Evaluating the Discrete D'Alembertian in N Dimensions

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Abstract

This report concerns the generalisation of Sorkin's candidate for a discretised D'Alembertian to curved spacetimes. It is shown here that for every dimensionality of spacetime, the first order correction to the flat space D'Alembertian is an additive factor equal to $\frac{(-1)^a}{2}R$, where R is the Ricci scalar and a is either odd or even depending on the dimensionality of the space. Such a result has implications concerning the writing of actions for causal sets.

Introduction

We begin by briefly reviewing a few of the basic aspects of causal set theory. Causal set theory is an approach to quantum gravity in which spacetime is treated not as a fundamental entity but as an emergent phenomenon. The deeper level of reality from which spacetime is supposed to emerge is known as a *causal set* or *causet* for short. A causet is a set C over which an binary relation $\prec$ is defined, which gives an ordering to the set. $\prec$ is endowed with the following properties:

$$\text{Irreflexivity} : \forall x \in C, x \not\prec x$$

$$\text{Transitivity} : \forall x, y, z \in C, x \prec y \prec z \implies x \prec z$$

$$\text{Local finiteness} : \forall x, z \in C \; s.t. \; x \prec z, \; \text{Card}\{y \,|\, x \prec y \prec z\} < \infty.$$

$\prec$ has exactly the proporties that one would expect for the relation of causal sucession between events. This is made clear by using the appropriate language to state the first two properties of $\prec$: "x cannot affect itself" and "if x can affect y and y can affect z, then x can affect z". Thus, a causal set can be thought of as representing the structure of the causal relations between a *discrete* set (see local finiteness) of spacetime events. The idea that macroscopic spacetime with all of its rich structure can emerge from such a seemingly simple underlying object as a causal set may appear at first to be an outrageous proposition. However, theorems have been proved (for instance the one found in [1]) which state that all of the information needed to construct a Lorentzian manifold, with the exception of information pertaining to the volume element $\sqrt{-g(x)}\mathrm{d}^d x$, is contained in its causal order. These theorems show that the structure of a causal set is in fact much richer than a first glance suggests.

The requirment of local finiteness has some phenomenological justification, but from the theorems we see that it must be imposed. The reason is as follows: if the causal relations between a set of events alone are to give rise to macroscopic spacetime, *all* of the information about the manifolds which represent macroscopic spacetime must be contained therein. One should not have to specify a local volume element in addition to whatever information can be obtained from C. If C were continuous, it simply would not contain information about volume. However, discreteness implies *countability*, and that turns the problem from impossible to very easy; we just associate the volume of a region with the number of points that region contains. Such considerations gave rise to the well known slogan coined by Sorkin: "Order+Number=Geometry".

Ultimately, we need intrinsic dynamical laws for C. Causal set theory seeks for a path integral approach to obtaining such laws; an action for C is constructed which, for each C, contributes to the probabaility amplitude for a particular evolution of the observed causal structure in some specified way, and the contributions from all possible C's are summed over. How might we go about finding the required action? When we quantize a classical field, the action that appears in the path integral is the same object as the classical action for that field. This action is constructed in such a way as to reproduce the observed classical equations of motion when minimized. What if we could find an action for C whose continuum limit, when minimized, produced (for instance) the Einstein field equations? Might this be a candidate for the action that is to appear in the sum over causets? It turns out that such an action can in fact be constructed; in order to do this, however, one must define a differential operator (a D'Alembertian, in fact) which can act on a field over a causal set.

Before turning to the question of this differential operator, we should mention an important result. C is a very general object; without any constriants or an action to tell us which causets will contribute to probablility amplitudes in a non-negligable way, a criteria must be established that specifies which kind of causets to study. If causal sets are to reproduce the manifold structures that appear in General Relativity in the macroscopic limit, it should be that a C which contributes to the dynamcis of the theory has a structure which corresponds (through the reconstruction theorems discussed above) to a Lorentzian manifold $\mathbf{M}$. The terminology used here is that "$\mathbf{M}$ furnishes a good approximation to C". It is well known [2] that causets generated by selecting points from $\mathbf{M}$ via a Poisson process are such that $\mathbf{M}$ furnishes a good approximation to C ("generated" here meaning that if two of the selected points are in causal contact in $\mathbf{M}$, this is reflected in the resluting C). The process of creating a causal set from a given manifold is known as *sprinkling*. For the present, then, we take as a rule that the causets which underlie spacetime are generated by a Poisson process in some manifold. Of course, this rule is only a temporary aid; we hope to discover dynamical laws for C which give us the correct causets without having to assume the preexistance of the manifold which is supposed to emerge from C. The rule can in fact be used to bring us closer to such laws, as we will soon see.

The discrete D'Alembertian

As mentioned above, the writing of a causal set action necessitates the creation of a discretised differential operator. In order to define such an operator, we need something for it to act on; we need the notion of a field over a causal set. This is not especially tricky. A scalar field, which turns out to be the required species of field, will be a map from the causet C to $\mathbb{R}$:

$$\begin{aligned} \phi : C &\to \mathbb{R} \\ x &\to \phi_x. \end{aligned} \tag{1}$$

Now for the differntial operator. What is required, in analogy with the usual derivative, is an operator which acts on ϕ_x in such a way that the result depends on the configuration of the field over C, meaning B will take the form of a matrix acting on a vector whose elements are the values of ϕ on each element of C. Moreover, as we zoom out and the discrete nature of the causet is lost, ϕ becomes the usual continuum field, $\phi(x)$. So, in the continuum limit, the the action of our operator must be the same as the action of the continuum operator we are trying to reproduce. Denote the candidate differential operator by B and the charactaristic separation between causet elemets by l. Then, if we are trying for a discrete D'Alembertian, for instance, we require:

$$\lim_{l \to 0} \sum_y B_{xy} \phi_y = \Box \phi(x). \tag{2}$$

There are a number of candidates for such a discrete D'Alembertian; a few are reviewed in Sorkin's paper [3]. That paper also introduces a new candidate, which has generated much interest. As we will see, it is the version of B which allows us to write our action.

