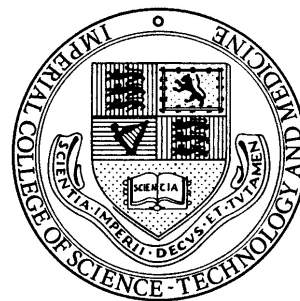# Imperial College
London

Majorana Fermions

QFFF MSc Dissertation

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Contents

1   Introduction

In 1937 Ettore Majorana suggested that chargeless fermions could be described by a real wave equation, leading to the possibility that this particle would be identical to its anti-particle[1]. These *Majorana fermions* seemed a mathematical quirk without any physical basis. If fermions and anti-fermions were indeed indistinguishable, they could co-exist without annihilating one another - not something that we seem to see in nature. However, recent experiments have indicated the existence of these particles within condensed matter systems. If further data bears this out, it could lead to myriad applications in fields as diverse as quantum computing and cosmology[2].

With its grounding in quantum field theory, Majorana fermions provide a useful insight into the utility of field theories, and the interplay between theoretical physics and its experimental cousin - with a 75 year-old theoretical prediction only recently gaining experimental credence. This dissertation will discuss some of the physics and applications of Majorana fermions, looking at the implications of their discovery. Firstly, this report will look at some of the basic concepts in field theory that underpin the theoretical aspects of these particles. As Majorana fermions could be integral to the description of massive neutrinos - an extension of the Standard Model, the next section will be devoted to this. Then, as recent results in condensed matter physics have shown glimpses of these particles, a qualitative look at Majorana fermions in this branch of research will be sought. Finally, the last section will briefly discuss some further applications of Majorana fermions and its utility in theoretical and experimental physics.

2   Particles and Anti-Particles

2.1   The Dirac Equation

Quantum field theory provides a convenient mathematical framework to view particles and their anti-particles. So, as with all discussions surrounding quantum field theory, it is probably best to start with looking at the *Dirac equation*[3]. This equation was Dirac's way of formulating a description of elementary spin-$\frac{1}{2}$ particles that was consistent with both quantum mechanics and special relativity.

2.1.1   Starting from Schrödinger

Dirac's famous equation was motivated by Schrödinger's equation for a free particle

$$-\frac{h^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi \tag{2.1}$$

The left hand side of this equation gives the non-relativistic kinetic energy. However, relativity treats space and time on the same footing. Therefore a relativistic treatment of Schrödinger's equation must have differential equations of the same order with respect to space and time. We can try to do this by starting with the relativistic invariance of 4-momentum

$$\frac{E^2}{c^2} - p^2 = m^2c^2 \tag{2.2}$$

If we express the energy and momentum as operators i.e. $\hat{E} = i\hbar\frac{\partial}{\partial t}$ and $\hat{p} = -i\hbar\nabla$, we can get the *Klein-Gordon* equation[1]

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = m^2\psi \tag{2.3}$$

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[1]From here on $c = \hbar = 1$

2

The wavefunction ψ is now a relativistic scalar and the space and time derivatives are both second order. However, although equation 2.3 is a valid wave equation, it is not a good relativistic generalisation of Schrödinger's equation. This is because the initial values of ψ and $\frac{\partial \psi}{\partial t}$ can be chosen freely, and as a result the probability density is no longer positive definite. This leaves open the possibility of negative probabilities - which is somewhat worrisome.

2.1.2   Enter Dirac

So making the connection between Schrödinger's equation and relativity was not so straight-forward. Equation 2.3 gives a relativistic wave equation, but with the possibility of negative probabilities. Dirac wondered if instead of getting a second order equation in space and time (as in the Klein-Gordon equation), he could find a first order equation that would be positive definite. He started by trying to take the square root of the wave equation

$$\nabla^2 - \frac{\partial^2}{\partial t^2} = \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + iD\frac{\partial}{\partial t} \right)^2 \tag{2.4}$$

when the right hand side of this equation is multiplied out, the cross terms ($\frac{\partial^2}{\partial x \partial y}$, $\frac{\partial^2}{\partial x \partial z}$ etc) cancel if the anti-commutators[2] of the coefficients vanish

$$\{A, B\} = \{C, D\} = \{A, D\} = \{B, C\} = \ldots = 0 \tag{2.5}$$

and they each square to give the identity

$$A^2 = B^2 = C^2 = D^2 = \mathbb{1}_{4 \times 4} \tag{2.6}$$

From this Dirac realised that to get a generalisation of quantum mechanics that was compatible with relativity you need to construct matrices that are at least 4×4 - leading to a

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[2]with $\{x, y\} = xy + yx$ being the usual anti-commutation relation.

