The Algebraic properties of black holes in higher dimension

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Abstract

we introduce the extension of algebraically special into higher dimensional asymtotically flat spacetimes using CMPP approach and investigate the behavior of known higher dimensional black holes under this algebraic classification.

Contents

1. Introduction

The motivation of the studying of black holes in higher dimension came from string theory which suggested that the physical spacetime might have more than four dimensions and from this idea we generalized the classical Einstein theory of gravity into higher dimension. Since we do not know what the structure of higher dimensional physical spacetime, many options of higher dimensional background spacetimes were studied. In this review we mainly use the $N+1$-dimensioanl asymptotically flat semi-Riemannian manifold (locally Minkowski spacetime). In addition one can choose Kaluza-Klein spacetime, Supersymmetric spacetime, AdS spacetime[16, 25] etc. which will yield different results. The expected consequence of allowing the spacetimes to possess extra dimension is that the gravity besomces much richer since metric tensors gain extra degree of freedoms, and also becomes more complicated. However, the interesting point of studying higher dimension general relativity is those unexpected new features of gravity that appear in new solutions such as the existence of black holes with their event horizons topology differ from the sphere.

Note that in 4D spacetimes the black holes horizon other than the sphere was forbidden [8] since the only unique black hole solution is Kerr-Newman black hole which has spherical topology. However, in higher dimensional spacetimes the gravity is weaker [21]; one can picture that when the same amount of flux from a gravitational source was emitted into extra dimensions the flux per area which we experience as gravitational field will become weaker. As a consequence, it might be possible for a higher dimensional rotating object to form a "black ring" which is a black hole with toroidal horizon. The intuitive explaination for the existence of this peculiar object is that a rotating black hole counters its own gravity with centrifugal force and prevents itself from collapsing into a spherical [27].

The black ring belongs to the family of stationary axisymmetric black holes in asymptotically flat spacetime and by terminology "black ring" generally means a black hole with horizon topology of $S^{N-2} \times S^1$ but for now the 5D black ring is the only known exact solution of this type. There is the other stationary axisymmetric solution that were dicovered earlier by Myers and Perry [5] as the extension of Kerr black hole to arbitrary higher dimension. The Myers-Perry(MP) solution describes the spherical rotating black holes which are certainly different from the black rings. However, in 5D spacetime there is a possibility for these two different kind of black holes to have the same mass and angular momentum (we will discuss this feature in more detals later) which implies the violation of uniqueness theorem of stationary black holes.

To fully understand these new features of higher dimensional black holes we need to find more exact solutions. However, solving Einstein equation is not such an easy task because in general it yields the system of non-linear partial differential equations which is either hard or impossible to solve. Therefore to achieve this goal one might look for some special class of spacetimes which in which the Einstein equation can be simplified, for example the five dimensional stationary axisymmetric spacetimes. The feature that makes this class of spacetimes interesting is that the symmetries of these spacetimes imply the integrability of vacuum Einstein equation. Moreover, given the known 5D black hole solution one can generate the new solution using Belinsky-Zakharov (BZ) method [21]. Therefore our knowledge of 5D black holes are greatly improved. Nevertheless, in the spacetimes of dimension higher than five the stationary axisymmetric does not necessarily imply integrability of vacuum Einstein equation and we need other method to find new exact solution.

Let us go back to the case of 4D spacetimes, many important solutions such as the Robinson-Trautman solution, Kerr solution and C-metric were found by assuming that the spacetimes are **Algebraically special**(AS) and with a help from the Goldburgh - Sachs theorem that simplifies Einstein filed equations. Therefore, it is natural to ask that can we generalize this useful notions to higher dimension? For the time being the answer is not clear, mainly because the structure of higher dimensional spacetimes is algebraically different from 4D spacetimes. Recently there are attempts to give the definition of AS in higher dimension in many different ways [14, 23, 27]. This review focuses on the classification scheme invented by A. Coley, R. Milson, V. Pravda and A. Pravdova [14], since it is applicable to the spacetimes of any dimensions and more importantly many known higher dimensional black holes are algebraically special in this approach.

In the next chapter we will give the introduction on the 4D algebraic classification which defines the notion of AS, and its application in general relativity which should be helpful to get the rough idea of how this AS notion works. Then in chapter 3 we will discuss the generalization of this algebraic classificaton. We also briefly introduce the Weyl solutions and BZ method in chapter 4 because some examples of 5D black holes given in the later chapter are constructed from these method. Finally in chapter 5, we investigate the algebraic types of some asymptotically flat black holes and then briefly discuss the general features of non-asymptotically flat black holes.

2. Algebraically special four dimensional spacetimes and black holes

In this chapter, the algebraic classification in four dimensions known as Petrov classification will be introduced. Then we discuss the connection of AS spacetimes with the Goldburg - Sachs theorem and its application in solving tne Einstein equation.

2.1 Petrov classification

In four dimensions, the classification of Weyl tensor known as Petrov classification and the Goldberg-Sachs theorem are the keys toward the discovery of important classes of solutions such as Kerr-Schild class and Robinson-Trautman class. In this introduction we used the spinor approach to derive Petrov classification rules [28]. Note that there are other equivalent approaches as well but this approach seems to be the shortest way. Consider a point p in a semi-Riemannian manifold M. The norm of tangent vector in the tangent space T_pM space (isomorphic to Minkowski space) is invariant under the action of Lorentz group $SL(2,C) \cong SO(4)$ which is homomorphic to $SU(2) \times SU(2) \cong Sp(2) \times Sp(2)$. For the complex symplectic space $Sp(2)$, let $(\ ,\ ) : Sp(2) \times Sp(2) \to \mathbb{C}$ be a symplectic form and $o = (0,1), \iota = (1,0)$ be the basis vector such that,

$$\begin{aligned} (o,o) &= \epsilon_{AB} o^A o^B = o_B o^B = 0 \\ (\iota,\iota) &= \epsilon_{AB} \iota^A \iota^B = \iota_B \iota^B = 0 \\ (o,\iota) &= \epsilon_{AB} o^A \iota^B = o_B \iota^B = -1, \end{aligned} \tag{2.1}$$

where,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Under group representation, any $SL(2,\mathbb{C})$ tensor can be decomposed into $Sp(2)\times Sp(2)$ tensor as follow,

$$\tau_{ab} = \tau_{AA'BB'}, \tag{2.3}$$

where a,b are Lorentz indices and A, B and A', B' are spinor and conjugate spinor indices respectively. Note that the a, b indices are defined locally in the neighbourhood of point p. Therefore bare in mind that every vectors and tensors in this chapter are all implicitly depend on point p and the Lorentz indices may not be the same on different tangent space (unless spacetime is flat). However, the pointwise set up is enough for algebraic classification purpose.
The important example of tensor in this spinor representation is the matric tensor which can be written as,

$$g_{ab} = g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'} \tag{2.4}$$

Lemma 2.1.1. *Let $\tau_{...CD...}$ be a multivalent spinor. Then*

$$\tau_{...CD...} = \tau_{...(CD)...} + \frac{1}{2}\epsilon_{AB}\tau_{...C}{}^{C}{}_{...} \tag{2.5}$$

Using this fact one can rewrite the Weyl tensor C_{abcd} in spinor form[28],

$$C_{abcd} + iC^{*}_{abcd} = 2\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} \tag{2.6}$$

where $C^{*}_{abcd} = \frac{1}{2}\epsilon_{cd}^{\ \ ef}C_{abef}$ and Ψ_{ABCD} is the totally symmetric spinor. Thus, we have a relatively simple form; the Weyl tensor is charecterized by five coefficients, $\Psi_{1111}, \Psi_{1110}, \Psi_{1100}, \Psi_{1000}$ and Ψ_{0000}. The next lemma will simplify Weyl tensor even further and will be the impotant ingredient for Petrov classification.

Lemma 2.1.2. *Let $\tau_{AB...C}$ be a totally symmetric spinor. Then there exist univalent spinors $\alpha_A, \beta_B, ..., \gamma_C$ such that*

$$\tau_{AB...C} = \alpha_{(A}\beta_{B}...\gamma_{C)} \tag{2.7}$$

From this lemma the spinor Ψ_{ABCD} is decomposed into four univalent spinors called principal spinors and we say that any one of these univalent spinors is "**repeated principal spinor**" (PNS) If it coincides with at least one other univalent spinor. Base on this property, any Weyl tensor can be classified as shown in the table below.

Table 1: Algebraic types of Weyl tensor

Algebraic type	multiplicity of spinors	equivalently
O	flat space	
N	4	$\Psi_{0000} = \Psi_{1000} = \Psi_{1100} = \Psi_{1110} = 0$
III	3,1	$\Psi_{0000} = \Psi_{1000} = \Psi_{1100} = 0$
II	2 ,1,1	$\Psi_{0000} = \Psi_{1000} = 0$
I	1,1,1,1	$\Psi_{0000} = 0$
D	2,2	$\Psi_{0000} = \Psi_{1000} = \Psi_{1111} = \Psi_{1110} = 0$

The last column of Table 1 obtained by contracting basis spinors o, ι to Ψ. For example consider a typeII spinor $\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$, then one can choose $o^A = \alpha^A$. Thus, when calculating $\Psi_{1000} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}\iota^A\alpha^B\alpha^C\alpha^D$ there will be at least one pair of contraction between α which yields zero ($\Psi_{0000} = 0$ is obvious).

Alternatively one can switch back to Lorentz group representation by define $l_a = \alpha_A\bar{\alpha}_{A'}$ a "**Principal Null Direction**" (one can check that, $l_a l^a = o_A\bar{o}_{A'}o^A\bar{o}^{A'} = o_A o^A\bar{o}_{A'}\bar{o}^{A'} = 0$) which is a null vector associated with PNS . Then using definition of Ψ_{ABCD} one can derive other useful form of the condition on the last column of the table,

$$O \qquad C_{abcd} = 0 \tag{2.8}$$

$$N \qquad C_{abcd}l^c = 0 \tag{2.9}$$

$$III \qquad C_{abc[d}l_{f]}l^c = 0 \tag{2.10}$$

$$II \qquad C_{abc[d}l_{f]}l^b l^c = 0 \tag{2.11}$$

$$I \qquad l_{[e}C_{a]bc[d}l_{f]}l^b l^c = 0 \tag{2.12}$$

$$D \qquad C_{abc[d}l_{f]}l^b l^c = C_{abc[d}n_{f]}n^b n^c = 0, \;\; l \neq n \tag{2.13}$$

From Table 1 we said that the non-vanishing Weyl tensor is algebraically general if it is Type$I$

and it is algebraically special if it is one of the other types. As we have seen that by assuming that spacetime is algebracally special some spinor components of Weyl tenosr vanish and there will be more important consequences which we will see in the next section. As a preparation for the next section let us choose the principal null direction $e_0^a = l^a$ to be our null tetrad basis and choose other real and complex tetrads such that they are quasi-orthonormal, meaning that

$$\begin{aligned} e_1^a = n^a = \bar{n}^a \quad &, \quad e_2^a = m^a = \bar{m}^a = \bar{e}_3^a, \\ e^a_{\ i} e_{aj} = \eta_{ij} \quad &, \quad e_a^{\ i} e_{bi} = g_{ab}. \end{aligned} \tag{2.14}$$

The bar on the basis denoted complex conjugation and η is the flat Minkowski metric in non-diagonal form,

$$\eta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If we regard η_{ij} as a metric, and $\mathbf{e}_i$ as basis, then this vector space will have its own Lorentz symmetry which does not imply the spacetime geometry but reflect the fact that one can choose different set of quasi-orthonormal basis. The indices i, j is called the tetrad indices which are different from spacetime indices. However, on the neighbourhood of point p one can choose the coordinates that g_{ab} and η_{ij} are equal. Therefore for the algebraic classification which only need to be done locally, one can use either Lorentz group that acts on tangent space or the Lorentz group that acts on tetrad basis (we will see the later case in the next chapter on CMPP classification).