Let us discuss Sorkin's idea for B. Consider the usual D'Alembertian, in one dimension for simplicity:

$$\Box \phi(x) = \lim_{\epsilon \to 0} \frac{\phi(x + 2\epsilon) - 2\phi(x + \epsilon) + \phi(x)}{\epsilon^2}. \tag{3}$$

The key features here are the $\frac{1}{\epsilon^2}$ dependence and the sum (with alternating signs) over the field at different distances from x. Both of these features can be carried over to a discrete operator on a causal set. Equations (2) and (3) together suggest that the $\frac{1}{\epsilon^2}$ factor in (3) should correspond to a factor of $\frac{1}{l^2}$ in (2). Further, the notion of "different distances from x" is most naturally replaced by the idea of "different numbers of elements of C lying causally between x and y", where x and y are elements of C, this being the simplest Lorentz invariant notion of separation between points in a causet. To clarify, the idea is to split the elements of C that causally preceed x into layers; in the first layer we have elements which are *directly linked* to x, meaning that $y \prec x$ and that there exists no z satisfying $y \prec z \prec x$; in the second layer we have elements which are one link away from x, meaning that $y \prec x$ and that there exists *exactly one* z satisfying $y \prec z \prec x$, and so on. The analogy is then between $\phi(x+\epsilon)$ in (3) and ϕ on the elements of layer 1, between $\phi(x+2\epsilon)$ and ϕ on the elements of layer 2, and of course between $\phi(x)$ and ϕ_x.

There is another feature of the continuum $\Box$ that should have an analogue in the discrete case. Given a sensible $\phi(x)$, $\Box\phi(x)$ is defined once and for all at each point. It manifestly does not matter in which order the limits appearing in the definition of $\Box$ are taken; $\Box$ is effectively "directionless", so to speak. In view of this, the discrete version should also not pick out any special "direction" in C. This gives us a clue as to how we should make the association between $\phi(x+n\epsilon)$ in the continuum and ϕ on the nth layer with respect to x in C: it should be done in such a way as to ensure that no particular path from x to elements of the nth layer is special. One way of doing this is simply to associate $\phi(x+n\epsilon)$ with an unweighted sum over ϕ on all the elements of the nth layer.

Taking all these things into account, a discrete D'Alembertian which preserves the key features of the continuum case then should, in d dimensions, take the general form

$$B\phi(x) = \frac{1}{l^2}\left(\alpha_d\phi(x) + \beta_d\sum_{i=1}^{n}(-1)^i c_i^{(d)} \sum_{y\in L_i}\phi(y)\right), \tag{4}$$

where $\sum_{y\in L_i}\phi(y)$ represents the sum of ϕ on all elements of the ith layer. By constructing a vector whose elements are the values of ϕ on each element of C, we can write B as a matrix, so that

$$B\phi(x) = \sum_y B_{xy}\phi_y. \tag{5}$$

This matrix will have $\frac{\alpha_d}{l^2}$ on the diagonal, and if the number of elements of C lying causally between x and y is i, we will have $B_{xy} = \frac{(-1)^i c_i^{(d)}}{l^2}$. It seems that now we are left only with the task of choosing n (the number of layers to be summed over), α_d, β_d, and the $c_i^{(d)}$ in such a way that (2) is satisfied. In fact, there is one further complication. The behaviour of B clearly depends critically on the structure of the causal set, and it would be very surprising indeed if we could choose the $c_i^{(d)}$ so that (2) were satisfied for any causal set. So rather than trying to find a B that always reproduces $\Box$, we should only aim to reproduce $\Box$ in the vast majority of cases for which C is well approximated by a Lorentzian manifold. As was mentioned in the introduction, such causal sets are generated by a Poisson process in $\mathbf{M}$. Thus, if we treat $B\phi(x)$ as a random variable generated by a Poisson process, the mean of this variable, denoted $\mathbb{E}B\phi(x)$, must obey:

$$\lim_{l\to 0} \mathbb{E}B_{xy}\phi_x = \Box\phi(x). \tag{6}$$

For d dimensional space, upon averaging, (4) will become

$$\int_{J^-(x)} \mathrm{d}^d y\, \bar{B}^{(d)}(x-y)\phi(y), \tag{7}$$

where $J^-(x)$ denotes the causal past of x, and $\bar{B}^{(d)}(x-y)$ is some kernal depending on the $c_i^{(d)}$. The form of $\bar{B}^{(d)}(x-y)$ can be determined by considering the Poisson process. The probability of finding n points sprinkled within some spacetime volume V is

$$P(n;V) = \frac{(Vl^{-d})^n \mathrm{e}^{-Vl^{-d}}}{n!}. \tag{8}$$

Now, $y \in C$ being in the ith layer with respect to $x \in C$ means that there are exactly $i-1$ elements of C in the interscection between the causal past of x and the causal future of y. Therefore, given $x, y \in C$; $y \prec x$, the probability of y being in the ith layer with respect to x is

$$P(i-1;V_d) = \frac{(V_d l^{-d})^{i-1} \mathrm{e}^{-V_d l^{-d}}}{(i-1)!}. \tag{9}$$

Here V_d is the volume of the interscetion between the causal past of x and the causal future of y in d dimensions (in future we will refer to this intersection as the *causal interval* between x and y). Further to this, it is clear that in flat space the probability to find a point sprinkled in the volume element

$\mathrm{d}^d y$ is $l^{-d}\mathrm{d}^d y$. To get $\mathbb{E}B\phi(x)$, then, we must integrate over all possible configurations of C weighted by their probabilities (with an element fixed at x with probability 1, since this is the point at which we wish to evaluate $\Box$), whence we obtain

$$\mathbb{E}B\phi(x) = \\ \alpha_d l^{-2}\phi(x) + \beta_d l^{-(d+2)} \int_{J^-(x)} \mathrm{d}^d y \, \frac{\sum_{i=1}^{n} c_i^{(d)} (V_d l^- d)^{i-1} \phi(y)}{(i-1)!} \mathrm{e}^{-V_d l^{-d}}. \quad (10)$$