wavefunction that has 4 components, known as a *spinor.*

Using equations 2.3 and 2.4 we can write

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + iD\frac{\partial}{\partial t} \right)\psi = X\psi \tag{2.7}$$

with X still to be determined. Applying X to both sides we have

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right)\psi = X^2\psi \tag{2.8}$$

by setting $X = m$ we have a first order equation in space and time

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + iD\frac{\partial}{\partial t} - m \right)\psi = 0 \tag{2.9}$$

setting $A = i\beta\alpha_1$, $B = i\beta\alpha_2$, $C = i\beta\alpha_3$ and $D = \beta$

$$i\beta\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z} + \frac{\partial}{\partial t} - m \right)\psi = m\psi \tag{2.10}$$

inserting ∇

$$i\beta\left(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \frac{\partial}{\partial t} \right)\psi = m\psi \tag{2.11}$$

From 2.6 it follows that $\beta^2 = \mathbb{1}$ and $\beta = \beta^{-1}$ and therefore we get

$$\left(i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + i\frac{\partial}{\partial t} \right)\psi = m\beta\psi \tag{2.12}$$

Then, if we insert the momentum operator $\hat{p} = -i\nabla$, we have the original form of the Dirac equation

$$i\frac{\partial}{\partial t}\psi = \left(m\beta + \boldsymbol{\alpha} \cdot \boldsymbol{\hat{p}} \right)\psi \tag{2.13}$$

By defining the gamma matrices γ^μ

$$\gamma^0 = \beta \tag{2.14}$$

$$\gamma^i = \beta\alpha_i \tag{2.15}$$

we get the more usual form of Dirac's equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{2.16}$$

with ψ being the four component spinor.

2.2   Field Theory and Anti-Matter

So equation 2.16 allows us to describe relativistic spin-$\frac{1}{2}$ particles, but what does this tell us about particles and their anti-particle partners? And what does this have to do with the Majorana fermion?

2.2.1   The Dirac Field

When building a quantum filed theory you can start with the Lagrangian for your fields then canonically quantise the Lagrangian by promoting the fields to operators. The Lagrangian of the free Dirac field ψ is given by

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{2.17}$$

with $\overline{\psi} \equiv \psi^\dagger \gamma^0$. It is easy to see that when this Lagrangian is varied we get back the Dirac equation, which is just what we want. The gamma matrices γ^μ follow the Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \tag{2.18}$$

$$= 2\eta^{\mu\nu} \tag{2.19}$$

Dirac found gamma matrices that were complex. From the form of equation 2.17 this means that the spinor field ψ must also be complex. This makes sense from the point of view of field theory as a complex field would create particles and annihilate anti-particles while its complex conjugate would create anti-particles and annihilate particles.

More explicitly we can expand ψ in terms of creation and annihilation operators $a_\mathbf{p}^s$, $a_\mathbf{p}^{s\,\dagger}$, $b_\mathbf{p}^s$, $b_\mathbf{p}^{s\,\dagger}$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_s (a_\mathbf{p}^s u^s(p)e^{-ip\cdot x} + b_\mathbf{p}^{s\,\dagger} v^s(p)e^{ip\cdot x}) \tag{2.20}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_s (b_\mathbf{p}^s \overline{v^s}(p)e^{-ip\cdot x} + a_\mathbf{p}^{s\,\dagger} \overline{u^s}(p)e^{ip\cdot x}) \tag{2.21}$$

With the creation and annihilation operators obeying the normal anti-commutation relations

$$\{a_\mathbf{p}^r, a_\mathbf{p}^{s\,\dagger}\} = \{b_\mathbf{p}^r, b_\mathbf{p}^{s\,\dagger}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs} \tag{2.22}$$

The explanation for these two types of operators is that $a_\mathbf{p}^{s\,\dagger}$ creates a fermion and $b_\mathbf{p}^{s\,\dagger}$ creates an anti-fermion, both with energy $E_\mathbf{p}$ and momentum $\mathbf{p}$. Therefore, one-particle states are

created by operating on the vacuum state $|0\rangle$

$$| e^-, \mathbf{p}, s \rangle \equiv \sqrt{2E_\mathbf{p}} a_\mathbf{p}^{s\,\dagger} |0\rangle \tag{2.23}$$

$$| e^+, \mathbf{p}, s \rangle \equiv \sqrt{2E_\mathbf{p}} b_\mathbf{p}^{s\,\dagger} |0\rangle \tag{2.24}$$

Here, e^- is the electron and e^+ is its anti-particle partner, the positron.

These anti-particles have the same mass as their particle partners but with opposite electric charge. When pairs meet they annihilate each other to produce photons, thus conserving total charge. Field theory gives a neat way to describe particles and their anti-particles, each being created by its own operator. However, as anti-particles are usually defined as having the same mass but opposite charge from their partners, what does this mean for neutral particles?

