The AdS/CFT duality and Black Holes

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Yet to be decided

I would like to dedicate this dissertation to my matchless parents and my great brother, whose loving guidance and keen advices drove me to where I am and whose eager expectations of my person infuse me with the determination I need to accomplish my goals.

To my ever-loving Cony whose tireless support and vehement comfort helped me in the best and worst times, without whom I would not have been able to bear the rushes of this year and to whom I owe my happiness.

And finally, to my four wonderful grandparents, in the hope that with this triumph I can be worth of their so longed pride.

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Chapter 1

Introduction

In this Introduction we briefly present what are conformal field theories, how do black holes on AdS space times behave and the connection between these 2 completely different subjects -the *AdS/CFT correspondence*-, and we will mention why physicists find this correspondence interesting and my motivation to study a very specific part of this correspondence.

1.1 The AdS/CFT Correspondence

Suppose we have a quantum field theory in d spatial dimensions and a gravity theory in $d+1$ dimensions which has a d dimensional asymptotic boundary (see figure 1.1), then there is an equality between these two theories which is named the *gauge/gravity duality*, because quantum field theories are gauge theories [1]. It can also be called the gauge/string duality, since gravity theories are string theories. The fact that a $d+1$ dimensional theory "projects" into a field theory of a lower dimension resembles the optical hologram, sometimes it is also identified as *holography*. Perhaps the most common name for this duality is the *AdS/CFT correspondence* which is just a name given to the simplest dualities -rather, the least complicated-, which are between anti-de Sitter spaces and conformal field theories. From now on, we shall always refer to the duality by this name, since we are only going to focus on the less complicated examples, but it is important to remark that the duality exist even for non conformal theories.

1.1.1 Black Holes in an AdS spacetime

As stated above, AdS stands for anti-de Sitter space, which is a space that solves Einstein's equations with a negative cosmological constant. A $d+1$ dimensional AdS space-time has a metric

$$ds^2_{AdS_{d+1}} = R^2 \left[-(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega^2_{d-1} \right] \tag{1.1}$$

Where R is the radius of curvature of our space-time and $d\Omega^2_{d-1}$ is the metric of the unit $d-1$ -sphere. This metric can be seen as a gas of gravitons and has a $so(2,d)$ symmetry algebra -once you compactify the time direction-. Another important property of the AdS space-time is that it has no 'centre', in other words, we can boost any massive particle to find a frame where the particle is at rest [1]. A black hole in AdS would have the form:

$$ds^2_{AdS_{d+1}} = R^2 \left[-(r^2+1-\frac{2gm}{r^d-2})dt^2 + \frac{dr^2}{r^2+1-\frac{2gm}{r^d-2}} + r^2 d\Omega^2_{d-1} \right] \tag{1.2}$$

where g is Newton's constant in terms of the radius of the AdS and can be seen as the effective gravitational coupling [1], i.e. a measure of the interactions between gravitons, and is given by

$$g \propto \frac{G_N^{d+1}}{R^{d-1}} \tag{1.3}$$

The value of r where the first term of 1.2 vanishes is called the horizon radius r_+. Bekenstein and Hawking ([2], [3]) proposed that one could associate a temperature $1/\beta$ to a black hole, which is -in case of a big black hole- proportional to its horizon radius. They also state that entropy of a black hole S is proportional to its horizon area A, in the case of big black hole on an AdS space the entropy would be [1]

$$S_{BH} \sim \frac{1}{g}\frac{1}{\beta^{d-1}} \tag{1.4}$$

We can study black holes from two statistically different points of view [4]. The micro-canonical and the canonical ensemble. In the micro-canonical one, the system with the largest entropy for a given energy is thermodynamically preferred. In the canonical ensemble the system with the smallest free energy for a given temperature would dominate.

1.1.2 Conformal Field Theories

A conformal field theory (CFT) is a quantum field theory that is invariant under conformal transformations. A conformal transformation is one that preserves angles, this transformations include: poincare transformations, dilatation processes and 'special conformal transformation' (the composition of a reflection and an inversion in a sphere). This CFT's have several properties but the most relevant for our discussion are [1]: they have no dimensionfull parameter, namely is scale invariant, and their stress energy momentum tensor is traceless, i.e.

$$T^a_a = 0 \tag{1.5}$$

Suppose we have a d-dimensional CFT, specifically a CFT on a space that goes like $R \times S^{d-1}$ with massless fields, then we can associate an effective temperature $1/\beta$ to this theory, and expect an entropy proportional to the temperature and to the volume of the $d-1$ sphere [1]. Then if we take the large temperature limit (large compared to the radius of the sphere) $\beta \ll 0$, we can anticipate that the entropy will be

$$S_{CFT} \propto c \frac{1}{\beta^{d-1}} \tag{1.6}$$

Where c is just a quantity that tells us the effective number of massless fields we have in our theory.

1.1.3 The duality

The AdS/CFT correspondence states that the physics of an *asymptotically* anti-de Sitter space-time can be described by a quantum field theory that lives in the boundary, boundary given by $R \times S^{d-1}$ [1] (see figure 1.1). The symmetries of the AdS space act also in the boundary, and take points in the boundary to points in the boundary, making those transformations conformal on the boundary, thus the quantum field theory in the boundary must be conformal. Note that the metric of the boundary field theory is *not* dynamical [1].

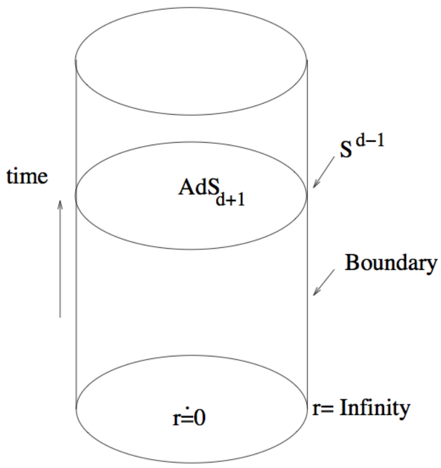

Figure 1.1: Penrose diagram of an AdS space. The boundary contains the time direction and the sphere S^{d-1} represented here as a circle. Image taken from [1]

For the CFT and the gravity theory to be equivalent, the entropy of the black hole 1.4 and the entropy of the CFT 1.6 must match [1], then

$$c \propto \frac{1}{g} \tag{1.7}$$

This equation is saying that the number of fields scales like the inverse of Newton's constant. To make things simpler, we expect the gravitons don't interact between each others, namely, that the coupling g is small, hence we need that the number N of fields on our CFT to be large.

Another feature that both theories must have, in order for them to match is the existence of a Fock space -or an analog- [1], namely, in string theory we can talk of single particles, two particles and so on. It is easy to see that a weakly coupled gravity theory does have this feature and that the space of operators in a large N CFT has a similar structure [1]. The AdS/CFT duality connects an operator in the

[1]For a brief explanation the reader is referred to [1]

field theory to a state in the bulk string theory, more specifically it associates single trace operators to single particle states and multitrace operators to multi particle states in the bulk in a one-to-one correspondence. The most direct example of such relation is the graviton-stress tensor duality.

We are now going to use a string theory point of view for the gravity theory. The lowest excitation of the string is the graviton, of size l_s -the string length-, and the coupling g explained above measures the strength of the interaction between strings. We want to ignore the massive states of this theory, and in order to ignore this we need that

$$\frac{R_{AdS}}{l_s} \gg 1 \tag{1.8}$$

Which is just stating that the radius of the AdS space must be much bigger that the string length in order for Einstein's gravity to be a good approximation. The massive states with higher spin ($S > 2$) of string theory are dual to single trace operators with higher spin. At weak *field theory* coupling, this operators give rise to particles in the bulk of mass comparable to the inverse of R_{AdS}, which contradicts 1.8, then we can conclude that in order to use gravity, the field theory should be strongly coupled [1].

Another relation that the AdS/CFT dictionary states is that, for a field ϕ in our gravity theory related to an operator $\mathcal{O}$ in the CFT, we have

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} Z_{Gravity}[\phi_0(x)] \tag{1.9}$$

Where $Z_{Gravity}$ is the partition function in the gravity theory [1].

An Additional interesting relation between this two theories is the case of a gauge field in the AdS which corresponds to a conserved current on the boundary CFT, in other words, a gauge symmetry in the gravity theory corresponds to a global symmetry on the boundary [1].

1.1.3.1 A -not so complicated- example

We will now revise a particular example of the AdS/CFT correspondence: the $\mathcal{N} = 4$ Super Yang Mills/AdS$_5 \times S^5$ example. We do this example with the solely purpose of showing an explicit relation and not because we will use it in the following chapters. If the reader is not interested in such relation, he should skip directly to next section and if he would like to see a more careful explanation, he should refer to [1].

Suppose that we have a quantum field theory and we want to make it a supersymmetric field theory, then the bosons and their fermonic partners would transform under the same representation of the gauge group, so if we add only one majorana fermion, we would get a $\mathcal{N} = 1$ supersymmetric theory. If instead we add four fermions and six scalars, with special couplings, we sill get a maximally supersymmetric $\mathcal{N} = 4$ theory [1], with a action given by

$$\begin{aligned} S = -\frac{1}{4g_{YM}^2}\int d^4x Tr\left[F^2 + 2(D_\mu\Phi^I)^2 + \chi \not{D}\chi + \chi[\Phi,\chi] - \sum_{IJ}[\Phi^I,\Phi^J]^2\right] \\ + \frac{\theta}{8\pi^2}\int Tr[F\wedge F] \end{aligned} \tag{1.10}$$

Where g_{YM} is the field theory coupling and θ is an angle. All the fields in 1.10 are in the same supermultiplet under supersymmetry transformations. This theory is classically and quantum mechanically conformally invariant, this means that the field coupling will remain fixed -constant- at all energies, which is a great difference with QCD -a $\mathcal{N} = 1$ CFT that is not quantum mechanically conformal invariant- that has a field coupling that varies depending on the energy one is working on. The -effective- coupling of the theory is then

$$\lambda = g_{YM}^2 N \tag{1.11}$$

Known as the 't Hooft coupling and can be thought of as proportional to 1.8 for this specific example [1].

We now study the gravitational part. Because we worked with a supersymmetric field theory it is natural to expect that it's gravitational dual will be supersymmetric as well. Type IIB supergravity (SUGRA) is a supersymmetric string theory that adjusts to our needs. This theory has a self dual five form, $F_5 = \star F_5$ and has electrically/magnetically charged black holes as solutions. The near horizon solution has the geometry of $\mathrm{AdS}_5 \times S^5$ [1], which is electrically charged if the five form is along the AdS_5 or magnetically if it is along the S^5. The action of motion of this spacetime can be written as

$$S = \frac{1}{(2\pi)^7 l_p^8} \int d^{10}x \sqrt{g}(R + F_5^2) \tag{1.12}$$

Where $l_p = l_s\, g_s^{1/4}$ is Planck's length and can be compared to the radius of the S^5, and R is the radius of the AdS. The relation between both radius is given by

$$4\pi N = \frac{R^4}{l_p^4} \tag{1.13}$$

With N the number of colours of out gauge theory. This theory has a compact massless scalar field called the axion $\chi \to \chi + 2\pi$, and so we find two parameters in type IIB SUGRA, χ and g_s, which can be related to our field theory's parameters, θ and g_{YM}. It is just natural to identify χ with θ and the coupling g_{YM} with the string coupling g_s, i.e. $g_{YM}^2 = 4\pi g_s$.

Then, in order to trust Einstein's gravity we need a large λ and to have a weakly coupled gravity theory we need large number of fields, $N \gg 1$. In [1], they explicitly compute the free energies of Super Yang Mills theory and of $\mathrm{AdS}_5 \times S^5$ showing that they agree up to a factor of 3/4 which accounts for the change in free energy when we go from weak to strong coupling. They also describe thoroughly the construction of states in the string theory using D-branes and compare the spectrum of states in the gravity sector and the spectrum of operators of the CFT.

1.2 Why the AdS/CFT correspondence?

As we shall see in this dissertation, many gravitational systems are very complicated and poorly understood, this translates into huge gaps in the information we have about them. The complications reside in the fact that many solutions cannot be find analytically and that the complexity of the numerical approximation is such, that the amount of physics we obtain from them is very limited. There was a point were it was thought that all it was to be understood about black holes and gravity was already done, and that the all the other knowledge on those systems was simply unreachable.

When the AdS/CFT correspondence was proposed a new path was created: Since the field theory has been long studied and far better understood than the gravity theories, we expect to be able to retrieve some of that gravitational lost knowledge by looking at it's conformal field dual. As seen in the previous section we saw that a large N strong coupled conformal field theory is dual to a weakly coupled black hole in AdS space-time, that a single traced operator in the field theory corresponds to a single particle state, in particular the metric/graviton is dual to the stress energy tensor and we also saw that the correlation functions of this CFT is related to the partition function of our gravitational system.

In chapter 2 we will see that this correspondence places constraints on the field theories that can be dual to the gravitational theories. The AdS/CFT matching tells us: how different temperature ranges in the field theory correspond to different gravity backgrounds, specifically what are the duals to the confinement/deconfinement phases of our field theory, also we will study other restrictions on our field theory, such as how the mass gap in the string theory translates to the dual field theory.

The advantages that we can take from the AdS/CFT correspondence go even further. So far, we have only talked about gravitational and field theory systems that do not evolve in time, but as a direct outcome of the duality, it turns out that a certain dynamics in one systems translates in a particular dynamical evolution in the other system. This relation between the evolution of systems is called the *fluid/gravity correspondence*, since the dynamics of the field theory can be thought of as the dynamics of fluid. This duality states that the dynamics of the stress energy tensor of our boundary/field theory are governed by Einstein's equations. This correspondence will also give us an algorithm to find the gravitational dual systems to a certain fluid, which is a powerful weapon to tackle the difficulties in the gravitational system. This fluid/gravity correspondence will be the main topic of Chapter 3.

There's a specific case of correspondence that is specially interesting: the equality between plasma ball and localised black holes, as we will see in Chapter 4. A plasma ball is just a spherically symmetric lump of matter that is in both the confinement and the deconfinement phases and depending the temperature of the ball is which phase dominates. A localised black hole is a solution to Einstein's equations which resembles a black brane over some critical temperature and a gas of gravitons at lower temperatures but, that at this critical temperature there is another solution called the *domain wall* that interpolates between these two previous solutions. We will also see, that after joining them via the AdS/CFT duality, we can learn from both systems that will help us to understand the other one.

In the last chapter we study one of the most important upshots of our duality and one of the main goals of this dissertation: the duality between the Rayleigh-Plaeau instability in a fluid tube and the Gregory-Laflamme instability on a black string. This means that the phase transitions in both the CFT part and the gravitational part are equivalent. It is almost unbelievable how a purely hydrodynamic instability is dual to a entirely gravitational instability, and is, I think, one of the most intriguing and exciting topics, since if there is a duality between a specific type of instabilities, then most surely there exists other dualities between other instabilities.

I would like to emphasise that there are many interesting topics regarding the AdS/CFT correspondence which I do not talk about in this dissertation, due to lack of both time and space. In addition I would like to stress that this dissertation is intended to give the reader a big picture of a small part of the current position on the AdS/CFT duality, and by no means it is to be taken as an exhaustive review, the reader who wishes to go further in the topics here analysed should turn to the references here cited and the citations within those references.

Chapter 2

Conformal Field Theories and Black Holes

The AdS/CFT correspondence revealed an amazing relation between conformal field theories and supergravity theories in an Anti-de Sitter (AdS) space, a space that has a cosmological constant with a negative value. There are many ways of describing this relationship and in this chapter we will be using the following: a conformal field theory on a n-dimensional manifold M can be studied by analysing a manifold B of dimension $n+1$ but that -at infinity- have the manifold M as a conformal boundary [5].

2.1 The *confinement* phase

There are two different background manifolds with which we can review an $\mathcal{N}=4$ field theory (a field theory with at finite temperature), S^3 and R^3. In this section section we will study the theory in S^3 and in the following sections we will study the R^3 by taking the limit where the radius of our S^3 goes to infinity.

For such study we need to examine the partition function on $S^3 \times S^1$, where S^1 represents the Euclidean time and we represent each circumference by β' and β. Because our theory is conformally invariant, and as such, it should have a dimensionless parameter, we could expect that the only parameter that really takes importance is the -dimensionless- ratio β/β'.

In a field theory with small N there is no phase transitions as function of β -or temperature in that case-. But if we take the large N limit of a field theory then it is possible to have a phase transition [5] as a function of the ratio β/β' [1] . If we take the radius of the S^1 to be large, i.e. large β with β' fixed, analogous to taking the low temperature limit, our theory has certain analogies with the confining phase in certain gauge theories, and, as we will see, if we take the small limit for β -high temperatures- we can also find analogies but to a deconfining phase of such gauge theories [4].

In the large N limit, a measure of the confinement of our theory is to check the order of such theories free energy: if it is of order N^2, it implies deconfinement, since this order shows the contributions given by the gluons themselves, whilst if it is of order one, it implies confinement, because it reflects the action of the hadrons (singlets under colour) [4].

In large N gauge theories on S^3, the solution with lowest energy can be made by fixing the gauge field, the scalars and the fermions to zero, which makes this configuration invariant to a global *SU(N)* gauge transformation, i.e. the state is unique up to a gauge transformation. The Gauss law in a finite volume limits the physical states -traces of fields- to be invariant under global *SU(N)* transformations. To calculate the free energy of the lowest energy states, we need to know its multiplicity, which is given by the number of traces we can make with the creation operators of the fields in our theory. Since such number of traces is independent of N, then the free energy is also independent of N [4] and therefore our theory living on S^3 is in the "confinement" phase.

2.1.1 AdS-Schwarzschild black holes and *confinement* phases

Suppose we have a classical, static, stationary black hole embedded on an n-dimensional AdS spacetime (with $n-1$ spatial dimensions), then we shall call it an AdS-

[1] More specifically this statement is valid for finite N, but when we take the large N limit this statement breaks due to the Mermin Wagner theorem [6]

Schwarzschild black hole -in n dimensions.

As stated above, to study an n-dimensional field theory on a manifold M, we need to solve Einstein's equations for a $n+1$-dimensional spacetime B. Since, in this section, we are primarily interested in studying the confinement phase of a gauge theory, then we should start with such a theory on a manifold M like $S^{n-1} \times S^1$.

This configuration can be identified with two different spacetimes B's [7]. One spacetime Y_1 is just AdS space (with Euclidean time), it's topology is $R^n \times S^1$ and the metric of such manifold is:

$$ds^2 = \left(\frac{r^2}{b^2} + 1\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2} + 1\right)} + r^2 d\Omega^2 \tag{2.1}$$

Which is the same as the metric 1.1 if we change $r \to r/b$. In equation 2.1 b is the radius of curvature of the Anti-de Sitter space, $d\Omega^2$ is the metic of a unit radius round sphere S^{n-1} and t is a periodic variable (the value of such a period will soon be clarified). Note there is no horizon in 2.1, so there's no black hole and, therefore, we shall refer to this manifold as the AdS spacetime. The second manifold Y_2, is the AdS-Schwarzschild black hole with metric:

$$ds^2 = \left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right)} + r^2 d\Omega^2 \tag{2.2}$$

where

$$w_n = \frac{16\pi G_N}{(n-1)\mathrm{Vol}(S^{n-1})} \tag{2.3}$$

is a constant factor included so that M has mass units, G_N is the $n+1$-dimensional Newton's constant and $\mathrm{Vol}(S^{n-1})$ is the volume of a unit radius $n-1$ sphere. It's a black hole because there is a horizon that can be by obtained taking the largest root r_+ of $g_{tt} = 0$. Note that because of the metric 2.2, our spacetime will be restricted to $r \geq r_+$ and this metric has Euclidean time as 2.1. In order to get a smooth metric 2.2, the period of t must be [5]:

$$\beta_0 = \frac{4\pi b^2 r_+}{n r_+^2 + (n-2)b^2} \tag{2.4}$$

Observe that β_0 has a maximum, so the AdS-Schwazschild black hole -manifold Y_2- will only be seen if β is small enough -or the temperature is high enough-. The topology of Y_2 is $R^2 \times S^{n-1}$, is simply connected and if we include the boundary points it would be $B^2 \times S^{n-1}$, where B^2 is the 2-ball .

