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Hitchin Functionals, Generalized Geometry and an exploration of their significance for Modern Theoretical Physics

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Contents

I. Introduction

1.1 Mathematical foundations

The two key geometrical concepts that provide the mathematical foundations for this dissertation are *Hitchin functionals* and *Generalized geometry*, both of which arose from the work of the mathematician Nigel Hitchin. [1] The common historical origin of both of these ideas lies in the observation that in dimensions 6 and 7 there exists a well-defined notion of non-degeneracy of a differential three-form by virtue of the fact that if V is a real vector space of dimension 6 or 7, then the group $GL(V)$ has an open orbit. In particular, in 6 dimensions there exists an open orbit with stabilizer conjugate to $SL(3,\mathbb{C})$ and in 7 dimensions there exists an open orbit with stabilizer conjugate to the exceptional Lie group G_2. A 3-form Ω that lies in either of these open orbits thus determines a reduction of the structure group of the manifold M on which it is defined to the corresponding stabilizer subgroup. Such a 3-form is said to be *stable* and it is this notion of stability that we take as the generalization of the non-degeneracy of a differential form.

The key point is that if the 3-form Ω lies in an open orbit everywhere and is also a critical point of certain diffeomorphism-invariant functionals defined on the space of 3-forms on the manifold M, then the reduction of the structure group is *integrable* in the sense that we obtain a complex threefold with trivial canonical bundle in 6 dimensions, and a Riemannian manifold with holonomy G_2 in 7 dimensions. These diffeomorphism-invariant functionals whose critical points yield such geometrical structures in this way are known as Hitchin functionals. Hitchin functionals have appeared in various contexts in the recent physics literature and part of this dissertation is devoted to an exploration of their significance as far as theoretical physicists are concerned. In particular, we shall consider their relation to topological string theories, black holes and Calabi-Yau moduli spaces.

Generalized geometry incorporates complex and symplectic geometry as special cases of a unifying geometrical structure. The starting point of generalized geometry is the replacement of the tangent bundle T with the direct sum of the tangent and cotangent bundles $T \oplus T^*$, and use of the *Courant bracket* on sections of the bundle $T \oplus T^*$ as the natural generalization of the Lie bracket defined on sections of T. This gives rise to a geometry with the novel feature that it transforms naturally under both the action of a closed 2-form B and the usual diffeomorphism group. It turns out that generalized geometry leads to a language ideally suited for the systematic description of supergravity backgrounds. In particular, we will show how the supersymmetry transformations and equations of motion of Type II supergravity can be written in a simple, elegant and manifestly $Spin(9,1) \times Spin(1,9)$ covariant form.

In light of the effectiveness of generalized geometry in describing supergravity backgrounds, it is interesting to note that its historical origin can be found in the very different context of Hitchin's work on characterizing special geometry in low dimensions using invariant functionals of differential forms. In particular, in six dimensions Hitchin constructs an invariant functional whose critical points yield *generalized Calabi-Yau manifolds* which belong to a particular class of geometrical structure that is compatible with the general framework of generalized geometry. Indeed, it was within this context that the basic structure of generalized geometry was discovered. This *generalized Hitchin functional* thus provides the link between Hitchin functionals and generalized geometry in the special 6-dimensional case. Unlike Hitchin functionals, however, generalized geometry is not restricted to specific low dimensions and can indeed be explained without any reference to invariant functionals of differential forms. Nonetheless, it is an interesting observation that generalized geometry essentially has its historical origin in work that relied on the simple mathematical fact known for more than a century [2] that if V is a real vector space of dimension 6 or 7, then the group $GL(V)$ has an open orbit.

1.2 Supersymmetry

Supersymmetry is a deep symmetry between boson and fermion fields. Consistency of string theory requires supersymmetry, without which one is only left with the nonsupersymmetric bosonic string theories which are clearly unrealistic since they do not incorporate fermions. Any string theory that includes fermions relies on local supersymmetry for its mathematical consistency. Supersymmetry is also perhaps the key concept that is a common feature of all of the theories from across the theoretical physics landscape that we shall consider in this dissertation.

1.3 Supergravity

Supergravity is an extension of Einstein's general relativity which inlcludes supersymmetry. Supergravity can be seen as naturally arising in the process of making supersymmetry transformations local. It turns out that this process requires the introduction of a graviton field and its supersymmetric partner the gravitino. This theory of local supersymmetry is known as supergravity.

1.4 Superstrings

Superstring theory is a version of string theory that incorporates supersymmetry. Consideration of both left-moving and right-moving modes using superstring formalism leads one to two different types of superstring theory known as type IIA and type IIB. A third type of superstring theory can be derived from type IIB using a procedure known as *orientifold projection*, which leaves only strings that are unoriented.

The *first superstring revolution* was sparked in the 1980s with the discovery that consistency of a ten-dimensional $N = 1$ supersymmetric theory with quantum mechanics requires a local Yang-Mills gauge symmetry based on one of two possible Lie algebras: $SO(32)$ or $E_8 \times E_8$. Using the formalism of the 26-dimensional bosonic string for the left-moving modes and the formalism of the 10-dimensional superstring theory for the right-moving modes leads to two other types of supersymmetric string theory known as *heterotic* theories.

Thus, in total we have five distinct types of ten-dimensional supersymmetric string theory. Type I and the two heterotic theories have $N = 1$ supersymmetry. The type I theory has the gauge group $SO(32)$, while the heterotic theories realize both $SO(32)$ and $E_8 \times E_8$. There are 16 conserved supercharges in each of these three theories. Type IIA and type IIB theories on the other hand have $N = 2$ supersymmetry with 32 supercharges.

Later it was discovered that some of these theories can be related via certain dualities. In particular, *T-duality* relates the two type II theories to each other as well as the two heterotic theories. As part of the *second superstring revolution* of the 1990s, *S-duality* was discovered as another kind of duality linking some of the different superstring theories at strong string coupling. For example, it was found that strongly coupled type I theory is equivalent under S-duality to weakly coupled $SO(32)$ heterotic theory. S-duality also provides a way of relating the type IIB superstring theory to itself, thereby essentially establishing a symmetry in this case.

1.5 M-theory

S-duality explains the behaviour of type IIB, type I and the $SO(32)$ heterotic superstring theory at strong string coupling g_s . For large g_s, the other two types of theory are understood to grow an eleventh dimension of size $g_s l_s$, where l_s is the fundamental string length scale. In type IIA this new dimension is a circle, whereas in the heterotic theory it is a line interval. When $g_s l_s$ is large, non-perturbative techniques are required which lead to a new type of quantum theory in 11 dimensions, known as M-theory. At low energies, M-theory is approximated by a classical field theory known as 11-dimensional supergravity. M-theory and the dualities described above provide a web of dualities that in effect unify the five superstring theories into a single theory.

II. Hitchin Functionals

In this chapter we define the notion of stability of a differential form as an extension of the non-degeneracy of bilinear forms and see how stable p-forms only exist for specific values of p in certain low dimensions. We then introduce Hitchin functionals as diffeomorphism-invariant functionals whose critical points on the space of stable forms yield interesting geometrical structures. In particular, Hitchin functionals are constructed in six and seven dimensions which generate Calabi-Yau algebraic three-folds and G_2 holonomy manifolds, respectively. We move on to introduce the idea of weak holonomy and consider its relation to certain constrained variational problems. Finally, we discuss a Hamiltonian flow construction by Hitchin that provides a bridge between $SU(3)$ structures and G_2 holonomy manifolds.

2.1 Special holonomy

In this section we introduce the notion of special holonomy and show how it naturally leads us to the idea of stable differential forms which provide the starting point for the construction of Hitchin functionals. Riemannian manifolds with special holonomy also play an important role in string theory compactifications as we shall explain shortly. We begin by defining the holonomy group of a Riemannian manifold at a point.

Definition 2.1: *Let (M, g) be an n-dimensional Riemannian manifold with an affine connection ∇. Let p be a point in (M, g) and consider the set of closed loops at p, $\{\gamma(t): 0 \le t \le 1,\ \gamma(0) = \gamma(1) = p\}$. By parallel transporting a vector $X \in T_P M$ along a loop γ we obtain a new vector $X_\gamma \in T_P M$. In this way the loop $\gamma(t)$ and the connection define a linear transformation on the tangent space at p. The set of these transformations is called the holonomy group at p.*

Berger's work on holonomy groups of Riemannian manifolds in the 1950's led to a classification of the holonomy groups of manifolds which are not locally a product space. It was essentially discovered that under this irreducibility assumption, the manifold is either locally a Riemannian symmetric space [5] or the holonomy group is given by one of the following:

(1) $SO(n)$

(2) $U(n/2)$: Kähler manifolds

(3) $SU(n/2)$: Calabi-Yau manifolds

(4) $Sp(1)Sp(n/4)$: quaternion Kähler manifolds

(5) $Sp(n/4)$: hyperkähler manifolds

(6) G_2: special manifolds in dimension 7

(7) $Spin(7)$: special manifolds in dimension 8. [6]

Among these the manifolds with holonomy groups $G_2, Spin(7), SU(n/2), Sp(n/4)$ are Ricci-flat in the sense that they have vanishing Ricci curvature.

In general, the holonomy group is a subgroup of $GL(n, \mathbb{R})$. If ∇ is a metric connection so that $g(X_\gamma, X_\gamma) = g(X, X)$ for all $X \in T_P M$, then the holonomy group is a subgroup of $SO(n)$ provided (M, g) is orientable and Riemannian. That is, for an n-dimensional Riemannian manifold M, we have $Hol(M) \subseteq SO(n)$.

Definition 2.2: *The manifolds with special holonomy are those whose holonomy groups satisfy the relation $Hol(M) \subset SO(n)$.*

These structures are characterized by the existence of covariantly constant spinors; that is, spinor fields ψ satisfying

$$\nabla\psi = 0. \tag{2.1}$$

It is the existence of such spinors that lies behind the significance of manifolds with special holonomy in string theory. The covariantly constant character of the spinor field ensures that the superstring compactification on M preserves a definite fraction of the original 32 supercharges.

It is also a property of special holonomy manifolds that they possess certain invariant p-forms know as calibrations. These are closed forms on M that define special kinds of minimal submanifold known as calibrated submanifolds and are non-trivial only for specific values of p. We may use the covariantly constant spinor ψ to construct a calibration on M as follows:

$$\omega^{(p)} = \psi^\dagger \gamma_{i_1 \ldots i_p} \psi. \tag{2.2}$$

Our interest in these objects lies in the fact that they provide an example of how geometric structure can be characterized by differential forms and action functionals. To elucidate this further, we recall that for a minimal submanifold $S \subset M$ of real dimension p, we may calculate the volume as $\mathrm{Vol(S)} = \int_S d^p\, x\sqrt{g}$ from knowledge of the metric g. On the other hand, via the calibration $\omega^{(p)}$ we may compute the volume as

$$\mathrm{Vol(S)} = \int_S \omega^{(p)} \tag{2.3}$$

using a differential form and without explicit use of the metric. As we shall see, the idea of characterizing geometric structure by means of invariant functionals of differential forms lies at the heart of this chapter.

2.2 Stable forms

We seek to arrive at a general notion of non-degeneracy of a *p*-form. To do this we turn to symplectic geometry for inspiration. A symplectic manifold of dimension n is a differentiable manifold endowed with a closed, non-degenerate 2-form ω where non-degeneracy means $\omega^{n/2} \neq 0$. An equivalent way of capturing this notion of non-degeneracy of $\omega \in \Lambda^2 V^*$ at each point is in terms of the action of $GL(V)$ on $\Lambda^2 V^*$, where V is the tangent space at the point. This alternative viewpoint is expressed by the requirement that the orbit of ω be open under the above action. This brings us in a position to define a suitable notion of non-degeneracy for a general *p*-form, which we shall refer to as *stability*, as described in [8].

Definition 2.3**:** *Let V be the tangent space at a point x, and $\Lambda^p V^*$ be the space of p-forms at x. Then a p-form $\rho \in \Lambda^p V^*$ is said to be stable at x if $\rho(x)$ lies in an open orbit of the $GL(V)$ action on $\Lambda^p V^*$. We say ρ is a stable form if it is stable at every point.*

The word "stability" in the above definition refers to stability of ρ in the sense of deformation invariance: all forms in a neighbourhood of ρ are equivalent to ρ by a local $GL(V)$ action.

How often do stable forms occur? To answer this question we look at the stabilizer subgroups of $\rho \in \Lambda^p V^*$ in $GL(V)$. For a stable 2-form in even dimension *n* the stabilizer is the symplectic group $Sp(n)$. This simply corresponds to the fact that the 2-form defines a presymplectic structure as one would expect. Denoting the stabilizer subgroup of ρ by G_ρ, so that $G_\rho = \{g \in GL(V): g(\rho) = \rho\}$, we have:

$$dim\, G_\rho = dim\, GL(V) - dim\, \Lambda^p V^*. \tag{2.4}$$

Since the growth of $dim\, \Lambda^p V^* = n!/p!\,(n-p)!$ is typically much faster than that of $dim\, GL(V) = n^2$, it is clear that in the vast majority of cases GL(V) cannot act locally transitively on the space of *p*-forms. Hence we see that stability of differential forms is indeed a rather unique phenomenon which is only seen non-trivially for certain combinations of values of *n* and *p*. In fact, other than the obvious cases of p = 1 and 2, we find that up to certain formal dualities there are only three other cases. These are when *p* = 3 and *n* = 6, 7, 8. As we will only be interested in the geometric structures that arise in the cases n = 6 and 7, we will not discuss the case n = 8 any further. With the combination *p* = 3, *n* = 6 we find for any 3-form ρ:

$$dim\ G_\rho = dim\ GL(V) - dim\ \Lambda^p V^*$$
$$= n^2 - n!/p!\,(n-p)!$$
$$= 36 - 20 = 16.$$

We note that the Lie group $SL(3,\mathbb{C})$ has real dimension 16 and it can be shown that indeed $SL(3,\mathbb{C})$ is the stabilizer group as the dimension counting argument above would suggest. A similar calculation for the case p = 3, n = 7 for a 3-form ρ gives dimG_ρ= 49 – 35 = 14. We note that the exceptional Lie group G_2 has dimension 14 and it turns out that in this case G_2 is the stabilizer of the three-form. [8]

Before continuing to the next section we briefly describe how we can deduce the existence of stable $(n-p)$-forms from knowledge of the existence of stable p-forms in n dimensions. If $GL(V)$ has an open orbit on $\Lambda^p V^*$ then it also does on the dual space $\Lambda^p V \cong \Lambda^{n-p} V^* \otimes \Lambda^n V$. As scalars act non-trivially for $p \neq 0$, it follows that there is an open orbit on $\Lambda^{n-p} V^*$. In particular, we can also consider stable 4-forms in 7 dimensions.

2.3 Geometric structures as critical points of functionals

The stabilizers of ρ discussed in the previous section, as well as the symplectic group $Sp(n)$, each preserve a volume element $\phi(\rho) \in \Lambda^n V^*$. Hence if we have a stable p-form ρ defined on a compact oriented manifold M, we can integrate the volume form $\phi(\rho)$ over M to obtain a volume $V(\rho)$. This provides a natural way of defining the action of a p-form on a manifold. The stability of the differential form ρ, in the sense that it lies in an open orbit U of $GL(V)$, ensures that nearby forms are also stable so that the volume functional is defined and smooth on an open set of forms.

The volume form $\phi(\rho)$ associated to ρ determines a $GL(V)$-invariant map $\phi : U \to \Lambda^n V^*$.

The derivative of ϕ at ρ is an invariantly defined element of $(\Lambda^p V^*)^* \otimes \Lambda^n V^*$. As $(\Lambda^p V^*)^* \cong \Lambda^{n-p} V^* \otimes \Lambda^n V$, the derivative lies in $\Lambda^{n-p} V^*$. Therefore, there exists a unique $\hat{\rho} \in \Lambda^{n-p} V^*$ such that

$$D\phi(\dot{\rho}) = \hat{\rho} \wedge \dot{\rho}. \qquad (2.5)$$

Now invariance applied to the action of the scalar matrices implies

$$\phi(\lambda^p \rho) = \lambda^n \phi(\rho). \qquad (2.6)$$

That is, ϕ is homogeneous of degree n/p. Thus, writing $\rho = \dot{\rho}$ in (2.5) and using Euler's formula for a homogeneous function we find that

$$\hat{\rho} \wedge \rho = \frac{n}{p}\phi(\rho). \quad (2.7)$$

Now let M be a closed, oriented n-dimensional manifold. If $\rho \in \Omega^p(M)$ is a globally defined stable p-form, we can define a volume functional as

$$V(\rho) = \int_M \phi(\rho). \quad (2.8)$$

By virtue of the stability of ρ it follows that forms in an open neighbourhood of ρ will also be stable and so we can differentiate the functional.

Theorem 2.1: *A closed stable form $\rho \in \Omega^p(M)$ is a critical point of $V(\rho)$ in its cohomology class if and only if $d\hat{\rho} = 0$.*

Proof: The first variation yields:

$$\delta V(\dot{\rho}) = \int_M D\phi(\dot{\rho}) = \int_M \hat{\rho} \wedge \dot{\rho},$$

where we have used (2.5). We consider variations restricted to closed stable p-forms within a fixed cohomology class so that $\dot{\rho} = d\alpha$. Hence,

$$\delta V(\dot{\rho}) = \int_M \hat{\rho} \wedge d\alpha = \pm \int_M d\hat{\rho} \wedge \alpha$$

by Stokes' theorem.

We see that the variation vanishes for arbitrary $d\alpha$ if and only if $d\hat{\rho} = 0$.

■

Recall that a symplectic manifold was defined as a differentiable manifold equipped with a closed, non-degenerate 2-form ω. The skew-symmetric character of differential forms combined with the fact that skew-symmetric matrices are singular in odd dimensions implies that symplectic manifolds only exist in even dimensions. Let M be a symplectic manifold of dimension $2m$. We take the Liouville volume form $\phi(\omega) = \frac{1}{m!}\omega^m$ and find that the volume $V(\omega) = \int_M \phi(\rho)$ is then constant on a fixed cohomology class $[\omega]$. Thus we see that we can encode the geometry of a symplectic manifold by means of a volume functional. Although this example may seem somewhat trivial since the functional we have constructed has turned out to be constant, it does nonetheless illustrate the manner in which geometric structures can arise as critical points. We now turn to the much more interesting cases of geometries defined by the action of three-forms in 6 and 7 dimensions. In the next two sections we will be exploring the work done in [8] in some detail.

2.4 The geometry of stable three-forms in 6 dimensions

Let W be a 6-dimensional real vector space and $\Omega \in \Lambda^3 W^*$. Let $w \in W$ and consider the interior product $\imath(w)\Omega \in \Lambda^2 W^*$. We have $\imath(w)\Omega \wedge \Omega \in \Lambda^5 W^*$. The exterior product pairing $W^* \otimes \Lambda^5 W^* \to \Lambda^6 W^*$ implies the isomorphism $\Lambda^5 W^* \cong W \otimes \Lambda^6 W^*$. We may now define a linear transformation $K_\Omega : W \to W \otimes \Lambda^6 W^*$ by

$$K_\Omega(w) = A(\imath(w)\Omega \wedge \Omega), \tag{2.9}$$

where $A: \Lambda^5 W^* \to W \otimes \Lambda^6 W^*$ denotes the isomorphism provided by the natural exterior product pairing.

Using K_Ω we define $\lambda(\Omega) \in (\Lambda^6 W^*)^2$ as

$$\lambda(\Omega) = \frac{1}{6} tr K_\Omega^2. \tag{2.10}$$

Now for a real one-dimensional vector space L, we define a vector $u \in L \otimes L = L^2$ as positive ($u > 0$) if $u = s \otimes s$ for some $s \in L$ and negative if $-u$ is positive. In what follows we shall make use of the following lemma, the details of the proof of which can be found in [8].

Lemma 2.1: *Assume $\lambda(\Omega) \neq 0$ for $\Omega \in \Lambda^3 W^*$. Then*

i) $\lambda(\Omega) > 0$ *if and only if* $\Omega = \alpha + \beta$ *where α, β are real decomposable 3-forms and* $\alpha \wedge \beta \neq 0$;

ii) $\lambda(\Omega) < 0$ *if and only if* $\Omega = \alpha + \bar{\alpha}$ *where* $\alpha \in \Lambda^3(W^* \otimes \mathbb{C})$ *is a complex decomposable 3-form and* $\alpha \wedge \bar{\alpha} \neq 0$.

Furthermore, these decomposable 3-forms are unique up to ordering.

The lemma ensures that the following is well-defined.

Definition 2.4: *Let W be oriented and $\Omega \in \Lambda^3 W^*$ be such that $(\Omega) \neq 0$. Writing Ω in terms of decomposable 3-forms ordered by the orientation, we define the complementary 3-form $\hat{\Omega} \in \Lambda^3 W^*$ by*

i) *if* $\lambda(\Omega) > 0$, *and* $\Omega = \alpha + \beta$ *then* $\hat{\Omega} = \alpha - \beta$;

ii) *if* $\lambda(\Omega) < 0$, *and* $\Omega = \alpha + \bar{\alpha}$ *then* $\hat{\Omega} = i(\alpha - \bar{\alpha})$.

The complementary 3-form $\hat{\Omega}$ has the defining property that if $\lambda(\Omega) > 0$ then $\Omega + \hat{\Omega}$ is decomposable and if $\lambda(\Omega) < 0$ then $\Omega + i\hat{\Omega}$ is decomposable. We also note that $\hat{\hat{\Omega}} = -\Omega$.

From here on we shall mainly be interested in the open set $U = \{\Omega \in \Lambda^3 W^* : \lambda(\Omega) < 0\}$.

If $\lambda(\Omega) < 0$, then Ω is the real part of the complex form of type (3,0):

$$\Omega^c = \Omega + i\widehat{\Omega}. \tag{2.11}$$

We can also show that

$$\Omega \wedge \widehat{\Omega} = 2\sqrt{-\lambda(\Omega)}. \tag{2.12}$$

This leads us to a natural symplectic interpretation of $\lambda(\Omega)$ which we shall explore a bit further before proceeding to state and prove the main theorem of this section.

Pick a non-zero vector $\epsilon \in \Lambda^6 W^*$ and note that ϵ is preserved by the group of linear transformations $SL(W) \subset GL(W)$. We endow $\Lambda^3 W^*$ with the symplectic form ω defined by

$$\omega(\Omega_1, \Omega_2)\epsilon = \Omega_1 \wedge \Omega_2 \in \Lambda^6 W^*. \tag{2.13}$$

By the invariance of ω under the action of $SL(W)$ we have the existence of the moment map μ: $\Lambda^3 W^* \to \mathfrak{sl}(W)^*$ where $\mathbf{\mathfrak{sl}}(W)$ is the Lie algebra of $SL(W)$. We may identify this Lie algebra with its dual via the bi-invariant form $tr(XY)$, which then allows us to write the linear map K_Ω defined by (2.9) in terms of the moment map simply as $\mu(\Omega) = K_\Omega$. We now see that $\lambda(\Omega)$ is an $SL(W)$- invariant function which by (2.10) can be expressed as

$$\lambda(\Omega) = \frac{1}{6} tr(\mu(\Omega)^2). \tag{2.14}$$

Restriction of the symplectic form to the open set $U = \{\Omega \in \Lambda^3 W^* : \lambda(\Omega) < 0\}$ defines the structure of a symplectic manifold. On U we define a smooth function ϕ as

$$\phi(\Omega) = \sqrt{-\lambda(\Omega)}. \tag{2.15}$$

We rewrite (2.12) using this smooth function as $\Omega \wedge \widehat{\Omega} = 2\phi\epsilon$ so that $2\phi(\widehat{\Omega})\epsilon = \widehat{\Omega} \wedge \widehat{\widehat{\Omega}}$. Recalling that $\widehat{\widehat{\Omega}} = -\Omega$ we find that $2\phi(\widehat{\Omega})\epsilon = -\widehat{\Omega} \wedge \Omega = \Omega \wedge \widehat{\Omega}$. Hence $\phi(\widehat{\Omega}) = \phi(\Omega)$.

The function ϕ defines a Hamiltonian vector field X_ϕ on U.

We now turn to our main result in this section. Let M be a closed, oriented 6-dimensional manifold. Let Ω be a 3-form on M which defines at each point a vector in $\Lambda^3 T^*$ and thus a global section $\lambda(\Omega)$ of $(\Lambda^6 T^*)^2$. As $|\lambda(\Omega)|$defines a non-negative continuous section of $(\Lambda^6 T^*)^2$, we may obtain a section of $\Lambda^6 T^*$ by taking the square root and indeed find that $\sqrt{|\lambda(\Omega)|}$ is a smooth continuous 6-form for $\lambda(\Omega) \neq 0$. We define a functional Φ on $C^\infty(\Lambda^3 T^*)$ by integrating this 6-form over M:

$$\Phi(\Omega) = \int_M \sqrt{|\lambda(\Omega)|}. \tag{2.16}$$

The $GL(6,\mathbb{R})$-invariance of λ implies that the functional Φ is invariant under the action of orientation-preserving differomorphisms.

Theorem 2.2: *Let M be a compact complex 3-dimensional manifold with trivial canonical bundle and Ω be the real part of a non-vanishing holomorphic 3-form. Then Ω is a critical point of the functional Φ restricted to the cohomology class $[\Omega] \in H^3(M,\mathbb{R})$.*

Conversely, if Ω is a critical point on a cohomology class of an oriented closed 6-dimensional manifold M and $\lambda(\Omega) < 0$ everywhere, then Ω is the real part of a non-vanishing holomorphic 3-form.

Proof: Let ϵ be a non-vanishing 6-form on M and Ω the real part of a non-vanishing holomorphic 3-form. Thus Ω is a closed 3-form with $\lambda(\Omega) < 0$ and writing $\lambda(\Omega) = -\phi(\Omega)^2\epsilon^2$ we have

$$\Phi(\Omega) = \int_M \phi(\Omega)\epsilon. \tag{2.17}$$

As $(\Omega) \neq 0$, ϕ is smooth enabling us to take the first variation of this functional:

$$\delta\Phi(\dot{\Omega}) = \int_M d\phi(\dot{\Omega})\epsilon. \tag{2.18}$$

Using the symplectic interpretation we discussed earlier we have $d\phi(\dot{\Omega}) = \omega(X_\phi,\dot{\Omega})$ and $X_\phi = -\widehat{\Omega}$ is the Hamiltonian vector field, so that $d\phi(\dot{\Omega})\epsilon = -\widehat{\Omega} \wedge \dot{\Omega}$. Therefore

$$\delta\Phi(\dot{\Omega}) = -\int_M \widehat{\Omega} \wedge \dot{\Omega}. \tag{2.20}$$

Since the variation is within a fixed cohomology class we may write $\dot{\Omega} = d\varphi$ for a 2-form φ. Hence

$$\delta\Phi(\dot{\Omega}) = -\int_M \widehat{\Omega} \wedge d\varphi = \int_M d\widehat{\Omega} \wedge \varphi \tag{2.21}$$

using Stokes' theorem. If follows that $\delta\Phi = 0$ at Ω for all φ if and only if $d\widehat{\Omega} = 0$.

If M is a complex manifold with a non-vanishing holomorphic 3-form $\Omega + i\widehat{\Omega}$, then (since a holomorphic 3-form is closed) $d\Omega = d\widehat{\Omega} = 0$ and so Ω is a critical point of Φ.