2.2.2   Neutral Particles

It is possible for neutrally charged particles to have a distinct anti-particle. For example the neutral neutron is made up of one up quark and two down quarks

$$n \rightarrow u + d + d \tag{2.25}$$

with the anti-neutron made from the corresponding anti-quarks.

$$\overline{n} \rightarrow \overline{u} + \overline{d} + \overline{d} \tag{2.26}$$

The up quark has charge $+\frac{2}{3}$ and the down quark has charge $-\frac{1}{3}$. With the anti-quarks having the equal but opposite charges, the overall charge of both the neutron and the anti-neutron

remains neutral. As expected, the anti-neutron acts like other anti-particles by annihilating the neutron and producing a photon.

Although the neutron has a distinct anti-particle, this is not true for all neutral particles. Some neutrally charged particles can be their own anti-particle. For example, bosons such as the photon (spin-1) and the graviton (spin-2). These particles are created by fields that are their own complex conjugate i.e. they are real with $\phi = \phi^*$. This is not a problem for bosonic fields which are formulated with real numbers that commute during their quantisation. However, fermionic fields use grassman numbers and anti-commute. Dirac's famous equation contains these fermionic fields along with complex variables, making the construction of real fields inherently difficult.

In 1937 Majorana questioned whether it was necessary for spin-$\frac{1}{2}$ particles to have equations that involved complex numbers. What was required were gamma matrices that still satisfy the Clifford algebra but were purely imaginary[2]. Majorana found such constructions as tensor products of Pauli matrices

$$\widetilde{\gamma}^0 = \sigma_2 \otimes \sigma_1 \tag{2.27}$$

$$\widetilde{\gamma}^1 = i\sigma_1 \otimes \mathbb{1} \tag{2.28}$$

$$\widetilde{\gamma}^1 = i\sigma_3 \otimes \mathbb{1} \tag{2.29}$$

$$\widetilde{\gamma}^0 = i\sigma_2 \otimes \sigma_3 \tag{2.30}$$

$$\tag{2.31}$$

This alters the Dirac equation (2.16) to give Majorana's equation

$$(i\widetilde{\gamma}^\mu \partial_\mu - m)\widetilde{\psi} = 0 \tag{2.32}$$

As $\widetilde{\gamma}^\mu$ is imaginary it follows that $\widetilde{\psi}$ must be real. With $\widetilde{\psi}$, being real there is no need to construct an analogy to $\overline{\psi} \equiv \psi^\dagger \gamma^0$. The field is therefore invariant to charge conjugation and can be constructed from one type of operator, as in real scalar field theory. Therefore, the particles created with this construction are fermions that are their own anti-particles, known as *Majorana fermions* or *Majorana particles*.

So in short the predicted Majorana fermion, taken from Majorana's formulation, is a neutral, spin-$\frac{1}{2}$ particle, whose field equations are invariant to charge conjugation. This seemingly simple statement of particles that are their own anti-particles, turned out to have many intriguing applications. The remainder of this dissertation will look at some of these applications in neutrino physics and condensed matter systems.

3   Neutrinos

Majorana suggested that the neutrino could be a spin-$\frac{1}{2}$ particle that was its own anti-particle. At the time, the neutrino had not been seen experimentally. When it was discovered its properties seemed to disprove Majorana's suggestion, as there was a clear difference between neutrinos and their anti-neutrino partners. However, further developments in neutrino physics reignited interest in the neutrino being a Majorana fermion.

3.1   Lepton Number Conservation

One major problem with the neutrino as a Majorana fermion is the fact that the Standard Model uses *Lepton Number* to distinguish ν from $\overline{\nu}$, which would mean $\nu \neq \overline{\nu}$. To illustrate this distinction we will look at pion decay[4]. When a π^+ decays a μ^+ is emitted along with a ν_μ

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \tag{3.1}$$

This muon neutrino interacts with matter to produce only a μ^- and never μ^+

$$\nu_\mu N \quad \rightarrow \quad \mu^- X \tag{3.2}$$

$$\nu_\mu N \quad \nrightarrow \quad \mu^+ X \tag{3.3}$$

For π^- decay a similar thing happens. A $\overline{\nu}_\mu$ is produced which only interacts with matter to produce a μ^+. This is normally explained by saying that $\overline{\nu}_\mu$ and ν_μ are distinct particles, and there is a conserved quantum number for these interactions.