Although we use the complex tetrad, the resulting metric is real and it can be checked by taking complex conjugation of the metric. Alternatively, we can define the quasi-orthonormal basis from the spinor basis i.e. choose $o^A = \alpha^A$ and the other basis ι, and then contruct the tetrad from them,

$$l^a = o^A \bar{o}^{A'}, \quad n = \iota^A \bar{\iota}^{A'}, \quad m = o^A \bar{\iota}^{A'}, \quad \bar{m} = \iota^A \bar{o}^{A'}. \tag{2.15}$$

It is easy to check that they satisfy condition (2.14). With this frame tetrads one reconstructs the metric in the form,

$$g_{ab} = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)} \tag{2.16}$$

The Lorentz transformation for the tetrad basis can be obtained from the transformations that preserve the symplectic structure of $Sp(2)$, i.e. $(o', \iota') = (zo, z^{-1}\iota)$ and $(\tilde{o}, \tilde{\iota}) = (o, \iota + xo)$ for any $z, x \in \mathbb{C}$. Hence let $z = \sqrt{\lambda} \exp i\theta/2$ then the transformation of tetrad basis associated with each spinor basis becomes,

$$l' = \lambda l, \quad n' = \lambda^{-1}, \quad m' = e^{i\theta} m, \tag{2.17}$$

$$\tilde{l} = l, \quad \tilde{n} = n + xm + \bar{x}\bar{m} + c\bar{c}l, \quad \tilde{m} = m + \bar{c}l. \tag{2.18}$$

If set $\lambda = 1$ in (2.17) it becomes spetial rotation and if set $\theta = 0$ then we have the boost. However, the transformation (2.18) is nothing like what we have experienced in coordinate frame, it is called null rotation which will keep one null direction unchanged. This transformation will become useful for choosing a convenient frame, moreover, the power (weight) of boost λ can be equivalently used to derive the rules of Petrov classification.

As mentioned earlier, by assuming the spacetime being algebraically special the number of algebraic equations are reduced and we have a better chance of solving Einstein equation. However, the number of equations is not the only problem; Einstein eqution is actually a system of non-linear PDE which in most of the cases are unsolvable.

2.2 Goldberg-Sachs theorem

The Petrov classification by itself is not very useful; it does not give much information about differential structure of the spacetime. However, solving Einstein equation is highly related to differentiability of the spacetime. To make use of this classification we need the connection between this algebraic property and the behavior of tetrad frame and their derivative in AS spacetime. This essential link between algebraic classification and differential geometry of the spacetime is given by Goldberg-Sach (GS) theorem. Before state the GS theorem, we shall develope the understanding of Newmann-Penrose (NP) scalars and define the notion of shear and divergence sacalar which are precisely the quantities that determine behavior of tetrad frame.

From the quasi-orthonormal frame defined in the previous section, at each point the directional derivative of the vectors(covectors) always lie within their tangent(cotangent) space, thus, $\nabla_{e^b} e^a = C^a_{\ cb} e^b$, for each $C^a_{\ cb} \in \mathbb{C}$. These scalars will play very important roles in solving Einstein equation, so we shall give them the names,

Table 2: NP scalar

e^a	∇_{e^a}	$C^a_{\ c0}$	$C^a_{\ c1}$	$C^a_{\ c2}$	$C^a_{\ c3}$
$l,$	$D \equiv \nabla_l$	$-(\epsilon + \bar{\epsilon})$	0	$\bar{\kappa}$	κ
	$\Delta \equiv \nabla_n$	$-(\gamma + \bar{\gamma})$	0	$\bar{\tau}$	τ
	$\delta \equiv \nabla_m$	$-(\beta + \bar{\alpha})$	0	$\bar{\rho}$	σ
$n,$	$D \equiv \nabla_l$	0	$\epsilon + \bar{\epsilon}$	$-\pi$	$-\bar{\pi}$
	$\Delta \equiv \nabla_n$	0	$\gamma + \bar{\gamma}$	$-\nu$	$-\bar{\nu}$
	$\delta \equiv \nabla_m$	0	$\beta + \bar{\alpha}$	$-\mu$	$-\bar{\lambda}$
$m,$	$D \equiv \nabla_l$	$-\bar{\pi}$	κ	$\bar{\epsilon} - \epsilon$	0
	$\Delta \equiv \nabla_n$	$-\bar{\nu}$	τ	$\bar{\gamma} - \gamma$	0
	$\delta \equiv \nabla_m$	$-\bar{\lambda}$	σ	$\bar{\alpha} - \beta$	0
	$\bar{\delta} \equiv \nabla_{\bar{m}}$	$-\bar{\mu}$	ρ	$\bar{\beta} - \alpha$	0

Note that some coefficients are zero since the frame fields are quasi-orthonormal and the epressions that are not listed in the table can be obtained by taking complex conjugation. The coefficients in Table 2 are the NP scalar and by contracting the frame fields to their derivatives each NP scalar can be written explicitly as follow,

Table 3: NP scalar

O	$m^a O l_a$	$\frac{1}{2}(n^a O l_a - \bar{m}^a O m_a)$	$-\bar{m}^a O n_a$
D	κ	ϵ	π
Δ	τ	γ	ν
δ	σ	β	μ
$\bar{\delta}$	ρ	α	λ

The Einstein equation written in this language [28] is called field equations. These field equations might look horrible at first glance, but there are great advantages from doing this. First of all, they are scalar equations and they can be simplified once the algebraically spacial condition is imposed. Let us now pay attention to three coefficients σ, ρ and κ which will be used more often later on. The scalar σ is called a shear and $\rho \equiv \Theta + i\omega$ is a complex divergence, where the real and imaginary parts of divergence scalar are called expansion and twist respectively. The last one *kappa* does not have particular name but it is easy to see that the congruence of vector l is geodesic if $\kappa = 0$.

Theorem 2.2.1. *(Goldburg-Sachs) A vacuum spacetime is algebraically special if and only if there exists a family of space-filling shear free geodesic null congruence i.e.* $\kappa = \sigma = 0$

The GS theorem simplifies Einstein equation by reducing the number of differential equations or makes them solvable. To get the clearer picture let us investigate some part of construction of well known AS solution (we will say the name later). Assuming that the spacetime is AS, twist-free but has non-vanishing divergence scalar and in addition $R_{00} = R_{02} = R_{22} = 0$, then let us try to find a solution for this particular spacetime. By GS theorem there exists family of null geodesic shear-free congruence, denoted by $u(r)$ where r is its affine parameter. Consider NP scalar $\tau = m^\mu l_{\mu;\nu} n^\nu$ which under null rotation, it transforms as $\tilde{\tau} = \tau + z\sigma + \bar{z}\rho + z\bar{z}\kappa = \tau + \bar{z}\rho$, [28] ,therefore we can set $\tau = 0$ because ρ does not vanish. Since all the scalars κ, σ and τ are now vanish, by the definition of connection one-form,

$$\mathbf{\Gamma}_{ij} = \Gamma_{kij}\mathbf{e}^k = e^\mu_{(i)} e_{(j)\mu;\nu} e^\nu_{(k)} \mathbf{e}^{(k)}, \tag{2.19}$$

it implies that $\Gamma_{020} = \Gamma_{120} = \Gamma_{220} = 0$, thus,

$$-\mathbf{\Gamma}_{02} = \mathbf{\Gamma}_{20} = \Gamma_{320}\mathbf{e}^{(3)} = \Theta\mathbf{m}. \tag{2.20}$$

The meaning of equation (2.20) is that the tetrad $\mathbf{m}$ will be immediately determined once the explicit form of $\mathbf{\Gamma}_{02}$ is known. Thus the next step is trying to solve for $\mathbf{\Gamma}_{02}$. From our assumption the Ricci components R_{00}, R_{02} and R_{22} vanish therefore,

$$0 = R_{00} = -R_{0001} - R_{0100} + R_{0203} + R_{0302} = 2R_{0203} \tag{2.21}$$

$$0 = R_{02} = -R_{0021} - R_{0120} + R_{0223} + R_{0322} = R_{0102} + R_{2032} \tag{2.22}$$

$$0 = R_{22} = -R_{2021} - R_{2120} + R_{2223} + R_{2322} = 2R_{2012}. \tag{2.23}$$

And the from Table 1, we also have,

$$0 = \Psi_{0000} = R_{2002}, \quad 0 = 2\Psi_{0001} = R_{2010} - R_{2032} \tag{2.24}$$

Using all these equations, the second Cartan equation becomes,

$$d\mathbf{\Gamma}_{02} + \mathbf{\Gamma}_{02} \wedge (\mathbf{\Gamma}_{32} + \mathbf{\Gamma}_{01}) = R_{0231}\bar{\mathbf{m}} \wedge \mathbf{n} = -\frac{R_{0231}}{\Theta}\mathbf{\Gamma}_{02} \wedge \mathbf{n}. \tag{2.25}$$

If we wedge (2.25) with $\mathbf{\Gamma}_{02}$ we finally obtain,

$$\mathbf{\Gamma}_{02} \wedge d\mathbf{\Gamma}_{02} = 0 \tag{2.26}$$

This last equation is the integrability condition; any one-form satisfies this condition can be integrated. To be more precise, the Frobenius theorem (differential form in complex manifold version) states that any one-form satisfies (2.26) if and only if there exist a complex value function P and a local coordinate ζ such that,

$$\mathbf{\Gamma}_{02} = \frac{d\zeta}{P}. \tag{2.27}$$

From (2.20) the tetrad $\mathbf{m}$ becomes,

$$m = -\frac{1}{P\Theta}d\zeta \tag{2.28}$$

So now we have convenient choice of coordinate for $\mathbf{m}$ (and $\bar{\mathbf{m}}$), then if we solve the remaining equations and choose the rest coordinates to be (u, r) the line element becomes [30],

$$ds^2 = -2H(r, u, \zeta, \bar{\zeta})du^2 - dudr + \frac{d\zeta d\bar{\zeta}}{P^2(u, \zeta, \bar{\zeta})\Theta^2} \tag{2.29}$$

Also note that one can always pick (by Lorentz transformation) the tetrad such that P is real and in twist-free case the scalar Θ can be chosen such that $\Theta = 1/r$, [30]. This solution was first dicovered by I. Robinson and A. Trautman [1] as the earliest significant application of AS and GS theorem in general relativity. The metric generally describes any spacetime with a pure radiation source and gravitational radiation in vacuum spacetime.

3. CMPP Classification

In the previous chapter, we have seen that the notion of AS together with GS theorem are useful for finding 4D exact solutions. Therefore in this chapter we will introduce one way to extend this notion into higher dimension. There are many attempts to extend the definition of AS to higher dimension, neither of them are equivalent[27](note that there are many non-equivalent ways to classify the Weyl tensor from its certain properties but only some of those classification will lead to the solutions of the Einstein equation). However, the classification by A. Coley, R. Milson, V. Pravda and A. Pravdova or CMPP classification is now recieved the most attention since it is applicable in spacetimes of arbitrary higher dimension and some well known class of higher dimensional solutions i.e. Kerr-Schild class (including Myers-Perry black hole), Robison-Trautman class were shown to be algebraically special in this classification scheme.

