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The AdS/CFT correspondence and condensed matter physics

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Contents

Chapter 1

Introduction

The AdS/CFT correspondence is one of the most influential conjectures that has been discovered recently in theoretical physics. It was first stated by Juan Maldacena [16] for a particular highly symmetric quantum field theory called $\mathcal{N} = 4$ super Yang-Mills in four space-time dimensions, and a string theory called type IIB formulated on a particular space-time background. At a basic level it states that a quantum theory of gravity, such as string theory, is dual to a lower dimensional quantum field theory without gravity.

Just the statement alone opens up many speculative ideas and many questions. For instance this could be a way of learning more about quantum gravity, a field that it notoriously challenging with many candidate theories [5]. Another possible use of the correspondence is to use it in the opposite direction: to use developments in string theory to describe certain classes of quantum field theories. This is the line of argument of this dissertation, which will consider quantum field theories which describe condensed matter systems under very specific physical conditions [26].

The condensed matter systems in question will be those that whose behavior can be described by the boson-Hubbard model, namely bosons whose dynamics are confined to sites of a lattice [28, section9]. The reason that the boson-Hubbard model is of interest is because at a certain quantum critical point between quantum phases of matter the system can be described by a quantum field theory with just the right symmetries to apply the AdS/CFT correspondence [29]. It is a remarkable duality between gravitational systems and ultra-cold atoms, which is only possible because of the loss of a physical length scale in the condensed matter system. It is this that allows the duality to be conjectured, and the reason that systems that naively have completely different physical distance scales are actually identical by the absence of scale.

The structure of this dissertation is to introduce the AdS/CFT correspondence in section 1 in a way that makes it easier to apply to condensed matter systems. Issues regarding the degrees of freedom, the Weinberg-Witten theorem [37] which appears to forbid such a duality, and the describing the extra dimensions of the gravity theory will be discussed.

Section 2 will introduce the boson-Hubbard model and show that the low energy limit of the model can be described by a particular type of quantum field theory amenable to a gravity description via AdS/CFT. Finally the duality will be used to predict physical properties of the system that other theories have been unable to provide [29]. The dissertation will conclude with additional developments that have been made, and scope for further research in this field.

Chapter 2

The AdS/CFT correspondence: a review

This section will serve to motivate the AdS/CFT correspondence by asking questions around the idea of a gauge-gravity duality. The first few sections will introduce the Weinberg-Witten theorem and the holographic principle which places restrictions on the form such a duality could take, such as the gravity theory having more space-time dimensions than the field theory. After introducing the 't Hooft large N limit the following sections will match the extra dimensions using the renormalisation group and the symmetries of the two theories by observation. The concluding section considers what conditions a QFT needs to have a gravity dual with an asymptotically AdS background besides conformal invariance.

2.1 The Weinberg-Witten theorem

When quantizing gravity it is inevitable that a spin 2 particle will emerge from the procedure commonly referred to as the graviton. When comparing a quantum gravity theory with a QFT a natural question would be is it possible to have a spin 2 particle in a QFT? In principle it seems one could form a bound state of say two gauge bosons such that they behave like a spin 2 particle. The theorem first proved by Steven Weinberg and Edward Witten [37] sheds light on this question:

Theorem 2.1.1 *A QFT with a Poincaré covariant conserved stress tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ which carry momentum (i.e. $P^\mu = \int d^3x \, T^{0\mu} \neq 0$.)*

An outline of the proof the Weinberg-Witten theorem will be provided here based on the arguments of the original paper [37] and a review by Florian Loebbert [15]. Additional details, especially those concerning normalization of one particle states can be found in the aforementioned review. The proof will be divided into two parts, the first will *(i)* show that the matrix elements corresponding to the the tensor T:

$$\langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle \tag{2.1.1}$$

cannot vanish in the limit in which $p' \to p$. The second part *(ii)* will show that the same matrix elements must vanish under the assumptions of the theorem if the helicity of massless particles and $j > 1$ for the matrix elements associated with T.

(i) To show the first result we will make use of the 4-momentum operator associated with $T^{\mu\nu}$ in the statement of the theorem. If the massless particles are charged under T then their eigenvalues with respect to T must be non-zero. This will be one of the crucial assumptions of the theorem.

We will define the 4-momentum eigenvalues of pure momentum 1-particle massless states as:

$$P^\mu | p, \pm j \rangle = p^\mu | p, \pm j \rangle. \tag{2.1.2}$$

All 1-particles states can be described as linear combinations of these pure momentum eigenstates with their momentum given by a linear combination of the same eigenstates' eigenvalues. We can then define the matrix elements above as follows:

$$\langle p', \pm j|P^{\mu}|p, \pm j\rangle = p^{\mu}\delta_a^{(3)}(\mathbf{p'} - \mathbf{p}) \qquad (2.1.3)$$

where we have normalized the momentum eigenstates using δ_a the nascent delta function defined in terms of the delta function as $\lim_{a\to 0}\delta_a^{(3)}(\mathbf{p'}-\mathbf{p}) = \delta^{(3)}(\mathbf{p'} - \mathbf{p})$. Normalizing using nascent delta functions 'smears out' the single particle state excitations allowing a more physical interpretation of results found using this convention as opposed to using ordinary δ-functions. For more details see [15]. The use of smeared out delta-functions corresponds to integrating not over all space, but finite volumes. The definition of the nascent delta-function above is defined by integrating it over a ball of radius $1/a$ which becomes an infinite sized ball in the limit $a \to 0$. We will use δ_a to define the 4-momentum operator as in the theorem over a finite volume:

$$\begin{aligned}
\langle p', \pm j|P^{\mu}|p, \pm j\rangle &= \int_{V_a} d^3x \langle p', \pm j|e^{i\mathbf{P}\cdot\mathbf{x}}T^{0\mu}(t, \mathbf{0})e^{-i\mathbf{P}\cdot\mathbf{x}}|p, \pm j\rangle \\
&= \int_{V_a} d^3x e^{i(\mathbf{p'}-\mathbf{p})\cdot\mathbf{x}}\langle p', \pm j|T^{0\mu}(t, \mathbf{0})|p, \pm j\rangle \\
&= (2\pi)^3\delta_a^{(3)}(\mathbf{p'} - \mathbf{p})\langle p', \pm j|T^{0\mu}(t, \mathbf{0})|p, \pm j\rangle . (2.1.4)
\end{aligned}$$

If we compare this result with our previous result **??** we obtain the identity:

$$\lim_{p'\to p}\langle p'|T^{0\mu}|p\rangle = \frac{p^{\mu}}{(2\pi)^3}. \qquad (2.1.5)$$

We can generalize this result using Lorentz covariance:

$$\lim_{p'\to p}\langle p'|T^{\mu\nu}|p\rangle = \frac{p^{\mu}p^{\nu}}{E(2\pi)^3}. \qquad (2.1.6)$$

Because we assumed that the eigenvalues of our one-particle states were non-zero these matrix elements must be non-zero.

(ii) To show the second result we will first consider this relation for light-like momenta p' and p:

$$\begin{aligned} (p'+p)^2 &= (p')^2 + p^2 + 2(p'\cdot p) = 2(p'\cdot p) \\ &= 2(\mathbf{p'}\cdot\mathbf{p} - |\mathbf{p'}||\mathbf{p}| \\ &= 2|\mathbf{p'}||\mathbf{p}|(cos(\phi) - 1) \leq 0. \end{aligned} \tag{2.1.7}$$

where ϕ is the angle between 3-momenta $\mathbf{p'}$ and $\mathbf{p}$. Because we are considering the limit $p^{'} \to p$ the case of $\phi = 0$ is not applicable, hence throughout the rest of this proof we will assume ϕ is non-zero. First by Poincaré covariance of our states we will choose a frame such that they have opposite 3-momentum: $p = (|\mathbf{p}|, \mathbf{p})$, $p' = (|\mathbf{p}|, -\mathbf{p})$. Next we will consider a rotation of θ about the 3-momentum $\mathbf{p}$ axis. The one-particle states transform as:

$$\begin{aligned} |p, \pm j\rangle &\to e^{\pm i\theta j}|p, \pm j\rangle \\ |p', \pm j\rangle &\to e^{\mp i\theta j}|p', \pm j\rangle. \end{aligned} \tag{2.1.8}$$

Thus the matrix elements transform as:

$$\langle p', \pm j|T^{\mu\nu}(t, \mathbf{0})|p, \pm j\rangle \to e^{\pm 2i\theta j}\langle p', \pm j|T^{\mu\nu}(t, \mathbf{0}|p, \pm j\rangle. \tag{2.1.9}$$

Another way we can define a rotation is as fundamental representations of the Lorentz group acting on the matrix T. We can compare the first and this description of the rotation in the same way we compare active and passive transformations. The first description of the rotation transforms the basis states, whilst the second description transforms the operators that act upon the states. The second description can be written as:

$$\langle p', \pm j|T^{\mu\nu}(t, \mathbf{0})|p, \pm j\rangle \to \Lambda(\theta)^{\mu}{}_{\rho}\Lambda(\theta)^{\nu}{}_{\sigma}\langle p', \pm j|T^{\rho\sigma}(t, \mathbf{0}|p, \pm j\rangle. \tag{2.1.10}$$

Equating both transformation descriptions:

$$e^{\pm 2i\theta j}\langle p', \pm j|T^{\mu\nu}(t, \mathbf{0}|p, \pm j\rangle = \Lambda(\theta)^{\mu}{}_{\rho}\Lambda(\theta)^{\nu}{}_{\sigma}\langle p', \pm j|T^{\rho\sigma}(t, \mathbf{0}|p, \pm j\rangle. \quad (2.1.11)$$

We now note that because Λ is a representation of a rotation in the Lorentz group its eigenvalues can only be either $e^{i\theta}$ or 1. This constrains the values j can possibly take to either 0, 1/2, or 1, for any other value the matrix elements of T vanish. We chose a reference frame for our analysis, but using Poincaré covariance of T and Poincaré invariance of helicities this result carries over to all Lorentz frames of reference.