This conserved quantum number, known as the lepton number, is the number of leptons (electron, muon or tau) minus the number of anti-leptons. The Standard Model relies on

the conservation of this lepton number in interactions. As an example, the interaction below shows how lepton number is conserved in muon decay.

$$\mu^- \quad \rightarrow \quad \nu_\mu + e^- + \overline{\nu}_e \tag{3.4}$$

$$L : 1 \quad = \quad 1 + 1 - 1 \tag{3.5}$$

Lepton number conservation alone explains why the interaction of equation 3.3 could not be allowed. ν_μ has a lepton number of $+1$ and μ^+ has a lepton number of -1. Therefore if this interaction were allowed, lepton number conservation would be violated.

The Lepton number became a useful tool in describing interactions. It led to the discovery of the different flavours of neutrino corresponding to each of three different leptons: the electron neutrino; muon neutrino; and tau neutrino. These properties showed there is a distinction between neutrinos and anti-neutrinos implying that they are not Majorana fermions.

However, this need not be the case. Another explanation for these interactions comes when looking at the helicity of the particles. In π^+ decay the neutrino produced is left-handed, while in π^- decay the anti-neutrino produced is right-handed. Supposing that every left-handed neutrino interacts giving a μ^- and every right-handed neutrino interacts to give a μ^+, then there is no need to appeal to lepton conservation to describe these interactions. The neutral particle produced in both the π^+ and π^- decays would then be the same particle, just with different helicities. In other words the neutrino is the same as the anti-neutrino i.e. it is a Majorana fermion.

Therefore, we have two explanations for the same process. If lepton number is conserved then $\nu \neq \overline{\nu}$ and the neutrino is the normal *Dirac Neutrino*. On the other hand, if lepton

number is not conserved then $\nu = \bar{\nu}$ and the neutrino is the *Majorana Neutrino.*

3.2 Massive Neutrinos and the Standard Model

The Standard Model describes neutrinos as being massless, as there are no gauge invariant interactions that can deliver neutrinos with non-zero mass. However, when trying to unify the weak, strong and electromagnetic interactions in Grand Unified Theories (GUTs), we construct a large multiplet containing a neutrino (ν), a lepton (l) and a positively and negatively charged quark (q^+ and q^-).

$$\begin{pmatrix} q^+ \\ q^- \\ l \\ \nu \end{pmatrix} \tag{3.6}$$

In this construction neutrinos are put on the same footing as (massive) leptons and quarks, and therefore it would seem reasonable to think that neutrinos are themselves massive[4]. Many experiments have revealed interesting aspects of the neutrino, which seem to contradict what is said in the Standard Mode - giving validity to this idea of neutrino mass.

3.2.1 Neutrino Oscillations

The Super-Kamiokande Experiment detects and observes solar, atmospheric and man-made neutrinos[5]. This has shown atmospheric muon neutrinos and solar neutrinos 'vanishing' from their detectors. As a neutrino's momentum and energy is unlikely to disappear, a explanation could be that the neutrino is changing into another type of neutrino, which we cannot detect. The idea is that muon neutrinos from the atmosphere would oscillate into

tau neutrinos, which would not be picked up experimentally - hence the vanishing. These *neutrino oscillations* mean that a neutrino of one flavour can change into a different one. However, if all three neutrinos have zero mass (or the same mass that is non-zero) this would not be possible[6].

As the neutrino travels, it oscillates between the different flavours - having a probability of being a particular type of neutrino at any particular time. To set up neutrino oscillations there has to be mixing between neutrino mass and weak interaction eigenstates. This can be written as

$$\begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{3.7}$$

There also needs to be a mass difference between the neutrino mass eigenstates i.e. $m_1 \neq m_2$. So, we have neutrinos of masses m_1 and m_2 which are allowed to propagate as matter waves of differing frequencies. From the time-dependent Schrödinger equation (2.1) we can construct the following

$$\begin{pmatrix} \nu_1(\mathbf{x}, t) \\ \nu_2(\mathbf{x}, t) \end{pmatrix} = e^{i\mathbf{p}\cdot\mathbf{x}} \begin{pmatrix} e^{-iE_1 t} |\nu_1(0)\rangle \\ e^{-iE_2 t} |\nu_2(0)\rangle \end{pmatrix} \tag{3.8}$$

$$= e^{i\mathbf{p}\cdot\mathbf{x}} \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} |\nu_1(0)\rangle \\ |\nu_2(0)\rangle \end{pmatrix} \tag{3.9}$$

If we use equation 3.7 we have

$$\begin{pmatrix} |\nu_\mu(\mathbf{x}, t)\rangle \\ |\nu_\tau(\mathbf{x}, t)\rangle \end{pmatrix} = e^{i\mathbf{p}\cdot\mathbf{x}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_\mu(0)\rangle \\ |\nu_\tau(0)\rangle \end{pmatrix} \tag{3.10}$$