3.1 Weyl aligned null direction (WAND)

As mentioned ealier, in arbitrary dimensional spacetimes the spinor classification is no longer effective (there is 5D spinor classification by De Smet [23], but it can not be extended to $N+1 > 5$); the Lorentz groups of higher dimensional spacetimes are not homomorphic to product of spin groups. As a consequence, the same spinor approach as in 4D can not be used. Instead, in CMPP approach the Weyl tensor is classified base on its transformation properities. Before we get into the classification, let us consider the transformation of tetrad basis which will imply the transformation of Weyl tensor. Let $l = m_0, n = m_1, m_i$, for $i = 2, ..., N-1$ (Note that the index i will be written as (i) when it might cause confusion with other indices) be the basis vectors for $N+1$ dimensional tangent space such that,

$$l^2 = n^2 = l \cdot m_i = n \cdot m_i = 0; \quad l \cdot n = 1; \quad m_i \cdot m_j = \delta_{ij}. \tag{3.1}$$

This basis vectors are similar to 4D quasi-orthonormal basis in the sense that l, n are null vectors but the spatial tetrads m are all real. The chosen basis is unique up to the symmetry group $SO(N+1)$ that is isomorphic to the Lorentz group. Similar to what we did in 4D case, the Lorentz transformation can be characterized by $\lambda, z_i \in \mathbb{C}, X \in SO(N-1)$ the boost, null rotation and spetial rotation respectively, then we have,

Boost	$l \mapsto \lambda l; \quad n \mapsto \lambda^{-}1n; \quad m_i \mapsto m_i$
Spatial rotation	$l \mapsto l; \quad n \mapsto n; \quad m_i \mapsto X_{ij}m_i$
Null rotation	$l \mapsto l; \quad n \mapsto n + z_i m_i - \frac{1}{2}z_i z_i l; \quad m_i \mapsto m_i - z_i l.$

Observe that the components of Weyl tensor in tetrad basis $m_{i'}$, for index $i' \in \{0, 1, i\}$ transform under the boost λ as $C_{i'j'k'l'} \mapsto \lambda^b C_{i'j'k'l'}$, the boost weight b is an integer and $2 \geq b \geq -2$. To see this explicitly, let $V_\mu, U_\mu, W_\mu, T_\mu$ be any covectors, define $V_{<\mu}U_\nu W_\sigma T_{\rho>} \equiv \frac{1}{2}(V_{[\mu}U_{\nu]}W_{[\sigma}T_{\rho]} + V_{[\sigma}U_{\rho]}W_{[\mu}T_{\nu]})$ which has the same symmetry as Weyl tensor. Then consider the following decomposition,

$$
\begin{aligned}
C_{\mu\nu\sigma\rho} \quad = \quad & 4C_{0i0j}n_{<\mu}m^i_\nu n_\sigma m^j_{\rho>} \\
& -8C_{010j}n_{<\mu}l_\nu n_\sigma m^i_{\rho>} - 4C_{0ijk}n_{<\mu}m^i_\nu m^j_\sigma m^k_{\rho>} \\
& +4C_{0101}n_{<\mu}l_\nu n_\sigma l_{\rho>} + C_{01ij}n_{<\mu}l_\nu m^i_\sigma m^j_{\rho>} + 8C_{0i1j}n_{<\mu}m^i_\nu l_\sigma m^j_{\rho>} + C_{ijkl}m^i_{<\mu}m^j_\nu m^k_\sigma m^l_{\rho>} \\
& -8C_{101i}l_{<\mu}n_\nu l_\sigma m^i_{\rho>} - 4C_{1ijk}l_{<\mu}m^i_\nu m^j_\sigma m^k_{\rho>} \\
& +4C_{1i1j}l_{<\mu}m^i_\nu l_\sigma m^j_{\rho>}.
\end{aligned} \tag{3.2}
$$

Each line in above equation correspond to components with boost weight $2, 1, 0, -1, -2$ from the top to the bottom line respectively.

Definition 3.1.1. *Let l be a null tetrad,*

i) l is Weyl aligned null direction(WAND) if the $b = 2$ components of Weyl tensor vanish.

ii) l is multiple WAND if all $b = 2, 1$ components vanish.

In four dimensions, the definition of WAND and principal null direction are equivalent [14]. Therefore one way to generalized the notion of AS into higher dimension is, roughly speaking, replaces repeated null direction with multiple WAND.

To complete the set up for AS in higher dimension, we need to generalized the definition of shear, divergence and other NP scalars. Form the tetrad basis, we define optical matrices [12],

$$
L_{i'j'} = l_{\mu,\nu}m^\mu_{i'}m^\nu_{j'}, \quad N_{i'j'} = n_{\mu,\nu}m^\mu_{i'}m^\nu_{j'}, \quad M^{(i)}_{i'j'} = m^{(i)}_{\mu,\nu}m^\mu_{i'}m^\nu_{j'}. \tag{3.3}
$$

From the definition of tetrad basis, one immediately see that,

$$
\begin{gathered}
L_{0i'} = N_{1j'} = M^{(i)}_{ij'} = 0, \\
N_{0i'} + L_{1i'} = 0, \quad M^{(i)}_{0i'} + L_{ii'} = 0, \quad M^{(i)}_{1i'} + N_{ii'} = 0, \quad M^{(i)}_{ji'} + M^{(j)}_{ii'} = 0
\end{gathered} \tag{3.4}
$$

Next consider the matrix $L_{ij} = L_{(ij)} + L_{[ij]} = S_{ij} + A_{ij}$ then define expansion and shear to be the trace and the traceless part of matrix S_{ij},

$$
\theta = \frac{1}{N-1} \sum_i S_{ii} = \frac{1}{N-1} l^{\mu}{}_{,\mu}, \tag{3.5}
$$

$$
\sigma_{ij} = S_{ij} - \theta \delta_{ij}. \tag{3.6}
$$

In addition, the anti-symmetric matrix A_{ij} is the twist metrix. In the next section we will use this set up to define the algebraic types and discuss some consequences of assuming that the spacetime is AS. As one might expected, there is also the relation between these optical matrix and the condition that the spacetime is AS, although the relation is not the same as in four dimension. It turns out that in higher dimensional AS spacetimes the full shear matrix no longer vanishes in general, as for exsampe the 5D MP black hole. Moreover, the type two WAND is not always geodesic, as in RT solution [20].

3.2 Algebraically Special in higher dimension

The CMPP approach generalizes the notion of algebraically special into higher dimension by replacing repeated PND with multiple WAND. The following table shows all types that the Weyl tensor can be, notice that there is one additional typeG in this family. TypeG spacetimes make the crucial different between AS in higher dimension and those in four dimension. In four dimensions there are always four pricipal null directions associated with principal spinors and one can pick two pricipal null directions to be the frame tetrads l, n, however, in higher dimension the WAND vector might not exist or only one WAND exists. The following table show all algebraic type of Weyl tensor in any N+1-dimensional spacetime.

Table 4: Algebraic types in higher dimension

Algebraic type	Vanishing Weyl components	type Constraints
O	$b = 2, 1, 0, -1, -2$	$C_{\mu\nu\sigma\rho} = 0$
N	$b = 2, 1, 0, -1$	$C_{\mu\nu\sigma\rho} l^{\sigma} = 0$
III	$b = 2, 1, 0$	$C_{\mu\nu\sigma[\rho} l_{\alpha]} l^{\sigma} = 0$
III_i	$b = 2, 1, 0, -2$	
II	$b = 2, 1$	$C_{\mu\nu\sigma[\rho} l_{\alpha]} l^{\nu} l^{\sigma} = 0(*)$
II_i	$b = 2, 1, -2$	
I	$b = 2$	$l_{[\beta} C_{\mu]\nu\sigma[\rho} l_{\alpha]} l^{\nu} l^{\sigma} = 0$
I_i	$b = 2, -2$	
D	$b = 2, 1, -1, -2$	n and l satisfy (*)
G	admit no WAND	

The algebraic type with subscript i mean that both l, n are WAND but n is not a multiple WAND. Observe that CMPP types are nothing like the four dimensional Petrov types although they share some properties. Due to this non-equivalence, we can not straightforwardly generalize Goldberg-Sachs theorem to higher dimension.

Since we now have more algebraic types comparing with 4D spacetime, it would be nice if we can eliminate some of these choices in the certain class of the black holes. Fortunately, in the familiar classes of black holes namely static black holes and subset of stationary black holes such elimination is possible [20].

Theorem 3.2.1. *Let (M, g) be a $N+1$ dimensional spacetime, the Weyl tensor in M can only be of types G, I_i, D or O if one of the following condition are fulfilled,*

i) *(M, g) is static spacetime.*

ii) *(M, g) is stationary with non-vanishing divergence scalar and metric g is invariant under the reflection map.*

The reflection map in *ii)* is any choice of transformation $(x^0, ..., x^N) \mapsto (\pm x^0, ..., \pm x^N)$ that leaves the metric invariant but changes the direction of WAND. Although we are not giving the rigorous proof here, it is not difficult to see why this theorem is true. First for the static case, if there exist a WAND, say $l = (l^0, l^i)$ (unless it will be typeG) then by time reversal one obtains different vecter $n = (-l^0, l^i)$ which is also a WAND satisfying the same type contriant as l. Thus, only possible choices of static spacetimes are types I_i, D and O. Similar argument for stationary case, one can use the reflection map instead of time reversal and non-vanishing divergence scalar to ensure that the transformation gives a distinct vector (not just proportional to l). This theorem excludes some possibilities of algebraic types for stationary and static black holes and shall be useful later.

Although the higher dimension GS theorem is not yet dicover or may not even exist, the attemp to extend notion of AS to higher dimension is not meaningless. Recently it was shown that in five dimensional AS spacetime, the optical matrix takes certain forms [26] so this can be regarded as a partial result of 5D Goldburg-Sachs theorem.

Theorem 3.2.2. *In a five dimensional AS Einstein spacetime that is not conformally flat, there exists a geodesic multiple WAND l and the orthonormal basis m_i such that the matrix L_{ij} takes* *one of the forms,*

$$i) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix}, \tag{3.7}$$

$$ii) \qquad b\begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.8}$$

$$iii) \qquad b\begin{pmatrix} 1 & a & 0 \\ -a & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

If the spacetime is type III or N then the form is ii).

The converse of this theorem is not generally true, the counter example is provided in [26]. So far we have discussed the detials of CMPP classification and should be ready to use it on black holes. Therefore, the readers who are interested to find out on how the extension of 4D algebraically special solutions behave under this classification scheme can skip to chapter 5. In the next chapter, we discuss the more general approach in finding 5D stationary axisymmetric solutions, which will be necessary for the construction of black saturn.

4. Inverse scattering method and stationary solution

In this chapter the other mothod for solving the Einstein equation, called inverse scattering method, will be briefly discussed. It is the most successful approach in dealing with five dimensional stationary axisymmetric black holes. In fact, the method can be applied for arbitrary high dimensional subclass of stationary axisymmetric black hole but only in 4D and 5D that this subclass is identical to the set of all stationary axisymmetric solutions.

4.1 Stationary axisymmetric solutions

Let us assume that a spacetime M^{N+1} is stationary axisymmetric and in addition posesses N-1 commuting smooth Killing vector fields. These Killing vector fields form a coordinate basis for a smooth $N-1$ dimensional tangent distribution, denoted by D (submanifold of tangent bundle). This tangent distribution is also *involutive* which means Lie bracket of any pair of basis vectors $\xi^a, \xi^b \in D$ takes its value in $D(0 \in D$ since D it is vector space at every point).

Definition 4.1.1. *Let M^n be the manifold and D be a k-dimensional smooth tangent distribution, we say D is completely integrable if for any $p \in M^n$ there exists a chart (U, φ) such that $\varphi(p) = (\xi^1(p), ..., \xi^k(p), x^{k+1}(p), ..., x^{n-k}(p)) \in W \times V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ and at every points in U the vector fields $\partial_{\xi^1}, ..., \partial_{\xi^k}$ span D.*

The meaning of this definition is that if D is completely integrable, then we know the submanifold that D belongs to and the coordinates on that submanifold are given by two independent sets of coordinates.

Theorem 4.1.1. *(Frobenius) A tangent distribution is completely integrable iff it is involutive.*

The proof of this theorem can be found in [29] (Note that is the same Frobenius theorem we saw in the previous chapter but in different version). As a consequence of this theorem one can find the coordinates for the stationary axisymmetric spacetime such that the metric can be written as,

$$g_{\mu\nu}dx^\mu dx^\nu = G_{ab}d\xi^a d\xi^b + e^{2\nu}(d\rho^2 + dz^2). \tag{4.1}$$

Since $\xi^0, ...\xi^{N-2}$ are all Killing vectors the matrix G and scalar ν only depend on ρ, z. we may also choose the coordinates ρ, z such that,

$$detG = -\rho^2, \tag{4.2}$$

(this is not necessary but then we will get this contraint afterward). Then substitutes this ansatz into the vacuum Einstein equation leads to a set of equations [18, 21],

$$\partial_\rho U + \partial_z V = 0, \tag{4.3}$$

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{1}{8\rho} Tr(U^2 - V^2) \quad , \quad \partial_z \nu = \frac{1}{4\rho} Tr(UV), \tag{4.4}$$

where $U = \rho(\partial_\rho G)G^{-1}, V = \rho(\partial_z G)G^{-1}$. Equations (4.2) and (4.3) imply that the scalar ν satisfies integrability condition $\partial_\rho \partial_z \nu = \partial_z \partial_\rho \nu$, thus, ν can be integrated once the matrix G is known.