We should make a couple of remarks now. Both manifolds -Y_1 and Y_2- contribute to the standard (canonical) thermal ensemble $\mathrm{Tr}e^{-\beta H}$. Also, note that Y_1 dominates at low temperatures while Y_2 dominates at -sufficiently- high temperatures.

2.2 Entropy of AdS-Schwarzschild black holes

We will now calculate the entropy of a Schwarzschild black hole on an n-dimensional AdS space and see that it agrees with Hawking's entropy which is proportional to the area of the horizon [3]. We will calculate the value of the action and then derive it by the inverse of the temperature to obtain the energy and thus work out the entropy. For the action of a spacetime with a negative cosmological constant can be written as follows

$$I = -\frac{1}{16\pi G_N}\int d^{n+1}x\sqrt{g}\left(R + \frac{n(n-1)}{2b^2}\right) \tag{2.5}$$

Where g is the determinant of the metric and R is Ricci scalar. Since the corrections to the AdS metric given by the black hole vanish very quickly at infinity, there is no surface term in the integral. When we solve Einstein's equations we see that $R = -n(n+1)/2b^2$ so the integrand of 2.5 simplifies to $-2n\sqrt{g}d^{n+1}x$ which is the volume form of a $n+1$ spacetime -times $-2n$-.

As both Y_1 and Y_2 have infinite volumes, [7] showed that we can subtract both volumes to get a finite value. The integrals will have the same form but different limits, i.e. the time upper limit of the volume of Y_1 is β' while the value of the period of the black hole for it to be smooth has to be β_0(2.4), also the black hole covers only the space where $r \geq r_+$ whilst the AdS spacetime covers the hole $r \geq 0$ region. Then, we have

$$V_1(R) = \int_0^{\beta'} dt \int_0^R dr \int_{S^{n-1}} d\Omega r^{n-1} \tag{2.6}$$

for the AdS spacetime, and for the black hole we have

$$V_2(R) = \int_0^{\beta_0} dt \int_{r_+}^R dr \int_{S^{n-1}} d\Omega r^{n-1} \tag{2.7}$$

Note that we have put an upper cutoff R to avoid problems. Seeing that the geometry of both manifolds has to be the same in the limit $r = R$, it is necessary to adjust β' and set it to [5]

$$\beta' \sqrt{\frac{r^2}{b^2} + 1} = \beta_0 \sqrt{\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}} \tag{2.8}$$

so at $r = R$

$$\beta' = \beta_0 \left(1 - \frac{w_n M}{R^{n-2}(R^2 + b^2)}\right)^{1/2} \tag{2.9}$$

after doing such adjustments, we can calculate the integrals 2.6 and 2.7

$$V_1(R) = \beta' \frac{R^n}{n} \mathrm{Vol}(S^{n-1}) = \beta_0 \frac{R^n}{n} \mathrm{Vol}(S^{n-1}) \left(1 - \frac{w_n M}{R^{n-2}(R^2 + b^2)}\right)^{1/2} \tag{2.10}$$

$$V_2(R) = \beta_0 \frac{R^n - r_+^n}{n} \mathrm{Vol}(S^{n-1}) \tag{2.11}$$

and so, subtracting these two volumes

$$V_2(R) - V_1(R) = \beta_0 \mathrm{Vol}(S^{n-1}) \left[\frac{R^n}{n} \left(1 - \left(1 - \frac{w_n M}{R^{n-2}(R^2+b^2)} \right)^{1/2} \right) - \frac{r_+^n}{n} \right] \tag{2.12}$$

taking the limit of this subtraction when $R \to \infty$ we get

$$\lim_{R\to\infty} (V_2(R) - V_1(R)) = \frac{\beta_0 \mathrm{Vol}(S^{n-1})}{n} \left(\frac{w_n M b^2}{2} - r_+^n \right) \tag{2.13}$$

But since we need the action solely as a function of r_+, we substitute β_0 with 2.4 and also the solution of $g_{tt} = 0$ (w_n as a function of r_+) into this last equation, then

$$\lim_{R\to\infty} (V_2(R) - V_1(R)) = \mathrm{Vol}(S^{n-1}) \frac{4\pi}{n(nr_+^2 + (n-2)b^2)} \left(\frac{w_n M b^2}{2} - r_+^{n+1} b^2 \right) \tag{2.14}$$

$$= \mathrm{Vol}(S^{n-1}) \frac{2\pi}{n} \left(\frac{b^2 r_+^{n-1} - r_+^{n+1}}{nr_+^2 + (n-2)b^2} \right) \tag{2.15}$$

finally multiplying this by $n/8\pi G_N$ we get the complete value of the action 2.5 after solving Einstein's equations, obtaining

$$I = \frac{\mathrm{Vol}(S^{n-1})(b^2 r_+^{n-1} - r_+^{n+1})}{4G_N(nr_+^2 + (n-2)b^2)} \tag{2.16}$$

From thermodynamics, we know that, in a standard Euclidean procedure, to compute the energy of our system, we need to derive the action by β_0, so

$$E = \frac{\partial I}{\partial \beta_0} = \frac{\partial I}{\partial r_+} \frac{\partial r_+}{\partial \beta_0} = \frac{(n-1)\mathrm{Vol}(S^{n-1})(r_+^n b^{-2} + r_+^{n+2})}{16\pi G_N} \tag{2.17}$$

$$= \frac{(n-1)\mathrm{Vol}(S^{n-1})w_n M}{16\pi G_N} = M \tag{2.18}$$

Where in the penultimate equality we have solved $g_{tt} = 0$ for M and in the last equality we have used 2.3. Now, the entropy is calculated (by analogy to thermodynamics second law) subtracting the action to the energy divided by the temperature $(1/\beta_0)$, then combining 2.17, 2.4 and 2.16

$$S = E\beta_0 - I = \frac{1}{4G_N}\mathrm{Vol}(S^{n-1})r_+^{n-1} \tag{2.19}$$

As we will note later, this is just Hawking's entropy [3]. Note that this discussion applies only to black holes whose Schwarzschild radius is much greater than the radius of curvature of the AdS space, i.e. $r_+ \gg b$. I would like to comment that the canonical ensemble uses the free energy rather than the entropy, and this can be easily calculated from the entropy calculated here.

2.3 The *deconfinement* phases

In this section we are going to study the relation between the deconfinement phase of gauge theories and the black holes. In previous sections, we studied a $\mathcal{N} = 4$ gauge theory on S^3 via the partition function on $S^3 \times S^1$ and declared that due to conformal invariance the only parameter of interest was the ratio β/β'. If we take the large β' limit, this changes our space to $R^3 \times S^1$ and here, once again we can scale out β on account of conformal invariance, giving us a $\mathcal{N} = 4$ gauge theory on R^3 which cannot have a phase transition.

When we go to $R^3 \times S^1$ by taking $\beta' \to \infty$ for fixed β, we get $\beta/\beta' \to 0$, which states that the only nonzero temperature phase of the gauge theory on R^3 is on high temperatures side of the phase transition, then it should be compared to the deconfining phase of the gauge theories.

As before, a measure of the confinement of our theory is to check the order of such theories free energy, so, in order for a theory to in a deconfinement phase it's free energy should scale as N^2. In $S^3 \times S^1$, if we take $\beta' \to \infty$ with fixed β, we are in the high temperature region, also, the fee energy is proportional to the volume of S^3 times the ground state energy density of the theory that is obtained by compactification on S^1. Doing this compactfication breaks supersymmetry making the cancellation of bosons and fermions unbalanced -even at one-loop level- and the one loop contribution is of order N^2, consequently the free energy of the compactified S^1 is proportional to N^2. The volume of S^3 is of order β'^3 and so the free energy on $S^3 \times S^1$ is of order $N^2\beta'^3$. Then, we have sketched the proof that the large β' limit is in the deconfinement phase of the gauge theory.

2.3.1 AdS-Schwarzschild black holes at large r

To understand the relation between black holes and deconfinement phase first we need to understand the behaviour of black holes at large r. For such goal we return to 2.1 and 2.2 and explain their behaviour.

The radius of S^1 is -approximately- $\beta = (r/b)\beta_0$ whilst the radius of the S^{n-1} is $\beta' = r/b$, then $\beta/\beta' = \beta_0$. At the limit where $\beta/\beta' \to 0$ we are in the $R^{n-1} \times S^1$ region, i.e. $\beta_0 \to 0$. If we observe 2.4, it is obvious that we can get to this limit either by taking $r_+ \to 0$ to $r_+ \to \infty$, but the last branch is thermodynamically preferred due to it's larger action. Observe that large r_+ also implies large M and we can take this limit of 2.4 and get

$$\beta_0 \sim \frac{4\pi b^2}{nr_+} \sim \frac{4\pi b^2}{n(w_n b^2)^{1/n} M^{1/n}} \tag{2.20}$$

Setting $r = (w_n M/b^{n-2})^{1/n}\rho$ and $t = (w_n M/b^{n-2})^{-1/n}\tau$ [1], taking the large M and substituting in 2.2 we get:

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}} + \left(\frac{w_n M}{b^{n-2}}\right)^{2/n} \rho^2 d\Omega^2 \tag{2.21}$$

[1]This substitution was taken from [5]

As the radius of the S^{n-1} gets larger, this makes $M \to \infty$ and makes our space look *locally* flat, which means that we can introduce -at a point on S^{n-1}- a change of coordinates, such that $d\Omega^2 = \sum dy_i^2$, then we can rewrite 2.21 as

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}} + \rho^2 \sum_{i=1}^{n-1} dx_i^2 \tag{2.22}$$

This is the solution wanted for a $R^{n-1} \times S^1$ instead of a $S^{n-1} \times S^1$. The topology of this manifold (including the boundary) is $R^{n-1} \times B^2$ where B^2 denotes the 2-ball.

In [3] it is shown that from the entropy density that can be calculated from 2.19, we find a relation to the volume of the horizon (hyper surface at $r = r_+$) of the black hole, i.e.

$$S = \frac{1}{4G_N} \text{Vol}(S^{n-1}) r_+^{n-1} = \frac{A}{4G_N} \tag{2.23}$$

Where A is the volume of the horizon. We need to compare this entropy with the entropy calculated from the dual conformal field theory living on $S^{n-1} \times S^1$ [5]. Conformal invariance implies that, when the limit $\beta_0 \to 0$ is taken -the high temperature region in the field theory and the large radius region in the black hole-, the entropy density on S^{n-1} scales as $\beta_0^{-(n-1)}$, then the entropy of the boundary field theory is of order r_+^{n-1} and, therefore, it is a multiple of the horizon volume A in equation 2.23. Hence, we have verified that the entropies of both the large N $\mathcal{N} = 4$ gauge theory and the AdS-Schwarzschild black hole match up to a factor in the large radius -high temperature- region. To give the exact factor more information about the quantum field theory in the boundary is needed. As in the previous section this discussion applies only to black holes whose Schwarzschild radius is much greater than the radius of curvature of the AdS space.

2.4 Consequences of the AdS/CFT correspondence

In the infinite volume limit -high temperature/large radius-, the AdS/CFT correspondence implies that the quantum filed theory should have, among others, three key properties: nonzero expectation value of the temporal Wilson loop (symmetry breaking), an area law for spatial Wilson lines (confinement) and a mass gap.

With this in mind, lets give a brief explanation about Wilson loops in $\mathcal{N} = 4$ gauge theories in *three spatial dimensions*. Suppose you want to study a field theory on a four-dimensional manifold M that is the boundary of a five-dimensional AdS manifold B (which complies with Einstein's equations). Remember that the $\mathcal{N} = 4$ gauge theories AdS/CFT duals are -in our case- type IIB string theories, hence to understand a Wilson loop associated to a contour $C \subset M$, we study strings on B with the characteristic that C is a boundary for the string's world sheet D i.e. $\partial D = C$. Since the area of D is infinite, we can define a regularized area $\alpha(D)$ [5]. Then the expectation value of the Wilson loop is

$$\langle W(C) \rangle = \int_K d\mu e^{-\alpha(D)} \tag{2.24}$$

Where K is the space of string world sheets D that obey the boundary conditions and $d\mu$ is the measure of the world sheet path integral. Equation 2.24 has two important outcomes: if C is not a boundary in B the expectation value $\langle W(C) \rangle$ will vanish and Wilson loops on R^4 will obey an area law, i.e. the minimum value of $\alpha(D)$ scales as a positive multiple of the area enclosed by C, in other words, the smallest of $\alpha(D)$ grows with C. For a much more detailed and fulfilling explanation, the reader should refer to [8].

2.4.1 Symmetry Breaking and temporal Wilson lines

We already mentioned that a good measure to recognise the phase of our gauge field is the way the free energy scales with respect to N. Another good order parameter for the deconfinement is the vacuum expectation value of the centre of the gauge group ([5]). In this section we will study how this symmetry breaking -with some help from Wilson lines- can define our deconfinement.

Suppose we have a gauge theory, which's gauge group is G, then the centre Γ of a group can be defined as follows:

$$\Gamma = \{g \in G \mid gz = zg \, \forall z \in G\} \tag{2.25}$$

In other words, the set of elements that commute with every element in G. Taking $G = SU(N)$ and the center to be $\Gamma = Z_N$, note that the centre acts trivially on all *fields* [5]. Consider a $SU(N)$ gauge theory on $Y \times S^1$, with Y any spatial manifold and take g to be a general gauge transformation that complies with $g(y,z) = g(y, z+\beta)$. As all fields transform trivially under the centre of G, we can write such a map as follows:

$$g(y,z) = g(y,z)h = g(y,z+\beta) = g(y,z+\beta)h \tag{2.26}$$

Where h can be any element of the centre of G. Observe that Γ acts trivially on fields but not in observables and that the action of Γ over G is a symmetry of our theory. An order parameter for the spontaneous breaking of the centre Γ is the vacuum expectation value of a temporal Wilson line [5], which is an operator defined as follows:

$$W(C) = \mathrm{Tr} P \exp \int_C A \tag{2.27}$$

Where C is an oriented closed path of the form $y \times S^1$, the trace is taken in the N-dimensional fundamental representation of $SU(N)$ and A is the gauge field. If we now do a transformation such as 2.26 this multiplies the holonomy of A around C by h, then

$$W(C) \rightarrow hW(C) \tag{2.28}$$

Then the expectation value $\langle W(C)\rangle$ of the Wilson line before and after the transformation will spontaneously break the symmetry of Γ unless, the such expectation value is zero, i.e.

$$\text{if } \langle W(C)\rangle \neq 0 \Rightarrow \langle W(C)\rangle \neq \langle hW(C)\rangle \tag{2.29}$$

Such spontaneous symmetry breaking will not happen in a finite volume since the expectation value of the Wilson loops will always vanish [5]. Now it is evident that, the expectation value of the *temporal* Wilson line is an order parameter for the spontaneous breaking of the Γ symmetry. But, how does the spontaneous breaking of the centre's symmetry in the *infinite* volume becomes a parameter for the deconfinement? The following can be used as a heuristic explanation [5]: inserting a Wilson line $W(C)$ is equal to include an external *static* quark to the system, and an expectation value of the Wilson line translates as the fact that the cost -in free energy- of perturbing the system with an external charge is finite, so if the system is in the confinement phase, the free energy cost to put an external charge in the system would be infinite, you can't have it there, so $\langle W(C)\rangle = 0$.

Lets talk about these temporal Wilson lines a bit more formally. For the sake of objectivity (we want to study three-dimensional $\mathcal{N} = 4$ gauge theories) we take our spacetime to be $R^3 \times S^1$ and take $C = P \times S^1$ where P is a point in R^3. For completeness we will consider also $S^3 \times S^1$ and in that case P is a point in S^3.

At low temperatures, field theories on $S^3 \times S^1$ are dominated by the topology of $R^4 \times S^1$ (recall 2.1), and the contour C that we have chosen is *not* a boundary of any whorldsheet in the Y_1 [5], thus, as a consequence of equation 2.24 at the beginning of the section, the expectation value of the Wilson loop vanishes which avoids symmetry breaking, meaning that we are in the confinement phase. This agrees with what we concluded in the previous section, so far, so good. At the high temperatures limit, the field theories on $S^3 \times S^1$ are dominated by Y_2 with a topology of $S^3 \times B^2$ -after including the boundary points at $r = \infty$-. In this phase our contour *is* a boundary of $D = P \times B^2$ [5], thus the expectation value will not vanish and there will be spontaneous symmetry breaking.

So at first glance it looks like we have a disagreement with our statements above, since we had asserted that there was no spontaneous symmetry breaking in finite volume. As it turns out, the integral 2.24 is not quite complete, we have to add a term $e^{i\psi}$ ([9]), where ψ is an "angle" that accounts for the freedom that comes from the fact that the two form field that sources the string worldsheet D has a symmetry -it can be shifted by 2π-. This term makes that the integral 2.24 vanishes on the high temperature limit of out field theory. And again, so far so good.

In the infinite volume regime -$R^3 \times S^1$-, ψ can be understood as a massless scalar field in the field theory on R^3 [5]. In this case, the integral 2.24 does not vanish and the expectation value is proportional, and hence dependant, to $e^{i\psi}$, so then there is symmetry breaking, throwing as result that we are in the deconfinement phase, which is in sympathy with what we concluded in previous sections.

2.4.2 The area law and spatial Wilson loops

The area law is an important property that the field theory -in the high temperature limit- should comply in order to fulfil the AdS/CFT correspondence.

A description of area law is the following: Let C be an oriented closed loop encircling an area A in R^3 at a fixed point on S^1, a *spatial* loop. The area law states that if C is scaled up, keeping its shape fixed and increasing A, then the expectation value of $W(C)$ vanishes exponentially with A.

We will start with the zero temperature region of a field theory on $S^3 \times S^1$. In this case, the area of D does not need to be proportional to the area enclosed by C and so the expectation value will not vanish when we scale up C and therefore the area law does not apply in this phase [5].

When the theory is at a nonzero temperature, the metric of our supergravity system is 2.22 with $n = 4$

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2}\right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2}} + \rho^2 \sum_{i=1}^{3} dx_i^2 \tag{2.30}$$

Since the area enclosed by the horizon is the last factor of the metric and ρ range is $[b, \infty]$, then the minimum value of the area enclosed by the contour $\alpha(D)$ is $b^2 A$, where A is the area enclosed by a -finite size- curve on flat 3D space [5]. Therefore, there *is* an area law for nonzero temperature field theories. Consequently the spatial Wilson loops obey an area law, which connects the field theory Wilson loop with the tension of the string in the supergravity theory -which translates as the area of the black hole horizon-.

2.4.3 The mass gap

The mass gap is the third property that the AdS/CFT correspondence asks for a $\mathcal{N} = 4$ three-dimensional gauge theory in the high temperature limit in order to concur with it's dual in string theory. One way to interpret the mass gap is that the correlation functions $\langle O(y,z), O'(y',z)\rangle$ vanish exponentially for $|y - y'| \to \infty$ [5]. The interpretation used in the following is not the simplest but a more direct than the correlation one, and we will identify quantum states in the supergravity theory with the ones in the boundary (in the field theory).