Then with $|\nu_\mu(0)\rangle = 1$ and $|\nu_\tau(0)\rangle = 0$

$$||\nu_\mu(\mathbf{x}, t)\rangle|^2 \;\; = \;\; \sin^2 2\theta \sin^2 \frac{(E_2 - E_1)\, t}{2} \tag{3.11}$$

$$\equiv \;\; P\,(\nu_\mu \to \nu_\tau) \tag{3.12}$$

which gives the probability of a muon neutrino turning into a tau neutrino. Experimentally we know that if the neutrino has a mass it must be (much) smaller than its kinetic energy, $E_1 E_2 \gg m_1 m_2$ and therefore we can write

$$E_2 - E_1 \;\; = \;\; \sqrt{m_2{}^2 + p^2} - \sqrt{m_1{}^2 + p^2} \tag{3.13}$$

$$\approx \;\; \frac{m_2{}^2 - m_1{}^2}{2p} \tag{3.14}$$

we can then make the approximations that $t \approx |\mathbf{x}| \equiv L$, where L is the distance travelled by the neutrino, and $p \approx E$. We then have

$$P\,(\nu_\mu \to \nu_\tau) \approx \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right) \tag{3.15}$$

where Δm^2 is the difference in the masses squared. From this final result, we can see that if the probability of a muon neutrino changing into a tau neutrino is non-zero, then $\Delta m^2 \neq 0$. To but it another way, if neutrino oscillations correctly describes the experimental 'vanishing' of atmospheric and solar neutrinos, then neutrinos would have some non-zero mass.

3.2.2   The Seesaw Mechanism

So where does the Majorana fermion fit in with all this? When creating a theory for neutrino mass you can have a *Dirac mass* term that conserves lepton number and involves transitions

between two different neutrinos ν_L and N_R[7][3]. So we can write the Lagrangian for the Dirac mass term as

$$-\mathcal{L}_{Dirac} = m_D \left(\overline{\nu}_L N_R + \overline{N}_R \nu_L \right) \tag{3.16}$$

$$= m_D \overline{\nu} \nu \tag{3.17}$$

where we have defined the Dirac field as $\nu \equiv \nu_L + N_R$. So the Dirac neutrino has four components, ν_L and N_R; and their CPT partners: $\nu^c{}_R$ and $N^c{}_L$[4]. In this case the ν_L is in an $SU(2)$ doublet and N_R is in an $SU(2)$ singlet. The mass is generated by a Yukawa coupling with $SU(2)$ symmetry breaking. However, there is a problem with this. The neutrino mass is thought to be of the order of 10eV, and this would require a very small Yukawa coupling of the order of 10^{-10}, which does not seem physical.

However, this is not the only mass term we can have in the Lagrangian. If the neutrino were a Majorana fermion, we would have $N_R \equiv \nu^c{}_R$. We can then combine our neutrino fields to give a *Majorana mass* term

$$-\mathcal{L}_{Majorana} = \frac{1}{2} m_M^L \left(\overline{\nu}_L \nu^c{}_L + \overline{\nu}^c{}_L \nu_L \right) + \frac{1}{2} m_M^R \left(\overline{\nu}_R \nu^c{}_R + \overline{\nu}^c{}_R \nu_R \right) \tag{3.18}$$

If both the Majorana and Dirac terms are present then matrix notation can be used to represent the terms

$$-\mathcal{L} = \frac{1}{2} \overline{\nu}_L \overline{\nu}^c{}_L \mathbf{M} \begin{pmatrix} \nu^c{}_L \\ \nu_R \end{pmatrix} + h.c \tag{3.19}$$

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[3]The L and R subscripts denote left and right handed states. It should also be noted that N_R is different from $\nu^c{}_R$, the later being the CPT partner of ν_L

[4]The superscript c denotes the charge conjugated field and is given by $\nu^c = \mathbf{C}\overline{\nu}^T = \mathbf{C}\gamma_0 \nu^*$, with $\mathbf{C}$ being the charge-conjugation matrix

where $\mathbf{M}$ is the known as the *seesaw matrix*

$$\mathbf{M} = \begin{pmatrix} m_M^L & m_D \\ m_D & m_M^R \end{pmatrix} \tag{3.20}$$

We can assume that the elements of this matrix have some hierarchy

$$m_M^R = M \gg m_D \gg m_M^L = \mu \tag{3.21}$$

with $\mu \ll 1$. When following this hierarchy $\mathbf{M}$ would have the eigenvalues

$$\lambda_\pm \approx \frac{M \pm \sqrt{M^2 + 4m_D{}^2}}{2} \tag{3.22}$$

and

$$\lambda_+ \quad \approx \quad M \tag{3.23}$$

$$\lambda_- \quad \approx \quad -\frac{m_D{}^2}{M} \tag{3.24}$$

so the determinate

$$det\mathbf{M} \quad \approx \quad \lambda_+ \lambda_- \tag{3.25}$$

$$\approx \quad -m_D{}^2 \tag{3.26}$$

Therefore if λ_+ goes up then λ_- goes down and vice versa - hence the name *seesaw mechanism*. So $\mathbf{M}$ gives the mass matrix for the right-handed neutrino, with the Majorana mass M at the GUT scale ($\sim 10^{16}$GeV), and the Dirac mass m_D being of the electroweak scale (~ 200GeV).