For the converse, we assume that $d(\Omega + i\widehat{\Omega}) = 0$ and $\lambda(\Omega) < 0$. For $\lambda(\Omega) < 0$ we may define an almost complex structure I_Ω by $I_\Omega = \frac{1}{\sqrt{-\lambda(\Omega)}} K_\Omega$ where K_Ω is the linear map defined in (2.9). Now a complex 1-form θ is of type $(1,0)$ with respect to I_Ω if and only if

$$\theta \wedge (\Omega + i\widehat{\Omega}) = 0. \tag{2.22}$$

Taking the exterior derivative of this equation yields

$$d\theta \wedge \left(\Omega + i\widehat{\Omega}\right) = 0, \tag{2.23}$$

implying that $d\theta$ has no (0,2) component. By the Newlander-Nirenberg theorem we deduce that I_Ω is integrable. As $\left(\Omega + i\widehat{\Omega}\right)$ is of type (3,0), we have

$$0 = d\left(\Omega + i\widehat{\Omega}\right) = \bar{\partial}\left(\Omega + i\widehat{\Omega}\right). \tag{2.24}$$

That is, $\Omega + i\widehat{\Omega}$ is a holomorphic 3-form. This completes the proof.

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We recall that a Calabi-Yau 3-fold can be defined as a Kähler manifold with a covariant constant holomorphic 3-form. Thus we see that the structure of a Calabi-Yau complex algebraic 3-fold can be encoded as a critical point of a diffeomorphism-invariant functional in dimension 6.

2.5 The geometry of stable three-forms in 7 dimensions

We now present a parallel treatment of the 7 dimensional case. Let W be a 7-dimensional real vector space and $\Omega \in \Lambda^3 W^*$. Define $B_\Omega : W \otimes W \rightarrow \Lambda^7 W^*$ by

$$B_\Omega(v, w) = -\frac{1}{6}\imath(v)\Omega \wedge \imath(w)\Omega \wedge \Omega, \qquad \forall v, w \in W \tag{2.25}$$

where the interior products $\imath(v)\Omega,\ \imath(w)\Omega \in \Lambda^2 W^*$. By the properties of the exterior product we see that the 2-forms $\imath(v)\Omega$ and $\imath(w)\Omega$ commute in the expression in (2.25) and so the equation is symmetric under the interchange of v and w. Thus B_Ω is a symmetric bilinear form on W with values in the one-dimensional space $\Lambda^7 W^*$ and so we may define a linear map $K_\Omega : W \rightarrow W \otimes \Lambda^7 W^*$ which satisfies

$$det\, K_\Omega \in (\Lambda^7 W^*)^9. \tag{2.26}$$

If $det\, K_\Omega \neq 0$, we find that an orientation is induced on W (as the exponent 9 is an odd number) with respect to which we take the positive root $(det\, K_\Omega)^{1/9} \in \Lambda^7 W^*$. This allows us to define a natural inner product

$$g_\Omega(v, w) = B_\Omega(v, w)(det\, K_\Omega)^{-1/9} \tag{2.27}$$

with volume form $(det\, K_\Omega)^{1/9}$.

Recall that in our earlier discussion of stability of differential forms we noted the existence of open orbits of forms under the action of $GL(V)$ as a rather special feature of a vector space V of dimension 6 or 7. Here we will need to define a particular open orbit which will play a significant role in the remainder of our analysis. We begin by choosing a basis $w_1, \cdots, w_7$ of W and its dual basis $\theta_1, \cdots, \theta_7$ in W^*. Define the 3-form

$$\varphi = (\theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4) \wedge \theta_5 + (\theta_1 \wedge \theta_3 - \theta_4 \wedge \theta_2) \wedge \theta_6 + (\theta_1 \wedge \theta_4 - \theta_2 \wedge \theta_3) \wedge \theta_7 + \theta_4\theta_6\theta_7.$$

The special significance of φ is that it has the property that when substituted in (2.27) it yields the Euclidean metric $g_\varphi = \sum \theta_i^2$. We denote the open orbit of φ by $U \subset \Lambda^3 W^*$and define a form to be *positive* if it lies in this orbit.

Definition 2.5: *Pick a basis vector $\epsilon \in \Lambda^7 W^*$and let Ω be a positive 3-form. Define $\phi(\Omega) \in \mathbb{R}$ by $\phi(\Omega)\epsilon = (det\ K_\Omega)^{1/9}$.*

Let $*: \Lambda^3 W^* \to \Lambda^4 W^*$ be the Hodge star operator associated with the metric g_Ω. Using the 3-form φ it can be shown that the invariant ϕ satisfies $\Omega \wedge * \Omega = 6\phi$.

The non-degenerate pairing provided by the exterior product sets up an isomorphism $(\Lambda^3 W^*)^* \cong \Lambda^4 W^*$. Hence $d\phi$ is a $\Lambda^4 W^*$-valued function.

The Hodge star identifies $\Lambda^3 W^*$ and $\Lambda^4 W^*$in an invariant way. It has been shown in the mathematics literature that G_2, which is the stabilizer group of Ω, has only the orthogonal projection onto Ω as an invariant of its action on $\Lambda^3 W^*$. [11] Therefore we must have $d\phi(\dot{\Omega}) = C(\Omega) * \Omega \wedge \dot{\Omega}$ for some real function C of Ω.

We note by *Definition 2.5* that since K_Ω is homogeneous of degree 3, ϕ is also homogeneous and of degree $3 \times \frac{7}{9} = \frac{7}{3}$. Taking $\Omega = \dot{\Omega}$ in $d\phi(\dot{\Omega}) = C(\Omega) * \Omega \wedge \dot{\Omega}$ and exploiting the homogeneity of ϕ, we obtain

$$C(\Omega)(* \Omega \wedge \Omega) = d\phi(\Omega) = \frac{7}{3}\phi(\Omega). \tag{2.28}$$

Comparison with $\Omega \wedge * \Omega = 6\phi$ shows that $C = \frac{7}{18}$. That is, C is a real constant.

We now define the relevant functional in dimension 7 and study its critical points. We have established that if Ω is a 3-form on a closed 7-dimensional manifold M, then $(det\ K_\Omega)^{1/9}$ defines a section of $\Lambda^7 T^*$ which is a volume form of the metric g_Ω whenever $det\ K_\Omega \neq 0$. We define a functional Φ on $C^\infty(\Lambda^3 T^*)$ as

$$\Phi(\Omega) = \int_M (det\ K_\Omega)^{1/9}. \tag{2.29}$$

Theorem 2.3: *Let M be a closed 7-dimensional manifold with a metric with holonomy G_2, with defining 3-form Ω. Then Ω is a critical point of the functional Φ restricted to the cohomology class $[\Omega] \in H^3(M, \mathbb{R})$.*

Conversely, if Ω is a critical point on a cohomology class of a closed oriented 7-dimensional manifold such that Ω is always positive, then Ω defines a metric with holonomy G_2 on M.

Proof: Let Ω be a positive closed 3-form and pick a non-vanishing 7-form ϵ on M. Using Definition 2.5 in the expression (2.29) for the functional Φ, we find

$$\Phi(\Omega) = \int_M \phi(\Omega)\epsilon. \tag{2.30}$$

The first variation gives:

$$\delta\Phi(\dot{\Omega}) = \int_M d\phi(\dot{\Omega})\epsilon = \frac{7}{18}\int_M * \,\Omega \wedge \dot{\Omega}\,. \tag{2.31}$$

Since the variation is within a fixed cohomology class we may rewrite $\dot{\Omega} = d\varphi$ for some 2-form φ. We see that the variation vanishes precisely when $\int_M d(*\,\Omega) \wedge \varphi = 0$ for all φ. That is, $\delta\Phi = 0$ if and only if $d(*\,\Omega) = 0$.

If Ω is covariant constant with respect to the metric g_Ω, so too is $*\,\Omega$. This implies that $*\,\Omega$ is closed and thus a G_2-manifold gives a critical point of Φ.

It is shown in [11] that if $d(\Omega) = d(*\,\Omega) = 0$ then Ω is covariant constant for g_Ω which proves the converse part of the theorem and so completes the proof.

■

2.6 Constrained critical points and weak holonomy

Recall that in our introduction to Riemannian manifolds with special holonomy, we noted that these manifolds are characterized by the existence of covariantly constant spinor fields ψ, satisfying $\nabla\psi = 0$. We now define a second category of special metrics known as metrics with *weak holonomy*.

Definition 2.6: *A Riemannian manifold is said to be of weak holonomy if there exist spinor fields ψ satisfying $\nabla_X\psi = \lambda X\psi$.*

These manifolds can be listed as:

(1) The spheres

(2) Einstein-Sasakian manifolds in odd dimensions

(3) 3-Sasakian manifolds in dimension $4k + 3$

(4) Weak holonomy $SU(3)$ manifolds in dimension 6

(5) Weak holonomy G_2 manifolds in dimension 7. [12]

We shall be interested in the 6-dimensional weak holonomy $SU(3)$ and the 7-dimensional weak holonomy G_2 manifolds. We discuss a natural approach to these structures involving invariant functionals of differential forms subject to certain constraints. This elegant characterization is independent of spinor fields and Riemannian metrics.

We begin by considering a compact oriented manifold M of dimension n. There exists a non-degenerate pairing between the spaces of forms $\Omega^p(M)$ and $\Omega^{n-p}(M)$ provided by

$$\int_M \alpha \wedge \beta, \tag{2.32}$$

for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{n-p}(M)$.

If α is exact so that $\alpha = d\gamma$ for some $\gamma \in \Omega^{p-1}(M)$, then Stokes' theorem gives

$$\int_M d\gamma \wedge \beta = (-1)^p \int_M \gamma \wedge d\beta \tag{2.33}$$

Thus,

$$\int_M d\gamma \wedge \beta = 0, \;\; \forall \gamma \in \Omega^{p-1}(M) \quad \Leftrightarrow \quad d\beta = 0, \tag{2.34}$$

which establishes the non-degenerate pairing

$$\Omega^p_{exact}(M) \cong \Omega^{n-p}(M)/\,\Omega^{n-p}_{closed}(M). \tag{2.35}$$

The exterior derivative d defines an isomorphism between $\Omega^{n-p}(M)/\,\Omega^{n-p}_{closed}(M)$ and $\Omega^{n-p+1}_{exact}(M)$ which finally yields the result

$$\Omega^p_{exact}(M)^* \cong \Omega^{n-p+1}_{exact}(M). \tag{2.36}$$

Recall that as part of our discussion of the stability of differential forms we showed how one can deduce the existence of stable $(n-p)$-forms from knowledge of the existence of stable p-forms in n dimensions. In particular, the existence of stable 3-forms in 7 dimensions implies the existence of stable 4-forms in 7 dimensions and indeed we may use this duality to translate our results regarding the geometry of 3-forms from the previous section to those of the geometry of 4-forms. With $n = 7$ and $p = 4$, (2.36) gives

$$\Omega^4_{exact}(M)^* \cong \Omega^4_{exact}(M). \tag{2.37}$$

This ensures that the non-degenerate quadratic form Q on $\Omega^4_{exact}(M)$ given by

$$Q(d\gamma) = \int_M \gamma \wedge d\gamma, \tag{2.38}$$

is well-defined. We can now state and prove our key theorem characterizing metrics with weak holonomy G_2.

Theorem 2.4: *Let σ be an exact stable 4-form on a compact 7-dimensional manifold M and $\phi(\sigma) \in \Lambda^7 V^*$ a volume form preserved by the stabilizer of σ. Following the analysis in section 2.3, we can define a volume functional as*

$$V(\sigma) = \int_M \phi(\sigma).$$

Now σ is a critical point of $V(\sigma)$ subject to the constraint $Q(\sigma) = constant$, if and only if σ defines a metric with weak holonomy G_2.

Proof: The derivative of ϕ at σ is a linear map $D\phi$ from $\Lambda^p V^*$ to $\Lambda^n V^*$. Thus,

$$D\phi \in (\Lambda^p V^*)^* \otimes \Lambda^n V^*. \tag{2.39}$$

The identification $(\Lambda^p V^*)^* \cong \Lambda^{n-p} V^* \otimes \Lambda^n V$ implies the existence of a unique

$$\hat{\sigma} \in \Lambda^{n-p} V^*, \tag{2.40}$$

such that

$$D\phi(\dot{\sigma}) = \hat{\sigma} \wedge \dot{\sigma}. \tag{2.41}$$

The analysis so far is valid for all admissible values of n and p. We note that in general the map

$$\hat{\sigma} : \sigma \longmapsto \hat{\sigma}(\sigma) \tag{2.42}$$

need not be linear. For $n = 7$, $p = 3$ or 4, however, this map coincides with the Hodge star operator $*$ for the inner product on V. That is, here we have

$$\hat{\sigma} = * \, \sigma. \tag{2.43}$$

The first variation of $V(\sigma)$ at $\sigma = d\gamma$ is

$$\delta V(d\dot{\gamma}) = \int_M D\phi(d\dot{\gamma}) \ = \int_M \hat{\sigma} \wedge d\dot{\gamma} \tag{2.44}$$

$$\Longrightarrow \quad \delta V(d\dot{\gamma}) = \int_M (* \, \sigma) \wedge d\dot{\gamma}. \tag{2.45}$$

The first variation of the quadratic form $Q(d\gamma) = \int_M \gamma \wedge d\gamma$ is

$$\delta Q(d\dot{\gamma}) = 2 \int_M \dot{\gamma} \wedge d\gamma. \tag{2.46}$$

Using a Lagrange multiplier, the constrained critical point is given by

$$d(*\,\sigma) = \lambda\sigma. \tag{2.47}$$

This is precisely equivalent to the structure of a manifold with weak holonomy G_2. [13]

■

We now consider a constrained variational problem in the case $n = 6$, $p = 3$. For this combination of n and p, (2.36) gives the non-degenerate pairing:

$$\Omega^3_{exact}(M)^* \cong \Omega^4_{exact}(M). \tag{2.48}$$

This pairing can be made explicit for an exact 3-form $\rho = d\alpha$ and an exact 4-form $\sigma = d\beta$ via

$$\langle\rho,\sigma\rangle = \int_M \alpha \wedge \sigma = -\int_M \rho \wedge \beta. \tag{2.49}$$

The stability of the 4-form σ implies the existence of a symplectic 2-form ω satisfying $\sigma = \frac{1}{2}\omega \wedge \omega$. As a consequence of its stability, σ defines a reduction of the structure group of the tangent bundle of the 6-dimensional manifold M to $Sp(6,\mathbb{R})$, and similarly ρ defines a reduction to $SL(3,\mathbb{C})$. Taken together ρ and σ provide a reduction to $SU(3)$, which may be viewed as an intersection of these two groups provided the following two compatibility conditions for ρ and σ are satisfied:

$$\omega \wedge \rho = 0, \tag{2.50}$$

and for some constant c,

$$\phi(\rho) = c\phi(\sigma). \tag{2.51}$$

Definition 2.7: *Let ρ be a stable 3-form and $\sigma = \frac{1}{2}\omega \wedge \omega$ a stable 4-form. We say that the pair (ρ,σ) is of positive type if the almost complex structure I_ρ defined by ρ turns $\omega(X, I_\rho X)$ into a positive definite form.*

We now have the following Theorem:

Theorem 2.5: *Let (ρ,σ) be a pair of exact stable forms of positive type defined on a compact 6-dimensional manifold M. The pair (ρ,σ) is a critical point of $3V(\rho) + 8V(\sigma)$ subject to the constraint $P(\rho,\sigma) = \langle\rho,\sigma\rangle = constant$, if and only if ρ and σ define a metric with weak holonomy $SU(3)$.*

Aside: We note that the coefficients 3 and 8 in the linear combination of $V(\rho)$ and $V(\sigma)$ have no special significance as far as the validity of the above statement is concerned, other than the fact that they are both positive real numbers. These specific values were chosen to simplify some of the expressions in the proof below.

Proof: Reasoning as in the first few lines of the proof of *Theorem 2.4*, the first variation of $3V(\rho) + 8V(\sigma)$ gives

$$3\int_M \hat{\rho} \wedge \dot{\rho} + 4\int_M \omega \wedge \dot{\sigma}. \tag{2.52}$$

The first variation of $P(\rho, \sigma)$ gives

$$\delta P(\dot{\rho}, \dot{\sigma}) = \int_M \dot{\rho} \wedge \beta + \int_M \dot{\sigma} \wedge \alpha, \tag{2.53}$$

for $\rho = d\alpha$ and $\sigma = d\beta$. Introducing a Lagrange multiplier 12λ, we find the constrained critical point is determined by

$$d\hat{\rho} = -2\lambda\omega^2 \tag{2.54}$$

and

$$d\omega = 3\lambda\rho. \tag{2.55}$$

The equations (2.54) and (2.55) together imply a metric of weak holonomy $SU(3)$, which is also known as a *nearly Kähler* metric. For the technical details of the last step of the proof deducing a metric of weak holonomy $SU(3)$ from equations (2.54) and (2.55), the interested reader is referred to [10].

■

Finally, we note that (2.54) and (2.55) taken together also imply the important compatibility conditions necessary for the combination of ρ and σ to give an $SU(3)$ structure on M.

2.7 A Hamiltonian Flow

Let $\mathcal{A}$ be a fixed cohomology class in $H^3(M, \mathbb{R})$ and $\mathcal{B}$ a fixed cohomology class in $H^4(M, \mathbb{R})$. $\mathcal{A}$ and $\mathcal{B}$ are affine spaces whose tangent spaces are naturally isomorphic to $\Omega^3_{exact}(M)$ and $\Omega^4_{exact}(M)$, respectively. We consider the Cartesian product $\mathcal{A} \times \mathcal{B}$ whose tangent space at each point is isomorphic to $\Omega^3_{exact}(M) \times \Omega^4_{exact}(M)$. We construct a symplectic form ω on $\mathcal{A} \times \mathcal{B}$ using the pairing $\langle \cdot, \cdot \rangle$ defined in (2.49) as follows

$$\omega\big((\rho_1, \sigma_1), (\rho_2, \sigma_2)\big) = \langle \rho_1, \sigma_2 \rangle - \langle \rho_2, \sigma_1 \rangle. \tag{2.56}$$

We now present our final theorem of this chapter which provides a bridge between $SU(3)$ structures and G_2 holonomy metrics.

Theorem 2.6: *Let $(\rho, \sigma) \in \mathcal{A} \times \mathcal{B}$ be a pair of exact stable forms of positive type that satisfy the compatibility conditions $\omega \wedge \rho = 0$ and $\phi(\rho) = 2\phi(\sigma)$ at time $t = t_0$. Define the functional H by*

$$H = V(\rho) - 2V(\sigma), \tag{2.57}$$

and allow the pair (ρ, σ) to evolve with respect to the Hamiltonian flow determined by the vector field X satisfying $i_X\omega = DH$, where $i_X\omega$ denotes an interior product. The 3-form

$$\varphi = dt \wedge \omega + \rho, \tag{2.58}$$

now defines a metric with holonomy G_2 on the 7-dimensional manifold $M \times (a, b)$.

Proof: We rewrite the equation $i_X\omega = DH$ as

$$\int_M \dot{\rho} \wedge \beta - \int_M \dot{\sigma} \wedge \alpha = DH. \tag{2.59}$$

Differentiating H, we find

$$DH(d\alpha, d\beta) = \int_M \hat{\rho} \wedge d\alpha - \int_M \omega \wedge d\beta, \tag{2.60}$$

which combines with (2.59) to give:

$$\int_M (\dot{\rho} - d\omega) \wedge \beta - \int_M (\dot{\sigma} + d\hat{\rho}) \wedge \alpha = 0, \tag{2.61}$$

for all α and β, following an application of Stokes' theorem. That is,

$$\partial\rho/\partial t = d\omega \tag{2.62}$$

and, using $\dot{\sigma} = \frac{\partial}{\partial t}(\frac{1}{2}\omega \wedge \omega) = \omega \wedge \dot{\omega}$,

$$\partial\sigma/\partial t = \omega \wedge \partial\omega/\partial t = -d\hat{\rho}. \tag{2.63}$$

A key step in the proof is to show that the compatibility conditions are preserved under time evolution. This ensures that the $SU(3)$ geometry on M is not altered as the time t is allowed to evolve. To do this for the first condition, we define

$$\mu_X(\rho, \sigma) = \int_M i_X\sigma \wedge \rho \tag{2.67}$$

as a function of the vector field X. We have,

$$\begin{aligned} d\mu_X(\dot{\rho}, \dot{\sigma}) &= \int_M i_X\dot{\sigma} \wedge \rho + \int_M i_X\sigma \wedge \dot{\rho} \\ &= -\int_M \dot{\sigma} \wedge i_X\rho + \int_M i_X\sigma \wedge \dot{\rho} \\ &= \int_M i_X\rho \wedge \dot{\sigma} + \int_M i_X\sigma \wedge \dot{\rho}, \end{aligned} \tag{2.68}$$

where we have used $\dot{\sigma} \wedge \rho = 0$ in the step from the first to the second lines.

Recalling that the Lie derivative $\mathcal{L}_X$ can be written as $\mathcal{L}_X = di_X + i_X d$ and using the fact that ρ and σ are closed, we have by the nilpotency of the exterior derivative that

$$\mathcal{L}_X(\rho, \sigma) = (di_X\rho, di_X\sigma) \tag{2.69}$$

and thus we can rewrite (2.68) in the symplectic form

$$d\mu_X(\dot{\rho}, \dot{\sigma}) = \langle \mathcal{L}_X(\rho, \sigma), (\dot{\rho}, \dot{\sigma}) \rangle. \tag{2.70}$$

The map $\mu: X \longmapsto \mu_X$ is thus the moment map of $Diff(M)$. Rewriting (2.67) in terms of the non-degenerate 2-form ω gives

$$\mu_X(\rho, \sigma) = \int_M i_X\omega \wedge \omega \wedge \rho, \tag{2.71}$$

which implies $\mu_X = 0$ for all X if and only if $\omega \wedge \rho = 0$.

The diffeomorphism invariance of the functionals $V(\rho)$ and $V(\sigma)$ ensures that H commutes with μ_X with respect to the Poisson bracket. It follows that the Hamiltonian flow of H and the subset of $\mathcal{A} \times \mathcal{B}$ consisting of the pairs (ρ, σ) for which μ_X uniformly vanishes for all X, are tangential. This subset of $\mathcal{A} \times \mathcal{B}$ is precisely the space characterized by $\omega \wedge \rho = 0$ and so we have established that if $\omega \wedge \rho = 0$ is satisfied at $t = t_0$, it remains satisfied for all values of t. It also follows that $\omega \wedge \hat{\rho} = 0$ for all t.

Recall that the equation $D\phi(\dot{\rho}) = \hat{\rho} \wedge \dot{\rho}$ uniquely determines $\hat{\rho}$ for a given ρ. As $\phi(\rho)$ is homogeneous of degree $n/p = \frac{6}{3} = 2$, its derivative $D\phi$ is homogeneous of degree one. Thus,

$$D^2\phi(\rho, \dot{\rho}) = D\phi(\dot{\rho}) = \hat{\rho} \wedge \dot{\rho}. \tag{2.72}$$

A key observation in proving the preservation of the second compatibility condition is that the derivative of $\hat{\rho}$ can be expressed in terms $D^2\phi$, since $\hat{\rho}$ itself is defined by $D\phi$. Hence,

$$\partial\hat{\rho}/\partial t \wedge \rho = D^2\phi(\rho, \dot{\rho}), \tag{2.73}$$

which combined with (2.72) gives $\partial\hat{\rho}/\partial t \wedge \rho = \hat{\rho} \wedge \partial\rho/\partial t$ and thus

$$\partial(\hat{\rho} \wedge \rho)/\partial t = 2\hat{\rho} \wedge \partial\rho/\partial t = 2\hat{\rho} \wedge d\omega, \tag{2.74}$$

where we have used the evolution equation $\partial\rho/\partial t = d\omega$. Taking the exterior derivative of $\omega \wedge \hat{\rho} = 0$ yields $\hat{\rho} \wedge d\omega = -d\hat{\rho} \wedge \omega$ and so

$$\begin{aligned}\partial(\hat{\rho} \wedge \rho)/\partial t &= -2d\hat{\rho} \wedge \omega \\ &= 2\omega \wedge \partial\omega/\partial t \wedge \omega\end{aligned}$$

$$= \frac{2}{3} \partial\omega^3/\partial t. \tag{2.75}$$

That is, $\partial\left(\hat{\rho} \wedge \rho - \frac{2}{3}\omega^3\right)/\partial t = 0$ from which it follows that if $\phi(\rho) = 2\phi(\sigma)$ holds at $t = t_0$, then it does so for all t. Therefore, the flow respects the $SU(3)$ structure on M.

Now consider $\varphi = dt \wedge \omega + \rho$. We have

$$d\varphi = dt \wedge d\omega = dt \wedge \partial\rho/\partial t = 0, \tag{2.76}$$

where we have used (2.62) and $d\rho = 0$. Using the Hodge star operator defined by the G_2 metric determined by φ, we have

$$* \varphi = dt \wedge \hat{\rho} - \sigma, \tag{2.77}$$

and thus

$$d * \varphi = dt \wedge d\hat{\rho} = -dt \wedge \partial\sigma/\partial t = 0, \tag{2.78}$$

using (2.63) and $d\sigma = 0$. We recall that $d\varphi = d * \varphi = 0$ is precisely the condition under which φ defines a metric of holonomy G_2 on a 7-dimensional manifold N which in this case is of the form $N = M \times (a, b)$ for some interval (a, b).

■

Finally, we note that the converse to the above theorem also holds true as the steps in the proof are essentially reversible. Specifically, if N is a 7-dimensional manifold of G_2 holonomy that is foliated by equidistant compact hypersurfaces diffeomorphic to a 6-dimensional manifold M, the defining closed forms ρ and σ restricted to each hypersurface evolve as the Hamiltonian flow of the functional $H = V(\rho) - 2V(\sigma)$.

III. Topological String Theory

We provide an introduction to topological strings and show how the A and B models naturally arise as the two distinct types of topological string theory. After briefly reviewing the relevant material from Chapter II, we construct the V_S and V_H Hitchin functionals which generate symplectic and holomorphic structures, respectively. We then consider certain conjectured relations between these functionals and the topological A and B models, which we shall support with classical arguments.

3.1 $\boldsymbol{\mathcal{N} = (2,2)}$ supersymmetry

The string theories with which we shall be concerned involve maps ϕ from a surface Σ to a target space X. In string theory we weigh each map ϕ by the Polyakov action and integrate over all such ϕ and over all metrics on Σ. If we only integrate over ϕ we obtain a 2-dimensional quantum field theory whose saddle points are locally area-minimizing surfaces in X. This 2-dimensional theory is known as a “sigma model into X” and has what is known as $\mathcal{N} = (2,2)$ supersymmetry. As we shall see, the definition of the topological string rests upon this $(2,2)$ supersymmetry.