Start with a spacetime that has a metric such as 2.30, we can see this spacetime as a warped product (since the last term is ρ dependant) of two different space times. One flat 3D space R^3 with coordinates x_i's and a space K parametrized by ρ and τ. The goal is to show that a free field propagating in 5D gives rise to a discrete spectrum of positive particle masses. We can do so (as done in [5]) by considering the propagation of a Type IIB dilation field in spacetime 2.30, where translation in τ is a symmetry and modes with different momentum in the τ direction decouple and their spectrum is discrete. We want a field of the form $\phi(\rho, x) = f(\rho)e^{ik\cdot x}$, thus the effective action is

$$I(f) = \frac{1}{2}\int_b^\infty d\rho\rho^3 \left(\left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2}\right)\left(\frac{df}{d\rho}\right)^2 + \rho^{-2}k^2 f^2 \right) \tag{2.31}$$

and the equation of motion for f is

$$-\rho^{-1}\frac{d}{d\rho}\left(\rho^{3}\left(\frac{\rho^{2}}{b^{2}}-\frac{b^{2}}{\rho^{2}}\right)\frac{df}{d\rho}\right)+k^{2}f=0 \tag{2.32}$$

A mode of momentum k has mass m -as in 3D where $m^2 = -k^2$-. The solutions to the equation of motion should obey two boundary conditions, at $\rho = b$, $\frac{df}{d\rho} = 0$ since at b, ρ behaves as an origin and smoothness requires the vanishing or the derivative. The second boundary condition is that for $\rho \to \infty$ then $f \sim \rho^{-4}$ [5]. As in quantum mechanical problems, for a given k^2, there is a unique normalizable solution they obeys both boundary conditions, which in turn determines a discrete set of values of k^2. There are not such normalizable solutions for $k^2 \geq 0$, so the discrete set of values of k^2 at which there are formalisable solutions are all negative, then the masses are all positive and thus we have confirmed the existence of a mass gap.

If instead of using 2.30 we used 2.1, the spectrum of normalizable solutions is continuous for all $k^2 < 0$, so as expected, there is no mass gap in the confinement phase of the field theory. Then as said at the beginning of the section, a large N, $\mathcal{N} = 4$ field theory in the deconfinement phase *does* comply with the three constraints that the AdS/CFT correspondence sets upon it.

Chapter 3

Fluids and Black Holes

3.1 The *fluid/gravity correspondence*

In the previous chapter we noticed that due to AdS/CFT correspondence we could associate the high temperature region of a large N three-dimensional $\mathcal{N} = 4$ gauge theory with the four-dimensional AdS-Schwarzschild space-time as we take $r \to \infty$, but this correspondence goes even further. As a very important outcome of this relation between gauge theories and gravity theories, there is a relation between dynamical equations in both the gauge and the gravity theories, the *fluid/gravity correspondence*. Lets set the table first, to properly describe this correspondence.

Recall that according to AdS/CFT correspondence there is a one-to-one correspondence between single particle states in classical string theories and the single trace operators in the gauge theory, e.g. the graviton -a single particle state- on the AdS space is dual to the stress energy tensor at the boundary field theory. The fluid gravity correspondence states, among other things, that the dynamics of the stress tensor in certain n-dimensional conformal quantum field theories (at strong coupling) are controlled by the dynamics of Einstein's equations in an $n+1$-dimensional AdS space-time.

Normally physicists study either the gauge theory or the supergravity theory to understand bits and parts that are not quite comprehended from the other theory. The problem is that -even classically- string theory dynamics are very complicated, but in the strong field theory coupling limit -and large N-, the massive string states decouple and our theory simplifies to Type IIB supergravity [10].

We are going to look for the dynamical equations that rule our gauge theory. To simplify our problem we need to look for a set of single traced operators that decouples from the rest of operators, taking at the same time the large λ limit, in order for the massive string states to decouple and we can reduce our whole string theory to Type IIB supergravity. For the rest of this Chapter we will be working with Type IIB supergravity theory on $AdS^5 \times S^5$ as our gravity theory, large N $\mathcal{N} = 4$ super Yang-Mills (SYM) theory as our gauge theory and in the 't Hooft limit ($N \to \infty$ keeping λ fixed). Note that this is *not* the only gauge theory that couples to an AdS space-time that follows Einstein's equations -actually, there's an infinite number of them [10] - but $\mathcal{N} = 4$ SYM is the simplest one.

The dynamics of Type IIB theory on $AdS^5 \times S^5$ are given by the solutions to Einstein's equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.1}$$

where, $R_{\mu\nu}$ is the Ricci tensor, Λ is the -negative- cosmological defined as follows

$$\Lambda \equiv -\frac{n(n-1)}{2} \tag{3.2}$$

Where in this case $n = 10$. Note that the formal definition of Λ is divided by R^2_{AdS}, which is the squared radius of curvature of the AdS and we have stetted it to one for convenience. Also observe that, for hydrodynamics, the S^5 sector is irrelevant from a gauge theory point of view [10]. The *hydrodynamics* of nearly equilibrated systems at high temperature is governed by the conservation of the stress energy tensor

$$\nabla_a T^{ab} = 0 \tag{3.3}$$

Now we are ready to formally define the *fluid/gravity correspondence*. As stated before, the AdS/CFT correspondence claims that the decoupled stress tensor dynamics is governed by 3.1, then we can conclude that, in suitable high temperature and long distance regime, equation 3.1 reduces to the equations of hydrodynamics controlled by 3.3. More specifically, given any solution to the fluid dynamical equations, the correspondence explicitly yields a solution to 3.1, which turns out to be nothing else but time dependant black holes with -slowly- varying horizons. An example of this is that in the bulk, a configuration will -in time- settle down to an AdS-Schwarzschild black hole, while in the boundary any excitation will eventually thermalise.

Now that we have stated the fluid/gravity correspondence, lets dig deeper to see if we can gather more information about this relationship. In the field theory, one characterises thermal equilibrium by a choice of static frame and a temperature field, while in the gravity side we characterise the equilibrium solution with stationary black holes. In addition , the temperature and the dynamical velocity of the fluid are given by the Hawking temperature and the horizon boost velocity of the black hole respectively.

This is a good point to stop to make some remarks. Note that the *bulk* stress tensor is different to the *boundary* stress tensor T^{ab}. The bulk one, is the one that appears in the right hand side of equation 3.1 and in this case is zero, whilst the boundary stress tensor is non-zero and is closely related to the bulk metric. Also note that -as in the previous chapter- we will take the radius of the horizon to be much larger than the AdS curvature radius, i.e. $r_+ > R$, so we will be talking about large AdS-Schwarzschild black holes.

3.2 The fluid point of view

In this section we will study only the boundary stress energy tensor part of our duality when the fluid is relativistic -moves at scales of light speed c-. We will find explicitly an expression for the stress tensor and a complete set of dynamical equations, in the process we will find out that the setting proposed demands the existence of a entropy current. At the end of the section we study the small velocity perturbation of a static fluid to see that it follows the Navier-Stokes equations.

As said before, at high temperatures, all quantum field theories equilibrates into a fluid phase, in other words, it equilibrates to a phase where you don't need force to adiabatic displace two adjacent particles. This equilibrium is parametrized by the temperature $T(x)$ and the fluid velocity field $u^a(x)$, but there are two scales worth mentioning, the mean free time t_m and the mean free length ℓ_m, which are simply the average time and length scale that a particle has before it collides with another particle. Note that both $T(x)$ and $u^a(x)$ are effective variables for dynamics at length and time scales large compared to t_m and ℓ_m, and that the system will *always* equilibrate *locally* over a finite time which is of the same order as t_m [10].

The stress tensor for a relativistic fluid in a local thermal equilibrium can be formulated as a function of $T(x)$ and $u^a(x)$. If the fluid is in a n-dimensional background with metric γ_{ab}, then the expression for the stress tender is

$$T^{ab} = [P(x) + \rho(x)]u^a(x)u^b(x) + P(x)\gamma^{ab} + \Pi^{ab}(x) \tag{3.4}$$

Where $P(x) = P(T(x))$ is the pressure, $\rho(x) = \rho(T(x))$ is the energy density and $\Pi^{ab}(x)$ represents the contributions of the derivatives of $T(x)$ and $u^a(x)$. Observe that 3.3 and 3.4 constitute a complete set of dynamical equations, which will govern the behaviour of our fluid. We can expand Π^{ab} as [10]

$$\Pi^{ab} = \sum_{k=1}^{\infty} \ell_m^k \Pi_{(k)}^{ab} \tag{3.5}$$

where $\Pi_{(k)}^{ab}$ are k^{th} order derivatives of the fluid's equilibrium parameters. The *constitutive* relations are relations between two dynamical variables that are specific for the fluid we are studying, e.g. pressure and shear modulus. In general it is possible to give an expression for some constitutive relations because of the constraints the $\Pi_{(k)}^{ab}$ puts on them, for example for the pressure of our fluid, we have [10]

$$\Pi_{(1)}^{\langle ab \rangle} \equiv P_c^a P_d^b \Pi_{(1)}^{cd} - \frac{1}{d-1} P^{ab} P_{cd} \Pi_{(1)}^{cd} = -2\eta\sigma^{ab} \tag{3.6}$$

$$\frac{1}{d-1}\Pi_{(1)}^{ab} P_{ab} - \frac{\partial P}{\partial \rho}\left(u_a u_b \Pi_{(1)}^{ab}\right) = -\zeta\theta \tag{3.7}$$

Where $\langle ab \rangle$ is the symmetric part of the expression, η and ζ are the shear and the bulk viscosity [1], here we will call them the transport coefficients. Also we have

$$P_{ab} \equiv u^a u^b + \gamma^{ab} \tag{3.8}$$

However, observe that the explicit form of $\Pi^{ab}_{(k)}$ depends on the dynamics of the specific system. In addition, all equations in fluid dynamics must be invariant under redefinitions of $T(x)$ and $u^a(x)$ that shrink to the identity at an equilibrium stage, we shall refer to such equations as "field redefinition invariant" [10]. Since equations 3.4 to 4.1 are local and thermodynamical, they should follow a local form of the second law of thermodynamics, i.e. there must be an entropy -or entropy current- which should always be non-negative. Due to the thermodynamical constraints and the fact that it has to be field redefinition invariant -at least at first order- the entropy current is forced to take the form [10]

$$J^a_s = su^a - \frac{1}{T} u_b \Pi^{ab}_{(1)}, \qquad \nabla J^a_s = -\nabla_a \left(\frac{u_a}{T}\right) \Pi^{ab}_{(1)} \tag{3.9}$$

Where in the last relation was obtained by using Euler and Gibbs-Duhem relations. Because $\Pi^{\langle ab \rangle}_{(1)} \sim -\eta$ and $\Pi^{ab}_{(1)} \sim -\zeta$, it is easy to verify that in order for the entropy to be positive that $\eta \geq 0$ and $\zeta \geq 0$. It is important to clarify that the entropy current and the constitutive relations here cited are results obtained in [10] by doing calculations at first order, in addition it can be noticed that in expansions at second order the calculations get highly complicated.

Our interest does not lie in any kind of field theory but in *conformal* field theories, for which the expressions obtained above simplify greatly. Because of the tracelessness of the stress tensor in a CFT, the constraint $\zeta = 0$ is imposed [10]. Furthermore, a CFT cannot have dimensionless parameter, the dependence of physical quantities on the temperature can be deduced using-dimensional analysis, i.e.

$$P = \alpha T^d, \qquad \rho = (d-1)\alpha T^d, \qquad \eta = \eta' T^{d-1} \tag{3.10}$$

Additionally, the stress tensor should transform covariantly under Weyl trans-

[1] The definitions of these quantities can be found in [10]

formations, then 3.4 reduces -again, to first order [10]- to

$$T^{ab} = \alpha T^d \left(n u^a u^b + \gamma^{ab}\right) - \eta' T^{d-1} \sigma^{ab} \tag{3.11}$$

So far we have figured out the properties of the stress tensor of a *relativistic* fluid, which in turn governs the dynamics of the fluid. A last property that we are going to see is that this equation (3.4) reduces to the Navier-Stokes equations when the velocity is scaled down and you fine tune some other aspects.

Consider a perturbation (parametrized by ϵ) of fluid at rest, where the fluctuation in the amplitude of the velocity is of order $\mathcal{O}(\epsilon)$, the fluctuations in the temperature are of order $\mathcal{O}(\epsilon^2)$ and the wavelength fluctuations are at least of order $\mathcal{O}(1/\epsilon)$. If we take the limit where $\epsilon \to \infty$, then the fluid will have four key properties [10]. It will be *non-relativistic* and *incompressible*, the temporal energy conservation equation reduces to $\vec{\nabla} \cdot \vec{v} = 0$ where $\vec{v}$ is the non-relativistic spatial velocity.

The last property is that the *spatial* energy conservation equations reduce -al leading order $O(\epsilon^3)-$, to the Navier-Stokes equations:

$$\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} P + \nu \nabla^2 \vec{v} \tag{3.12}$$

where ν is the kinematic viscosity given by

$$\nu = \frac{\eta}{\rho_0 + P_0} \tag{3.13}$$

with ρ_0 and P_0 are the background values of the density and pressure of the fluid. One of the corollaries of equation 3.13 is that the pressure is *not* an independent degree of freedom, this is because pressure may be solved in terms of the fluid velocity at any time which can be seen by taking the divergence of 3.13 [10]. There is much to be understood about the fluids that follow the Navier-Stokes equations 3.13, hence, the importance of the fact that this equations can be seen as a reduction of the dynamics of a relativistic fluid. This is one of many reasons of why the fluid/gravity correspondence is generating expectation, in the hope that by studying certain types of black holes in AdS space times we are able to throw some light into the dynamics of non-relativistic fluids, or viceversa.

3.3 The gravitational point of view

In this section has the aim to construct gravitational solutions dual to fluids by making an algorithm to construct -slowly- varying black hole space times that solve 3.1.

Lets consider a CFT on Minkowski space ($\gamma_{ab} = \eta_{ab}$), the AdS/CFT correspondence tells us that its gravity dual is the planar AdS$_{n+1}$-Schwarzschild black hole, with metric

$$ds^2 = -r^2 f(r/T)dt^2 + \frac{dr^2}{r^2 f(r/t)} + r^2\delta_{ij}dy^i dy^j, \qquad f(x) \equiv 1 - \left(\frac{4\pi}{nx}\right)^n \tag{3.14}$$

This space-time has a horizon at $r_+ \equiv \frac{4\pi T}{n}$ and is a n-parameter family of solutions, after a boost along y^i, with parameters T and u^a. As seen in the previous section, this solution to Einstein's equations will induce on its Minkowski boundary a stress tensor which will have the form of an ideal fluid stress tensor and will have parameters spelled out by 3.10 with the constant scaling as $\alpha \sim G_N^{(n+1)}$ [10]. All the planar AdS$_{n+1}$-Schwarzschild black holes of this family of solutions are dual to a conformal (ideal) fluid living on $R^{n-1,1}$ supplied with a Minkowski metric.

This fluid is in equilibrium and one should be able to describe the long wavelength fluctuations using the hydrodynamics just studied. This class of fluctuations in the fluid translates into AdS$_{n+1}$ black holes which are inhomogeneous and dynamical.

3.3.1 The gradient expansion and the *corrected* metric

We will now describe the algorithm needed to construct black holes such as the described in the previous section, such an algorithm is obtained in [10] via a gradient expansion. An approximate solution to Einstein's equations is obtained only if u^a and T are slowly varying and satisfy the conservation equations of fluid dynamics. Take the metric 3.14 and rewrite it in ingoing coordinates, replace parameters u^a and T by functions of the boundary x^a and fix the gauge by setting $g_{rr} = 0$ and

$g_{ra} = -u_a$, then it would yield

$$ds^2 = -2u_a dx^a dr - r^2(f(r/T)u_a u_b + P_{ab})dx^a dx^b \tag{3.15}$$

We are going to call this metric $g^{(0)}_{\mu\nu}$, which is not a solution to equation 3.1 but has two peculiar attributes: it is regular for all $r > 0$ and if u^a and T have small derivatives, then it can be expanded to be locally approximated by boosted black hole solutions. The book-keeping parameter for a gradient expansion is ϵ (is useful to keep track of the derivatives with respect to the boundary coordinates x^a), and after such expansion we have this expressions

$$g_{\mu\nu} = \sum_{k=0}^{\infty} \epsilon^k g^{(k)}_{\mu\nu}, \qquad u^a = \sum_{k=0}^{\infty} \epsilon^k u^{a(k)}(\epsilon x), \qquad T = \sum_{k=0}^{\infty} \epsilon^k T^{(k)}(\epsilon x) \tag{3.16}$$

Where $g^{(k)}_{\mu\nu}$ depends on $u^a(\epsilon x)$ and $T(\epsilon x)$ and it should be determined, at the same time as $u^{a(k)}$ and $T^{(k)}$ by solving equation 3.1 to the k^{th} order in gradient expansion. What follows is to find the associated partial differential equations that solve 3.16 at the desired order, say j.

Note that -after the gauge fixing- if we expand to order n, then there are $\frac{n(n+1)}{2}$ equations but only $\frac{n(n-1)}{2}$ variables. Such behaviour can be explained in the following manner [10]. You can divide equations 3.1 in three parts, E_{rr}, E_{ra} and E_{ab}, the E_{ra} sector are momentum constraint equations for evolution in the radial directions which have only to be satisfied at a given r, while the E_{ab} equations are the dynamical equations and have to be solved in r "by slices". Finally the E_{rr} equations provides a constraint int the radial direction related to the Hamiltonian. It is natural to study the E_{ab} sector at the boundary region, where they are simply reduced to the conservation of the stress tensor equations 3.3 and since the metric $g^{(n)}_{\mu\nu}$ is not involved in those equations, it can be seen as a constraint on the solution.

Once you have the constraints at order j you can calculate the "corrected" metric at order $j + 1$, next you use this corrected metric to find the constraints at order $j + 1$ that will help you solve the partial differential equations and then again you calculate the corrected metric at order $j+2$ by solving such PDE's, and so on, and so forth. This algorithm will render you the solution to Einstein's equations at any order.

In summary, a sector of Einstein's equations actually becomes the stress tensor conservation equations and are considered constraints on our system, with such constraints you can solve the partial differential equations that gives you a corrected metric and so the space-time that was not a solution for Einstein's equations, after gradient expansion, becomes a solution -at order j- since it complies with the boundary stress tensor conservation equations. Note that even though in this case we have used obtained a result for a flat boundary metric, this procedure can be generalised to any slowly varying curved boundary metric due to locality of the perturbation theory [10].

3.4 Expansion at second order

We now present a second order expansion that in contrast with what we did in the last section it is generalised for a non-flat boundary metric. The results here shown were obtained in [10] and they use a rather different but simpler structure, the *Weyl covariant formalism*.