Note for future reference that in null coordinates [4]

$$V_d(u,v) = c(uv)^{\frac{d}{2}}, \quad (11)$$

where

$$c = \frac{2^{\frac{4-d}{2}} \pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2}) d(d-1)}. \quad (12)$$

(10) can be put into a slightly different form by acting with some differential operator in $\frac{\partial}{\partial l}$ on the exponential, in lieu of explicitly including the $\frac{\sum_{i=1}^{n} c_i^{(d)} (V_d l^- d)^{i-1}}{(i-1)!}$ terms. This operator will be denoted by $\hat{O}_d$. It turns out that the resulting expression is much more convenient for what follows hereafter. Carrying out this step,

$$\mathbb{E}B\phi(x) = \alpha_d l^{-2}\phi(x) + \beta_d l^{-(d+2)} \hat{O}_d \int_{J^-(x)} \mathrm{d}^d y \, \phi(y) \mathrm{e}^{-V_d l^{-d}}. \quad (13)$$

Thus, the problem of selecting α_d, β_d, $c_i^{(d)}$, and n so that (6) is satisfied has been transformed into the problem of selecting α_d, β_d, and $\hat{O}_d$ so that

$$\lim_{l \to 0} \left(\alpha_d l^{-2}\phi(x) + \beta_d l^{-(d+2)} \hat{O}_d \int_{J^-(x)} \mathrm{d}^d y \, \phi(y) \mathrm{e}^{-V_d l^{-d}} \right) = \Box\phi(x). \quad (14)$$

The correct choises are derived in [4]; they are:

$$\alpha_d = \begin{cases} -\frac{2c}{\Gamma(\frac{d}{2}+1)} & : d \, even \\ -\frac{c}{\Gamma(\frac{d}{2}+1)} & : d \, odd, \end{cases} \quad (15)$$

$$\beta_d = -\alpha_d \left(\lim_{l \to 0} l^{-d} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \hat{O}_d \int_{-\infty}^{0} \int_{u}^{0} \mathrm{d}u \mathrm{d}v \, \left(\frac{v-u}{\sqrt{2}} \right)^{d-2} \mathrm{e}^{-V_d l^{-d}} \right)^{-1}, \quad (16)$$

$$\hat{O}_d = \begin{cases} \frac{(H+2)(H+4)...(H+d+2)}{2^{\frac{d}{2}+1}(\frac{d}{2}+1)!} & : d\, even \\ \frac{(H+2)(H+4)...(H+d+1)}{2^{\frac{d+1}{2}}(\frac{d+1}{2})!} & : d\, odd, \end{cases} \tag{17}$$

where H is the homogoneity operator, $H = -l\frac{\partial}{\partial l}$. With these definitions in place, (13) is the form of the discrete D'Alembertian for flat space which we will use going forward.

Before we continue, it is worth mentioning briefly that equation (6), although being a necessary condition for B to be a good D'Alembertian, is not sufficent. The mean of a random variable alone does not tell us much about the values the variable actually takes; we also need to consider the magnitude of fluctuations about the mean. The upshot of this is that even if (6) is satisfied B can give wildly incorrect answers for most causets taken individually. According to [3] this does in fact seem to be the case; however in the same paper a solution to the issue is presented which involves only a slight modification of (13). We will not consider this modified version here, although our result should be applicable to the modified version.

We will now look at $\mathbb{E}B\phi(x)$ in general spacetimes. In order to do this, we begin with the flat space version (13) and make the necessary corrections, that is, we allow $\sqrt{-g}$ to become different from one, and we correct V_d. If $V_{\diamond d}$ denotes the volume of the causal interval in curved space, then the curved space version of (n) is as follows:

$$\mathbb{E}B\phi(x) = \alpha_d l^{-2}\phi(x) + \beta_d l^{-(d+2)}\hat{O}_d \int_{J^-(y)} \mathrm{d}^d y\, \sqrt{-g(y)}\phi(y)\mathrm{e}^{-l^{-d}V_{\diamond d}(y)}. \tag{18}$$

We have seen that in flat space the limit as $l \to 0$ of $\mathbb{E}B\phi(x)$ is $\Box\phi(x)$. Since one can smoothly go between flat space and curved space, for sufficently small curvature this limit will become $\Box\phi(x)+X\phi(x)$. Now, $\Box$ contributes to the dimensionality of whatever expression it appears in with L^{-2}. Therefore, whatever the contribution to the limit from the curvature corrections is, it must have dimensions of $L^{-2}[\phi(x)]$. Put another way,

$$\lim_{l\to 0} \mathbb{E}B^{(d)}\phi(x) = \Box\phi(x) + aR(x)\phi(x), \tag{19}$$

a being some number, since $[R] = L^{-2}$ (a more rigourous argument for (19) can be found in [5]). Whether the limit above actually exists is a matter of ongoing investigation; for the present we will assume that it does. It would be nice to know what a is; the main purpose of this report is to prove that it is in fact $\frac{(-1)^a}{2}$, at least when only first order curvature corrections are taken into account. Here, a is odd when d is odd or when $\frac{d}{2}$ is even,

otherwise it is a is even. The claim made in the introduction, namely that the discrete D'Alembertian leads to an action which reproduces the Einstein-Hilbert action when the continuum limit is taken, is seen to be correct here, since $\mathbb{E}B(Const) = aR(x)$. Such an action is $S[C] = B(-1)$ [5].
Since the writing of an action is the most interesting use of B at present, and that action has constat ϕ, we will treat ϕ as a constant in what follows.