$\mathcal{N} = 2$ supersymmetry means there are 4 worldsheet currents and prescribed operator product relations. These worldsheet currents are

$$J, G^+, G^-, T \tag{3.1}$$

and have spins $1, \frac{3}{2}, \frac{3}{2}, 2$ respectively. In the case of the sigma model into X, these are analogous to the operators $deg, \bar{\partial}, \bar{\partial}^\dagger, \Delta$ acting on the space of differential forms on the loop space of X. This analogy suggests the following important operator product relations which do in fact hold true.

$$(G^\pm)^2 \sim 0, \tag{3.2}$$

$$G^+G^- \sim T + J. \tag{3.3}$$

We will now explain the structure of $\mathcal{N} = (2,2)$ supersymmetry which will be vital to the rest of our discussion. Let X be Calabi-Yau so that the sigma model is conformal. Here we can write each of the currents J, G^+, G^-, T as a sum of two commuting holomorphic and antiholomorphic copies, effectively producing two copies of the $\mathcal{N} = 2$ algebra. We write these as (J, G^+, G^-, T) and $(\bar{J}, \overline{G^+}, \overline{G^-}, \bar{T})$ and refer to this split structure as $\mathcal{N} = (2,2)$ supersymmetry. The $\mathcal{N} = (2,2)$ structure of superconformal field theory should be viewed as an invariant of the Calabi-Yau manifold X. [15]

3.2 The “twisting” procedure and the topological string

We consider a procedure referred to as “twisting” whereby we can produce a topological theory from the $\mathcal{N} = (2,2)$ superconformal field theory discussed in the previous section. We can think of the transition from our ordinary field theory to its topological version as analogous to the move from the de Rham complex $\Omega^*(X)$ to its cohomology $H^*(X)$. Inspired by this analogy and in view of $(G^+)^2 \sim 0$, we encounter an immediate obstacle when we attempt to form the cohomology of the zero mode of G^+: to obtain a scalar zero mode we must begin with an operator of spin 1, whereas G^+ has spin $3/2$.

The resolution to this problem lies in “twisting” the sigma model as described in detail by Edward Witten in [16] and [17]. The twist can be described as a shift in the operator $T \to T'$ which has the effect of changing the spins S of all the operators $S \to S'$ by an amount proportional to their $U(1)$ charge q:

$$T' = T - \frac{1}{2}\partial J \tag{3.4}$$

and

$$S' = S - \frac{1}{2}q. \tag{3.5}$$

Following this twisting procedure the operators (G^+, J) have spin 1 and (T, G^-) have spin 2. We note that although $G^\pm$ now possess integer spin they of course continue to retain their fermionic nature. We also point out that in (3.4) we have made a choice between one of two possibilities in the direction of shift. We could equally well-have chosen to shift $T \to T'$ in the opposite direction: $T' = T + \frac{1}{2}\partial J$. For now we will suffice it to say that this degree of freedom will have a profound consequence for the number of ways we can construct a topological string from a given Calabi-Yau manifold. We continue with the choice made in (3.4) and take $Q = G_0^+$. We note that Q is well-defined on arbitrary Σ, fermionic and nilpotent: $Q^2 = 0$. We restrict our attention to observables which are annihilated by Q.

Here we note that the twisted $\mathcal{N} = 2$ algebra is isomorphic to a subalgebra of the symmetry algebra of the bosonic string. This isomorphism can be written as

$$(G^+, J, T, G^-) \leftrightarrow (Q, J_{ghost}, T, b) \tag{3.6}$$

where (Q, J_{ghost}) and (T, b) are currents of spin 1 and 2 respectively, and (Q, b) are the BRST current and antighost corresponding to the diffeomorphism symmetry on the bosonic string worldsheet.

By performing the Faddeev-Popov procedure we can reduce the integral over metrics on Σ to an integral over the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces, where the measure is provided by the b ghosts. The genus g free energy of the bosonic string in the $g > 1$ case is found in this way to be

$$\int_{\mathcal{M}_g} \langle \left| \prod_{i=1}^{3g-3} b(\mu_i) \right|^2 \rangle. \tag{3.7}$$

This expression requires only slight modifications for the $g = 0, 1$ cases related to the fact that the sphere and torus admit nonzero holomorphic vector fields. In (3.7), the $\langle \cdots \rangle$ denotes a conformal field theory correlation function and the μ_i are "Beltrami differentials". These are antiholomorphic one-forms on Σ with values in the holomorphic tangent bundle.

We can now define a "topological string" from the correlation functions of the $\mathcal{N} = (2,2)$ superconformal field theory on fixed Σ by replacing b by G^- in (3.7) due to the isomorphism (3.6). That is,

$$F_g = \int_{\mathcal{M}_g} \langle \left| \prod_{i=1}^{3g-3} G^-(\mu_i) \right|^2 \rangle. \tag{3.8}$$

The full topological string free energy is then defined to be

$$\mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2-2g} F_g. \tag{3.9}$$

This expression is to be understood as an asymptotic series in λ, where λ is the "string coupling constant" weighing contributions at each genus. Thus we arrive at a definition of the topological string partition function $Z = \exp \mathcal{F}$.

3.3 The topological A and B models

Recall that when defining the shift $T \to T'$ by (3.4) we noted that we could just as well have performed the shift in the opposite direction so that

$$T' = T + \frac{1}{2} \partial J \tag{3.10}$$

and

$$S' = S + \frac{1}{2} q. \tag{3.11}$$

Following this twist, G^- will have spin 1 and G^+ will have spin 2 so that we may now take G^- as our BRST operator Q. Thus we see that we find two distinct paths to the construction of a topological string theory from a given Calabi-Yau X. We refer to these topological string theories as the A and B models corresponding to the choices $(G^+, \overline{G^+})$ and $(G^+, \overline{G^-})$ for the BRST operators, respectively. We also note the existence of $\overline{\mathrm{A}}$ and $\overline{\mathrm{B}}$ models arising from a similar freedom in the choice of shift direction in the antiholomorphic sector. However, since these are simply related to the A and B models via overall complex conjugation we are essentially only left with two distinct types of topological string theory.

In the A model, the combination $Q + \bar{Q}$ is the d operator on X and its cohomology is the de Rham cohomology. It is natural to impose a further "physical state" constraint which leads to considering only the degree $(1,1)$ part of this cohomology which correspond to a deformation of the Kähler form on X. The observables of the A model which we are considering are deformations of the Kähler moduli of X.

In the B model, the complex is the $\bar{\partial}$ cohomology with values in $\Lambda^* TX$. Imposing the "physical state" constraint once again leads us to considering only the degree $(1,1)$ part of the cohomology. We find that the observables are $(0,1)$-forms taking values in TX. That is, the observables in this case are the Beltrami differentials on X. These differentials span the space of infinitesimal deformations of the $\bar{\partial}$ operator on Σ and so the observables of the B model are deformations of the complex structure of X.

3.4 Hitchin's V_S and V_H functionals in six dimensions

Recall that in Chapter II we defined the notion of stability of a differential p-form in n dimensions as an extension of the non-degeneracy of 2-forms. We also explained how the existence of stable p-forms in dimension n implies the existence of stable $(n-p)$-forms. We then noted in particular the obvious existence of stable 2-forms in even dimension $n = 2m$ as well as the more interesting fact that stable 3-forms exist in dimensions 6 and 7. Thus in six dimensions we find that we can have stable 4-forms by virtue of the existence of stable 2-forms via the duality mentioned above. We now consider in six dimensions the action functionals $V_S(\sigma)$ and $V_H(\rho)$, depending on a stable 4-form σ and a stable 3-form ρ, which generate symplectic and holomorphic structures respectively.

We begin with $V_S(\sigma)$. For a 4-form σ the stability condition can be expressed as the requirement that there exists a 2-form k such that $\sigma = \pm\frac{1}{2} k \wedge k$. Take the $+$ case and define V_S by

$$V_S = \frac{1}{6} \int_M k \wedge k \wedge k, \qquad (3.12)$$

where M is our 6-dimensional manifold. We may express this functional in terms of σ as

$$V_S = \frac{1}{6}\int_M \sigma^{3/2}, \tag{3.13}$$

or in components as:

$$V_S = \frac{1}{6}\int_M \sqrt{\frac{1}{384}\sigma_{a_1a_2b_1b_2}\sigma_{a_3a_4b_3b_4}\sigma_{a_5a_6b_5b_6}\epsilon^{a_1a_2a_3a_4a_5a_6}\epsilon^{b_1b_2b_3b_4b_5b_6}}, \tag{3.14}$$

where $\epsilon^{a_1\cdots a_6}$ is the Levi-Civita tensor in six dimensions.

Keeping the cohomology class $[\sigma] \in H^4(M,\mathbb{R})$ fixed so that $\sigma = \sigma_0 + d\alpha$, where $d\sigma_0 = 0$ we have

$$V_S = \frac{1}{3}\int_M \sigma \wedge k = \frac{1}{3}\int_M (\sigma_0 + d\alpha) \wedge k \tag{3.15}$$

Invoking *Theorem 2.1* we find that the condition that V_S is extremal under variations of α is

$$dk = 0. \tag{3.16}$$

Hence we see that when extremized, V_S generates a symplectic structure on M.

We now consider the functional $V_H(\rho)$ which was in fact studied in detail in Chapter II. Here we will use *Theorem 2.1* to arrive at the result that a holomorphic structure on M can be obtained as a critical point of this functional. Suppose ρ is a stable 3-form and that it satisfies the condition $\lambda(\rho) < 0$ where λ was defined in section 2.4 of Chapter II. Here we will refer to such ρ as a positive stable 3-form. The important point is that such ρ is fixed by a subgroup of $GL(6,\mathbb{R})$ isomorphic to (two copies of) $SL(3,\mathbb{C})$ and so it determines a reduction of $GL(6,\mathbb{R})$ to $SL(3,\mathbb{C})$, which is equivalent to an almost complex structure on M. This means that we can find complex 1-forms $\zeta_i, i =1, 2, 3$ such that

$$\rho = \frac{1}{2}(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \overline{\zeta_1} \wedge \overline{\zeta_2} \wedge \overline{\zeta_3}), \tag{3.17}$$

where the almost complex structure is determined by the ζ_i. Construct a (3,0)-form Ω on M using the complex 1-forms ζ_i as

$$\Omega = \zeta_1 \wedge \zeta_2 \wedge \zeta_3. \tag{3.18}$$

We may rewrite Ω in terms of ρ as

$$\Omega = \rho + i\hat{\rho}(\rho), \tag{3.19}$$

where $\hat{\rho}$ is given by

$$\hat{\rho} = -\frac{i}{2}(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 - \overline{\zeta}_1 \wedge \overline{\zeta}_2 \wedge \overline{\zeta}_3). \qquad (3.20)$$

If there exist local complex coordinates z_i such that $\zeta_i = dz_i$, then the almost complex structure is integrable and so defines a complex structure. We note that this integrability condition is clearly equivalent to $d\Omega = 0$.

Define $V_H(\rho)$ to be the holomorphic volume

$$V_H(\rho) = -\frac{i}{4}\int_M \Omega \wedge \overline{\Omega} = \frac{1}{2}\int_M \hat{\rho}(\rho) \wedge \rho. \qquad (3.21)$$

This can be written in components as

$$V_H(\rho) = -\frac{i}{4}\int_M \sqrt{-\frac{1}{6}K_a^{\ b}K_b^{\ a}}, \qquad (3.22)$$

where $K_a^{\ b} = \frac{1}{12}\rho_{a_1a_2a_3}\rho_{a_4a_5a}\epsilon^{a_1a_2a_3a_4a_5b}$. The "positivity" condition on the 3-form ρ that was defined earlier ensures that the square root in (3.22) gives a real number.

Once again we keep the cohomology class fixed so that $\rho = \rho_0 + d\beta$ with $d\rho_0 = 0$ and perform the first variation of $V_H(\rho) = \frac{1}{2}\int_M \hat{\rho}(\rho) \wedge \rho$. By *Theorem 2.1* we find that the variation vanishes precisely when

$$d\hat{\rho} = 0. \qquad (3.23)$$

From $\rho = \rho_0 + d\beta$ we also have $d\rho = 0$. Combining these two results we arrive at $d\Omega = 0$, which was found to be the condition for the integrability of the almost complex structure. Thus we see how $V_H(\rho)$ generates a complex structure with holomorphic 3-form.

In summary, we have found that extremizing the two functionals $V_S(\sigma)$ and $V_H(\rho)$ respectively yield a symplectic form k and a closed holomorphic (3,0)-form Ω on a 6-dimensional manifold M. This seems to suggest a possible relation to the topological A and B models. For the remainder of this chapter, we will aim to establish a link between the Hitchin functionals V_S, V_H and these topological models.

3.5 The V_S functional as the A model

It is argued in [18] that the A model can be reformulated in terms of a topologically twisted $U(1)$ gauge theory on M, whose bosonic action contains the observables

$$S = \frac{g_s}{3}\int_M F \wedge F \wedge F + \int_M k_0 \wedge F \wedge F. \qquad (3.24)$$

In this theory the partition function is a function of the fixed class k_0 and the path integral can be expressed as a sum over Kähler geometries with quantized Kähler class

$$k = k_0 + g_s F. \tag{3.25}$$

Now for some coefficients α and β consider the action

$$\mathrm{S} = \alpha \int_M \tilde{F} \wedge \tilde{F} \wedge \tilde{F} - \beta \int_M \sigma \wedge \tilde{F}, \tag{3.26}$$

where $\tilde{F}$ is a 2-form and σ is a 4-form restricted to a fixed cohomology class.

The equation of motion for $\tilde{F}$ is

$$3\alpha \tilde{F} \wedge \tilde{F} - \beta\sigma = 0, \tag{3.27}$$

which immediately implies that σ is a stable 4-form. Substituting this back into the action (3.26) we find that it is equal to the Hitchin functional $V_S(\sigma) = \frac{1}{6}\int_M \sigma^{3/2}$.

We now return to the action (3.26) and instead consider the equations of motion for σ.

We note that $[\sigma] \in H^4(M, \mathbb{R})$ is fixed so that we can write $\sigma = \sigma_0 + d\gamma$ to find

$$\mathrm{S} = \alpha \int_M \tilde{F} \wedge \tilde{F} \wedge \tilde{F} - \beta \int_M \sigma_0 \wedge \tilde{F} - \beta \int_M d\gamma \wedge \tilde{F}. \tag{3.28}$$

It is clear that for $\beta \neq 0$, $\delta S = 0$ if and only if $\int_M d(\delta\gamma) \wedge \tilde{F} = -\int_M \delta\gamma \wedge d\tilde{F} = 0$; i.e.

$$d\tilde{F} = 0. \tag{3.29}$$

Hence the action for $\tilde{F}$ can be written as

$$\mathrm{S} = \alpha \int_M \tilde{F} \wedge \tilde{F} \wedge \tilde{F} - \beta \int_M \sigma_0 \wedge \tilde{F}. \tag{3.30}$$

By introducing a change of variables $F = \tilde{F} - \xi k_0$, where ξ is a parameter and k_0 is the background Kähler form satisfying $\sigma_0 = k_0 \wedge k_0$, we can rewrite (3.30) as

$$\mathrm{S} = \alpha \int_M F \wedge F \wedge F + 3\xi\alpha \int_M F \wedge F \wedge k_0 + \int_M (3\xi^2 \alpha k_0 \wedge k_0 \wedge F - \beta\sigma_0 \wedge F). \tag{3.31}$$

Comparing with (3.24) we see that for $\alpha = g_s/3$ and $\xi = \beta = 1/g_s$ this action is exactly equal to that of the $U(1)$ gauge theory action (3.24). Furthermore, the change of variables $F = \tilde{F} - \xi k_0$ with $\xi = 1/g_s$ implies

$$\delta k = g_s F, \tag{3.32}$$

which is equivalent to the quantization condition (3.25).

Now consider the rescaling operation $\tilde{F} \to \gamma\tilde{F}$. Under this rescaling (3.26) becomes

$$S = \alpha\gamma^3 \int_M \tilde{F} \wedge \tilde{F} \wedge \tilde{F} - \beta\gamma \int_M \sigma \wedge \tilde{F}. \tag{3.33}$$

For $\gamma = \sqrt{3}/g_s$ and α, β as above , we find that both terms of the action S have the same coefficient $1/\hbar = \sqrt{3}/g_s^2$ and thus the semiclassical limit $\hbar \to 0$ corresponds precisely to the weak coupling limit $g_s \to 0$. Therefore, the gauge theory action (3.24) is equivalent to $V_S(\sigma)$ in the weak coupling limit.

Finally we turn to the connection between $V_S(\sigma)$ and the topologicallt twisted version of the supersymmetric $U(1)$ gauge theory with action (3.26) which was just shown to be equivalent to the $U(1)$ gauge theory with action (3.24). In the topological $U(1)$ gauge theory on M, the partition sum over Kähler geometries with quantized Kähler class (3.25) can be written as a vacuum expectation value

$$\langle \exp\left(\frac{g_s}{3} \int \mathcal{O}_1 + \int \mathcal{O}_2\right)\rangle, \tag{3.34}$$

where $\mathcal{O}_i$ are the topological observables

$$\begin{aligned} \mathcal{O}_1 &= F \wedge F \wedge F, \\ \mathcal{O}_2 &= k_0 \wedge F \wedge F, \\ &\vdots \end{aligned} \tag{3.35}$$

It was shown by Baulieu and Singer in [19] how the action of this 6-dimensional topological theory can be reconstructed by considering the BRST symmetries that preserve the expectation value (3.34) and the observables (3.35). This construction leads to two gauge fields A, F and three ghost fields c, ψ and ϕ. The action of the BRS operator s on these fundamental fields is then given by

$$sA = \psi - Dc,$$

$$sc + \frac{1}{2}[c, c] = \phi,$$

$$s\psi + [c, \psi] = -D\phi,$$

$$s\phi + [c, \phi] = 0,$$

$$sF + [c, F] = -D\psi,$$

where $D = d + [A, .]$ is the gauge covariant derivative. [20]

For the purposes of our analysis we begin by writing F locally as the field strength of a gauge potential A,

$$F = dA. \tag{3.36}$$

Now (3.34) and (3.35) are not only invariant under the gauge transformation $\delta A = d\epsilon_0$ but also under the more general transformation

$$\delta A = \epsilon_1 \tag{3.37}$$

where the parameter ϵ_1 is an infinitesimal 1-form on M. Gauge fixing with respect to (3.37) leads to a 1-form ghost field ψ and consideration of the additional symmetry under $\epsilon_1 \to \epsilon_1 + d\lambda$ gives rise to a bosonic 0-form ϕ. The only such 6-dimensional topological theory containing a gauge field and a scalar is a maximally supersymmetric theory with $\mathcal{N}_T = 2$ topological supersymmetry.

We now return to the functional $V_S(\sigma) = \frac{1}{6}\int_M \sigma^{3/2}$. It was shown how in the weak coupling limit the field F is identified with the Kähler form via $\delta k = g_s F$. We thus find that

$$\delta k = d\epsilon_0, \tag{3.38}$$

where $\sigma = k \wedge k$. Once more varying σ within a fixed cohomology class so that $\sigma = \sigma_0 + d\gamma$ for $d\sigma_0 = 0$ we see that V_S is indeed invariant under the symmetry (3.38):

$$\begin{aligned} \delta V_S &= \frac{3}{2}\int_M k \wedge \delta\sigma \\ &= 3\int_M k \wedge k \wedge \delta k \\ &= 3\int_M \sigma \wedge d\epsilon_0 \\ &= -3\int_M d\sigma_0 \wedge \epsilon_0 - 3\int_M d^2\gamma \wedge \epsilon_0 = 0. \end{aligned} \tag{3.39}$$

3.6 The V_H functional as the B model

The B model on a Calabi-Yau 3-fold M can be constructed by topologically twisting the physical theory with a fixed "background" complex structure given by a holomorphic 3-form Ω. As defined in [21] the partition function Z_B of the theory is the generating functional of the correlation functions of the marginal operators ϕ_i, $i = 1, \cdots, h_{2,1}$. The marginal operators ϕ_i are the topological observables of the theory and represent infinitesimal deformations of the complex structure. Introducing the variables x^i, $i = 1, \cdots, h_{2,1}$, where the x^i label infinitesimal deformations, we find that for fixed Ω_0 $Z_B(x, g_s, \Omega_0)$ is a holomorphic function of x from the holomorphic tangent space $T_{\Omega_0}\mathcal{M}$ to the moduli space $\mathcal{M}$ of complex structures. Closer examination of the Ω_0 dependence of Z_B leads us to a wavefunction interpretation of Z_B as is explained in [21]. Beginning by combining g_s and x into a "phase space" of dimension $h_{2,1} + 1$ we arrive at the result that from $Z_B(x, \Omega_0)$ we can obtain any other $Z_B(x, \Omega_0{}')$ by taking an appropriate Fourier transform. In this interpretation Z_B is a wavefunction arising from the quantization of the symplectic phase space $H^3(X, \mathbb{R})$.

Now consider the partition function $Z_H([\rho])$ of the 6-dimensional theory with action given by the Hitchin functional V_H. We may write this partition function in terms of the formal expression

$$Z_H([\rho]) = \int D\beta \exp\big(V_H(\rho + d\beta)\big), \tag{3.40}$$

which makes clear in particular the dependence of Z_H on $[\rho] \in H^3(X, \mathbb{R})$. Here we note that Z_H is independent of the particular choice of symplectic coordinates for $H^3(X, \mathbb{R})$, whereas Z_B has different representations for different choices of symplectic coordinates. Thus we see that Z_H and Z_B cannot be equal.

To overcome this obstacle we combine the B and $\overline{\text{B}}$ models, where we recall that these two are related via overall complex conjugation. Consider $\Psi = Z_B \otimes \overline{Z_B}$, where $\overline{Z_B}$ is of course the $\overline{\text{B}}$ model partition function. This product state exists in the doubled Hilbert space generated by the quantization of the phase space $H^3(X, \mathbb{R})$, which is itself doubled to $H^3(X, \mathbb{C})$. By dividing $H^3(X, \mathbb{C})$ into real and imaginary parts we can find a representation of Ψ as a function of the real parts of all the periods: $\Psi(\text{Re } X_I, \text{Re } F^I)$. This once again yields a function on $H^3(X, \mathbb{R})$ which is now independent of the choice of symplectic coordinates. The function Ψ which we have arrived at via the above construction coincides with the "Wigner function" of Z_B which expresses the phase-space density associated with Z_B.

Now choose a basis of A and B cycles $\{A_I, B^I\}$ in $H_3(X, \mathbb{Z})$ and denote the A and B cycle periods by X_I and F^I, respectively. Z_B can now be written as a function of the A cycle periods: $Z_B = Z_B(X_I)$. For $P_I = \text{Re } X_I$ and $Q^I = \text{Re } F^I$, the Wigner function of Z_B is given by

$$\Psi(\text{Re } X_I, \text{Re } F^I) = \int d\Phi_I \exp(-Q^I \Phi_I) |Z_B(P_I + i\Phi_I)|^2. \tag{3.41}$$

It is this Wigner function which we seek to identify with $Z_H([\rho])$. This can be achieved if we identify P_I and Q^I as the periods of $[\rho] \in H^3(X, \mathbb{R})$. We will conclude this chapter by showing the validity of this identification for large ρ which is equivalent to demonstrating agreement between Z_H and the Wigner function of Z_B at the string tree level. In a later chapter we shall look deeper into this identification to see whether these connections persist at one loop.

At the string tree level (large ρ limit) Z_B is dominated by the tree level free energy F_0. Setting $Z_B = \exp\,(-\frac{i}{2} F_0)$ we can rewrite (3.41) as

$$\Psi(\text{Re}\,X_I, \text{Re}\,F^I) = \int d\Phi_I \exp(-Q^I\Phi_I) Z_B(P_I + i\Phi_I)\overline{Z_B(P_I + \iota\Phi_I)}$$

$$= \int d\Phi_I \exp(-Q^I\Phi_I) \exp\left(-\frac{i}{2}F_0(P_I + i\Phi_I)\right) \exp\left(\frac{i}{2}\overline{F_0(P_I + \iota\Phi_I)}\right)$$

$$= \int d\Phi_I \exp\left(-\frac{i}{2}F_0(P_I + i\Phi_I) + \frac{i}{2}\overline{F_0(P_I + \iota\Phi_I)} - Q^I\Phi_I\right). \tag{3.42}$$

Now the integral over Φ_I is in a form which allows us use the method of steepest descent to approximate it. The function $S(\Phi) = -\frac{i}{2}F_0(P_I + i\Phi_I) + \frac{i}{2}\overline{F_0(P_I + \iota\Phi_I)} - Q^I\Phi_I$ has its critical point where $Q^I = \text{Re}\,\frac{\partial F_0}{\partial X_I} = \text{Re}\,F^I(P + i\Phi)$. At this Φ, we have:

$$S = -\frac{i}{4}X_I F^I + \frac{i}{4}\bar{X}_I\overline{F^I} - (\text{Re}\,F^I)(\text{Im}\,X_I)$$

$$= \frac{i}{4}X_I\overline{F^I} - \frac{i}{4}\bar{X}_I F^I. \tag{3.43}$$

We recognize this as the Hitchin functional $V_H = -\frac{i}{4}\int_M \Omega \wedge \bar{\Omega}$ evaluated at the value of Ω for which $\text{Re}\,X_I = P_I$ and $\text{Re}\,F^I = Q^I$.

Finally, we note that the arguments presented here have only been rigorous at the classical level. In Chapter VIII, we return to the conjectured relation between the topological B-model and the Hitchin functional V_H. There we will consider a suitable reformulation of this connection after incorporating the one-loop contributions to the partition functions.

IV. BPS Black Holes

In the early 1970s a striking and beautiful correspondence was discovered between the laws of thermodynamics and the laws of black hole dynamics. In particular the Bekenstein-Hawking entropy was found to exhibit behaviour identical to that of thermodynamic entropy. A central question in black hole physics concerns the precise statistical mechanical interpretation of black hole entropy. Derivation of the Bekenstein-Hawking entropy via counting black hole microstates would promote the sharp analogy existing between the laws of thermodynamics and the dynamics of black holes to a deeper and more precise identification of the two. In this chapter we discuss some relevant ideas and in particular the entropy of so-called BPS black holes. In the process we give a brief introduction to N=2 supergravity and towards the end of the chapter we consider an important connection between black holes and Hitchin functionals.

4.1 Black hole entropy

We begin by considering the Reissner-Nordsrom solutions of the classical Einstein-Maxwell field theory of gravity and electromagnetism in four space-time dimensions:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \qquad (4.1)$$

These solutions are static, spherically symmetric black holes with a flat space-time geometry at spatial infinity and are characterized by a charge Q and a mass M. For $Q = 0$ we see a reduction to Schwarzschild solutions. The black hole horizon has associated with it an area A that is defined by the area of the two-sphere determined by the horizon, and a surface gravity κ_s which is interpreted as the force measured at spatial infinity required to hold a unit test mass in place.

Under the celebrated analogy between the laws of thermodynamics and black hole dynamics, the first law

$$\delta E = T\delta S - p\delta V, \qquad (4.2)$$

translates into

$$\delta M = \frac{\kappa_s}{2\pi}\frac{\delta A}{4} + \mu\delta Q + \Omega\delta J. \qquad (4.3)$$

The term $\frac{\kappa_s}{2\pi}$ in (4.3) gives the Hawking temperature which leads us to the identification of the black hole entropy with one quarter the area of the event horizon:

$$S = \frac{1}{4}A. \qquad (4.4)$$

4.2 BPS black holes

The N-extended four-dimensional supersymmetry algebra is given by

$$\{Q^A_\alpha, Q^{\dagger B}_\beta\} = 2\sigma^\mu_{\alpha\beta}\delta^{AB}P_\mu, \quad \{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB}, \tag{4.5}$$

where $A, B = 1, \dots, N$ label the supercharges which are taken to be Majorana spinors. Z^{AB} is a complex antisymmetric matrix of central operators which can be skew-diagonalized using R-symmetry rotations. By Schur's lemma the eigenvalues Z_i are constant on irreducible representations and the same holds true for the mass $M^2 = -P_\mu P^\mu$. Using (4.5) we can show that the mass is bounded by the eigenvalues Z_i, which are known as the central charges. We have:

$$M^2 \geq |Z_1|^2 \geq |Z_2|^2 \geq \cdots \geq 0. \tag{4.6}$$

If at least one of the inequalities for M in (4.6) is saturated, then part of the supercharges operate trivially and consequently the corresponding representation will be smaller than a generic massive representation and will have a different spin content. These representations are known as BPS representations. The special case in which all the central charges are equal so that $M^2 = |Z_i|^2$, for all i, is known as a ½-BPS representation.

