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Semiclassical strings and spin chains in the AdS/CFT correspondence

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Declaration

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.

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Abstract

In this MSc dissertation, we explore the improvement of our understanding of the AdS/CFT correspondence during the last years from studies of the by far most explored case of the duality, which is the one in which the gravity or string theory lives in a curved background consisting on 5 dimensional anti de Sitter spacetime times a 5-sphere and the dual conformal field theory is the maximally supersymmetric QFT, that is, the one with 4 supercharges. A further restriction takes us to the limit in which the duality appears to be more reliable, namely the planar limit for the conformal field theory and the correspondingly free string theory on the gravity side.

After a very general, brief and focused review of the AdS/CFT duality, we present the techniques than one uses to put the correspondence to work and test it by direct comparison of what are expected to be equivalent results on both sides of the duality. The duality states that the energy of string states should be equal to the scaling dimension of the dual gauge operators. For the time being, this can only be reasonably computed under certain conditions and by means of the techniques that we expose in this dissertation, namely the semiclassical approach to the string states to calculate their energy and the spin-chain analogy to obtain the scaling dimension of gauge operators.

The underlying reason why so much progress has been made in this area of research in the last decade is the discovery of some hidden symmetries that simplify the resolution of the problems. These hidden symmetries stem from the integrability of the theories

under certain limits and could be an open door to the full solution of the theory on both sides of the correspondence. In other words, for the first time we could be on the verge of completely solving a quantum interacting theory, and this would probably have some consequences as far as interesting cases of them, such as quantum electrodynamics, are concerned.

The general aim is to check and better understand the duality by finding which gauge operators correspond to which string states. The community would like to use this to learn more about how the duality works and how it might be extended to less symmetric cases.

Contents

1 Introduction

String theory was originally formulated as a theory for the Physics of the strong interaction [1]. After the advent of gauge field theories and in particular of quantum chromodynamics to model the interactions between quarks, the especticism about the possibilities of success of string theory quickly fell and the theory was mostly abandoned for a while. Later on string theory started being seen by many as a promising candidate for a unifyied theory of everything, that is of quantum gravity along with existing gauge theories that describe all other interactions in nature. In this interpretation quantum chromodynamics is conceived as a low energy limit of its dynamics. The gauge fields though are understood as secondary objects which follow from the fundamental strings. A crucial step into a rather novel path was made by t'Hooft [2]. T'Hooft's insight was that perturbative expansions of $SU(N)$ gauge field theories in the limit of very large N (number of colours) take after the genus expansion that string theorists were familiar with. In other words, it looked like for $N \to \infty$ with a given parameter fixed (the so-called t'Hooft parameter $\lambda \equiv g_{YM}^2 N$), expansions in terms of Feynman diagrams turn out to be expansions in different topologies, which is exactly what one has when considerin a string perturbative expansion over Riemann surfaces, whereby $1/N$ plays the role of the string coupling. At the same time, works on lattice gauge theory by Wilson [3] and Polyakov [4] also pointed out that the strong coupling expansions can be seen as propagating closed strings.

All this paved the way to the strong belief that gauge theories and string theories

are nothing but two sides of the same coin and hence gave birth to the concept of gauge/string dualities. By that time this was nothing but a well grounded physical intuition but a mathematical formalism, let alone a formal proof, were lacking. Nowadays, most of the community shares the opinion that further exploring this exciting duality might give us new insights into both of its sides. On the one hand, we could profit from our knowledge about perturbative string theory to learn more about strongly coupled quantum field theories. On the other hand, our techniques in perturbative quantum field theries might help us understanding how to include gravity consistently in a quantum field theory.
Another crucial impulse to the field was given by Witten and his realization that a stack of N coincident D-branes gives place to a $U(N)$ symmetric gauge theory [5]. This incredible idea of gauge/string dualities would then be established in a more formal way in the late 1990s by [6]. Maldacena presented a concrete conjecture realization of such a duality. He showed that the four dimensional $\mathcal{N} = 4$ supersymmetric gauge theory with a gauge group $SU(N)$ is equivalent to type IIB string theory on an $AdS_5 \times S^5$ background. In particular, he showed that if considering the geometry of a stack of D3-branes, the gauge theory emerging for open strings stretching between the branes is $\mathcal{N}$=4 SYM , while the geomtry sourced by the branes is nothing but $AdS_5 \times S^5$ spacetime. Now a four dimensional scenario can be envisaged as the boundary of AdS_5 , where the gauge theory would live, while the S^5 part of the geomtry can be seen to arise due to the internal supersymmetry of the gauge theory. Hence the relation between gauge/gravity dualities and the concept of holography.

This puts string theory and quantum gauge theories on the same footing as fundamental theories that arise at a time. Ever since all these discoveries took place, lots of efforts have been devoted to the improvement of our understanding of this marvelous duality between some gravity theories and some quantum field theories. The AdS/CFT correspondence, whereby gravitational theories in an Anti de Sitter background are dual to quantum field theories with conformal symmetry, is by far the most known and studied case of such a duality. In this dissertation, we explore the advances that have been made in this field by using the semiclassical approach to a theory of strings on the so called "AdS side" and the spin chain analogy that arises when trying to compute scaling dimensions on the gauge side.

The semiclassical approach to strings in the context of the AdS/CFT duality is based on the simple idea of finding perturbative corrections around classical solutions of the string equations of motion in the $AdS_5 \times S^5$ background. Given the technical problems involved in the solution and the quantization of the string action in $AdS_5 \times S^5$ this technique gave the first formal checks of the duality (in the sense of computing a quantity in both sides of the duality) beyond the supergravity approximation. This conceptually simple approach was precluded by the works of Berenstein, Maldacena and Nastase in [7] and also of Gubser, Klebanov and Polyakov in [8] and formally launched by Frolov and Tseytlin in [9]. The technique uses classical solutions as a starting point, about which perturbative corrections are calculated. In particular, an expression for the energy in terms of other quantum numbers (spins) can be found, first at a classical level and then in a quantum corrected version. This quantum corrected energy should then be compared to the dual anomalous dimension for the corresponding gauge operator and a proper check of the duality would be given by an agreement of both quantities. The key assumption of the semiclassical approach relies on the fact that in the $N \to \infty$ approximation spins must be large ($J \to \infty$), while keeping the ratio J^2/N fixed. This limit can make the semiclassical perturbative approach or even the classical one quantum exact, since in this limit quantum corrections can get cancelations (see [10], [11]). This means that all-loop predictions for the scaling dimension of the dual gauge theory can be made in some cases.

Studies in this field of research aim at a quantitative check of the duality beyond the well known supersymmetric sector as well as at an improvement of our understanding of the role played by the integrable structures which have been recently discovered on both sides of the correspondence. Integrability has now posed serious hopes of making considerable steps forwards in our search for the complete solution of the theory. Integrability manifests itself in the weak coupling limit via the analogy established between the dilatation operator sof $\mathcal{N}$=4 SYM and the hamiltonian of a 1-dimensional spin chain, which is an integrable structure and which we shall be reviewing further below. In the strong coupling limit, integrability can be seen in the sigma model formulation of the string worldsheet action in $AdS_5 \times S^5$. As a result of this integrability on both sides of the duality, it seems like several important features of the theory can be learn by means of the so-called Bethe ansatz.

In more concrete terms, on the gauge side of the duality, the first traces of integrability were revealed by Minahan and Zarembo [12]. They related for the first time the one-loop dilatation operator to a spin-chain, which is a well-known integrable system. This was nothing but the reflection on the theory side of the classical integrability of the string sigma model of Metsaev and Tseytlin, which would shortly after be discovered by Bena, Polchinski and Roiban [13]. Integrability was then extended to all one-loop operators and later on to two-loop and three-loop orders in $\mathcal{N}$=4 SYM . Soon it was conjectured that integrability could be an all-loop feature [14]

An additional technique which was launched by Kazakov, Marshakov, Minahan and Zarembo in [15] is that of the algebraic curve. They realised that classical solutions in the $su(2)$ subsector can be expressed using algebraic curves, which is a general and an elegant formal way of solving the problem. Since algebraic curves are also obtained on the gauge side approach as a continuum limit of the Bethe ansatz equations, it looks once more like we are looking at different sides of the same coin. So integrability has brought new hopes of a complete solution of the theory, which might be perceived both as a way of understanding how strongly coupled quantum field theories behave and as a way of understanding quantum free strings in curved backgrounds.

Some important aspects of this research field have been intentionally let aside on behalf of conciseness and brevity. Those are for example the pure spinor formalism originally presented in [16] and didactically explained and related to the content of this dissertation in [17] and [18]. Also the Yangian symmetry structure related to $N = 4$ SYM, [19], [20]. Other cases will be mentioned below.

2 The $AdS_5 \times S^5$ / $\mathcal{N} = 4$ SYM correspondence

The AdS/CFT correspondence ([6], [21], [22]) is the best known example of a gauge/gravity duality, which states that $\mathcal{N}$=4 SYM theory is dynamically equivalent to type IIB superstring theory on $AdS_5 \times S^5$, as long as the parameters of both theories are related by the relations

$$g_s = \frac{4\pi g^2}{N}, \qquad \frac{R^2}{\alpha'} = 4\pi g \tag{2.1}$$

whereby R stands for the radi of AdS_5 and of S^5 , g_s denotes the string coupling constant and g is the coupling of the gauge theory. From the point of view of symmetries it can be shown that both theories share the same symmetry group, $PSU(2,2|4)$, which contains the maximal bosonic subgroup $SO(2,4) \times SO(6)$. $SO(2,4)$ can be understood on the gauge side as the conformal group in four spacetime dimensions, while on the string side it is seen as the group of isometries of the AdS_5 metric. Meanwhile, $SO(6)$ may be seen as the R-symmetry group for a theory with four supercharges[1] and as the group of isometries of S^5 respectively. Taking the supersymmetri generators into account does indeed enchance the symmetry grop to the full $PSU(2,2|4)$ for $\mathcal{N}$=4 SYM . For the string side this enhancement stems from the fact that 16 of the 32 Poincaré supersymmetries are preserved by the array of N D3-branes, which are supplemented by another 16 supersymmetries [23]. The formal proof of this equality of the full symmetries was given by Metsaev and Tseytlin in [24], where they used the sigma-model

[1] $\mathcal{N}$=4 SYM has four fermionic supercharges for which the symmetry group is given by $SU(4) \cong SO(6)$

approach to show that its symmetry group is in fact $PSU(2,2|4)$.

Both $SO(2,4)$ and $SO(6)$ are rank three groups, so each gauge operator or string state will be labelled by six quantum numbers. In the case of a string state, the three quantum numbers corresponding to $SO(2,4)$ will be the energy (recall that the time direction in AdS_5 haas been decompactified) and two spins and the three quantum numbers corresponding to $SO(6)$ will be three spins accounting for the rotation of the string in S^5

$$|\text{string state}\rangle = |E, J_1, J_2; S_1, S_2, S_3\rangle \tag{2.2}$$

For a gauge operator, the three labels given by the $SO(2,4)$ symmetry are understood as the scaling dimension and two spins, while the three S^5 labels are the R-charges:

$$\mathcal{O} = \mathcal{O}(\Delta, s_1, s_2; R_1, R_2, R_3) \tag{2.3}$$

From this analogy, it is readily seen as a consequence of the duality, that the energy of a given string state should conincide with the scaling dimension of the corresponding dual gauge operator. Gauge-invariant local operators are just traces taken over the colour indices of products of the fundamental fields contained in the theory.
We know that the energy is nothing but the eigenvalue of the Hamiltonian operator, which generates time translations in AdS_5 . The scaling dimension Δ is the parameter that measures how an operator transforms under scale transformations

$$x \rightarrow \Lambda x \;\Rightarrow\; \mathcal{O}(x) \rightarrow \Lambda^{-\Delta}\mathcal{O}(x)$$

and it normally gets quantum corrections under renormalization. So in general we have $\Delta = \Delta_0 + g\gamma$, whereby Δ_0 is the bare scaling dimension and γ contains the quantum corrections and goes under the name of anomalous dimension. g is just the coupling. In particular the scaling dimension can be identified as the eigenvalue of the dilatation operator D of $\mathcal{N}$=4 SYM . So in other words, if $|\mathcal{O}\rangle$ is a string state (eigenstate of the corresponding Hamiltonian) and $\mathcal{O}(x)$ is its corresponding gauge dual operator with scaling dimension Δ, the AdS/CFT correspodence establishes [25], [26]

$$H_{\rm str}|\mathcal{O}\rangle = E\left(\frac{R^2}{\alpha'}, g_S\right)|\mathcal{O}\rangle \qquad D\mathcal{O}(x) = \Delta\left(g, 1/N\right)\mathcal{O}(x) \tag{2.4}$$

so assuming the duality and hence the correspondence of both elements amounts to the assumption

$$E\left(\frac{R^2}{\alpha'}, g_S\right) = \Delta\left(g, 1/N\right) \tag{2.5}$$

In principle, checking this correspondence should look like a hopeless enterprise. Computing the spectrum of a given string state for an arbitrary value of N to all-loop orders is almost technically impossible. The planar limit, where $N \to \infty$ provides a convenient limit where things turn out to be more tractabel and the duality can indeed be tested. In this limit, strings become free strings while our quantum field theory reduces to a theory consisting of planar Feynman diagrams alone. From the point of view of topology, a Riemann sphere can be rapidly seen on both sides by thinking of the worldsheet of strings or of the topology of the diagrams in this limit.
We will restrict ourselves to the planar limit all along this work. So our duality will be actually that between free type IIB string theory in AdS_5 and planar $\mathcal{N}$=4 SYM .