Calculating a

Looking at (18) and (19), it is clear that our task is to show that the variation under curvature in $\bar{B}^{(d)}\phi$ is equal to $-\frac{1}{2}R\phi$. If we define $f(l^{-d}V)\mathrm{e}^{-l^{-d}V_d} := \hat{O}_d\mathrm{e}^{-l^{-d}V_d}$ and take Riemann normal coordiantes centered at the origin, then to first order in the curvature we have

$$\begin{aligned}
&\delta\bar{B}^{(d)}\phi(0) = \\
&\qquad \beta_d l^{-(d+2)}\int_{J^-(y)} \mathrm{d}^d y \left\{\delta\sqrt{-g}\phi f(l^{-d}V)\mathrm{e}^{-l^{-d}V_d}\right. \\
&\qquad\qquad \left. + \delta V\phi\left(\frac{\partial}{\partial V}f(l^{-d}V) - l^{-d}f(l^{-d}V)\right)\mathrm{e}^{-l^{-d}V_d}\right\} \qquad (20)
\end{aligned}$$

In order to proceed, we need to know what the corrections induced by curvature are. [6] deals with this task, and before going any further we should write down their results, which in Reimann normal coordinates centered at the origin and to first order in the curvature are

$$\delta\sqrt{-g} = -\frac{1}{6}R_{\mu\nu}x^\mu x^\nu, \qquad (21)$$

$$\delta V_d = V_d\frac{d}{24(d+1)}(R_{\mu\nu}x^\mu x^\nu - \frac{R\tau^2}{d+2}), \qquad (22)$$

where $R_{\mu\nu}$ denotes a component of the Ricci tensor evaluated at the origin, so it is understood that $R_{\mu\nu}$ is a constant with respect to the integration in (18). Now, if we take spherical coordinates for the spatial part of our manifold, we can easily do the angular integrals in (18), leaving only r, t dependence. The procedure is found in appendix I, the result is

$$\begin{aligned}
\int \mathrm{d}\Omega\, R_{\mu\nu}x^\mu x^\nu &= \frac{2\sqrt{\pi}^{d-1}}{\Gamma(\frac{d-1}{2})}\left(R_{00}t^2r^{d-2} + \frac{(R+R_{00})}{d-1}r^d\right) \\
&:= I^{(\Omega)}(R_{00}t^2r^{d-2} + \frac{(R+R_{00})r^d}{d-1}), \qquad (23)
\end{aligned}$$

$$\int \mathrm{d}\Omega\, R = I^{(\Omega)} R r^{d-2}, \tag{24}$$

so that, after switching from the (r,t) coordinates to null coordinates (u,v),

$$\begin{aligned}
\delta\bar{B}^{(d)}\phi(0) = \beta_d l^{-(d+2)}\phi \int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}u\mathrm{d}v \Bigg\{ -\frac{1}{6}\ \{\ I^{\Omega} R_{00}\left(\frac{u+v}{\sqrt{2}}\right)^2\left(\frac{u-v}{\sqrt{2}}\right)^{d-2} \\
+\frac{I^{(\Omega)}}{d-1}(R+R_{00})\left(\frac{u-v}{\sqrt{2}}\right)^{d}\ \}\, f(l^{-d}V)\mathrm{e}^{-l^{-d}V_d} \\
+\frac{d}{24(d+1)}\ \{\ I^{(\Omega)} R_{00}\left(\frac{u+v}{\sqrt{2}}\right)^2\left(\frac{u-v}{\sqrt{2}}\right)^{d-2} + \frac{I^{(\Omega)}}{d-1}(R+R_{00})\left(\frac{u-v}{\sqrt{2}}\right)^{d} \\
-\frac{d}{d+2} I^{(\Omega)} 2Ruv\left(\frac{u-v}{\sqrt{2}}\right)^{d-2}\ \}\left(V_d\frac{\partial}{\partial V} f(l^{-d}V) - l^{-d}V_d f(l^{-d}V)\right)\mathrm{e}^{-l^{-d}V_d}\Bigg\}.
\end{aligned} \tag{25}$$

This expression can be tamed to a degree; our strategy will be to cast it into a more convienent form, then to solve the generic integral over u and v that we encounter. To begin with, because $\delta\bar{B}^{(d)}\phi(x)$ must be proportionl to $R\phi(x)$, one sees from (25) that the relation

$$\int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, I^{(\Omega)}\left(\frac{u-v}{\sqrt{2}}\right)^{d} = -\int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, \frac{I^{(\Omega)}}{d-1}\left(\frac{u+v}{\sqrt{2}}\right)^{2}\left(\frac{u-v}{\sqrt{2}}\right)^{d-2} \tag{26}$$

must hold. Furthermore, by noticing that $V\frac{\partial}{\partial V} f(l^{-d}V) = -\frac{l}{d}\frac{\partial}{\partial l} f(l^{-d}V)$, we can write

$$\left(V_d\frac{\partial}{\partial V} f(l^{-d}V) - l^{-d}V_d f(l^{-d}V)\right)\mathrm{e}^{-l^{-d}V_d} = -\frac{l}{d}\frac{\partial}{\partial l}\left(f(l^{-d}V)\mathrm{e}^{-l^{-d}V_d}\right). \tag{27}$$

By using (26) and (27), we see that

$$\begin{aligned}
\delta\bar{B}^{(d)}\phi(0) = \\
-R\beta_d l^{-(d+2)}\phi\left(\frac{1}{6}+\frac{l}{24(d+1)}\frac{\partial}{\partial l}\right)\hat{O}_d \int_{-\infty}^{0}\int_{u}^{0}\mathrm{d}u\mathrm{d}v\,\frac{I^{(\Omega)}}{d-1}\left(\frac{u-v}{\sqrt{2}}\right)^{d}\mathrm{e}^{-l^{-d}V_d} \\
+R\beta_d l^{-(d+2)}\phi\frac{l}{24(d+1)(d+2)}\frac{\partial}{\partial l}\hat{O}_d \int_{-\infty}^{0}\int_{u}^{0}\mathrm{d}u\mathrm{d}v\, I^{(\Omega)} 2uv\left(\frac{u-v}{\sqrt{2}}\right)^{d-2}\mathrm{e}^{-l^{-d}V_d}.
\end{aligned} \tag{28}$$