Hence for a Poincaré covariant tensor $T^{\mu\nu}$ it follows that:

$$\lim_{p'\to p}\langle p', \pm j|T^{\mu\nu}|p, \pm j\rangle = 0 \quad for \;\; j > 1. \qquad (2.1.12)$$

Hence the first result shows that $\langle p'|T^{\mu\nu}|p\rangle$ cannot vanish in the limit $p^{'} \to p$ and the second result show that $\langle p'|T^{\mu\nu}|p\rangle$ must vanish if $j > 1$, hence we obtain the result that massless particles with helicity $j > 1$ vanish in a Poincaré covariant QFT with covariant $T^{\mu\nu}$.

This appears to forbid any spin-2 particle, even bound states, which rules out the graviton, since any particle with spin-2 shares all the properties of the graviton [15, page 18]. However, there is a window provided by general relativity. For any stress-energy tensor of a gravity theory that obeys Einstein's field equations $T^{\mu\nu} \propto \frac{\delta S}{\delta g_{\mu\nu}}$ where S is the Einstein-Hilbert action. This vanishes by the equations of motion of the gravity theory. Thus the stress energy tensor for the full gravity theory vanishes and so the Weinberg-Witten theorem no longer applies since a non-vanishing stress-energy tensor was assumed.

We can still have massless particles with momentum since the stress-energy tensor is the sum of the stress-energy tensor of the QFT plus the stress-energy of just pure gravity, $T^{\mu\nu} = T^{\mu\nu}_{QFT} + T^{\mu\nu}_{QG}$. From the holographic principle we will assume that the QFT lives on the boundary of the whole space-time with gravity living in the 'bulk' of the space-time. $T^{\mu\nu}_{QFT}$ is Poincaré covariant on the boundary, but need not be in the bulk, hence the Weinberg-Witten theorem applies on the boundary and so there can be no massless particles of $spin > 2$ there, but in the bulk, since there is no Poincaré covariant tensor except the total one which vanishes, the Weinberg-Witten theorem does not apply. This means a graviton composite particle can exist, but not on the boundary of the space-time where the QFT resides. It must live in the bulk with at least one extra dimension necessary to describe it, corresponding to the radial distance from the boundary.

2.2 The holographic principle

The holographic principle was first proposed by Gerard 't Hooft [35] with additional contributions to bring it into the field of string theory by Leonard Susskind [33]. Also many of the ideas have been discussed by Charles Thorn [33]. The result stems from the work of Bekenstein and Hawking that states that black holes not only have a temperature [12], but also an entropy [2] that is proportional to the black hole's surface area A:

$$S_{BH} = \frac{1}{4}\frac{c^3 k_B}{G\hbar}A \tag{2.2.1}$$

where c is the speed of light, k_B is Boltzmann's constant, G is Newton's gravitational constant, and $\hbar$ is Planck's constant. The only other result that the holographic principle requires is simply that if enough energy accrues within a small enough volume the matter will undergo gravitational collapse and form a black hole. This result can be seen in the canonical theoretical example of the Schwarzschild solution, but also the more general cases of rotating Kerr and charged Reissner-Nordström solutions of Einstein's field equations, see [36, pages 312, 158, and 317] for more details.

Both of these results together imply the holographic principle, that the maximum entropy in a space-time volume is proportional to its surface area.

This is exactly the entropy of the largest black hole that can occupy this volume. To see why this is consider a volume of space-time V with surface area A and entropy $S > S_{BH} \propto A$. Assume also that the energy in this volume is less than that of a black hole (otherwise the energy density would cause gravitational collapse and a black hole would be formed). Now add energy in any form to the volume so that the energy density reaches the value at which a black hole is formed. The configuration we have now has less entropy than the one we started with, since $S > S_{BH}$. This violates the second law of thermodynamics, so rather than do that, 't Hooft assumed that a black hole is the configuration with the highest entropy per unit volume, and thus the maximum entropy in a space-time volume is proportional to the surface area of the volume.

This is far smaller than the entropy of a local QFT on the same space, even with some UV cutoff equivalent to a length scale such as the Planck length l_P. Such a theory would have a number of states $N_s \propto n^{V/l_P^3}$ (where V is the spatial volume of the QFT and n is the number of states per site) with maximum entropy $\propto ln(N_s)$ [18, 33]. Hence for a quantum theory of gravity to be dual to a QFT, the QFT would have to be formulated on a a lower number of dimensions, at least one dimension lower, so that the degrees of freedom can at least in principle be equated.

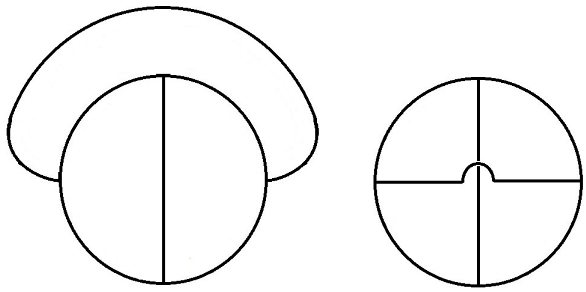
2.3 The 't Hooft limit

The 't Hooft large N limit was first discovered by Gerard 't Hooft in 1974 [34] whilst trying to find a way of simplifying the calculations of quantum chromodynamics. He used the rank of the gauge group as a variable of the theory, hence instead of $SU(3)$ he used $SU(N)$ for N an integer parameter. As the name suggests by taking N to approach infinity the theory drastically simplifies calculations of the theory in an illuminating fashion. Consider the Lagrangian of the theory for a gauge field A^a_μ in the adjoint representation of $SU(N)$:

$$\mathscr{L} = -\frac{1}{4g_{YM}^2} tr(F_{\mu\nu}F^{\mu\nu}) \tag{2.3.1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - i\epsilon^{abc} A^b_\mu A^c_\nu$ such that $\mu, \nu = 0, 1, 2, 3$ and $a = 1, \ldots N^2 - 1$.

The field A has been defined differently from the usual form in the following way $A^a_\mu \rightarrow g_{YM}^{-1} A^a_\mu$ so that in this form the field strength is independent of the gauge coupling. This form will become useful later in this section. If we expand the Lagrangian out we see that the theory has 3-point and 4-point interaction vertices. Now consider the two vacuum diagrams (i) and (ii):



(a) (b)

Figure 2.1: The connected vacuum diagrams associated with four 3-point interactions.

To work out their dependence on N and the coupling g_{YM} we need to establish some rough Feynman rules. From the Lagrangian the kinetic term has a factor of g_{YM}^{-2} in front of it, hence by inverting it to get the propagator we infer that the propagator of the theory is proportional to g_{YM}^2. Each of the interaction vertices has a factor of g_{YM}^{-2} in front of it, hence each interaction vertex is proportional to g_{YM}^{-2}. Finally there is a factor of N given for each sum over colour indices. One can work this out by splitting the diagrams into propagators and vertices, use the Feynman rules given by the Lagrangian, and count the number of sums. Using these

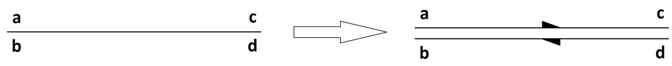
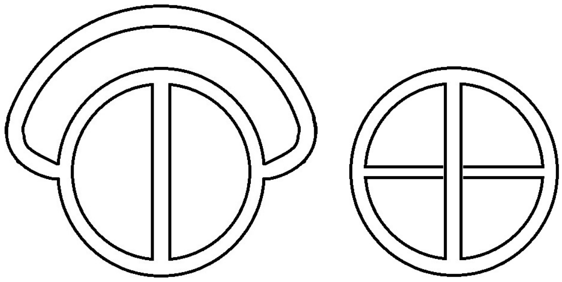


Figure 2.2: A single line representing an adjoint field propagator can be split into a pair of oppositely oriented lines representing a fundamental-anti-fundamental product.

rules diagram (i) has associated with it a factor of $g_{YM}^4 N^4$ and diagram (ii) has a factor of $g_{YM}^4 N^2$.