4.2 Weyl solution and BZakharov method

Although, we have reduced the Einstein equation to simple set of matrices and scalar equations, it is still not easy to solve. Let us consider the special case when all Killing vector are mutually orthogonal, the metic then takes diagonal form,

$$ds^2 = -e^{2U_0} dt^2 + \sum_{i=1}^{N-2} e^{2U_i} (d\xi^i)^2 + e^{2\nu}(d\rho^2 + dz^2). \tag{4.5}$$

Then substitute the diagonal matrix G into equations (4.2)-(4.4) we have,

$$(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2) U_a = 0, \tag{4.6}$$

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{\rho}{2} \sum_{a=0}^{N-2} [(\partial_\rho U_a)^2 - (\partial_z U_a)^2] \quad , \quad \partial_z \nu = \rho \sum_{a=0}^{N-2} \partial_\rho U_a \partial_z U_a, \tag{4.7}$$

and also a contraint on metric G becomes,

$$\sum_{a=0}^{N-2} U_a = \ln \rho. \tag{4.8}$$

One can see that the sum diverges at $\rho = 0$ along the z-axis. This sum also called an infinite rod since it can be pictured as a long thin rod stretched along z-axis.

Euations (4.6) can be thought of as the set of Laplace equations in three dimensional flat space i.e. (ρ, ϕ, z) where U_a are not the function of unphysical coordinate ϕ. This set of equations is solvable if any sufficient boundary conditions for each U_a are given. Suppose the boundary conditions at infinity and some points on z-axis, say $a_1, ..., a_k$ are specified then the solutions that satisfy Laplace equations and constraint (4.8) are given by [21],

$$U_j = \frac{1}{2} \ln \mu_j \tag{4.9}$$

$$U_i = \frac{1}{2} \ln \left(\frac{\mu_i}{\mu_{i+1}} \right) \tag{4.10}$$

$$U_0 = \frac{1}{2} \ln \left(\frac{\rho^2}{\mu_1} \right) = \frac{1}{2} \ln \bar{\mu}_1 \tag{4.11}$$

where,

$$\mu_i = \sqrt{\rho^2 + (z - a_i)^2} - (z - a_i), \tag{4.12}$$

$$\bar{\mu}_i = \sqrt{\rho^2 + (z - a_i)^2} + (z - a_i) \tag{4.13}$$

the factor 1/2 or rod density is needed to make the sum of U_a satisfies (4.8). If U_k is a constant then one can set it to zero and shift the all indices $i > k$ down but the form of $U_{i,j}$ stay the same so that they still satistfy (4.8). The functions $\mu, \bar{\mu}$ are called soliton and anti-soliton, the symbol $\tilde{\mu}$ will stand for either of them. Notice that the funcion $U_j = \frac{1}{2} \ln \mu_j$ diverges at $\rho = 0$ as well as original infinite rod but only when $z \geq a_j$, on the other hand, U_0 blows up when $\rho = 0, \ z \leq a_1$ both are called semi-infinite rod. The rest U_i are finite rod which only blow up on the finite interval of z-axis i.e. $z \in [a_i, a_{i+1}]$. The metric obtained from this method is called Weyl solution which is completely characterized by the set of functions U_a. and they can be written in the diagram called rod structures.

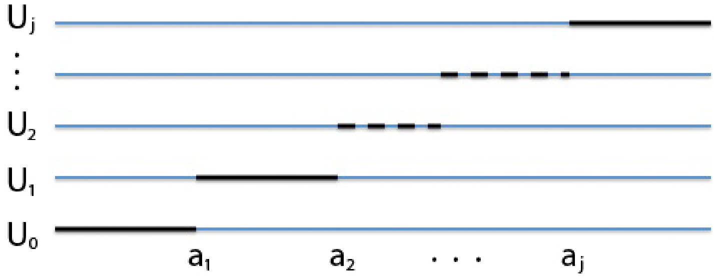

Figure 1: Rod structure of Weyl solution

Since Weyl solutions require all commuting Killing vectors to be orthogonal, it is even more restricted subclass of stationary axisymmetric solutions. However, the known Weyl solutions are very useful as a "seed" for finding the more general solutions in this class. There is the method introduced by Belinsky and Zakharov (BZ) by extending (4.3) into linear spectral equations [4, 21] and use the solutions of this spectral equation to generate new matrix G from the known one. To see how this method works, let $G_0(\rho, z), G_1(\rho, z)$ be two different solutions of (4.3), then define the functions $\Psi_0(\lambda, \rho, z), \Psi_1(\lambda, \rho, z)$ such that,

$$\Psi_0(0, \rho, z) = G_0(\rho, z), \tag{4.14}$$

$$\Psi_1(0, \rho, z) = G_1(\rho, z), \tag{4.15}$$

where λ is a complex parameter. In doing so equation (4.3) needs to be modified as well,

$$D_1\Psi \equiv (\partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\partial_\lambda)\Psi = \frac{rV - \lambda U}{r^2 + \lambda^2}\Psi \tag{4.16}$$

$$D_2\Psi \equiv (\partial_z + \frac{2\lambda^2}{\lambda^2 + r^2}\partial_\lambda)\Psi = \frac{rU + \lambda V}{r^2 + \lambda^2}\Psi. \tag{4.17}$$

Suppose G_0 is a known solution, for example a Weyl solution, then one can generate new solution by given a relation, $\Psi_1(\lambda, \rho, z) = \chi(\lambda, \rho, z)\Psi_0(\lambda, \rho, z)$, for $\chi(\lambda, \rho, z)$ is a complex matrix called the dressing matrix. By substituting this expression back in system of equations(4.16)-(4.17) the matrix χ can be solved and the most interesting solution is *n-solitonic solution* or simply a solution with n poles on the real axis,

$$\chi = 1 + \sum_{i=1}^{n} \frac{R_i}{\lambda - \tilde{\mu}_i}, \tag{4.18}$$

where R_i are residue matrices and once they are determined the new solution is achieved i.e.,

$$G_1 = \Psi_1(0, \rho, z) = \chi(0, \rho, z) G_0 = G_0 + \sum_{i=1}^{n} \frac{R_i G_0}{\lambda - \tilde{\mu}_i}. \tag{4.19}$$

Each matrix R_i can be determined by introducing n constant vectors, m_0^i (N-1 dimensional vectors), and then evaluate a matrix Γ,

$$\Gamma_{ij} = \frac{(m_1^i)^T G_0 m_1^j}{\rho^2 + \tilde{\mu}_i \tilde{\mu}_j}, \tag{4.20}$$

where vector $m_1^i = m_0^i \Psi_0^{-1}(\tilde{\mu}_i, \rho, z)$. The residue R_i then given by,

$$(R_i)_{ab} = m_a^i \sum_{j=1}^{n} \frac{(\Gamma^{-1})_{ji} m_c^j (G_0)_{cb}}{\tilde{\mu}_j}. \tag{4.21}$$

The resulting matrix might not satisty (3.2), but it can be normalized,

$$G' = \pm \left(\frac{\rho^2}{\pm \det G} \right)^{1/(N-2)} G. \tag{4.22}$$

This method is very useful for solving the five dimensional Einstein equation. Since we know many stationary axisymetric solution comparing with dimension higher than five, thus, one can feed these seed metrics to BZ method to find new solutions. In next chapter we will encounter one of the new solution obtained by this method.

5. Black holes in higher dimension and their algebraic types

Most of higher dimensional black hole solutions were achieved by extended some properties of known solutions in four dimensional spacetimes into higher dimension. Those properties determine the coordinate in which the metric take a convenient form, then we use the ansatz metric to solve the Einstein equation. In this review we focus on the asymptotically flat solutions which competible with CMPP classification. Since CMPP approach was constructed from the Lorentz group of asymtotically flat spacetimes, it is not clear that the application on non-asymtotically spacetimes is possible. The De Smet approach on the other hand, [23] can be used for some non-asymptotically flat black holes but it is not equivalent to CMPP approach. However, at the end of this chapter the non-asymtotically flat solutioin will be briefly disscused.

5.1 Static spherical symmetric black holes

This type of black hole is the extension of Schwarzschild solution (and Reissner-Nordstrom solution in presence of EM field) the neutral case also called Schwarzschild-Tangherlini (ST) solution which is the earliest and the simplest solution found for higher dimensional black hole [2]. In any $N+1$ dimensional spacetimes of this type with $N > 1$, the metric takes the same form,

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 d\Omega_{N-1}^2. \tag{5.1}$$

By using this metric to solve vacuum Einstein equation and source free Einstein-Maxwell equations, one obtains the explicit form of $f(r)$, $g(r)$,

$$\begin{array}{ll} \text{vacuum} & f = g^{-1} = \left(1 - \frac{C}{r^{N-2}}\right)^{1/2}, \\ \text{EM background} & f = g^{-1} = \left(1 - \frac{C}{r^{N-2}} + \frac{B^2}{r^{2(N-2)}}\right)^{1/2}, \end{array}$$

where C, B are constants. Consider the neutral case, the uniqueness of this static asymptotically flat non-degenerated horizon black hole is still hold even in higher dimension [24]. Moreover, as the continuous limit of Myer-Perry black hole and Robinson-Trautman black hole, Schwarzschild-Tangherlini soluiton is typeD (will be shown in next section).

5.2 Robinson-Trautman black hole

The higher dimensional version of RT solution was first accomplished in [17] by assuming the existence of a geodesic, shear-free (shear matrix vanished) null congruence. The null congruence form the family of null hypersurfaces $u(x) = constant$, then define, $l_\mu = -u_{,\mu}$, the normal null covector to this surface (Note that because of nullity the normal vector is also the tangent vector). Let r be the affine parameter on each hypersurface, sience $u(r)$ is constant and l^μ is null, for any coordinates system $\{x^\mu\}$ one immediately see that,

$$\frac{\partial u}{\partial x^\mu}\partial_r x^\mu = u_{,r} = 0, \tag{5.2}$$

and hence, we have $l^\mu = \partial_r x^\mu$. Suppose we choose $u = x^0, r = x^1$ and $\tilde{x}^i$ the spatial coordinates that are transverse to u and r and constant along the geodesic, then,

$$\begin{aligned} l_\mu = -\delta^u_\mu,\ l^\mu = \delta^\mu_r \quad \rightarrow \quad & g_{ur} = -1,\ g_{rr} = g_{ri} = 0, \\ & g^{ru} = -1,\ g^{uu} = g^{iu} = 0. \end{aligned} \tag{5.3}$$

Note that we have used $l^\nu = g^{\nu\mu} l_\mu$. From these equations we can then reduce the metric into the simpler form,

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= g_{uu}du^2 + 2g_{ur}dudr + g_{rr}dr^2 + g_{ij}d\tilde{x}^i d\tilde{x}^j \\ &= -g^{rr}du^2 - 2dudr + g_{ij}(d\tilde{x}^i + g^{ri}du)(d\tilde{x}^j + g^{rj}du) \\ &= -g^{rr}du^2 - 2dudr + g_{ij}dx^i dx^j \end{aligned} \tag{5.4}$$

where we redefined $\tilde{x}^i = dx^i + g^{ri}du$. Also observe that the covariant derivative of vector l is,

$$\begin{aligned} l_{\mu;\nu} &= l_{\mu,nu} - \Gamma^\alpha_{\mu\nu} l_\alpha \\ &= -\frac{1}{2}g^{u\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\nu\mu,\beta}) \\ &= \frac{1}{2}g_{\mu\nu,r}. \end{aligned} \tag{5.5}$$

Therefore in addition to vanishing shear matrix, the twist metix also vanishes i.e. if one chooses spatial tetrad m_i^μ then $A_{ij} = l_{\mu;nu} m^\mu_{[i} m^\nu_{j]} = \frac{1}{2} g_{\mu\nu} m^\mu_{[i} m^\nu_{j]} = 0$.