Expression (25) is now in a nicer form. If we solve the integral

$$\int_{-\infty}^{0}\int_{u}^{0}\mathrm{d}v\mathrm{d}u\, u^{d-m}v^{m}\mathrm{e}^{-l^{-d}c(uv)^{\frac{d}{2}}}, \tag{29}$$

the result can be easily plugged into (28), bringing us close to what we are looking for. So let us do this now. We begin by performing a simple change of variables on the above in order to obtain

$$-\frac{2}{d}\int_{-\infty}^{0}\int_{0}^{l^{-d}cu^d} \mathrm{d}w\mathrm{d}u\,(l^{-d}c)^{\frac{2m+2}{d}}u^{d-2m-1}w^{\frac{2m+2}{d}-1}\mathrm{e}^{-w}, \tag{30}$$

which, by definition of the lower incomplete gamma function γ, is

$$-\frac{2}{d}\int_{-\infty}^{0}\mathrm{d}u\,(l^{-d}c)^{-\frac{2m+2}{d}}u^{d-2m-1}\gamma((2m+2)d^{-1}, l^{-d}cu^d). \tag{31}$$

With another change of variables (31) may be expressed as

$$(-1)^d\frac{2}{d^2}\int_{0}^{\infty}\mathrm{d}w\,(l^{-d}c)^{-\frac{d+2}{d}}w^{-\frac{2m}{d}}\gamma((2m+2)d^{-1}, w). \tag{32}$$

Now, γ has the following series representation:

$$\gamma(x,w) = \sum_{k=0}^{\infty}\frac{w^x\mathrm{e}^{-w}w^k}{(x)_{k+1}} \tag{33}$$

where $(x)_n$ is the *Pochhammer symbol* or *rising factorial*:

$$(x)_n = (x)(x+1)(x+2)...(x+n-1). \tag{34}$$

Thus, (32) may be written as

$$(-1)^d\frac{2}{d^2}\sum_{k=0}^{\infty}\int_{0}^{\infty}\mathrm{d}w\,\frac{(l^{-d}c)^{-\frac{d+2}{d}}w^{\frac{2}{d}+k}\mathrm{e}^{-w}}{(\frac{2m+2}{d})_{k+1}} \tag{35}$$

$$= (-1)^d\frac{2}{d^2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{2}{d}+1+k)}{(\frac{2m+2}{d})_{k+1}}(l^{-d}c)^{-\frac{d+2}{d}} \tag{36}$$

$$= \frac{(-1)^d}{d}\sum_{k=0}^{\infty}\frac{(\frac{2}{d}+1)_k}{(\frac{2m+2}{d}+1)_k(m+1)}\Gamma(\frac{2}{d}+1)(l^{-d}c)^{-\frac{d+2}{d}}, \tag{37}$$

where in the last step we used the recursion relation for Γ, namely

$$\Gamma(x+n) = (x)_n\Gamma(x). \tag{38}$$

Clearly, (37) is nothing but

$$\frac{(-1)^d(l^{-d}c)^{-\frac{d+2}{d}}}{d(m+1)}\Gamma(\frac{2}{d}+1)_2F_1(1,\frac{2}{d}+1;\frac{2m+2}{d}+1;1) \tag{39}$$

by definition of the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$.
${}_2F_1(a,b;c;1)$ converges only for $c > a+b$, so we see that the integral (29) converges only if $m > d/2$. We will return to this issue shortly.
For ease of future manipulation, it is desirable to express the result of (29) in a form that does not contain a hypergeometric function. Fortunately there exists a relation known as Kummer's first formula:

$$ {}_2F_1(\frac{1}{2}+m-k,-n;2m+1;1) = \frac{\Gamma(2m+1)\Gamma(m+\frac{1}{2}+k+n)}{\Gamma(2m+1+n)\Gamma(m+\frac{1}{2}+k)}. \tag{40} $$

From the above, (39) is seen to be equal to

$$ \frac{(-1)^d(l^{-d}c)^{-\frac{d+2}{d}}}{d(m+1)} \frac{\Gamma(\frac{2}{d}+1)\Gamma(\frac{2m+1}{d}+1)\Gamma(\frac{2m}{d}-1)}{\Gamma(\frac{2m+2}{d})\Gamma(\frac{2m}{d})}. \tag{41} $$

Using (38), and with a small amount of algebra, this can be simplified considerably, leaving us with the result

$$ \int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, u^{d-m}v^{m}\mathrm{e}^{-l^{-d}c(uv)^{\frac{d}{2}}} = \begin{cases} \frac{(-1)^d(l^{-d}c)^{-\frac{2+d}{d}}}{d(m-\frac{d}{2})}\Gamma(\frac{2}{d}+1) & : m > d/2 \\ \infty & : m \leq d/2. \end{cases} \tag{42} $$

The divergence is not problematic as long as we assume that our operator $\hat{O}_d$ acts on the integral (29) in such a way as to give a finite result. Were this not the case, then clearly our candidate for $\Box$ would be unacceptable (it would always give infinity) so the assumption is not outrageous; it is implicit in the assumption that the discrete D'Alembertian we are considering is a good one.
Now, the asymmetry in (29) between $m > d/2$ and $m \leq d/2$ arises from the convergence conditions on ${}_2F_1(a,b;c;1)$. Suppose ${}_2F_1(a,b;c;1)$ were always convergent; then the symmetry would be restored and we could write

$$ \int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, u^{d-m}v^{m}\mathrm{e}^{-l^{-d}c(uv)^{\frac{d}{2}}} = \frac{(-1)^d(l^{-d}c)^{-\frac{2+d}{d}}}{d(m-\frac{d}{2})}\Gamma(\frac{2}{d}+1). \tag{43} $$