The seesaw mechanism allows for the neutrino to exist both as Majorana and Dirac fermions. In this scenario the small neutrino mass that has already been detected is the Dirac fermion, with the larger mass being a Majorana fermion - indicating that lepton number is violated at scales of $\sim 10^{16}$GeV.

3.3   Neutrinoless Double Beta Decay

Currently, many different processes are being used to delve into the true nature of the neutrino. One example is *Double Beta Decay*[8], which releases two beta rays in a single process. This can appear in two different modes

$$A(Z, N) \quad \rightarrow \quad A(Z + 2, N - 2) + 2e^- + 2\overline{\nu}_e \tag{3.27}$$

$$A(Z, N) \quad \rightarrow \quad A(Z + 2, N - 2) + 2e^- \tag{3.28}$$

The first mode emits two anti-neutrinos and is allowed by the Standard Model. However, the second mode violates lepton number conservation by two units. As no neutrinos are produced this is known as *Neutrinoless Double Beta Decay*, or $0\nu\beta\beta$.

With its violation of lepton number conservation, the discovery of $0\nu\beta\beta$ would imply that the neutrino is a Majorana fermion.

3.3.1   In Search of $0\nu\beta\beta$

$0\nu\beta\beta$ not only provides proof for Majorana fermions, but also underpins many other aspects of physics, so experiments are trying to catch a glimpse of this elusive interaction. Isotopes that are good candidates for $0\nu\beta\beta$ (equation 3.28) are also liable to decay as $2\nu\beta\beta$ (equation 3.27) so the difficulty is in distinguishing between the two decay processes.

One such experiment will search for $0\nu\beta\beta$ in ^{76}Ge[9]. The double beta decay of ^{76}Ge produces two electrons that usually deposit ionisation energy at the detectors in a range of a couple of millimetres. However, the gamma rays produced from the process can undertake Compton-scattering with energy scales of over a centimetre. The geometry of the detectors is such that it can isolate the $0\nu\beta\beta$ from not just the $2\nu\beta\beta$ process but also the Compton-scattering of the gamma rays. Therefore the $0\nu\beta\beta$ can be picked out from the other decay processes.

3.3.2   Theory vs. Experiment

This discussion surrounding neutrino physics gives a concrete example of the interplay between theoretical and experimental physics. The success brought from the theoretical view of the Standard Model led to the neutrino being thought of as massless. This was put in doubt with the observations made of atmospheric neutrinos. It was then necessary to find a new theoretical framework, which is possibly given by neutrino oscillations and the seesaw mechanism. This in turn re-opened the possibility of the Majorana neutrino, leading to the need for more experimental data sought for in processes such as $0\nu\beta\beta$.

4   Condensed Matter Physics

Studies have recently suggested that particle excitations in some condensed matter systems are Majorana fermions[10]. Generally, excitations in these systems are bound states of electrons and holes, created by combinations of electron and hole operators. These pairs are invariant under charge conjugation, and therefore are indeed their own anti-particle. However, they are in fact bosons, and are therefore not the sought after Majorana fermion.

Superconductors however, work differently to other condensed matter systems. Here the distinction between holes and electrons is not as clear cut. For electrons to occupy the lowest energy level without violating Pauli's exclusion principle they form *Cooper Pairs*, and in effect electron number is no longer conserved. In addition to this, a superconductor's charge is unobservable - making the charge conjugation property of the Majorana fermion irrelevant. Therefore, superconductor physics looks like a good place to find these particles. In this section we will review some of the relevant condensed matter physics that underpins the search for Majorana fermions.

4.1   Electrons and Holes

A hole is the absence of an electron in an energy band, which is theoretically described as a single positive charge. This is done to make a convenient mathematical interpretation of condensed matter systems, and is distinct from the positron, which is the electron's anti-matter partner.

To help with the formulation of these systems, the notion of a bound state of an electron and a hole is needed. These bound states are called *excitons* - quasi-particles held together by

the coulomb force. Using a field theory construction, they are created by electron and hole creation operators of the form $a_j^\dagger a_k + a_k^\dagger a_j$ which are invariant under charge conjugation i.e. when $a_j^\dagger \to a_j$. Therefore, excitons are their own anti-particles, but again these are bosons with integer spin.