As N is taken towards infinity it will be useful to take the coupling g_{YM} to zero such that the quantity $g_{YM}^2 N = \lambda$ remains fixed. When we do this 2.1(a) with coefficient $\lambda^2 N^2$ dominates over (b) with coefficient λ^2. The dominance of planar diagrams such as that of (a) over (b) occurs for all diagrams with similar number of vertices and propagators. This is the 't Hooft large N limit. To see how this limit relates to string theory we'll introduce a technique invented by 't Hooft called double line notation to keep track of which classes of diagrams we keep and which we ignore. We consider each gauge field or 'gluon' propagator that transforms in the adjoint representation of $SU(N)$ as a "quark anti-quark" pair that transforms in the product of the fundamental and the anti-fundamental of $SU(N)$, since the adjoint can be thought of as roughly the product of the fundamental and anti-fundamental. Diagrammatically this corresponds to replacing each gluon propagator with two parallel lines with opposite orientations, the fundamental and anti-fundamental going forwards and backwards in time respectively.

This alteration also changes the interaction vertices in the obvious way, so that the vacuum diagrams we drew above can be drawn using double line notation as in 2.3(a) and (b). In this notation we see explicitly that the factors of N correspond to the number of loops in the diagram: there are four in (b), but only two in (b).



(a) (b)

Figure 2.3: The connected vacuum diagrams associated with four 3-point interactions drawn using double line notation.

This notation is said to 'polygonize' the Feynman diagrams, the Feynman diagrams provide a 'polygonization' of a 2-dimensional surface. To make this relation between diagrams and polygons more precise we will relate exactly the vertices of the Feynman diagrams to the vertices of the polygons, the propagators to the edges, and the loops in the double line notation to the faces. This means that any polygon with V vertices, E edges, and F faces corresponds to a Feynman diagram, which according to the Feynman rules above is proportional to:

$$(g_{YM}^2)^E (g_{YM}^{-2})^V (N)^F = \lambda^{E-V} N^{V-E+F}. \tag{2.3.2}$$

Note that the exponent of N, $V - E + F$ is the Euler character χ of the polygon, a well-known topological invariant used for classifying 2-dimensional surfaces. For closed Riemann 2-surfaces $\chi = 2 - 2g$ where g is the genus, the number of holes, associated with the Riemann surface. Hence any perturbation expansion can be written as

$$\sum_{g=0}^{\infty} N^{\chi(g)} \sum_{i=0}^{\infty} c_{g,i} \lambda^i = \sum_{g=0}^{\infty} f_g(\lambda) N^{\chi(g)}. \tag{2.3.3}$$

The expansion is ordered by the Euler character of each contribution. In the large N limit the planar diagrams, diagrams that can be drawn on a sphere, with $g = 0$ will dominate all higher genus contributions. If we compare this expansion to string theory it too involves a double sum in terms of string theory's two parameters: the string coupling g_s and the string tension α'. By comparing both sums we identify g_s with $1/N$ and α' with λ giving a formal analogy between string theory and $SU(N)$ QCD.

A subtlety to point out is that it has been assumed that the gauge group was $SU(N)$, but the expansion has been carried out as if the gauge group is U(N) and not SU(N). This is valid up to leading order since $U(N)$ has N^2 generators whereas $SU(N)$ has N^2-1 generators which can be equated for large N at leading order, but at sub-leading order the difference must be taken into account. Note also that this procedure can be applied for any other gauge group, for example $SO(N)$ or $Sp(N)$, however since these groups do not have conjugate representations the approach used to obtain the 't Hooft limit will differ. The natural question that arises when studying the 't Hooft limit is can $SU(N)$ QCD be constructed as a string theory? So far all efforts have been unsuccessful, but attempting to do so has led to the AdS/CFT correspondence.

2.4 Extra dimensions and the renormalisation group

The Weinberg-Witten theorem seems to effectively forbid the notion of a spin 2 particle in a QFT with Poincaré covariance. To get around it we concluded in accordance with the holographic principle that the spin 2 'graviton' does not live in the QFT, but in a gravity theory formulated in a higher number of space-time dimensions. But how does this extra dimension manifest itself in the QFT? If a QFT and a quantum theory of gravity are dual to each other this additional degree of freedom required by the graviton must have a description in the QFT. The answer to this question makes use of the renormalisation group (RG).

The renormalisation group is related to the procedure known as renormalisation. Renormalisation is a means of dealing with the infinities or ultraviolet (UV) divergences present and seemingly intrinsic to QFTs whenever physical quantities are computed. The problem involves space-time integrals that are necessarily evaluated over all of space-time where quantum fields are defined. By introducing a UV cut-off, usually denoted Λ, it is possible to truncate all energy/momentum space integrals used to the calculate physical observables of the theory. By systematically noting the dependence of all correlation functions on the cut-off Lambda it can be shown whether the QFT depends upon Λ in a finite or infinite number of ways. If the QFT is renormalisable not only is the Λ dependence finite, but also the number of times Λ appears is less than or equal to the number of couplings of the theory. If this is the case the UV divergences can be 'absorbed' into the definitions of the couplings, now called 'bare' couplings, and a new set of finite couplings can be determined called 'renormalised' or 'physical' couplings. These couplings can be measured, so that calculations which depend upon the physical couplings rather than the bare couplings yield finite results for physical quantities. Not only are they finite, in the case of QED they are very accurate [23, page 198].

The cut-off scale Λ is key to the concept of the renormalisation group. Because momentum and position are conjugate variables in quantum theory plus energy and momentum are related in the same way as space and time are related by relativity, an energy scale defines a conjugate length scale. The length scale associated with the cut-off can be imagined as a resolution scale used to describe a physical system, much like a digital camera used to observe physical objects has a resolution associated with the number of pixels of data the camera uses. If the cut-off scale of the theory is changed then the parameters (the couplings) change as well as the degrees of freedom in the form of fields. This change is described by a group action, in that any energy scale is accessible by any other via one of these transformations; hence the name renormalisation group.

One of the most important equations associated with the renormalisation group is the rate of change of the couplings with respect to a change in the energy scale which is called the Beta function of the theory:

$$\beta(g(\Lambda)) = \frac{dg}{d\Lambda}(\Lambda) \tag{2.4.1}$$

An important thing to note about this relation is that it depends locally on the value of the energy scale Λ. This means that to know how the coupling changes we don't need to know about the coupling in the high energy UV or in the low energy infra-red (IR), all we need is local knowledge to work out the beta function locally. This locality is reminiscent of space-time coordinates, since in general relativity to compute local quantities only the local geometry of space-time needs to be taken into account. Hence it seems possible to associate the extra dimensions of the gravity theory with the energy scale of the renormalisation group.

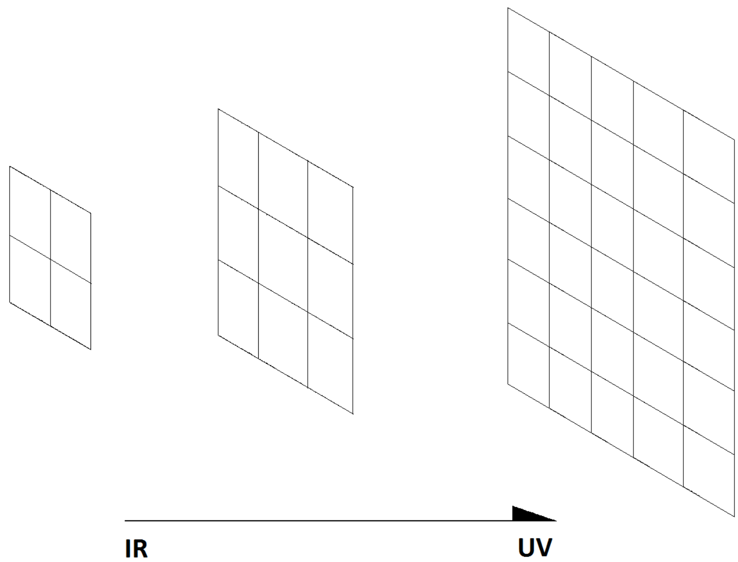


Figure 2.4: The energy scale can be viewed as a resolution scale of the theory, hence there is a higher resolution at the ultra-violet energy scale and lower at the infra-red.

A natural theory to consider is one for which the couplings don't change at all with energy scale, i.e. the beta function is zero. Such QFTs that realize this condition are thus scale invariant. A special subset of the scale invariant QFTs called conformal field theories (CFTs) have a much stronger set of conformal symmetries. As the name suggests are an essential part of the AdS/CFT correspondence as the following section will discuss.

2.5 Anti-deSitter space and conformal field theories

An important concept in the AdS/CFT correspondence and gauge-gravity duality in general is how the geometry of the gravity theory gives rise to additional symmetries and how these manifest themselves in the quantum field theory. In the particular case of the AdS/CFT correspondence the gravity theory has a space-time background can be described by asymptotically anti-deSitter (AdS) space. This is dual to a special subset of quantum field theories call conformal field theories (CFTs). These terms will be explained in the following section and the relation between their symmetries will be motivated.