With this in mind, we see that by making the assumption mentioned above, namely that the divergence is removed from (29) by acting with $\hat{O}_d$, we obtain

$$ \hat{O}_d\int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, u^{d-m}v^{m}\mathrm{e}^{-l^{-d}c(uv)^{\frac{d}{2}}} = \hat{O}_d\frac{(-1)^d(l^{-d}c)^{-\frac{2+d}{d}}}{d(m-\frac{d}{2})}\Gamma(\frac{2}{d}+1), \tag{44} $$

since there is now no difference between the $m > d/2$ and $m \leq d/2$ cases other than the value of m, and we just proved (n) for $m > d/2$.
It is easy to calculate the action of $\hat{O}_d$ here. Looking at (17) tells us that

$$\hat{O}_d \frac{(-1)^{d+1}(l^{-d}c)^{\frac{2+d}{d}}}{d(m-\frac{d}{2})}\Gamma(\frac{2}{d}+1) = \begin{cases} (-\frac{d}{2})_{1+\frac{d}{2}} \frac{(l^{-d}c)^{-\frac{2+d}{d}}}{(\frac{d}{2}+1)!d(m-\frac{d}{2})}\Gamma(\frac{2}{d}+1) & : d\,even \\ -(-\frac{d}{2})_{\frac{d+1}{2}} \frac{(l^{-d}c)^{-\frac{2+d}{d}}}{(\frac{d+1}{2})!d(m-\frac{d}{2})}\Gamma(\frac{2}{d}+1) & : d\,odd. \end{cases} \tag{45}$$

For the odd-dimensional case this result requires no further manipulation; it will be used in the above form to prove our conjecture. The even dimensional case is slightly more subtle, so let us consider it further. To begin with, notice that $(-\frac{d}{2})_{1+\frac{d}{2}} \equiv 0$. Thus, in even dimensions, (29) vanishes unless $m = \frac{d}{2}$. When $m = \frac{d}{2}$, we have the ambiguous result $\frac{0}{0}$. One way of dealing with this is to evaluate the following:

$$\lim_{\epsilon\to 0} \hat{O}_d \int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, u^{\frac{d}{2}} v^{\frac{d}{2}+\epsilon} \mathrm{e}^{-l^{-d}c(uv)^{\frac{d}{2}}} \tag{46}$$

$$= \lim_{\epsilon\to 0} \hat{O}_d \frac{2(l^{-d}c)^{-\frac{2+d+\epsilon}{d}}}{d\epsilon}\Gamma(\frac{2}{d}+1+\epsilon), \tag{47}$$

where the second line is obtained by the same procedure as was used to deduce (44). Defining $\hat{O}'$ via $\hat{O}_d = \hat{O}'(-l\frac{d}{dl} + (d+2))$, we can recast the above as

$$= \lim_{\epsilon\to 0} \hat{O}' \frac{2c^{-\frac{2+d+\epsilon}{d}}}{d\epsilon}\Gamma(\frac{2}{d}+1+\epsilon)(-l\frac{d}{dl}+(d+2))l^{2+d+\epsilon} \tag{48}$$

$$= \hat{O}' \frac{2(l^{-d}c)^{-\frac{2+d}{d}}}{d}\Gamma(\frac{2}{d}+1) \tag{49}$$

$$= (-\frac{d}{2})_{\frac{d}{2}} \frac{(l^{-d}c)^{-\frac{2+d}{d}}}{d(\frac{d}{2}+1)!}\Gamma(\frac{2}{d}+1). \tag{50}$$

In summary, then,

$$\hat{O}_d \int_{-\infty}^{0}\int_{u}^{0} \mathrm{d}v\mathrm{d}u\, u^{d-m} v^{m} \mathrm{e}^{-l^{-d}c(uv)^{\frac{d}{2}}} =$$

$$\begin{cases} 0 & : d\,even,\, m \neq \frac{d}{2} \\ (-\frac{d}{2})_{\frac{d}{2}} \frac{(l^{-d}c)^{-\frac{2+d}{d}}}{d(\frac{d}{2}+1)!}\Gamma(\frac{2}{d}+1) & : d\,even,\, m = \frac{d}{2} \\ -(-\frac{d}{2})_{\frac{d+1}{2}} \frac{(l^{-d}c)^{-\frac{2+d}{d}}}{d(m-\frac{d}{2})(\frac{d+1}{2})!}\Gamma(\frac{2}{d}+1) & : d\,odd. \end{cases} \tag{51}$$

The final part of the proof consists (aside from some algebra) of plugging this result into equation (28) and evaluating the resulting sum. Since the result for odd and even dimensions differs, we will have to treat each case separately. Before we begin, let us for convenience define:

$$A(d) := -(-\frac{d}{2})_{\frac{d+1}{2}} \frac{c^{-\frac{2+d}{d}}}{d(\frac{d}{2}+1)!} \Gamma(\frac{2}{d}+1), \tag{52}$$

$$B(d) := (-\frac{d}{2})_{\frac{d}{2}} \frac{c^{-\frac{2+d}{d}}}{d(\frac{d+1}{2})!} \Gamma(\frac{2}{d}+1). \tag{53}$$

Odd dimensions

For odd dimensions, if we plug (51) into (28) and use the binomial theorem, we see that

$$\begin{aligned}
&\delta \bar{B}^{(d)} \phi(0) = \\
&- R\beta_d l^{-(d+2)} I^{(\Omega)} \phi \Big\{ \left(\frac{1}{6} + \frac{l}{24(d+1)} \frac{\partial}{\partial l} \right) l^{d+2} A(d) \sum_{i=1}^{d} \binom{d}{i} \frac{(-1)^{i+1}}{(d-1)(i-\frac{d}{2})} \\
&\qquad + \frac{l}{12(d+1)(d+2)} \frac{\partial}{\partial l} l^{d+2} A(d) \sum_{i=1}^{d-2} \binom{d-2}{i} \frac{(-1)^{i}}{(i+1-\frac{d}{2})} \Big\} \\
&\quad = R\beta_d \phi I^{(\Omega)} A(d) \Big\{ \left(\frac{1}{6} + \frac{d+2}{24(d+1)} \right) \sum_{i=1}^{d} \binom{d}{i} \frac{(-1)^{i}}{(d-1)(i-\frac{d}{2})} \\
&\qquad + \frac{1}{12(d+1)} \sum_{i=1}^{d-2} \binom{d-2}{i} \frac{(-1)^{i+1}}{(i+1-\frac{d}{2})} \Big\}.
\end{aligned} \tag{54}$$