If we are interested in the hydrodynamics of a CFT living on a different manifold $\mathcal{B}_n$, rather that in flat manifold, then, to simplify the results for conformal fluids, we should focus on the conformal class of metrics ($\mathcal{B}_n$,$\mathcal{C}$), which has a naturally associated derivative defined with the Weyl connection. A Weyl transformation and a Weyl covariant tensor are defined as follows

$$\gamma_{ab} = e^{2\phi}\tilde{\gamma}_{ab}, \qquad \mathcal{Q} = e^{w\phi}\tilde{\mathcal{Q}} \tag{3.17}$$

Where w is the weight of the tensor. A Weyl connection is a class of torcionless connection, which captures the fact that every metric in the conformal class $\mathcal{C}$ transforms homogeneously under conformal transformations

$$\nabla_a^{\mathbf{Weyl}}\gamma_{bc} = 2\,\mathcal{A}_a\gamma_{bc} \tag{3.18}$$

Where $\mathcal{A}_a$ is the one form that characterises the connection, and which together with the connection defines the Weyl covariant derivative

$$\mathcal{D}_a = \nabla_a^{\mathbf{Weyl}} + w\mathcal{A}_a \tag{3.19}$$

This derivative will act on a conformally covariant tensor in such a way that the derivative has the same weight as the tensor its actin upon, i.e.

$$\begin{aligned}\mathcal{D}_c \mathcal{Q}^{a\dots}_{b\dots} \equiv \nabla_c + w\mathcal{Q}^{a\dots}_{b\dots} + w\mathcal{A}_c\mathcal{Q}^{a\dots}_{b\dots} \\ + (\gamma_{cd}\mathcal{A}_a - \delta^a_c\mathcal{A}_d - \delta^a_d\mathcal{A}_c)\mathcal{Q}^{b\dots}_{d\dots} + \cdots \\ - (\gamma_{cb}\mathcal{A}_d - \delta^d_c\mathcal{A}_b - \delta^d_b\mathcal{A}_c)\mathcal{Q}^{d\dots}_{a\dots} - \cdots\end{aligned} \tag{3.20}$$

The relation between the Weyl covariant derivative and the “usual” covariant derivative is given by

$$\mathcal{D}_a T^{ab} = \nabla_a T^{ab} + T^a_a \mathcal{A}^b \tag{3.21}$$

Because in a conformal fluid the stress tensor has to be traceless, then it is clear that we can rewrite 3.3 as

$$\mathcal{D}_a T^{ab} = 0 \tag{3.22}$$

We are now ready to redefine the bulk metric 3.15 in order to get a different boundary metric that Minkowski,

$$ds^2 = -2u_a(x)dx^a dr + (-2u_a(x)\mathcal{G}_b(r,x) + \mathcal{J}_{ab}(r,x))dx^a dx^b \tag{3.23}$$

where $\mathcal{G}_b(r,x)$ and $\mathcal{J}_{ab}(r,x)$ are fields that admit an expansion in the boundary derivatives. In [10; 11] they parametrize both fields at second order which contain Weyl covariant derivatives ($\mathcal{G}_b(r,x)$) or Weyl covariant basis tensors ($\mathcal{J}_{ab}(r,x)$). The Weyl Covariant basis tensors are functions of the vorticity and shear tensors and the velocity vector of the fluid

$$\begin{aligned}&\mathcal{T}_1^{ab} = 2u^c\mathcal{D}_c\sigma^{ab}, \quad \mathcal{T}_2^{ab} = C^{acbd}u_c u_d, \\ &\mathcal{T}_3^{ab} = 4\sigma^{c\langle a}\sigma_c^{b\rangle}, \quad \mathcal{T}_4^{ab} = 4\sigma^{c\langle a}w_c^{b\rangle}, \quad \mathcal{T}_5^{ab} = w^{c\langle a}w_c^{b\rangle}\end{aligned} \tag{3.24}$$

where

$$\sigma^{ab} = \mathcal{D}^{(a}u^{b)}, \qquad w^{ab} = -\mathcal{D}^{[a}u^{b]} \tag{3.25}$$

are the shear and vorticity tensors [10]. we can define our boundary metric -used to lower and raise indices of boundary tensors- as

$$\gamma_{ab} = \lim_{r\to\infty} \frac{1}{r^2}(\mathcal{J}_{ab} - 2u_{(a}\mathcal{G}_{b)}) \tag{3.26}$$

We now have a metric (3.23) that solves Einstein's equations to second order in gradient expansion, provided that 3.11 satisfies equation 3.22. Note, that since it is possible to determine the location of the black hole horizon within our gradient expansion, the metric 3.23 is regular and we then can calculate an entropy current like 3.9 for our fluid [10]. Remember, that for the planar AdS$_{n+1}$-Schwarzschild black hole, the horizon was located at $r_+ = \frac{4\pi T}{n}$. Then *locally* the gradient expanded horizon radius is

$$r_{\mathcal{H}} = \frac{4\pi T(x)}{n} + \sum_k \epsilon^k r_{(k)}(x) \tag{3.27}$$

the functions $r_{(k)}(x)$ are computed by solving the null condition

$$g^{\mu\nu}\partial_\mu S_{\mathcal{H}}\partial_\nu S_{\mathcal{H}} = 0 \tag{3.28}$$

where $S_{\mathcal{H}} = r - r_{\mathcal{H}}(x) = 0$

Knowing the event horizon of a black holes implies being able to calculate the entropy of such black hole [2]. This can be done by taking the bulk area $(n-1)$-form $a_{\mathcal{H}}$ that characterises the horizon and pulling it back to the boundary where the dual fluid lives [10]. Since a $(n-1)$ form has a 1-form dual, the area form has a dual 1-form -or current- which is the entropy current in the boundary. Because of the area law that rules black holes, we are granted the non negativity of such an entropy current which satisfies the thermodynamic's second law. A general expression for this current is

$$J_s^a = su^a + \frac{sn^2}{(4\pi T)^2}u^a\left(A_1\sigma_{cd}\sigma^{cd} + A_2 w_{cd}w^{cd} + A_3 R\right) \\ + \frac{sn^2}{(4\pi T)^2}u^a\left(B_1\mathcal{D}_c\sigma^{ac} + B_2\mathcal{D}_c w_{ac}\right) \tag{3.29}$$

Where s is the entropy density

$$s = \frac{1}{4\pi G_N^{n+1}}\left(\frac{4\pi T}{n}\right)^{n-1} \tag{3.30}$$

and the coefficients $A_{1,2,3}$, $B_{1,2}$ are defined a-priori. Note that 3.29 doesn't constraint such numerical coefficients but $\mathcal{D}_c J_s^c = 0$ and equations 3.23 do and fix completely their values, given in [10].

Given an asymptotically locally AdS$_{d+1}$ metric, one can construct a "local" boundary tensor which is conserved and is *associated* to the stress tensor of the CFT which is dual to such space-time [10]. We now want to state this association and by doing so, tie some knots between this section and the previous one. The boundary stress tensor is given by -at second order- [10]

$$T_{ab} = \lim_{r\to\infty}\frac{-r^d}{8\pi G_N^{(n-1)}}\left(K_{ab} + (n-(K+1))\gamma_{ab} - \frac{1}{n-2}\left(R^\gamma_{ab} - \frac{1}{2}R^\gamma\gamma_{ab}\right)\right) \tag{3.31}$$

Where R^γ are the curvatures of the boundary metric and K are the extrinsic curvatures of the surface, from 3.31 we can obtain explicit expressions for $\Pi^{ab}_{(1)}$ and $\Pi^{ab}_{(2)}$, namely

$$\Pi^{ab}_{(1)} = -2\eta\sigma^{ab}, \qquad \Pi^{ab}_{(2)} = \tau_\pi\eta\mathcal{T}_1^{ab} + \kappa\mathcal{T}_2^{ab} + \sum_{i=1}^{3}\lambda_i\mathcal{T}^{ab}_{i+2} \tag{3.32}$$

where the six transport coefficients for conformal in n-dimensional boundary are

$$\begin{aligned} \eta &= \frac{1}{16\pi G_N^{n+1)}}\left(\frac{4\pi T}{n}\right)^{n-1}, \qquad \tau_\pi = \frac{n}{4\pi T}\left(1+\frac{1}{n}\text{Harmonic}\left(\frac{2-n}{n}\right)\right), \\ \kappa &= \frac{n}{2\pi(n-1)}\frac{\eta}{T}, \qquad \lambda_1 = \frac{n}{8\pi}\frac{\eta}{T}, \qquad \lambda_2 = \frac{1}{2\pi}\text{Harmonic}\left(\frac{2-n}{n}\right)\frac{\eta}{T}, \end{aligned} \tag{3.33}$$

and $\lambda_3 = 0$. Here Harmonic is the harmonic number function. If we take $n = 4$ one can obtain the transport coefficients for $SU(N)$ $\mathcal{N} = 4$SYM theory. There is a bound between viscosity and entropy density [10; 12] given by $\frac{\eta}{s} \geq \frac{1}{4\pi}$ and an immediate consequence from equations 3.30 and 3.33 we see that such an inequality is saturated [10].

3.5 Different fluid flows and their gravitational duals

Conformal theories are rather special fluids and the "common" fluids deviate notably from this behaviour. Thankfully the construction presented above can be generalised for many cases. In this section we shall discuss some special and interesting cases, stopping in some of them to solve explicitly the fluid equations rather than just proving the existence of a map from the dynamics of a fluid to the dynamics of gravity.

Before going into different fluid cases, an important point but somehow detached from the aim of this section is worth mentioning. Due to the presence of a horizon, no normal modes are admitted in the black holes, but because the future horizon has to be regular, one can find perturbations/quasi normal modes -modes with complex frequencies that characterise the decay of perturbations- [10]. This modes delineate the timescale for the return of thermal equilibrium of a field theory once it has been perturbed. Asymptotically AdS black holes can have an infinite family of perturbations, specifically planar black holes admit only a certain (finite) amount of massless perturbations. These massless perturbation are called hydrodynamical modes and since they can have arbitrarily long spatial wavelengths, they fall in the long wavelength regime.

3.5.1 AdS Rotating black holes

Besides the planar AdS-Schwarzschild black hole, there is a second class of examples with explicit solutions corresponding to *stationary* configurations in hydrodynamics. This class is a bit more interesting than the past one, because it has non-trivial fluid flows. Suppose a conformal relativistic n-dimensional fluid on a spatial $\mathbf{S}^{n-1}$, then its fluid flows conserve both angular momentum and energy. Since the $\mathbf{S}^{n-1}$ has a $SO(n)$ symmetry, the number of independent parameters will be $[n/2]$. For each choice of these $d = [n/2]$ momentum and energy there will be an arbitrary fluid that will settle into an equilibrium stationary configuration. The metric of the $\mathbf{S}^{2d-1}$ will be

$$ds^2_{\mathbf{S}^{2d-1}} = \sum_{i=1}^{d} \mu_i^2 d\phi_i^2 + d\mu_i^2, \qquad \text{where} \qquad \sum_{i=1}^{d} \mu_i^2 = 1 \tag{3.34}$$

The complete metric of such field theory would be [10]

$$ds^2 = \gamma^2(-dt^2 + ds^2_{\mathbf{S}^{2d-1}}), \qquad \text{where} \qquad \gamma = \left(1 - \sum_{i=1}^{d} w_i^2 \mu_i^2\right)^{-\frac{1}{2}} \tag{3.35}$$

where w_i is the chemical potential for angular potential. According to the AdS/CFT correspondence, the dual description of a CFT on $R \times \mathbf{S}^{n-1}$ is asymptotically $\mathrm{Ad}(S)_{n+1}$. But if enough energy and angular momentum are injected to this gravitational system we would eventually reach equilibrium in the form of a large rotating black hole. If we follow the algorithm for the fluid/gravity map considered in the above sections, it should yield a metric for large rotating AdS_{n+1} black hole -to second order-. It is possible [13] to transform the metric of a rotating AdS_{n+1}

black hole into a form as 3.23, by taking the fields as

$$\mathcal{G}_a(r,x) = r\mathcal{A}_a - \mathcal{S}_{ac}u^c - \frac{r^2 u_a (4T\pi)^n}{2n^2 det[r\delta^b_c - w^b_c]}, \qquad (3.36)$$
$$\mathcal{J}_{ab}(r,x) = r^2 P_{ab} - w^c_a w_{cb}$$

Where S_{ab} is the Schouten tensor, a function of the Ricci scalar, the tensor of the bulk and the metric in the boundary. It is rather remarkable and not a generality that when expanded -at second order- this metric in derivatives the exact metric dual to the fluid flow 3.35 is recovered.

3.5.2 Non-relativistic fluids

We are interested in non-relativistic fluids, because the majority of the fluids we handle on a daily basis aren't relativistic. As stated in the previous section, the non-relativistic limit of the fluid/gravity correspondence yields the Navier-Stokes equations 3.13. The following is just a flavour of a deep and not at all understood subject, for more details the reader should address to the original paper from where the following equations where taken, namely [14].

Since the Navier-Stokes equations solve the stress energy conservation equations and are obtained in the limit of small amplitude on the velocity and the temperature perturbation, it is natural to suspect that the gravity dual of these equations will also be a small perturbation on the gravitational theory which will solve Einstein's equations -to cubic order-. Suppose we have a metric $g_{\mu\nu}$ and we perturb it with a "perturbation metric" $H_{\mu\nu}$, i.e. $\tilde{g}_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$. We can write the perturbed metric of such a gravitational theory -up to first order in derivatives- as $ds^2 = ds^2_0 + ds^2_1$ (the zeroth and the first order expansion metrics) where

$$ds^2_0 = -2u_a dx^a dr + \left(\frac{1}{b^n r^{n-2}} u_a u_b + r^2 g_{ab}\right) dx^a dx^b \qquad (3.37)$$

$$ds^2_1 = \left(-2ru_a(u^c\vec{\nabla}_c)u_b - 2br^2 rF(br)\sigma_{ab} + \frac{2}{n-1} r(\vec{\nabla}_c u^c)u^a u^b\right) dx^a dx^b \qquad (3.38)$$

with

$$F(x) = \int_x^\infty dy \, \frac{y^{n-1} - 1}{y(y^n - 1)}, \qquad \sigma_{ab} = \sigma_{ab}(u^a, \nabla_a u^a, g_{ab}), \qquad b = \frac{n}{4\pi T} \tag{3.39}$$

We can express the spatial sector of the metric -g_{ij}- as follows

$$b_0 r^2 F(b_0 r)(\vec{\nabla}_i v_j + \vec{\nabla}_j v_i) + \frac{1}{r^{n-2}}(A_i + v_i)(A_j + v_j) \tag{3.40}$$

Where $A_i = H_{0i}$ is part of the perturbing metric and v^i are the spatial components of the velocity u^a . The first part of 3.40 is the dual of the viscous part of the Navier-Stokes equations 3.13, whilst the second factor is dual to the convective sector of the 3.13. Thus, we see that even in the non-relativistic limit the duality holds.

3.5.3 Non-conformal fluids

Until this point all the cases we have examined, even the Navier-Stokes non-relativistic limit have been all conformal field theories. When exploring *some* non-conformal field theories there is an advantage of the AdS/CFT correspondence that we can one in our favour, in theories that naturally arise on the world volume of Dp-branes, one finds classes of non-conformal fluids [10].

Suppose we are analysing the M-Theory -11dimensional supergravity theory -. In such a theory the near horizon geometry of the M5 brane is $\text{AdS}_7 \times \mathbf{S}^4$, and Einstein's equations 3.1 in 11 dimensions, accept a cut shot to 7-dimensional equations [10]. If we compactify M-theory on an $\mathbf{S}^1$ we get to a 10-dimensional supergravity theory known as Type IIA Theory. If the M5 brane is wrapped the $\mathbf{S}^1$, then it's dimensional reduction is the D4-brane. Also when we compactify, the 7-dimensional Einstein equations truncate even further to a -consistent- 6-dimensional set of equations known as the Einstein-dilaton equations of the D4-brane background. It turns out that the fluid dynamics dual to fluctuations on the thermal M5 brane solution is the solution computed above when $n = 6$.

A very interesting fact is that the fluid dynamics of the D4-brane world volume theory is a dimensional reduction of the conformal fluid dynamics on the world volume of an M5 brane, but note that the dimensional reduction of a conformal theory is non-conformal [10]. Also the gravitational duals to the flows on the D4-brane world volume are obtained from the Kaluza-Klein reduction of the results obtained in Section 3.4. These features generalise to Dp-branes for all p, i.e. you can always find an Einstein-dilaton system that can be thought of as a dimensional reduction of the -negative cosmological constant- Einstein's equations in a higher dimension [10].

3.5.4 Charged fluids

Due to the fact that most of the "common" fluids carry one or more conserved charges, it makes sense to study a CFT with a global symmetry which has a gravitational dual that is a gauge symmetry in the bulk. This leads us to think that the dual of charged AdS black holes are charged fluids. In fact, if one performs a gradient expansion -as done in Section 3.4 - in a charged fluid stress energy tensor conservation equations, an analogous algorithm to that presented in Section 3.3 constructs the gravitational solutions to Einstien-Maxwell-Chern-Simons theory dual to the charged fluid flows order by order [10].

Together with equation 3.3 a charged fluid has a charge current conservation equation, namely

$$\nabla_a J^a = 0 \tag{3.41}$$

As done for the stress tensor and for the entropy current in Section 3.1, we have to perturb also the charge current since it is a conserved quantity, i.e.

$$J^a = qu^a + J^a_{diss} \tag{3.42}$$

Where J^a_{diss} represents the contribution of the derivatives of $T(x)$ and $u^a(x)$ to the charge current, then -as before 3.5- we gradient expand the charge current

$$J^a_{diss} = \sum_{k=1}^{\infty} \ell_m^k j^a_{(k)} \tag{3.43}$$

and so the constitutive relations are 3.6 in addition to

$$P^a_c\left(j^c_{(1)}+\frac{q}{\rho+P}(u_b\Pi^{ab}_{(1)})\right)=\kappa V^a_1,\quad \text{where}\quad V^a_1\equiv -P^{ab}\nabla_b\frac{\mu}{T}+\frac{F^{ab}u_b}{T} \tag{3.44}$$

$$\text{and}\quad \frac{1}{n-1}\Pi^{ab}_{(1)}P_{ab}-\frac{\partial P}{\partial\rho}(u_au_b\Pi^{ab}_{(1)}+\frac{\partial P}{\partial q}(u_aj^a_{(1)})=-\beta\delta_cu^c \tag{3.45}$$

Where μ is the chemical potential of the fluid, and F^{ab} is the background electromagnetic field that couples to J^a in 3.41. As before, the non negativity of the entropy current,

$$J^a_s=su^a-\frac{1}{T}u_b\Pi^{ab}_{(1)}-\frac{\mu}{T}j^a_{(1)} \tag{3.46}$$

then

$$\nabla_aJ^a_s=-\nabla_a\frac{u_b}{T}\Pi^{ab}_{(1)}-\left(\nabla_a\frac{\mu}{T}-\frac{F_{ab}u^b}{T}\right)j^a_{(1)} \tag{3.47}$$

implies that η,κ, and $\beta\geq 0$. Further research ([10]) revealed that the actual constitutive relations of the fluid dual to AdS-Einstien-Maxwell-Chern-Simons are different that 3.44 and 3.45. This is because when a theory is anomalous -as the Chern-Simons theory is- there should be an additional term in 3.47 that accounts for such anomaly. One could think that this compromises the non negativity of the entropy current but corrections to 3.44 and 3.45 due to the anomaly ensures the positivity of the current.

3.5.5 Superfluids

When a charged scalar field, interacts with a charged asymptotically AdS_5 black hole, it makes the black hole unstable which's endpoint is the hairy black hole -a black hole immersed in a charged scalars condensate. The CFT dual of a hairy black hole is a phase in which there is a spontaneous breaking of a global $U(1)$ symmetry by the presence of an expectation value of a charged scalar [10; 15]. In condensed matter this is known as a superfluid.

The variables of a relativistic superfluid are: a normal fluid velocity u^a, a superfluid velocity u_s^a, the temperature and the chemical potential. The fact that ξ_a -the gradient of the phase condensate- is curl free, together with the stress tensor and share current conservation equations are the equations of a superfluid's dynamics and in conjunction with the respective constitutive relations form a closed dynamical system. Note that the superfluid velocity u_s^a is the unit vector in the direction of $-\xi_a$ and that the phase of the condensed is an "analogue" to a Goldstone modes, since it is the result of spontaneous symmetry breaking.

In [15] it has been proved that it is possible to apply the fluid/gravity correspondence for hairy black holes and get the holographic superfluid. They have found solutions to a 4-dimensional gravity theory which is dual to holographic superfluids with a super current density and shown that the dynamics of such superfluid is dictated by the relativistic Landau-Tiszca two fluid model. Here, we will merely write down the results obtained in that paper, for o much more detailed description of the construction of such equations the reader should address [15].