Anti-deSitter space-time (AdS) is a space-time with constant negative curvature. It is the Lorentzian analogue of a hyperbolic space just as Minkowski space-time is the Lorentzian analogue of Euclidean space. To describe AdS space in (d+1)-dimensions one can embed it as a hyperbolic hyper-surface in (d+2)-dimensions with embedding coordinates $(X_{-1}, X_0, \boldsymbol{X} = X_1, \ldots, X_d)$ as:

$$-X_{-1}^2 - X_0^2 + \boldsymbol{X} \cdot \boldsymbol{X} = -R^2 \tag{2.5.1}$$

where R is the AdS radius. This equation makes the symmetry group of AdS space apparent, which is the $SO(2, d)$. This equation can be solved using Poincaré coordinates:

$$
\begin{aligned}
X_{-1} &= \frac{1}{2z}(z^2 + R^2 + x_i^2 - t^2)) \\
X_0 &= \frac{R}{z}t \\
X_i &= \frac{R}{z}x_i \quad, i \in (1, \ldots, d-1) \\
X_d &= \frac{1}{2z}(z^2 - R^2 + x_i^2 - t^2))
\end{aligned}
\tag{2.5.2}
$$

which yields the metric:

$$
ds^2 = (\frac{R}{z})^2[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2]. \tag{2.5.3}
$$

An advantage of describing AdS space locally using Poincaré coordinates is that the constant z hypersurfaces are just copies of Minkowksi space, a useful feature that motivates the AdS/CFT correspondence statement that the CFT can be said to live on the bounding surface of AdS space.

The boundary of AdS space at $z = 0$ is an important thing to notice. To get to the boundary from a finite value of z requires traversing an infinite proper distance, however massless particles can still reach the boundary in a finite time. For flat Minkowski space the boundary can only be reached for time-like and null geodesics in an infinite proper time, so for most practical situations boundary conditions can be trivial, i.e. all degrees of freedoms such as fields vanish at infinity. However in the case of AdS space the boundary conditions cannot always be trivial, they need to be specified when setting up an physical problem, as well as initial conditions of the degrees of freedom, on AdS space. These boundary conditions are a crucial aspect of the AdS/CFT correspondence. A possible boundary condition is that the space-time is asymptotically AdS, i.e. as one approaches the boundary in the limit $z \to 0$ the space looks locally like it has constant negative curvature.

Conformal field theories have the usual Poincaré group symmetries of translations, rotations, and boosts, but are also invariant under scale transformations such as $x \to x/b$ and what are called special conformal transformations. It can be shown by considering the Lie algebra of CFTs that the special conformal transformations act like the translations, and together with scale transformations CFTs are invariant under local scale transformations of the form

$$x^\mu \to \frac{x^\mu}{b(x)} \tag{2.5.4}$$

where $b(x) \in \mathbb{R}$ depends upon space-time position. Scale invariance is a subset of these local symmetries when b is constant. If one considers the Lie algebra of a conformal field theory one finds that it is the Lie algebra of $SO(2, d)$. As stated in the previous section, because CFTs are invariant under spatial scaling they are also invariant under changes in the energy scale, hence the beta function of any CFT vanishes, $\beta(g) = 0$.

Motivated by the previous section let's consider a QFT that is scale invariant, i.e. with a vanishing beta function, and use the energy scale of the theory u as an additional coordinate to construct a metric for the gravity theory. Using scale invariance the metric has to be invariant under scale transformations $x \to x/b$, with u transforming $u \to ub$ by dimensional analysis. The metric must also preserve Poincaré invariance. Imposing these constraints results in the metric:

$$ds^2 = (u\tilde{L})^2 \eta_{\mu\nu} dx^\mu dx^\nu + (\frac{du}{u})^2 L^2 \tag{2.5.5}$$

where L and $\tilde{L}$ are lengths. Under the scaling transformation $u \to \frac{L}{\tilde{L}} u$ we remove the dependence on $\tilde{L}$ and the metric takes the form:

$$ds^2 = (uL)^2 \eta_{\mu\nu} dx^\mu dx^\nu + (\frac{du}{u})^2 L^2. \tag{2.5.6}$$

We can make one more final coordinate change to bring the metric into a form that makes the scale invariance more apparent. If we swap u with a length scale $z = 1/u$ then the metric takes the form:

$$ds^2 = (\frac{L}{z})^2[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2]. \tag{2.5.7}$$

This is in exactly the same form as the AdS metric in Poincaré coordinates with the identification $R = L$. Another thing to note is that the symmetry group of AdS space in (d+1) dimensions and a CFT in d dimensions are both $SO(2, d)$. This motivates the result that a scale invariant quantum field theory with a gravity dual can also be described by a conformal field theory.

2.6 CFT conditions for duality

As the previous section clearly demonstrates, a CFT is a strong indicator that a field theory has a dual description in terms of a quantum theory of gravity formulated on an asymptotically AdS background. But are there any additional conditions on the field theory side for an AdS/CFT type correspondence? This question was considered by Polchinski et al [24] using bulk locality to set conditions on the CFT. Arguing backwards they came up with a conjecture based on the maximal set of necessary conditions for a gauge-gravity duality.

The conjecture is: *any CFT that has a large-N expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual* , [24].

The first condition we have already encountered in the form of the 't Hooft limit. This particular kind of large-N limit allows correlation function contributions to be written as a perturbative expansion in terms of the genus of the corresponding Feynman diagrams. The above condition requires that any CFT must have some form of large-N limit such that this is the case. The large-N limit on the gravity theory side is the limit in which the theory becomes classical, which in the case of string theory means that the theory is well understood.

The second condition refers to single-trace operators which are dual to single particle states in the bulk. Because no field theory we have encountered so far encourages operators at low-energy with spin greater than two it would seem necessary that their presence in our field of experimental vision must be suppressed. If these states had a large number of dimensions beyond the four we observe at low energy, the presence of these operators would be sufficiently suppressed.

The second condition can be motivated using an important result from the duality and considerations from string theory. The AdS/CFT correspondence relates single trace operators in the CFT with scaling dimension Δ to single particle field excitations in the bulk gravity theory of mass m via this relation:

$$\Delta(\Delta - 4) = m^2 R^2 \tag{2.6.1}$$

where R is the AdS radius. This is shown in [17]. For string excitations in the bulk the mass scale will be proportional to the inverse string scale by dimensional analysis $1/l_s = \lambda^{1/4}/R$ where λ is the 't Hooft coupling, hence the operators in the CFT have dimensions which scale as $\lambda^{1/4}$. In order that there be a hierarchy between the AdS radius and the string scale, namely $R \gg l_s$ it follows that $\lambda >> 1$. Hence this implies that all CFT operators that are dual to string excitations will have large scaling dimensions, just as the conjecture states. The string excitations correspond to $spin > 2$ operators in the CFT whilst Kaluza-Klein excitations correspond to spin ¡ 2. These have a mass of the order $1/R$, and so have scaling dimension of order 1. This dimension gap is a very interesting feature of the AdS/CFT correspondence since it is difficult to find field theories that have such a gap.

Chapter 3

Condensed matter systems and holography

The AdS/CFT correspondence has a great deal of applications in theoretical physics. One of them was to gain insights into quantum gravity, especially string theory in the non-classical regime. Other applications involve using the duality in the reverse direction, using string theory to make prediction about systems that are strongly coupled. This approach has led to a great deal of research in heavy ion physics [3], and more recently in condensed matter systems [29], amongst others. As the title of this dissertation suggests this section will focus on the condensed matter applications, particularly a subclass of systems that can all described by the same theoretical model: the Bose-Hubbard model [28, section 9]

3.1 The Bose-Hubbard model

The Bose-Hubbard (B-H) model can be used to describe a wide range of systems such as Cooper pairs of electrons that can move via Josephson tunneling between different superconducting 'islands', Helium atoms moving on a substrate, or ultracold atoms such as Rb^{87} confined to an optical lattice (laser beam interference forming a periodic potential that can trap neutral atoms). The B-H model can be described by the Hamiltonian:

$$H_b = -\omega \sum_{\langle ij \rangle} b_i b_j + \frac{U}{2} \sum_i n_i(n_i - 1) - \mu \sum_i n_i \qquad (3.1.1)$$

where the indices $\{i, j\}$ label the lattice sites of the system, and $b_i^\dagger$, b_i represent creation and annihilation operators respectively that satisfy the usual commutation relations:

$$[b_i^\dagger, b_j] = \delta_{ij} \quad ; \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0. \qquad (3.1.2)$$

The first term proportional to the constant ω is often called the 'hopping' term describing the energy in the system transferred as kinetic energy as a boson tunnels from one lattice site to another. The $\langle \ldots \rangle$ represents summing over neighboring sites only, so tunneling only occurs between adjacent sites on the lattice. The second strictly positive term proportional to U is the repulsive potential which depends on the boson number operator $n_i = b_i^\dagger b_i$. Since the sum is over individual sites the potential only acts between bosons on the same site modeling a short range interaction between bosons. Hence this term is zero if there is either zero or one boson per site and increases with each additional boson thereafter. The third and final term is proportional to the chemical potential of the system, this term is the energy contribution from each boson that is added or removed from the system. Changing the chemical potential corresponds to changing the number of bosons in the system.