But even after this convenient approximation has been assumed, we still have to face the crude reality of this correspondence having a weak/strong nature. After all, our calculations are mostly based on perturbative methods and this weak/strong character of the duality means that perturbative regimes on both sides of the duality are non-overlapping, which in princple would make any check of the duality by means of real computations hopeless. An important exception is given by states that are protected from getting quantum corrections by the high amount of symmetry available. They are the so-called BPS states. String energies of BPS states do not get quantum corrections and correspondingly, the scaling dimension of the dual gauge operators do not depend on the coupling. All the early checks of the duality were restricted to such states, since they posed the only possible way of performing real calculations on both sides of the duality. As we will see below, this situation changed drastically in the early 2000s thanks to a series of discoveries that we will be covering in this work.

It is worth pointing out now that the duality goes far beyond an equality of symmetries and the corresponding duality of representations. Its dynamical statement foresees that to each string state in $AdS_5 \times S^5$ there exists a gauge dual in $\mathcal{N}$=4 SYM such that all associated physical quantites coincide [23]. Still symmetries will catch most of our atention here, since we will be mainly interested in the spectral correspondence 2.5.

3 Integrability

We would like to use this chapter as a brief introduction to the concept of integrability both at a classical and at a quantum level. Physics is all about symmetries. Global symmetries lead to conserved quantities and these allow for the establishment of general physical laws, like the conservation of energy for time invariant systems or angular momentum for rotation invariant theories. The symmetries of a theory play a very relevant role in its resolution, since they significantly simplify the problem and the computations by reducing the amount of independent degrees of freedom. So the more symmetries a system possesses, the simpler its resolution. A perfect situation is that in which symmetry allows for a complete resolution of the theory, and this only happens when the number of symmetries is large enough. Some theories have the privilege to be in this situation and they go under the name of integrable theories.

The concept of integrability has been extensively used in classical theories for a very long time in Physics. In plain words, a theory is said to be classically integrable when there is a sufficient number of conserved charges, making it exactly solvable. Integrability is a phenomenon typically restricted to two-dimensional theories.

For a very good review on everything related to integrability in the AdS/CFT correspondence we refer the reader to [27].

3.1 Classical integrability

In particular, we will be interested in the method used to solve integrable systems[1] involving nonlinear partial differential equations, the so called inverse scattering method. In two dimensions (τ, σ), imagine we start form an overdetermined system of partial differential equations of the form [28]

$$\frac{\partial \Psi}{\partial \tau} = L_\tau(\sigma, \tau, z)\Psi \qquad \frac{\partial \Psi}{\partial \sigma} = L_\sigma(\sigma, \tau, z)\Psi \tag{3.1}$$

Now differentiating the first equation with respect to σ and the second equation with respect to τ and equating the results

$$\frac{\partial^2 \Psi}{\partial \sigma \partial \tau} = \partial_\sigma L_\tau(\sigma, \tau, z)\Psi + L_\tau(\sigma, \tau, z)\partial_\sigma \Psi = \partial_\tau L_\sigma(\sigma, \tau, z)\Psi + L_\sigma(\sigma, \tau, z)\partial_\tau \Psi$$

which taking the previous equation into account implies

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau + [L_\sigma, L_\tau] = 0 \tag{3.2}$$

So if L_α, $\alpha = \tau, \sigma$ is understood as a non-Abelian connection, the previous equation is just the requirement that it be flat for all values of z, known as the spectral parameter. This is also known as the zero curvature condition of the connection. Furthermore, the matrices L_α must be chosen in such a way that the fulfilment of the zero curvature condition implies the validity of the initial equations for all values of z. Such a connection is called a Lax connection or a Lax pair and is defined up to gauge transformations. Note that equation 3.2 can be reestated in the language of differential forms as

$$dL - L \wedge L = 0 \tag{3.3}$$

The Lax connection is a key element to realize the integrability of a theory since it provides a way of finding all conservation laws (related to the equations of motion) by

[1] For a good review of the different uses and techniques of integrable systems, see the lecture notes by G. Arutyunov.

defining the so called monodromy matrix

$$T(z) = \mathcal{P} \exp \int_0^{2\pi} d\sigma L_\sigma(\sigma, z) \tag{3.4}$$

whereby $\mathcal{P}$ signals the path-ordering. We assume that the σ coordinate is circular $0 \geq \sigma \geq 2\pi$, but this does not necessarily need to be the case [29]. Let us now derive the previous equation with respect to our time coordinate

$$\begin{aligned} \partial_t T(z) &= \int_0^{2\pi} d\sigma\, \mathcal{P} e^{\int_\sigma^{2\pi} d\chi L_\sigma} \left(\partial_t L_\sigma\right) \mathcal{P} e^{\int_0^\sigma d\chi L_\sigma} \\ &= \int_0^{2\pi} d\sigma\, \mathcal{P} e^{\int_\sigma^{2\pi} d\chi L_\sigma} \left(\partial_\sigma L_\tau + [L_\tau, L_\sigma]\right) \mathcal{P} e^{\int_0^\sigma d\chi L_\sigma} \\ &= \int_0^{2\pi} d\sigma\, \partial_\sigma \left(\mathcal{P} e^{\int_\sigma^{2\pi} d\chi L_\sigma}\, L_\tau\, e^{\int_0^\sigma d\chi L_\sigma}\right) \end{aligned} \tag{3.5}$$

which summing up means

$$\partial_\tau T(z) = [L_\tau(0, \tau, z), T(z)] \tag{3.6}$$

which means that the eigenvalues of the monodromy matrix $T(z)$, which are given by

$$\Gamma(z, \mu) = \det(T(z) - \mu \mathbb{I}) = 0 \tag{3.7}$$

form an infinite set of conserved quantities (one for each value of the spectral parameter), or in other words, an infinite set of equations of motion. So the monodromy matrix contains all the information about the spectrum of the theory. The equation 3.7 defines an algebraic curve in $\mathbb{C}^2$ called the spectral curve [28].

So we have seen that if a theory admits the definition of a Lax connection, whose zero curvature condition is equivalent to the equations of motion of the system, the theory has an infinite number of conserved quantities and so it is classically integrable.

Very recently Beisert and Lücker [30] found a new method for generally constructing a Lax connection of a 2d relativistic integrable sigma model on coset space which also poses a test for integrability for a theory formulated in terms of a coset sigma model. Normally the Lax connection of a theory is found by formulating a sensible ansatz in terms of the fields and then impose the flatness condition. The authors of [30] propose the construction of the Lax connection in a more general way:

$$L(x,t,\lambda) = \exp(\lambda\Sigma)J \tag{3.8}$$

whereby J is the Maurer-Cartan form, taking values in the Lie algebra corresponding to the group manifold in which a sigma model for the theory can be formulated and Σ is a newly defined operator whose significance is explored in the paper, and it turns out to act on the Lax connection as a shift operator. After spliting J into its chiral components

$$J^{\pm} = \frac{1}{2}(J \pm *J), \qquad J = J^{+} + J^{-} \tag{3.9}$$

and from here define

$$\Sigma(J^{\pm}) = \Sigma^{\pm}(J^{\pm}) \tag{3.10}$$

then from the last two equations and from 3.8, the connection takes now the form

$$L(\lambda) = \exp(\lambda\Sigma^{+})(J^{+}) + \exp(\lambda\Sigma^{-})(J^{-}) \tag{3.11}$$

using now the flatness condition the authors of [30] show that it the connection is flat to order $\mathcal{O}(\lambda^3)$ is exactly flat for all λ. They also show that the map $\Sigma^{\pm}$ defines an action for a sigma model.

3.2 Quantum integrability

At a classical level, we have seen that an integrable theory is that for which N conserved charges can be found for a phase space that is N-dimensional. In other words, for an integrable theory we would be able to find N quantities $\mathcal{Q}_1, ..., \mathcal{Q}_N$ satisfying

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0 \quad \forall i,j, \qquad \{H, \mathcal{Q}_i\} = 0 \quad \forall i \tag{3.12}$$

whereby H is the Hamiltonian of the theory. The normal prodedure to translate all this into the quantum language is usually the substitution of Poisson brackets by common commutators, but in this case this procedure turns out not to be enough [31]. A better and more intuitive description of integrability at a quantum level can be obtained from the point of view of scattering of particles. These particles need not be physical at all. Several physica problems can be understood in terms of virtual particles scattering against one another. Imagine we have two particles with asymptotic (far from their intersection point) momenta p_1 and 'p_2 respectively. They might be represented by an incoming wavefunction

$$\Psi_{\text{in}}(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} \qquad x_1 \ll x_2 \tag{3.13}$$

We may know call upon the conservation of energy and momentum, implying

$$E = \frac{1}{2}(p_1^2 + p_2^2) = \text{constant} \qquad P = p_1 + p_2 = \text{constant} \tag{3.14}$$

So if both particles scatter against each other, the momenta after the collision can only be a permutation of the original momenta

$$p_1' = p_2 \qquad p_2' = p_1 \tag{3.15}$$

All the above can be summed up using an S-matrix by

$$\Psi(x_1, x_2) \propto e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) e^{i(p_2 x_1 + p_1 x_2)} \tag{3.16}$$

Now if instead of dealing with just two particles, we are scattering three or more, things change. The predictability of the above model was based on the fact that in a two-particle collision respecting the conservation of energy and momentum, momenta can be at most exchanged between the two particles. With three or more particles involved in the interaction this is no longer true, so we can split up the asymptotic wave function and epxress it as a sum of a wave function for which only two-particle interactions are taken into account and a second part for which many-particle interactions do play a role [31] (again, consider the asymptotic region $x_n \gg x_{n-1} \gg ...$):

$$\Psi(x_1, x_2, ..., x_n) \propto \sum_{\mathcal{P}} \left(\Psi_{\text{two}} e^{i(p_{\mathcal{P}(1)}x_1) + p_{\mathcal{P}(2)}x_2 + ... + p_{\mathcal{P}(n)}x_n} \right) + \Psi_{\text{many}}(x_1, x_2, ..., x_n) \tag{3.17}$$

where we have sum over all possible permutations of the particles for the first part of the wave function, since it describes a sequence of two-particle interactions. This first part of the wave function is therefore completely determined by the two-body S-matrix. This opens the door to a new definition of quantum integrability. A system is said to be quantum integrable when only two-particle interactions are relevant. In other words, the asymptotic wavefunction can be expressed as

$$\Psi(x_1, x_2, ..., x_n) \propto \sum_{\mathcal{P}} \Psi_{\text{two}} e^{i(p_{\mathcal{P}(1)}x_1) + p_{\mathcal{P}(2)}x_2 + ... + p_{\mathcal{P}(n)}x_n} \tag{3.18}$$

which is equivalent to the statement that the $n \to n$ S-matrix factorises into a product of $2 \to$ S-matrices.

It is very intuitive to consider which consequences this has for the resolution of the system in terms of conserved charges. For the two-particle interaction, we have seen that the theory is integrable given the existence of the two conserved charges related to the equations of motion, namely the energy and the momentum. If we assumed the conservation of a third-charge, say for example $\sum_i p_i^2$, this would turn a three-body interaction into a permutation of the initial momenta again. The existence of the third conserved charge then implies the solvability (integrabiltiy) of the model. So when the requirement of integrability is satisfied, and interactions are only pairwise for a set of n particles, n conserved charges must exist. Of course the additional $n-2$ charges need not be related to the equations of motion and that is why sometimes they are refered to as "hidden" conserved charges. They are eigenvalues of the generators of the hidden symmetries and no general method is known to find them for a general integrable system[2].