We are interested in black holes with flat horizon sections, then the metric will have the form

$$ds_0^2 = -2hu_a dx^a dr - \frac{r^2}{\ell^2}(fu_a u_b - \Delta_{ab})dx^a dx^b \tag{3.48}$$

where, h and f are bulk functions that will define the holographic superfluid. Also

$$\Delta_{ab} = u_a u_b + \lim_{r\to\infty} \frac{\ell^2}{r^2}\gamma_{ab} \tag{3.49}$$

with γ_{ab} is the boundary's metric. After "turning" the super current on, the modified metric will be

$$ds^2 = ds_0^2 + \frac{r^2}{\ell^2}(2\mathcal{C}u_{(a}n_{b)} - \mathcal{B}n_a n_b)dx^a dx^b + \frac{2\mathcal{C}h}{f}n_a dx^a dr \tag{3.50}$$

where n_a is a constant vector of unit magnitude that obeys

$$n_a u_b \lim_{r\to\infty} \frac{\ell^2}{r^2}\gamma^{ab} = 0 \tag{3.51}$$

and $\mathcal{C}$ and $\mathcal{B}$ are functions are bulk functions that depend on r only. Using 3.50, setting the constraints on the boundary stress tensor, [15] was able to find the stress energy tensor and the super current that describes the dynamics of the holographic superfluid, namely

$$\begin{aligned} T_{ab} &= (\epsilon + P)u_a u_b + P\eta_{ab} - \mathcal{B}^{(3)} n_a n_b + 2\mathcal{C}^{(3)} u_{(a} n_{b)} \\ J_a &= \rho u_a - J_s n_a \end{aligned} \tag{3.52}$$

where ρ is the fluid density.

3.5.6 Fluids with a confinement phase

This subsection was placed until the very end of this Chapter because it contains interesting features with will be central in the following chapters. There are various field theories that have a confinement phase at a finite temperature and they can be distinguished form one to another because -in general- they have different geometries -as seen in the previous chapter-. The AdS/CFT correspondence leads us to think that we can find a different gravitational dual to each of these theories. Furthermore the fluid/gravitiy correspondence gives us the relation between the dynamics of each of the field theories and it's dual and that instabilities in one theory lead to instabilities in is dual.

As we saw in Chapter 2 when a we take a field theory at high enough temperatures it will be in a deconfinement phase, and so far in this Chapter we have only considered theories which are "deconfined" at all temperatures. As we did previously , if we consider a pure Yang Mills which has a first order phase transition at finite temperature and is the deconfinement is dual to a black hole only above the deconfinement temperature. The low temperature phase is given by glue balls and indistinguishable from the vacuum, has free energy of order one and therefore is in the confinement phase.

The Scherk-Schwarz reduction -antiperiodic boundary conditions for fermions- of $\mathcal{N} = 4$ Yang Mills on a circle of radius R undergoes a deconfinement phase transition when $TR = 2\pi$, this theory at high temperatures -much higher that the transition phase- the effective 3-dimensional theory is a dimensional reduction of the

4-dimensional conformal field theory. Its gravitational match at high temperatures is the $\mathbf{S}^1$ compactification of the AdS_5 black hole, but at *low temperatures* its dual is the AdS-soliton [10].

So, something new happened here, from a gravitational point of view there are two possible solutions at the transition temperature, and as we will see in future chapters, there is a domain wall between the AdS-soliton and the compactification and the AdS_5 black hole. such intermediate solution -the domain wall- has been constructed numerically in [16]. From the CFT point of view this translates as a fluid with a boundary and can be parametrized by the surface tension of the boundary -if it is an ideal fluid-. There are solutions to the fluid equations and with the right boundary conditions which are stationary called plasma-balls and plasma-rings, which have duals as black holes and black rings among others [16; 17] and we shall study them in detail later.

One of this configurations is the fluid tube which is a two-dimensional analogue of the three-dimensional cylindrical tube of fluid, which undergoes a dynamical instability called the Rayleigh instability. At the same time, the dual of the fluid tube, the infinite black string in 5 dimension also has a well known instability called the Gregory-Laflamme instability, which is said to be dual to the Rayleigh instability [18]. This will be the main topic of Chapter 5.

Chapter 4

Matching the Plasma-Ball and the Black Hole

In the first chapter we have already seen that a field theory can be in the confinement or deconfinement phase, depending on its temperature and that, thanks to the AdS/CFT correspondence, this theories have their own gravity duals, depending on which field theory we are studying. In this chapter we will study large N gauge theories with a confining phase and propose it's gravitational dual, that will turn out to be static black holes with a domain wall.

4.1 Large N gauge theories and plasma-balls

In the first sections of Chapter 1, we superficially explained the reasons and consequences of having a gauge theory in the confinement or deconfinement phase. We now proceed to do a much formal review on the subject. Recall that it is believed that large $NSU(N)$ gauge theories are dual to string theories with $g_s \,\alpha\, 1/N$ and that the duals of the strings can be in different states, each of which we shall call glue balls.

Consider a large N gauge theory, then as seen in the first chapter, to agree with the AdS/CFT correspondence, it *must* have a mass gap Λ_{gap} at high temperatures with a first order phase transition -from confined to deconfined- and with long lived excitations. This configuration is a spherical homogenous bulge of deconfined plasma fluid, it's energy density is just above the density -called the critical energy density- at the deconfinement temperature (the temperature mentioned in the first chapter at which the deconfinement starts), in other words, our configuration is just above the energy necessary to deconfine, we call to this configuration the *plasma-ball*, it is static because the pressure of the plasma recedes at the critical energy density and most importantly it contains the confined and deconfined phases at the same time.

However, these plasma balls are meta-stable, which means that they will eventually decay into a gas of hadrons [16]. As seen before, the energy density of the deconfinement phase of a gauge theory is of order N^2 and therefore $1/N^2$ of all the glue balls formed in the gluon-gluon collisions can escape the plasma ball, causing it's -slow- decay. The thermal nature of our configuration indicates that the decay rate obeys the Boltzmann factor, yielding a lifetime for a plasma ball of order $N^2 R$ where R is the radius of the plasma ball and it complies that $R \gg 1/\Lambda_{gap}$ [16].

4.1.1 The Plasma ball as a phase of the gauge theory

We said that the plasma ball is a phase of some large N gauge theories, which is a static, meta stable lump of plasma fluid, but what constraints does this system have? In this subsection we will try to clarify this point.

Suppose a p-dimensional spherical ball of plasma fluid of radius $R \gg 1/\Lambda_{gap}$ with uniform bulk pressure and temperature. In order for it to be static, the surface tension of the 'domain wall' that separates the plasma ball from the gas of hadrons must be balanced with the pressure of the plasma in the ball, i.e.

$$P = \frac{(p-1)\Sigma}{R} \tag{4.1}$$

with

$$\Sigma = \frac{1}{p-1}\int_0^\infty dr(g^{ij}T_{ij} - pT_{rr}) \tag{4.2}$$

Where Σ is the surface tension, T is the stress energy tensor, g is the metric and i, j are summed over all spatial coordinates. Since the ball is in the deconfined phase, both its pressure and the surface tension are both of order N^2, this is due to the fact that the pressure is just minus the free energy. From equation 4.1 we can infer that static plasma balls at large R can only exist if the pressure of the deconfined phase vanishes at finite temperature.

There are some important thermodynamical considerations to ensure the plasma ball is static [16]: the pressure increases with temperature (if the specific heat is positive), it is continuous throughout the phase transition and it is of order one at the deconfinement temperature T_d. Since it is a first order transition, the specific heat, hence dP/dT, are positive at T_d and of order N^2, and so, the pressure of the plasma ball vanishes at $T = T_d - \mathcal{O}(1/N^2)$.

We can then conclude that, in large N gauge theories with a first order transition, the uniform plasma balls are static at the temperature T_d. The large plasma balls can also live at temperatures slightly higher than T_d, provided that the surface tension is positive and will have a slightly greater energy density than the critical one. Observe that 4.1 says that the pressure with positive surface tensions is a decreasing function of its radius, and therefore will have negative specific heat. This is not a contradiction with the stated in the previous paragraph, since this is the specific heat of the plasma ball as an object and the previous was the specific heat per unit volume of the deconfined phase, of which the plasma ball is composed.

In addition to being static, plasma balls are also hydrodynamically stable. Since we have assumed the transition to be of first order, this means that the deconfinement phase remains even at temperatures lower than T_d. Then the pressure at temperatures higher that T_d is positive and negative when the temperature drops below the critical temperature, making the plasma balls hydrodynamically stable, i.e. stable against contractions and expansions.

So far we have only considered plasma balls that don't have conserved charges, but the generalisation to stationary plasma balls is simple and will be reminded towards the end of this chapter. Consider a spherically symmetric bulge of plasma rotating at angular velocity w, then the force balance implies that

$$\frac{dP}{dr} = \rho(r) w^2 r \tag{4.3}$$

Where $P(r)$ and $\rho(r)$ are the pressure and density at r and the velocity w is non-relativistic. Using the equation of state for $P(r)$ we obtain

$$\int_{P(r)}^{\Sigma/R} \frac{dP}{\rho(P)} = \frac{w^2}{2}(R^2 - r^2) \tag{4.4}$$

This equation may have two distinct classes of solutions: a solution where the variable r has a range $(0, R)$ which may be thought of as a rotating plasma ball and a solution where r has the range (R_1, R) which can be imagined as a rotating plasma ring. In this dissertation we will focus on the first kind of solutions, if the reader wishes to learn about the rotating plasma rings and their duals, he should refer to [17].

4.1.2 The decay of the plasma balls

In this section we will determine the dependence of the decay -hadronization- of the plasma ball on N (using [16] as reference), such hadronization happens when a glue ball escapes from the plasma ball. We will model the decay of a gas of gluons which don't interact with each other unless they collide, this collisions can sometimes produce glue balls. To determine the dependence on N we first need to understand this dependence form a point of view of Feynman diagrams that produce a glue ball.

Consider a Feynman graph that has n initial gluons scattering into m gluons and k glue balls. This graph will the have $n + m$ external gluon lines and will include k insertions of glue ball creation operators, each of the insertion of this operators will give add an extra factor $1/N$ compared to an insertion of a normal interaction vertex. Suppose this Feynman graph and its CPT conjugate, if we 'sew' this two graphs together (each free gluon line together with its corresponding CPT conjugate) we will get a Feynman diagram such as 4.1

This graph has no free gluons, then the dependence on N of the contribution of

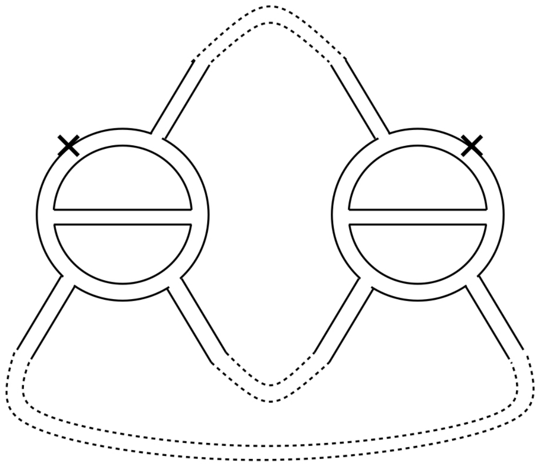

Figure 4.1: Feynman diagram of colliding gluons, using a double line notation. Taken from [16]

this graph to a glue ball production can be obtained by adding over all indices and it is proportional to

$$N^{2-2g-2k} \tag{4.5}$$

The leading behaviour of the production of glue balls is driven by the planar Feynman diagrams ($g = 0$) which is the type of diagram drawn in 4.1, then

$$N^{2-2k} \tag{4.6}$$

We now study what happens for different cases of k

$k = 0$, This value of k means that there is no glue ball production and therefore it will tell us about the interaction of gluons and how often they collide. According to 4.6, the number of gluon-gluon collisions in a plasma ball of volume V is of order is of order N^2, then the rate at which a gluon undergoes a collition is of order N^0. Therefore the time scale of a plasma fluid's relaxation is N^0.

$k = 1$, With $k = 1$, equation 4.6 will tell us how often is a glue ball produced. There are two mechanisms of how a glue ball can be produced and radiated: the first is when two gluons with the same -but alternated- indices collide *near* the surface, form a glue ball and then this shoot out form the plasma ball.

The second one consists on a really energetic gluon which has a great velocity on the radial direction of our plasma ball, enough to go through the surface of the ball, at that point the string that attaches the gluon to the plasma ball breaks, letting the gluon scape as a glue ball. The glue ball creation is dominated [16] -at large N- is of order one according to 4.6.

It is important to note, that as well as a glue ball can escape from a plasma ball, an external glue ball can get into the plasma ball, dissolve inside of it and obey 4.6. If a glue ball is produced far away from the surface of the plasma ball will disintegrate before it reaches the surface, then the decay rate is not proportional to the volume of the plasma ball but to its surface area.

$k = 2$, This value of k will have a term that indicates the interaction between two -already formed- glue balls, as well as a term that describes the collisions of such glue balls with the gluons in the plasma ball. These collisions will eventually slow down the glue ball and deflect it's path and are interactions of order $1/N^2$ [16].

4.2 Localised Black Holes

Once we have discussed the physics of the plasma balls, we now start to study the physics of their gravitational dual, the localised black holes. To this end let's first consider Einstein's equations with coupled matter fields, i.e. a metric like

$$ds^2 = \alpha' L^2 (W^2(u) dx_\mu^2 + ds_{int}^2) \tag{4.7}$$

Where $\mu = 0, 1, \cdots, p$, ds_{int}^2 is the metric of an internal manifold one of which's coordinates is u (compact and with range $u_0 < u < \infty$) and $\alpha' L^2$ is a constant. In the rest of this chapter we will make some assumptions: the metric 4.7 is smooth and without boundaries, including $u = u_0$, and that the space-time we work in has a well defined thermodynamics and the fluctuations about the metric is gapped, i.e. it can include confined backgrounds.

In addition we expect 4.7 to have another *stable* solution -when its energy density is above the critical ρ_c-, called a black brane, which has finite energy. If we call T_d to the temperature of the black brane with energy density ρ_c, then the graviton gas 4.7 dominates for $T < T_d$, the black brane dominates for $T > T_d$ and the system will undergo a first order phase transition for $T = T_d$ (assuming only these two phases exist) [16].

The spaces that obey the conditions stated above, have a family of - p-dimensional spherical- black hole solutions which are completely defined by their mass, and in the large mass limit will have the following attributes: their volume in p dimensions is proportional to their mass, i.e. $r \sim (m/\rho_c)^{1/p}$ [16], away from the edges of the R^p (in the interior) these solutions approximate the black brane at energy density ρ_c and near the edges of the R^p the solutions reduce to a domain wall in the R^p directions that mediates between the critical black brane and the graviton gas.

4.2.1 Black holes in warped backgrounds

At low energies, the ten-dimensional Schwarzschild solution black hole of radius $R_s = \sqrt{\alpha'}L$ is a solution for 4.7, but at higher energies, the story goes a bit different. It is a natural step to assume that a black hole with a very large energy (not infinite as before) is a large spherical bulge in R^p which in the interior closely resembles the translationally invariant black brane. Then taking the energy to infinity imposes the same three conditions on the black hole as it did for the black brane [16] stated in the above paragraph and we shall call to this configuration the *localised* black hole , this can be better depicted as it is in figure 4.2.

In the previous section we saw that the plasma ball can exist at different temperatures but that is a static meta-stable phase if and only if the the force between the pressure and the surface tension on the domain wall are precisely balanced. In this case, the same must happen if we want the homogeneous black brane to be static, i.e. the pressure of the domain wall of the black brane solution must vanish at large radius and this happened precisely at the deconfinement temperature T_d. Note that these arguments apply since it is a warped background that contains a well defined boundary stress tensor. For these reasons, we expect that large black

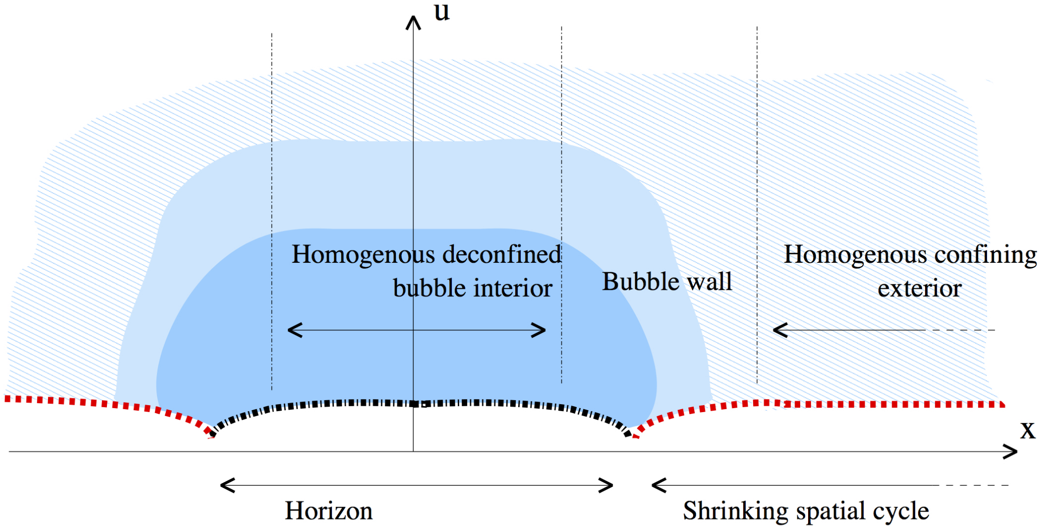

Figure 4.2: The localised black holes also referred as the "pancake" model. The vertical axis is the radial coordinate and the horizontal is any of the other spatial coordinates. Taken from [16]

holes will reduce -at the interior- to a homogeneous black brane at the deconfinement temperature.

It's is a good moment to do a few notes and remarks about the above. We anticipate that the width of the domain wall is of order of the mass gap Λ_{gap}. Warped backgrounds of the form seen here always host a pancake-shaped black hole of the form of figure 4.2. The temperature T_d of the black holes here described is finite, even in the large mass limit. Additionally, if we assume the surface tension as positive, the black hole has negative specific heat, whilst if the tension was negative, we would expect it to have a instability of the type Gregory-Laflamme [19] which is the centre of next chapter.

As we did in for the plasma ball, we can also generalise the black hole solution for solutions that host a non vanishing angular momentum -or other charges-. This could be rotation black holes or black rings, with the purpose of keeping this dissertation not so long, the reader is referred to [20] for rotating black holes and to [21] for black ring further explanations.

4.2.2 A specific case

Since in the past chapters we have talked about space times with a Scherk-Schwarz compactifications, we now -very palely- construct the solutions to Einstein's equations that also comply with the constrictions we need on our space and we do so taking [16] as a reference.

Suppose a space-time warped in such a way that it's metric looks like

$$ds^2 = \alpha' L^2 \left(e^{2u}(-dt^2 + T_{2\pi}(u)d\theta^2 + dw_i^2) + \frac{1}{T_{2\pi}(u)}du^2 \right) \tag{4.8}$$

with

$$T_x(u) = 1 - \left(\frac{x}{4\pi}(d+1)e^u \right)^{-(d+1)} \tag{4.9}$$

and $i = 1, \cdots, d-1$, $\theta \equiv \theta + 2\pi$. This metric is known as the *AdS Soliton* and is a solution $d+2$-dimensional Einstein's equations with a cosmological constant

$$R_{\mu\nu} = -\frac{d+1}{L^2\alpha'} g_{\mu\nu} \tag{4.10}$$

It can also be thought as an AdS_{d+2} with a Scherk-Schwarz compactification on a a circle. Recall that a Scherk-Schwarz compactification is just an extremely special kind of compactification of one dimension in which the spacetime fermions are chosen antiperiodic along one circle of the compactification manifold [16]. So, the metric 4.8 at large u reduces to AdS_{d+2} in Poincaré-patch coordinates, taking u as a radial coordinate and with θ compactified in a circle, this can be easily seen in this limit, $T_x(u) \simeq 1$. Note that this metric has a cutoff of the IR region, since at finite u, the θ circle shrinks to zero size.