Now consider once more the Reissner-Nordstrom solution (4.1). We note that for the Reissner-Nordstrom black hole we need to distinguish three different cases based on the relative values of the charge Q and the mass M. For $M > Q$ we find both an interior and an exterior event horizon. When $M = Q$ the solution is said to be an extremal black hole, for which the surface gravity vanishes and the two horizons coalesce into one. In the $M < Q$ case we find a physically unacceptable solution where the electromagnetic energy exceeds that of the total energy and the event horizon is replaced by a naked singularity. Thus extremal black holes saturate the bound $M \geq Q$ for physically acceptable black hole solutions. In the context of $N = 2$ supergravity, the extremal Reissner-Nordstrum black hole is a standard example of a so-called BPS soliton. In fact, as it has four Killing spinors, it is a ½-BPS solution

We now discuss an important property of BPS states which is relevant to the significance of these states in the physics literature. There is a natural embedding of Einstein-Maxwell theory into $N = 2$ supergravity. In the next section we present a brief introduction to $N = 2$ supergravity, but here we point out that when performing the embedding into supergravity theory, the classification of the Reissner-Nordstrom black hole solutions acquires an interpretation in terms of the supersymmetry algebra. This algebra has a central extension proportional to the black hole charge(s) and the unitary representations of the algebra must have masses that are greater than or equal to the charge(s) in accordance with (4.6). When this bound is saturated, we obtain BPS supermultiplets as was defined earlier. Because of the fact that these BPS supermultiplets are smaller than the generic massive supermultiplets, they are stable under (adiabatic) changes of the coupling constants and the relation between charge and mass remains preserved. Finally we emphasize that as defined above, the BPS black hole solutions saturate a bound implied by the supersymmetry algebra and are characterized by their possession of some residual supersymmetry.

4.3 $N = 2$ supergravity

Supergravity is an extension of supersymmetry which arises naturally when one attempts to make supersymmetry transformations local following the approach taken in gauge theories. Starting with global supersymmetry, we find that we necessarily have to introduce a graviton field (and its supersymmetric partner the gravitino) in the process of making supersymmetry transformations local, thus incorporating gravity. Here we briefly introduce the $N = 2$ supermultiplets and then move on to discuss the construction of supersymmetric actions. Some familiarity with the basic concepts of supersymmetry is assumed in what follows.

4.3.1 Supermultiplet Decomposition

Of particular interest are the vector and the Weyl supermultiplets. The tensor supermultiplets and the hypermultiplets will not be considered in detail as they play a subsidiary role. The covariant fields and field strengths of the gauge fields associated with the vector or Weyl supermultiplet make up a chiral multiplet which is described in superspace by a chiral superfield. Scalar chiral fields in $N = 2$ superspace have 16 bosonic and 16 fermionic field components which are reduced to 8 bosonic and 8 fermionic field components by a constraint which expresses higher-θ components of the superfield in terms of lower-θ components or their space-time derivatives. The vector supermultiplet and the Weyl supermultiplet are related to these reduced chiral multiplets.

The vector supermultiplet is described by a scalar reduced chiral superfield, whose lowest-θ component X is a complex field. There is also a doublet of chiral fermions Ω_i, where i is an $SU(2)$ R-symmetry doublet index. We recall that the R-symmetry group is by definition the maximal group that rotates the supercharges in a way that commutes with Lorentz symmetry and is compatible with the supersymmetry algebra. The R-symmetry group in $N = 2$ supersymmetry is $SU(2) \times U(1)$, which acts on the spinors. The chirality of the spinor field is determined by the position of the index i, such that Ω_i has positive chirality and Ω^i negative. The fields $\bar{X}$ and Ω^i are lowest-θ components in the anti-chiral superfield, related to the chiral superfield via complex conjugation. At the θ^2-level, we find the field strength $F_{\mu\nu}$ of the gauge field and an auxiliary field written as a symmetric real tensor $Y_{ij} = Y_{ji} = \varepsilon_{ik}\varepsilon_{jl}Y^{kl}$. In this way we have obtained precisely the 8 + 8 independent field components or off-shell degrees of freedom.

For the Weyl supermultiplet, the reduced chiral superfield is an anti-selfdual Lorentz tensor rather than a scalar. In the case of extended supersymmetry the Weyl superfield is assigned to the antisymmetric representation of the R-symmetry group so that its lowest order θ-component is written as $T_{ab}{}^{ij}$, where the a, b indices denote the components of space-time tangent space tensors. Its complex conjugate corresponds to the anti-chiral superfield and has a corresponding selfdual tensor T_{abij}. In light of its tensorial nature, the Weyl supermultiplet has 24 + 24 off-shell degrees of freedom. We now consider the covariant components of the Weyl multiplet. At the linear θ-level we find that the fermions decompose into the field strength of the gravitini and a doublet spinor χ^i. The gravitini field strengths account for 16 degrees of freedom and combined with the 8 independent components corresponding to the spinors χ^i, we count 24 off-shell degrees of freedom. At the θ^2-level we find the Weyl tensor, field strengths of the gauge fields associated with R-symmetry and a real scalar field D. These respectively account for 5, 4×3 and 1 off-shell degrees of freedom, which combine with the 6 degrees of freedom associated with $T_{ab}{}^{ij}$ to give the full 24 bosonic degrees of freedom.

The Weyl multiplet contains the fields of $N = 2$ conformal supergravity and one can write an invariant action that is quadratic in its components. We also note that the gravitini transform into $T_{ab}{}^{ij}$ and in locally supersymmetric Lagrangians of vector multiplets that are at most quadratic in space-time derivatives, $T_{ab}{}^{ij}$ acts as an auxiliary field and couples to a field-dependent linear combination of the vector multiplet field strengths. For this class of Lagrangians, all the fields of the Weyl multiplet with the exception of the graviton and gravitini fields act as auxiliary fields.

As was discussed, our aim is to move towards a theory of local supersymmetry. For this, the vector multiplets must first be formulated in a supergravity background, which leads to extra terms in the supersymmetry transformation rules as well as in the superfield components which are dependent on the supergravity background. Some of these additional terms correspond to the introduction of covariant derivatives. As mentioned before, we consider only vector and Weyl multiplets here. The Weyl multiplet describes the supermultiplet of conformal supergravity and so to achieve consistency we need to formulate the vector supermultiplet in a superconformal background, and indeed the vector supermultiplet is a representation of the full superconformal algebra. Therefore all the superconformal symmetries can be realized as local symmetries.

4.3.2 Supersymmetric Lagrangians

As both the vector and Weyl supermultiplets are described by chiral superfields even beyond their linearized form, we construct supersymmetric actions by taking some function of the vector superfield and the square of the Weyl superfield. Expansion of the superfields in θ components generates multiple derivatives of this function which depend on the lowest-θ components, X and $A = (T_{ab}{}^{ij}\varepsilon_{ij})^2$. As the function $F(X, A)$ is holomorphic (i.e. it is independent of the complex conjugates $\bar{X}$ and $\bar{A}$) we take the imaginary part of the resulting expression.

Here we focus on a few characteristic terms in the supersymmetric action. The scalar kinetic terms are accompanied by a coupling to the Ricci scalar and the scalar field D of the Weyl multiplet as follows

$$\mathcal{L} \propto i\left(\partial_\mu F_I \partial^\mu \bar{X}^I - \partial_\mu \bar{F}_I \partial^\mu X^I\right) - i\left(\frac{1}{6}R - D\right)(F_I \bar{X}^I - \bar{F}_I X^I), \qquad (4.7)$$

where F_I denotes the derivative of F with respect to X^I. When F depends on $T_{ab}{}^{ij}$, this will generate interactions between the kinetic term for the vector multiplet scalars and the tensor field of the Weyl multiplet. This pattern continues for other terms.

The kinetic term of the vector fields is proportional to the second derivative of the function F and is given by

$$\mathcal{L} \propto \frac{1}{4} i F_{IJ}\left(F_{ab}{}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}{}^{ij}\varepsilon_{ij}\right)\left(F^{-abJ} - \frac{1}{4}\bar{X}^J T^{abkl}\varepsilon_{kl}\right) + \cdots \qquad (4.8)$$

Finally we consider a term involving the square of the Weyl tensor, contained in the tensor $\mathcal{R}(M)$,

$$\mathcal{L} \propto 16 i F_A\left[2\mathcal{R}(M)^{cd}{}_{ab}\mathcal{R}(M)_{cd}{}^{-ab} - 16 T^{abij} D_a D^c T_{cbij}\right] + \cdots, \qquad (4.9)$$

where D_a denotes a superconformally covariant derivative which also contains terms proportional to the Ricci tensor. [25]

4.4 The microscopic description of black holes

As was alluded to at the start of this chapter, a key challenge in the study of black hole thermodynamics has been the development of a statistical mechanical interpretation of the black hole entropy. Ideas from string theory have led to progress in this direction, which have made possible the identification of the black hole entropy with the logarithm of the degeneracy of states $d(Q)$ of charge Q belonging to a certain system of microstates. In string theory these microstates are provided by the states of wrapped brane configurations of given momentum and winding. Calculation of the black hole solutions in the corresponding effective field theory with charges specified by the brane configuration leads us to a remarkable correspondence: the black hole area is equal to the logarithm of the brane state degeneracy, at least in the limit of large charges. [25]

To understand the relation between microscopic and field-theoretic descriptions of black holes we need to recall that strings exist in more dimensions than the familiar four space-time dimensions. In most situations the extra dimensions are compactified on some internal manifold X and one has the standard Kaluza-Klein scenario leading to effective field theories in four dimensions, describing low-mass modes of the fields associated with appropriate eigenfunctions on the internal manifold. The original space-time is then locally written as a product $M^4 \times X$, where M^4 is our four-dimensional space-time, so that at each point x^μ of M^4 we have a corresponding space X. In this picture, the size of X is such that it will not be directly observable. We emphasize that the space X need not be the same at every point and in principle the internal spaces defined at various points of M^4 belong to some well-defined class of fixed topology parameterized by certain moduli. These moduli appear as fields in the four-dimensional effective field theory.

We now return to look at our black hole through this higher-dimensional lens. Note that in this perspective the internal spaces X as well as the four-dimensional space-time fields (such as the metric $g_{\mu\nu}$) will vary over M^4. In the effective field-theoretic context only the local degrees of freedom of strings and branes are captured. But extended objects may also carry global degrees of freedom since they can also wrap themselves around non-trivial cycles of the internal space X. This wrapping tends to occur at a particular point in M^4 and so in the context of the four-dimensional effective field theory will manifest itself as a pointlike object. This wrapped object is the string theory representation of the black hole.

We have thus found two complementary descriptions of the black hole. One is the usual story based on general relativity where a point mass generates a global space-time solution to Einstein's equation and is known as the so-called macroscopic description. The alternative picture which we have just discussed is based on the internal space where an extended object is entangled in one of its cycles and is referred to as the microscopic description. We note that the microscopic picture does not immediately involve gravitational fields and so can easily be described in flat space-time.

We will not delve deeper into the structure of this complementary picture by discussing the technical aspects of deriving the black hole entropy by microscopic arguments. We will suffice it to underline the important fact that the microscopic expression for the black hole entropy will depend only on the charges Q and not on other quantities, such as the values of the moduli fields at the horizon.

4.5 The attractor mechanism

Clearly the microscopic description of black holes would lose much of its appeal if one could not find a way of connecting it to the macroscopic description. It is expected that any description of the black hole microstates would require some version of a quantum theory of gravity. String theory, which is of course one such theory of quantum gravity, has in recent decades been shown to provide a link between these two complementary pictures. In particular, the fact that in string theory gravitons are closed string states which interact with the wrapped branes ensures that such a link does exist. These interactions are governed by the string coupling g_s and we are thus faced with an interpolation in the coupling constant.

Unfortunately, relating the two pictures in practice has turned out to be an exceedingly difficult proposition, so much so that a realistic comparison of results between the two is usually not possible. However, in the case of extremal BPS black holes it has been demonstrated that reliable predictions are indeed possible such that the microscopic and macroscopic approaches can be successfully compared to create a deeper understanding of black holes.

When introducing the microscopic description of black holes we noted that the expression for the black hole entropy in this picture depends only on the charges. The corresponding field-theoretic calculation on the other hand may also depend on the values of the moduli fields at the horizon, for instance. Thus, arriving at any reasonable agreement between these two pictures necessitates that the moduli take fixed values at the horizon which may depend only on the charges. This is indeed the case for extremal solutions as was shown in the context of $N = 2$ supersymmetric black holes.[26][27][28] The values taken by the fields at the horizon are independent of their asymptotic values at spatial infinity and this fixed point behaviour is determined by so-called attractor equations.

In the presence of higher-derivative interactions where explicit construction of black hole solutions proves to be a challenge, one can usually find the fixed-point values without the need for interpolation between the horizon and spatial infinity, by focusing on the near-horizon region. Where the symmetry of the near-horizon region is sufficiently strong, the attractor mechanism can be expressed through a variational principle for a suitably defined entropy function. [29]

The $N = 2$ BPS attractor equations can be determined by classifying all possible $N = 2$ supersymmetric solutions through use of supersymmetric variations of the fermions in an arbitrary bosonic background. Setting these variations equal to zero implies strong restrictions on this background, as was shown in [28] for $N = 2$ supergravity with an arbitrary number of vector multiplets and hypermultiplets, including higher-order derivative couplings proportional to the square of the Weyl tensor. It was found that the horizon geometry and the values of the fields of interest are completely determined by the charges. In particular, it was found that the extremal value of the square of the central charge provides the area of the horizon, which depends only on electric and magnetic charges. Furthermore, supersymmetry also determines the entropy as given by the Noether charge definition of Wald [30]. In the absence of charges we have a flat Minkowski space-time with constant moduli and ${T_{ab}}^{ij} = 0$.

It is not feasible to present a detailed and comprehensive explanation of the black hole attractor phenomenon in this section. Our aim is only to convey a sense of its crucial role in providing a link between the microscopic and macroscopic descriptions of black holes and hopefully pass on enough of an understanding of the attractor mechanism to make it possible for the reader unfamiliar with this phenomenon to experience some appreciation of the results in the remainder of this chapter.

Earlier we saw how geometric structures can be constructed using p-forms defined on a manifold M. Here we wish to emphasize a somewhat analogous way of looking at the attractor mechanism in superstring theory compactified on M. In this picture, given a black hole charge Q, which can be represented by an integral cohomology class on M, the attractor mechanism fixes certain metric data $g_{\mu\nu}$ on M at the black hole horizon. That is, it provides a map

$$Q \longmapsto g_{\mu\nu}. \tag{4.10}$$

We now turn our attention to the link existing between black holes and Hitchin functionals. Following [9] we suggest that the map (4.10) can be seen as arising from these functionals. In a sense, the metric flow of the internal manifold from spatial infinity to the black hole horizon can be viewed as a geodesic flow as determined by the Hitchin functional. Indeed this new approach is more powerful in the sense that the Calabi-Yau form of the metric follows from the action principle which gives the relation between the charge and the metric, whereas it is an underlying assumption in the other picture. We shall now consider how far we can take these ideas in the case of BPS black holes in dimensions 4 and 5.

4.5 BPS black holes in four dimensions

We consider four-dimensional BPS black holes in Type IIB string theory compactified on a Calabi-Yau 3-fold M. Recall that in Chapter II we defined Hitchin's V_H functional as

$$V_H(\rho) = \frac{1}{2}\int_M \hat{\rho} \wedge \rho, \tag{4.11}$$

where ρ is a stable 3-form in six dimensions. We also saw that we could rewrite (4.11) in terms of $\Omega = \rho + i\hat{\rho}(\rho)$ as

$$V_H(\rho) = -\frac{i}{4}\int_M \Omega \wedge \overline{\Omega}. \tag{4.12}$$

By varying ρ within a fixed cohomology class $[\rho]$, the critical points of V_H generate holomorphic 3-forms on M with real part ρ. So by considering variations of V_H, we find the imaginary part of Ω as a function of its real part.

Now we consider the effect of the attractor mechanism. For a given fixed charge Q in the theory on $M = \mathbb{R}^4$, the attractor mechanism gives the value of Ω at the black hole horizon and the real part of Ω is equal to Q^*, where Q^* is the Poincare dual of Q. So we notice the natural identification

$$[\rho] = Q^*. \tag{4.13}$$

Here we note that under the above identification, restricting ρ to variations within a fixed cohomology class corresponds precisely to fixing the black hole charge. We have thus established a connection between V_H and the attractor mechanism at least classically.

The low-energy theory of Type IIB string theory compactified on a Calabi-Yau 3-fold M is $N = 2$ supergravity in four dimensions. Let X^I, $I = 1, 2, \cdots, h^{2,1}$, be the scalar components of the $h^{2,1}(M)$ vector multiplets of $N = 2$ supergravity. The vector multiplet part of the effective action is determined by the holomorphic prepotential $\mathcal{F}(X)$ which defines a special Kähler geometry of the moduli space of M, with Kähler potential K. Let $\{A^I, B_I\}$ be a basis of $H_3(M, \mathbb{Z})$ and $\{\alpha_I, \beta^I\}$ be the dual basis of $H^3(M, \mathbb{Z})$. Writing $F_I(X) = \partial_I \mathcal{F}$ we have

$$X^I = \int_{A^I} \Omega, \tag{4.14}$$

$$F_I = \int_{A^I} \Omega, \tag{4.15}$$

$$\Omega = X^I \alpha_I - F_I(X)\beta^I \tag{4.16}$$

and

$$K = -\log i(\bar{X}^I F_I - X^I \bar{F}_I). \tag{4.17}$$

In this context, V_H is related to the Kähler potential K by

$$V_H(\rho) = -\frac{i}{4}\int_M \Omega \wedge \overline{\Omega} = \frac{1}{2} Im(X^I \bar{F}_I) = \frac{1}{4} e^{-K}. \tag{4.18}$$

Now let $Q = [\gamma] \in H_3(M)$ and denote its Poincare dual by Q^*. Then there exist p^I, q_I such that

$$Q^* = p^I \alpha_I - q_I \beta^I. \tag{4.19}$$

The central charge field is then given by

$$Z(Q) = e^{K/2} \int_\gamma \Omega = e^{K/2} \int_M \Omega \wedge Q^* = e^{K/2}(p^I F_I - q_I X^I). \tag{4.20}$$

For a static spherically symmetric extremal BPS black hole, the semiclassical black hole entropy can be expressed as

$$S = \pi |Z|^2 = \pi \, |p^I F_I - q_I X^I| / 2 I\mathrm{m}(X^I \bar{F}_I). \tag{4.21}$$

The scalar components X^I of the vector multiplets describe the complex structure moduli of M. The equation (4.21) is valid for values of the moduli fields at the black hole horizon. The attractor mechanism specifies the values of the moduli fields at the horizon as a function of the charges through the following attractor equations [32]:

$$\mathrm{Re}(CX^I) = p^I, \tag{4.22}$$

$$\mathrm{Re}(CF_I) = q_I, \tag{4.23}$$

$$C = -2i\bar{Z}e^{\frac{K}{2}}. \tag{4.24}$$

Due to invariance of the entropy with respect to dilatations, we may set $C = 1$ to find that the charges p^I and q_I are simply given by the real parts of X^I and F_I, respectively. Substituting these into (4.21) we obtain

$$S = \frac{\pi}{2} \mathrm{Im}(X^I \bar{F}_I). \tag{4.25}$$

Combining this result with the expression given for V_H in (4.18), we find that at a critical point ρ_c of V_H:

$$S_{BH} = \pi V_H(\rho_c), \qquad [\rho] = Q^*, \tag{4.26}$$

where S_{BH} is the semiclassical black hole entropy.

We now discuss a conjecture put forth in [9] which would suggest a deep and powerful connection between Hitchin's theory and black holes. In [15] and [33] the "mixed" partition function Z_{BH} is defined by

$$Z_{BH}(\phi^I, p^I) = \textstyle\sum_{q_I} \Omega(\, p^I, q_I) e^{-\phi^I q_I}, \tag{4.27}$$

where p^I and q_I are in the forms (4.22) - (4.23), $\phi^I = \frac{\pi}{i} \mathrm{Im}(CX^I)$ and the $\Omega(p^I, q_I)$ are integer black hole degeneracies. A key point is that $Z_{BH}(Q)$ performs a "mixed" counting of the states of the black hole. Now we formally define the partition function Z_H by the path integral

$$Z_H(Q) = \int_{[\rho]=Q^*} D\rho \exp(V_H(\rho)). \quad (4.28)$$

The conjecture can be stated as

$$Z_{BH}(Q) = Z_H(Q). \quad (4.29)$$

A key point supporting this remarkable statement is that provided the path integral defining $Z_H(Q)$ converges, it would define a function of the charge Q which agrees with the black hole entropy on leading asymptotics.

Another interesting observation is that splitting $H^3(M)$ into A, B cycles and the charge Q into electric and magnetic charges, $Q = (p, q)$, we can identify Z_{BH} with a Wigner function derived from the B model partition function Z_B:

$$Z_{BH}(Q) = \int d\Phi \exp(iQ^I\Phi_I)|Z_B(p + i\Phi)|^2. \quad (4.30)$$

This is essentially the same as the link existing between V_H and the B model which was discussed in Chapter III. There we also discussed how the functional V_H is related to a combination of the B and $\bar{B}$ models which is consistent with the role that this same B and $\bar{B}$ combination plays in the counting of black hole entropy. [15]

The conjecture directly connects Hitchin's theory to the microscopic description of black holes.

4.5 BPS black holes in five dimensions

In this section we will argue that for a stable 4-form σ defined on the 6-manifold M, the Hitchin functional $V_S(\sigma) = \frac{1}{6}\int_M \sigma^{3/2}$ can be related to black hole entropy in the 5-dimensional theory obtained by compactifying M-theory on M. The BPS black holes in this 5-dimensional theory are constructed by wrapping M2-branes over 2-cycles of M, and are characterized by a charge $Q \in H_2(M,\mathbb{Z}) = H^4(M,\mathbb{Z})$ and a spin j.

First we consider the $j = 0$ case. Making a parallel identification to the one made in (4.13) during our analysis of 4-dimensional BPS black holes, we write

$$[\sigma] = Q^*. \quad (4.31)$$

The value of the moduli at the horizon, as determined via the attractor mechanism ([34], [35]) are then given by a Kähler form k satisfying $\frac{1}{2}k^2 = \sigma$. We then have the following relation between the black hole entropy S_{BH} and the classical value of $V_S(\sigma)$:

$$S_{BH} \sim \int_M k \wedge k \wedge k = \int_M \sigma^{3/2}\,. \tag{4.32}$$

For non-zero black hole spin j, we introduce a 6-form field J and identify j with the integral cohomology class of J:

$$j = [J] \in H^6(M, \mathbb{Z}).$$

Noting that the entropy of the spinning black hole in five dimensions scales as $\sqrt{Q^3 - J^2}$, we define the modified V_S functional

$$V_S(\sigma, J) = \int_M \sqrt{\sigma^3 - J^2}, \tag{4.33}$$

where

$$\sigma^3 - J^2 = \left(\sigma_{i_1 i_2 i_3 i_4}\sigma_{j_1 j_2 j_3 j_4}\sigma_{k_1 k_2 k_3 k_4} - J_{i_1 i_2 i_3 i_4 k_1 k_2} J_{j_1 j_2 j_3 j_4 k_3 k_4}\right)\epsilon^{i_1 i_2 j_1 j_2 k_1 k_2}\epsilon^{i_3 i_4 j_3 j_4 k_3 k_4}.$$

We emphasize that the Kähler form k is invariant under the introduction of the 6-form J.

In the case of 5-dimensional BPS black holes, the conjecture analogous to (4.29) is that

$$Z_S(Q) = \int_{[\sigma]=Q^*} D\sigma \exp(V_S(\sigma, J)) \tag{4.34}$$

should count the degeneracies of the BPS black hole characterized by charge Q and spin j, where $j = [J] \in H^6(M, \mathbb{Z})$.

V. The Geometry of Calabi-Yau Moduli Spaces

In this chapter we introduce the notion of Calabi-Yau moduli spaces as parameter spaces describing deformations of Calabi-Yau manifolds. We shall see how the moduli space admits a metric whose form implies a product structure on the moduli space, consisting of two factors corresponding to complex-structure deformations and Kähler-structure deformations. We then consider the geometry of these spaces in some detail and find, in particular, that in either case the Kähler potentials are given by Hitchin functionals.

5.1 Calabi-Yau moduli spaces

The Betti number b_r associated with a manifold M is defined as the dimension of the rth de Rham cohomology $H^r(M)$ and is a fundamental topological number. In the case of Kähler manifolds, the Betti numbers decompose as

$$b_r = \sum_{p=0}^{k} h^{p,k-p}, \tag{5.1}$$

where $h^{p,q}$ are *Hodge numbers* which count the number of harmonic (p,q)-forms on the manifold. The Hodge numbers do not provide a complete topological characterization of the manifold and some Calabi-Yau manifolds with the same Hodge numbers can be smoothly related via deformations of the parameters characterizing their size and shape. These parameters are known as the *moduli*. We consider a Calabi-Yau manifold with fixed Hodge numbers and study the fluctuations arising from deformations of the metric.

In the context of superstring compactifications, the zero modes of the graviton field generate the four-dimensional metric $g_{\mu\nu}$ together with a set of massless scalar fields arising from the internal components of the metric g_{mn}. In Calabi-Yau compactifications, the metric does not generate any massless vector fields since $b_1 = 0$. Requiring the background to remain Calabi-Yau under a small variation of the metric on the internal space leads to differential equations whose solutions determine the number of ways that the background metric can be deformed in a manner that preserves supersymmetry and the topology. The moduli can then be interpreted as the coefficients of the independent solutions to these differential equations and are given by the expectation values of massless scalar fields known as *moduli fields*. The moduli provide a topologically invariant parametrization of the size and shape of the internal Calabi-Yau manifold.

5.2 Deformations of Calabi-Yau three-folds

We perform a small variation

$$g_{mn} \longrightarrow g_{mn} + \delta g_{mn}, \tag{5.2}$$

and demand that the Calabi-Yau conditions are satisfied both before and after the variation. Thus, in particular, both g_{mn} and $g_{mn} + \delta g_{mn}$ must be Ricci-flat

$$R_{mn}(g) = 0 = R_{mn}(g + \delta g). \tag{5.3}$$

Writing $\delta g_m^m = g^{mp}\delta g_{mp}$, we impose a gauge fixing condition

$$\nabla^m \delta g_{mn} = \frac{1}{2}\nabla_n \delta g_m^m \tag{5.4}$$

that removes the metric deformations that correspond simply to coordinate transformations. Now the expansion of $R_{mn}(g + \delta g) = 0$ yields the following differential equations for δg:

$$\nabla^k \nabla_k \delta g_{mn} + 2R_m{}^p{}_n{}^q \delta g_{pq} = 0, \tag{5.5}$$

where we have used $R_{mn}(g) = 0$. It can be shown that the differential equations for the pure components δg_{ab} and the mixed components $\delta g_{a\bar{b}}$ decouple. We interpret $g + \delta g$ as a Kähler metric and use it to define the Kähler form

$$J = i g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}. \tag{5.6}$$

Each of the ten-dimensional superstring theories contain an NS-NS 2-form B. Following compactification on a Calabi-Yau 3-fold, the internal (1,1)-form $B_{a\bar{b}}$ will have $h^{1,1}$ zero modes generating massless scalar fields in four dimensions. The *complexified Kähler form* $\mathcal{J}$ whose variations give rise to $h^{1,1}$ massless complex scalar fields in four dimensions is defined by combining the real closed 2-form B with the Kähler form J:

$$\mathcal{J} = B + iJ. \tag{5.7}$$

Although the purely holomorphic and purely antiholomorphic metric components vanish, we may consider variations of these to nonzero values producing a variation in the complex structure. Each such variation is associated with a complex $(2,1)$-form

$$\Omega_{abc} g^{c\bar{d}} \delta g_{\bar{d}\bar{e}} dz^a \wedge dz^b \wedge d\bar{z}^{\bar{e}}, \tag{5.8}$$

which is harmonic if the variations respect (5.3).