[2]Some exceptions to this regarding spin chains are covered in [32]

Of course the quantum regime should take us to the classical regime for the corresponding (continuum) limit. This means that it must be possible to recover the idea of classical integrability discussed in the previous section from this quantum picture of integrability. In other words, we should be able to rephrase our definition of quantum integrability in terms of a Lax connection. This can indeed be done. We want to end this chapter by noting that it could be interesting to explore the possibility of using the new method shown by Beisert and Lücker in [30] for finding the Lax connection to find a general method of computing the hidden conserved charges of an integrable system.

Before closing this chapter on integrability, it is mandatory to point out the most comprehensive reference on this topic within the framework of AdS/CFT, which was written by Beisert et. al. in 2010: [33].

4 Strings on $AdS_5 \times S^5$

Type IIB superstring theory in a curved spacetime is normally described using either the NSR or the GS approach (see [34] for an introduction to both formalisms). For type IIB superstrings in $AdS_5 \times S^5$ the presence of the self-dual five form makes the NSR approach unfeasible. So type IIB superstrings in $AdS_5 \times S^5$ are best described using the Green-Schwarz formalism ([35], [24]), which makes spacetime supersymmetry manifest (in the NSR approach, supersymmetry is restricted to the string worldsheet). It turns out though, that constructing a GS action for superstrings in an arbitray background is terribly hard [28]. But fortune happens to be on our side once again. The special symmetries of the flat background and of $AdS_5 \times S^5$ make the situation much better. In such backgrounds, a non-linear sigma model can be used on the coset superspace that reflects the symmetry of the background. For the flat background Henneaux and Mezincescu [36] used a coset space that was nothing but the 10-dimensional super Poincaré group modulo the Lorentz group $SO(9,1)$. Based on this, Metsaev and Tseytlin managed to build a string sigma model of the type IIB superstring in $AdS_5 \times S^5$ [24] using the coset superspace

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$$

and which fulfilled the required conditions they had in mind. Those were that the bosonic part of their action should be the standard sigma model for $AdS_5 \times S^5$, that the theory should be invariant under a global $SU(2,2|4)$ symmetry, that κ-symmetry should be verified and that the sigma model should reduce to the flat space one in the limit $R \to \infty$. And not only did they succeed in finding such an action, but also showed that it is unique.

To understand their construction, which is very well explained in [28], we need some previous concepts (see also [27]). The superalgebra $sl(4|4)$ can be thought of as the set of $4|4 \times 4|4$ complex matrices with vanishing supertrace. That is, matrix of the form

$$M = \begin{pmatrix} a & \theta \\ \eta & b \end{pmatrix} \tag{4.1}$$

whereby a, b are bosonic matrices (even) and θ, η are fermionic grassmanian (odd) matrices and

$$\text{STr } M = \text{Tr } a - \text{Tr } b = 0 \tag{4.2}$$

The non-compact real form of $sl(4|4)$, which is $su(2,2|4)$ is a restriction to the former by the condition

$$M^\dagger H + HM = 0 \tag{4.3}$$

with

$$H = \begin{pmatrix} \mathbb{I}_2 & 0 & 0 \\ 0 & -\mathbb{I}_2 & 0 \\ 0 & 0 & \mathbb{I}_4 \end{pmatrix} \tag{4.4}$$

It is worth mentioning that the algebra $PSU(2,2|4)$is what we obtain if the identity is removed from $su(2,2|4)$ [27]. Let us now introduce the supertransposition of matrices.

$$M^{st} = \begin{pmatrix} a^T & -\theta^T \\ \eta^T & b^T \end{pmatrix} \tag{4.5}$$

If we now introduce the matrix

$$K = \text{ diag}(\sigma, \sigma, \sigma, \sigma) \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.6}$$

which allows us to build the fourth-order auromorphism

$$M \to \Omega(M) = -KM^{st}K^{-1} \tag{4.7}$$

which endows $sl(4|4)$ with a $\mathbb{Z}_4$ grading, since $\Omega^4 = 1$. We can now define

$$\mathcal{G}^{(k)} = \{M \in sl(4|4)\,|\,\Omega(M) = i^k M\} \tag{4.8}$$

which automatically allows us to decompose the algebra vector space as a sum of graded subspaces

$$sl(4|4) = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)} \oplus \mathcal{G}^{(2)} \oplus \mathcal{G}^{(3)} \tag{4.9}$$

Note that $\mathcal{G}^{(0)}$ is the fixed point of the automorphism Ω and it coincides with with the algebra $so(4,1) \oplus so(5)$ [27]. Given any matrix $M \in sl(4|4)$, tis projection $M^{(k)} \in \mathcal{G}^{(k)}$ can be built as [28]

$$M^{(k)} = \frac{1}{4}\left(M + i^{3k}\Omega(M) + i^{2k}\Omega^2(M) + i^k\Omega^3(M)\right) \tag{4.10}$$

which makes manifest that while $M^{(0)}$ and $M^{(2)}$ are even, $M^{(1)}$ and $M^{(3)}$ are odd.

Let us now assume that $g \in SU(2,2|4)$. We can define an associated one-form A that will live in the corresponding algebra $su(2,2|4)$

$$A = -gdg^{-1} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)} \tag{4.11}$$

the reason why we define A this way is that it automatically has the property of being flat

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0 \qquad \alpha, \beta = \tau, \sigma \tag{4.12}$$

The Lagrangian density describing a type IIB superstring in $AdS_5 \times S^5$ is then postulated to be

$$\mathcal{L} = -\frac{\sqrt{\lambda}}{4\pi} \left[\sqrt{-h} h^{\alpha\beta} \ \mathrm{STr}(A_\alpha^{(2)} A_\beta^{(2)}) + \kappa \epsilon^{\alpha\beta} \ \mathrm{STr}(A_\alpha^{(1)} A_\beta^{(3)}) \right] \tag{4.13}$$

with $\epsilon^{\sigma\tau} = 1$ and $h^{\alpha\beta}$ being the world-sheet induced metric. In the conformal gauge $h^{\alpha\beta} =$ diag$(-1, 1)$. The first term in the action is just a non-linear sigma model on $AdS_5 \times S^5$. The second term corresponds to a Wess-Zumino term steming from the $\mathbb{Z}_4$-grading. The coefficient κ is required to be ± 1 by the κ symmetry [29].

Let us now consider a transformation of our postulated action of the form $g \to gh$, with $h \in so(4,1) \times so(5)$. The effect of this transformation on our one-form A is

$$A^{(i)} \to h^{-1} A^{(i)} h \qquad i = 1, 2, 3 \tag{4.14}$$

that is they undergo a similarity transformation that leaves their contributions to the action invariant. But

$$A^{(0)} \to h^{-1} A^{(0)} h - h^{-1} dh \tag{4.15}$$

which can be seen as a gauge transformation. This means that $SO(4,1) \times SO(5)$ acts as a stabilizer so rather than depending on $g \in SU(2,2|4)$, our action depends on a coset element of the group $\frac{SU(2,2|4)}{SO(4,1) \times SO(5)}$. Moreover, if we take into account that the Lagrangian is also invariant under a shift by the identity matrix, we get the final coset group that reflects the symmetry of the sigma model

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \tag{4.16}$$

It should be no surprise that the stabilizer of the AdS part of the symmetry group be $SO(1,4)$, since there is only 1 physical timelike direction and we embedded AdS_5 in $\mathbb{R}^{2,d-1}$ by introducing an additional timelike direction, pretty much the same way we normally introduce an additional spacelike direction to embed spheres. So for example it is widely known that S^2 can be described as $SO(3)/SO(2)$. In other words, S^2 is what we get by counting all $SO(3)$ rotations that are not different by just an $SO(2)$ rotation. Thus equivalent AdS configurations with respect to the additional timelike direction must be counted only once and this implied taking the modulo with respect to the left $SO(1,4)$ symmetry, which can be understood as the gorup of local Lorentz transformations. It is not then hard to visualise that AdS_5 can be seen as the coset $\frac{SO(4,2)}{SO(4,1)}$ whereas S^5 can be seen as $\frac{SO(6)}{SO(5}$, which together build the bosonic partialof the full symmetry group.

A natural question to ask is hwo to formulate the equations of motion corresponding to our Lagrangian. An elegant way of formulating them is by defining [28]

$$W^\alpha = \frac{\sqrt{\lambda}}{2\pi}\left[\sqrt{h}h^{\alpha\beta}A^{(2)}_\beta - \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(A^{(1)}_\beta - A^{(3)}_\beta\right)\right] \tag{4.17}$$

The equations of motion can then be shown to be

$$\partial_\alpha W^\alpha - [A_\alpha, W^\alpha] = 0 \tag{4.18}$$

Furthermore, it can also be shown that the conserved Noether current corresponding to the global $PSU(2,2|4)$symmetry of the theory is given by

$$J^\alpha = gW^\alpha g^{-1}, \qquad \partial_\alpha J^\alpha = 0 \tag{4.19}$$

which allow us to define the conserved charges

$$Q = \int d\sigma J^\tau = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma g\left(A^{(2)}_\tau - \frac{\kappa}{2}(A^{(1)}_\sigma - A^{(3)}_\sigma)\right) \tag{4.20}$$

where the conformal gauge has been assumed. Also, the equation of motion for the world-sheet metric delievers the Virasoro constraints, which take the form

$$\mathrm{STr}(A^{(2)}_\alpha A^{(2)}_\beta) - \frac{1}{2}h_{\alpha\beta}h^{\rho\delta}\ \mathrm{STr}(A^{(2)}_\rho A^{(2)}_\delta) = 0 \tag{4.21}$$

Up to now our considerations have been very formal and general to present the sigma model for type IIB superstrings in $AdS_5 \times S^5$. Of course when one has to make physical computations it is mandatory to pick a given parametrization, in this case of the coset representative, to get an explicit form of the action. This is done in [24], where the authors present an explicit action. that takes the form

$$I = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\xi \left[L_B(X,Y) + L_F(X,Y,\theta) \right], \qquad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'} \tag{4.22}$$

whereby $\xi^a = (\tau, \sigma)$, $\sigma = \sigma + 2\pi$ and given the conformal gauge convention $\sqrt{-h}h^{ab} = \eta^{ab}$. Also

$$L_B = \frac{1}{2}\sqrt{-h}h^{ab}\left[G^{(AdS_5)}_{mn}(X)\partial_a X^m \partial_b X^n + G^{(S^5)}_{ij}(Y)\partial_a Y^i \partial_b Y^j\right] \tag{4.23}$$

and the fermionic part is

$$L_F = i\left(\sqrt{-h}h^{ab}\delta^{IJ} - \varepsilon^{ab}s^{IJ}\right)\hat{\theta}^I \rho_a D_b \theta^J + O(\theta^4) \tag{4.24}$$

with $I, J = 1, 2$, $s^{IJ} =$ diag(1,-1) and ρ_a are the projected 10-dimensional Dirac matrices

$$\rho_a = \Gamma_M e^M_N \partial_a x^N = (\Gamma_i e^i_N + \Gamma_j e^j_N)\partial_a x^M \tag{4.25}$$

e^M_N being the vielbein of the target space metric, $x^M = (X^m, Y^{m'})$ are the coordinates and i, j are tangent space indices for AdS_5 and S^5 . D_b is the projection of the 10-dimensional derivative, which can be put into the form

$$D_b \theta^J = \left(\delta^{JK} D_b - \frac{i}{2}\epsilon^{JK}\Gamma_* \rho_b\right)\theta^K, \qquad \Gamma_* = i\Gamma_{01234}, \qquad \Gamma_*^2 = 1 \tag{4.26}$$

The metric of $AdS_5 \times S^5$ has a direct product structure. As a consequence of this, the bosonic part of the action is a sum of the AdS_5 and S^5 sigma models ([37]).