If we change to Euclidean coordinates, compactifying the time $\tau \equiv \tau + \beta$ on the metric 4.8 (after Euclidean continuation), we get the solution known as the *thermal gas* solution, a thermal gas of gravitons at temperature β^{-1}, i.e.

$$ds^2 = \alpha' L^2 \left(e^{2u}(d\tau^2 + T_{2\pi}(u)d\theta^2 + dw_i^2) + \frac{1}{T_{2\pi}(u)}du^2 \right) \tag{4.11}$$

Another solution to the Euclidean space is the *black brane* at temperature β^{-1}, and can be obtained by doing a continuation to Lorentzian space, where the metric has a horizon, namely

$$ds^2 = \alpha' L^2 \left(e^{2u}(T_\beta(u) d\tau^2 + d\theta^2 + dw_i^2) + \frac{1}{T_\beta(u)} du^2 \right) \tag{4.12}$$

It is obvious that if we take the temperature of the black brane to be $\beta = 2\pi$ it has the same free energy as the thermal gas 4.11, since we can identify the τ circle with the θ circle.

In [16] they calculate the free energy at every temperature and find that the thermal gas is thermodynamically preferred for $\beta > 2\pi$ whilst the black brane has lower free energy for higher temperatures and that -as anticipated- the pressure of the black brane vanishes at the phase transition temperature $T_d = 1/2\pi$. In that same paper, they show that the thickness of the brane -at T_d- is of order one.

One of the principal attributes of a metric like 4.8 is that it is a simple warped background that appears when some string theories are compactified (by 'appears' we mean that there are solutions of string theory of the form $M \times$4.8). This is of notable importance, since these string theories are dual to $d+1$dimensional CFT's with one dimension compactified on a Scherk-Schwarz circle. Related to what we saw in Chapter 1, when we make $d = 3$ and take $M = S^5$ is $\mathcal{N} = 4$ supersymmetric Yang Mills theory -with a Scherk-Schwarz compactification-.

So far, we have talked about the black brane and the graviton gas metrics, but we haven't said anything about the domain wall. As seen in the previous sections we need a solution that interpolates between 4.11 and 4.12 at the deconfinement temperature. When we take the energy and the radius of this lumps to be very large, the edge should resemble a domain wall joining these to solutions. To date, there has not been found an analytic solution to this problem, and all progress done so far has been numerical [16].

Specifically we are searching for a solution to 4.10 that has translational and rotational symmetry in $(d-2)$ spatial dimensions w_i which we call r_a $(a = 1, \cdots, (d-2))$ and a translational symmetry as well as a reflection symmetry in the θ and τ

directions. This can be done in with a metric like

$$ds^2 = A^2 d\tau^2 + B^2 d\theta^2 + e^{2C} dr_a^2 + e^{2D}(dx^2 + dy^2) \tag{4.13}$$

where x and y are combinations of the direction of the domain wall and the radial coordinate in the bulk space. We should tweak the functions A, B, C, D -functions of x and y- to make 4.13 interpolate between the black brane and the thermal gas. Note the the actual radii of θ and τ are not of physical relevance. Much more details and formal explanations can be found in [16]. At large x and y the geometry should tend to one of the homogeneous solutions 4.11 and 4.12 when the appropriate circle shrinks (figure 4.3).

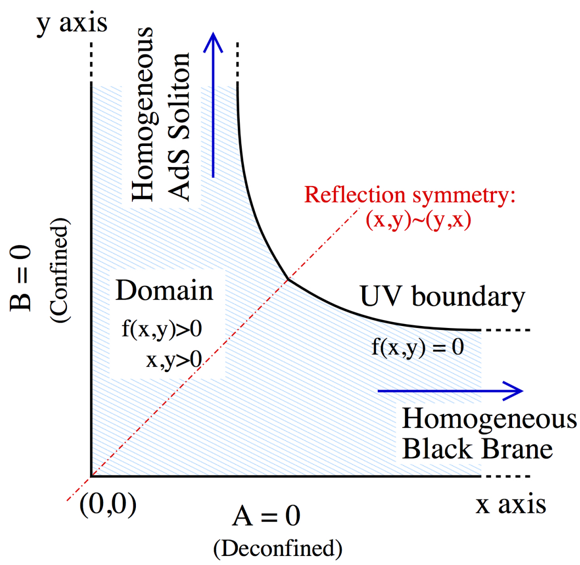

Figure 4.3: x, y plane sketching the domain wall situation. Taken from [16]

If we concentrate on as small enough region near the origin, we can ignore the bulk cosmological constant. Then the metric 4.13 is simply a $(d+2)$-dimensional flat space written as a product of four-dimensional flat space (with coordinates A, B, τ and θ) and the flat $(d-2)$-dimensional space (with coordinates r_a), namely

$$ds^2 = \left[A^2 d\tau^2 + B^2 d\theta^2 + dA^2 + dB^2\right] + e^{2C} dr_a^2 + \mathcal{O}(A^2, B^2) \tag{4.14}$$

This is the form of the metric of our domain wall.

4.3 The Plasma-Ball/Localised Black Hole duality

The AdS/CFT correspondence tells us, as seen in the Introduction, that there are several confined gauge theories that have a gravitational duals, such as the one treated in the Section 5.1 and according to this correspondence we know that the duals will be localised black brane solutions which approximate the deconfined black brane at it's centre [16]. Therefore, the plasma balls-localised lumps of plasma at T_d- are dual to the localised black holes referred in the past section.

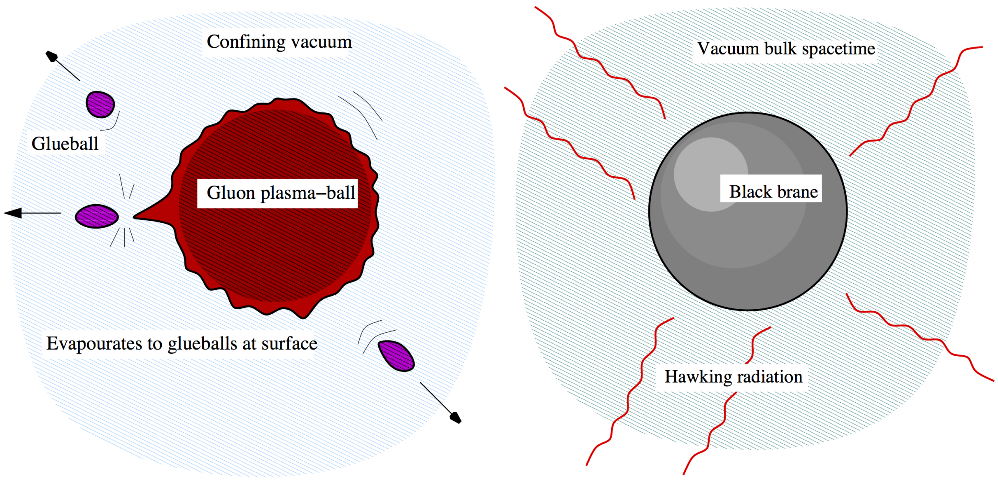

Figure 4.4: The plasma-ball and its decay via hadronization and its dual, the localised black hole and its decay via Hawking radiation. Taken from [16]

Let us explain this duality more explicitly rather than just state it. Suppose a plasma ball that is scattered by a glue ball, the glue ball can then be either reflected, absorbed or transmitted along the surface of the plasma ball. The gravitational counterpart of this is easy to imagine, consider a localised black hole in a p-dimensional space and the scattering of the black hole and a graviton in certain state. The wave function of the scattering has four components: the incident part, the transmitted part, the absorbed part (into the black hole) and the reflected part. We can then conclude that the gravitons outside the horizon of the black brane sector map to glue balls inside the plasma ball, whilst gravitons near the horizon map to gluons -which are deconfined- [16].

Now, at finite -but large- N, the lumps of gluon plasma overlap with the glue balls, this explains the decay of plasma-balls by hadronization. This can be seen from a gravitational point of view as the fact that at finite g_s the gravitons and the black holes mix and therefore the non-extremal black hole decays due to Hawking radiation.

4.3.1 From gravity to the field theory

In this subsection we will see two aspects of plasma balls that are better understood when studied from their gravitational counterpart: the hadronization of a plasma ball viewed as Hawking radiation of the localised black hole and the production plasma balls as collisions of gravitons.

As seen in Section 4.1, a plasma ball can decay via hadronization, which means that hadrons -glueballs- escape form the ball. From a gauge theory point of view, this process is very difficult to study even numerically, therefore we appeal to the large λ AdS/CFT dual of the plasma ball decay: the Hawking radiation.

Classically a localised black hole is a stable solution to Einstein's equations, but quantum mechanically it decays via Hawking radiation [16]. This radiation from the "flat" part of the pancake solution (figure 4.2) which goes in the radial direction is simply bounced back into the black hole, in other words, there is no radiation in this direction, it only loses energy form Hawking radiation at the edge. In the same manner, when a glue ball is created via the collision of gluons in the bulk it is dissolved before it can escape out of the plasma ball, i.e. it only decay via the glue balls produced in a collision near the surface of the plasma ball [16].

So far, in this chapter we have only talked of the geometry, the physics and specially evolution of the plasma ball but we have not mentioned how the actually create, the most simple way of creating this plasma balls is via hadron-hadron collisions.

Consider a collision of two glue balls with centre of mass energy large compared to $N^2\Lambda_{gap}$. In the large λ string theory these glue balls map to gravitons, hence we shall study the collision of two light gravitons because this phenomena is much better understood than the dual glue ball collision. It is easy to set an upper bound of the scattering cross section, by assuming that the gravitational force dominate at high energies [16]. The force between the two gravitons will ten be of the form

$$\frac{G_4 E^2}{b^2} e^{-\Lambda_{gap} b} \tag{4.15}$$

Where b is an impact parameter, which is the separation between the particles colliding. Then, the incident particles would pass each other if b is larger than a number of order $\ln{(E)}/\Lambda_{gap}$ with a cross section of the form

$$\sigma \sim \frac{\ln^2 E}{\Lambda_{gap}^2} \tag{4.16}$$

With this knowledge and the AdS/CFT duality, this implies that at large N and λ, two glue balls will most likely (with probability close to one when $E \gg N^2 \Lambda_{gap}$) merge into a plasma ball if the impact parameter is smaller than $\ln{(E)}/\Lambda_{gap}$ [16]. Such plasma ball will have a very long lifetime and will decay as pointed out throughout this chapter.

At small λ, the glue balls will be formed by partons, each of which carries a good part of the energy of the glue ball. In the large N limit, the string that makes the gluons stick together cannot be broken, so one may think that when two partons collide they would form a plasma ball. As it turns out ([16]), since the smallest energy to form a plasma ball is of order N^2, then the partons actually carry enough energy to snap the string and do not form a plasma ball. So, in this limit the very energetic partons interact very weakly and their interactions will not form a plasma.

4.3.2 From the field theory to gravity

In contrast with what the previous subsections talks, we now invert the order, meaningly, we go from the things we know about in the plasma ball sector and try to understand some obscure parts about the localised black holes.

As the localised black holes (figure 4.2) at large λ starts to lose some energy due to radiation, they start shrinking in size until they reach a size of the same scale as the length of -negative- curvature of the space-time of the form 4.7. At this point they localise on the internal manifold by a transition of the type Gregory-Laflamme. If they still continue to radiate, they keep on shrinking until their size is much smaller than the scale of the background curvature, and in that moment, it would resemble the simple ten-dimensional Schwarzschild solution [16] and it's evolution is the same as the black holes in a flat ten-dimensional space.

This ever contracting black holes have duals as plasma balls in large λ gauge theories that are hadronizing. Provided that the black hole further keeps radiating and loosing energy, they would eventually contract to string scale, here the black hole's dual would be a small plasma ball [16]. Here is were the plasma balls and the AdS/CFT correspondence would -and probably will- play a very important role, as this part of the gravity sector is not well understood.

One of the most straight forward consequences of the existence of black holes, is that any particle shot into a black hole is always absorbed regardless of its energy. But if we look at it's dual, at first sight it looks like a particle incident on the plasma ball at an energy much larger than T_d would just go across it, challenging it's AdS/CFT dual.

What happens is that we are not taking into account one important fact. The only objects one can shoot to a plasma ball are glue balls. At large λ glue balls may be thought as consisting of many low energy partons [16], where the energy of each parton is given by

$$E_{parton} \approx E \left(\frac{\Lambda_{gap}}{E} \right)^{\lambda} \tag{4.17}$$

Observe that even in the limit where $E \to \infty$, $E_{part} \ll \Lambda_{gap}$, therefore glue balls may always be absorbed by plasma balls, regardless its energy. As explained before, at small λ the glue balls can be thought as consisting of a small number of high energetic partons, and we could anticipate that they would pass throughout the plasma ball, then the plasma balls at low λ cannot be dual to a black hole, since it does not absorb all glue balls that are shot at them [16]. An interesting point in this subject is if this can be translated into a decrease in the absorption of particles of a black hole when it's curvatures are of string scale order.

Finally, there's a subtlety that we have ignored, but that has been a constant question that physicist have failed to answer. The fact that at quantum mechanical level Hawking's radiation is present in the black holes, is something that is called the information paradox, since the fact of the existence of Hawking's radiations breaks unitarity, one of the most important restriction that mathematics puts on physics as a price for using it's commodities.

Let's explain this a bit further. The AdS/CFT correspondence checks that the black hole evaporation is a unitary process, this is because the production of a plasma ball in a gluball-glueball collision and its hadronization is clearly a unitary process. So even though Hawking's radiation dual predicts unitarity, the actual precept of Hawking breaks this same rule. This can be thought from two points of view: Either we use plasma ball's physics to understand the flaw in Hawking's argument, or Hawking's radiation has another dual in the field theory which is non-unitary.

To finalise this chapter, I would like to emphasise that this chapter is meant to be brief explanation, to show the landscape of the plasma ball-localised black hole duality and by no means it is to be taken as a complete explanation of the subject, for much more information, the reader is remitted to [16] and [22]. There is, as well, a duality between the plasma ring and the black black ring [23] and between the rotating black hole and the axially symmetric rotating plasma ball [17], which is a very interesting topic as well.

Chapter 5

The Rayleigh-Plateau/Gregory Laflamme duality

In this chapter we are going to find out that due to the fluid/gravity correspondence studied in the previous chapter, there is a relation between the instabilities in the conformal field theory (the Rayleigh-Plateau instability) and the instabilities in the gravitational theory (the Gregory-Laflamme instability). We do so by finding solutions to the *relativistic* Navier-Stokes equations and see which of them have an axisymmetric equilibrium configuration. Next, we study the phase transition between solutions and prove their instabilities, which we find to be the Rayleigh-Plateau instability. Afterwards we look for solutions for Einsteins equations on a higher-dimensional AdS space that has certain compactifications, and study the phase transition between this solutions. Finally we study the instabilities of the solutions, to find that they are the Gregory-Laflamme instability.

5.1 The Rayleigh-Plateau instability

We will first study the *relativistic* Navier Stokes equations. The dynamics of a given fluid is given the the conservation of the stress tensor of a conformal field theory that lives in a boundary, i.e. equation 3.3. Suppose our fluid lives on a d-dimensional flat space time and that the fluid configuration we want is axisymmetric, then it is

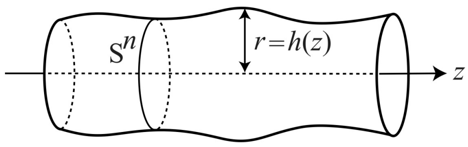

Figure 5.1: Axisymmetric static equilibrium of a fluid in a d-dimensional flat spacetime. Taken from [4]

convenient to express it cylindrical coordinates, namely

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + r^2 \gamma_{ij} d\phi^i d\phi^j \tag{5.1}$$

Where $x^\mu = (t, z, r)$, γ^{ji} is the metric of $d-3$ unit sphere. The boundary stress tensor is composed of three parts: a perfect fluid part, a surface part and a dissipative part. It can be proven ([4]) that for static fluids, the dissipative part of the stress tensor does not contribute to the equations of motion, then

$$T^{ab} = T^{ab}_{perf} + T^{ab}_{surf} \tag{5.2}$$

With

$$T^{ab}_{perf} = (\rho + P)u^a u^b + P g^{ab}, \qquad T^{ab}_{surf} = \sigma(n^a n^b - g^{ab})\sqrt{\partial\Phi \cdot \partial\Phi}\,\delta(\Phi) \tag{5.3}$$

Where -as always- P is the pressure, ρ is the energy density, u^a is the fluid velocity g_{ab} is the metric 5.1, also σ is the tension of the boundary. The surface is given by the equation $\Phi(r, z) = r - h(z) = 0$, with a normal unit vector $n_a := \partial_a \Phi(\partial\Phi \cdot \partial\Phi)^{-1/2}$ [4]. After substitution in 5.3, the perfect fluid component of the stress tensor is

$$T^{ab}_{perf} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & r^{-2} P \gamma^{ij} \end{pmatrix} \tag{5.4}$$

and

$$\nabla_a T^{ab}_{perf} = \left(0\ ,\ \frac{\partial P}{\partial z}\ ,\ \frac{\partial P}{\partial r}\ ,\ 0\right), \qquad \text{using} \qquad \tilde{\nabla}_i \gamma_{jk} \equiv 0 \tag{5.5}$$

To calculate the surface part of the stress tensor we first need to work out the unit normal vectors n^a for a single valued height function $h(z)$, so

$$n^a = \frac{1}{\sqrt{1+h'^2}}(\delta^a_r - h'\delta^a_z), \qquad \text{where} \qquad h' :\, = \partial_z h \tag{5.6}$$

then

$$T^{ab}_{surf} = \frac{\sigma\delta(r-h)}{\sqrt{1+h'^2}} \begin{pmatrix} 1+h'^2 & 0 & 0 & 0 \\ 0 & -1 & -h' & 0 \\ 0 & -h' & -h'^2 & 0 \\ 0 & 0 & 0 & r^{-2}(1+h'^2)\gamma^{ij} \end{pmatrix} \tag{5.7}$$

and a divergence that goes as

$$\nabla_a T^{ab}_{surf} = \sigma\delta(r-h)\,(0, -(d-2)h'H, (d-2)H, 0) \tag{5.8}$$

Where H is the mean curvature of the surface. From equations 3.3 we obtain [4]

$$\begin{aligned} \frac{\partial P}{\partial z} &= (d-2)\sigma\delta(r-h)h'H \\ \frac{\partial P}{\partial r} &= -(d-2)\sigma\delta(r-h)H \end{aligned} \tag{5.9}$$

From here it is obvious that in the bulk -away from the boundary- the pressure is constant. If we integrate 5.9, we get an expression for the pressure -just- inside P_- and outside P_+ the boundary

$$P_+ - P_- = -(d-2)\sigma H(z) \tag{5.10}$$

If, for simplicity, P_+ is set to zero

$$\frac{P_-}{(d-2)\sigma} = \frac{1}{d-2}\left(-\frac{h''}{(1+h'^2)^{3/2}} + \frac{n}{h\sqrt{1+h'^2}}\right) = H_0 \tag{5.11}$$

Where in the first equality, the definition of H was substituted and H_0 is a constant mean curvature. It is worth indicating that there is an alternate way of getting equation 5.11. Suppose a d-dimensional space, and consider the z direction as compactified on a circle, $z \in [-L/2, L/2]$, then you can get the equations of motion ([4]) of the configuration you are looking for by varying the action $I[h] = \sigma A[h] - P\_V[h]$ where $A[h]$ and $V[h]$ are the surface area and the interior volume of a fluid body. The Euler-Lagrange equation obtained by varying $I[h]$ is equivalent to 5.11 and is the governing equation to determine the axisymmetric equilibrium states of the fluid, there are two *trivial* functions that solve this equation, the *uniform tube* and the *spherical ball*

$$\begin{aligned} h_{UT} = r_0 \qquad H_{UT} = \frac{d-3}{(d-2)r_0} \\ h_{SB} = \sqrt{R_0^2 - z_0^2} \qquad H_{SB} = \frac{1}{R_0} \end{aligned} \tag{5.12}$$

Where R_0 is the radius of the spherical ball and r_0 is the radius of the cylindrical solution. Observe that the spherical ball's mean curvature has no dependance in the dimensions whilst the uniform tube's has. The Rayleigh-Plateau instability is the instability in a -4-dimensional spacetime- that overcomes a cylindrical configuration when it's linear dimension is longer than its circumference [4]. This instability is easily generalised to higher dimensions. Suppose a perturbation on the function $h(z)$ of the uniform tube of the form

$$h(z) = r_0 + \varepsilon h_1(z) + \mathcal{O}(\varepsilon^2) \tag{5.13}$$

Then -at order $\mathcal{O}(\varepsilon)$- substituting on equation 5.11 and with the boundary condition that $h_1'(0) = 0$ (meaning that the boundary of the fluid at z=0 will remain constant) we get the function

$$h_1(z) = h^{(1)} \cos(k_{RP} z), \qquad \text{with} \qquad k_{RP} = \frac{\sqrt{d-3}}{r_0} \tag{5.14}$$

If we substitute 5.14 in 5.13 we get the marginally stable mode of the Rayleigh-Plateau instability and then the uniform tube is unstable if the length of the cylinder L satisfies $L > L_{RP} = 2\pi/k_{RP}$, i.e. if the radius $r_0 < r_{RP} = \sqrt{d-3}L/2\pi$ [4]. Observe that if we make $d = 4$ this condition gives rise to the definition of the Rayleigh-

Plateau instability given before and that there is a -dimensional dependence of the critical mode.