3.2 The phases of the Bose-Hubbard model

The Bose-Hubbard model realizes two distinct quantum phases when the physical parameters of the model are varied. To demonstrate this the Hamiltonian H_b can be described by a single site Hamiltonian by replacing the hopping term with terms that depend on a complex number $\Psi_B \in \mathbb{C}$:

$$H_{MF} = \sum_i -\mu n_i + \frac{U}{2} n_i(n_i - 1) - \Psi_B^* b_i - \Psi_B b_i^\dagger \tag{3.2.1}$$

The advantage doing this is that H_{MF} is a sum of single site Hamiltonians so its energy eigenvalues will be sums of single site eigenvalues and its eigen-states products of single-site eigen-states. The task now is to find a value of Ψ_B such that H_{MF} is as close to H_b as possible, which can be done using mean field theory methods. First determine the ground state wave-function of H_{MF} as a function of Ψ_B which will be a product of single particle wave-functions. Next evaluate the expectation value of H_b for this wave-function.

$$\begin{aligned} H_b \quad + \quad & w \sum_{\langle ij \rangle} [b_i^\dagger b_j + h.c.] = H_{MF} + \sum_i \Psi_B b_i^\dagger + \Psi *_B b_i \\ \rightarrow \quad & E_0 = \langle H_{MF} \rangle - ZMw \langle b^\dagger \rangle \langle b \rangle + M \langle \hat{b} \rangle \Psi_B^* + M \langle \hat{b}^\dagger \rangle \Psi_B \quad (3.2.2) \end{aligned}$$

By adding and subtracting H_{MF} from H_b as in 3.2.2 above the approximation of the ground state energy of the B-H model E_0 is given by

$$\frac{E_0}{M} = \frac{E_{MF}(\Psi_B)}{M} - Zw \langle \hat{b}^\dagger \rangle \langle \hat{b} \rangle + \langle \hat{b} \rangle \Psi_B^* + \langle \hat{b}^\dagger \rangle \Psi_B \tag{3.2.3}$$

where M is the number of sites on a lattice, $E_{MF}(\Psi_B)$ is the ground state energy of H_{MF} and Z is the number of neighboring sites to each site on the lattice. Finally by varying Ψ_B the right hand side of the above equation can be minimized resulting in the best approximation for the ground state energy of H_b. This can be done numerically giving the following mean field phase diagram:

The M.I.n parts of the phase diagram for which $n \in \mathbb{Z}$ are Mott insulator phases with n denoting the number density $n_0(\mu/U)$ in that region.

Note that when using numerical methods a possible problem is the fact that even for each individual site there is an infinite number of possible states corresponding to $m \geq 0$, the number of bosons occupying the site. This has to be truncated for numerical methods to work, but fortunately the errors that result from this approximation are not hard to suppress.

in the limit of $w = 0$ all sites on the lattice decouple from one another and so for $\Psi_B = 0$ the mean field theory approximation is trivially exact. Because the B-H Hamiltonian only has n dependent terms $\langle H_{MF} \rangle$ is minimized by finding the right boson occupation number. This gives the ground state wave-function as:

$$|m_i = n_0 \left(\frac{\mu}{U} \right) \rangle \tag{3.2.4}$$

Thus for $w = 0$ the ground state consists of each site having the same integer number of bosons with this value jumping discontinuously whenever the ratio μ/U takes on an integer value. Note also that for $\mu/U \in \mathbb{Z}$ there is a twofold degeneracy of the energy eigen-states such that for $\mu/U = i - 1$ or i, n_0 takes on the same value and so correspond to the same eigen-state with the same energy eigenvalue. Thus there is a 2^M-fold degeneracy for the entire lattice whenever $\mu/U \in \mathbb{Z}$. This large degeneracy implies a large macroscopic entropy with is necessarily lifted when a small $w \neq 0$ is turned

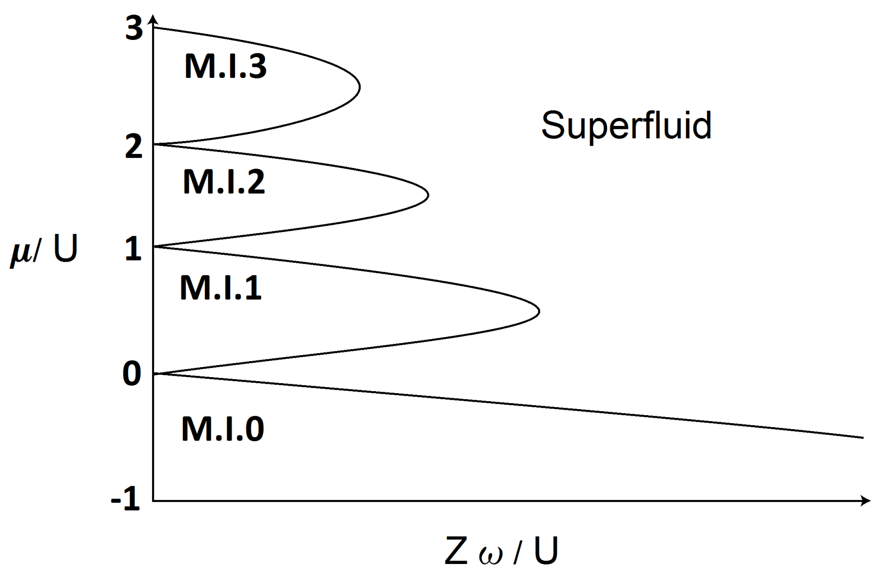


Figure 3.1: The phase diagram obtained using mean field theory methods.

on.

The above discussion concerns only a small part of the phase diagram. In order to explore the rest of the phase diagram bosons need to be allowed to 'hop' between neighboring sites which is equivalent to introducing a non-zero w term. Looking at the figure it can be seen that even for a small non-zero w there exist regions of the phase diagram in which $\Psi_B = 0$. Only at points where the ratio μ/U is an integer does $w \neq 0$ automatically imply that $\Psi_B \neq 0$.

For the $w \neq 0$, $\Psi_B = 0$ region, its states given by mean field theory have wave-functions given by $|m_i = n_0(\mu/U)\rangle$ even though $w \neq 0$. However another prediction beyond mean field theory is possible: that the expected number of bosons per site is given by

$$\langle b_i^\dagger b_i \rangle = n_0 \left(\frac{\mu}{U} \right) \tag{3.2.5}$$

the same result obtained by using the product state above for $w = 0$. The two assumptions used are that there is an energy gap between the ground state and excited states, and also that the boson number operator N_b commutes with H_b.

First note that for $w = 0$ and $\mu/U \notin \mathbb{Z}$ the ground state is unique and there is an energy gap between the ground state and excited states. It then follows that by turning on a small w, the ground state will be moved adiabatically, however no level crossings will occur between the ground state and other states. The $w = 0$ states is an exact eigen-state of N_b with an eigenvalue of $M.\ n_0(\mu/U)$ and the perturbation from $w \neq 0$ commutes with N_b. Thus the ground state remains an eigen-state of N_b with the same eigenvalue for small values of w. Translational invariance then implies that this result is the same for all sites giving the final result in the equation above.

Note also that the same argument holds not only for small values of

w, but for all points in the lobes of the mean field phase diagram. The boson density in these regions remains quantized and there is an energy gap between the ground state and excited states. Systems with these properties are called Mott insulators, with ground states that are similar, but not equal to $|m_i = n_0(\mu/U)\rangle$. The is made by additional fluctuations of bosons between pairs of neighboring sites that create particle-hole pairs. Mott insulators are also known to be 'incompressible' since varying parameters such as the chemical potential doesn't affect the particle density:

$$\frac{\partial \langle N_b \rangle}{\partial \mu} = 0 \tag{3.2.6}$$

The result shown before this is very unusual in classical critical phenomena. It is not often the value of an observable is quantized not just at isolated points but over an entire region of the phase diagram, but as will be shown there is a particular subclass of quantum field theories for which this is common.

The boundary of the Mott insulating phases is a second-order quantum phase transition. By using this assumption it is possible to determine the location of the phase boundaries. One can use a standard argument based on Landau theory, that the ground state energy can be expanded in powers of the order parameter Ψ_B:

$$E_0 = E_{00} + r|\Psi_B|^2 + O\left(\Psi_B^4\right) \tag{3.2.7}$$

then compute the value of r using second order perturbation theory and previous results:

$$\begin{aligned} r &= \chi_0(\mu/U)[1 - Zw\chi_0(\mu/U)]; \\ \text{where } \chi_0(\mu/U) &= \frac{n_0(\mu/U)+1}{Un_0(\mu/U)-\mu} + \frac{n_0(\mu/U)}{\mu - U(n_0(\mu/U)-1)}. \end{aligned} \tag{3.2.8}$$

Note that because of n_0's dependence on μ/U the denominators of both terms making up χ_0 are positive, except at the points of n_0's discontinuity at $w = 0$. By solving $r = 0$ one obtains the phase boundary shown in the figure above.

Thats a substantial portion of the quantum phase diagram for the B-H model, now the remaining section in which $\Psi_B \neq 0$ needs to be considered. Because Ψ_B varies smoothly as the physical parameters of the B-H model are varied the density is no longer quantized in this phase. In fact the density can be varied continuously between any two real positive values. This corresponds to a compressible state for which

$$\frac{\partial \langle N_b \rangle}{\partial \mu} \neq 0 \tag{3.2.9}$$

Recall from the form of the mean field theory Hamiltonian that a non-zero Ψ_B breaks the U(1) symmetry the remaining part of the Hamiltonian preserve. Rotations of the order parameter in this broken symmetric phase are also accompanied by non-zero stiffness. It can be shown that this broken U(1) symmetry corresponds to a superfluid state, with the stiffness equated with the superfluids density.