4.1 Classical strings

Classical strings can be described by just taking bosonic fields into account, most conveniently taking a given parametrization of the coordinates that accounts for the $AdS_5 \times S^5$ geometry. A five sphere can be easily described by embedding it in R^6

$$\delta_{AB} X^A X^B = X_A X^A = X_0^2 + ... + X_5^2 = 1 \tag{4.27}$$

Five dimensional AdS space can be thought of as an hyperboloid in R^6

$$-\eta_{MN} Y^M Y^N = Y_0^2 - Y_1^2 - ... - Y_4^2 + Y_5^2 = 1 \tag{4.28}$$

with

$$\eta = (-1, +1, ..., +1, -1) \tag{4.29}$$

Note that the radius of both the sphere and the hyperboloid have been set to 1 on behalf of simplicity (even if the fact that they be equal is given by the correspondence). It is also worth mentioningf how explicit the symmetry groups $SO(6)$ for S^5 and $SO(2,4)$ for AdS_5 are made in this parametrization. Now a parametrization of the coordinates is just a given solution to the two last equations. The parametrization which is most often used and shown in the literature is given by 5+5 independent global coordinates:

$$\begin{aligned} Y_1 + iY_2 &= \sinh\rho\cos\theta e^{i\phi_1}, & Y_3 + iY_4 &= \sinh\rho\sin\theta e^{i\phi_2}, \\ Y_5 + iY_0 &= \cosh\rho e^{it}, & X_0 + iX_1 &= \sin\gamma\cos\psi e^{i\varphi_1} \\ X_2 + iX_3 &= \sin\gamma\sin\psi e^{i\varphi_2}, & X_4 + iX_5 &= \cos\gamma e^{\varphi_3} \end{aligned} \tag{4.30}$$

With such a parametrization, equations 4.27 and 4.28 can be rewritten in a new form

$$ds^2_{AdS_5} = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\left(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2\right) \tag{4.31}$$

$$ds^2_{S^5} = d\gamma^2 + \cos^2\gamma\, d\varphi_3^2 + \sin^2\gamma\left(d\psi^2 + \cos^2\psi\, d\varphi_1^2 + \sin^2\psi\, d\varphi_2^2\right) \tag{4.32}$$

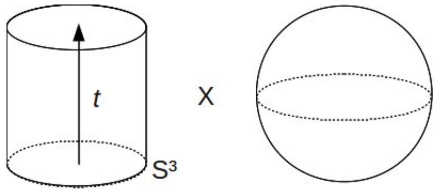

Figure 4.1: Taking 4.33 into account, $AdS_5 \times S^5$ can be thought of as the product space of a cylinder and a five-sphere, whereby the boundary of the cylinder is $R \times S^3$.

The t coordinate in the hyperboloid is normally decompactified to avoid closed time-like curves, so instead of having $0 < t < 2\pi$ we assume $-\infty < t < \infty$. So rather than the hyperboloid described by 4.28, this modified version of it or universal cover is used. Of ocurse, this parametrization is just a convenient choice of the many we might have taken. Other coordinate descriptions and parametrizations might be found in the literature. Note that keeping 4.31 in mind, two important limits can be noticed for big and small AdS radius. For big AdS radius, namely far from the center of AdS the geometry is perceived by a local observer as that of $R^1 \times S^3$

$$\lim_{\rho\to\infty} ds^2_{AdS_5} = \underbrace{-\cosh^2\rho dt^2}_{R^1} + \underbrace{\sin^2\rho d\Omega_3^2}_{S^3} \tag{4.33}$$

where as for an observer close to the center of AdS, the geometry observed is $R^1 \times R^4$

$$\lim_{\rho\to 0} ds^2_{AdS_5} = \underbrace{-dt^2}_{R^1} + \underbrace{d\rho^2 + \rho^2 d\Omega_3^2}_{R^4} \tag{4.34}$$

Bearing this in mind, it may be useful to picture $AdS_5 \times S^5$ as a product space, the AdS_5 part being a bulk cylinder with a boundary which is $R \times S^3$ as depicted in figure 4.1.

Let us now focus on the bosonic part of the above Lagrangian for reasons that will later become clear. As a consequence of the conformal gauge, the so called Virasoro constraints which are requried for the independent auxiliary metric to coincide with the induced metric (see [38]), apply

$$\dot{X}^M X'_M + \dot{Y}^N Y'_N = 0 \tag{4.35}$$

$$\dot{X}^M \dot{X}_M + \dot{Y}^N \dot{Y}_N + X'^M X'_M + Y'^N Y'_N = 0 \tag{4.36}$$

where the contractions are to be made using the metrics in the embeddings 4.27 and 4.28. A dot signals a time derivative whereas the prime stands for a ∂_σ derivative. These Virasoro constraints play a crucial role when it comes to computing the energy of a given string state in terms of the other quantum numbers (spins) as we will see below. Now recovering the embeddings shown above the bosonic part of the action may be written as (see [37] for more details)

$$\frac{T}{2}\int d\tau \int_0^{2\pi} d\sigma \left(-\partial_a Y_M \partial^a Y^M - \Theta(Y_M Y^M + 1) - \partial_a X^N \partial^a X^N + \Lambda(X_N X^N - 1)\right) \tag{4.37}$$

whereby $T = R^2/2\pi\alpha' = \sqrt{\lambda}/2\pi$ note that the embedding in a six-dimensional space-time makes calculations simpler by introducing plane metrics but also requires the introduction of the Lagrange multipliers Θ and Λ which impose the corresponding hypersurface conditions. Also, notice that in our integration over σ we have assumed that our strings are closed and hence periodic. The equations of motion that follow from this action are

$$\partial_a \partial^a Y_M - \Theta Y_M = 0, \qquad Y_M Y^M = -1 \tag{4.38}$$

$$\partial_a \partial^a X_N + \Lambda X_N = 0, \qquad X_N X^N = 1 \tag{4.39}$$

and from them one finds

$$\Theta = \partial^a Y_M \partial_a Y^M, \qquad \Lambda = \partial^a X_N \partial_a X^N \tag{4.40}$$

So for the equations of motion the starting point is non-linear (nothing like $\Box\, x = 0$ reflecting the strong coupling situation. So canonical quantization procedures are far from being trivial or straightaway.

Now looking agian at equations 4.32 and 4.31 one identifies that the cyclic coordinates of the action are $(t, \phi_1, \phi_2, \varphi_1, \varphi_2, \varphi_3)$ from which looking at 4.30 we see that this cyclicity corresponds to pairwise rotations in the (X, Y)-coordinates. It is then not hard to see that the conserved charges corresponding to the six Cartan generators are of the form

$$J_{IJ} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(Y_I \dot{Y}_J - Y_J \dot{Y}_I \right), \qquad S_{MN} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(X_M \dot{X}_N - X_N \dot{X}_M \right) \tag{4.41}$$

and in particular correspond to the choices (recall 2.2)

$$J_0 \equiv J_{05} = E, \quad J_1 \equiv J_{12}, \quad J_2 \equiv J_{34}, \quad S_1 \equiv S_{01}, \quad S_2 \equiv S_{23}, \quad S_3 \equiv S_{45} \tag{4.42}$$

So in particular we have

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} G_{tt}^{AdS_5}\, \partial_\tau t = sqrt\lambda \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(Y_5 \dot{Y}_0 - Y_0 \dot{Y}_5 \right) = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho\, \partial_\tau t \tag{4.43}$$

Note that for a classical string solution to have a consistent interpretation, we whould be able to find a dual gauge operator that carries the same quantum numbers, and in particular that has a definite scaling dimension. We will see below that this means that the gauge operator has to be a highest weight state of the symmetry algebra.

4.2 The semiclassical approach

Quantization of the action 4.22 is beyond current methods. Hence our best approach to quantum strings in AdS_5 is then derived from a perturbative approach around classical solutions, expanding in powers of the effective string tension $\sqrt{\lambda}$. For our purporses, the semiclassical approach departs from a classical expression for the energy of a given string state, $E(J_i, S_j, k_r)$[1], and then computes the quantum corrections to it as an expansion in large tension for the limit of large spins and fixed rescaled charges of the

[1] k_r stand for additional hidden charges like number of folds, spikes or winding numbers

form $J/\sqrt{\lambda}$. The energy is then understood as

$$E = \sqrt{\lambda}\left(E_0(J_i, S_j, k_r) + \frac{1}{\sqrt{\lambda}}E_1(J_i, S_j, k_r) + \frac{1}{\lambda}E_2(J_i, S_j, k_r) + ...\right) \tag{4.44}$$

Problems derived from gauge fixing, kappa gauge invariance and the conformal anomaly are avoided thanks to cancelation of UV divergences, as explained in [39]. The mission of checking the AdS/CFT duality beyond the most symmetric cases looked almost hopeless given its weak/strong. The situation changed drastically in 2002, when new limits were explored in which quantum numbers become large in a given way and the large N limit is kept.

In general to compute quantum corrections around a definite classical solution, one can fix a physical gauge, solve the corresponding constraints and quantise the remaining degrees of freedom. In particular, to compute quantum corrections to the energy at 1-loop for a given classical string state we first try to find an expression for the energy in terms of the fluctuation fields. The gauge constraints then should allow to express the energy in terms of the other quantum numbers (momenta and other conserved charges) and a term quadratic in the fluctuation fields. If the fluctuation fields are quantised using the quadratic part of the strin action, then the computation of the expectation value of the energy might be possible. This is nicely explained in appendix A of [9] for the conformal gauge case, which we follow now for a while.

Recall the general expression for the classical energy of a string 4.43, which we are trying to find quantum corrections for. Let us now introduce the density of the 2d Hamiltonian

$$\begin{aligned}\mathcal{H}(X,Y) &= \frac{1}{2}G^{(AdS_5)}_{mn}(X)\left[\partial_\tau X^m \partial_\tau X^n + \partial_\sigma X^m \partial_\sigma X^n\right] \\ &+ \frac{1}{2}G^{(S^5)}_{ij}(Y)\left[\partial_\tau Y^i \partial_\tau Y^j + \partial_\sigma Y^i \partial_\sigma Y^j\right] \\ &= -\frac{1}{2}G_{tt}\left(\partial_\tau t \partial_\tau t + \partial_\sigma t \partial_\sigma t\right) + ...\end{aligned} \tag{4.45}$$

from the conformal gauge condition 4.36, which is proportional to the hamiltonian density above, we get the condition $\mathcal{H} = 0$. So if we let the field t fluctuate and rewrite it as $\kappa\tau + \tilde{t}$ we get (time translations are an isometry)

$$G_{tt}\partial_\tau t = \frac{1}{2}\kappa G_{tt} + \frac{1}{\kappa}\mathcal{H}(t \to \tilde{t}) \tag{4.46}$$

Of course our aim is to get the corrected version of the energy in terms of the other charges. In particular, if we stick to [9] and consider a string which has 1 spin in AdS_5 and 1 spin in S^5 , which are given by

$$J = \sqrt{\lambda}\int \frac{d\sigma}{2\pi} G_{\phi\phi}\partial_\tau\phi \qquad S = \sqrt{\lambda}\int \frac{d\sigma}{2\pi} G_{\varphi\varphi}G_{\varphi\varphi}\partial_\tau\varphi \tag{4.47}$$

whereby the metric components are the ones in 4.30-4.32. We now let the fields ϕ and φ fluctuate around their classical values as we did for t, substitute in the expressions, taking into account again that the metric is invariant under translations of these fields

$$\phi \to \omega\tau + \tilde{\phi} \qquad \varphi \to \nu\tau + \tilde{\varphi} \tag{4.48}$$

So considering now the full expression for the Hamiltonian density up to relevant order in the fields and the last expressions for E, J, S one is able to find

$$E = \frac{\omega}{\kappa}J + \frac{\nu}{\kappa}S + \frac{\sqrt{\lambda}}{\kappa}\int \frac{d\sigma}{2\pi}\left(\frac{1}{2}\kappa^2 G_{tt} - \frac{1}{2}\omega^2 G_{\phi\phi} - \frac{1}{2}\nu^2 G_{\varphi\varphi} + \tilde{\mathcal{H}}\right) \tag{4.49}$$

where by $\tilde{\mathcal{H}}$ we mean the hamiltonian density evaluated for the fluctuation fields considered $(\tilde{t}, \tilde{\phi}, \tilde{\varphi})$. All other fields (the ones that do not couple to conserved charges) can now be expanded around their classical values and substituted in the previous expression. By doing this, one obtains the final expression (see [9] for a complete derivation)

$$E = E_0 + \frac{\omega}{\kappa}(J - J_0) + \frac{\nu}{\kappa}(S - S_0) + \frac{1}{\kappa}\int \frac{d\sigma}{2\pi}\tilde{\mathcal{H}}_2 d \tag{4.50}$$

and by $\tilde{\mathcal{H}}_2 d$ we mean the 2-d hamiltonian density corresponding to the quadratic fluctuation action evaluated. Note that classical relations can provide us with an expression for ν, κ, ω in terms of the charges J, S.