5.1.1 Three phases of an axisymmetric fluid in flat spacetime

We need to derive the equations of state of each configuration to learn about the thermodynamic properties of the fluid, with that in mind we obtain the equations of state by making a Scherk-Schwarz compactification (a compactification in which the spacetime fermions are chosen antiperiodic along one circle of the compactification manifold) of a $(d+1)$ -dimensional CFT and define the characteristic length scale, the temperature and entropy density

$$l_0 = \frac{\sigma}{\rho_0}, \quad T_c = \left(\frac{\rho_0}{\alpha}\right)^{1/(d+1)}, \quad s_0 = \left(\alpha\rho_0^d\right)^{1/(d+1)} \tag{5.15}$$

Where ρ_0 is the vacuum energy, α is a constant and T_c is the critical temperature of the deconfinement that we saw in Chapter 2. The energy density and entropy density will be defined as [4]

$$T^{tt} = \rho + \sigma\delta(r-h)\sqrt{1+h'^2} \tag{5.16}$$

$$s = (d+1)s_0\left(\frac{T}{T_c}\right)^d \tag{5.17}$$

When we integrate the above densities we can get the energy, entropy and (Helmholtz) free energy [1]. This quantities for the uniform tube case are functions of a -dimensionless- parameter r_0/L and can be written as

$$E_{UT} = \rho_0\frac{\Omega_n L^{n+2}}{n}\left[(n^2+4n+1)\tilde{T}_{UT}^{n+4} - 1\right]\left(\frac{r_0}{L}\right)^{n+1}, \quad T_{UT} = T_c\left[1 + n\tilde{l}_0\left(\frac{L}{r_0}\right)\right]^{1/(n+4)}$$
$$F_{UT} = \rho_0\frac{\Omega_n L^{n+2}}{n}\left(\tilde{T}_{UT}^{n+4} - 1\right)\left(\frac{r_0}{L}\right)^{n+1}, \quad S_{UT} = s_0(n+4)\Omega_n L^{n+2}\tilde{T}_{UT}^{n+3}\left(\frac{r_0}{L}\right)^{n+1} \tag{5.18}$$

[1] The following expressions (5.18 - 5.23) were taken from [4]

Where $\tilde{l}_0 = l_0/L$ is a dimensionless parameter and for simplicity we have taken $n = d - 3$. For the spherical ball we use a -dimensionless- parameter R_0/L, so we have

$$S_{SB} = s_0(n+4)\Omega_{n+1}L^{n+2}\tilde{T}_{SB}^{n+3}\left(\frac{R_0}{L}\right)^{n+2}, \quad T_{SB} = T_c\left[1 + (n+1)\tilde{l}_0\left(\frac{L}{R_0}\right)\right]^{1/(n+4)}$$
$$F_{SB} = \rho_0\frac{\Omega_{n+1}L^{n+2}}{n+1}\left(\tilde{T}_{SB}^{n+4} - 1\right)\left(\frac{R_0}{L}\right)^{n+2}$$
$$E_{SB} = \rho_0\frac{\Omega_{n+1}L^{n+2}}{n+1}\left[(n^2+5n+5)\tilde{T}_{SB}^{n+4} - 1\right]\left(\frac{R_0}{L}\right)^{n+2} \tag{5.19}$$

Note that, as one would find natural, the uniform tube thermodynamical variables depend of the dimensions, while the spherical ball's ones don't and since $R_0 \leq L/2$, the spherical ball has an upper energy bound and a lower temperature bound. For convenience -when drawing phase diagrams- we define four new "normalised" quantities and a dimensionless parameter,

$$\hat{E} = \frac{E}{E_{RP}}, \quad \hat{T} = \frac{T}{T_{RP}}, \quad \hat{S} = \frac{S}{S_{SB}}, \quad \hat{F} = \frac{F}{F_{SB}}, \quad \tilde{l}_0 = \frac{l_0}{L} \tag{5.20}$$

Where the subscripts "RP" stand for critical uniform tube and "SB" for spherical ball and this quantities are given by

$$E_{RP} = \rho_0\frac{\Omega_n L^{n+2}}{n}\left(\frac{\sqrt{n}}{2\pi}\right)^{n+1}\left[(n^2+4n+1)\tilde{T}_{RP}^{n+4} - 1\right] \quad \text{with} \quad \tilde{T}_{RP} = \frac{T_{RP}}{T_c}$$
$$T_{RP} = T_c\left(1 + 2\pi\sqrt{n}\tilde{l}_0\right)^{1/(n+4)} \tag{5.21}$$

Observe, that for each value of $\tilde{l}_0$, we'll get the thermodynamical relations $\hat{S} = \hat{S}(\hat{E})$ and $\hat{F} = \hat{F}(\hat{T})$. In the previous subsection we saw that there were only two *trivial* solutions to equation 5.11, the spherical ball and the uniform tube, but there is one constant mean curvature nontrivial solution called the *non-uniform tube* [4]. When solving the PDE's from equation 5.11, the first integral can be written in a potential form, with potential

$$U(w) = 1 - \left(\frac{w^n}{w^{n+1} + K}\right)^2 \tag{5.22}$$

where K is an integration constant and $w = H_0 h(z)$. Such potential has two -positive- roots w_- and w_+ and correspond to the smallest and largest radius of the non-uniform tube, respectively. We can use a dimensionless parameter $\lambda = w_-/w_+$ to parametrize our solution (in analogy with r_0/L and R_0/L) [4]. The -normalised- thermodynamical variables of this fluid configuration are

$$\begin{aligned}
\hat{S}_{NUT} &= \frac{\Omega_n}{\Omega_{n+1}} \left[\frac{1 + (n+1)\tilde{l}_0 \tilde{L}}{1 + (n+1)\tilde{l}_0 L/R_0}\right]^{(n+3)/(n+4)} \frac{\tilde{V}}{\tilde{L}^{n+2}} \left(\frac{L}{R_0}\right)^{n+2} \\
\hat{T}_{NUT} &= \frac{\Omega_n}{(n+1)^{n+1}\Omega_{n+1}} \left(\tilde{T}_{NUT}^{n+4} - 1\right)^{n+2} \frac{\tilde{A} - \tilde{V}}{\tilde{l}_0^{n+2} \tilde{L}^{n+2}} \\
\hat{F}_{SB} &= \frac{n+1}{n^n} \frac{\left(\tilde{T}_{NUT}^{n+4} - 1\right)^n (\tilde{A} - \tilde{V})}{\tilde{l}_0^n \tilde{L}^{n+1}} \\
\hat{E}_{SB} &= n \left(\frac{2\pi}{\sqrt{n}}^{n+1}\right) \frac{[(n+3)\tilde{V} + \tilde{A}][1 + (n+1)\tilde{l}_0 \tilde{L}] + \tilde{V} - \tilde{A}}{[(n^2 + 4n + 1)(1 + 2\pi\sqrt{n}\tilde{l}_0 - 1)]\tilde{L}^{n+2}}
\end{aligned} \tag{5.23}$$

where $\tilde{T}_{NUT} = T_{NUT}/T_c$ and $\tilde{L}(\lambda)$, $\tilde{A}(\lambda)$ and $\tilde{V}(\lambda)$ are dimensionless parameter [4]. Before going to next subsection we would like to remind the reader that the range of the parameters of each solution. For the uniform tube we have $r_0/L \in [0, \infty)$, the sphere has a parameter $R_0/L \in [0, 1/2]$ and the non-uniform tube has a parameter $\lambda \in (0, 1)$.

5.1.2 Phase Transitions

To study phase transitions we first need to specify l_0. Up to this point we have not mentioned the thickness of the boundary of the fluid surface and we have assumed it to be sharp, but it isn't. So, to be able to use the variables here derived, the thickness of the boundary needs to be in a limit where it can be ignored, and since it is of order $T_c^- 1 \sim l_0$, then we need to work in the limit $l_0 \ll L$ [4]. Observe that

-in this limit- l_0 can be regarded as measure of the thickness of the boundary.

We will study the microcanonical ensemble, where we study the function $\hat{S} = \hat{S}(\hat{E})$ or the $(\hat{E}, \hat{S})$ phase diagrams and the preferred phase is the one with the largest value of entropy for a given energy, specifically we are going to study what happens when we let the energy decrease. We will also study the canonical ensemble, where we study the function $\hat{F} = \hat{F}(\hat{T})$ or the $(\hat{T}, \hat{F})$ phase diagrams and the preferred phase is the one with the smallest value of free energy for a given temperature, in particular the case where we increase the temperature.

As shown before, the functions we are talking about depend on the dimension we are considering. Instead of showing each dimension, we will study $d = 5$, $d = 11$ and $d = 13$ since are the most outstanding and we shall comment about the other dimensions along the discussion. The figures displayed below were taken from [4].

5.1.2.1 $d = 5$

In the microcanonical ensemble, we can observe (figure 5.2) that the non-uniform branch begins at $\hat{E} = 1$, i.e. at the Rayleigh-Plateau critical point. The most important feature of this dimension is that the non-uniform branch always has smaller entropy than the other two, thus it is not thermodynamically preferred. If we start with a uniform tube, when we decrease the energy, at the point A of figure 5.2, it will transform into a spherical ball, thus going under a phase transition. In this case is a first order transition because the transition between both phases is not smooth, it's discontinuous.

In the canonical ensemble, we begin with a uniform tube and increase the temperature, the uniform tube transits discretily into a spherical ball at point A in figure 5.3, making a first order transition, again the non-uniform phase transition don't play any part in this phase transitions (in this $d = 5$). Fluids living in spactimes with dimensions $4 \leq d \geq 9$ have qualitatively the same phase diagrams as figure 5.2 and 5.3

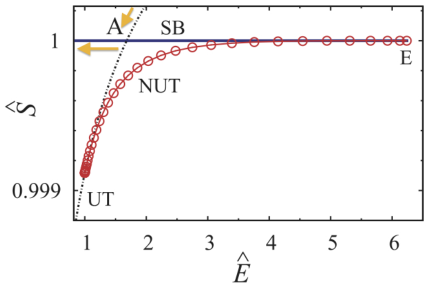

Figure 5.2: The energy-entropy diagram for $d = 5$, containing the phases of the spherical ball (SB, solid line), uniform tube (UT, dashed curve), and non-uniform tube (NUT, dotted line). The arrows show the evolution of the system when we start with an uniform tube and decrease the energy. The maximum entropy state for a given energy is favoured. Taken from [4].

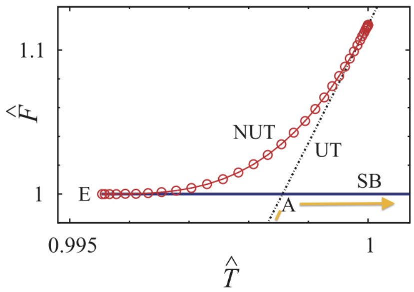

Figure 5.3: The temperature-free energy diagram for $d = 5$, containing the phases of the spherical ball (SB, solid line), uniform tube (UT, dashed curve), and non-uniform tube (NUT, dotted line). The arrows show the evolution of the system when we start with an uniform tube and increase the temperature. The minimum free energy state for a given temperature is favoured. Taken from [4].

5.1.2.2 $d = 11$

In the microcanonical, the first thing we notice is that figure 5.4 has two cusps, namely, we can divide the non-uniform branch in to three parts: The first leaves the uniform tube at the Rayleigh-Plateau critical point, the second one is connected to the endpoint of the energy of the spherical ball and the third one is the one that connect the previous two branches, the one "in between". We also note that it is the first time where the entropy of the non-uniform branch is -in some regions- greater than both of the other branches, meaning that it is the thermodynamically favoured. Observe that in this dimension, the uniform tube and the spherical ball don't intersect anymore, this is the lowest dimension where this can be seen for this case .

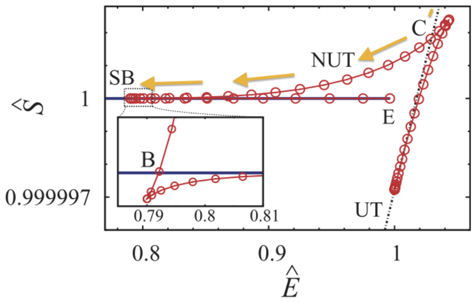

Figure 5.4: The energy-entropy diagram for $d = 11$, note the appearance of a second cusp. Taken from [4].

If we begin with a uniform tube and decrease the energy, the uniform tube transits to the non-uniform branch at the point C in figure 5.4 with a discrete tube, if we continue to decrease the energy, the non-uniform tube eventually transits to a spherical ball at the point B with other discrete jump.

In the canonical ensemble if we begin with a uniform tube and increase the temperature, the uniform tube transits smoothly into a non-uniform tube at the Rayleigh-Plateau critical point . As it transits in a smooth manner, this phase transition is called a second order or higher. When we keep increasing the temperature, the non-uniform transforms into a spherical ball at point B in figure 5.5 in a discrete way, being a first order phase transition. The first dimension where we see a second order phase transition is called the critical dimension, in the canonical ensemble in this case, the critical dimension is $d_*^{can} = 11$.

In $d = 10$ the canonical ensemble the non-uniform branch has no part in the phase transition and in the microcanonical ensemble the spherical ball and the uniform tube do intersect, whilst in $d = 11$ and $d = 12$ they don't.

5.1.2.3 $d = 13$

In the microcanonical ensemble the cusp near to the Rayleigh-Plateau critical point disappears from $d = 13$ on (the one near point B in figure 5.6 disappears from $d = 15$ on). If we begin with a uniform tube and decrease the energy, the uniform

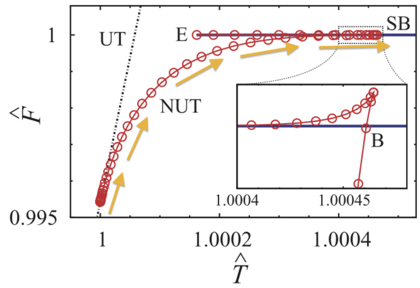

Figure 5.5: The temperature-free energy diagram for $d = 11$, this dimension presents a second order transition for the first time (in the canonical ensemble). Taken from [4].

tube transits to a non-uniform bench at the Rayleigh-Plateau critical point, this phase transition is not accompanied by a discrete jump, therefore is of second order or higher. If we keep decreasing the energy the non-uniform tube branch intersects the ball at the point B in a first order phase transition. Thus we infer that the critical dimension in the microcanonical ensemble is $d_*^{can} = 13$.

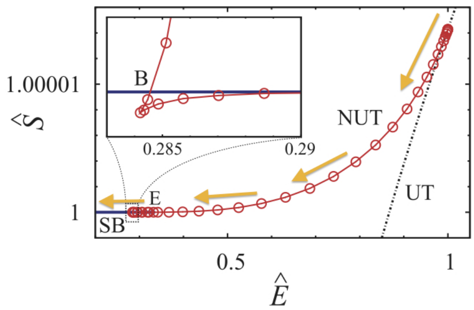

Figure 5.6: The energy-entropy diagram for $d = 13$, this dimension presents a second order transition for the first time (in the microcanonical ensemble). Taken from [4].

In the canonical ensemble $d = 11$ is the canonical ensemble, therefore $d \geq 12$ are not of interest as well as $d \geq 14$ for the microcanonical ensemble.

Finally I would like to make clear the conditions that must be fulfilled to make the calculations here presented valid [4]. Firstly the radius of curvature of the fluid surface in all directions must be much larger that the thickness of the surface, i.e. $L \gg l_0$. Secondly, the temperature needs to be in the neighbourhood of the critical temperature, if not, the surface tension σ can not be regarded as a constant. Lastly, the temperature and pressure must vary in a length scale larger than the mean free path of the 'particles' of the theory, which in the large 't Hooft limit is $\sim T_c^{-1}$.

5.2 The Gregory-Laflamme instability

In this section we are going to study the Gregory-Laflamme instability and in later sections try to connect it with the Rayleigh-Plateau instability. The Gregory-Laflamme was originally described in a *flat* space-time that has more than 4 dimensions and it has been extensively studied in such space [24]. The reader might wonder how are this and the Rayleigh-Plateau instabilities connected if the Gregory-Laflamme instability is not on and AdS space-time. As it turns out an unusual property about the black hole-black string system in flat space is *qualitatively* similar to the black hole-black string system in AdS with a Scherk-Schwarz compactification [17]. As so, we will, for simplicity, study the black hole-black string in a flat space-time.

Suppose we are in a 5-dimensional flat spacetime, then Einstein's equation's 3.1 are re-expressed as

$$R_{ab} = 0 \tag{5.24}$$

The simplest and most forward solution for this equations is the 5-dimensional spherical symmetric Schwarzschild black hole [24] with metric

$$ds^2 = -\left(1 - \frac{r_5^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_5^2}{r^2}\right)} + r^2 d\Omega_3^2 \tag{5.25}$$

Where the r_5 is the horizon radius with definition

$$r_5^2 = \frac{8G_5 M_5}{3\pi} \tag{5.26}$$

But suppose that *nothing* depends in the extra dimension, then another solution to 5.24 is the metric

$$ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu + dz^2 \tag{5.27}$$

Where $g_{\mu\nu}$ is such that it solves $R_{5a} = 0$, more explicitly

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)} + r^2 d\Omega_2^2 + dz^2 \tag{5.28}$$

Where the first three terms are the solution to the 4-dimensional black hole and the last factor accounts for the extra dimension. We will call 5.28 the black string metric. So now, we have two different metrics that solve 5.24, the black hole 5.25 and the black string 5.28, and it can be concluded that in higher dimensions event horizons need not be spherical. When this was first realised, it came as a shock to many physicist, because it was believed that -in flat space- the only static stable non charged black hole was the Schwarzschild solution (the uniqueness theorem). It is now known that, this is uniqueness a rather remarkable feature of the 4-dimensional flat space time, and that it those not follow through as we go to higher dimensions [25].