This analysis has used the mean field theory phase diagram almost exclusively to extract properties of the B-H model, however this is only an approximation to the actual phase diagram. Corrections have been made to this phase diagram by Freericks and Monien [8] in which they find singularities shaped like the Mott lobes at the places where z = 1 transitions (?) occur. In addition to this there have been Monte Carlo simulations done which agrees with all of the conclusions discussed in this section.

3.3 Quantum phase transitions

The previous section describes the two quantum phases of the boson-Hubbard model which can occur at zero temperature and in absence of disorder.

When these two phases meet a phase transition occurs between the Mott insulating phase and the superfluid phase. This phase transition cannot be classical, since classical systems at zero temperature have no entropy. Thus this is what is known as a quantum phase transition (QPT). For a transition between two quantum phases at non-zero temperature the transition can be described using classical thermodynamics, hence in order to differentiate QPTs from classical phase transitions, QPTs only occur at zero temperature. Hence the transition cannot be reached by varying the temperature, but by varying some other physical parameter such as the pressure or the particle density. Without loss of generality let's consider the parameter to be a dimensionless 'coupling' denoted by g.

There are two types of QPT; the first is one in which the QPT occurs at a thermodynamic singularity for $T = 0$ and $g = g_c$, some critical value of the coupling, whilst for $T \neq 0$ all physical quantities are analytic with respect to g. This point is called the quantum critical point and is the point at which the QPT occurs. The second type has this same singular point at the same location, however in addition to this there is a curve of what are known as second order phase transitions which terminates at the quantum critical point. The order of a phase transition depends upon the first discontinuous derivative of some thermodynamic potential of the system. A phase transition in which the first derivative of said potential has a discontinuity when evaluated at some point in phase space has a first order phase transition at this point. Hence a second order phase transition has a discontinuity in the second derivative of this thermodynamic potential.

Systems with such a set of second order phase transitions will usually have a characteristic energy scale Δ associated with them at zero temperature. This energy scale is either the energy difference between the ground state and first excited state, or if there is no energy gap between these states then it is the difference between the ground state and an energy at which there is a significant change in the energy spectrum. This energy scale has the property of vanishing as the critical point $g = g_c$ is approached. This energy scale has associated with it a length scale called the correlation length which diverges as $g \to g_c$. The correlation length represents the typical distance scale at which the degrees of freedom are correlated. For example, for a system of fermions of spin- each confined to a point on a periodic lattice the spin orientations would be degrees of freedom that experience correlation.

Since this length scale diverges close to the critical point it means that at the critical point there is no length scale associated with the system. If there is no scale then the system is scale invariant, and based upon the discussion in 2.5 the AdS/CFT correspondence would apply at the quantum critical point.

3.4 The coherent state path integral

To demonstrate that the B-H Hamiltonian H_B at low energies describes a conformal field theory for fixed values of the parameters the relationship between the partition function associated with H_B and the path integral needs to be used. Rather than use the conventional path integral derivation however, which integrates over all possible quantum trajectories of the configuration space, this section will derive instead the coherent state path integral which integrates over phase space. This is because there is no configuration space which does not break at least one of the Hamiltonian's symmetries.

Coherent states form an infinite set with each being uniquely defined by a real vector $\mathbf{N}$ (for the B-H model $\mathbf{N}$ is a 2-dimensional vector, or a complex number). These states are normalized such that

$$\langle \mathbf{N} | \mathbf{N} \rangle = 1 \tag{3.4.1}$$

and they have a completeness relation associated with them

$$C_N \int d\mathbf{N}|\mathbf{N}\rangle\langle\mathbf{N}| = 1 \qquad (3.4.2)$$

such that C_N is some normalization constant. In contrast to the usual path integral formalism in quantum field theory, none of these states are orthogonal to one another $\langle\mathbf{N}|\mathbf{N}'\rangle \neq 0$ for $\mathbf{N} \neq \mathbf{N}'$. This property means that these states can be considered to be 'over-complete'. Finally these states are chosen to have the additional property that the diagonal expectation values of operators $\hat{\mathbf{S}}$ in the Hamiltonian $H_B(\hat{\mathbf{S}})$ take the form

$$\langle\mathbf{N}|\hat{\mathbf{S}}|\mathbf{N}\rangle = \mathbf{N}. \qquad (3.4.3)$$

This property implies that $\mathbf{N}$ is the classical approximation of the operator(s) $\hat{\mathbf{S}}$. Assuming 3.4.1, 3.4, and 3.4.3 is sufficient to define the set of coherent states $\{|\mathbf{N}\rangle\}$. Using 3.4.3 it is usually possible to order the operators in H_B such that the following is satisfied

$$\langle\mathbf{N}|H_B(\hat{\mathbf{S}})|\mathbf{N}\rangle = H_B(\mathbf{N}). \qquad (3.4.4)$$

It is possible to show that the order which is used to obtain this result is normal ordering, which involves moving all the creation operators to the left and the annihilation operators to the right, for example

$$: \hat{b}_i\hat{b}_j^\dagger\hat{b}_k\hat{b}_l^\dagger := \hat{b}_j^\dagger\hat{b}_l^\dagger\hat{b}_i\hat{b}_k. \qquad (3.4.5)$$

Within this section normal ordering will be assumed when evaluating such expectation values. The derivation of the coherent state path integral starts with the partition function

$$\mathcal{Z}_B = Tr\, exp(-H_B(\hat{\mathbf{S}})/T) \qquad (3.4.6)$$

where Boltzmann's constant k_B has been set to 1. The next couple of steps resemble the derivation of the path integral in quantum field theory which can be found in many textbooks, for example [23, page 275]. The derivation for this case will nonetheless be briefly outlined here. First the exponential is broken up into a large number of exponentials of infinitesimally small time operators

$$\mathcal{Z}_B = \lim_{M\to\infty} \prod_{i=1}^{M} exp(-\Delta\tau_i H_B(\hat{\mathbf{S}})). \tag{3.4.7}$$

such that $\Delta\tau_i = 1/MT$. Next by inserting a set of coherent states between each pair of exponentials using 3.4 and labeling the inserted state with a time τ so that it is denoted $|\mathbf{N}(\tau)\rangle$. Then each expectation value can be evaluated using 3.4.4 in the following way

$$\begin{aligned}\langle\mathbf{N}(\tau)|exp(- \Delta & \tau H_B(\hat{\mathbf{S}}))|\mathbf{N}(\tau-\Delta\tau)\rangle \\ &\approx \langle\mathbf{N}(\tau)|(1-\Delta\tau H_B(\hat{\mathbf{S}}))|\mathbf{N}(\tau-\Delta\tau)\rangle \\ &\approx 1-\Delta\tau\langle\mathbf{N}(\tau)|\frac{d}{d\tau}|\mathbf{N}(\tau)\rangle - \Delta\tau H_B(\mathbf{N}) \\ &\approx exp(-\Delta\tau\langle\mathbf{N}(\tau)|\frac{d}{d\tau}|\mathbf{N}(\tau)\rangle - \Delta\tau H_B(\mathbf{N})). \end{aligned} \tag{3.4.8}$$

It has been assumed in anticipation of the limit $M\to\infty$ that $\Delta\tau \ll 1$. In the third line it has been assumed that states can be expanded in time derivatives such that $|\mathbf{N}(\tau-\Delta\tau)\rangle \approx (1-\Delta\tau\frac{d}{d\tau})|\mathbf{N}(\tau)\rangle$. This assumption is not entirely trivial for coherent states, since even a small relative time difference between coherent states can result in completely different orientations. The book by Negele and Orland [22] discusses this issue very carefully and comes to the conclusion that, excepting the case of the 'tadpole' diagram which involves point-splitting of time, the time-derivative expansion is always valid.

Inserting 3.4.8 into the previous expression for the partition function 3.4.7 and taking the $M\to\infty$ limit results in the following expression for

$\mathcal{Z}_B$

$$\mathcal{Z}_B = \int_{\mathbf{N}(0)=\mathbf{N}(1/T)} \mathcal{D}\mathbf{N}(\tau) exp\left[-\mathcal{S}_{Berry} - \int_0^{1/T} d\tau H(\mathbf{N}(\tau))\right] \tag{3.4.9}$$

where

$$\mathcal{S}_{Berry} = \int_0^{1/T} d\tau \langle \mathbf{N}(\tau)| \frac{d}{d\tau} |\mathbf{N}(\tau)\rangle. \tag{3.4.10}$$

The additional term in the exponent $\mathcal{S}_{Berry}$ called a Berry phase term and describes how coherent states at infinitesimally separated times overlap. Using the above normalization condition 3.4.1 one can show that the Berry phase term is purely imaginary.