5.3 The moduli space metric

The moduli space is essentially a parameter space describing complex and Kähler deformations of Calabi-Yau manifolds. This space is endowed with a natural metric [3] consisting of a sum of two pieces corresponding to deformations of the complex structure and deformations of the complexified Kähler form

$$ds^2 = \frac{1}{2V}\int g^{a\bar{b}} g^{c\bar{d}}[\delta g_{ac} g_{\bar{b}\bar{d}} + (\delta g_{a\bar{d}} \delta g_{c\bar{b}} - \delta B_{a\bar{d}} \delta B_{c\bar{b}})]\sqrt{g}\, d^6 x, \tag{5.9}$$

where V is the volume of the Calabi-Yau manifold M. The form of the moduli space metric implies a local splitting of the moduli space structure into a product

$$\mathcal{M}(M) = \mathcal{M}^{2,1}(M) \times \mathcal{M}^{1,1}(M). \tag{5.10}$$

We now take a deeper look into the geometry of each of these two spaces.

5.4 The complex-structure moduli space

First we consider the space of complex-structure metric deformations. Let t^α, $\alpha = 1, \dots, h^{2,1}$, be local coordinates for the complex-structure moduli space and define a set of (2,1)-forms $\chi_\alpha = \frac{1}{2}(\chi_\alpha)_{ab\bar{c}} dz^a \wedge dz^b \wedge d\bar{z}^{\bar{c}}$ by

$$(\chi_\alpha)_{ab\bar{c}} = -\frac{1}{2}\Omega_{ab}{}^{\bar{d}} \partial g_{\bar{c}\bar{d}}/\partial t^\alpha, \tag{5.11}$$

where we raise and lower indices using the hermitian metric. We may invert these relations to find that

$$\delta g_{\bar{a}\bar{b}} = -\frac{1}{\|\Omega\|^2}\bar{\Omega}_{\bar{a}}{}^{cd}(\chi_\alpha)_{cd\bar{b}}\delta t^\alpha, \tag{5.12}$$

where $\|\Omega\|^2 = \frac{1}{6}\Omega_{abc}\bar{\Omega}^{abc}$. Writing the moduli space metric in the form

$$ds^2 = 2G_{\alpha\bar{\beta}}\delta t^\alpha \delta \bar{t}^{\bar{\beta}}, \tag{5.13}$$

we find that

$$G_{\alpha\bar{\beta}}\delta t^\alpha \delta \bar{t}^{\bar{\beta}} = -\left(\frac{i\int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{i\int \Omega \wedge \bar{\Omega}}\right)\delta t^\alpha \delta \bar{t}^{\bar{\beta}}$$

on substituting (5.12) into (5.9).

Under a change in complex structure, dz becomes a linear combination of dz and $d\bar{z}$, and the holomorphic (3,0)-form can be expressed as a linear combination of a (3,0)-form and the (2,1)-forms χ_α defined in (5.11). Differentiating $\Omega = \frac{1}{6}\Omega_{abc} dz^a \wedge dz^b \wedge dz^c$, we find that

$$\partial_\alpha \Omega = K_\alpha \Omega + \chi_\alpha, \tag{5.14}$$

where $\partial_\alpha = \partial/\partial t^\alpha$. Towards the end of this section we derive the precise form of K_α, but for now we note that it depends only on the coordinates t^α of the complex structure moduli space and not on the coordinates of the Calabi-Yau manfold M.

Noting that $G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}^{2,1}$ and using $G_{\alpha\bar{\beta}} \delta t^\alpha \delta \bar{t}^{\bar{\beta}} = -\left(\frac{i \int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{i \int \Omega \wedge \bar{\Omega}}\right) \delta t^\alpha \delta \bar{t}^{\bar{\beta}}$ together with (5.14), we find that the metric on the complex-structure moduli space is Kähler with the Kähler potential given in terms of the V_H Hitchin functional by

$$\mathcal{K}^{2,1} = -\log(i \int \Omega \wedge \bar{\Omega}). \tag{5.15}$$

Now choose a basis of three-cycles A^I, B_J $(I, J = 0, \dots, h^{2,1})$ satisfying

$$A^I \cap B_J = -B_J \cap A^I = \delta^I_J \qquad \text{and} \qquad A^I \cap A^J = B_I \cap B_J = 0. \tag{5.16}$$

The dual cohomology basis α_I, β^J satisfies

$$\int_{A^J} \alpha_I = \int \alpha_I \wedge \beta^J = \delta^J_I \qquad \text{and} \qquad \int_{B_J} \beta^I = \int \beta^I \wedge \alpha_J = -\delta^I_J. \tag{5.17}$$

These properties are preserved under the action of the symplectic modular group $Sp(2h^{2,1} + 2; \mathbb{Z})$.

We define a set of coordinates t^α $(\alpha = 1, \dots, h^{2,1})$ of the complex-structure moduli space by $t^\alpha = X^\alpha / X^0$ where X^I are the A periods of the holomorphic 3-form Ω:

$$X^I = \int_{A^I} \Omega, \qquad \text{for} \qquad I = 0, \dots, h^{2,1}. \tag{5.18}$$

As the moduli space is spanned by the X^I, the B periods

$$F_I = \int_{B_I} \Omega \tag{5.19}$$

are functions $F_I = F_I(X)$ and we can express the holomorphic 3-form Ω as

$$\Omega = X^I \alpha_I - F_I(X) \beta^I. \tag{5.20}$$

The relation $\partial_\alpha \Omega = K_\alpha \Omega + \chi_\alpha$ implies the condition $\int \Omega \wedge \partial_I \Omega = 0$, which can be used to show that

$$F_I = X^J \, \partial F_J / \partial X^I = \frac{1}{2} \partial\left(X^J F_J\right) / \partial X^I. \tag{5.21}$$

Thus, we find that the B periods are given by the derivatives of the *prepotential* $F = \frac{1}{2} X^J F_J$. It is clear from (5.21) that the prepotential is homogenous of degree 2.

Using the expression of the Kähler potential in terms of the Hitchin functional we have

$$e^{-\mathcal{K}^{2,1}} = i \int_M \Omega \wedge \bar{\Omega} = -i \sum_{I=0}^{h^{2,1}} (\int_{A^I} \Omega \int_{B_I} \bar{\Omega} - \int_{A^I} \bar{\Omega} \int_{B_I} \Omega)$$

$$\Rightarrow e^{-\mathcal{K}^{2,1}} = -i \sum_{I=0}^{h^{2,1}} (X^I \bar{F}_I - \bar{X}^I F_I). \tag{5.22}$$

Therefore, the prepotential F is seen to completely determine the Kähler potential.

Finally we note that for a function f of the moduli space coordinates X^I that is independent of the Calabi-Yau coordinates, the transformation

$$\Omega \longrightarrow e^{f(X)}\Omega \tag{5.23}$$

is a symmetry of the theory since Ω is uniquely defined only up a function $f = f(X)$. The Kähler metric is also invariant under (5.23).

Differentiating $\mathcal{K}^{2,1} = -\log(i \int \Omega \wedge \overline{\Omega})$ and using $\partial_\alpha \Omega = K_\alpha \Omega + \chi_\alpha$, we find the explicit form of K_α to be $K_\alpha = -\partial_\alpha \mathcal{K}^{2,1}$. We can now define the covariant derivative

$$D_\alpha = \partial_\alpha + \partial_\alpha \mathcal{K}^{2,1} \tag{5.24}$$

satisfying

$$D_\alpha \Omega \longrightarrow e^{f(X)} D_\alpha \Omega. \tag{5.25}$$

5.5 The Kähler-structure moduli space

Define a cubic form on the space of real (1,1)-forms by

$$\kappa(\rho, \sigma, \tau) = \int_M \rho \wedge \sigma \wedge \tau, \tag{5.26}$$

and note that the volume of the compact 6-dimensional manifold M can be written as

$$V = \frac{1}{6}\int_M J \wedge J \wedge J = \frac{1}{6}\kappa(J, J, J). \tag{5.27}$$

The space of (1,1) cohomology classes is endowed with a natural inner product

$$G(\rho, \sigma) = \frac{1}{2V}\int_M \rho \wedge \star\, \sigma, \tag{5.28}$$

where $\star$ is the Hodge star operator on the Calabi-Yau manifold. Using the identity $\star\, \sigma = -J \wedge \sigma + \frac{1}{4V}\kappa(\sigma, J, J) J \wedge J$, we can rewrite the metric as

$$G(\rho, \sigma) = -\frac{1}{2V}\kappa(\rho, \sigma, J) + \frac{1}{8V^2}\kappa(\rho, J, J)\kappa(\sigma, J, J). \tag{5.29}$$

Choosing a real basis e_α ($\alpha = 1, \ldots, h^{1,1}$) of harmonic (1,1)-forms, the equation

$$\mathcal{J} = B + iJ = w^\alpha e_\alpha, \tag{5.30}$$

defines the coordinates w^α. The moduli space metric can now be written as

$$G_{\alpha\overline{\beta}} = \frac{1}{2} G\left(e_\alpha, e_\beta\right) = \frac{\partial}{\partial w^\alpha}\frac{\partial}{\partial \overline{w}^{\beta}}\, \mathcal{K}^{1,1}, \tag{5.40}$$

where the Kähler potential $\mathcal{K}^{1,1}$ is related to the volume of the Calabi-Yau manifold according to

$$e^{-\mathcal{K}^{1,1}} = \frac{4}{3}\int_M J \wedge J \wedge J = 8V. \tag{5.41}$$

The space spanned by w^α is a Kähler manifold. We notice, of course, that the Kähler potential is once again given by one of Hitchin's volume functionals.

Finally, we introduce a new coordinate w^0 and define a homogeneous function $G = G(w)$ of degree 2 which acts as the analogue to the complex structure moduli-space prepotential:

$$G(w) = \frac{1}{6}\frac{w^\alpha w^\beta w^\gamma}{w^0}\int e_\alpha \wedge e_\beta \wedge e_\gamma. \tag{5.42}$$

The Kähler potential $\mathcal{K}^{1,1}$ is then related to this function as

$$e^{-\mathcal{K}^{1,1}} = i\sum_{I=0}^{h^{1,1}}(w^I \partial\bar{G}/\partial\bar{w}^I - \bar{w}^I\,\partial G/\partial w^I), \tag{5.43}$$

where $w^0 = 1$.

VI. Applications of Hitchin's Hamiltonian Flow

After a brief review of the relevant mathematical ideas from Chapter II, we consider several applications of Hitchin's Hamiltonian flow construction. In particular, we consider the implications of the Hamiltonian flow for topological M-theory on $M^6 \times S^1$ and the canonical quantization of topological M-theory. Towards the end of the chapter we introduce a particularly interesting class of manifolds known as half-flat manifolds which arise in the context of $\mathcal{N} = 1$ string compactifications. We will demonstrate how these manifolds can be identified with the phase space of Hitchin's Hamiltonian flow.

6.1 Hitchin's V_7 functionals

We begin with a review of the main result from Chapter II on the geometry of stable 3-forms in 7 dimensions. Specifically, we construct a functional $V_7(\Phi)$ on a manifold X of real dimension 7 which endows X with a metric of holonomy G_2, where Φ is a stable 3-form. We are aware, of course, that stable 4-forms also exist in 7 dimensions and indeed, as we shall see, we can construct a functional $V_7(G)$ depending on a 4-form G which endows X with the same structure. In essence, these V_7 functionals are different sides of the same mathematical coin.

Let Φ be a stable 3-form. Recall that Φ generates a G_2 structure on X simply because the exceptional Lie group G_2 is the stabilizer of Φ. Define a symmetric tensor B by

$$B_{jk} = -\frac{1}{144}\Phi_{ji_1i_2}\Phi_{ki_3i_4}\Phi_{i_5i_6i_7}\epsilon^{i_1\cdots i_7}, \tag{6.1}$$

using the Levi-Civita tensor in seven dimensions. We now construct a metric g on X in terms of Φ using the tensor B

$$g_{jk} = B_{jk}\det(B)^{-1/9}. \tag{6.2}$$

The Hitchin functional $V_7(\Phi)$ now has a simple interpretation as the volume of X as defined by the metric g. That is,

$$V_7(\Phi) = \int_X \sqrt{g\Phi} = \int_X \det(B)^{1/9}. \tag{6.3}$$

We can rewrite (6.3) as

$$V_7(\Phi) = \int_X \Phi \wedge * \Phi, \tag{6.4}$$

where $*$ denotes the Hodge star operator associated with the metric g.

We now repeat our familiar analysis by fixing the cohomlogy class of Φ so that

$$\Phi = \Phi_0 + dB \quad \text{where} \quad d\Phi_0 = 0,$$

and B is some 2-form on X. Performing the first variation of $V_7(\Phi)$ and using homogeneity we find that

$$\delta V_7(\Phi)/\delta\Phi = \tfrac{7}{3} * \Phi. \tag{6.5}$$

Thus, for a critical point we must have

$$d\Phi = d * \Phi = 0, \tag{6.6}$$

which holds if and only if Φ defines a metric of holonomy G_2 on X.

We now turn to an interesting construction of $V_7(G)$ where G is a stable 4-form. We begin by choosing a basis $\{e_a^{ij}\}$ for the 21-dimensional space of alternating bilinear vectors $\Lambda^2 V$, so that $e_a^{ij} = -e_a^{ji}$ for $i, j = 1, \cdots, 7$ and $a = 1, \cdots, 21$. We now define a 21 × 21 matrix Q by

$$Q_{ab} = \tfrac{1}{2} e_a^{ij} e_b^{kl} G_{ijkl}, \tag{6.7}$$

in terms of which the functional V_7(G) is written as

$$V_7(G) = \int_X \det(\mathrm{Q})^{1/12}.$$
(6.8)

This functional turns out to be equal to the one we would have obtained had we instead performed a construction parallel to (6.3), in effect expressing $V_7(G)$ as a volume functional based on a metric derived from the 4-form G. In either case we end up with the functional

$$V_7(G) = \int_X G \wedge * G, \tag{6.9}$$

where we emphasize that here $*$ is the Hodge star operator defined by the metric derived from the stable 4-form G.

Considering the first variation of $V_7(G)$ with G restricted to a fixed cohomology class, we once again find

$$dG = d * \mathrm{G} = 0, \tag{6.10}$$

implying that the critical points of $V_7(G)$ are in precise correspondence to the metrics of holonomy G_2 on X.

We shall return to this $V_7(G)$ functional later when we consider the effective action of the 7-dimensional topological M-theory. For now we move on to a brief review of the construction of Hitchin's Hamiltonian flow.

6.2 $SU(3)$ structures and metrics with holonomy G_2

Recall that for a 6-dimensional manifold M we defined the Hitchin functionals $V_S(\sigma)$ and $V_H(\rho)$ by

$$V_S(\sigma) = \frac{1}{6}\int_M k \wedge k \wedge k = \frac{1}{6}\int_M \sigma^{3/2}, \tag{6.11}$$

and

$$V_H(\rho) = -\frac{i}{4}\int_M \Omega \wedge \bar{\Omega} = \frac{1}{2}\int_M \hat{\rho}(\rho) \wedge \rho, \tag{6.12}$$

where ρ is a stable 3-form and σ is a stable 4-form. We saw how when extremized $V_S(\sigma)$ and $V_H(\rho)$ respectively generate a symplectic structure (with symplectic form k) and a complex structure (with holomorphic 3-form Ω) on M. Provided we can combine these two structures in a consistent manner, we can think of k as the Kähler form on M, with respect to the complex structure defined by Ω. As discussed in Chapter II, the symplectic and complex structures on M are compatible in this way provided the following two conditions are satisfied:

$$k \wedge \rho = 0, \tag{6.13}$$

and

$$V_H(\rho) = 2V_S(\sigma). \tag{6.14}$$

These two consistency equations encode the basic compatibility requirements that k be of type (1,1) in the complex structure given by Ω and that there be agreement between the volume forms defined by the symplectic and holomorphic structures. Under these conditions, k and Ω combine to generate an $SU(3)$ structure on M. Furthermore, if k and Ω satisfy $dk = 0$ and $d\Omega = 0$, as well as the compatibility conditions, then M is a Calabi-Yau 3-fold.

In *Theorem 2.6* of Chapter II, we saw how for closed ρ and σ defined on M, we can always uniquely find a metric of holonomy G_2 on the manifold $M \times (a, b)$, where (a, b) is some interval. This was achieved by allowing ρ and σ to time evolve with respect to the flow governed by the equations

$$\partial\rho/\partial t = dk, \tag{6.15}$$

and

$$\partial\sigma/\partial t = k \wedge \partial k/\partial t = -d\hat{\rho}, \qquad (6.16)$$

where t is interpreted in as the "time" variable. As was demonstrated, these flow equations are equivalent to the characteristic G_2 holonomy equations $d\Phi = d * \Phi = 0$ for the 3-form Φ defined by

$$\Phi = \rho(t) + k(t) \wedge dt. \qquad (6.17)$$

6.3 Topological M-theory on $M^6 \times S^1$

Let ρ and k be a 3-form and 2-form, respectively, on the 6-dimensional manifold M and consider the 7-manifold $X = M \times S^1$. Define a 3-form Φ on X by

$$\Phi = \rho + k \wedge dt. \qquad (6.18)$$

If we now assume that Φ is stable on X, it follows that ρ and k are stable on M and so generate an $SU(3)$ structure on M, provided they satisfy the compatibility conditions (6.13) and (6.14). The integrability conditions of this $SU(3)$ structure are

$$dk = 0, \qquad (6.19)$$

$$d\Omega = d\big(\rho + i\hat{\rho}(\rho)\big) = 0, \qquad (6.20)$$

and these are precisely equivalent to

$$d\Phi = d * \Phi = 0. \qquad (6.21)$$

Interpreting (6.21) as the equations of motion of topological M-theory on X, we see a reduction to a 6-dimensional theory whose classical solutions are Calabi-Yau 3-folds.

Recalling the link existing between the functionals V_S, V_H and the topological A and $B + \bar{B}$ models respectively, it perhaps seems natural to interpret the 6-dimensional reduction of the topological M-theory on X as corresponding in some sense to a combination of the A and B models. This might provide some insight into the nonperturbative completion of the topological string, which is expected to involve a combination of the A and B models. Under the above construction, the compatibility conditions between the symplectic and holomorphic structures on M provide the coupling between the A and B models.

6.4 Canonical quantization of topological M-theory

In this section we consider the canonical quantization of the 7-dimensional theory whose action is given by the functional $V_7(G) = \int_X G \wedge * G$, where G is a stable 4-form on the manifold $X = M \times S^1$. We fix the cohomology class $[G] \in H^4(X, \mathbb{R})$ so that

$G = G_0 + d\Gamma$, where Γ is a 3-form gauge field and $dG_0 = 0$. Inspired by the ideas in the previous section, we express G in the form

$$G = \sigma + \hat{\rho} \wedge dt, \tag{6.22}$$

so that it's Hodge dual takes the form

$$* G = \rho + k \wedge dt. \tag{6.23}$$

Writing the $V_7(G)$ action explicitly in terms of $G = \sigma + \hat{\rho} \wedge dt$, we find that after imposing the first compatibility condition $k \wedge \rho = 0$, the action can be written simply as a linear combination of V_H and V_S

$$V_7(G) = \int_{X=M\times S^1} dt(2V_H + 3V_S) = 2V_H(\rho) + 3V_S(\sigma). \tag{6.24}$$

We now write the gauge field Γ in the same general form using a 3-form γ and a 2-form β whose components lie only along M,

$$\Gamma = \gamma + \beta \wedge dt. \tag{6.25}$$

Assuming that G and Γ take the forms given above, we find that $(\sigma, \hat{\rho}(\rho))$ and (γ, β) are related by

$$\hat{\rho} = \hat{\rho}_0 + d\beta + \dot{\gamma}, \tag{6.26}$$

$$\sigma = \sigma_0 + d\gamma, \tag{6.27}$$

so that (γ, β) spans the configuration space. Variation of the action with respect to β yields

$$\delta V_7/\delta\beta = d\rho = 0, \tag{6.28}$$

which generates the gauge transformation

$$\gamma \to \gamma + d\lambda. \tag{5.29}$$

Thus following canonical quantization of the theory defined by Hitchin's $V_7(G)$ action, the reduced phase space is parameterized by (γ, ρ) for

$$\gamma \in \Omega^3(M)/\Omega^3_{exact}(M), \tag{6.30}$$

$$\rho \in \Omega^3_{closed}(M). \tag{6.31}$$

Using homogeneity, we calculate the conjugate momentum π_γ to be

$$\pi_\gamma = \frac{7}{4}\rho, \tag{6.32}$$

which immediately suggests the canonical commutation relation

$$\{\delta\gamma, \delta\rho\} = \int_M \delta\gamma \wedge \delta\rho, \tag{6.33}$$

where we interpret the canonical variable γ as "position" and ρ as "momentum".

This commutation relation suggests an uncertainty principle for the quantum Calabi-Yau, whereby it would be impossible to perform simultaneous measurements of the complex and Kähler structures at the same point in M. In this way, the A and B models will not be independent in the conjectured reduction of topological M-theory to a combination of the A and B models.

Now under a variation of γ that satisfies $d(\delta\gamma) = 0$, a variation is induced in the nonperturbative A model partition function via the coupling to Lagrangian branes while the Kähler form is preserved. These closed variations of γ parameterize an $H^3(M, \mathbb{R})$ in the phase space which is canonically conjugate to the cohomology class of ρ by (6.33). In this sense we see a mixing of the A and B model variables, with the parameter γ representing the Lagrangian D-brane tension in the A model being conjugate to the B model 3-form ρ.

It is argued in [9] that this conjugacy of the degrees of freedom of the A and B models within the context of topological M-theory might be related to the conjectured S-duality relating the A and B models. The connection of the nonperturbative amplitudes of the A model to Lagrangian D-branes and the γ field, together with the connection of the nonperturbative amplitudes of the B model to the D1-brane and the B-field, is viewed as supporting the idea that the full nonperturbative topological string can be interpreted as a single object consisting of a combination of the A and B models.

6.5 Half-flat manifolds in $\mathcal{N} = 1$ string compactifications

We now turn to the use of Hitchin's Hamiltonian flow construction in the context of the study of so-called *half-flat* manifolds. It is known that 6-dimensional manifolds with $SU(3)$ structure have a significant connection to the space of string vacua with minimal ($\mathcal{N} = 1$) supersymmetry. An $SU(3)$ structure on M is defined by the existence of a non-degenerate almost hermitian structure $J_m{}^n$ together with a globally defined 3-form Ω of type $(3,0)$ with respect to $J_m{}^n$. Such a structure determines a Riemannian metric g_{mn} on M, which can in turn be used to define a 2-form J of type $(1,1)$ in terms of the almost complex structure $J_m{}^n$:

$$J_{mn} = J_m{}^p g_{pn}. \tag{6.34}$$

A characteristic property of an orientable 6-manifold M with $SU(3)$ structure is that it admits a single globally defined Majorana spinor η which is nonzero everywhere. The existence of such a spinor implies a reduction of the structure group of M from $SO(6)$ to $SU(3)$. We can construct the differential forms J_{mn} and Ω_{mnp} using the spinor η as

$$\begin{aligned} J_{mn} &= i\bar{\eta}\Gamma_7\Gamma_{mn}\eta \\ \mathrm{Re}\,\Omega_{mnp} &= -\bar{\eta}\Gamma_7\Gamma_{mnp}\eta \\ \mathrm{Im}\,\Omega_{mnp} &= i\eta\Gamma_{mnp}\eta, \end{aligned} \tag{6.35}$$

where Γ_7 is the 6-dimensional chirality operator and Γ_{mn}, Γ_{mnp} are antisymmetrized products of the corresponding pair and triplet of gamma matrices.

The $SU(3)$ invariant spinor η satisfies

$$\nabla'\eta = 0, \tag{6.36}$$

for some connection ∇' which may be different from the Levi-Civita connection ∇ of the metric g_{mn}. When the structure group is reduced to $SU(3)$, the difference between ∇' and ∇ defines the intrinsic torsion W. In the special case when ∇' coincides with the Levi-Civita connection, the spinor η is covariantly constant ($\nabla\eta = 0$) and the holonomy group $Hol(M,\nabla)$ is thus given by $SU(3)$. That is, M is a Calabi-Yau manifold when $\nabla \equiv \nabla'$. The manifolds with special holonomy $SU(3)$ are Kähler with vanishing first Chern class and have metrics that are Ricci flat. Hence, we see that the intrinsic torsion W is a measure of deviation from the Calabi-Yau condition.

The intrinsic torsion W lies in $\Lambda^1 \oplus \mathfrak{su}(3)^\perp$ with $\mathfrak{su}(3)^\perp$ defined by the decomposition $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp$. Using the decompositions $\mathfrak{su}(3)^\perp = \mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$ and $\Lambda^1 \cong \mathbf{3} \oplus \mathbf{3}$ under $SU(3)$, it can be shown that the intrinsic torsion W decomposes into five torsion classes W_i corresponding to the following $SU(3)$ modules

$$\begin{aligned} &W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \\ &\qquad = (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \overline{\mathbf{6}}) \oplus (\mathbf{3} \oplus \overline{\mathbf{3}}) \oplus (\mathbf{3}' \oplus \overline{\mathbf{3}}') \end{aligned}$$

Now dJ and $d\Omega$ are in the $\mathbf{20}$ and $\mathbf{24}$ representations of $SO(6)$, respectively, and have the following decompositions ([36], [37]) under $SU(3)$:

$$dJ = -\tfrac{3}{2}\mathrm{Im}(W_1\overline{\Omega}) + W_4 \wedge J + W_3 = (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{3} \oplus \overline{\mathbf{3}}) \oplus (\mathbf{6} \oplus \overline{\mathbf{6}}) = \mathbf{20} \tag{6.37}$$

and

$$d\Omega = W_1 J \wedge J + W_2 \wedge J + \overline{W_5} \wedge \Omega = (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \overline{\mathbf{8}}) \oplus (\mathbf{3}' \oplus \overline{\mathbf{3}}') = \mathbf{24}, \tag{6.38}$$

where

$$W_3 \wedge J = W_3 \wedge \Omega = W_2 \wedge J \wedge J = 0. \tag{6.39}$$

We note that

$$W_1 \in \Omega^0(M),$$
$$W_2 \in \Omega^2(M),$$
$$W_3 = \overline{W_3} \in \Omega^{2,1}_{prim}(M) \oplus \Omega^{1,2}_{prim}(M),$$
$$W_4 = \overline{W_4} \in \Omega^1(M),$$
$$W_5 \in \Omega^{1,0}(M). \tag{6.40}$$

Distinct classes of $SU(3)$ structure manifolds can be defined by requiring various combinations of the torsion classes to vanish. For example, Calabi-Yau manifolds are characterized as $SU(3)$ structures whose five torsion classes uniformly vanish.