4.2 Superconductors

4.2.1 Cooper Pairs

In superconductors, two electrons pair together in what is known as a *Cooper Pair*. This allows them to form a condensate without violating Pauli's exclusion principle (by being bound together they can occupy the lowest energy level). In these pairings electron number is not conserved. In other words a Cooper Pair can be added or taken away from the condensate without the system changing in any substantial way.

Another property of a superconductor is that charge is no longer observed. The superconductor in effect shields the electric field and confines the magnetic field. Therefore, it seems plausible that particles or quasi-particles in the superconductor would be invariant to charge conjugation - leading to the Majorana fermion.

However, as a Copper Pair partners two electrons, it is bosonic in nature. So how are fermionic modes created in the superconductor? An electron mode can lower its energy by mixing with a hole that is attached to a Cooper Pair. Creation operators for the superconducting state are of the form $a_j cos\theta + a_k^\dagger sin\theta$. But these are only their own anti-particles in the case when $k = j$ and $\theta = \pm\frac{\pi}{4}$, so are not Majorana fermions. Therefore to get the fermionic modes we require, we need to look at the idea of a vortex.

4.2.2 Vortices

Superconductors expel magnetic fields, which if strong enough quenches the superconductive state. However, it may be favourable for the superconductor to form a lattice of quantum vortices that carry quantised magnetic flux through the superconductor.

These vortices may trap spin-$\frac{1}{2}$ excitons, known as *zero modes*[11]. These are equal numbers of particles and holes, with the associated quasi-particles being known as *partiholes*. Unlike the excitons, they are created by operators of the form $\gamma_j = c_j + c_j^\dagger$, which is indeed invariant under charge conjugation, as required. Therefore, γ_j, creates spin-$\frac{1}{2}$ particles that are their own anti-particles - and we have the Majorana fermion.

These zero modes (also known as *Majorana modes*) have the property that $\gamma_j^2 = 1$, and not zero - which means that the partihole is neither a normal fermion, nor a normal boson. Adding a second partihole recreates the state of zero occupancy, and does not annihilate the original state or create a new state which is occupied by both.

γ_j could be seen as a *fractionalised zero-mode* consisting of 'half' a fermion[12]. In other words, a full fermionic state, with a well-defined occupation number, is made from a combination of zero modes, which can be described as $f_j = \frac{1}{2}(\gamma_1 + i\gamma_2)$. This operator f, now represents the 'normal' fermion with $f \neq f^\dagger$, obeying the usual anti-commutation relations.

However, f is non-trivial. $\gamma_{1,2}$ can be localised arbitrarily far from each other, implying that f encodes highly non-local entanglement. Secondly, you can empty or fill the non-local state of f with no energy cost, meaning the ground state is degenerate. This leads to the emergence of non-abelian statistics for Majorana fermions.

Consider $2N$ vortices binding Majorana zero modes: $\gamma_{(1,2,3\ldots2N)}$. You can combine Majorana fermions in pairs to define N fermion operators, $f_j = \frac{1}{2}(\gamma_{(2j-1)} + i\gamma_{2j})$, which corresponds to the zero-energy states that can be either full or empty. The vortices then create $2N$ degenerate groundstates that can be labelled in terms of occupation numbers $n_j = f_j f_j^\dagger$ giving $|n_1, n_2, \ldots, n_N\rangle$. If a system is then put into one of the ground states, with a pair of vortices then being exchanged, the system generally ends up in a different ground state from the original one. This is as the process swaps two Majorana modes, each being 'half' a fermion. As these Majorana zero modes are bounded by vortices, the vortices themselves exhibit non-abelian statistics.

4.3   Non-Abelian Symmetries

As has been noted, non-abelian symmetries are closely linked to the appearance of Majorana zero modes in the vortex - and therefore the Majorana fermion itself. In general terms, these exchange statistics codify the way that multi-particle wavefunctions transform under the interchange of indistinguishable particles[13]. In 3+1 dimensions the only possible wavefunctions are the bosonic wavefunction, being symmetric under such an interchange; and the fermionic wavefunction, which is anti-symmetric.

With a system that has no input of energy (i.e. adiabatic), a process where two particles are interchanged twice is equivalent to a process where one particle is taken around the other. In 3 dimensions this wrapping of one particle around another is topologically equivalent to no particles moving at all. As a result the wavefunction is unchanged by one particle moving around another and by implication, two particles interchanging twice. Therefore, for one exchange of particles the wavefunction can only change by a sign (+ for bosonic, and −

for fermionic).