To obtain the coherent state path integral from the B-H Hamiltonian the first thing to do is find a set of coherent states which depend upon the bosonic creation/annihiliation operators denoted by $\hat{b}^\dagger/\hat{b}$, and a complex number ψ representing $\mathbf{N}$. One possible set of states that satisfies all the coherent states properties 3.4.1, 3.4, and 3.4.3 consists of states of the form

$$|\psi\rangle = e^{-|\psi|^2/2} e^{\psi \hat{b}^\dagger} |0\rangle \tag{3.4.11}$$

such that $|0\rangle$ is the boson vacuum state with zero boson number. The states are normalized as

$$\begin{aligned}
\langle\psi|\psi\rangle &= e^{-|\psi|^2} \langle 0| e^{\psi^* \hat{b}} e^{\psi \hat{b}^\dagger} |0\rangle \\
&= e^{-|\psi|^2} \langle 0| exp(\psi^* \hat{b} + \psi \hat{b}^\dagger + |\psi|^2/2[\hat{b}, \hat{b}^\dagger] + \dots)|0\rangle \\
&= e^{-|\psi|^2/2} \langle 0| exp(\psi^* \hat{b} + \psi \hat{b}^\dagger)|0\rangle \\
&= e^{-|\psi|^2/2} \langle 0| e^{|\psi|^2/2} |0\rangle = 1
\end{aligned} \tag{3.4.12}$$

where the commutation relations $[\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{b}, \hat{b}] = [\hat{b}^\dagger, \hat{b}^\dagger] = 0$ were used as well as the Baker-Campbell-Hausdorff formula which states that for operators $\hat{A}$ and $\hat{B}$,

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+\ldots} \tag{3.4.13}$$

where the dots denote additional terms that include nested commutators such as $[\hat{A}, [\hat{A}, \hat{B}]]$ which will be zero if $\hat{A} = \psi^* \hat{b}$ and $\hat{B} = \psi \hat{b}^\dagger$. Additionally it follows that

$$\begin{aligned} \langle \psi | \hat{b} | \psi \rangle &= e^{-|\psi|^2} \frac{\partial}{\partial \psi^*} \langle 0 | e^{\psi^* \hat{b}} e^{\psi \hat{b}^\dagger} | 0 \rangle \\ &= e^{-|\psi|^2} \frac{\partial}{\partial \psi^*} e^{|\psi|^2} = \psi \end{aligned} \tag{3.4.14}$$

satisfying 3.4.3. The completeness relation follows from

$$\begin{aligned} \int d\psi d\psi^* |\psi\rangle\langle\psi| &= \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int d\psi d\psi^* |\psi|^{2n} e^{-|\psi|^2} \\ &= \pi \sum_{n=0}^{\infty} |n\rangle\langle n| \end{aligned} \tag{3.4.15}$$

where $|n\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n}} |0\rangle$ are the number states, and only diagonal terms have been picked in the double sum over the number states since off diagonal terms will vanish due to the angular ψ integral. Identifying the sum in the bottom line with the identity matrix requires that $C_N = 1/\pi$ so that is satisfied.

The only thing that is left to be determined is the Berry phase that needs to be added. Since this path integral is being integrated over the complex plane due to the choice of ψ, the Berry phase is given by

$$\langle \psi(\tau) | \frac{d}{d\tau} | \psi(\tau) \rangle = e^{-|\psi|^2} \langle 0 | e^{\psi^*(\tau)\hat{b}} \frac{d}{d\tau} e^{\psi(\tau)\hat{b}^\dagger} | 0 \rangle = \psi^* \frac{d\psi}{d\tau}. \tag{3.4.16}$$

Inserting 3.4.16 into 3.4.9 completes the derivation of the coherent state path integral for the boson-Hubbard model.

3.5 Extracting the conformal field theory

In the previous section the Hamiltonian of the B-H model was related via the partition function to the coherent state path integral defined as

$$\mathcal{Z}_B = \int \mathcal{D}b_i(\tau)\mathcal{D}b_i^\dagger(\tau) exp\left[-\int_0^{1/T} d\tau \mathcal{L}_b\right] \tag{3.5.1}$$

such that

$$\mathcal{L}_b = \sum_i \left(b_i^\dagger \frac{db_i}{d\tau} - \mu b_i^\dagger b_i + (U/2) b_i^\dagger b_i^\dagger b_i b_i\right) - \omega \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + h.c.\right). \tag{3.5.2}$$

The only way in which this differs from 3.4.9 is the replacement $\psi(\tau) \rightarrow b(\tau)$ as a reminder of the bosonic nature of the degrees of freedom. Because the remainder of this dissertation concerns only continuum theories there should be no confusion between this notation and the $\hat{b}$ operators of the Hamiltonian formalism.

It is possible to express the coherent state path integral in terms of another field $\Psi_B(x, \tau)$ which has a role analogous to that of the mean-field parameter Ψ_B used in describing the phases of the B-H model in an earlier section. This field can be incorporated by using a Hubbard-Stratanovich transformation applied to the coherent path integral which decouples the hopping term proportional to ω from the fields $\hat{b}$ and $\hat{b}^\dagger$ such that

$$\mathcal{Z}_B = \int \mathcal{D}b_i(\tau)\mathcal{D}b_i^\dagger(\tau)\mathcal{D}\Psi_{Bi}(\tau)\mathcal{D}\Psi_{Bi}^\dagger(\tau) exp\left(-\int_0^{1/T} d\tau \mathcal{L}_b'\right) \tag{3.5.3}$$

such that

$$\mathcal{L}_b' = \sum_i \left(b_i^\dagger \frac{db_i}{d\tau} - \mu b_i^\dagger b_i + (U/2) b_i^\dagger b_i^\dagger b_i b_i - \Psi_{Bi} b_i^\dagger - \Psi_{Bi}^* b_i \right) + \sum_{i,j} \Psi_{Bi}^* \omega_{ij}^{-1} \Psi_{Bj}. \tag{3.5.4}$$

The matrix ω_{ij} is symmetric with entries that are only non-zero if i and j are nearest neighbors. One can show that 3.5.3 and 3.5.4 are equal by doing the Gaussian integral over Ψ_B. Some constant factors get created in doing this, however they can be removed by redefining the measure $\mathcal{D}\Psi_B$. One important point to note is that the inversion of the matrix ω_{ij} is only possible if all of its eigenvalues are positive. In the case where ω_{ij} is symmetric with non-zero nearest neighbor entries this will not be the case, some of the eigenvalues will be negative. To fix this it is possible to add a positive constant along the diagonal of ω_{ij} and subtract the same constant from the on-site part of the Lagrangian. This will not be necessary in this case however, since the only modes of Ψ_B that need be considered in the low energy limit are the long wavelength modes which correspond to positive eigenvalues of ω_{ij}.

Note that the above Lagrangian is invariant under the following symmetry

$$\begin{aligned} b_i &\to b_i e^{i\phi(\tau)} \\ \Psi_{Bi} &\to \Psi_{Bi} e^{i\phi(\tau)} \\ \mu &\to \mu + i\frac{d\phi}{d\tau}. \end{aligned} \tag{3.5.5}$$

Note that this transformation results in μ being time dependent, taking the parameters out of the physical parameter regime. Despite this, imposing this symmetry because places useful restrictions on the ways in which $\mathcal{Z}_B$ is manipulated.

It is possible to integrate out both of the fields b_i and $b_i^\dagger$ so that the Lagrangian can be expressed in powers of Ψ_B and Ψ_B^*. The Ψ_B^* part can be determined in a closed form since the Ψ_B independent part of $\mathcal{L}_b'$ is just a sum of single-site Hamiltonians for the b_i which is easily diagonalised, the eigenstates are given in equation 3.2.5. By re-exponentiating the series in powers of Ψ_B, Ψ_B^* and then expand the terms in spatial and temporal gradients of Ψ_B, then $\mathcal{Z}_B$ can the written as in [6]

$$\mathcal{Z}_B = \int \mathcal{D}\Psi_{Bi}(\tau)\mathcal{D}\Psi_{Bi}^{\dagger}(\tau) exp\left(-\frac{V\mathcal{F}_0}{T} - \int_0^{1/T} d\tau \int d^d x \mathcal{L}_B\right),$$

$$\mathcal{L}_B = K_1 \Psi_B^* \frac{\partial \Psi_B}{\partial \tau} + K_2 \left|\frac{\partial \Psi_B}{\partial \tau}\right|^2 + K_3 |\nabla \Psi_B|^2 + \bar{r}|\Psi_B|^2 + \frac{u}{2}|\Psi_B|^4 + \dots \quad (3.5.6)$$

where $V = Ma^d$ is the total lattice volume. $\mathcal{F}_0$ is the free energy of density of the system with decoupled sites. When differentiated with respect to the chemical potential it gives the density of the Mott insulating state

$$-\frac{\partial \mathcal{F}_0}{\partial \mu} = \frac{n_0(\mu/U)}{a^d}. \quad (3.5.7)$$

All the other unknown parameters can be expressed in terms of the physical parameters μ, U, and ω, however the only important parameter that need be considered here is $\bar{r}$ which is

$$\bar{r}a^d = \frac{1}{Z\omega} - \chi_0(\mu/U) \quad (3.5.8)$$

such that χ_0 is as defined in 3.2.8. Note that $\bar{r}$ is directly proportional to the mean field parameter r in 3.2.8. This means that $\bar{r}$ will vanish when r does which is precisely at the critical point between the Mott insulator and the superfluid.