We need to evaluate the sum

$$S = \sum_{i=1}^{d} \binom{d}{i} \frac{(-1)^i}{(i-\frac{d}{2})}. \tag{55}$$

This can be done by realising that the factorial and the gamma function share the same recursion relation, so for $n \in \mathbb{N}, n! = \Gamma(n+1)$ and

$$\binom{d}{i} = \frac{d!}{i!(d-i!)} = \frac{\Gamma(d+1)}{i!\Gamma(d-i+1)}. \tag{56}$$

Then,

$$S = \sum_{i=1}^{d} \frac{\Gamma(d+1)}{i!\Gamma(d-i+1)} \frac{(-1)^i}{(i-\frac{d}{2})} \tag{57}$$

$$= \sum_{i=1}^{d} \frac{1}{i!(d+1)_{-i}} \frac{(-1)^i}{(i-\frac{d}{2})} \tag{58}$$

by equation (38). But, a property of the Pochhammer symbol is

$$(x)_{-n} = \frac{(-1)^n}{(1-x)_n}, \tag{59}$$

so that

$$S = \sum_{i=1}^{d} \frac{(-d)_i}{i!} \frac{1}{(i-\frac{d}{2})} = -\frac{2}{d} \sum_{i=1}^{d} \frac{(-d)_i}{i!} \frac{(-\frac{d}{2})_i}{(1-\frac{d}{2})_i}$$

$$= -\frac{2}{d} \sum_{i=1}^{\infty} \frac{(-d)_i}{i!} \frac{(-\frac{d}{2})_i}{(1-\frac{d}{2})_i} = -\frac{2}{d}\, {}_2F_1(-d, -\frac{d}{2}; 1-\frac{d}{2}; 1). \tag{60}$$

So, after making use of Kummer's first formula (40) as well as equation (38) and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we finally obtain

$$\sum_{i=1}^{d} \binom{d}{i} \frac{(-1)^i}{(i-\frac{d}{2})} = \frac{2^{d+1}\Gamma(-\frac{d}{2})\Gamma(\frac{d-1}{2})}{(d-1)\sqrt{\pi}}. \tag{61}$$

By the same procedure, we can also show that

$$\sum_{i=1}^{d-2} \binom{d-2}{i} \frac{(-1)^i}{(i+1-\frac{d}{2})} = -\frac{2^{d-3}d\Gamma(-\frac{d}{2})\Gamma(\frac{d-1}{2})}{\sqrt{\pi}}. \tag{62}$$

Substituting these sums into (54), we find

$$\delta\bar{B}^{(d)}\phi(0) = \phi \frac{R\beta_d A(d) 2^{d-3}\Gamma(-\frac{d}{2})\Gamma(\frac{d+1}{2})(2d+13)I^{(\Omega)}}{6\sqrt{\pi}(d^2-1)}. \tag{63}$$

At this point, the only remaining task is to calculate β_d. This is easily done: by the method used to evaluate (29) it can be shown that, in odd dimensions,

$$\hat{O}_d \int_{-\infty}^{0} \int_{u}^{0} \mathrm{d}u\mathrm{d}v\, u^{d-2-m} v^m \mathrm{e}^{-l^{-d}V_d} = \frac{l^{-d}c}{(m+1-\frac{d}{2})(\frac{d+1}{2})!} \left(1-\frac{d}{2}\right)_{\frac{1+d}{2}}. \tag{64}$$

Then, by substituting this result into (16) and evaluating the sum that results using (62), we get

$$\beta_d = \frac{-\sqrt{\pi}c^{\frac{2}{d}}\left(\frac{d+1}{2}\right)!}{2^{d-3}\Gamma(1+\frac{2}{d})d(d-1)(1-\frac{d}{2})_{\frac{1+d}{2}}\Gamma(-\frac{d}{2})\Gamma(\frac{d-1}{2})}. \tag{65}$$

In light of this last equation and with some simplification, (63) says

$$\delta\bar{B}^{(d)}\phi(0) = \phi\frac{2^{-\frac{d}{2}}\pi^{\frac{d-1}{2}}R}{cd\Gamma\left(\frac{1+d}{2}\right)}. \tag{66}$$

Plugging in expression (12) for c gives

$$\delta\bar{B}^{(d)}\phi(0) = -\frac{R}{2}\phi \tag{67}$$

as required.

Even dimensions

The analogue of (54) for even dimensions is

$$\begin{aligned}
&\delta\bar{B}^{(d)}\phi(0) = \\
&\quad R\beta_d l^{-(d+2)}\phi\Big\{\left(\frac{1}{6}+\frac{l}{24(d+1)}\frac{\partial}{\partial l}\right)l^{d+2}B(d)\binom{d}{\frac{d}{2}}\frac{I^{(\Omega)}}{(d-1)} \\
&\qquad -\frac{l}{12(d+1)(d+2)}\frac{\partial}{\partial l}l^{d+2}B(d)\binom{d-2}{\frac{d}{2}-1}I^{(\Omega)}\Big\} \\
&\qquad = R\beta_d\phi B(d)\Big\{\left(\frac{1}{6}+\frac{d+2}{24(d+1)}\right)\binom{d}{\frac{d}{2}}\frac{I^{(\Omega)}}{(d-1)} \\
&\qquad\qquad + \frac{1}{12(d+1)}\binom{d-2}{\frac{d}{2}-1}I^{(\Omega)}\Big\}.
\end{aligned} \tag{68}$$

