_z}\cdot\lambda_z^2) \\
= &\ \text{sgn}\left(-\text{sin}^2(\hat{\eta})\text{sin}^2(\mu)\text{cos}^2(\tau) + \text{cos}^2(\hat{\eta})\text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(-\text{sin}^2(\mu)\text{cos}^2(\tau) + \text{cot}^2(\hat{\eta})\text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\text{cot}^2(\hat{\eta})\left(\frac{-\text{sin}^2(\mu)\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\right)\right) \\
= &\ \text{sgn}\left(\text{cot}^2(\hat{\eta})\right)\text{sgn}\left(\frac{-\text{sin}^2(\mu)\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\frac{-\text{sin}^2(\mu)\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\frac{-\text{sin}^2(\mu)\text{cos}^2(\tau) - \text{cos}^2(\mu)\text{cos}^2(\tau) + \text{cos}^2(\mu)\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\frac{-\text{cos}^2(\tau)\left(\text{sin}^2(\mu) + \text{cos}^2(\mu)\right) + \text{cos}^2(\mu)\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\frac{-\text{cos}^2(\tau) + \text{cos}^2(\mu)\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\frac{-\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \frac{\text{cos}^2(\mu)\text{cos}^2(\tau) + \text{cos}^2(\mu)\text{cot}^2(\hat{\eta})}{\text{cot}^2(\hat{\eta})}\right) \\
= &\ \text{sgn}\left(\frac{-\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})} + \text{cos}^2(\mu)\left(\frac{\text{cos}^2(\tau) + \text{cot}^2(\hat{\eta})}{\text{cot}^2(\hat{\eta})}\right)\right) \\
= &\ \text{sgn}\left(\left(\frac{\text{cos}^2(\tau) + \text{cot}^2(\hat{\eta})}{\text{cot}^2(\hat{\eta})}\right)\left(\frac{\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta})}\left(\frac{\text{cot}^2(\hat{\eta})}{\text{cot}^2(\hat{\eta}) + \text{cos}^2(\tau)}\right) + \text{cos}^2(\mu)\right)\right) \\
= &\ \text{sgn}\left(\frac{\text{cos}^2(\tau) + \text{cot}^2(\hat{\eta})}{\text{cot}^2(\hat{\eta})}\right)\text{sgn}\left(\frac{\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta}) + \text{cos}^2(\tau)} + \text{cos}^2(\mu)\right) \\
= &\ \text{sgn}\left(\frac{\text{cos}^2(\tau)}{\text{cot}^2(\hat{\eta}) + \text{cos}^2(\tau)} + \text{cos}^2(\mu)\right)
\end{aligned}$$

Since $\text{cot}^2(\hat{\eta})$ and $\frac{\text{cot}^2(\hat{\eta})}{\text{cos}^2(\tau)+\text{cot}^2(\hat{\eta})}$ are always positive.

Finally from equations 65 and 66

$$\begin{aligned}\chi_1 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{\kappa_1(\eta)}{2})}} = \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{\pi \sin^2(\eta)}{2})}} \\ = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\hat{\eta})}}\end{aligned} \tag{86}$$

Hence

$$\mathrm{sgn}(\hat{\boldsymbol{a}}\cdot\lambda)\mathrm{sgn}(\hat{\boldsymbol{b}}\cdot\lambda) = \mathrm{sgn}\left(\chi_1^2-\cos^2(\mu)\right)$$

QED

## Numerical example for $\left|Q_1^{\psi,\tau}\right|$

From equations 65 and 66 $\chi_n$ takes the following form

$$\begin{aligned}\chi_1 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{\pi \sin^2(\eta)}{2})}}, & \chi_2 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{\pi \sin^2(3\eta)}{2})}} \\ \chi_3 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{4\eta+\pi \sin^2(\eta)}{2})}}, & \chi_4 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{4\eta-\pi \sin^2(\eta)}{2})}}\end{aligned}$$

Equations 71 and 72can be written in the following concise form

$$Q_1^{\psi,\tau} = \begin{cases} 2\left(|\chi_1|-|\chi_2|+|\chi_3-\chi_4|-1\right) & \text{for} \quad 0 \le \eta \le \tilde{\eta} \\ 2\left(|\chi_1|-|\chi_2|-|\chi_3+\chi_4|+1\right) & \text{for} \quad \tilde{\eta} \le \eta \le \frac{\pi}{4} \end{cases} \tag{87}$$

It was mentioned in section 4.2 that $Q_1^{\psi,\tau} \sim 4$ and $Q_1^{\psi,\tau} \sim -4$ in neighbourhood of $(\eta,\tau) = (\tilde{\eta},\frac{\pi}{2})$ and $(\eta,\tau) = (\frac{\pi}{6},\frac{\pi}{2})$ respectively where where $\hat{\eta} \simeq 0.562$ [14]. $Q_1^{\psi,\tau}$ reaches those values asymptotically because when $\tau = \frac{\pi}{2}$, $\cos(\frac{\pi}{2}) = 0$ resulting in $\chi_n = 0$ and $Q_1^{\psi,\tau} = \pm 2$.

Example for $Q_1^{\psi,\tau} \sim -4$ case:

Choose $(\eta,\tau) = (0.521, 1.57)$

$$\begin{aligned}\chi_1 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{\pi \sin^2(\eta)}{2})}} \sim \frac{\cos(1.57)}{\sqrt{\cos^2(1.57)+\cot^2(\frac{\pi \sin^2(0.521)}{2})}} \\ \sim & \ 0\end{aligned}$$

$$\begin{aligned}\chi_2 = & \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{\pi \sin^2(3\eta)}{2})}} \sim \frac{\cos(1.57)}{\sqrt{\cos^2(1.57)+\cot^2(\frac{\pi \sin^2(1.563)}{2})}} \\ \sim & \ 0.998\end{aligned}$$

$$\chi_3 = \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{4\eta+\pi\sin^2(\eta)}{2})}} \sim \frac{\cos(1.57)}{\sqrt{\cos^2(1.57)+\cot^2(\frac{4\times 0.521+\pi\sin^2(0.521)}{2})}}$$
$$\sim 0$$

$$\chi_4 = \frac{\cos(\tau)}{\sqrt{\cos^2(\tau)+\cot^2(\frac{4\eta-\pi\sin^2(\eta)}{2})}} \sim \frac{\cos(1.57)}{\sqrt{\cos^2(1.57)+\cot^2(\frac{4\times 0.521-\pi\sin^2(0.521)}{2})}}$$
$$\sim 0$$

$$Q_1^{\psi,\tau} \sim 2\left(|\chi_1| - |\chi_2| - |\chi_3+\chi_4| + 1\right) \sim 2(0 - 0.998 + 0 - 1) \sim -4$$