In 2 dimensions things are different. There is no topological equivalence with moving particles around one another. Their trajectories involve a non-trivial winding and the system does not necessarily return to the same state. So if two exchanges of a pair of particles may not return the wavefunction to the original state, it follows that one exchange does not necessarily change the wavefunction by $\pm$. This change in wavefunction due to one exchange of particles may be written as: $\phi(r_1, r_2) \rightarrow e^{i\pi\theta}\phi(r_1, r_2)$. So after a second exchange we have: $\phi(r_1, r_2) \rightarrow e^{2i\pi\theta}\phi(r_1, r_2)$. With the cases of $\theta = 0, 1, 2 \ldots n$ corresponding to the bosons and fermions. Particles with different values of θ are called *anyons*.

As a result, 2 dimensional systems allow for particles with statistics that are neither fermionic nor bosonic. Anyons under an interchange can change to a totally different quantum state. This means the exchanges do not commute i.e. they are non-abelian.

4.3.1   Discovering the Majorana Fermion?

Using these non-abelian symmetries in condensed matter systems, researchers have recently found signatures of the Majorana fermion in a one-dimensional nanowire[10]. This was made of semiconducting material that has a strong spin-orbit interaction. When a magnetic field is placed along the axis of this wire a gap opens up between the two spin-orbit bands. If the Fermi energy is within the gap then the degeneracy is two-fold, while if the Fermi energy is outside this gap, then it is four-fold.

Connecting this nanowire to a superconductor causes electron states of opposite momentum and spins to pair together, which in turn induces a gap. Combining the degeneracy

with a induced gap creates what is known as a *topological superconductor*. On the ends of this construction zero-energy bound states (as outlined in Section 4.2.2) appeared, which are thought to be the Majorana fermion.

4.3.2 Quantum Computing

This section outlined the possible non-abelian nature of Majorana particles in condensed matter physics. One possible application of this is in quantum computing[14].

A normal computer works by handling *bits* that are in one of two states (0 or 1). In comparison, quantum states can be a superposition. So instead of using bits of only two states, a quantum computer manipulates information as *qubits*, which can be a superposition of many states. As a quantum computer can simultaneously contain multiple states it can theoretically perform millions of times more operations than a normal computer.

However, as with all quantum systems, if you try and measure a qubit the wavefunction will collapse and it will go into one of its allowed states - no longer being in superposition. This in effect returns the quantum computer into the operation of a normal computer. One way round this *decoherence* is to use *entangled states*. Another option is to encode quantum information in a ground state of Majorana fermions. As has been noted these can display non-abelian statistics that can simulate the actions of an entangled state[15].

5   Discussion

This dissertation shows a brief view of the physics behind particles that are their own anti-particles, concentrating on condensed matter and neutrino physics with its importance in practical applications such as quantum computing. However, there are many other areas of physics that use the idea of the Majorana fermion to advance current research.

5.0.3   Supersymmetry

Supersymmetry relates every fermion with a corresponding boson, and every boson with a fermion. Each superpartner has the same mass and internal quantum numbers. Although there is currently no experimental evidence for this, it does help with solutions for many of the varied problems in theoretical physics, such as the hierarchy problem.

If supersymmetry holds, then a boson such as the photon must have a spin-$\frac{1}{2}$ superpartner. As this has the same internal quantum numbers as the photon, it too must be its own anti-particle. Therefore, the supersymmetric partner of the photon, the *photino*, has to be a Majorana fermion. This reasoning holds for other boson and superpartner pairs, which means that supersymmetry is loaded with Majorana fermions.

5.0.4   Dark Matter

Dark matter is thought to constitute a large part of the mass of the universe. It is used to explain the discrepancies between the gravitational effects of large objects, such as galaxy clusters, with their observed mass.

One candidate for dark matter is a Weakly Interacting Massive Particle (WIMP)[16], this

could be one of the superpartners to ordinary matter that we have mentioned above. Therefore, it could be possible that these WIMPs that dominate the Universe's mass are Majorana fermions themselves.

5.1 Conclusion

Majorana fermions provide an interesting insight into the formulism and utility of field theories. Its development show how theory can feed into experiment, which in turn feeds back into theory. Originally thought of as a mathematically nicety, but with no basis in physical systems, more and more branches of physics are starting to rely on the existence of these elusive particles. Recent experiments have shown a glimpse of their quarry in a condensed matter system[10], with further research honing in on the possibly that it is a massive neutrino[17]. How these experiments will inform our view of physics in the future is still an open question. However, the insights they have already yielded show the power of field theories to describe the physical world - with ideas and techniques developed 80 years ago giving fresh perspectives on cutting edge physics.

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