Subtitute this ansatz into the Einstein equation without matter filed (allow radiation and cosmological constant) one obtains the result,

$$
\begin{aligned}
ds^2 &= \frac{r^2}{P^2}\gamma_{ij} dx^i dx^j - 2dudr - 2H du^2, \\
H &= \frac{\tilde{R}(u)}{(N-1)(N-2)} - 2r(lnP)_{,u} - \frac{\Lambda}{(N-1)(N)} r^2 - \frac{\mu(u)}{r^{N-2}},
\end{aligned} \tag{5.6}
$$

where $\gamma_{ij}(x,u)$ is unimodular spatial metric i.e. $\det\gamma = 1$, $P \equiv P(x,u)$, $\mu(u)$ are functions and $\tilde{R}$ is the Ricci scalar associated with $h_{ij} \equiv \gamma_{ij}/P^2$. From the remaining unsolved equations, the spatial metric and function P also need to satisfy the constraints,

(case $N > 3$)

$$
\tilde{R}_{ij} = \frac{\tilde{R}}{(N-1)P^2}\gamma_{ij} \quad , \quad \left(\frac{\gamma_{ij}}{P^2}\right)_{,u} = -2\frac{\gamma_{ij}}{P^2}(\ln P)_{,u} \tag{5.7}
$$

$$
N\mu(\ln P)_{,u} - \mu_{,u} = \frac{16\pi n^2(x,u)}{N-1}. \tag{5.8}
$$

For $N = 3$ the constraint (5.8) is more complicated,

$$
\frac{1}{2}(\tilde{R}_{,i}\, h^{ij})_{,j} - (\tilde{R}_{,i}\, h^{ij})(\ln P)_{,j} + 6\mu(\ln P)_{,u} - 2\mu_{,u} = 16\pi n^2(x,u). \tag{5.9}
$$

Since in 4D $\tilde{R}$ depend on both u and x while in higher dimension it is only the function of u [17]. In non-vacuum case ,the function $n(x,u)$ corresponds to energy-momentum tensor of pure radiationin. The vacuum RT solution can be obtained by setting $n(x,u) = 0$, and with appropriated continuous limit one also recovers ST solution.

Algebraic type

All solutions in higher dimensional RT class are type$D$ since they have two multiple WANDs $l = \partial_r, n = \partial_u - H\partial_r$. The non-vanishing Weyl tensor components are given by [17],

$$C_{ruru} = -\mu(u)\frac{(N-1)(N-2)}{2r^N}, \quad C_{iuuj} = 2HC_{iruj} = -\mu(u)\frac{(N-2)}{2r^{N-2}}h_{ij}, \tag{5.10}$$

$$C_{ijkl} = r^2\tilde{R}_{ijkl} - 2r^2\left(\frac{R(u)}{(N-1)(N-2)} - \frac{\mu(u)}{2r^{N-2}}\right)h_{i[k}h_{l]j}, \tag{5.11}$$

by straightforward calculation (details in appendix A) one can see that,

$$n^\mu n^\nu C_{\sigma\mu\nu[\rho}n_{\alpha]} = l^\mu l^\nu C_{\sigma\mu\nu[\rho}l_{\alpha]} = 0. \tag{5.12}$$

Hence all RT spacetimes are type$D$ and by take continuous limit $n(x,u) \to 0$, equation (5.12) is preserved and the vacuum RT solutions (black holes) are also type$D$. Before proceed to the next example, there is an important remark here. Instead of become more complicated the RT metrics are simpler in higher dimension, since $\tilde{R}$ can only depend on u. This may seem contradicted with what we say earlier that"gravity is much richer in higher dimension" but this statement will surely be confirmed in the next examples.

5.3 Myers-Perry black hole

Originally, Myers-Perry (MP) solution was obtained by assume that the metric in higher dimension has the Kerr-Schild form,

$$g_{\mu\nu} = \eta_{\mu\nu} - 2h(x)l_\mu l_\nu, \tag{5.13}$$

where l is a null vector. Then solving the vacuum Eistien equation using this anzat metric. The solution describes rotating black holes in any dimensions, which indeed reduces to the Kerr solution in 4D. The difference between Kerr solution and its higher dimensional extensions is that in higher dimension the black holes are allowed to rotate independently in more than one plan. One can think of simple classical example by considers a rotating point mass, M, in $\mathbb{R}^4$ space with coordinates (x_1, x_2, x_3, x_4) then transforms them into a pair of polar coordinates,

$$x_1 = r_1 \cos\phi_1 \quad , \quad x_2 = r_1 \sin\phi_1, \tag{5.14}$$

$$x_3 = r_2 \cos\phi_2 \quad , \quad x_4 = r_2 \sin\phi_2. \tag{5.15}$$

In the new coordinates one can define the angular momentum, $J_1 = Mr_1^2\dot{\phi}_1$ and $J_2 = Mr_2^2\dot{\phi}_2$, which are obviously independent quantities. In general spacetimes the number of rotation planes is $\lfloor N/2 \rfloor$, the integer value of a half of spatial dimension because each plane consists of two coordinates, and for space with odd dimensions, there will be one coordinate lelf. From this observation one might have guessed that the form of solutions are different in odd and even dimension,

for even N

$$l_\mu dx^\mu = dt + \sum_{i=1}^{N/2} \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dx^i - y^i dy^i)}{r^2 + a_i^2}, \tag{5.16}$$

for odd N

$$l_\mu dx^\mu = dt + \sum_{i=1}^{(N-1)/2} \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dx^i - y^i dy^i)}{r^2 + a_i^2} + \frac{zdz}{r}, \tag{5.17}$$

In both cases the function $h(x) = -\frac{\mu r}{2\Pi F}$, where

$$F = 1 - \sum_{i=1}^{\lfloor N/2 \rfloor} \frac{a_i^2(x^{i2} + y^{i2})}{(r^2 + a^2)^2} \tag{5.18}$$

$$\Pi = \prod_{i=1}^{\lfloor N/2 \rfloor} (r^2 + a_i^2). \tag{5.19}$$

Note that, for five dimensional case the solution can be obtained from inverse scattering method [21].The MP black holes are stationary with parameter a_i related to rotation on each plane. All MP black holes have event horizons of S^{N-1} topology, but only in cases of $N = 3, 4, 5, 6, 8$, that the closed formed of horizons are known [5]. In four dimensional spacetime MP black holes reduce to Kerr black holes, the only general solution for stationary(neutral) black hole which uniquely characterized by mass and angular momentum. However, in higher dimension, for given mass and angular momentum there may be black holes that have different horizon topology.

Algebraic type

The MP black holes were verified to be type$D$ [20]. To demonstrate this fact, we will need theorem 2.2.1. to shorten the calculation. First we need to find the reflection symmetry. Let us choose the coordinate [5] such that transform the MP metrics into the following form,

D is even

$$ds^2 = -dt^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r^2}{\Pi F}(dt + a_i \mu_i^2 d\phi_i)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \tag{5.20}$$

D is odd

$$ds^2 = -dt^2 + r^2 d\alpha^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r^2}{\Pi F}(dt + a_i \mu_i^2 d\phi_i)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2. \tag{5.21}$$

It is now easy to see that the metrics admit reflection symmetry $(t, \phi) \to (-t, -\phi)$. The non-vanishing divergence is just to ensure that the new null vector we got is different from the previous one which is necessary in case the exact form of null vector is not known (in our case it is easy to see that we really get a different vector). Therefore, by theorem 3.2.1, our choices are now reduced to four. Next we will show that the Kerr-Schild spacetimes is always typeII

(it can be more special) then together with our reduced choices, the MP black holes have to be type$D$, unless it is conformally flat. To prove this we need to calculate $C_{0i0j}, C_{010i}, C_{0ijk}$ explicitly. The nice feature of Kerr-Schild form is that the null vector l^{μ}, is null in both flat and curved geometry i.e. $g_{\mu\nu}l^{\mu}l^{\nu} = \eta_{\mu\nu}l^{\mu}l^{\nu} = 0$ and that $\nabla_l l^{\mu} = l^{\mu}{}_{,\nu}l^{\nu}$. Also note that in general Kerr-Schild spacetimes l is not always geodesic, one can show that,

$$R_{\mu\nu}l^{\mu}l^{\nu} = 2hg^{\mu\nu}(l_{\mu;\sigma}l^{\sigma})(l_{\nu;\rho}l^{\rho}). \tag{5.22}$$

Therefore l can not be geodesic unless the null component of energy-momentum tensor $T_{\mu\nu}l^{\mu}l^{\nu}$ vanishes [22] but we will not have to worry about that here. Let us consider the curvature tensor,

$$\begin{aligned} R_{\mu\nu\sigma\rho} &= \frac{1}{2}\left(g_{\mu\sigma},_{\nu\rho} + g_{\rho\nu},_{\mu\sigma} - g_{\sigma\nu},_{\mu\rho} - g_{\mu\rho},_{\nu\sigma}\right) + 2\Gamma_{\lambda\mu[\sigma}\Gamma^{\lambda}_{\rho]\nu} \\ &= [-\left(hl_{\mu}l_{\sigma}\right),_{\nu\rho} - \left(hl_{\nu}l_{\rho}\right),_{\mu\sigma} + \left(hl_{\sigma}l_{\nu}\right),_{\mu\rho} + \left(hl_{\mu}l_{\rho}\right),_{\nu\sigma}) + 2\Gamma_{\lambda\mu[\sigma}\Gamma^{\lambda}_{\rho]\nu}, \end{aligned} \tag{5.23}$$

where,

$$g^{\sigma\alpha}\Gamma_{\alpha\mu\nu} = \Gamma^{\sigma}_{\mu\nu} = -(hl^{\sigma}l_{\mu}),_{\nu} - (hl^{\sigma}l_{\nu}),_{\mu} + \eta^{\sigma\alpha}(hl^{\mu}l_{\nu}),_{\alpha} + 2hl^{\sigma}l^{\alpha}(hl_{\mu}l_{\nu}),_{\alpha}. \tag{5.24}$$

All expression above will be transform in to the tetrad basis and after some tedious caculation (details in appendix B), one can verify that $R_{0i0j} = R_{01i0} = R_{0ijk} = 0$. From this we can immediately calculate some components of Ricci tensor,

$$R_{00} = -R_{0001} - R_{0100} + R_{0i0j}\delta^i_j = 0 \tag{5.25}$$

$$R_{0i} = -R_{01i0} - R_{00i1} + R_{0i0j}\delta^i_j = 0. \tag{5.26}$$

From the information we have here, it is enough to show that $C_{0i0j} = C_{0ijk} = C_{010j} = 0$ and consequently, the Kerr-Schild spacetimes are typeII. Therefore, the MP black hole must be type$D$. Alternatively, instead of direct use of theorem 3.2.1; if one transform $l \to n$ using reflection, then the metric is in the same form but has n in place of l. Then as n and l switch their role, the calculation will finally yields $C_{1i1j} = C1ijk = C_{101j} = 0$, therefore, we can conclude that n is also typeII WAND. Note that this is not true for arbitrary choice of energy momentum tensor but only those make n a geodesic i.e. $T_{\mu\nu}n^{\mu}n^{\nu} = 0$, which is automatically satisfied for vacuum spacetime. However, from this observation we can further conclude that the Kerr-Schild spacetime with $T_{\mu\nu}n^{\mu}n^{\nu} = T_{\mu\nu}l^{\mu}l^{\nu} = 0$ is type$D$.

5.4 Black rings

Black rings are the type of black holes that have toroidal horizon's topology. The idea of extension is motivated from black string in 5D Kaluza-Klein space time(see later section on non-asymptoticall flat solution); by performing Wick rotation on the compact direction and time direction one obtains a metric for asymptotically flat spacetime. Alternatively, the black ring can be generated by BZ method and its rod structure is given shown in the following figure.