We consider a particularly interesting class of manifolds of $SU(3)$ structure which appear in the study of mirror symmetry on compactifications of string theories on Calabi-Yau manifolds with NS fluxes. These are known as *half-flat* manifolds and play a significant role in the construction of realistic vacua with minimal ($\mathcal{N} = 1$) supersymmetry. Half-flat manifolds can be viewed as mirrors of Calabi-Yau manifolds with a certain kind of NS-NS fluxes [38] and are characterized by the vanishing of the following torsion components:

$$\text{Re}\, W_1 = \text{Re}\, W_2 = W_4 = W_5 = 0. \tag{6.41}$$

In terms of J and Ω, this defining property of half-flatness is expressed as

$$d(J \wedge J) = 0,$$
$$d(\text{Re}\, \Omega) = 0. \tag{6.42}$$

Defining σ and ρ by $\sigma = \frac{1}{2} J \wedge J$ and $\rho = \text{Re}\, \Omega$, we can rewrite (5.42) as

$$d\sigma = 0,$$
$$d\rho = 0, \tag{6.43}$$

together with the constraint $\rho \wedge J = 0$. We notice that this precisely corresponds to the structure induced on a 6-dimensional hypersurface embedded in a G_2 manifold. Thus we see that Hitchin's Hamiltonian flow allows us to uniquely extend a half-flat $SU(3)$ manifold to a manifold $X = M \times (a, b)$ with a metric of holonomy G_2. So the half-flat manifolds arising in the context of $\mathcal{N} = 1$ string compactifications can be precisely identified with the phase space of Hitchin's Hamiltonian flow. Furthermore, the ground states are in correspondence with Calabi-Yau manifolds, which are the stationary solutions of the flow equations.

It remains an open question whether all $\mathcal{N} = 1$ string vacua can be realized in topological M-theory.

VII. Generalized Geometry

In this chapter we introduce the basic mathematical structure of generalized geometry. The key feature of generalized geometry is the replacement of the tangent bundle T with the direct sum of the tangent and cotangent bundles $T \oplus T^*$, and use of the Courant bracket on sections of the bundle $T \oplus T^*$ as the natural generalization of the Lie bracket defined on sections of T. As we shall see, this gives rise to a geometry with the novel feature that it transforms naturally under both the action of a closed 2-form B and the usual diffeomorphism group. After setting up this basic structure, we proceed to define the more sophisticated concepts of generalized tangent bundles and generalized metrics.

7.1 The linear algebra of $V \oplus V^*$

Let V be an n-dimensional vector space and V^* be its dual space. The $2n$-dimensional space $V \oplus V^*$ admits a non-degenerate symmetric inner product of signature (n,n) defined by

$$\langle v+\xi, w+\eta\rangle = \frac{1}{2}\big(\xi(w)+\eta(v)\big), \tag{7.1}$$

for $v,w \in V$ and $\xi,\eta \in V^*$. The Lie group preserving this inner product together with the canonical orientation existing on $V \oplus V^*$ is $SO(V \oplus V^*) \cong SO(n,n)$. The Lie algebra of $SO(V \oplus V^*)$ is given by $\mathfrak{so}(V \oplus V^*) = \{T: \langle Tx, y\rangle + \langle x, Ty\rangle = 0;\ \forall x,y \in V \oplus V^*\}$ and based on the $V \oplus V^*$ structure, we have the decomposition

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}, \tag{7.2}$$

where $A \in \mathrm{End}(V)$, $B: V \to V^*$ satisfies $B^* = -B$ and $\beta: V^* \to V$ satisfies $\beta^* = -\beta$. Thus via $B(v) = i_v B$ we think of B as a 2-form in $\Lambda^2 V^*$, which leads us to the decomposition

$$\mathfrak{so}(V \oplus V^*) = \mathrm{End}(V) \oplus \Lambda^2 V^* \oplus \Lambda^2 V. \tag{7.3}$$

By considering

$$\exp\begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \tag{7.4}$$

for $B \in \Lambda^2 V^*$, we find a particular type of orthogonal symmetry of $T \oplus T^*$ in the identity component of $SO(V \oplus V^*)$, known as a B-transform and given by

$$v + \xi \longmapsto v + \xi + i_v B. \tag{7.5}$$

Now let $CL(V \oplus V^*)$ be the Clifford algebra defined by

$$\langle v, v\rangle = v^2,\ \forall v \in V \oplus V^*, \tag{7.6}$$

and consider the action of $v + \xi \in V \oplus V^*$ on the exterior algebra $\Lambda^* V^*$, given by

$$(v + \xi) \cdot \varphi = i_v \varphi + \xi \wedge \varphi. \tag{7.7}$$

We find that

$$(v + \xi)^2 \cdot \varphi = i_v(i_v\varphi + \xi \wedge \varphi) + \xi \wedge (i_v\varphi + \xi \wedge \varphi) = (i_v\xi)\varphi = \langle v + \xi, v + \xi \rangle \varphi, \tag{7.8}$$

thus defining an algebra representation on $\Lambda^* V^*$, known as the spin representation $Spin(V \oplus V^*)$. Since in signature (n, n) the volume element ω of a Clifford algebra satisfies $\omega^2 = 1$, the spin representation splits into two irreducible half-spin representations S^+ and S^- corresponding to the ± 1 eigenspaces of ω.

We note that the group $Spin(V \oplus V^*)$ is a double cover of $SO(V \oplus V^*)$ as seen by the homomorphism

$$\rho: Spin(V \oplus V^*) \longrightarrow SO(V \oplus V^*)$$

$$\rho(x)(v) = xvx^{-1} \qquad x \in Spin(V \oplus V^*),\ v \in V \oplus V^*. \tag{7.9}$$

Now $\exp B$, obtained by exponentiating $B \in \Lambda^2 V^*$ in the Lie group $Spin(V \oplus V^*)$, acts on spinors as

$$\exp B\,(\varphi) = \left(1 + B + \frac{1}{2} B \wedge B + \cdots\right) \wedge \varphi. \tag{7.10}$$

For even dimensional V, there exists an invariant bilinear form $\langle \cdot, \cdot \rangle$ on $S^\pm$ which is symmetric if $\dim V = 4k$ and skew-symmetric if $\dim V = 4k + 2$. Defining $\sigma: \Lambda^{ev/odd} V^* \longrightarrow \Lambda^{ev/odd} V^*$ by

$$\sigma(\varphi_{2m}) = (-1)^m \varphi_{2m}, \qquad \sigma(\varphi_{2m+1}) = (-1)^m \varphi_{2m+1}$$

we can write the bilinear form on $S^\pm$ as

$$\langle \varphi, \psi \rangle = (\sigma\varphi \wedge \psi)_n. \tag{7.11}$$

The pairing defined in (7.11) is known as the *Mukai pairing* between two spinors.

We end this section by presenting the definition of a *pure spinor*. First note that for a nonzero spinor φ, the null space $L_\varphi = \{v \in V \oplus V^* : v.\varphi = 0\}$ depends equivariantly on φ under the spin representation, in the sense that for all $g \in Spin(V \oplus V^*)$

$$L_{g.\varphi} = \rho(g) L_\varphi. \tag{7.12}$$

Now $\langle v, w \rangle = \frac{1}{2}(vw + wv).\varphi = 0$ for $v, w \in L_\varphi$, so that

$$\langle v, w \rangle = 0 \quad \forall v, w \in L_\varphi. \tag{7.13}$$

That is, null spaces are *isotropic*.

Definition 7.1: *A spinor φ is said to be pure if L_φ is maximally isotropic, i.e. it has dimension equal to $dim\, V$.*

7.2 The Courant bracket

We consider a bracket acting on pairs (X, ξ) of a vector field X and a p-form ξ on a manifold M, defined by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi). \tag{7.14}$$

The skew-symmetric operator $[.,.]$ is known as the Courant bracket, which unlike the Lie bracket fails to satisfy the Jacobi identity.

A feature of the Lie bracket on sections of T is that it only has automorphisms which are trivial, in the sense that they correspond simply to diffeomorphisms of the manifold M. One of the key properties of the Courant bracket is that it unlike the Lie bracket it also possesses non-trivial automorphisms which are defined by forms. To see this, let $\alpha \in \Omega^{p+1}$ be a closed form and consider the vector bundle automorphism F of $T \oplus \Lambda^P T^*$ defined by

$$F(X + \xi) = X + \xi + i_X \alpha. \tag{7.15}$$

A simple calculation confirms that indeed

$$F([X + \xi, Y + \eta]) = [F(X + \xi), F(Y + \eta)]. \tag{7.16}$$

Our interest in the Courant bracket will be limited to the case where $p = 1$.

7.3 The structure of generalized geometries

We now translate the results from our earlier study of the linear algebra of $V \oplus V^*$ to the general setting of a manifold M of dimension n. Denote the tangent bundle of M by T and consider the direct sum of the tangent and cotangent bundles $T \oplus T^*$, which has a natural inner product of signature (n, n) determined by

$$(X + \xi, X + \xi) = i_X \xi\ , \tag{7.17}$$

where X and ξ are tangent and cotangent vectors, respectively. The decomposition (7.3) translates here to the result that the skew-adjoint endomorphisms of $T \oplus T^*$ are sections of the bundle $\mathrm{End}(T) \oplus \Lambda^2 T^* \oplus \Lambda^2 T$. As before, we focus on the action of a 2-form B by taking the exponential to obtain an orthogonal transformation on $T \oplus T^*$, yielding

$$X + \xi \longrightarrow X + \xi + i_X B \tag{7.18}$$

Consider the action of $X + \xi \in T \oplus T^*$ on spinors φ defined by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi. \tag{7.19}$$

Mirroring the ideas in section 6.1, we think of the bundle of differential forms $\Lambda^* T^*$ as a bundle of Clifford modules over the Clifford algebra defined by this action of $T \oplus T^*$.

Thus, in this context we can think of forms as spinors. The 2-form B then exponentiates into the spin group to act on a form as $\varphi \longmapsto e^B \varphi$.

We are now in a position to see how the Courant bracket defined in (7.14) arises naturally when we attempt to endow the direct sum $T \oplus T^*$ with a suitable differential structure. The Lie bracket and the exterior derivative d are related by the equation

$$2 i_{[X,Y]} = d([i_X, i_Y]\alpha) + 2 i_X d(i_Y \alpha) - 2 i_Y d(i_X \alpha) + [i_X, i_Y] d\alpha \tag{7.20}$$

If we replace the vector fields X, Y by sections u, v of $T \oplus T^*$, and the interior products by Clifford products, we arrive at a new bracket $[u, v]$ defined on sections of $T \oplus T^*$ and satisfying the identities

$$[u, fv] = f[u, v] + (\pi(u)f)v - (u, v)df \tag{7.21}$$

$$\pi(u)(v, w) = ([u, v] + d(u, v), w) + (v, [u, w] + d(u, w)) \tag{7.22}$$

where π is the projection onto the tangent bundle: $\pi(X + \xi) = X$. When written out explicitly in terms of $u = X + \xi$ and $v = Y + \eta$, we find that this new bracket is precisely the Courant bracket defined in the previous section as

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi).$$

We note that if B is a closed 2-form, the Courant bracket is invariant under the map $X + \xi \longrightarrow X + \xi + i_X B$.

We have now developed all the elements necessary to define generalized geometry. Following Hitchin [39] we define "generalized geometries" as consisting of the data on $T \oplus T^*$ which are compatible with the $SO(n, n)$ structure and satisfy an integrability condition expressed by the Courant bracket. The symmetry group of the geometry is then provided by the semi-direct product $\mathrm{Diff}(M) \ltimes \Omega^2_{closed}$ which replaces the usual diffeomorphism group.

7.4 The generalized tangent bundle

Our definition of the generalized tangent bundle will rely on ideas related to a topological construct known as a *gerbe* on a manifold M. We briefly review these ideas before discussing the generalized tangent bundle. Taking an open cover $\{U_\alpha\}$ of M, we think of a gerbe on M as a collection of functions

$$g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow S^1 \tag{7.23}$$

satisfying $g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = \cdots$ and $\delta g = g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$.

In the language of homological algebra, this simply means that a gerbe on M is thought of as a 2-cocycle. We define a *connective structure* on a gerbe as a collection of 1-forms $A_{\alpha\beta}$ in $\Omega^1(U_\alpha \cap U_\beta)$ satisfying $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Definition 7.2: *Let $\{A_{\alpha\beta}\}$ be a connective structure on a gerbe on M. A curving of this connective structure is a collection of 2-forms $B_\alpha \in \Omega^2(U_\alpha)$ satisfying $B_\beta - B_\alpha = dA_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. This defines a global closed 3-form $H = dB_\alpha = dB_\beta$, known as the curvature.*

Now given a connective structure $\{A_{\alpha\beta}\}$, it follows from $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$ that $dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = 0$ so that we have a cocycle with values in the space of closed 2-forms. Through identification of $T \oplus T^*$ on U_α with $T \oplus T^*$ on U_β via the B-field action $X + \xi \longrightarrow X + \xi + i_X dA_{\alpha\beta}$, this defines a vector bundle E as an extension

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0. \tag{7.24}$$

Since the $SO(n,n)$ structure of $T \oplus T^*$ is preserved by the action of the 2-form $dA_{\alpha\beta}$, the vector bundle E is also endowed with this structure. Moreover, the preservation of the Courant bracket by the action of the closed 2-form $dA_{\alpha\beta}$ implies the existence of an induced bracket on sections of E satisfying (7.21) and (7.22), where π is the bundle projection $\pi: E \longrightarrow T$. This structure defines the *generalized tangent bundle E*.

7.5 Generalized metrics

Suppose g is a Riemannian metric on the manifold M. We can think of g as a homomorphism $g: T \longrightarrow T^*$ and consider its graph V as a subset of $T \oplus T^*$. $V \subset T \oplus T^*$ is then a subbundle on which the indefinite inner product is positive definite. This observation motivates the following definition of a *generalized metric*, which is not explicitly based on the existence of a Riemannian metric g on M.

Definition 7.3: *Let E be the generalized tangent bundle for a gerbe with connective structure. A generalized metric is a subbundle $V \subset E$ of rank n on which the induced inner product is positive definite.*

Since the inner product is positive definite on the generalized metric V and zero on the cotangent bundle T^*, we must have $V \cap T^* = 0$. Hence, V defines a splitting of the exact sequence $0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0$. Similarly, a second splitting is defined by the orthogonal complement $V^\perp$ since the (n, n) signature of the inner product on E implies a negative definite inner product on $V^\perp$.

Locally, the splitting V provides a family of sections $\{C_\alpha\}$ of $T^* \otimes T^*$ satisfying $C_\beta - C_\alpha = dA_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. A tangent vector X is then lifted locally to $X + i_X C_\alpha$. Define $C_\alpha(X, Y) = i_X C_\alpha(Y)$ and note that

$$C_\alpha(X, Y) = (X + i_X C_\alpha, X + i_X C_\alpha) \tag{7.25}$$

is positive definite by the positive definiteness of the induced inner product. We have,

$$(Y - i_Y C_\alpha^T, X + i_X C_\alpha) = \frac{1}{2}\left(C_\alpha(X, Y) - C_\alpha^T(Y, X)\right) = 0, \tag{7.26}$$

showing that $Y - i_Y C_\alpha^T$ is orthogonal to V. Half the difference of the two splittings $X + i_X C_\alpha$ and $X - i_X C_\alpha^T$ is the symmetric part of C_α:

$$\frac{1}{2}(i_X C_\alpha + i_X C_\alpha^T), \tag{7.27}$$

which is positive definite. Therefore, we see that half the difference of the splittings defined by V and $V^\perp$ yields a metric g on M.

Another interesting observation is that the average of the two splittings yields the skew symmetric part of C_α, which is a 2-form B_α defining a curving of the connective structure as can be seen by the relation $C_\beta - C_\alpha = dA_{\alpha\beta}$ satisfied on $U_\alpha \cap U_\beta$. This generates a curvature H.

VIII. Generalized Calabi-Yau Manifolds

Now that we have developed a solid foundation for generalized geometry, we introduce a generalized complex structure as a particular kind of structure that is compatible with this general background. We show how symplectic and complex structures are included as extremal special cases of this higher geometric construct. We also define the notion of generalized Calabi-Yau manifolds and confirm that these manifolds are special cases of generalized complex manifolds.

8.1 Linear generalized complex structures

Consider a real finite-dimensional vector space V of dimension n. We know that a nondegenerate bilinear form $\omega \in \Lambda^2 V$ defines a *symplectic structure* on V. Using the interior product, a symplectic form ω can be thought of as a mapping $\omega: v \longmapsto i_v\omega$ from V to its dual space V^*. This gives an equivalent way of defining a symplectic structure on V as an isomorphism $\omega: V \longrightarrow V^*$ satisfying

$$\omega^* = -\omega, \tag{8.1}$$

where ω^* is the linear dual of the mapping ω.

A *complex structure* on V is an endomorphism $J: V \longrightarrow V$ such that

$$J^2 = -1. \tag{8.2}$$

We now take advantage of the structure of the $2n$-dimensional direct sum $V \oplus V^*$ to embed these two very different structures into a higher geometric construct. This higher structure is known as the *generalized complex structure* on V.

Defininition 8.1: *A generalized complex structure on V is an endomorphism $\mathcal{J}$ of $V \oplus V^*$ satisfying both $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$.*

Now let J be a complex structure on V and consider the endomorphism

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \tag{8.3}$$

written with respect to $V \oplus V^*$. It is easy to see that $\mathcal{J}_J^2 = -1$ and $\mathcal{J}_J^* = -\mathcal{J}_J$, confirming that $\mathcal{J}_J$ is a generalized complex structure. Similarly if ω is a symplectic structure on V, the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{8.4}$$

is a generalized complex structure since it satisfies $\mathcal{J}_\omega^2 = -1$ and $\mathcal{J}_\omega^* = -\mathcal{J}_\omega$. Thus we see that complex and symplectic structures on a finite-dimensional vector space appear as extremal special cases of linear generalized complex structures. Specifically, complex and symplectic structures on V correspond to diagonal and anti-diagonal linear generalized complex structures, respectively.

8.2 Generalized complex structures on manifolds

In this section we define a generalized complex structure on a smooth manifold of even dimension. The definition that we present here is the one given by Hitchin in [40]. Through a sequence of propositions in Gualtieri's thesis [41], one can see that this definition is a natural extension of the notion of a generalized complex structure on a finite-dimensional vector space as defined in the previous section.

The following definition is inspired by the observation that in the case of an even dimensional manifold M, an almost complex structure $J: T \longrightarrow T$ with $J^2 = -1$ has a $+i$-eigenspace in $T \otimes \mathbb{C}$ which is a subbundle E whose sections are closed under the Lie bracket and satisfies $E \oplus \bar{E} = T \otimes \mathbb{C}$. The Newlander-Nirenberg Theorem then implies that the subbundle E defines an integrable complex structure on M. Replacing T by $T \oplus T^*$ and the Lie bracket by the Courant bracket, we arrive at Hitchin's definition after introducing an isotropic condition.

Definition 8.2: *Let M be a smooth manifold of dimension $2n$ and consider the bundle $T \oplus T^*$. A generalized complex structure on M is a subbundle $E \subset (T \oplus T^*) \otimes \mathbb{C}$ such that*

$i)$ $E \oplus \bar{E} = (T \oplus T^*) \otimes \mathbb{C}$

$ii)$ *the space of sections of E is closed under the Courant bracket*

$iii)$ *E is isotropic.*

The isotropic condition ensures the compatibility of the almost complex structure $J: T \oplus T^* \longrightarrow T \oplus T^*$ with the indefinite metric on $T \oplus T^*$, in the sense that the metric $(X + \xi, Y + \eta) = \frac{1}{2}(i_X\eta + i_Y\xi)$ is pseudo-hermitian. Furthermore, the obstruction to condition $ii)$ requiring closure under the action of the Courant bracket is tensorial only if E is isotropic.

This definition of a generalized complex structure yields a reduction of the structure group of $T \oplus T^*$ from $SO(2n, 2n)$ to $U(n, n) \subset SO(2n, 2n)$, together with an integrability condition. In the special case where $J = I$ on T and $J = -I$ on T^*, we recover an ordinary complex manifold.

8.3 Spinors

We recall from the previous chapter that due to the linear structure of $T \oplus T^*$, there exists a decomposition of the spin representation into two irreducible half-spin representations S^+ and S^-. This decomposition corresponds to a splitting of the bundle of formal sums of differential forms $\Lambda^* T^*$ on M into the bundle $\Lambda^{ev} T^*$ of even forms and the bundle $\Lambda^{odd} T^*$ of odd forms. Considering that the action of $GL(n, \mathbb{R}) \subset Spin(T \oplus T^*)$ on $\Lambda^* T^*$ is

$$\varphi \longmapsto |\det M|^{1/2} M^* \varphi, \tag{8.5}$$

where M^* denotes the standard action of $M \in GL(n, \mathbb{R})$ on $\Lambda^* T^*$, we note the following relation between the spin bundle S and the bundle of forms $\Lambda^* T^*$:

$$S = \Lambda^* T^* \otimes (\Lambda^n T)^{1/2}. \tag{8.6}$$

Thus, we have

$$S^+ = \Lambda^{ev} T^* \otimes (\Lambda^n T)^{1/2}, \tag{8.7}$$

$$S^- = \Lambda^{odd} T^* \otimes (\Lambda^n T)^{1/2}. \tag{8.8}$$

Recall also that the spin representations S^+ and S^- have an associated invariant bilinear form $\langle \varphi, \psi \rangle$ which takes values in the one-dimensional space $\Lambda^n T^*$. As defined in (8.11), the bilinear form is equivalent to the exterior product pairing up to a choice of sign.

The action of $X + \xi$ is determined by Clifford multiplication to be

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi, \tag{8.9}$$

and in this representation the action of $B \in \Lambda^2 T^*$ is

$$\exp B(\varphi) = \left(1 + B + \frac{1}{2} B \wedge B + \cdots \right) \wedge \varphi. \tag{8.10}$$

By removing the trivial line bundle $(\Lambda^n T)^{1/2}$, we can think of forms as spinors. It is easy to see that the null space $L_\varphi = \{X + \xi \in (T \oplus T^*) \otimes \mathbb{C} : (X + \xi). \varphi = 0\}$ of a spinor φ is an isotropic subspace. A *pure* spinor was defined earlier by the condition that L_φ is maximally isotropic. In the case of the pure spinor $1 \in \Lambda^0 T^*$, its null space is the maximal isotropic subspace $U = T \subset T \oplus T^*$. Transforming by $B \in \Lambda^2 T^*$ we obtain the pure spinor $\exp B(1) = 1 + B + \frac{1}{2} B^2 + \cdots$, whose maximal isotropic subspace is $U = \exp B(T)$.

Therefore a generalized complex structure is defined in the neighbourhood of each point by a complex pure spinor field, up to multiplication by a scalar. In general these locally defined pure spinors are local trivializations of a complex line bundle. However, for a particular type of generalized complex structure known as a *generalized Calabi-Yau manifold*, the pure spinor is global. We now turn to the study of these structures.

8.4 Generalized Calabi-Yau manifolds

Definition 8.3: *A generalized Calabi-Yau manifold is an even-dimensional manifold M with a closed form $\varphi \in \Omega^{ev/odd} \otimes \mathbb{C}$ which is a complex pure spinor for the orthogonal vector bundle $T \oplus T^*$ satisfying $\langle \varphi, \bar{\varphi} \rangle \neq 0$ at each point.*

The pure spinor φ associated with a generalized Calabi-Yau manifold (M, φ) defines a maximal isotropic subbundle $E_\varphi \subset (T \oplus T^*) \otimes \mathbb{C}$ where

$$E_\varphi = L_\varphi = \{X + \xi \in (T \oplus T^*) \otimes \mathbb{C} : (X + \xi) \cdot \varphi = 0\} \tag{8.11}$$

is the annihilator of φ. The condition $\langle \varphi, \bar{\varphi} \rangle \neq 0$ implies

$$E_\varphi \cap E_{\bar{\varphi}} = E_\varphi \cap \bar{E}_\varphi = 0 \tag{8.12}$$

and thus

$$E_\varphi \oplus \bar{E}_\varphi = (T \oplus T^*) \otimes \mathbb{C}. \tag{8.13}$$

Now all that remains to show that a generalized Calabi-Yau manifold is indeed a special case of a generalized complex manifold is to confirm that sections of E_φ are closed under the Courant bracket. To see this, assume that $X + \xi$ and $Y + \eta$ lie in E_φ so that

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi = 0 \tag{8.14}$$

and

$$(Y + \eta) \cdot \varphi = i_Y \varphi + \eta \wedge \varphi = 0. \tag{8.15}$$

Recalling that the Lie derivative is defined by $\mathcal{L}_X = d i_X + i_X d$ and using $d\varphi = 0$, $i_X \varphi = -\xi \wedge \varphi$ and $i_Y \varphi = -\eta \wedge \varphi$, we find that

$$\begin{aligned} i_{[X,Y]}\varphi &= \mathcal{L}_X(i_Y\varphi) - i_Y \mathcal{L}_X \varphi \\ &= -\mathcal{L}_X \eta \wedge \varphi - \eta \wedge \mathcal{L}_X \varphi - i_Y d(i_X \varphi) \\ &= -\mathcal{L}_X \eta \wedge \varphi - \eta \wedge d i_X \varphi + i_Y (d\xi \wedge \varphi) \\ &= -\mathcal{L}_X \eta \wedge \varphi + \eta \wedge d\xi \wedge \varphi + i_Y(d\xi) \wedge \varphi - d\xi \wedge \eta \wedge \varphi \end{aligned}$$

$$\Rightarrow \quad i_{[X,Y]}\varphi = -\mathcal{L}_X \eta \wedge \varphi + i_Y(d\xi) \wedge \varphi \tag{8.16}$$

By skew-symmetry,

$$
\begin{aligned}
[X,Y]\varphi &= \tfrac{1}{2}(i_{[X,Y]}\varphi - i_{[Y,X]}\varphi) \\
&= \tfrac{1}{2}(-\mathcal{L}_X\eta + i_Y d\xi + \mathcal{L}_Y\xi - i_X d\eta) \wedge \varphi \\
&= \tfrac{1}{2}(-i_X d\eta - d i_X\eta + i_Y d\xi + i_Y d\xi + d i_Y\xi - i_X d\eta) \wedge \varphi \\
&= \tfrac{1}{2}[2(d i_Y + i_Y d)\xi - 2(d i_X + i_X d)\eta - d(i_Y\xi - i_X\eta)] \wedge \varphi \\
&= [\mathcal{L}_Y\xi - \mathcal{L}_X\eta - \tfrac{1}{2}d(i_Y\xi - i_X\eta)] \wedge \varphi.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&[[X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi)].\,\varphi = 0 \\
&\Rightarrow \quad [X+\xi, Y+\eta]\cdot\varphi = 0, \qquad (8.17)
\end{aligned}
$$

so that $[X+\xi, Y+\eta] \in E_\varphi$ as claimed.

We will now show how complex and symplectic manifolds appear as special cases of generalized Calabi-Yau manifolds. First consider an m-dimensional complex manifold M_C with a non-vanishing holomorphic form Ω of degree m. For any $v \in T \otimes \mathbb{C}$ of type (0,1) and $\xi \in T^* \otimes \mathbb{C}$ of type (1,0), we have

$$
(v+\xi)\cdot\Omega = i_v\Omega + \xi \wedge \Omega = 0. \qquad (8.18)
$$

Hence, Ω is pure and we have a maximal annihilator subspace of real dimension $2m$. Since Ω is closed and satisfies

$$
\langle\Omega, \overline{\Omega}\rangle = (-1)^m \Omega \wedge \overline{\Omega} \neq 0, \qquad (8.19)
$$

the pair (M_C, Ω) defines a generalized Calabi-Yau manifold.