However this procedure can get very tedious for non simple string configurations or just for higher loop orders. In such situations, there is a more convenient method based on a worldsheet effective action that resembles thermodynamic computations (see [40] for more details on this thermodynamic analogy). In any quantum field theory we know that the expectation value of a source term $J(x)$ conjugated to a field $\varphi(x)$ is given by the functional derivative of the corresponding effective action $\Gamma[\varphi(x)]$, defined as the Legendre transform

$$\Gamma[\varphi(x)] = -E(J) - \int d^4x J(x)\varphi(x) \qquad E(J) = i\ln[Z(J)] \tag{4.51}$$

with $Z(J)$ being the generating functional. In our cases, the sources are the conserved charges E, S, J (or rather their densities), which are conjugate to time derivatives of the corresponding fields, so we have [39]

$$\frac{1}{T}\Gamma(\kappa,\omega,\nu) = -\frac{i}{T}\ln\left\langle e^{iH_{2d}T}\right\rangle + \kappa\left\langle E\right\rangle + \omega\left\langle J\right\rangle + \nu\left\langle S\right\rangle \tag{4.52}$$

whereby T is just the worldsheet time interval (which we let go to infinity)[2]. As we have seen some lines further up, the first of Virasoro conditions implies the vanishing of the 2d-Hamiltonian. Furthermore not all paramters are independent, so we can have for example $\kappa = \kappa(\omega, \nu)$. Taking this into accounta allows us to find expressions of the form

$$\frac{1}{T}\frac{\partial\Gamma}{\partial\nu} = \frac{\partial\kappa(\omega\nu)}{\partial\nu}\left\langle E\right\rangle - \left\langle S\right\rangle \tag{4.53}$$

So knowledge of the effective worldsheet action Γ suffices to compute quantum corrections to the energy. In particular the leading quantum correction Γ_1 is normally found by expanding the Lagrangian about a classical solution (introducing again fluctuating

[2] In the thermodynamic analogy, this is equivalent to taking the zero temperature limit.

fields like we did before, $\varphi \to \varphi + \tilde{\varphi}$) and performing the Gaussian integral (see for example the lecture notes [41] for many more intermediate steps leading to this result)

$$\Gamma_1 = -\frac{i}{2} \log \det \left[-\frac{\delta^2 L}{\delta\tilde{\varphi}\delta\tilde{\varphi}} \right] = -\frac{i}{2} \text{Tr } \log \left[-\frac{\delta^2 L}{\delta\tilde{\varphi}\delta\tilde{\varphi}} \right] \tag{4.54}$$

Note that when our quantum field theory is a string theory we will also have to consider fermonic fields and ghosts.

Semiclassical string energies can potentially yield information about the quantum spectrum of the string in the limit of very large spins and hence give an all-loop prediction for the scaling dimension of the corresponding dual gauge operators, which are eigenvalues of the dilatation operator. Furthermore, the semiclassical analysis can in some cases be applied to short strings (dual to "short" gauge operators). Expansions for large t'Hooft coupling can be made while keeping $J/\sqrt{\lambda}$ fixed. After this, we can try to expand our results in the limit of small spin $J \ll \sqrt{\lambda}$. We will come back to both situation further below.

4.3 Integrability of the classical sigma model

Using now what we have learnt about integrability in the previous chapter and the notions of classical strings reviewed in the preceeding sections, we are now ready to show that the string sigma model for classical strings is indeed an integrable model and accepts a formulation in terms of a Lax connection that satisfies the required conditions. This was first shown by Bena, Polchinski and Roiban in [13]. See also [29] for a good review.

The method is based on an educated guess for the form of the Lax connection

$$L(z)_\alpha = c_0(z) A_\alpha^{(0)} + c_1(z) A_\alpha^{(2)} + c_2(z) * A_\alpha^{(2)} + c_3(z) A_\alpha^{(1)} + c_4(z) A_\alpha^{(3)} \tag{4.55}$$

The flatness condition for this connection is equivalent to the equations of motion as long as the kappa symmetry is respected ($\kappa = \pm 1$) [27] and the coefficients are

$$c_0 = 1 \qquad c_1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) \qquad c_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right), \qquad c_3 = z, \qquad c_4 = \frac{1}{z} \tag{4.56}$$

Note that the requirement that κ be $\pm$ encodes the striking result that the fermionic kappa symmetry is a result of the integrability of the equations of motion, namely of the existence of a suitable Lax connection. Another relevant observation made in [28] is that when performing a κ-symmetry transformation, the Lax connection continues to be flat if and only if the Virasoro constraints are satisfied. This confirms that the local symmetries of the model are very thightly related to the existence of the Lax connection.

Once it has been shown that the sigma model for strings on $AdS_5 \times S^5$ is equivalent to a classical two-dimensional integrable system, we can expect to use all the techniques from integrable systems for its resolution.

4.4 The spectral curve

As we have just seen, classical integrability of the string sigma model implies the existence of a Lax pair or connection, which is flat, and from which an infinite number of conserved charges can be found. So it looks like a complete classical description of the theory should be possible but in practice computing particular solutions is a very involved task and happens to be restricted to a reduced number of simple configurations. However, sometimes we do not worry that much about explicit solutions. What we are mostly interested in in this dissertation is the so-called spectral AdS/CFT, namely the existing duality between the energy of string states in $AdS_5 \times S^5$ and the scaling dimension of the dual conformal gauge operators in $\mathcal{N}$=4 SYM . The possibility of finding spectral curves is a direct consequence of integrability and plays a very interesting role as long as this area of research is concerned, since it can provide us with a shortcut towards our final aim at least for classical and semi-classical string states. Specifically, using spectral curve techniques, the 1-loop energy for string states can be computed without having to solve explicitly the equations of motion.

Using the eigenvalues of the monodromy matrix, which go under the name of quasi-momenta, a spectral curve can be constructed that helps characterising the different possible classical solutions. This turns out to be a very powerful tool when it comes to describe the space of possible classical solutions. Furthermore, the semiclassical approach can also be taken here. The quasi-momenta can be semiclassically quantised in order to get corrections to interesting quantities related to the strings such as the energy and what is more, this can be done in a very powerful way.
Recall that given an integrable theory, a Lax connection can be found which is planar and from which the equations of motion of the system can be recovered. Using the Lax connection we can build a monodromy matrix, whose eigenvalues (equation 3.7) are a continuum set of conserved charges, namely a spectral (complex) curve. Since the Lax connection for classical string theory in $AdS_5 \times S^5$ contains singular points (recall equations 4.55, 4.56), it is to expect, that the eigenvalues of the monodromy matrix $\xi_i(z)$ do exhibit singularities in the spectral parameter as well. Therefore we prefer to work with the so-called quasimomenta, $\xi_i(z) = e^{ip_i(z)}$. [25] presents lots of details about the way we can relate quasimomenta to the eigenvalues of the monodromy matrix. The last expression will be enough for us here.
Given the symmetry group of the string worldsheet 4.16, we know that the Lax connection must be super-traceless and hence the monodromy matrix

$$T(z) = \mathcal{P} \exp \left(\oint L(z) \right) \tag{4.57}$$

will have SDet $T(z) = 1$. So if we think of diagonalising $T(z)$, we can think of it as

$$T(z) \propto \text{ diag} \left(e^{ip_1(z)}, e^{ip_2(z)}, e^{ip_3(z)}, e^{ip_4(z)} | e^{iq_5(z)}, e^{iq_6(z)}, e^{iq_7(z)}, e^{iq_8(z)} \right) \tag{4.58}$$

So the condition SDet $T(z) = 1$ can be rewritten as

$$\sum_{i=1}^{4}(p_i - q_i) = 2\pi n, \qquad n \in \mathbb{Z} \tag{4.59}$$

So we see that the quasimomenta define an algebraic curve consisting of cuts connecting eight sheets of a Riemann surface. Note that the number of sheets is given by the degree of the characteristic polynomial, which in this case of strings in $AdS_5 \times S^5$ is eight . So the algebraic curve can be thought of as of cuts connecting the eight Riemann sheets. The quasimomenta will hence have discontinuities along such cuts, so if let $\mathcal{C}_i j$ denote a cut between the sheets i and j, we have for example

$$P_i(z + i\epsilon) - Q_j(z - i\epsilon) = 2\pi n_{ij} \tag{4.60}$$

Since four of the quasimomenta or sheets correspond to the AdS_5 part of the string target metric p_i and four correspond to the S^5 part q_i, P_i and Q_j take the following values in the previous equation:

$$P_i = (p_1, p_2, q_1, q_2) \qquad Q_j = (p_3, p_4, q_3, q_4) \tag{4.61}$$

Cuts stretching between sheets of various types can be associated to different polarizations [42]:

$$\begin{aligned} S^5: &\quad (q_1, q_3), (q_1, q_4), (q_2, q_3), (q_2, q_4) \\ AdS_5: &\quad (p_1, p_3), (p_1, p_4), (p_2, p_3), (p_2, p_4) \\ \text{Fermions}: &\quad (p_1, q_3), (p_1, q_4), (p_2, q_3), (p_2, q_4) \\ &\quad (q_1, p_3), (q_1, p_4), (q_2, p_3), (q_2, p_4) \end{aligned} \tag{4.62}$$

where we have used the quasimomenta to denote the corresponding cuts. This is depicted in figure 4.2.

As we had already anticipated, the singularities in the Lax connection are reflected as simple poles of the quasimomenta for $x = \pm 1$. The corresponding residues are

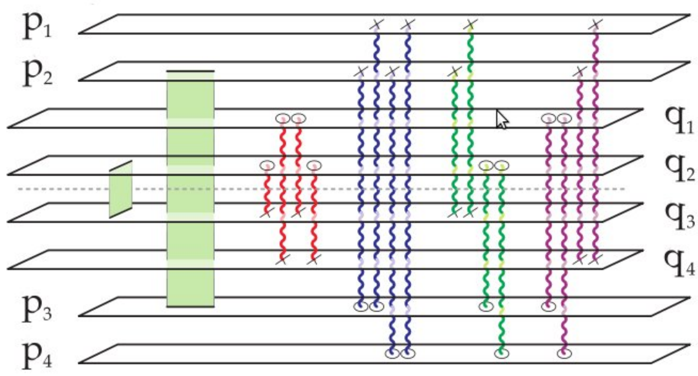

Figure 4.2: (From [42]). The picture shows the spectral curve of superstrings on $AdS_5 \times S^5$. The green cuts connecting sheets characterise classical solutions. In red are depicted polarization of bosonic fluctuations in S^5 . In blue are depicted bosonic fluctuations in AdS_5 . In green and purple are shown fermionic fluctuations.

correlated via the Virasoro constraint:

$$(p_1, p_2, p_3, p_4|q_1, q_2, q_3, q_4) = \frac{(\alpha_\pm, \alpha_\pm, \beta_\pm, \beta_\pm|\alpha_\pm, \alpha_\pm, \beta_\pm, \beta_\pm)}{x \pm 1} \tag{4.63}$$

An additional constaint on the quasimomenta is given by the automorphism in 4.7 which defines an inversion symmetry ([42], [27]).

$$\begin{aligned} p_{1,2}(z) &= -p_{2,1}(1/z) - 2\pi m \\ p_{3,4}(z) &= -p_{4,3}(1/z) - 2\pi m \\ q_{1,2,3,4} &= -q_{2,1,4,3}(1/z) \end{aligned} \tag{4.64}$$

Finally if we focus on the limit $z \to \infty$ one can see ([43]) that the asymptotics of the quasimomenta are given by global charges. This is no surprise if we take into account that in this limit the Lax connection happens to be the Noether current associated to those charges:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \hline q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{2\pi}{z\sqrt{\lambda}} \begin{pmatrix} E - J_1 + J_2 \\ E + J_1 - J_2 \\ -E - J_1 - J_2 \\ -E + J_1 + J_2 \\ \hline S_1 + S_2 - S_3 \\ S_1 - S_2 + S_3 \\ -S_1 + S_2 + S_3 \\ -S_1 - S_2 - S_3 \end{pmatrix} \tag{4.65}$$

And note the possibility of expressing the energy as

$$E = \frac{\sqrt{\lambda}}{4\pi} \lim_{z\to\infty} z\left(p_1(z) + p_2(z)\right) \tag{4.66}$$

which is of course very useful for the spectral problem.

Another important concept is that of the filling fraction, which is defined as []

$$S_{ij} = \pm\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{C}_{ij}} \left(1 - \frac{1}{x^2}\right) P_i(x)dx \tag{4.67}$$

whereby the plus sign is used when P_i is a p_i and the minus sign is used for q_i. The filling fractions are integers that roughly measure the length of the cut. They are also identified as the action angle variables of the theory [44].

All the considerations above led to the realization by the authors of [15] that the quasi-momenta (namely their algebraic curve) can be used to characterise classical string solutions. This means a subtle change in our perspective. Earlier we have characterised classical string solutions by means of their Cartan charges, that is $(E, J_1, J_2; S_1, S_2, S_3)$. Now the characterisation is made via asymptotical constraints on the quasimomenta, which will be given by the Cartan charges.

As we anticipated above, the spectral curve does not only help in characterising the classical string solutions of the theory. It also allows us to compute quantum corrections in a semiclassical way. Ssee [45] for detailed description.