The sums disappear since the only surviving terms in (29) have $m = \frac{d}{2}$. In even dimensions, we have

$$\hat{O}_d\int_{-\infty}^{0}\int_{u}^{0}\mathrm{d}u\mathrm{d}v\, u^{d-2-m}v^m\mathrm{e}^{-l^{-d}V_d} = \begin{cases} 0 & : m \neq \frac{d}{2}-1 \\ \frac{2}{d}l^d c^{-1} & : d\, odd, \end{cases} \tag{69}$$

where, again, one should use the same method as for (29). This implies that in even dimensions

$$\beta_d = \frac{2c^{\frac{2}{d}}(\frac{d}{2}+1)!}{(d-1)\binom{d-2}{\frac{d}{2}-1}\left(1-\frac{d}{2}\right)_{\frac{d}{2}-1}\Gamma(1+\frac{2}{d})} \tag{70}$$

plugging (70) into (68) and doing some algebra gives the desired result:

$$\delta \bar{B}^{(d)}\phi(0) = (-1)^{\frac{d}{2}+1}\frac{R}{2}\phi. \tag{71}$$

Conclusion

We have shown that, to first order in R, the discrete D'Alembertian B acting on a constant scalar field gives the following:

$$\lim_{l\to 0} \bar{B}^{(d)}\phi = \begin{cases} -\frac{R}{2}\phi & : d\, odd \\ (-1)^{\frac{d}{2}+1}\frac{R}{2}\phi. & : d\, even \end{cases} \tag{72}$$

assuming that the limit exists. The logical next step would be to try to prove this for higher orders in R and ultimately to all orders. If this could be done, we would certianly have an action for causal sets for arbitrary dimensions which would recover classical gravity in the macroscopic limit. The implications of such an action deserve to be thoroughly investigated.

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Appendix I: Integration over spatial angles

We wish to calculate

$$\int \mathrm{d}\Omega\, R_{\mu\nu}x^{\mu}x^{\nu} \tag{73}$$

which, explicitly in spherical coordinates, is

$$\int_0^{2\pi}\int_0^{\pi}...\int_0^{\pi} \mathrm{d}\phi_1...\mathrm{d}\phi_{d-2}\, R_{\mu\nu}x^{\mu}x^{\nu}r^{d-2}\prod_{i=1}^{d-3}\sin^{d-2-i}(\phi_i), \tag{74}$$

where

$$x^0 = t,$$

$$x^1 = r,$$

$$x^n = r\sin(\phi_1)...\sin(\phi_{n-2})\cos(\phi_{n-1}) \;\; : (d-1 > n > 1),$$

$$x^{d-1} = r\sin(\phi_1)...\sin(\phi_{d-2}). \tag{75}$$

In light of the fact that

$$\int_0^{\pi} \mathrm{d}x\, \cos(x) = \int_0^{2\pi} \mathrm{d}x\, \sin(x) = 0, \tag{76}$$

(74) reduces to

$$\begin{aligned} 2\pi \int_0^{\pi} ... \int_0^{\pi} \mathrm{d}\phi_1 ... \mathrm{d}\phi_{d-3} \Bigg\{ & R_{00} t^2 r^{d-2} \prod_{i=1}^{d-3} \sin^{d-2-i}(\phi_i) \\ & + r^d \sum_{j=1}^{d-2} R_{jj} \cos^2(\phi_j) \left(\prod_{i=1}^{j-1} \sin^{d-i}(\phi_i) \right) \left(\prod_{i=j}^{d-3} \sin^{d-2-j}(\phi_i) \right) \\ & + \frac{1}{2} R_{(d-1)(d-1)} \prod_{i=1}^{d-3} \sin^{d-i}(\phi_i) \Bigg\}. \end{aligned} \tag{77}$$

Taking note of the following integrals

$$\int_0^{\pi} \cos^2(x) \sin^n(x) = \frac{\sqrt{\pi}\,\Gamma(\frac{1+n}{2})}{2\Gamma(2+\frac{n}{2})} \tag{78}$$

$$\int_0^{\pi} \sin^n(x) = \frac{\sqrt{\pi}\,\Gamma(\frac{1+n}{2})}{\Gamma(1+\frac{n}{2})}, \tag{79}$$

(77) is

$$\begin{aligned} & 2\pi \left(\prod_{i=1}^{d-3} \sqrt{\pi} \frac{\Gamma(\frac{d-i}{2}-1)}{\Gamma(\frac{d-i}{2})} \right) R_{00} t^2 r^{d-2} \\ & + 2\pi r^d \sum_{j=1}^{d-2} R_{jj} \left(\prod_{i=1}^{j-1} \sqrt{\pi} \frac{\Gamma(\frac{d+1-i}{2}-1)}{\Gamma(1+\frac{d-i}{2})} \right) \left(\prod_{i=j+1}^{d-3} \sqrt{\pi} \frac{\Gamma(\frac{d-i}{2}-1)}{\Gamma(\frac{d-i}{2})} \right) \frac{\sqrt{\pi}\Gamma(\frac{d-1-j}{2})}{2\Gamma(1+\frac{d-j}{2})} \\ & + \pi R_{(d-1)(d-1)} \prod_{i=1}^{d-3} \sqrt{\pi} \frac{\Gamma(\frac{d+1-i}{2}-1)}{\Gamma(1+\frac{d-i}{2})} \\ & = \frac{2\sqrt{\pi}^{d-1} R_{00}}{\Gamma(\frac{d-1}{2})} t^2 r^{d-2} + \frac{2\sqrt{\pi}^{d-1}(R+R_{00})}{\Gamma(\frac{d-1}{2})(d-1)} r^d, \end{aligned} \tag{80}$$

as required. Having seen this, it is evident also that

$$\int \mathrm{d}\Omega\, R = \frac{2\sqrt{\pi}^{d-1} R}{\Gamma(\frac{d-1}{2})} r^{d-2}. \tag{81}$$

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