Now let M_S be a symplectic manifold of dimension m with symplectic form ω and recall that $1 \in \Lambda^0 T^*$ is a pure spinor. We obtain another pure spinor

$$
\varphi = \exp i\omega \in \Omega^{ev} \otimes \mathbb{C} \qquad (8.20)
$$

by exponentiating the 2-form $i\omega$. Note that $d\varphi = 0$ and

$$
\langle\varphi, \overline{\varphi}\rangle \sim \omega^m \neq 0 \qquad (8.21)
$$

since ω is closed and non-degenerate. Thus, (M_S, φ) is a generalized Calabi-Yau manifold.

8.5 B-field transforms

Let (M, φ) be a generalized Calabi-Yau manifold and consider the action of a real closed 2-form $B \in \Lambda^2 T^*$ on φ given by

$$\varphi' = e^B \varphi. \tag{8.22}$$

As noted earlier, φ' is both closed and pure. Since the B-field acts via the Spin group, we have

$$\langle \varphi', \overline{\varphi'} \rangle = \langle e^B \varphi, e^B \bar{\varphi} \rangle = \langle \varphi, \bar{\varphi} \rangle \neq 0, \tag{8.23}$$

implying that we obtain another generalized Calabi-Yau manifold (M, φ') through the action of the B-field on φ. The B-transform provides a way of interpolating between symplectic and Calabi-Yau structures as we shall see.

For a symplectic manifold with symplectic form ω, the pure spinor is $\varphi = e^{i\omega}$ and the transform takes the form

$$\varphi' = c e^{B+i\omega}, \tag{8.24}$$

where c is a nonzero complex constant.

Let $\omega^c = \omega_1 + i\omega_2$ be the holomorphic non-degenerate (2,0)-form of a holomorphic symplectic manifold M of complex dimension $2k$ and note that each of ω_1 and ω_2 defines a real symplectic structure on M. For $t \neq 0$, the action of $B = \omega_1/t$ on the generalized Calabi-Yau manifold $(M, e^{i\omega_2/t})$ yields the generalized Calabi-Yau manifold $(M, \varphi(t))$, where

$$\varphi(t) = e^{(\omega_1 + i\omega_2)/t}. \tag{8.25}$$

The limit

$$\varphi = \lim_{t \to 0} t^k \varphi(t) = \frac{1}{k!}(\omega_1 + i\omega_2)^k \tag{8.26}$$

gives the $(2k, 0)$-form of a Calabi-Yau structure via the B-field transform of the symplectic structure defined by ω_2/t.

8.6 Structure group reductions

As was mentioned earlier, a generalized complex structure is a reduction of the structure group of $T \oplus T^*$ from $SO(2n, 2n)$ to $U(n, n,)$ together with an integrability condition. Here we consider the reduction of the structure group defined by a Calabi-Yau manifold (M, φ). As $\langle \varphi, \bar{\varphi} \rangle \neq 0$, $\varphi / \sqrt{|\langle \varphi, \bar{\varphi} \rangle|}$ defines a non-vanishing section

$$S^+ \otimes \mathbb{C} \cong \Lambda^{ev} T^* \otimes (\Lambda^n T)^{1/2} \otimes \mathbb{C} \tag{8.27}$$

of the spinor bundle S. We can rewrite this as

$$S^+ \otimes \mathbb{C} \cong \Lambda^{ev} {E_\varphi}^* \otimes (\Lambda^n E_\varphi)^{1/2} \otimes \mathbb{C}, \tag{8.28}$$

since $E_\varphi \subset (T \oplus T^*) \otimes \mathbb{C}$ is a maximally isotropic subbundle. Thus φ defines a non-vanishing section of $(\Lambda^n E_\varphi)^{1/2}$ and so determines a complex volume form on E, implying a reduction of the structure group to $SU(n,n) \subset U(n,n) \subset SO(2n,2n)$.

Although we will note pursue this line of thought much deeper, we would like to note that by considering further reductions of the structure group to even smaller subgroups, one is led to other generalized geometries. For example, fixing 2, 3 or 4 spinors by subgroups of $Spin(4,4)$ in 4 dimensions yields the special double covers

1) $SU(2,2) \longrightarrow SO(4,2)$

2) $Sp(1,1) \longrightarrow SO(4,1)$

3) $Sp(1) \times Sp(1) \longrightarrow SO(4)$

corresponding to a generalized Calabi-Yau stucture, a *generalized hyperkähler structure* and a *generalized hyperkähler metric*, respectively. [12] [41]

IX. The Generalized Hitchin Functional

In this chapter we present a variational approach to generalized Calabi-Yau manifolds in the 6-dimensional case, thereby reconnecting with many of the key ideas we started with. After setting up the necessary algebraic background, we use a certain invariant quartic form to construct a volume functional which yields generalized Calabi-Yau manifolds as its critical points. Equipped with this generalized Hitchin functional, we take a deeper look into the connection between the topological B-model and the Hitchin action discussed in Chapter III.

9.1 An invariant quartic function

To set up the necessary algebra, we work with the group $Spin(12, \mathbb{C})$ instead of $Spin(6,6)$ and focus on a complex 6-dimensional vector space V. The spin spaces $S^{\pm}$ are of real dimension 32 and constitute symplectic representations due to the skew symmetry of the spinor bilinear form $\langle \varphi, \psi \rangle$ in this dimension. Set $S = S^{+} \cong \Lambda^{ev} V^* \otimes (\Lambda^6 V)^{1/2}$ and note that a symplectic action of a Lie group G on a vector space S defines a moment map $\mu: S \longrightarrow \mathfrak{g}^*$ by

$$\mu(\rho)(a) = \frac{1}{2} \langle \sigma(a)\rho, \rho \rangle, \tag{9.1}$$

where $\rho \in S$, $a \in \mathfrak{g}$ and $\sigma: \mathfrak{g} \longrightarrow \text{End}\, S$ gives the representation. By identifying the Lie algebra $\mathfrak{so}(12, \mathbb{C})$ with its dual via the bi-invariant form $tr(XY)$, $\mu(\rho)$ can be seen as taking values in the Lie algebra.

Let $\nu \in (\Lambda^6 V)^{1/2}$ be a basis vector and write $a = A + B + \beta$ with respect to the decomposition $\mathfrak{so}(V \oplus V^*) = \text{End}(V) \oplus \Lambda^2 V^* \oplus \Lambda^2 V$. We find that

$$\mu(\nu)(a) = \frac{1}{2} \langle \sigma(a)\nu, \nu \rangle = \langle -(trA/2)\nu + B \wedge \nu, \nu \rangle = 0, \tag{9.2}$$

so that $\mu: \nu \longmapsto 0$ for any $\nu \in (\Lambda^6 V)^{1/2}$. Hence, the moment map vanishes on any pure spinor. We now construct an invariant quartic function which will provide a key link to pure spinors.

Definition 9.1: *Using the moment map μ of the spin representation S of $Spin(12, \mathbb{C})$, we define an invariant quartic function q on S by*

$$q(\rho) = tr\, \mu(\rho)^2. \tag{9.3}$$

A crucial property of this quartic form is that $q(\rho) \neq 0$ if and only if there exists a unique pair of pure spinors $\{\alpha, \beta\}$ satisfying $\langle \alpha, \beta \rangle \neq 0$ such that $\rho = \alpha + \beta$. [40]

Furthermore, it can be shown that a pair of pure spinors α, β satisfy

$$q(\alpha + \beta) = 3\langle \alpha, \beta \rangle^2. \tag{9.4}$$

If $\rho = \alpha + \beta$ is real, $\{\alpha, \beta\}$ forms a complex conjugate pair. In the case that α is real, we have $\langle \alpha, \beta \rangle = \langle \alpha, \bar{\alpha} \rangle = \langle \alpha, \alpha \rangle \in \mathbb{R}$ so that $q(\rho) > 0$ by (8.4). If α is not real, then $\langle \alpha, \beta \rangle = \langle \alpha, \bar{\alpha} \rangle$ is imaginary and $q(\rho) < 0$. We thus arrive at the following lemma which connects the quartic form q to the pure spinors φ satisfying $\langle \varphi, \bar{\varphi} \rangle \neq 0$.

Lemma 9.1: *If $\rho \in S$ is a pure spinor with $q(\rho) < 0$, then ρ is the real part of a complex pure spinor φ satisfying $\langle \varphi, \bar{\varphi} \rangle \neq 0$.*

9.2 Generalized geometry from a variational perspective

Consider the real group $\mathbb{R}^* \times Spin(6, 6)$ and note that it acts transitively on the open set

$$U = \{\rho \in S: q(\rho) < 0\}, \tag{9.5}$$

when we take the vector space V to be real. We now use the quartic form to construct an $SO(6,6)$-invariant homogeneous function ϕ of degree 2 on U.

Definition 9.2: *Let $U = \{\rho \in S: q(\rho) < 0\}$ and define $\phi: U \longrightarrow \mathbb{R}^+$ by*

$$\phi(\rho) = \sqrt{-q(\rho)/3}. \tag{9.6}$$

According to *Lemma 8.1*, $\rho \in U$ can be written as $\rho = \varphi + \bar{\varphi}$ in terms of a pure spinor φ that satisfies $\langle \varphi, \bar{\varphi} \rangle \neq 0$. From (8.4) we see that $\langle \varphi, \bar{\varphi} \rangle^2 = q(\rho)/3 < 0$, so that the function ϕ is related to the purely imaginary number $\langle \varphi, \bar{\varphi} \rangle$ by

$$i\phi(\rho) = \langle \varphi, \bar{\varphi} \rangle. \tag{9.7}$$

Although for definiteness we fixed $S = S^+ \cong \Lambda^{ev} V^* \otimes (\Lambda^6 V)^{1/2}$ in our analysis, we could just as easily have worked with $S^- = \Lambda^{odd} V^* \otimes (\Lambda^n V)^{1/2}$. Thus, we have effectively defined an invariant homogeneous function $\phi: U \longrightarrow \Lambda^6 V^*$ of degree 2 on an open set U in $\Lambda^{ev/odd} V^*$.

Now let M be a compact oriented 6-dimensional manifold and note that a form $\rho \in \Omega^{ev/odd}(M)$ is stable if it lies in the open set U at each point of M. We now construct the generalized Hitchin functional $\mathbb{H}_{\mathcal{G}}$ as a volume functional

$$\mathbb{H}_{\mathcal{G}}(\rho) = \int_M \phi(\rho), \tag{9.8}$$

defined on the space of even or odd differential forms. Note that $\mathbb{H}_{\mathcal{G}}$ is invariant under both diffeomorphisms and the action of the B-field $\varphi \longmapsto e^B \varphi$.

Before considering the critical points of this functional, we note that the derivative of ϕ at ρ is related to the bilinear symplectic form according to

$$D\phi(\dot{\rho}) = \langle \hat{\rho}, \dot{\rho} \rangle. \tag{9.9}$$

With $\hat{\rho}$ as above, we state and prove the key result regarding the generalized Hitchin functional $\mathbb{H}_{\mathcal{G}}$.

Theorem 9.1: *A closed stable form $\rho \in \Omega^{ev/odd}(M)$ is a critical point of $\mathbb{H}_{\mathcal{G}}(\rho)$ in its cohomology class if and only if $\rho + i\hat{\rho}$ defines a generalized Calabi-Yau structure on M.*

Proof: The first variation yields

$$\delta\mathbb{H}_{\mathcal{G}}(\dot{\rho}) = \int_M D\phi(\dot{\rho}) = \int_M \langle \hat{\rho}, \dot{\rho} \rangle. \tag{9.10}$$

As the variation is restricted to a fixed cohomology class, we have $\dot{\rho} = d\alpha$. That is,

$$\delta\mathbb{H}_{\mathcal{G}}(\dot{\rho}) = \int_M \langle \hat{\rho}, d\alpha \rangle. \tag{9.11}$$

Recall that the bilinear form $\langle .,. \rangle$ is equivalent to an exterior product pairing together with a choice of sign determined by the function $\sigma: \Lambda^{ev/odd}V^* \longrightarrow \Lambda^{ev/odd}V^*$, where $\sigma(\psi_{2m}) = (-1)^m \psi_{2m}$ and $\sigma(\psi_{2m+1}) = (-1)^m \psi_{2m+1}$. Thus, we have

$$\delta\mathbb{H}_{\mathcal{G}}(\dot{\rho}) = \int_M \sigma(\hat{\rho}) \wedge d\alpha \tag{9.12}$$

and by an application of Stokes' theorem we find that

$$\delta\mathbb{H}_{\mathcal{G}}(\dot{\rho}) = \pm \int_M d\sigma(\hat{\rho}) \wedge \alpha = \pm \int_M \sigma(d\hat{\rho}) \wedge d\alpha = \pm \int_M \langle d\hat{\rho}, \alpha \rangle. \tag{9.13}$$

The variation vanishes for all α precisely when $d\hat{\rho} = 0$. That is, the closed form ρ is a critical point if and only if $d\varphi = 0$, where $2\varphi = \rho + i\hat{\rho}$. Since $\langle \varphi, \bar{\varphi} \rangle \neq 0$, we see that (M, φ) is a Calabi-Yau manifold.

■

9.3 The topological B-model at one loop

In Chapter III, we noted that the Hitchin functional $V_H(\rho)$ defined for a real stable 3-form ρ on a 6-dimensional manifold M is related to the topological B-model with target space M. The relation that we suggested was an identification of the partition function of the Hitchin functional with the Wigner transform of the partition functions of the B and $\overline{\mathrm{B}}$ topological strings on M, which we supported with classical arguments. In this section, we present a modification to that conjecture after considering the quantization of the Hitchin functional at the quadratic or one-loop order. [42]

First we consider the one-loop contribution to the B-model partition function Z_B. The holomorphic Ray-Singer torsion for the $\bar{\partial}$-complex of the vector bundle of holomorphic p-forms $\Omega^{p,0}(M)$ on M is defined as

$$I_{\bar{\partial},p}^{RS} = \sqrt{\prod_{q=0}^{n} (\det{}' \Delta_{pq})^{(-1)^{q+1}q}}, \tag{9.14}$$

where $\det{}' \Delta_{pq}$ denotes the zeta function regularized product of nonzero eigenvalues of the Laplacian Δ_{pq} acting on (p,q)-forms.

It is shown in [21] that the one-loop contribution F_1^B to the B-model partition function is given by the product of the holomorphic Ray-Singer $\bar{\partial}$-torsions $I_{\bar{\partial},p}^{RS}$ as

$$F_1^B = \log \prod_{p=0}^{n} (I_{\bar{\partial},p}^{RS})^{(-1)^{p+1}p}. \tag{9.15}$$

Expansion in terms of the determinants $\det{}' \Delta_{pq}$ yields

$$F_1^B = \frac{1}{2} \log \prod_{p=0}^{n} \prod_{q=0}^{n} (\det{}' \Delta_{pq})^{(-1)^{p+q}pq}\,. \tag{9.16}$$

Using the Hodge duality relation $I_p = I_{3-p}$, we can write the one-loop factor in the B-model partition function as

$$Z_{B,1-loop} = \exp(-F_1^B) = I_1/I_0^3. \tag{9.17}$$

Now recall that V_H can be expressed as a volume functional

$$V_H(\rho) = \int_M \phi(\rho), \tag{9.18}$$

where $\phi(\rho)$ defines a measure on M. We saw how a critical point of this functional defines a complex structure I on M together with a holomorphic (3,0)-form $\rho + i\hat{\rho}$. We refer to this structure as a Calabi-Yau 3-fold. We fix the cohomology class of ρ by writing $\rho = \rho_0 + d\beta$ for a 2-form β which needs to be quantized in order to test the conjecture at one-loop. The partition function of the Hitchin functional can be written as a path integral over the 2-form β:

$$Z_H([\rho_0]) = \int D\beta \exp\big(-V_H(\rho_0 + d\beta)\big). \tag{9.19}$$

To perform the quantization at one-loop, we need to consider the second variation of V_H. This second variation is a non-degenerate function of β modulo the action of diffeomorphisms $\mathrm{Diff}(M)$ combined with gauge transformations $\beta \longrightarrow \beta + d\lambda$. If the cohomology class $[\rho_0]$ lies in a certain open set in $H^3(M,\mathbb{R})$, then V_H has a unique critical point modulo gauge transformations and thus generates a local mapping from $H^3(M,\mathbb{R})$ into the moduli space of Calabi-Yau structures on M. [8]

Recall that the first variation of V_H is given by

$$\delta V_H(\rho) = \int_M \delta\rho \wedge \hat{\rho}. \tag{9.20}$$

Let ρ_c be a critical point of V_H defining a complex structure I on M. The second variation evaluated at ρ_c is

$$V_H^{(2)}(\rho_c, \delta\rho) = \delta_\rho^2 V_H(\rho) = \int_M \delta\rho \wedge J\delta\rho, \tag{9.21}$$

where J is a certain complex structure on the space of 3-forms $W = \Lambda^3(\mathbb{R}^6) \otimes \mathbb{C}$ which commutes with the Hodge decomposition of W with respect to I and has eigenvalues $J = +i$ on $W^{3,0}$, $W^{2,1}$ and $J = -i$ on $W^{1,2}$, $W^{0,3}$. [8]

The one-loop quantum partition function for the Hitchin theory can now be written as

$$Z_{H,1-loop}([\rho_c]) = \int D\beta \exp\left(-V_H^{(2)}(\rho_c, d\beta)\right). \tag{9.22}$$

Writing the 2-form β as $\beta = \beta_{20} + \beta_{11} + \beta_{02}$ in terms of the Hodge decomposition with respect to the complex structure I on M, we find that

$$\begin{aligned} V_H^{(2)}(\rho_c, \delta\rho) &= \int_M d\beta \wedge Jd\beta \\ &= \int_M d(\beta_{20} + \beta_{11} + \beta_{02}) \wedge Jd(\beta_{20} + \beta_{11} + \beta_{02}) \\ &= \int_M d(\beta_{20} + \beta_{11} + \beta_{02}) \wedge (id\beta_{20} + Jd\beta_{11} - id\beta_{02}) \end{aligned}$$

$$\Rightarrow\ V_H^{(2)}(\rho_c, \delta\rho) = \int_M \partial\beta_{11} \wedge \bar{\partial}\beta_{11}, \tag{9.23}$$

where we have used the fact that the form $d\beta_{20}$ lies entirely in the i-eigenspace of J to write $Jd\beta_{20} = id\beta_{20}$ and then $\int_M d\beta \wedge d\beta_{20} = 0$ by Stokes' theorem. The $d\beta_{02}$ term in the second variation vanishes in a similar manner. The β_{20} and β_{02} terms do not appear in the intergral as they can both be removed by a diffeomorphism which leaves the action invariant. Indeed, for each β there exists a unique vector field which generates a diffeomorphism that simultaneously removes β_{20} and β_{02}. We are thus left with a path integral over β_{11} with the remaining gauge invariance given by $\delta\beta_{11} = \bar{\partial}\lambda_{10} + \partial\lambda_{01}$.

We will not go through the technical details of the quantization of the $\int_M \partial\beta_{11} \wedge \bar{\partial}\beta_{11}$ term which is skilfully presented by V.Pestun and E.Witten in [42]. Here we note that the one-loop path integral of the V_H-theory was calculated in [42] to be given by a product of Ray-Singer torsions as

$$Z_{H,1-loop} = I_1/I_0, \tag{9.24}$$

which disagrees with the B-model partition function $Z_{B,1-loop} = I_1/I_0^3$. That is, the genus one free energy of the topological B-model disagrees with the one-loop free energy of the Hitchin functional V_H.

Fortunately, we can escape this discrepancy by replacing V_H with the generalized Hitchin functional $\mathbb{H}_{\mathcal{G}}(\rho)$, where this time $\rho \in \Omega^{odd}(M)$ is a formal sum

$$\rho = \rho_1 + \rho_3 + \rho_5 \tag{9.25}$$

of odd forms $\rho_i \in \Omega^i(M)$. We saw that if ρ lies in the open set U defined in the previous section, then it generates a generalized almost complex structure on M determined by a pure spinor φ satisfying $2\varphi = \rho + i\hat{\rho}$. Furthermore, if $d\rho = d\hat{\rho} = 0$ then this generalized almost complex structure is integrable and we have a critical point of the functional $\mathbb{H}_{\mathcal{G}}(\rho)$. If $b_1(M) = 0$, then the generalized complex structure coincides with an ordinary complex structure and the critical points of the generalized Hitchin functional correspond to Calabi-Yau structures on M. Therefore, we see that in this context the generalized Hitchin functional $\mathbb{H}_{\mathcal{G}}(\rho)$ is an extension of the functional V_H and the classical agreement between the B-model partition function and that of Hitchin theory will continue to hold when we replace V_H by $\mathbb{H}_{\mathcal{G}}$. *However, since the fluctuating degrees of freedom are different, these two theories are not equivalent on a quantum level*.

For trivial $b_1(X)$, the quadratic expansion of $\mathbb{H}_{\mathcal{G}}$ is given by

$$\int_M \delta\rho \wedge J\delta\rho, \tag{9.26}$$

where J is a complex structure on $\Omega^{odd}(M)$. With respect to Hodge decomposition, J has $+i$ eigenvalues on the spaces $\Omega^{1,0}$, $\Omega^{3,0}$, $\Omega^{2,1}$, $\Omega^{3,2}$ and the one-loop expansion of the classical generalized Hitchin functional near a critical point ρ_c is

$$\mathbb{H}_{\mathcal{G}}^{(2)}(\rho_c, \delta\rho) = \int_M \left(\beta_{11} \wedge \partial\bar{\partial}\beta_{11} + \beta_{00} \wedge \partial\bar{\partial}\beta_{22}\right). \tag{9.27}$$

Note the additional $\int_M \beta_{00} \wedge \partial\bar{\partial}\beta_{22}$ term compared to (9.23). Carrying out the quantization of (9.27) as in [42], we find that the partition function Z_{GH} of the generalized Hitchin model at one loop is given by

$$Z_{GH,1-loop} = I_1/I_0^3, \tag{9.28}$$

in agreement with the partition function of the topological B-model (9.17).

Thus we see that the one-loop contribution to the partition function of the B-model is equal to the one-loop contribution to the partition function of generalized Hitchin theory:

$$Z_{GH,1-loop} = Z_{B,1-loop}. \tag{9.29}$$

Finally, we note that the precise relation of the Hitchin functional to the full partition function of the topological B-model, if indeed such a link exists, has yet to be determined.

X. Supergravity in the language of Generalizeed Geometry

We present a reformulation of ten-dimensional type II supergravity in the language of generalized geometry by considering an $O(9,1) \times O(1,9) \subset O(10,10) \times \mathbb{R}^{+}$ structure on the generalized tangent space. Following the construction of torsion-free, compatible generalized connections, we are led to the definition of the generalized Ricci tensor and generalized Ricci scalar. This brings us to a position from which we can express the supersymmetry variations and equations of motion in an elegant and manifestly $Spin(9,1) \times Spin(1,9)$-covariant form.

10.1 Type II supergravity

We begin with a brief introduction to the structure of $d = 10$ type II supergravity in the democratic formalism [44]. The type II fields are

$$\left\{g_{\mu\nu}, B_{\mu\nu}, A^{(n)}_{\mu_1 \dots \mu_n}, \psi^{\pm}_{\mu}, \lambda^{\pm}\right\}, \tag{10.1}$$

where $g_{\mu\nu}$ is the metric, $B_{\mu\nu}$ is the 2-form potential and ϕ is the dilaton. The RR potentials are given by $A^{(n)}_{\mu_1 \dots \mu_n}$ where n is odd for type IIA and even for type IIB. In type IIA, $\psi^{\pm}_{\mu}$ and $\lambda^{\pm}$ are chiral components of the gravitino and dilatino

$$\psi_{\mu} = \psi^{+}_{\mu} + \psi^{-}_{\mu} \qquad \text{and} \qquad \lambda = \lambda^{+} + \lambda^{-}, \tag{10.2}$$

where $$\gamma^{(10)}\psi^{\pm}_{\mu} = \mp\psi^{\pm}_{\mu} \qquad \text{and} \qquad \gamma^{(10)}\lambda^{\pm} = \pm\lambda^{\pm}. \tag{10.3}$$

In type IIB, we have

$$\psi_{\mu} = \begin{pmatrix}\psi^{+}_{\mu} \\ \psi^{-}_{\mu}\end{pmatrix} \qquad \text{and} \qquad \lambda = \begin{pmatrix}\lambda^{+} \\ \lambda^{-}\end{pmatrix}, \tag{10.4}$$

where $$\gamma^{(10)}\psi^{\pm}_{\mu} = \psi^{\pm}_{\mu} \qquad \text{and} \qquad \gamma^{(10)}\lambda^{\pm} = -\lambda^{\pm}. \tag{10.5}$$

As we shall see, it is natural to replace $\lambda^{\pm}$ with the combination

$$\rho^{\pm} = \gamma^{\mu}\psi^{\pm}_{\mu} - \lambda^{\pm}. \tag{10.6}$$

We write the field strengths as a sum of even or odd forms $F^{(B)} = \sum_n F^{(B)}_{(n)}$, where the n-form RR field strength is $F^{(B)}_{n} = e^{B} \wedge dA_{(n-1)}$. Letting $H = dB$, the bosonic "pseudo-action" [43] can be written as

$$S_B = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[e^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2\right) - \frac{1}{4}\sum_n \frac{1}{n!}\left(F^{(B)}_{(n)}\right)^2\right].$$

The term "pseudo-action" refers to the failure of S_B to imply the self-duality relation $F_{(n)}^{(B)} = (-1)^{[n/2]} * F_{(10-n)}^{(B)}$ satisfied by the RR fields. Here $*$ denotes the Hodge star operator and $[n]$ is the integer part of n.