5 $\mathcal{N}$=4 super Yang-Mills and spin chains

$\mathcal{N}$=4 super Yang-Mills is the gauge theory with the maximal possible amount of supersymmetry in four spacetime dimensions. It also has the very relevant characteristic of being a conformal theory, namely a theory with no physical scale. This conformal symmetry does not only hold at the classical level but also at the quantum level, as can be seen from the vanishing of its β-function. As widely known $\mathcal{N}$=4 SYM has a simple field content consisting of the gauge field A_μ, four chiral fermions λ_α, four anti-chiral fermions $\bar{\lambda}^{\dot{\alpha}}$ and six real scalars ϕ. All fields transform in the adjoint representation of the gauge group, which for us will always be SU(N). Given the existence of four fermionic supersymmetry generators that can be rotated into one another an SU(4) R-symmetry is also present. So summing up the theory satisfies a conformal symmetry in four spacetime timensions given by the conformal group $SO(2,4)$, an $SU(4)$ R-symmetry, the Poincaré supersymmetry generated by the supercharges and the conformal supersymmetries, which are all enclosed in the superconformal group $PSU(2,2|4)$, which contains the bosonic subgroup $SO(2,4) \times SO(6)$. The theory is completely described by its Lagrangian, which is [27]

$$\begin{aligned} \mathcal{L} = &\frac{1}{4}\mathrm{Tr}\ F^2 + \frac{1}{2}\mathrm{Tr}\ D_\mu\phi_i D^\mu\phi^i - \frac{g_{YM}^2}{4}\mathrm{Tr}\ [\phi_i,\phi_j][\phi^i,\phi^j] + \mathrm{Tr}\ \bar{\psi}^a\sigma^\mu D_\mu\psi_a \\ &- \frac{ig_{YM}}{2}\mathrm{Tr}\ \sigma_i^{ab}\psi_a[\phi^i,\psi_b] - \frac{ig_{YM}}{2}\mathrm{Tr}\ \sigma^i_{ab}\bar{\psi}^a[\phi_i,\bar{\psi}^b] \end{aligned} \tag{5.1}$$

Many excellent reviews on $\mathcal{N}$=4 SYM exist in the bibliography, among which we may point out the presentations contained in [23] and in [46]. We will now follow the latter for a brief presentation of the superconformal algebra. For a more detailed description of $PSU(2,2|4)$ we refer the reader to [47]

5.1 The algebra of $\mathcal{N}$=4 SYM

The bosonic symmetry group of $\mathcal{N}$=4 SYM , $SO(2,4) \times SO(6)$, is generated by 15 generators, 10 of which correspond to the Poincaré generators in four dimensions, (4 generators for spacetime translations P_μ + 6 generators for Lorentz transformations, $M_{\mu\nu}$), 4 being the generators of special conformal transformations, K_μ and one of them generating dilatations, D. This last generator plays a crucial role in the spectral study of $\mathcal{N}$=4 SYM . For a comprehensive paper on the dilatation operator see Niklas Beisert's PhD Thesis, [48]. As already pointed out above $\mathcal{N}$=4 SYM can be shown to be a fully conformal theory, even at the quantum level and at all-loop level by computing its β-function (see [49] for formal review and [46] for a more simplistic and intuitive one). The algebra of $PSU(2,2|4)$can summarised by having a look at the commutation relations between its generators

$$\begin{aligned} &[D, P_\mu] = -iP_\mu \qquad [D, M_{\mu\nu}] = 0 \qquad [D, K_\mu] = iK_\mu \\ &[M_{\mu\nu}, P_\lambda] = -i\left(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu\right) \qquad [M_{\mu\nu}, K_\lambda] = -i\left(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu\right) \\ &[P_\mu, K_\nu] = 2i\left(M_{\mu\nu} - \eta_{\mu\nu}D\right) \end{aligned} \tag{5.2}$$

If we now assume that $\mathcal{O}(x)$ is a local operator in our gauge theory $\mathcal{N}$=4 SYM with scaling dimension Δ, we are implicitly assuming that under a coordinate rescaling the behaviour of the operator is given by this parameter [46]

$$x \to \lambda x \Rightarrow \mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x) \tag{5.3}$$

Recall that coordinate rescalings are generated by the scaling operator D

$$\mathcal{O}(x) \to \lambda^{-iD}\mathcal{O}(x)\lambda^{iD} \tag{5.4}$$

so when acting on $\mathcal{O}(x)$ (transforming in the adjoint representation), the scaling operator gives place to

$$[D, \mathcal{O}(x)] = i\left(-\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x) \tag{5.5}$$

5.2 Primary operators

Let us now consider the operator resulting from the action of K_μ upon $\mathcal{O}(0)$[1] and let D act on it using the Jacobi identity

$$\begin{aligned}
&[D, [K_\mu, \mathcal{O}(0)]] = [[D, K_\mu], \mathcal{O}(0)]] + [K_\mu, [D, \mathcal{O}(0)]] \\
&= i[K_\mu, \mathcal{O}(0)] - i\Delta[K_\mu, \mathcal{O}(0)]
\end{aligned} \tag{5.6}$$

If one compares this last expression to 5.5 at $x = 0$, we see that the effect of K_μ upon the gauge operator has been a lowering of its scaling dimension by 1. We can now use our "holographic intuition" and think that if Δ is to be dual to an energy, it should always remain a positive quantity. Still, by consequtive actions of the special conformal operators K_μ it looks like their scaling dimension would always become lower. It is then easy to believe that there must be an operator that stops the trend by satisfying

$$[K_\mu, \mathcal{O}(0)] = 0 \tag{5.7}$$

such operators are the basis of the tower (they have the lowest possible scaling dimension) and are usually called primary operators. Operators than do not satisfy the previous equation and which can be obtained from a primary operator via the action of K_μ are called descendants. This resembles the role played by a highest weight state in a representation of a group. Well indeed primary operators are highest weights of irreducible representations of $PSU(2,2|4)$, the descendants being all other weights. Of course we are free to increase Δ as many times as we want, which means that such representations are inifinite dimensional, as we could have expected from the non-compactness of $PSU(2,2|4)$.

[1]As pointed out by [46] the gauge operator must be evaluated at the origin. Were it not, commutation relations would have to be different.

It is revealing for our purposes to consider what happens when we further restrict our operators and pick just those that satisfy the requirement

$$[Q^a_\alpha, \mathcal{O}(0)] = 0 \qquad \text{for some } \alpha, a \tag{5.8}$$

namely that it commute with some of the supercharges. It turns out that the theory does not only contain the generators listed in 5.2 and the supercharges. From the action of the special conformal generators K^μ upon the supercharges (that is from their commutation relations), new charges are found, which go under the name of superconformal charges. So from

$$[K^\mu, Q_{\alpha a}] = \gamma^\mu_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{S}_{\dot{\beta}a} \qquad [K^\mu, \bar{Q}^a_{\dot{\alpha}}] = \gamma^\mu_{\alpha\dot{\alpha}}\epsilon^{\alpha\beta}S^a_\beta \tag{5.9}$$

From it, it can be shown [46] that the commutators between the supercharges and the superconformal charges are

$$\begin{aligned}
\{Q_{\alpha a}, S^b_\beta\} &= -i\epsilon_{\alpha\beta}\sigma^{AB}{}_a{}^b R_{AB} + \gamma^{\mu\nu}_{\alpha\beta}\delta_a{}^b M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta_a{}^b D \\
\{\bar{Q}^a_{\dot{\alpha}}, \bar{S}_{\dot{\beta}b}\} &= +i\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{AB a}{}_b R_{AB} + \gamma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\delta^a{}_b M_{\mu\nu} - \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\delta^a{}_b D \\
\{Q_{\alpha a}, \bar{S}_{\dot{\beta}b}\} &= \{\bar{Q}^a_{\dot{\alpha}}, S^b_\beta\} = 0
\end{aligned} \tag{5.10}$$

whereby R_{AB} stands for the R-symmetry generators, which recall that for $\mathcal{N}$=4 SYM are the generators of $SO(6)$. Of course the supersymmetry generators transform under the two spin representations of $SO(6)$, which are also the fundamental representation of $SU(4)$. The σ-matrices carry indices in both representations. We have also used $\gamma^{\mu\nu}_{\alpha\beta} = \gamma^{[\mu}_{\alpha\dot{\alpha}}\gamma^{\nu]}_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}$.

Now from the relations 5.9 and from

$$[D, Q_{\alpha a}] = -\frac{i}{2}Q_{\alpha a} \qquad [D, \bar{Q}^a_{\dot{\alpha}}] = -\frac{i}{2}\bar{Q}^a_{\dot{\alpha}} \tag{5.11}$$

we learn that since the supercharges Q have dimension $1/2$, the superconformal charges must have dimension -1/2. This automatically means that successive application of the superconformal charges to any gauge operator will lower its scaling dimension by 1/2. We could now repeat the same argumentation as for the definition of primary operators in the previous section to realise that there must be a lower bound on the tower of states. Hence we define a conformal primary operator to be an operator that satisfies

$$[S^a_\alpha, \bar{\mathcal{O}}(0)] = [\bar{S}_{\dot{\alpha}a}, \bar{\mathcal{O}}(0)] = 0 \tag{5.12}$$

Note that this is a stronger restriction than that of the operator being a primary operator. All superconformal primary operators are also primary operators, but the opposite is not true. As with the primary operators, other operators, also called descendants, can be obtained from a superconformal primary operator by letting the supercharges Q act upon. The superconformal primary operator an its descendants belong to the same irreducible infinite dimensional representation of $PSU(2,2|4)$.

Primary operators (operators that commute with all K_μs) are highest weights of irreducible representations of $PSU(2,2|4)$. All other members of the representations are descendants that can be obtained from the primary operator. As a consequence of the non-compactness of $PSU(2,2|4)$, such representations are infintely dimensional.

5.2.1 BPS operators

Of course we are free to further restrict the kind of gauge operators on which we focus our attention by placing new restrictions. A remarkable case is that of operators that commute with some of the supercharges, that is, operators which fulfill

$$[Q, \mathcal{O}(0)] = 0 \tag{5.13}$$

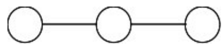

Figure 5.1: Dynkin diagram of $SO(6)$ and hence of $SU(4)$ too.

for some of the supercharges. It then follows from the relations 5.10 that such an operator must also satisfy [46]

$$\sigma^{AB}{}_{a}{}^{b}[R_{AB}, \mathcal{O}(0)] = \Delta \delta_{a}{}^{b} \mathcal{O}(0) \tag{5.14}$$

We now that $SO(6)$ is a rank three group (see the corresponding Dynkin diagram in figure 5.1), so we can take three of the R-generators to be the mutuallty commuting Cartan generators, say R_{12}, R_{34} and R_{56}. Let us now take J_i with $i = 1, 2, 3$ to be the corresponding charges. As a consequence of the conrete form of the σ matrices (see [46]), one can see that for those operators that have Cartan charges equal to their scaling dimension (that is for example $(J_1 = \Delta, J_2 = 0, J_3 = 0)$) the relation 5.14 is fulfilled for half of the supercharges. Such operators go under the name of BPS operators and will be relevant for us. The most relevant feature of such operators is that their scaling dimension Δ is independent of the gauge theory coupling g_{YM}. In other words, they have no anomalous dimension. This follows from the fact that the R-charges do not depend on the coupling. Furthermore, this extends to all descendants, since if for a BPS primary operator we have $\Delta = \Delta_0 + \lambda\gamma$, for the descendants we will have $\Delta = \Delta_0 + \lambda\gamma + n(1/2)$. So if $\gamma = 0$ for the BPS primary operator, so must it be for all descendants.

In principle primary gauge operators could be made up of any of the fundamental fields present in the theory. Still, if we take into account the condition 5.13 and the effect of the supercharges on the different fundamental fields [23]

$$\begin{aligned} &\{Q, \lambda\} = F^{+} + [X, X] \qquad \{Q, \bar{\lambda}\} = DX \\ &[Q, X] = \lambda \qquad [Q, F] = D\lambda \end{aligned} \tag{5.15}$$

Since primary operators cannot contain commmutators of a supercharge with any other field (per definition), we now that they cannot contain any of the fields on the right hand sides of the previous expressions, which leaves us just with symmetrised[2] combinations of scalars. So for example the simplest primary operators will be of the form

$$\text{str}\ \left(X^{i_1}X^{i_2}...X^{i_n}\right) \tag{5.16}$$

where "str" denotes a symmetrised trace. One of the biggest insights in this research field stems from the realisation that we can think of such operators as of states of a spin chain, where each spin takes valuees from 1 to 6, corresponding to the number of different scalar in $\mathcal{N}$=4 SYM . We will come back to this soon.