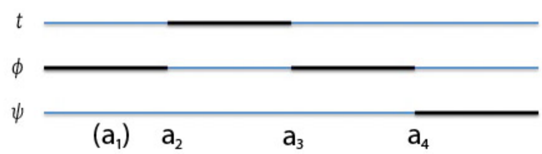

Figure 2: The rod structure of black ring

As we mentioned earlier that, in principal, the black ring is the definition for any black hole with horizon of $S^1 \times S^n$ topology but the only known exact solution is in five dimensional spacetime. The line element for the 5D neutral one plane rotating black ring is given in ring coordinates,

$$\begin{aligned} ds^2 = & - \frac{F(\tilde{y})}{F(\tilde{x})}\left(d\tilde{t} - \tilde{R}C\frac{(1+\tilde{y})}{F(\tilde{y})}d\tilde{\psi}\right)^2 \\ & + \frac{\tilde{R}^2}{(\tilde{x}-\tilde{y})^2}F(\tilde{x})\left[-\frac{G(\tilde{y})}{F(\tilde{y})}d\tilde{\psi}^2 - \frac{d\tilde{y}^2}{G(\tilde{y})} + \frac{d\tilde{x}^2}{G(\tilde{x})} + \frac{G(\tilde{x})}{F(\tilde{x})}d\tilde{\phi}^2\right], \end{aligned} \tag{5.27}$$

where $F(z) = 1 + \tilde{\lambda}z$, $G(z) = (1-z^2)(1+\tilde{\mu}z)$ and R is the raduis of the ring. The range of the coordinates are,

$$-1 \leq \tilde{x} \leq 1, \quad -\infty < \tilde{y} \leq -1, \quad 0 \leq \tilde{\psi}, \tilde{\phi} < 2\pi. \tag{5.28}$$

This line element is the original form dicovered by R. Emparan and H. Reall [16]. In general, the black ring is allowed to have two different angular momentums [21] but it is more difficult to demonstrate the algebraic classification. There is other more convenient form [11, 15], which are related by coordinates transformation and altering some parameters,

$$\begin{aligned}
\tilde{t} = t, \quad (\tilde{x}, \tilde{y}) = \left(\tfrac{x-\lambda}{1-\lambda x}, \tfrac{y-\lambda}{1-\lambda y}\right), \quad (\tilde{\psi}, \tilde{\phi}) = \frac{1-\lambda\nu}{\sqrt{1-\lambda^2}}(\psi, \phi), \\
\tilde{\lambda} = \lambda, \qquad \tilde{\nu} = \tfrac{\lambda-\nu}{1-\lambda\nu}, \qquad \tilde{R} = R\sqrt{\frac{1-\lambda^2}{1-\lambda\nu}}.
\end{aligned} \tag{5.29}$$

The resulting line element is,

$$\begin{aligned}
ds^2 &= -\frac{F(x)}{F(y)}\left(dt + R\sqrt{\lambda\nu}(1+y)d\psi\right)^2 \\
&\quad + \frac{R^2}{(x-y)^2}\left[-F(x)\left(G(y)d\psi^2 + \frac{F(y)}{G(y)}dy^2\right) + F^2(y)\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right)\right],
\end{aligned} \tag{5.30}$$

where $F(z) = 1 - \lambda z$, $G(z) = (1 - z^2)(1 - \nu z)$. The black ring in form (5.30) will be mainly used in this review. Note that, there are sigularities at $y = x = -1$ and $x = 1$, called canonical singularity which can be eliminated by setting [11],

$$\Delta\psi = \Delta\phi = 2\pi\frac{\sqrt{1+\lambda}}{1+\nu}, \tag{5.31}$$

$$\lambda = \frac{2\nu}{1+\nu^2}. \tag{5.32}$$

The quantities $\Delta\psi, \Delta\phi$ are the peroid of black ring which is simply the range of angular coordinates. Thus, there are only two free parameters in the metric, say R, the radius and ν(or alternatively λ), the parameter that determines the mass of black ring. These can be interpreted as balancing the force in the ring; if the radius and the mass are fixed then the angular momentum need to be adjusted to keep the black ring in balance. The event horizon of the black ring is the surface $y = 1/\nu$.

General properties and limit

Although Black ring belongs in stationary axisymmetric class of solutions as well as MP black hole, they are two distinct solutions i.e. by any continuous change of parameters black ring can not turn into MP solution. There is only one common situation that both become naked singularity (see figure 3). However, The form of metric (5.30) can be used as MP metric if one sets $\lambda = 1$ but leaves μ as a free parameter. As mentioned previously, the stationary black holes are not unique in higer dimension, the black ring is an example of this violation. Suppose we have MP black hole and black ring, both rotate in one plane, then the area (reduced area) of horizon and the angular momentum per unit mass of them are given by [16],

$$\text{MP black hole,} \qquad a = 2\sqrt{2(1-j^2)},$$
$$\text{Black ring,} \qquad a = 2\sqrt{\nu(1-\nu)}, \quad j = \sqrt{\tfrac{(1+\nu)^3}{8\nu}}$$

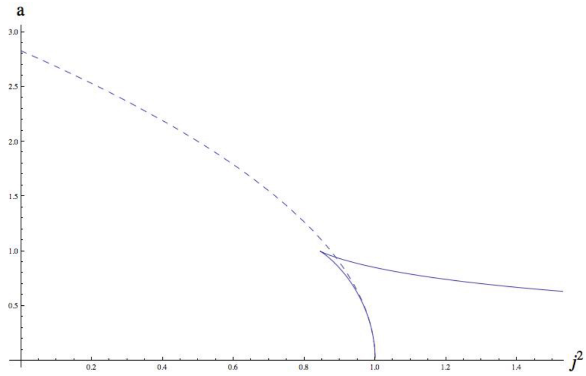

Figure 3: MP black hole and and black ring

The dashed line represents the MP black holes and the dotted one represents the black rings. If assuming both MP black hole and balck ring have the same mass, the plot showed that they can have the same angular momentum. Notice that the black ring exists near the maximum momentum of the MP black hole, moreover, from figure 3 there is the situation when black 5D one plane rotating MP black hole and black ring have the same entropies (which are proportional to their area). Therefore it was suggested that at some point, when the MP black hole spin fast enough, it might turn in to black ring [9, 13], although the change of horizon topology is not classically allowed. However, we can continue adding angular momentum, which have to be conserved, if the cosmic censorship conjecture is correct there must be the way for the black hole to avoid naked singularity.

Algebraic type

The spacetimes of rotating black ring do not admit any WAND , yet if consider only on the event horizon, its Weyl tensor is typeII. However, when the ring is static ($\nu = 0$ which is unlikely physical), it can have typeI_i Weyl tesor[15]. Let us invatigate some more details on typeII

horizon. First of all, we need to make coordinates transformation,

$$
\begin{aligned}
d\chi &= d\psi + \frac{\sqrt{-F(y)}}{G(y)}dy \qquad (5.33)\\
dv &= dt - R\sqrt{\lambda\nu}(1+y)\frac{\sqrt{-F(y)}}{G(y)}dy, \qquad (5.34)
\end{aligned}
$$

where the remaining coordinates stay the same. We transform the metric into coordinates (v, χ, y, x, ϕ), so it is regular on the horizon,

$$
\begin{aligned}
ds^2 &= -\frac{F(x)}{F(y)}\left(dv + R\sqrt{\lambda\nu}(1+y)d\chi\right)^2 \\
&+ \frac{R^2}{(x-y)^2}\left[-F(x)(G(y)d\chi^2 - 2\sqrt{-F(y)}d\chi dy) + F(y)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right)\right], \qquad (5.35)
\end{aligned}
$$

Then it can be shown that the null Killing vector, $l = \partial_v - \sqrt{\frac{\nu}{\lambda}}\frac{1}{R(1+\nu)}\partial_\chi$, is a WAND. To verify that l satisfies $K_{\mu\rho\alpha} = l^\nu l^\sigma C_{\mu\nu\sigma[\rho}l_{\alpha]} = 0$, one needs to check 15 independent components of $K_{\mu\nu\rho}$ which require powerful computing effort since the Weyl tensor is very complicated even on the event horizon (list of some manipulable curvature tensor components is in apendix C). The full calculation of Weyl tensor and its algebraic type were performed in Mathematica using the package from [31]. However, there are some calculation that can be carried out by hand,

$$
\begin{aligned}
K_{y\phi y} &= (C_{yvv\phi} + l^\chi C_{y\chi v\phi} + l^\chi C_{yv\chi\phi} + (l^\chi)^2 C_{y\chi\chi\phi})l_y, \qquad (5.36)\\
K_{v\phi y} &= (C_{vvv\phi} + l^\chi C_{v\chi v\phi} + l^\chi C_{vv\chi\phi} + (l^\chi)^2 C_{v\chi\chi\phi})l_y, \\
&= (l^\chi C_{v\chi v\phi} + (l^\chi)^2 C_{v\chi\chi\phi})l_y, \qquad (5.37)\\
K_{vxy} &= (C_{vvvx} + l^\chi C_{v\chi vx} + l^\chi C_{vv\chi x} + (l^\chi)^2 C_{v\chi\chi x})l_y, \\
&= (l^\chi C_{v\chi vx} + (l^\chi)^2 C_{v\chi\chi x})l_y, \qquad (5.38)\\
K_{\chi xy} &= (C_{\chi vvx} + l^\chi C_{\chi\chi vx} + l^\chi C_{\chi v\chi x} + (l^\chi)^2 C_{\chi\chi\chi x})l_y, \\
&= (C_{\chi vvx} + l^\chi C_{\chi v\chi x})l_y. \qquad (5.39)
\end{aligned}
$$

The components $C_{\chi vvx}, C_{\chi v\chi x}$ and those with ϕ appear only once vanish, thus all (5.39)-(5.39) vanish everywhere(not only on event horizon). The rest 11 terms only vanish at $y = 1/\nu$. Note that the divergence scalar vanish since the WAND is a constant vector, thus, this result does not contradict to theorem 3.2.1. Therefore, the rotating black rings are algebraically special only in certain area.

5.5 Black Saturn

Black saturn is the black hole with disconnected event horizon, one is spherical and another is toroidal. The saturn is stationary axisymmetric solution botained from inverse scattering method, its rod structure is shown in figure 4. Note that the parameters a_i was rescaled for convenience, since it does not change the solution, from now we will assume that $a_1 = 0 \leq a_5 \leq a_4 < a_3 \leq 1 = a_2$(in this case the choice $a_3 = a_4$ is forbiden to avoid canonical singularity in the solution).

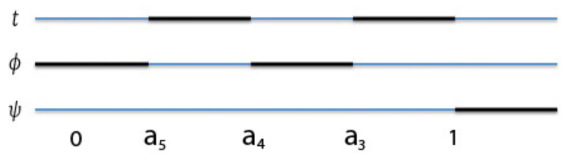

Figure 4: The rod structure of black saturn

The line element of black saturn is given by [18],

$$ds^2 = -\frac{H_x}{H_y}\left[dt - \left(\frac{\omega_\psi}{H_y} + q\right) d\psi\right]^2 + H_x\left[\frac{G_y}{H_y} d\psi^2 + k^2 P\left(d\rho^2 + dz^2\right) + \frac{G_x}{F_x} d\phi^2\right], \qquad (5.40)$$

where,

$$H_x = F^{-1}\left(M_0 + c_1^2 M_1 + c_2^2 M_2 + c_1 c_2 M_3 + c_1^2 c_2^2 M_4\right), \qquad (5.41)$$

$$H_x = F^{-1}\left(M_0 \frac{\mu_1}{\mu_2} + c_1^2 M_1 \frac{\rho^2}{\mu_1 \mu_2} + c_2^2 M_2 \frac{\mu_1 \mu_2}{\rho^2} + c_1 c_2 M_3 + c_1^2 c_2^2 M_4 \frac{\mu_2}{\mu_1}\right), \qquad (5.42)$$

$$G_x = \frac{\rho^2 \mu_4}{\mu_3 \mu_5}, \qquad (5.43)$$

$$G_y = \frac{\mu_3 \mu_5}{\mu_4}, \qquad (5.44)$$

$$P = (\mu_3 \mu_4 + \rho^2)(\mu_1 \mu_5 + \rho^2)(\mu_4 \mu_5 + \rho^2), \qquad (5.45)$$

$$\omega = \frac{c_1 R_1 \sqrt{M_0 M_1} + c_2 R_2 \sqrt{M_0 M_2} + c_1^2 c_2 R_2 \sqrt{M_1 M_4} - c_1 c_2^2 R_1 \sqrt{M_2 M_4}}{F\sqrt{G_x}} \qquad (5.46)$$