Using the Levi-Civita connection ∇, the fermionic action takes the form

$$
\begin{aligned}
S_F = -\frac{1}{2k^2}\int \sqrt{-g}\,[e^{-2\phi}\Big(&2\bar{\psi}^{+\mu}\gamma^\nu\nabla_\nu\psi_\mu^+ - 4\bar{\psi}^{+\mu}\nabla_\mu\rho^+ - 2\bar{\rho}^+ \not{\nabla}\,\rho^+ - \frac{1}{2}\bar{\psi}^{+\mu}\not{H}\;\psi_\mu^+ \\
&- \bar{\psi}_\mu^+ H^{\mu\nu\lambda}\gamma_\nu\psi_\lambda^+ - \frac{1}{2}\rho^+ H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi_\lambda^+ + \frac{1}{2}\rho^+\not{H}\;\rho^+ \\
&+ e^{-2\phi}\Big(2\bar{\psi}^{-\mu}\gamma^\nu\nabla_\nu\psi_\mu^- - 4\bar{\psi}^{-\mu}\nabla_\mu\rho^- - 2\bar{\rho}^-\not{\nabla}\,\rho^- + \frac{1}{2}\bar{\psi}^{-\mu}\not{H}\;\psi_\mu^- \\
&+ \bar{\psi}_\mu^- H^{\mu\nu\lambda}\gamma_\nu\psi_\lambda^- + \frac{1}{2}\rho^- H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi_\lambda^- - \frac{1}{2}\rho^-\not{H}\;\rho^-\Big) \\
&- \frac{1}{4}e^{-\phi}\Big(\bar{\psi}_\mu^+\gamma^\nu\not{F}^{(B)}\gamma^\mu\psi_\nu^- + \rho^+\not{F}^{(B)}\rho^-\Big)\Big],
\end{aligned}
$$

up to and including quadratic fermionic terms. [43]

Now setting the fermions to zero, the bosonic equations of motion can be written in the form

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi - \tfrac{1}{4}e^{2\phi}\textstyle\sum_n \frac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1\ldots\lambda_{n-1}}F_\nu^{(B)\lambda_1\ldots\lambda_{n-1}} = 0, \quad (10.7)$$

$$\nabla^\mu\big(e^{-2\phi}H_{\mu\nu\lambda}\big) - \tfrac{1}{2}\textstyle\sum_n \frac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} = 0, \quad (10.8)$$

$$\nabla^2\phi - (\nabla\phi)^2 + \tfrac{1}{4}\mathcal{R} - \tfrac{1}{48}H^2 = 0, \quad (10.9)$$

with a Bianchi identity for F:

$$dF^{(B)} - H \wedge F^{(B)} = 0. \quad (10.10)$$

Variation of the fermionic action given above yields the fermionic equations of motion with leading-order fermionic terms:

$$
\begin{aligned}
\gamma^\nu\Big[\Big(\nabla_\nu \mp \tfrac{1}{24}H_{\nu\lambda\rho}\gamma^{\lambda\rho} - \partial_\nu\phi\Big)\psi_\mu^\pm \pm \tfrac{1}{2}H_{\nu\mu}{}^\lambda\psi_\lambda^\pm\Big] - \Big(\nabla_\mu \mp \tfrac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\Big)\rho^\pm \\
= \tfrac{1}{16}e^\phi \textstyle\sum_n(\pm 1)^{[(n+1)/2]}\gamma^\nu\not{F}^{(B)}_{(n)}\gamma_\mu\psi_\nu^\mp, \quad (10.11)
\end{aligned}
$$

$$
\begin{aligned}
\Big(\nabla_\mu \mp \tfrac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - 2\partial_\mu\phi\Big)\psi^{\mu\pm} - \gamma^\mu(\nabla_\mu \mp \tfrac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu\phi)\rho^\pm \\
= \tfrac{1}{16}e^\phi \textstyle\sum_n(\pm 1)^{[(n+1)/2]}\not{F}^{(B)}_{(n)}\rho^\mp. \quad (10.12)
\end{aligned}
$$

We now consider the supersymmetry transformations in type II supergravity. In both type IIA and type IIB, supersymmetry variations can be parameterized by a pair of chiral spinors $\epsilon^{\pm}$. [43] Choosing an orthonormal frame e_μ with respect to the metric $g_{\mu\nu}$, the bosonic supersymmetry transformations in type IIA can be expressed as

$$\delta e_\mu^a = \bar{\epsilon}^+ \gamma^a \psi_\mu^+ + \bar{\epsilon}^- \gamma^a \psi_\mu^-,$$

$$\delta B_{\mu\nu} = 2\bar{\epsilon}^+ \gamma_{[\mu} \psi_{\nu]}^+ - 2\bar{\epsilon}^- \gamma_{[\mu} \psi_{\nu]}^-,$$

$$\delta\phi - \tfrac{1}{4}\delta \log(-g) = -\tfrac{1}{2}\bar{\epsilon}^+ \rho^+ - \tfrac{1}{2}\bar{\epsilon}^- \rho^-,$$

$$\begin{aligned}(e^B \wedge \delta A)^{(n)}_{\mu_1 \dots \mu_n} &= \tfrac{1}{2}\left(e^{-\phi} \bar{\psi}_\nu^+ \gamma_{\mu_1 \dots \mu_n} \gamma^\nu \epsilon^- - e^{-\phi} \bar{\epsilon}^+ \gamma_{\mu_1 \dots \mu_n} \rho^-\right) \\ &\quad - \tfrac{1}{2}\left(e^{-\phi} \bar{\epsilon}^+ \gamma^\nu \gamma_{\mu_1 \dots \mu_n} \psi_\nu^- + e^{-\phi} \bar{\rho}^+ \gamma_{\mu_1 \dots \mu_n} \epsilon^-\right). \qquad (10.13)\end{aligned}$$

The type IIB bosonic supersymmetry transformations are also given by the above after a change in sign in the last line, so that the final equation becomes

$$\begin{aligned}(e^B \wedge \delta A)^{(n)}_{\mu_1 \dots \mu_n} &= \tfrac{1}{2}\left(e^{-\phi} \bar{\psi}_\nu^+ \gamma_{\mu_1 \dots \mu_n} \gamma^\nu \epsilon^- - e^{-\phi} \bar{\epsilon}^+ \gamma_{\mu_1 \dots \mu_n} \rho^-\right) \\ &\quad + \tfrac{1}{2}\left(e^{-\phi} \bar{\epsilon}^+ \gamma^\nu \gamma_{\mu_1 \dots \mu_n} \psi_\nu^- + e^{-\phi} \bar{\rho}^+ \gamma_{\mu_1 \dots \mu_n} \epsilon^-\right). \qquad (10.14)\end{aligned}$$

The fermionic supersymmetry transformations are

$$\delta\psi_\mu^{\pm} = \left(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda} \gamma^{\nu\lambda}\right)\epsilon^{\pm} + \tfrac{1}{16} e^{\phi} \textstyle\sum_n (\pm 1)^{[(n+1)/2]} \not{F}^{(B)}_{(n)} \gamma_\mu \epsilon^{\mp},$$

$$\delta\rho^{\pm} = \gamma^\mu \left(\nabla_\mu \mp \tfrac{1}{24} H_{\mu\nu\lambda} \gamma^{\nu\lambda} - \partial_\mu \phi\right)\epsilon^{\pm}. \qquad (10.15)$$

Finally, we note that the symmetry group of the NSNS bosonic sector is given by the semi-direct product $\mathrm{Diff}(M) \ltimes \Omega^2_{closed}$. [43] This is, of course, precisely the group of transformations that preserve generalized geometries.

10.2 Conformal split frames

Let M be a 10-dimensional spin manifold. The potential B is a locally defined 2-form patched as

$$B_{(i)} = B_{(j)} - d\Lambda_{(ij)} \qquad (10.16)$$

across the overlap $U_i \cap U_j$ of coordinate patches, where the $d\Lambda_{(ij)}$ satisfy

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = d\Lambda_{(ijk)}. \qquad (10.17)$$

Thus, according to our definition in Chapter VII, B defines a *connective structure on a grebe* on M. Recall that the generalized tangent bundle E was defined as an extension

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0, \tag{10.18}$$

where T and T^* denote the tangent and cotangent bundles, respectively. Consider the bundle

$$\tilde{E} = \det T^* \otimes E \tag{10.19}$$

which has a natural $O(10,10) \times \mathbb{R}^+$ structure. Now choose a basis $\{\hat{e}_a\}$ for T and let $\{e^a\}$ be the dual basis on T^*. Define the *split frame* $\{\hat{E}_A\}$ for $\tilde{E}$ by the construction given in [43] as

$$\hat{E}_A = \hat{E}_a = (\det e)(\hat{e}_a + i_{\hat{e}_a} B) \quad \text{for} \quad A = a,$$

$$\hat{E}_A = E^a = (\det e) e^a \qquad \text{for} \quad A = a + d. \tag{10.20}$$

A split frame is one that is defined by a splitting $T \longrightarrow E$ of the generalized tangent bundle. Using (10.20), we have

$$\langle \hat{E}_A, \hat{E}_B \rangle = (\det e)^2 \, \eta_{AB},$$

where η is the $O(10,10)$ metric $\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$. Since $\Phi = \det e \in \det T^*$ is a frame-dependent conformal factor, we see that $\{\hat{E}_A : A = 1, \dots, 20\}$ defines what is known as a *conformal basis* on $\tilde{E}_x$. It is useful to conformally rescale (10.20) by a function ϕ so that

$$\hat{E}_A = \hat{E}_a = e^{-2\phi}(\det e)(\hat{e}_a + i_{\hat{e}_a} B) \quad \text{for} \quad A = a,$$

$$\hat{E}_A = E^a = e^{-2\phi}(\det e) e^a \qquad \text{for} \quad A = a + d. \tag{10.21}$$

10.3 Generalized Connections

Following [46], we define the *Dorfman derivative* $L_V W$ for $V = v + \lambda \in \Gamma(E)$ and $W = w + \zeta \in \Gamma(\tilde{E})$ by

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \zeta - i_w d\lambda, \tag{10.22}$$

as a generalization of the Lie derivative which captures the symmetries of the bosonic NSNS sector. A *generalized connection* is defined as a first-order linear operator D that is compatible with the $O(10,10) \times \mathbb{R}^+$ structure and satisfies

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B - \Lambda_M W^A, \tag{10.23}$$

using frame indices. We can construct such a generalised connection using the split frame defined in (10.21) by lifting a connection ∇ to an action on $\tilde{E}$:

$$\begin{aligned} \left(D_M^{\nabla} W^A\right)\hat{E}_A &= \left(\nabla_\mu w^a\right)\hat{E}_a + (\nabla_\mu \zeta_a)E^a \quad \text{for} \quad M=\mu, \\ \left(D_M^{\nabla} W^A\right)\hat{E}_A &= 0 \quad \text{for} \quad M=\mu+10. \end{aligned} \tag{10.24}$$

A generalized connection generates a corresponding *generalized torsion* T as a linear map $T\colon \Gamma(E) \longrightarrow \Gamma(\Lambda^2 E \oplus \mathbb{R})$ defined by

$$T(V).W = L_V^D W - L_V W, \tag{10.25}$$

where L_V^D is the Dorfman derivative with ∂ replaced by D.

10.4 The $O(9,1)\times O(1,9)$ structure

Following [41] we consider an $O(9,1)\times O(1,9)$ principal subbundle of the $O(10,10)\times \mathbb{R}^+$ bundle. This is equivalent to specifying a metric g of signature (9,1), a B-field and a dilaton ϕ. Thus it provides the appropriate generalised structure to describe the NSNS supergravity fields. Geometrically, the $O(9,1)\times O(1,9)$ structure defines a splitting of E into two 10-dimensional subbundles

$$E = C_+ \oplus C_-, \tag{10.26}$$

such that the natural $O(10,10)$ metric restricts to give a metric of signature (9,1) on C_+ and a metric of signature (1,9) on C_-. Following [43] we construct a frame $\{\hat{E}_a^+\}\cup\{\hat{E}_{\bar{a}}^-\}$ where $\{\hat{E}_a^+\}$ defines an orthonormal basis for C_+ and $\{\hat{E}_{\bar{a}}^-\}$ defines an orthonormal basis for C_-. That is,

$$\begin{aligned} \langle \hat{E}_a^+, \hat{E}_b^+\rangle &= \Phi^2 \eta_{ab}, \\ \langle \hat{E}_{\bar{a}}^-, \hat{E}_{\bar{b}}^-\rangle &= -\Phi^2 \eta_{\bar{a}\bar{b}}, \\ \langle \hat{E}_a^+, \hat{E}_{\bar{a}}^-\rangle &= 0, \end{aligned} \tag{10.27}$$

where $\Phi \in \Gamma(\det T^*)$ is a fixed density which gives an isomorphism between $\tilde{E}$ and E. Note that in (10.27), η_{ab} and $\eta_{\bar{a}\bar{b}}$ are flat metrics of signature (9,1).

We can explicitly construct a generic $O(9,1)\times O(1,9)$ structure as

$$\begin{aligned} \hat{E}_a^+ &= e^{-2\phi}\sqrt{-g}\left(\hat{e}_a^+ + e_a^+ + i_{\hat{e}_a^+}B\right), \\ \hat{E}_{\bar{a}}^- &= e^{-2\phi}\sqrt{-g}\left(\hat{e}_{\bar{a}}^- - e_{\bar{a}}^- + i_{\hat{e}_{\bar{a}}^-}B\right), \end{aligned} \tag{10.28}$$

where the fixed conformal factor in (10.27) is $\Phi = e^{-2\phi}\sqrt{-g}$ and $\{\hat{e}^+_a\}$, $\{\hat{e}^-_{\bar{a}}\}$, and their duals $\{e^+_a\}$, $\{e^-_{\bar{a}}\}$, are two independent orthonormal frames for the metric g. That is,

$$g_{ab} = \eta_{ab} e^{+a} \otimes e^{+b} = \eta_{\bar{a}\bar{b}} e^{-\bar{a}} \otimes e^{-\bar{b}},$$
$$g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab}, \qquad g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}. \tag{10.29}$$

One can also construct the invariant *generalised metric*

$$G = \Phi^{-2}\left(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}\right), \tag{10.30}$$

where $\Phi = e^{-2\phi}\sqrt{-g}$ is the fixed conformal density. In the coordinate frame, we have

$$G_{MN} = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}_{MN}. \tag{10.31}$$

The pair (G, Φ) parameterizes the coset $(O(10,10) \times \mathbb{R}^+)/O(9,1) \times O(1,9)$.

10.5 Torsion-free, compatible generalized connections

A generalized connection D is said to be compatible with the $O(9,1) \times O(1,9)$ structure if $DG = 0$. Any generalised connection D can be written in the form

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B \tag{10.32}$$

in terms of D^∇. Imposing the compatibility condition and setting the torsion of D to zero, we find that

$$D_a w^b_+ = \nabla_a w^b_+ - \frac{1}{6}H_a{}^b{}_c w^c_+ - \frac{2}{9}(\delta_a{}^b\partial_c\phi - \eta_{ac}\partial^b\phi)w^c_+ + A^{+b}_a{}_c w^c_+$$
$$D_{\bar{a}} w^b_+ = \nabla_{\bar{a}} w^b_+ - \frac{1}{2}H_{\bar{a}}{}^b{}_c w^c_+,$$
$$D_a w^{\bar{b}}_- = \nabla_a w^{\bar{b}}_- + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} w^{\bar{c}}_-,$$
$$D_{\bar{a}} w^{\bar{b}}_- = \nabla_{\bar{a}} w^{\bar{b}}_+ + \frac{1}{6}H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} w^{\bar{c}}_- - \frac{2}{9}\left(\delta_{\bar{a}}{}^{\bar{b}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi\right)w^{\bar{c}}_- + A^{-\bar{b}}_{\bar{a}}{}_{\bar{c}} w^{\bar{c}}_-, \tag{10.33}$$

where the $A^{\pm}$ are undetermined tensors that do not contribute to the torsion. [43]

Thus, we can always construct torsion-free compatible connections D. We also note that in contrast to the Levi-Civita connection in ordinary Riemannian geometry, the conditions defining a torsion-free, compatible generalized connection are not strong enough to uniquely determine D.

Since M is assumed to be a spin manifold, we can promote the local structure to $Spin(9,1) \times Spin(1,9)$. Let $S(C_\pm)$ be the spinor bundles associated with the subbundles $C_\pm$ and denote the corresponding gamma matrices by γ^a, $\gamma^{\bar{a}}$. By definition, a generalized connection acts on spinors as

$$D_M \epsilon^+ = \partial_M \epsilon^+ + \frac{1}{4}\Omega_M{}^{ab}\gamma_{ab}\epsilon^+,$$

$$D_M \epsilon^- = \partial_M \epsilon^- + \frac{1}{4}\Omega_M{}^{\bar{a}\bar{b}}\gamma_{\bar{a}\bar{b}}\epsilon^-, \qquad (10.34)$$

where $\epsilon^\pm \in \Gamma(S(C_\pm))$. We can uniquely build a set of four operators from these derivatives:

$$D_{\bar{a}}\epsilon^+ = \left(\nabla_{\bar{a}} - \frac{1}{8}H_{\bar{a}bc}\gamma^{bc}\right)\epsilon^+,$$

$$D_a\epsilon^- = \left(\nabla_a + \frac{1}{8}H_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\right)\epsilon^-,$$

$$\gamma^a D_a \epsilon^+ = \left(\gamma^a\nabla_a - \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon^+,$$

$$\gamma^{\bar{a}} D_{\bar{a}} \epsilon^- = \left(\gamma^{\bar{a}}\nabla_{\bar{a}} + \frac{1}{24}H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{a}}\partial_{\bar{a}}\phi\right)\epsilon^-. \qquad (10.35)$$

We now define a traceless *generalized Ricci tensor* $R^0_{a\bar{b}}$ by either of

$$\frac{1}{2}R^0_{a\bar{b}}\gamma^a\epsilon^+ = [\gamma^a D_a, D_{\bar{b}}]\epsilon^+,$$

$$\frac{1}{2}R^0_{\bar{a}b}\gamma^{\bar{a}}\epsilon^- = [\gamma^{\bar{a}} D_{\bar{a}}, D_b]\epsilon^-, \qquad (10.36)$$

and a *generalized curvature scalar R by*

$$-\frac{1}{4}R\epsilon^+ = (\gamma^a D_a \gamma^b D_b - D^{\bar{a}}D_{\bar{a}})\epsilon^+, \qquad (10.37)$$

or alternatively,

$$-\frac{1}{4}R\epsilon^- = \left(\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a\right)\epsilon^-. \qquad (10.38)$$

10.6 Fermionic degrees of freedom

The type II fermionic degrees of freedom correspond to spinor and vector-spinor representations of $Spin(9,1) \times Spin(1,9)$. The spinor bundles $S(C_\pm)$ decompose into spinor bundles $S^\pm(C_+)$ and $S^\pm(C_-)$ of definite chirality under $\gamma^{(10)}$. For the gravitino degrees of freedom we have

$$\psi^{+}_{\bar{a}} \in \Gamma(C_{-} \otimes S^{\mp}(C_{+})), \qquad \psi^{-}_{a} \in \Gamma\big(C_{+} \otimes S^{+}(C_{-})\big), \tag{10.39}$$

and the dilatino degrees of freedom correspond to

$$\rho^{+} \in \Gamma\big(S^{\pm}(C_{+})\big), \qquad \rho^{+} \in \Gamma\big(S^{+}(C_{-})\big). \tag{10.40}$$

Finally, for the supersymmetry parameters, we have

$$\epsilon^{+} \in \Gamma\big(S^{\pm}(C_{+})\big), \qquad \epsilon^{-} \in \Gamma\big(S^{+}(C_{-})\big). \tag{10.41}$$

Note that in (10.39)-(10.41), the upper signs correspond to type IIA and the lower signs to type IIB supergravity.

10.7 RR fields

Let $S^{\pm}_{(1/2)}$ be the spin bundles of definite chirality under $\Gamma^{(20)}$, where Γ denotes the Cliff$(10{,}10;\mathbb{R})$ gamma matrices. The RR field strengths $F = \sum_n F_{(n)}$ transform as spinors of positive (negative) chirality for type IIA (type IIB). The generalized metric structure leads to the decomposition of Cliff$(10{,}10;\mathbb{R})$ into Cliff$(9{,}1;\mathbb{R}) \otimes$ Cliff$(9{,}1;\mathbb{R})$, and thus the identification of $S_{(1/2)}$ with $S(C_{+}) \otimes S(C_{-})$. Using the spinor norm on $S(C_{-})$, we interpret $F \in S_{(1/2)}$ as a map from sections of $S(C_{-})$ to sections of $S(C_{+})$. We denote the image under this isomorphism by

$$F_{\#}\colon S(C_{-}) \longrightarrow S(C_{+}), \tag{10.42}$$

and its conjugate map by

$$F^{T}_{\#}\colon S(C_{+}) \longrightarrow S(C_{-}). \tag{10.43}$$

Finally, we note that the $O(9{,}1) \times O(1{,}9)$ structure provides two chirality operators $\Gamma^{(\pm)}$ on $Spin(10{,}10) \times \mathbb{R}^{+}$ spinors given by

$$\Gamma^{(+)} = \frac{1}{10!}\epsilon^{a_1 \dots a_{10}}\Gamma_{a_1} \dots \Gamma_{a_{10}}, \quad \text{and} \quad \Gamma^{(-)} = \frac{1}{10!}\epsilon^{\bar{a}_1 \dots \bar{a}_{10}}\Gamma_{\bar{a}_1} \dots \Gamma_{\bar{a}_{10}}. \tag{10.44}$$

The self-duality condition $F^{(B)}_{(n)} = (-1)^{[n/2]} * F^{(B)}_{(10-n)}$ of the RR field strengths can now be expressed as a chirality condition under $\Gamma^{(-)}$:

$$\Gamma^{(-)}F = -F, \tag{10.45}$$

where $F \in S^{\pm}_{(1/2)}$.

10.8 The equations of motion

The supersymmetry variations can be expressed in an elegant, locally $Spin(9,1) \times Spin(1,9)$ covariant form using the torsion-free compatible generalized connection D. Consider the fermionic variations (10.15). Using the uniquely determined spin operators (10.35), these can be written as

$$\delta\psi_{\bar{a}}^{+} = D_{\bar{a}}\epsilon^{+} + \frac{1}{16}F_{\#}\gamma_{\bar{a}}\epsilon^{-},$$

$$\delta\psi_{a}^{-} = D_{a}\epsilon^{-} + \frac{1}{16}F_{\#}^{T}\gamma_{a}\epsilon^{+},$$

$$\delta\rho^{+} = \gamma^{a}D_{a}\epsilon^{+},$$

$$\delta\rho^{-} = \gamma^{\bar{a}}D_{\bar{a}}\epsilon^{-}. \tag{10.46}$$

For the NSNS bosonic fields, the supersymmetry variations can be express in terms of the generalized metric as

$$\delta G_{a\bar{a}} = \delta G_{\bar{a}a} = 2(\bar{\epsilon}^{+}\gamma_{a}\psi_{\bar{a}}^{+} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi_{a}^{-})\mathrm{vol}_{G}^{2} + 2(\bar{\epsilon}^{+}\rho^{+} + \bar{\epsilon}^{-}\rho^{-})G_{a\bar{a}}, \tag{10.47}$$

where $\mathrm{vol}_{G} = e^{-2\phi}\sqrt{-g} = \Phi$. Finally, the variation of the RR potential A can be written as a bispinor

$$\frac{1}{16}(\delta A_{\#}) = (\gamma^{a}\epsilon^{+}\bar{\psi}_{a}^{-} - \rho^{+}\bar{\epsilon}^{-}) \mp (\psi_{\bar{a}}^{+}\bar{\epsilon}^{-}\gamma^{\bar{a}} + \epsilon^{+}\bar{\rho}^{-}), \tag{10.48}$$

where the upper (lower) sign corresponds to type IIA (Type IIB).

We now write the supergravity equations of motion in the language of generalized geometry. In terms of the generalized Ricci tensor $R_{a\bar{b}}^{0}$, the generalized curvature scalar R, and the Mukai pairing $\langle\cdot,\cdot\rangle$ introduced in Chapter VII, the equations of motion for g, B and ϕ become

$$R_{a\bar{b}}^{0} = -\frac{1}{16}\mathrm{vol}_{G}^{-1}\langle F, \Gamma_{a\bar{b}}F\rangle,$$

$$R = 0, \tag{10.49}$$

in direct analogy to the Einstein equations. The RR supergravity fields equations of motion take the compact form

$$\frac{1}{2}\Gamma^{A}D_{A}F = dF = 0. \tag{10.50}$$

Finally, the bosonic pseudo-action S_B and the fermionic action S_F are written as

$$S_B = \frac{1}{2\kappa^2} \int (\mathrm{vol}_G R + \frac{1}{4} \langle F, \Gamma^{(-)} F \rangle). \tag{10.51}$$

and

$$S_F = -\frac{1}{2\kappa^2} \int 2\mathrm{vol}_G \, [\bar{\psi}^{+\bar{a}} \gamma^b D_b \psi^+_{\bar{a}} + \bar{\psi}^{-a} \gamma^{\bar{b}} D_{\bar{b}} \psi^-_a + 2\bar{\rho}^+ D_{\bar{a}} \psi^{\bar{a}} + 2\bar{\rho}^- D_a \psi^{-a}$$

$$-\bar{\rho}^+ \gamma^a D_a \rho^+ - \bar{\rho}^- \gamma^{\bar{a}} D_{\bar{a}} \rho^- - \frac{1}{8} (\bar{\rho}^+ F_\# \rho^- + \bar{\psi}^+_{\bar{a}} \gamma^a F_\# \gamma^{\bar{a}} \psi^-_a)]. \tag{10.52}$$

XI. Final Remarks

11.1 Related topics in the existing physics literature

In this section we draw attention to *some* parts of the existing physics literature that are related to the ideas covered in this dissertation. In [31] a connection is established between the Hitchin functionals used to define the actions in form theories of gravity and entanglement measures for systems with a small number of constituents. It is then argued that this connection lies behind the Black Hole/Qubit Correspondence (BHQC) relating certain black hole entropy formulas in supergravity to multipartite entanglement measures of quantum information theory.

In our development of generalized geometry, we defined a generalized complex structure as a particular kind of geometrical structure that is compatible with the basic set up of generalized geometry. We could also consider generalized versions of other geometries such as generalized G_2 structures. In [47] the quantization of the effective target space description of topological M-theory is considered in terms of the Hitchin functional whose critical points yield 7-dimensional manifolds with G_2 holonomy. The one-loop partition function for this theory is calculated and a generalized version of it is shown to be related to generalized G_2 geometry. This is then compared to the topological G_2 string. The Hitchin functional description of the topological string in six dimensions is then related to the reduction of the effective action for the generalized G_2 theory.

In [48] Pestun extends the OSV conjecture $Z_{BH} = \left|Z_{top}\right|^2$ [33] to topological strings on generalized Calabi-Yau manifolds. It is shown that the classical black hole entropy is given by the generalized Hitchin functional introduced in Chapter IX. The geometry differs from an ordinary geometry if $b_1(X) \neq 0$, where X is the generalized Calabi-Yau manifold.

In [49] the structure of "exceptional generalized geometry" (EGG) is explained and shown to provide a unified geometrical description of 11-dimensional supergravity backgrounds. On a d-dimensional background, the action of the $O(d,d)$ symmetry group of generalised geometry is replaced in EGG by the exceptional U-duality group $E_{d(d)}$ as originally described in [50]. The metric and form-field degrees of freedom combine into a single geometrical object and the differential structure of EGG is given by an analogue of the Courant bracket. Exceptional generalized geometry is ideally suited for a systematic description of supergravity backgounds.

In [52] a unified description of bosonic 11-dimensional supergravity restricted to a d-dimensional manifold is presented for all $d \leq 7$. The theory is based on a natural $E_{d(d)} \times \mathbb{R}^+$ action on an extended tangent space. The bosonic degrees of freedom as well as the diffeomorphism and gauge symmetries are unified as a “generalized metric”. The analogues of the Levi-Civita connection and Ricci tensor are also introduced and the bosonic action is simply given by the generalized Ricci scalar. The equations of motion are then elegantly encoded by the vanishing of the generalized Ricci tensor. The significance of the formulation for M-theory is also considered. Finally, the completion of the description of supergravity by inclusion of the fermionic degrees of freedom and supersymmetry transformations (at least to leading order) will be presented as the main result of [53].

11.2 Directions for future research

Deeper investigations into many of the bold claims made in “*Topological M-theory as Unification of Form Theories of Gravity*” [9] should provide fertile ground for original research. Certainly a more rigorous treatment of some of the key conjectures contained in that paper would be desirable. Some of these include the conjectured relations of Hitchin functionals to the models of topological string theory and BPS black holes. In particular, the relation of the topological B model to Hitchin’s V_H functional was seen to break down at one loop. Although we saw that we could achieve agreement between the topological B model and Hitchin theory by replacing V_H with the generalized Hitchin functional, the precise form of the full relationship between the two remains an open question. Attempts to test this link at higher loops might prove to be worthwhile.

In the context of the use of generalized geometry in the description of supergravity backgrounds, one could consider the description of higher-derivative correction terms to the theory, assuming that the generalized structure is preserved. One could also consider explicit constructions of supergravity backgrounds with particular special structures on the generalised tangent space. A deeper question is whether there can be any extension of the generalized geometrical description for exotic string backgrounds. Such a description would have to retain concepts such as connections and curvature as some sensible limit of the full string theory while moving away from the usual 10-dimensional manifold.

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