5.3 The anomalous dimension

The scaling dimension takes a relevant role in conformal QFTs, since it enters the two-point function. Consider a two-point function or propagator of two operators of the CFT. Poincaré symmetry dictates that the function can only depend on the interval between the two spacetime points in which the operators are defined. Furthermore, invariance under scaling dimensions and under special conformal transformations, which complete the conformal symmetry, force the propagator to take the form

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta(g)}} \tag{5.17}$$

When renormalizing, we must recall the presence in the action of a quartic term in the scalars and notice that as long as the bosonic part is concerned, only the scalar vertex needs to be taken into account. Gauge bosons do not carry a colour index and hence their effect on the anomalous dimension can be determined a posteriori by exploiting the strength of the superconformal algebra [46]. In the weak coupling limit we can

[2] Commutators are antisymmetrisations

perform an expansion of the above propagator to get

$$\langle \mathcal{O}_A(x)\bar{\mathcal{O}}_B(y)\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}\left(\delta_{AB} - g^2\gamma_{AB}\ln\Lambda^2|x-y|^2\right) + ... \tag{5.18}$$

whereby Λ is the cut-off. With the definitions

$$\Delta = \Delta_0 + g\gamma \qquad \mathcal{O}^{\text{ren}} = \Lambda^{-g\gamma} \tag{5.19}$$

5.18 can be rewritten as

$$\langle \mathcal{O}_A^{\text{ren}}(x)\bar{\mathcal{O}}_B^{\text{ren}}(y)\rangle = \frac{1}{|x-y|^{2\Delta}} \tag{5.20}$$

which basically is the statement that renormalization and anomalous dimension mean the same thing. If we know how to renormalize (namely what power of the cut-off Λ $\mathcal{O}$ must be multiplied by) the guage operator we are dealing with, we then know its scaling dimension Δ and viceversa.

In general we will be interested in computing the correction to the bare scaling dimension given by the quantum (interacting) nature of the theory. So the question we must ask ourselves is how to compute Δ for a given configuration in the quantum theory rhather than how to individualise particular states that solve the corresponding equations of motion. In particular we know that the scaling dimension is closely related to renormalization. Let us elaborate a to-do list when it comes to computing the scaling dimension and the corrections it receives, from what we have just reviewed in this section:

1. Compute the renormalized 2-point function or propagator.

2. Find the logarithmic divergence. Its coefficient is the correction to the scaling dimension.

Let us now turn to $\mathcal{N}$=4 SYM and notice first of all that different fields can share the same Δ_0 without actually being the same field. Think for example of the following gauge operators, made up of the complex scalar operators $Z = \phi_1 + i\phi_2$ and $X = \phi_3 + i\phi_4$

$$\text{Tr}\,(ZXZX) \qquad \text{Tr}\,(ZZXX) \tag{5.21}$$

They will both have $\Delta_0 = 4$. So we can actually recover expression 5.18 and note that the indices A and B run over the different possibilites when picking a field, in this case either Z or X. At this step, diagonalising the operators is a sensible thing to do. So we can linearly combine our different fields, which share the same bare scaling dimension

$$\mathcal{O}_A^{\text{diag}} = T_{AB}\mathcal{O}_B \tag{5.22}$$

so that

$$\gamma\mathcal{O}_B^{\text{diag}} = \gamma_B\mathcal{O}_B^{\text{diag}} \tag{5.23}$$

whereby the first γ is meant to be a matrix, while the second one is a number. So we have

$$\langle\mathcal{O}_A^{\text{diag}}(x)\mathcal{O}_B^{\text{diag}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}\left(\delta_{AB} - g^2\gamma_A \ln\Lambda^2|x-y|^2\right) + ... \tag{5.24}$$

So in other words, we are working in the vector space of operators that share the same scaling dimension and diagonalise the matrix γ. Its eigenvalues are the corrections to each scaling dimension and the eigenvectors are the operators whose correction to the scaling dimension are in each case the corresponding eigenvalue.

5.4 Spin chains

Let us try to compute the scaling dimension of a primary operator of the kind in 5.16

$$\Psi = \frac{(4\pi^2)^{L/2}}{\sqrt{L}N^{L/2}}\text{Tr}\,(\underbrace{ZZ...ZZ}_{L}) \tag{5.25}$$

where we have chosen the right prefactors for normalization purposes. $\mathcal{N}$=4 SYM is a conformal field theory and so we know that the scalar propagator at tree-level will be

$$\langle Z^A_B(x)\bar{Z}^C_D(y)\rangle_0 = \frac{\delta^A_D\delta^C_B}{4\pi^2|x-y|^2} \tag{5.26}$$

so for the correlator of the Ψ operator then we will have to contract all component fields Z in all different ways (pairings), including

$$...Z^{A'}_A Z^A_{A'} Z^{B'}_B Z^B_{B'} Z^{C'}_C Z^C_{C'}... \tag{5.27}$$

but also pairings like

$$...Z^{A'}_A Z^A_{B'} Z^{C'}_B Z^B_{A'} Z^{B'}_C Z^C_{C'}... \tag{5.28}$$

and with more crossings but is a good point to observe that whereas the first kind of pairings will scale like N^L the second kind will scale like N^{L-2} and so this and more complicated pairings will be supressed in the large N limit we are interested in. Contractions of the first structure are called planar and they are the only ones to consider in the large N limit[3] So in the large N limit the propagator for our gauge operator Ψ will be given by

$$\langle\Psi(x)\bar{\Psi}(y)\rangle = \frac{1}{|x-y|^{2L}} \tag{5.29}$$

The expression can be generalised for any properly normalized scalar operator (based on the trace of a combination of scalars)

$$\langle\mathcal{O}_{I_1,\dots,I_L}(x)\bar{\mathcal{O}}^{J_1,\dots,J_L}(y)\rangle \propto (\delta^{J_1}_{I_1}...\delta^{J_L}_{I_L} + \text{ shifts })\frac{1}{|x-y|^{2L}} \tag{5.30}$$

where by "shifts" we mean the $L-1$ cyclic planar contractions of the deltas, which can be represented schematically by figure 5.2. We now want to take a step further and look at the 1-loop contribution to the propagator. For this purpose, we need to take a second and closer look at the Lagrangian 5.1. We are focusing our atention on the

[3]We are assuming thought that $L \ll N$, see [46]

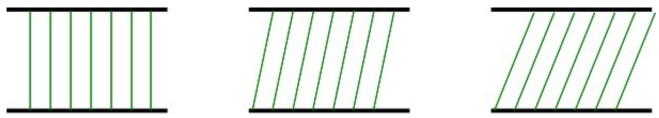

Figure 5.2: Planar contraction of the scalar fields and cyclic shifts. There are $L-1$ such shifts. Horizontal lines represent the operatos and vertical lines the correlators that contract them.

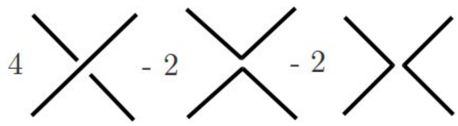

Figure 5.3: Contraction of the two incoming and two outgoing scalar fields at the scalar vertex given by 5.32. The figures stand for the possible permutation of the fields, where as the relative sign are given by the commutator in 5.31.

scalars of the theory, so we will only need the part of the Lagrangian involving only scalars, namely

$$\frac{1}{2}\mathrm{Tr}\ D_\mu\phi_i D^\mu\phi^i - \frac{g_{YM}^2}{4}\mathrm{Tr}\ [\phi_i,\phi_j][\phi^i,\phi^j] \tag{5.31}$$

other fields will also have their contribution at 1-loop level but the superconformal algebra allows us to derive it from just the scalar contribution. What is more, the symmetry allows to consider just the scalar vertex, namely the second term in the previous equation. So to get the contribution to the 1-loop correlator, we must take the scalar vertex and Wick contract it with two incoming scalar and two outgoing scalars. The scalar interaction vertex can be expanded as

$$\phi_i\phi_j\phi^i\phi^j - \phi_i\phi_j\phi^j\phi^i \tag{5.32}$$

which schematically and taking into account the possible permutations of incoming and outgoing the fields in each diagram can be displayed as in figure 5.3. In more formal terms, what we have represented in figure 5.3 can be written, taking propagators into account but forgetting about prefactors as

$$\langle\phi(x)\phi(x)| \underbrace{\phi(z)^4}_{\text{scalar vertex}} |\phi(y)\phi(y)\rangle \propto \underbrace{\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}...\delta^{j_L}_{i_L}}_{\text{other legs}}$$
$$\times\left(4\delta^{j_{k+1}}_{i_k}\delta^{j_k}_{i_{k+1}} - 2\delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}} - 2\delta^{j_k}_{i_k}\delta^{j_{k+1}}_{i_{k+1}}\right)\frac{1}{|x-y|^{2(L-2)}}\lambda\int\frac{dz}{|x-z|^4|y-z|^4} \tag{5.33}$$

where the integral is originated by the propagators and the powers of 4 in the denominator come from each propagator involving two scalars and from the two copies of each propagator. Of course the integral has UV divergences at $x = z$ and at $y = z$. Imagina for example that we are at $x = z$, then we get from the integrals above

$$\left(\int\frac{dz}{|x-z|^4}\right)\frac{1}{|x-y|^4} \tag{5.34}$$

so that the total power of $|x - y|$ in the denominator is $2L$, which is exactly what we could have expected according to 5.24, since in this case of course $\Delta_0 = L$. So we are now left with the task of computing the coefficient of the logarithmic divergence in 5.24. It will be given by the integral in 5.34, which can be computed by performing a change of variables $(x - z) \to \xi$ and inserting an UV cutoff Λ to integrate out short distances

$$\int\frac{dz}{|x-z|^4} = \int_{1/\Lambda}\frac{\xi^3 d\xi}{\xi^4} \propto \ln\Lambda \tag{5.35}$$

We have been leaving lots of prefactors apart. Doing everything rigurously, we would have got for 5.33

$$\langle\phi(x)\phi(x)|\phi(z)^4|\phi(y)\phi(y)\rangle \propto -\frac{\lambda}{8\pi^2}\left(2\delta^{j_{k+1}}_{i_k}\delta^{j_k}_{i_{k+1}} - \delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}} - \delta^{j_k}_{i_k}\delta^{j_{k+1}}_{i_{k+1}}\right)\frac{\ln\Lambda^2|x-y|^2}{|x-y|^{2L}} \tag{5.36}$$

And from this we can immediately extract the matrix γ in 5.24 and which gives us the 1-loop corrections to the scaling dimension of our operator.

There is still an important thing to notice, which is that we cannot neglect one-loop planar diagrams with internal gluons and fermions, like the ones in figure 5.4. Such diagrams cannot add new terms to our computation, since they leave the R-charges of the scalars unchanged. So they just contribute by shifting the third term in the diagram 5.3. This means that a constant α must replace the 2 coefficient in this term and it will have to be determined. Fortunately it is not hard to determine that α has to be exaclty 2, [46].

So summing up we have found that the γ matrix in 5.24 we were looking form is

$$\gamma = \frac{1}{16\pi^2} \sum_{l=1}^{L} (K_{l,l+1} - 2P_{l,l+1} + 2) \tag{5.37}$$

whereby we have defined the operators $P_{l,l+1}$, which is the exchange operator that exchanges the flavour indices of the l and $l+1$ sites of the trace and $K_{l,l+1}$, which is the trace operator contracting the flavours of neighbouring sites in the trace.

This is what inspired Minahan and Zarembo to change the way we look at our gauge operator. They suggested taking the change of perspective

$$\text{Tr } (\phi_{i_1}...\phi_{i_L}) \rightarrow |i_1, ..., i_L\rangle \tag{5.38}$$

and then interpreting gauge operators as states in a spin chain. Then they went on to suggest

$$\gamma\mathcal{O} \rightarrow H|\mathcal{O}\rangle \tag{5.39}$$

that is of considering the gamma operator for the gauge operators as a Hamiltonian operator for the spin chain. Where now we have

$$H = \frac{1}{16\pi^2} \sum_{l=1}^{L} (K_{l,l+1} - 2P_{l,l+1} + 2) \tag{5.40}$$

with

$$P|i,j\rangle = |j,i\rangle \qquad K|i,j\rangle = \delta_{ij} \sum_{k=1}^{6} |kk\rangle \tag{5.41}$$

So our hope of finding the eigenvalues of the γ matrix to get the anomalous dimensions of the corresponding gauge operators has now translated in the problem of diagonalising the Hamiltonian of a spin chain! Diagonalizing the Hamiltonian of $\mathcal{N}$=4 SYM at 1-loop

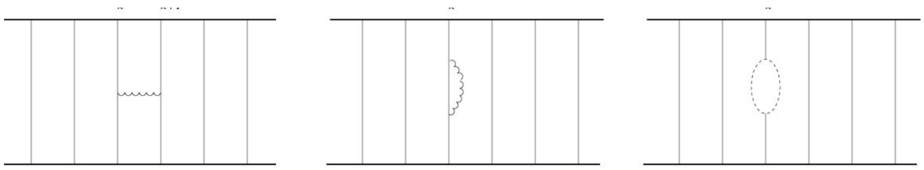

Figure 5.4: Such diagrams must be taken into account, since they do contribute to the anomalous dimension, but given that they leave flavors unchanged, the only think they can do is shifting the term in 5.3 that is proportional to the identity by a fixed amount.

for the SO(6) sector would in principle require the diagonalization of a $6^L \times 6^L$ matrix, which might be quite involved. So only a miracle could prevent us from abandoning any hopes of going further on into this. Well it turns out this miracle does indeed happen and it goes under the name of integrability!