The polynomails $F, M_0, M_1, M_2, M_3, M_4$ are defined by,

$$M_0 = \mu_2\mu_5^2(\mu_1-\mu_3)^2(\mu_2-\mu_4)^2(\rho^2+\mu_1\mu_2)^2(\rho^2+\mu_1\mu_4)^2(\rho^2+\mu_2\mu_3)^2, \quad (5.47)$$

$$M_1 = \mu_1^2\mu_2\mu_3\mu_4\mu_5\rho^2(\mu_1-\mu_2)^2(\mu_2-\mu_4)^2(\mu_1-\mu_5)^2(\rho^2+\mu_2\mu_3)^2, \quad (5.48)$$

$$M_2 = \mu_2\mu_3\mu_4\mu_5\rho^2(\mu_1-\mu_2)^2(\mu_1-\mu_3)^2(\rho^2+\mu_1\mu_4)^2(\rho^2+\mu_2\mu_5)^2, \quad (5.49)$$

$$\begin{aligned} M_3 &= 2\mu_1\mu_2\mu_3\mu_4\mu_5(\mu_1-\mu_3)(\mu_1-\mu_5)(\mu_2-\mu_4)(\rho^2+\mu_1^2)(\rho^2+\mu_2^2), \\ &\quad \times(\rho^2+\mu_1\mu_4)(\rho^2+\mu_2\mu_3)(\rho^2+\mu_2\mu_5) \quad (5.50) \end{aligned}$$

$$(5.51)$$

$$M_4 = \mu_1^2\mu_2^2\mu_3^2\mu_4^2(\mu_1-\mu_5)^2(\rho^2+\mu_1\mu_2)^2(\rho^2+\mu_2\mu_5)^2, \quad (5.52)$$

$$(5.53)$$

$$\begin{aligned} F &= \mu_1\mu_5(\mu_1-\mu_3)^2(\mu_2-\mu_4)^2(\rho^2+\mu_1\mu_3)(\rho^2+\mu_1\mu_4)(\rho^2+\mu_2\mu_3) \\ &\quad \times(\rho^2+\mu_2\mu_4)(\rho^2+\mu_2\mu_5)(\rho^2+\mu_3\mu_5)\prod_{i=1}^{5}(\rho^2+\mu_i^2). \quad (5.54) \end{aligned}$$

Parameters c_1, c_2 came from the constant vector introduced in BZ method. The real parameter q is there to ensure that the metric is asymtotically flat and k is an integration constant. Note that q and k are not free parameters and can be determined from asymptotic behavior of the metric [18] but we do not need their exact form here. However, it is important to know that c_1 need to be fixed, i.e. $|c_1| = \sqrt{\frac{2a_3a_4}{a_5}}$ unless there will be leftover singularity in z coordinate from the seed metric.

General properties and limits

As mentioned earlier the black saturn is a combination of two different horizons, H_1 for spherical horizon and H_2 for toroidal horizon but not a combination of perfect MP black hole and black ring; horizon H_i are distorted. However, in appropriated parameter limits that eliminate one singularity black saturn metric can turn into both MP black hole and Black ring with one angular momentum,

$$\begin{aligned} &\text{MP } a_5 \to a_4 \quad \text{and then} \quad a_4 \to 0, \\ &\text{BR } c_2 = 0 \quad \text{and then} \quad a_2 = a_3. \end{aligned}$$

The details are shown in Appendix of [18]. Note that both limits are not continuous, for MP black hole limit, if we take $a_5 \to 0$ then c_1 blows up and then the metric blows up before we can get any sensible solution. In the case of black ring after the limits were taken, the region that used to be singularity just become smooth (MP black hole singularity).

On each horizon there is a null killing vector $\partial_t + \Omega_i \partial_\psi$,

$$\Omega_1 = (1+a_4)\sqrt{\frac{a_4 a_5}{2a_3}}\frac{a_5(1-a_3)-a_3(1-a_4)(1-a_5)\bar{c}_2}{a_5(1-a_3)+a_3a_4(1-a_4)(1-a_5)\bar{c}_2^2} \tag{5.55}$$

$$\Omega_2 = (1+a_4)\sqrt{\frac{a_3 a_5}{2a_4}}\frac{a_5-a_4(1-a_5)\bar{c}_2}{a_5-a_5(a_3-a_4)\bar{c}_2+a_3a_4(1-a_5)\bar{c}_2^2}, \tag{5.56}$$

where $\bar{c}_2 = \frac{c_2}{c_1(1-a_4)}$. Althogh this metric is very complicated, from the agular velocity Ω we can see that this solution only rotates in one plane so it is not the most general solution for black saturn in 5D.

Algebraic type

For the time being, there is no literature on algebraic type of black saturn possibly because of its complication; one needs to use numerical analysis only to verify that it is Ricci flat. Therefore We can only make some observation. Since neither of parameter limits of black saturn to MP black hole and black ring are continuous, we can't apply the results from previous calculations. However, recall that in Myers-Perry black hole and Black ring cases, the null Killing are WAND on the horizon hence it might be a good start to show that whether or not Killing vectors of black saturn is WAND, even so that might require some appropriate numerical method.

5.6 Non-asymptotically flat solutions

The well-known example of non-asymptotically flat vacuum solution are black strings/p-branes which are the product of the black hole and a p dimensional space, such as $\mathbb{R}^p, S^p$ etc. Hence the objects possess the string/brane-like singularity. One example is the product of Schwarzschild spacetime M_{BH} and real line $\mathbb{R}$ (static black string) which the line element can be written as,

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\Omega_2^2 + dy^2, \tag{5.57}$$

where y is added coordinate. From this metric, it is clear that the sigularity at $r = 0$ is now extended along y direction and makes this spacetime non-asymtotically flat.

The Kaluza-Klein black string can be obtained by compactify $\mathbb{R}$ and the spacetime becomes $M_{BH} \times S^1$. Although, the existence of black string breaks the Lorentz symmetry (boost in y direction), the compactness of Kaluza-Klein spacetime provides us new $U(1)$ symmetry. The breaking symmetry can be restored by introducing a gauge field, as the representation of $U(1)$ Lie algebra, in the metric [25]. Consider the boost $(t, y) \mapsto (t', y') = (t\cosh\alpha - y\sinh\alpha, y\cosh\alpha - t\sinh\alpha)$ define,

$$A_t = -\frac{m\cosh\alpha\sinh\alpha}{r + 2m\sinh^2\alpha} \tag{5.58}$$

$$e^{-4\phi/\sqrt{3}} = 1 + \frac{2m\sinh^2\alpha}{r}. \tag{5.59}$$

Scalar field ϕ is the result of dimensional reduction of form to 5D standard volume form into (4+1)D standard volume (wedge of all dx^μ with factor $\sqrt{-g}$) . The line element then can be written in the invariant form,

$$ds^2 = e^{-4\phi/\sqrt{3}}(dy + 2A_\mu dx^\mu)^2 + e^{2\phi/\sqrt{3}}\tilde{g}_{\mu\nu}dx^\mu dx^\nu. \tag{5.60}$$

The metric $\tilde{g}_{\mu\nu}$ form is similar to the Schwarzschild metric,

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -\frac{r - 2m}{R}dt^2 + (\frac{r - 2m}{R})^{-1}dr^2 + Rrd\Omega_2^2, \tag{5.61}$$

where $R = \sqrt{r^2 + 2m(\cosh^2 \alpha - 1)}$. The Wick rotation $y \mapsto iy$, $A_\mu \mapsto iA_\mu$ of this metric yields "bubbles" which are point-like regular solutions in Kaluza-Klein spacetime [7]. If instead one generalized 4D dilation C-metric into Kaluza-Klein spacetime[3, 6, 7] and then performing the transformation $t \mapsto i\psi$, $\psi \mapsto it$ one obtians rotating black ring [9, 27]. This is the original approach toward the discovery of black ring.

Although, CMPP classification can not be used on this kind of black holes, it is worth to have some ideas about them. There is an attemp to classify the Weyl tensor on five dimension Kaluza-Klein spacetime [10] however the De Smet approach only works for five dimension and does not seem to lead to generalized version of Goldburg-Sachs theorem [27].

6. Summary and Discussion

We have investigated many higher dimensional black holes constructed from different methods. We have seen that the extension of AS solutions in 4D i.e. Schwarzschild-Tangherlini black hole, RT black hole and MP black are all algebraically special of TypeD in CMPP classification. While only some variation of black ring, namely the static black ring admit WAND at every points and was classified as typeI_i. In case of one plane rotating black ring, there is no global WAND but locally on event horizon the Weyl tensor is typeII. The black saturn is too complicated for the exact calculation but an observation was made that it might be useful to determin the algebraic type of the horizon first. The algebraic property of disconnected black hole might help us understand how the null vector behave on different horizon. In the last part of chapter 4, we briefly discuss the construction of Kaluza-Klein black string, which gave the idea of how the back ring was originally discovered.

Acknowledgement

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A WAND of RT black hole

From the null vectors we chose in section 5.2, let us first look at the equation for vector l. The only non zero component of l is $l^r = 1$(and $l_u = 1$ for covector),

$$C_{\sigma rr[\rho}l_{u]} = C_{\sigma rri} = 0. \tag{A1}$$

For vector n the equation get slightly more complicated because $n = \partial_u - H\partial_r$ and covector $n^* = -Hdu - dr$, both have two non-vanishing componets,

$$\begin{aligned} K_{\sigma\rho\alpha} &\equiv n^\mu n^\nu C_{\sigma\mu\nu[\rho}n_{\alpha]} \\ &= C_{\sigma uu[\rho}n_{\alpha]} - H\left(C_{\sigma ur[\rho}n_{\alpha]} + C_{\sigma ru[\rho}n_{\alpha]}\right) + H^2 C_{\sigma rr[\rho}n_{\alpha]}. \end{aligned} \tag{A2}$$

Observe that K is anti-symmetric in the last two indices, thus $K_{\mu\alpha\alpha} = 0$. Furthermore, all Weyl component with only one spatial index vanish and $n_i = 0$, hence,
$K_{iur} = K_{iru} = K_{uri} = K_{rui} = K_{rij} = K_{uij} = K_{ijk} = 0$. Let us consider other non-trivial components separately,
$i)$ $\sigma = i, \rho = j$

$$\begin{aligned} K_{ij\alpha} &= C_{iuu[j}n_{\alpha]} - 2HC_{iur[j}n_{\alpha]} \\ &= 2HC_{iur[j}n_{\alpha]} - 2HC_{iur[j}n_{\alpha]} = 0. \end{aligned} \tag{A3}$$

Then from these vanishing components imply, $K_{iuj} = K_{irj} = 0$.

$ii)$ $\sigma = u, r$ and $\rho = r, u$.

$$\begin{aligned} K_{uu\alpha} &= -HC_{uru[u}n_{\alpha]} + H^2 C_{urr[u}n_{\alpha]}, \\ &= -HC_{uru[u}n_{r]} + H^2 C_{urr[u}n_{r]}, \\ &= HC_{urur}n_u + H^2 C_{urru}n_r = 0. \end{aligned} \tag{A4}$$

$$\begin{aligned} K_{rr\alpha} &= C_{ruu[r}n_{\alpha]} - HC_{rur[r}n_{\alpha]}, \\ &= C_{ruu[r}n_{u]} - HC_{rur[r}n_{u]}, \\ &= C_{ruur}n_u + HC_{ruru}n_r = 0. \end{aligned} \tag{A5}$$

From above result we also have $K_{uru} = K_{uiu} = K_{rur} = K_{rir} = K_{uir} = K_{riu} = 0$. Therefore n is typeII WAND.