If our extra dimension -z- is compact with length L, the black string then corresponds simply to a Kaluza-Klein black hole, which does not depends on the geometry of the extra dimension, and from a 4-dimensional point of view it just looks like a Schwarzschild black hole. At energies of order L^{-1} this extra dimension geometry is expected to play a part, so we can ask if -at this level of energy- there are more solutions to 5.24.

Let us begin with a 5-dimensional black holes, metric 5.25, since z is finite, there is no 5-dimensional spherical symmetry. When $r_+ \ll L$ the black hole is still a good solution for Einsteins equations, but as the mass of the black hole increases further it can no longer fit inside the extra dimension and must become a string-like solution. Tis solutions are not analytic and have to be found numerically, and are called *nonuniform string* and *caged black holes* [24]. To see which of the solutions is preferred we need to study their thermodynamics and examine their instabilities, if they have any.

When looking for the thermodynamics of the black hole-black string system in a Kaluza-Klein theory, one needs to take into account the corrections to the Planck mass depending on the dimensions one is working on, i.e.

$$M_p^2 = V_{d-4} M_d^{d-2} \tag{5.29}$$

With d the number of dimensions and V_{d-4} is the volume of the internal space on which we compactify. Then

$$G_5 = LG_4 = L \quad \text{with} \quad G_4 = 1 \tag{5.30}$$

Then, since $S = A/4$ ([3; 26]) then we can calculate the entropies of the black hole and the black string

$$S_{BH} = \pi^2 r_5^3 / 2L \quad \text{and} \quad S_{BS} = \pi r_+^2 \tag{5.31}$$

With r_+ and r_5 the radius of the horizon of the black string and the black hole respectively. To compare the we must take their masses to be equal, i.e.

$$M_{BH} : = \frac{3\pi^2 r_5^2}{8L} = \frac{r_+}{2} =: M_{BS} \tag{5.32}$$

Making the entropies

$$S_{BH} = 4\pi M^2 \sqrt{\frac{8L}{27\pi M}}, \qquad S_{BS} = 4\pi M \tag{5.33}$$

So, for a sufficiently small L the black strings will be thermodynamically preferred, but as we increase L the energy will decrease, causing an IR instability to this configuration. Note that such an instability will appear when we decrease the mass (energy) and the point where the non-uniform black string and the uniform black string merge will be called M_{crit}, also it is important to remark that for a black hole in a Kaluza-Klein theory, thermodynamics second law reads as follows [27]

$$dM = TdS + nM \frac{dL}{L} \tag{5.34}$$

Where n is the dimensionless tension of the string.

5.2.1 The black string perturbation

We already said that the string will have an instability at certain energy levels, but now we want to understand further the configuration that takes the black string to transit to the black hole, to do so we will need to perturb the black string. There is a very important fact to bear in mind when perturbing a gravity theories, gravity has an infinite gauge group, so there are an infinite different transformations and as such there will be many of these transformations which will be just pure gauge. This means that we will need to check if our perturbation, which at the end can be realised as a change of coordinates, gives a physical change on our theory or if it is just a pure gauge transformation.

With this in mind, consider a perturbation

$$g_{ab} \;\to\; g_{ab} + h_{ab} \tag{5.35}$$

which solves Einstein's equations in the vacuum, this transformation changes the Ricci tensor to [24]

$$R_{ab} \;\to\; R_{ab} - \frac{1}{2}\Delta_L h_{ab} \tag{5.36}$$

Where Δ_L is the Lichnerowicz operator. After taking into account that, $\Delta_L h_{ab} = 0$ -because Ricci tensor must vanish in the vacuum-, that there are no z components of the Riemann tensor and that we can choose asa transverse trace free perturbation h_{ab}, i.e. $\nabla_a h^a_b = 0 = h$, the Lichnerowicz operator simplifies [24] to

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} = 0 \tag{5.37}$$

Note, however, that the black string has z and time translation symmetries that tell us that we can separate two factors: an oscillatory factor e^{imz} and a growing factor $e^{\Omega t}$. Furthermore, this system has a $SO(3)$ symmetry that comes form the first three terms in 5.28 and as a result we now that the perturbation will not have any cross terms with the angular sector [24], i.e.

$$h_{ab} = \begin{bmatrix} h_{tt} & h_{tr} & 0 & 0 & h_{tz} \\ h_{tr} & h_{rr} & 0 & 0 & h_{rz} \\ 0 & 0 & h_{\theta\theta} & 0 & 0 \\ 0 & 0 & 0 & h_{\theta\theta}\sin^2\theta & 0 \\ h_{tz} & h_{rz} & 0 & 0 & h_{zz} \end{bmatrix} \tag{5.38}$$

The next thing to do is to find the explicit components of h_{ab}, which is rather lengthy and is beyond the objectives of this dissertation[1]. We will however, sketch how it should be calculated. Observe that the z-components of the perturbation decouple from the rest of perturbation, after taking the boundary conditions into account (vanishing at the horizon and at infinity for regularity purposes) it can be shown ([28]) that the hole z sector vanishes. We are then left with a four-dimensional perturbation of the form

$$h_{\mu\nu} = e^{imz} e^{\Omega t} H_{\mu\nu}(r) \tag{5.39}$$

where $H_{\mu\nu}$ is a solution of

$$H' = \frac{\Omega(H_+ = H_-)}{2V} - \frac{(1+V)H}{rV} \tag{5.40}$$

$$H'_- = \frac{m^2 H}{\Omega} - \frac{H_+}{r} + \frac{(1-5V)H_-}{2rV} \tag{5.41}$$

with

$$H = H_{tr}, \qquad H_\pm = \frac{H_{tt}}{V} \pm V H_{rr} \tag{5.42}$$

Then the *instability* is a solution of 5.40 and 5.41 which is regular at the horizon and at infinity, and we must integrate numerically this equations. There is not going to be solutions for all m and Ω, but we can expect a single characteristic frequency Ω_m for a given m (see figure 1.3 in [24]).

Once you have -numerically- found the instability, there is still the issue we

[1] The explicit calculations can be found in [24]

mention before of checking the physicality of such transformation. This can be confirmed, noting that, due to the decoupling of h_{za} and the vanishing of R_{zabc}, equation 5.37 reduces to the 4-dimensional Lichnerowicz operator with a mass term [24], namely

$$\Delta_L^{(5)} h_{\mu\nu} = \Delta_L^{(4)} h_{\mu\nu} + \frac{\partial^2}{\partial z^2} h_{\mu\nu} = \left[\Delta_L^{(4)} - m^2\right] h_{\mu\nu} \tag{5.43}$$

The fact that the Lichnerowicz operator does not reduce solely to the four-dimensional Lichnerowicz operator, tells us that $h_{\mu\nu}$ is a physical mode and not a purely gauge mode. In other words, as long as a mass term appears in the operator 5.43, any solution to such operator will be a physical Kaluza-Klein instability [24]. To close these subsection, it is worth mentioning that in higher dimensions the solutions to Einstein's equations -called black branes- also have this instability, of course the exact functions will change, but qualitatively they are the same as the solutions to equation 5.43. An interesting fact is that in higher dimension a decoupling also happens [24], i.e.

$$h_{ab} \ \rightarrow \ h_{\mu\nu} = u_m(z^i) e^{\Omega t} H_{\mu\nu}(r) \tag{5.44}$$

Where z^i are the extra dimensions.

5.2.2 Aftermath of the instability

We have already stated how the instability is going to behave and how to find it, but now we want to take physics out of it and for that we need to study the effect of such perturbation in the horizon ($r \rightarrow r_+$).

For this kind of calculations it is more convenient to work in Kruskal coordinates and see what happens to the outgoing light rays near the original horizon. In these coordinates, in the unperturbed space-time, the null outgoing geodesics of the string configuration obey $R = T + R_0$ and $R = T$ is the future horizon [24]. The perturbed geodesics become

$$\left(\frac{dR}{dT}\right)^2 = 1 + \epsilon \cos mz T^{2r_+\Omega-2} \left(1 + \frac{dR}{dT}\right)^2 \tag{5.45}$$

With ϵ is the perturbation's amplitude parameter. Equation 5.46 tells us that the horizon will start rippling, (figure 5.7) meaning that -in Schwarzschild coordinates- the horizon would be at

$$r = r_+ + \epsilon T^{2\Omega} \cos mz \tag{5.46}$$

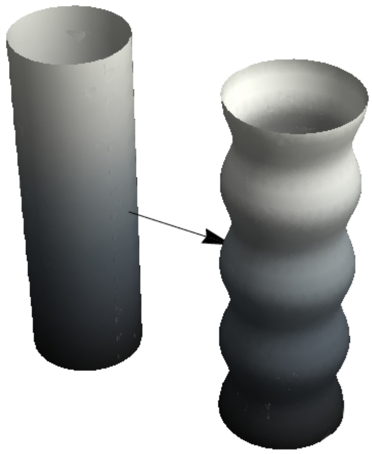

Figure 5.7: The effect of the Gregory-Laflamme instability on the black string horizon. Taken from [24]

Now, as seen above, the four-dimensional event horizon cannot shrink without breaking the positivity of energy when undergoing any classical process (as the one we have just studied). Thus, even though at some points the horizon is actually shrinking, at some other points it is expanding, and then the overall process has a growing horizon area, agreeing with the positivity of the entropy change [24]. In addition if we follow the instability to an end point, it is believed that the unstable black string will eventually form a black holes joined by thin strings [29]. This 'joining' strings eventually keep shrinking, forming -smaller- black holes, generating thus a stream of black holes, until in the end at infinite asymptotically time, the string like segments become a naked singularity. This, however, violates the cosmic censorship principle and is better discussed in [29].

In [27] they -numerically- prove that the non-uniform black string does evolve into a black hole in static 5 or 6-dimensional flat space when you decrease the mass of a non-uniform string. Recall that the black holes in Kaluza-Klein theories follow equation 5.34, and therefore their mass/energy will be determined not only by the temperature and entropy but also by the tension n and the length L of the compact coordinate z. The tension and temperature of the string when it is at the point where the instability becomes relevant are called n_{crit} and T_{crit}, respectively. In [27] they calculated the temperature as a function of the tension (normalised) of a 6-dimensional string (figure 5.8) and confirmed that the uniform string transits to a non-uniform one and then to a black hole.

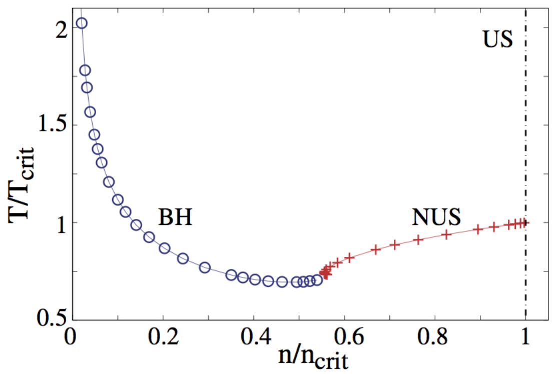

Figure 5.8: The temperature of the spherical black hole, non-uniform string and uniform string as a function of the tension. Taken from [27]

But from figure 5.8 we can't appreciate which branch is thermodynamically preferred at a given mass or energy of the black hole, however in figure 5.9 it easy to see that if -in 6 dimensions- we start with a high energy black string and decrease the energy we will eventually transit to a black hole with a first order phase transition (because there is not a smooth continuation of the phases) and that at no point the non-uniform black string is thermodynamically preferred in comparison with the other two branches.

Other phase transitions (and their phase diagrams) for different systems with different starting points can be found in [30], where they study the fate of black holes when it starts radiating and "evaporating", and in [31], where they plot phase diagrams for different scenarios in which the system can or cannot go through the non-uniform black string phase.

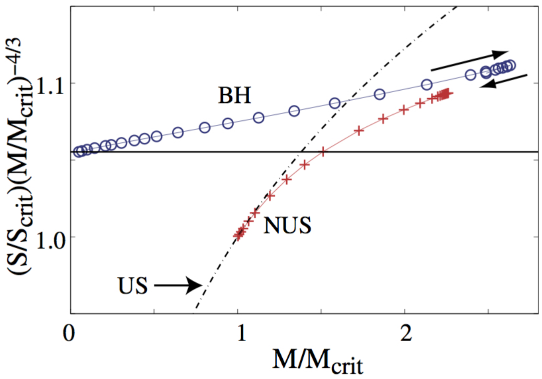

Figure 5.9: The mass-entropy diagram for $d = 6$, note that the non-uniform string does not play any important roll in this dimension. Taken from [27]

5.3 Similarities between fluid tubes and black strings

We have separately developed the phase transitions between a fluid tube and a spherical ball of a field theory due to the Rayleigh-Plateau instability and the phase transition between a black string and a black hole in a flat space-time with a Kaluza-Klein compactification due to the Gregory-Laflamme instability. We now mention how they are similar and how we can -as a result of the fluid/gravity correspondence- connect both of them.

The first indication we can take of a duality between the Gregory-Laflamme and the Rayleigh-Plateau instabilities is that figures 5.2 and 5.9 are qualitatively the same, i.e. in both the non-uniform branch is not thermodynamically preferred, this branch eventually takes the same entropy value as the spherical branch at a higher energy position than the transit of the spherical and cylindrical branches. And there is a first order transition in both systems. Owing to the fact that the black hole-black string system on a AdS with a Scherk-Schwarz compactification is qualitatively alike to a flat space-time with Kaluza-Klein compactification black hole-black string system ([17]) then we can see that the instabilities on the AdS gravity theory are dual to the instabilities on the CFT theory.

In [32] they define the critical dimension of a gravity theory as the greatest dimension at which the transition from black string to black hole is of first order and found that for this system, the critical dimension -in the microcanonical ensemble- is $d = 13$, meaning that the first dimension where there is a second order phase transition is $D_{**}^{microcan} = 14$. In [33] they found that in the canonical ensemble of a Kaluza-Klein system, the first dimension where a second order transition is obtained is $D_{**}^{can} = 13$. From a point of view of the filed theory part, one would expect that such critical dimensions would be $D_{*}^{microcan} = 15$ and $D_{*}^{can} = 13$ respectively ([4]).

Another similarity between this systems is their evolution at large times. When we start with the uniform tube and perturb it we obtain a nonuniform tube phase which is unstable, if we follow it until it's endpoint, then it will fragment in little spheres/droplets, whilst the black string ones perturbed will evolve and become a non-uniform black string at which's endpoint it "breaks" and form small -spherically symmetric- black holes([29]).

The similarity between the Scherk-Schwarz and the Kaluza-Klein systems is vaguely comprehended, and so is the fact of the agreement of the critical dimensions predicted from both the gravity and the field theories. It is very important to remark that this is a part of physics that is in current development an that there are many gaps yet to be understood. Much more information relating this topics can be found in [34], [35] and the references given along this chapter.

Chapter 6

Conclusions

We have seen that the AdS/CFT correspondence is a very powerful weapon we can use to approach the problems we face when studying complicated gravitational systems and their evolution. We have reviewed how a $d + 1$ dimensional gravity theory is dual to a d-dimensional field theory that lives on it's boundary and we have done so comparing their entropies and observing which constraints this duality imposes upon our theories.

We have explicitly examined how a conformal field theory with high temperature living on R^3 is analogous to a theory in a deconfinement phase whilst the confinement base of such theory is equivalent to the same CFT but living on an S at low temperatures. Then we used the AdS/CFT correspondence to establish the connection between this two-phased conformal field theory and a black hole in AdS space, specifically matching up the entropies of the deconfinement phase of our field theory and the Hawking entropy of our gravitational theory in the large r limit. We also observed that -at high temperatures- the CFT has a mass gap, an spontaneous symmetry breaking and an area law for spatial Wilson lines, features that, according to the gauge/gravity duality, are the constraints placed over the CFT to be dual to a black hole on AdS space at large r.

In the beginnings of Chapter 3 we said that the dynamics of a field theory can be studied as hydrodynamics, we have also inspected the relation between the dynamics of a black hole in AdS space and the dynamics of a conformal field theory, and we saw that they are associated via the fluid/gravity correspondence. It was concluded that this duality is a consequence that the graviton is dual to the stress energy tensor and that the conservation of the latter can be seen as a sector of Einstein's equations, i.e. the fluid/gravity correspondence is nothing but consequence of the AdS/CFT correspondence. In addition we verified that the fluid/gravity duality is so powerful that it gives us an algorithm with which given a particular field theory (stress energy tensor) we can find the metric of our gravity spacetime up to some order. Even further, we spelled out relations between several different field theories - rotating fluid, non-relativistic fluids, non-conformal theories- and their gravitational duals.

Returning to conformal field theories with deconfinement phases, we did an extensive study of the relation of plasma balls and localised black holes. We saw that a large N gauge theory at its deconfiment temperature can form a plasma ball a mixture of gluons and glue balls , which's can be studied using planar Feynman diagrams, predicting the ball's decay time and evolution. The localised black holes are solutions to Einstein's equations which have both a graviton gas configuration and a black brane living at the same range of energy density, plus a domain wall which interpolates between both of these backgrounds. We saw that with the AdS/CFT correspondence we can relate an AdS space with a Scherk-Schwarz compactification with a $d+1$ conformal field theory with a Scherk-Schwarz compactification and wrote down the metric for the domain wall in the specific case of the $\mathcal{N}=4$ SYM theory. We also commented that gluons are dual to gravitons near the horizon of the black brane and that glue balls inside the plasma ball are dual to gravitons outside the horizon and that the AdS/CFT correspondence told us that the plasma ball only decays via glue balls produced *near* the surface of the plasma ball and that the formation of a plasma ball is consequence the merging of glue balls that pass by each other at a certain distance (impact parameter).

One of the most upshots of the gauge/gravity duality is that -some- instabilities on the field theory can translate into instabilities in the gravity theory and we studied that explicitly for the Rayleigh-Plateau and the Gregory-Laflamme instabilities. We revised the fact that the Gregory-Laflamme instability in flat space is very much alike to the instabilities present in a AdS space with a Scherk-Schwarz compactification, making it possible to use the AdS/CFT correspondence. We saw that both the fluid tube and the black string have three possible phases: a spherical symmetric solution, a uniform tube/ black string solution and a non-uniform tube/black string solution. Morevover, we saw that the Rayleigh-Plateau and the Gregory-Laflamme instabilities take you from one configuration to the other one, when you reduce the size of the tube or black string respectively. We studied the phase transitions of the fluid tube in the canonical and microcanonical ensembles for different dimensions as well as the phase transition of the black string. We observed that the critical dimensions of both theories agree -to some extent- and that the phase diagrams are qualitatively the same for the field theory and for the gravitational theory.

We expect the gauge/string duality can cast some light over some big gaps of information we have such as: what happens when a black hole radiates so much energy for so long time that its size is comparable to the string size, the breaking of unitarity due to Hawking's radiation and many others. There are many other questions that can arise without changing so much the topic, one of them being: A black hole with only one Killing vector field in AdS is unstable when a scalar field is scattered ([36; 37]), this is called turbulence, so what would this instability's field theory dual be? In [37; 38] they try to give a flavour of this duality. This topic was intended to be a chapter in this dissertation but I decided to cut it out, not because it is not interesting but with the intention of keeping the document short.

In this dissertation we have only superficially discussed the topics, but it is important to remark that there is yet a lot to be understood about the dualities and relations here stated. For example, topics such as the one discussed in [34] come into conflict with the fluid/gravity duality and the connection and similarities between the Scherk-Schwarz and Kaluza-Klein compactifications. Is of great importance to be able to answer these questions in order to understand thorough fully the dualities.

The aim of this dissertation was to describe the power of the AdS/CFT correspondence and show the "tip of the iceberg" of what is becoming one of the most important breakthrough in theoretical physics in the last years -quite a few now-. I hope that in the past 5 chapters I have been able to transmit to the reader the importance of understanding the duality, how it has modify our way of thinking of black holes and fluids and specially all the work that is to be done in the subject.

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