5.5 The SU(2) sector and the Bethe ansatz

If we restrict our spin chains to the $SO(6)$ sector, this amounts on the string side to letting the string move just on the S^5 part. If we further restrict to an $SU(2)$ subsector, taking the effect of the supercharges rotating into one another, we see that this is equivalent to taking just two complex scalars into account and forgetting about the rest of the field content of $\mathcal{N}$=4 SYM . On the gravity side this can be understood as letting our strings having just two spins in S^5 . Now since only two different type os scalar fields are present in the gauge theory, the two scalar fields can be represented via a shorthand notation that resembles the one usually employed to deal with spin chains:

$$|\uparrow\rangle \equiv Z = \phi_1 + i\phi_2 \qquad |\downarrow\rangle \equiv X = \phi_3 + i\phi_4 \tag{5.42}$$

We can now realise how the operators present in the Hamiltonian affect the different possible pairings

$$
\begin{aligned}
&P|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle \\
&K|\uparrow\downarrow\rangle = 0 \\
&K|\uparrow\uparrow\rangle = K|\phi_1\phi_1\rangle - K|\phi_2\phi_2\rangle = 0
\end{aligned}
\tag{5.43}
$$

which means that the Hamiltonian is reduced to an effective version for the SU(2) sector of $\mathcal{N}$=4 SYM

$$
H_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}(\mathbb{I}_{l,l+1} - P_{l,l+1}) \tag{5.44}
$$

which is the Hamiltonian of the Heisenberg spin-chain with L lattice sites. The solution to this spin-chain can be found by means of the Bethe ansatz, which was first introduced by Hans Bethe in [50]. We start with the ground state of such a spin chain, which is $|\uparrow\uparrow ... \uparrow\rangle$ and from which the whole Hilbert space can be generated by reversing an arbitray number of spins. Such a spin chain represents a BPS gauge operator of the form Tr Z^L Each fliped spin is called a magnon. The SU(2) Hamiltonian conserves the number of spins and so it can be diagonalised for a fixed number of magnons in the spin chain. Let us think first of 1-magnon states

$$
\Psi(p) = \sum_{k=1}^{L} e^{ipl}|\uparrow ... \overset{l}{\downarrow} ... \uparrow\rangle \tag{5.45}
$$

such states are eigenstates of the SU(2) Hamiltonian as far as $p = 2\pi n/L$, which implies the periodicity of the spin chain, with

$$
H_{SU(2)}|\Psi(p)\rangle = \varepsilon(p)|\Psi(p)\rangle, \qquad \varepsilon(p) = \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} \tag{5.46}
$$

but recall that our spin chain describes a trace gauge operator, and this introduces the requirement that the spin chain be invariant under shifts of all fields by one position, given the ciclicity if the trace. This forces the 1-magnon state to have $p = 0$ and hence $\varepsilon = 0$. This implies that 1-magnon states do not have corrections to the anomalous dimensions as we could have expected from their being descendants of the BPS operator represented by the ground state of the spin chain.

So the first non-trivial case is that of the 2-magnon state, which we define by (neglecting normalization factors)

$$\Psi(p_1, p_2) = \sum_{1<l_1<l_2<L} \psi(l_1, l_2)|l_1, l_2\rangle \tag{5.47}$$

where $|l_1, l-2\rangle$ stands for the state where the spins at sites l_1 and l_2 have been fliped. A sensible guess for the two particle wave function $\psi(l_1, l_2)$ is

$$\psi(l_1, l_2) = e^{i(p_1 l_1 + p_2 l_2)} + s(p_1, p_2) e^{i(p_2 l_1 + p_1 l_2)} \tag{5.48}$$

where the second therm, that includes the S-matrix must be included to allow for the two magnons to interact and the only possible way they can interact is by exchanging their momenta. Since energy is conserved in the interaction, it can be measured far away from the interaction and hance it is clear that the total energy will be the sum of the individual 1-magnon energies.

$$E(p_1, p_2) = \varepsilon(p_1) + \varepsilon(p_2) \tag{5.49}$$

Nonetheless note that when we try to solve the eigenvalue equation for the Hamiltonian, the presence of the P operator forces us to consider two different possibilities. If l_1 and l_2 are not next to each other we have

$$H\psi(l_1, l_2) = 2\psi(l_1, l_2) - \psi(l_1 - 1, l_2) - \psi(l_1 + 1, l_2) + 2\psi(l_1, l_2) - \psi(l_1, l_2 - 1) - \psi(l_1 - 1, l_2) \tag{5.50}$$

whereas in case they are indeed next to each other we have

$$H\psi(l_1, l_2) = 2\psi(l_1, l_2) - \psi(l_1 - 1, l_2) - \psi(l_1, l_2 - 1) \tag{5.51}$$

Combining the last equation with 5.49, requiring that the two-magnon state be an eigenfunction of the Hamiltonian everywhere, one finds a defining condition for the S-matrix:

$$S(p_1, p_2) = \frac{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} + i}{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} - i} = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \tag{5.52}$$

where we have introduced in the second equality the so-called Bethe rapidities $u_i = \frac{1}{2}\cot\frac{p_i}{2}$ on behalf of comodity. The 1-magnon state energy can also be redefined in terms of u

$$\varepsilon(u) = \frac{\lambda}{8\pi^2}\frac{1}{u^2 + 1/4} \tag{5.53}$$

Now the invariance under translations that we must impose on the spin chain bring up the requirements

$$e^{ip_1 L} = S(p_1, p_2) \qquad e^{ip_2 L} = S(p_2, p_1) \tag{5.54}$$

which pose algebraic equations whose solutions determine the corresponding energies when plugged into 5.49.

Let us now move on to the N-magnon state, which will not be hard to understand once we have understood the 2-magnon case. Again, since scattering does not change momenta, we can measure the energy of the N-magnon state when particles are far apart and will get

$$E = \sum_{j=1}^{N} \varepsilon(p_j) = \sum_{j=1}^{N} \frac{\lambda}{8\pi^2}\frac{1}{u_j^2 + 1/4} \tag{5.55}$$

This is were the magic of integrability shows up. We see that since the interactions in the SU(2) Hamiltonian can at most exchange pairwise the momenta and any permutation can be built from a product of transpositions, what we have here is exactly the situation of factorised scattering that we described in our chapter on integrability as the definition of what we think quantum integrability is. and in this case the ciclicity of the spin chain will amount to the requirement

$$e^{ip_j L} = \left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{N} \frac{u_j - u_k + i}{u_j - u_k - i} \tag{5.56}$$

and again this is a set of algebraic equations that can be solved. They are called the Bethe equations. This is a much better situation that when one has to diagonalise a $6^L \times 6^L$ matrix like the one we were afraid of! When the set of algebraic equations is solved, we can plug the Bethe rapidities into the corresponding expression for the energy and hence get the 1-loop correction to the anomalous dimension of the corresponding gauge operator

$$\Delta = L + \frac{\lambda}{8\pi^2} \sum_{j=1}^{N} \frac{\lambda}{8\pi^2} \frac{1}{u_j^2 + 1/4} + O(\lambda^2) \tag{5.57}$$

The momentum constraint now takes the form

$$\prod_{k=1}^{N} \frac{u_k + i/2}{u_k - i/2} = 1 \tag{5.58}$$

So we have seen in the $SU(2)$ subsector the problem of finding the eigenvalues of the dilatation operator in $\mathcal{N}$=4 SYM can be reformulated as the problem of solving the energy eigenvalues of a Heisenberg spin chain.

All along the last lines we have focused on the $SU(2)$ subsector. Well it turns out that the whole $SO(6)$ was shown to be integrable by Reshetikhin in [51]. Later on Beisert computed the 1-loop dilatation operator for the full $PSU(2,2|4)$sector in [52].

5.5.1 Thermodynamic Bethe ansatz

If we want to aim at contrasting results of the Bethe ansatz for gauge operators with results obtained from the semiclassical calculations to the correction of the energy of the dual strings we must make both regimes compatible. This can be achieved by letting the number of fields in our gauge operators, and consequently the lenght of our spin chains to grow to infinity so as to fulfill the requirements of the semiclassical regime for strings. This takes us to the so-called thermodynamic Bethe ansatz, for which both the number of fields L and the number of magnons N are large. This thermodynamic limit is obtained by taking the logarithm [26] of 5.56

$$L \ln \left(\frac{u_j + i/2}{u_j - i/2} \right) = \sum_{k \neq j}^{N} \left(\frac{u_j - u_k + i}{u_j - u_k - i} \right) - 2\pi i n_i \tag{5.59}$$

with a set of arbitrary integers n_i associated to the different rapidities. Since momenta scale as $p \sim 1/L$, the Bethe roots u scale as $u \sim L$, so in the $L \to \infty$ limit the equation reduces to

$$\frac{1}{u_i} = 2\pi n_j + \frac{2}{L} \sum_{k \neq j}^{N} \frac{1}{u_j - u_k} \tag{5.60}$$

In this limit, the Bethe roots condensate on contours in the complex plane which transform the set of algebraic Bethe equations into an integral equation (see for example [53].

6 Conclusion and outlook

We now could set up to put everything we have reviewed this far into work, that is to really make calculation on both sides of the duality with the above elucidated tools and testing the expected matchings.
Remember that the main problem we come across when trying to test the AdS/CFT duality is its weak/strong nature. String calculations can only be trusted for a large value of the 't Hooft coupling, while the gauge theory calculations we can do are only realiable for small values of this same coupling. There are two basic sets of operators that escape this trouble. One the one side, we have seen that BPS operators (chiral primaries and their descendants) have a scaling dimension that does not get quantum corrected. On the ohter side, states that are dual to semiclassical string states can be used to check whether quantities match. Since the semiclassical approach requires large global charges, gauge operators dual to semiclassical string states must contain a large number of fields. This normally would make our life harder, since a large number of operator normally means operator mixing (given the degeneracy of classical scaling dimensions). Of course the larger the operator the more mixing of operators we expect to have. Therefore the situation would be completely hopeless were it not for the fact that as we have seen in the previous chapter, the mixing matrix can be seen as the Hamiltonian of an integrable spin chain. So it all looks like precisely what we have reviewed in this dissertation this far enables us to test the duality! The trick consists in realising that whereas our semiclassical calculations with strings can be done when both the global charge (dual to the bare dimension of the gauge operator, i.e. the length

of the spin chain) and the 't Hoof coupling are large but the ratio λ/J^2 is finite. This ratio acts as an effective coupling λ'. The good thing is that we can make calculations for strong coupling and still remain in a regime of small λ', which sometimes makes direct comparisons to results from the gauge theory possible.

We would like to point out that the techniques reviewed in this dissertation are not only applicable in the context of spectral AdS/CFT. Fields that are currently being intesively researched like that of scattering amplitudes do profit from all the things we have seen in this work and in particular it looks like a hidden integrability symmetry has been unvailed in them [54].
Also, our skills of integrability and of the AdS/CFT correspondence could turn out to be useful for other theories that appear to be dual to each other. In more concrete terms, $\mathcal{N} = 6$ super-conformal Chern-Simons theory in 2+1 dimensions seems to be dual to string theory in $AdS_4 \times CP^3$ [55]. Very recently, the same techniques as in using the algebraic curve to compute the spectrum of configurations in $AdS_5 \times S^5$ have been used to quantize the classical solution of a folded type IIA string in this background. In more concrete terms, a string spinning in the AdS part and with angular momentum in CP^3 is analysed. This configuration is relevant because its gauge dual are twist operators in the ABJM superconformal theory, as the authors point out. In that paper they also find the first semiclassical correction to the energy and the slope function for short string configurations. Other suspected dualities are the one between type IIB strings in $AdS_3 \times S^3 \times T_4$ or $AdS_3 \times S^3 \times S^3 \times S^1$ backgrounds and some 2-dimensional conformal field theories [56]. Such theories could also contain integrable structures in the large N limit [57], in which case our efforts in the case of $AdS_5 \times S^5$ would be highgly rewarded.

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