B Curvature tensor and Ricci tensor of Myers-Perry Black hole

First of all consider the optical matrix L_{ij}, the Christoffel term of covariant derivative of l^μ vanish when contracting with m_i^μ, m_j^ν hence,

$$L_{ij} = l_{\mu};_{\nu}\, m_i^\mu m_j^\nu = l_{\mu},_{\nu}\, m_i^\mu m_j^\nu \tag{B1}$$

Ricci tensor

Let us substitute metric (5.13) into equation (5.23), from Christoffel connection define in (5.24), observe that $\Gamma^\nu_{\mu\nu} = 0$, hence, the Ricci tensor takes the form,

$$\begin{aligned} R_{\mu\nu} &= \Gamma^\sigma_{\mu\nu},_\sigma - \Gamma^\sigma_{\alpha\nu}\Gamma^\sigma_{\mu\sigma} \\ &= \eta^{\sigma\rho}(hl_\mu l_\nu),_{\sigma\rho} - (hl^\sigma l_\mu),_{\nu\sigma} - (hl^\sigma l_\nu),_{\mu\sigma} \\ &\quad +2h\left[D^2 h + Dh(l^\sigma,_\sigma) - h(l^\sigma,_\rho\, l^\rho_, \sigma - \eta^{\rho\alpha} l^\sigma,_\rho\, l_\sigma,_\alpha)\right] \end{aligned} \tag{B2}$$

Curvature tensor

From this expression in (5.23), the second derivative terms are quite easy to deal with, so let us begin by consider the connection term. In term of $h(x)$ and l^α we have,

$$\begin{aligned}\Gamma_{\lambda\mu\sigma}\Gamma^{\lambda}_{\rho\nu} \quad = \quad & -Dhl_\mu l_\rho\,(h,_\nu l_\sigma + h,_\sigma l_\nu + hl_{\sigma},_{\nu} + hl_{\nu},_{\sigma} - 2hDhl_\sigma l_\nu) \\ & + h^2(l^\lambda,_\nu l_\sigma + l^\lambda,_\sigma l_\nu)(l_\lambda,_\mu l_\rho + l_\lambda,_\rho l_\mu) \\ & - h(hl_\mu l_\rho),_\lambda (l^\lambda,_\nu l_\sigma + l^\lambda,_\sigma l_\nu) - h(hl_\mu l_\sigma),_\lambda (l^\lambda,_\nu l_\rho + l^\lambda,_\rho l_\nu) \\ & - Dhl_\sigma l_\nu\,(h,_\mu l_\rho + h,_\rho l_\mu + hl_{\rho},_{\mu} + hl_{\mu},_{\rho}) + (hl_\sigma l_\nu),_\lambda (hl_\mu l_\rho),^\lambda .\end{aligned} \tag{B3}$$

The expression might looks tedious but observe that,

$$\Gamma_{\lambda\mu[\sigma}\Gamma^{\lambda}_{\rho]\nu} l^\mu = 0. \tag{B4}$$

Therefore we can drop this Christoffel connection terms for the Curvature tensor components in tetrad basis that containing ‘0’ index. This connection term also vanishes when contract with tensor $m^\sigma_i m^\rho_j m^\mu_k$. Hence the only curvature tensor component that contains non-vanishing connection term is R_{1i1j},

$$2\Gamma_{\lambda\mu[\sigma}\Gamma^{\lambda}_{\rho]\nu} n^\mu m^\rho_j n^\sigma m^\nu_i = 2hDhl_{\rho,\nu} m^\nu_{(i} m^\rho_{j)} + h^2(l^\lambda_{,\nu} l_{\lambda,\rho} - 2l^\lambda_{,(\nu} l_{\rho),\lambda} + l_{\nu,\lambda} l_{\rho,}{}^{\lambda}) m^\nu_i m^\rho_j. \tag{B5}$$

Next consider the second derivative terms and expand one of them explicitly,

$$\begin{aligned}(hl_\mu l_\sigma)_{,\nu\rho} \quad = \quad & h_{,\nu\rho} l_\mu l_{sigma} + h,_\nu (l_\mu,_\rho l_\sigma + l_\mu l_\sigma,_\rho) + h,_\rho (l_\mu,_\nu l_\sigma + l_\mu l_\sigma,_\nu) \\ & + h\,(l_\mu,_{\nu\rho} l_\sigma + l_\mu l_\sigma,_{\nu\rho} + l_\mu,_\nu l_\sigma,_\rho + l_\mu,_\rho l_\sigma,_\nu) .\end{aligned} \tag{B6}$$

The other three terms are obtianed by permuting the indices. Now consider curvature tensor in tetrad basis, using (3.1) and fact that l^μ is null geodesic vector which yeilds $l_\mu,_\nu l^\mu = l_\mu,_\sigma l^\sigma =$

$l^2 = 0$, then,

$$\begin{aligned}
\left(hl_\mu l_\sigma\right)_{,\nu\rho} l^\mu l^\sigma m_i^\nu m_j^\rho &= m_i^\nu m_j^\rho h_{,\nu\rho} l^2 l^2 + h_{,\nu} l^2 \left(l_{\mu,\rho} + l_{\sigma,\rho}\right) + h_{,\rho} l^2 \left(l_{\mu,\nu} + l_{\sigma,\nu}\right) \\
&\quad + h\left(l_{\mu,\nu\rho} l^2 + l_{\sigma,\nu\rho} l^2 + 2(l_{\mu,\nu} l^\mu)(l_{\sigma,\rho} l^\sigma)\right) \\
&= 0. \qquad \text{(B7)}
\end{aligned}$$

By permuting indices, one also finds that,

$$\left(hl_\rho l_\nu\right)_{,\mu\sigma} l^\mu l^\sigma m_i^\nu m_j^\rho = \left(hl_\nu l_\sigma\right)_{,\mu\rho} l^\mu l^\sigma m_i^\nu m_j^\rho = \left(hl_\mu l_\rho\right)_{,\nu\sigma} l^\mu l^\sigma m_i^\nu m_j^\rho = 0. \qquad \text{(B8)}$$

All together we have,

$$R_{0i0j} = R_{\mu\nu\sigma\rho} l^\mu m_i^\nu l^\sigma m_j^\rho = 0. \qquad \text{(B9)}$$

. Likewise, $R_{0ijk} = R_{010i} = 0$.

$$\begin{aligned}
R_{0101} &= -D^2 h \qquad \text{(B10)} \\
R_{01ij} &= 2DhL_{[ij]} + 2hl^\mu l_{\rho,\mu\sigma} m^\sigma_{[i} m^\rho_{j]} \qquad \text{(B11)} \\
R_{0i1j} &= -DhL_{ij} - 2hl^\mu l_{[\sigma,\mu]\rho} m_i^\sigma m_j^\rho \qquad \text{(B12)} \\
R_{011i} &= m_i^\rho l^\mu h_{,\mu\rho} - Dhn^\sigma m_i^\rho (2l_{[\sigma,\rho]} + l_{\sigma,\rho}) - hl^\mu n^\sigma m_i^\rho (2l_{\sigma,\mu\rho} - l_{\rho,\mu\sigma} - l_{\mu,\sigma\rho}) \qquad \text{(B13)} \\
R_{1ijk} &= 2\delta_i hL_{[jk]} + 2\delta_{[k} hL_{j]i} \\
&\quad + 2h(l_{\sigma,\nu\rho} m_i^\nu m^\sigma_{[j} m^\rho_{k]} + l_{\mu,\nu} n^\mu m_i^\nu L_{[jk]} + l_{\mu,\nu} n^\mu m_j^\nu L_{[ik]}) \qquad \text{(B14)} \\
R_{ijkl} &= 4h(L_{[ji]} L_{[kl]} + m^\nu_{[j} m^\mu_{i]} l_{\nu,\rho} l_{\sigma,\mu} m^\sigma_{[l} m^\sigma_{k]}) \qquad \text{(B15)} \\
R_{1i1j} &= -h_{\nu,\rho} m_i^\nu m_j^\rho - 2\Delta hL_{(ij)} + 4L_{1(j}\delta_{i)} h - 2\delta_{(i} hL_{j)1} + h[2l_{\mu,\nu\rho} n^\mu m_i^\nu m_j^\rho \\
&\quad - 2l_{\rho,\mu\sigma} n^\mu m^\rho_{(i} m^\sigma_{j)} - 2L_{11} L(ij) + 2L_{1(i} L_{j)1} - 2L_{1i} L_{1j} - 2L_{(j|1|} L_{i)1} \\
&\quad + 2DhL(ji) + hm_i^\nu m_j^\rho (l^\lambda_{,\nu} l_{\lambda,\rho} - 2l^\lambda_{,(\nu} l_{\rho),\lambda} + l_{\nu,\lambda} l^\lambda_\rho)] \qquad \text{(B16)}
\end{aligned}$$

C Curvature tensor of Black ring

The linearly independent nonvanishing curvature tensor components (which is also Weyl tensor) of neutral black ring are the following,

$$R_{vyvx} = \frac{\lambda(1-\lambda x)}{(x-y)(1-\lambda y)^2} \quad , \quad R_{vx\chi y} = R\sqrt{\lambda\nu}\frac{(1+\lambda)(1-\lambda x)}{(x-y)(1-\lambda y)^2} \tag{C1}$$

$$R_{v\chi yx} = -R\sqrt{\lambda\nu}\frac{(1-\lambda x)}{2(x-y)(1-\lambda y)} \quad , \quad R_{vy\chi x} = R\sqrt{\lambda\nu}\frac{(1-\lambda x)(1-\lambda(2+y))}{2(x-y)(1-\lambda y)^2} \tag{C2}$$

$$R_{y\phi x\phi} = R^2\lambda\frac{(1-x^2)(1-\nu x)(1-\lambda y)}{2(x-y)^3(1-\lambda x)} \quad , \quad R_{y\phi y\phi} = -R\lambda\frac{(1-x^2)(1-\nu x)}{2(x-y)^3} \tag{C3}$$

$$R_{vyvy} = -\frac{\lambda(1-\lambda x)^2}{(x-y)(1-\lambda y)^3} \quad , \quad R_{yxyx} = R^2\frac{\lambda(1-\lambda x)}{2(x-y)^3(1-x^2)(1-\nu x)} \tag{C4}$$

$$R_{\chi yvy} = -R\sqrt{\lambda\nu}\frac{(1+\lambda)(1-\lambda x)^2}{(x-y)(1-\lambda y)^3} \quad , \quad R_{\chi yyx} = -R^2\frac{\lambda(1-\lambda x)}{2(x-y)^3\sqrt{-1+\lambda y}} \tag{C5}$$

$$R_{yxxv} = R^2\frac{\sqrt{\lambda\nu}(1-\lambda x)}{2(1-x^2)(1-\nu x)(x-y)\sqrt{-1+\lambda y}} \quad , \quad R_{v\phi\phi y} = R\frac{\sqrt{\lambda\nu}(1-x^2)(1-\nu x)}{2(x-y)\sqrt{-1+\lambda y}} \tag{C6}$$

$$R_{v\chi v\chi} = -\lambda\frac{(1-y^2)(1-\nu y)(\nu-\lambda+2x-(\lambda+3\nu)x^2+2\lambda\nu x^3)}{4(1-\lambda y)^3} \tag{C7}$$

$$R_{yvv\chi} = -\lambda\frac{(\nu-\lambda+2x-(\lambda-3\nu)x^2+2\lambda\nu x^3)}{4(1+\lambda y)^{5/2}} \tag{C8}$$

$$R_{v\chi\chi y} = R\frac{\lambda\sqrt{\lambda\nu}(1+y)(\nu-\lambda+2x-(\lambda-3\nu)x^2+2\lambda\nu x^3)}{4(1+\lambda y)^{5/2}} \tag{C9}$$

$$\begin{aligned} R_{v\phi v\phi} &= \frac{\lambda(1-x^2)(1-\nu x)}{4(1-\lambda x)^2(1-\lambda y)^2}[(\lambda-\nu)(1-2\lambda x+x^2)-y(2+\lambda(\nu-\lambda)-2x(\nu+\lambda)+\lambda(\nu+\lambda)x^2) \\ &\quad +2\nu(1-\lambda x)^2y^2] \end{aligned} \tag{C10}$$

$$\begin{aligned} R_{\chi y\chi x} &= R^2\frac{\lambda(1+y)(1-\lambda x)}{2(x-y)^3(1-\lambda y)^2}[-1+\nu(3+2\lambda)x^2+y(1+\lambda+\nu+2\nu(3+2\lambda)x+\lambda\nu x^2) \\ &\quad y^2(2\nu+\lambda(-1+\nu+2\nu x))] \end{aligned} \tag{C11}$$

The other non-vanishing linearly independent terms are, $R_{\chi\phi\chi\phi}, R_{v\phi\chi\phi}, R_{\chi\phi y\phi}, R_{\chi y\chi y}, R_{\chi y\chi x}$, $R_{xvxv}, R_{\chi xvx}, R_{x\chi x\chi}, R_{xyx\chi}$ but they are too complicated to write down. Other terms that is not related to above component by symmetry of curvature tensor vanish.

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