This is almost a conformal field theory, the only term that is the exception is the first term with coefficient K_1 which includes a first order time derivative. If the symmetry in 3.5.5 is imposed for small ϕ then by noting that $\bar{r}$ can be seen to transform as

$$\bar{r}|\Psi_B|^2 = \left[\bar{r}(\mu=0) + \frac{\partial \bar{r}}{\partial \mu}\mu + \ldots\right]|\Psi_B|^2 \rightarrow \left[\bar{r}(\mu=0) + \left(\frac{\partial \bar{r}}{\partial \mu}\right)'(\mu + i\partial_\tau \phi) + \ldots\right]|\Psi_B|^2 \tag{3.5.9}$$

where $(\ldots)'$ indicates a transformed term, and the K_1 term transforms as

$$K_1 \Psi_B^* \frac{\partial \Psi_B}{\partial \tau} \rightarrow K_1' \left[\Psi_B^* \frac{\partial \Psi_B}{\partial \tau} + i\frac{\partial \phi}{\partial \tau}|\Psi_B|^2\right] \tag{3.5.10}$$

the symmetry is maintained by setting $K_1 = -\partial\bar{r}/\partial\mu$. This means that K_1 vanishes when $\bar{r}$ is independent of μ, which due to $\bar{r} \propto r$ only occurs when the insulator-superfluid phase boundary has a vertical tangent, i.e. at the tips of the Mott insulator lobes in figure 3.2. Thus it is these points on the phase diagram which can be described by a conformal field theory, and so are possible candidates field theories for the AdS/CFT correspondence.

3.6 Motivation for holographic methods

The previous section shows that the Bose-Hubbard model certainly fits all the requirements for a dual gravity description, but if the model is entirely described by theories that are well understood there is no need to resort to speculative methods like the AdS/CFT correspondence. However, as we will soon see all other tools that are traditionally used to study the B-H model break down when the quantum critical point is approached.

In this section the conductivity or charge transport of this model will be considered. This is the conductivity induced by placing the system in a 'electric' field E that has associated with it a frequency ω as considered in [29, 20]. To start with let's consider the conductivity in both of the two quantum phases of the B-H model. The first, the Mott insulating phase can be described using particle and hole excitations. We can describe the motion

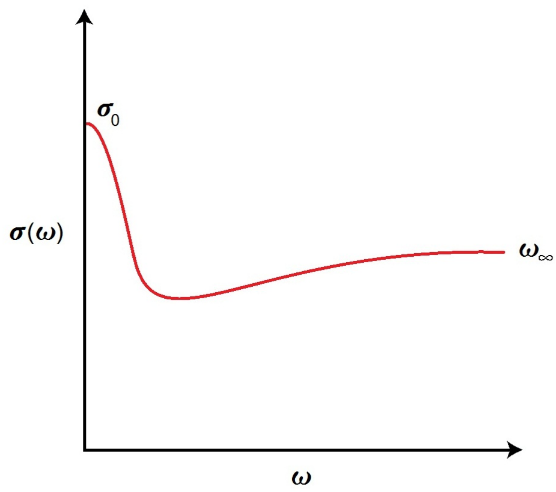


Figure 3.2: The real part of the frequency dependent conductivity predicted by assuming particle-like excitations.

of such a system using a Boltzmann type equation for particles moving in this external field E at an average velocity v by the following:

$$\frac{dv}{dt} + \frac{v}{\tau_c} = qE \tag{3.6.1}$$

where τ_c is the average time between collisions of the particles. This equation is valid when the coupling is small $g \ll 1$, but let us attempt to see what this equation says about physics at the critical point $g = g_c$. Assuming that the time scale τ_c is given by the energy scale, i.e. the temperature near the critical point $\tau_c\, \hbar/(k_B T)$, then the above equation is solvable. Predicting a 'Drude' form for the conductivity gives

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau_c} \tag{3.6.2}$$

From the large ω behavior noted earlier, the real part of this function looks like:

Because the critical point lies between two quantum phases it should also be possible to approach the critical point from a superfluid state. The superfluid phase has vortex-like excitation rather than the particle and hole-like

Mott insulator phase. However, in 2-d all vortices have a center described by a point and so can be nonetheless viewed as 'particles'. Thus another Boltzmann equation can be used to describe the motion of the excitations. The only property necessary to obtain the conductivity is that it is equal to the resistivity of the 'particles' that enter the system [7]. Based on this one would assume that the conductivity in the superfluid phase would be given by the inverse of the conductivity in the Mott insulating phase.

The critical point appears to have completely different descriptions of the frequency dependent conductivity depending on from which quantum phase one approaches the point. At the time of writing it is not clear which if either is the correct description. Methods such as ϵ and large N vector expansions [30, 25] both suggest that the insulator description is closer. The validity of these procedures however, is not certain in this case.

3.7 Holographic analysis

Now that the premise for AdS/CFT correspondence has been discussed holographic methods can be applied to the field theory at the quantum critical point of the B-H model. The way that will be used here is to begin with a solvable model, then modify this so that it generalizes to a broader class of physical models. The model in question will be the only model in $D = 3$ space-time dimensions to have been solved explicitly: the ABJM model [1]. As described in the introduction this AdS/CFT type duality in a 'large-N' type limit maps a $D = 3$ CFT to a supersymmetric gravity theory in $D = 4$ space-time dimensions. Following the previous section a current associated with globally conserved charge will be considered, specifically its correlations. Current correlations are described simply by $D = 4$ Einstein-Maxwell (E-M) theory with a negative cosmological constant and action [1]

$$\mathcal{S}_{EM} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4e^2}F_{ab}F^{ab}\right] \tag{3.7.1}$$

where $x \equiv (t, u, r)$ such that t represents time, r the two spatial coordinates of the CFT, and u is the 'emergent' coordinate needed to describe the gravity theory. The constant κ is related to Newton's constant by $\kappa^2 = 8\pi G$. The gravity side has a metric g as its degrees of freedom with associated Riemann curvature tensor R, whilst the $U(1)$ gauge theory has a 2-form field strength $F = F_{ab}\, dx^a \wedge dx^b$ with $F_{ab} = \partial_a A_b - \partial_b A_a$, and $a, b, = 0, \ldots, D$ the $D+1$ coordinates of the gravity theory. The vacuum solution of the equations of motion obtained from varying the action with respect to the metric and the U(1) gauge field A are the anti-deSitter metric in 4 space-time dimensions:

$$ds^2 = g_{ab} dx^a dx^b = \left(\frac{L}{u}\right)^2 du^2 + \left(\frac{u}{L}\right)^2 \left(-dt^2 + dr^2\right) \tag{3.7.2}$$

and $F_{\mu\nu} = 0$ where $\mu, \nu = 0, \ldots, D-1$, the coordinates of the D-dimensional space-time. L is the radius of curvature of AdS space. As was shown in the previous section this metric is invariant under not only Poincaré transformations, but also the broader class of conformal transformations of the form 2.5.4 as well. It is possible to describe the dual to the U(1) gauge field A: the conserved current of the CFT denoted J_μ here. The conductivity in question is dual to the two-point correlator of J_μ. To explicitly identify the correlator the CFT action will be sourced by a term K_μ

$$\mathcal{S}_{CFT} \rightarrow \mathcal{S}_{CFT} - \int d^{D-1} r dt K^\mu J_\mu. \tag{3.7.3}$$

From holography considerations it follows that the source is equal to the boundary value of the vector potential A_a [18]

$$A_\mu(t, r, u \rightarrow \infty) = K_\mu(t, r) \tag{3.7.4}$$

Next, to find the conductivity itself one needs to solve the equations of motion of the gravity theory bearing in mind the above relation, The solution will show how the gravitational action depends upon the source K_μ, which by AdS/CFT corresponds to the dependence of the CFT action on K_μ as well. From there the conductivity can be obtained in the usual manner by taking functional derivatives of the action with respect to K_μ.

Before this can be done there are two important things to discuss beforehand. Firstly the analysis that has been done so far has been in the zero temperature regime. Clearly however, to make testable predictions this treatment must be extended to $T \neq 0$ also. This can be accomplished by considering black hole solutions to the equations of motion so far discussed. These will bear some resemblance to the Schwarzchild solution [36, chapter 6]

$$ds^2 = \left(\frac{L}{u}\right)^2 \frac{du^2}{f(u)} + \left(\frac{u}{L}\right)^2 \left(-f(u)dt^2 + dr^2\right) \tag{3.7.5}$$

such that

$$f(u) = 1 - \left(\frac{R}{u}\right)^3 \tag{3.7.6}$$

for $D = 3$. In this case R labels the location of the black hole event horizon. Strictly speaking this is in fact a black brane with a spatially infinite event horizon in the flat 2-d r-space. As discussed earlier in the holographic principle section black holes have been shown to have a temperature T which can be shown to be dual to the temperature of the CFT. The temperature can be computed by performing a Wick rotation on the time so that the signature of the metric is Euclidean $(+ \cdots +)$. Then by demanding that the space is periodic in the new time with period $\hbar/k_B T$ results in a definition of the temperature

$$k_B T = \frac{3\hbar v R}{4\pi L^2}. \tag{3.7.7}$$

The v, that plays the role of the speed of light in the CFT is making a brief reappearance. One can also see this equation as a constraint on R.

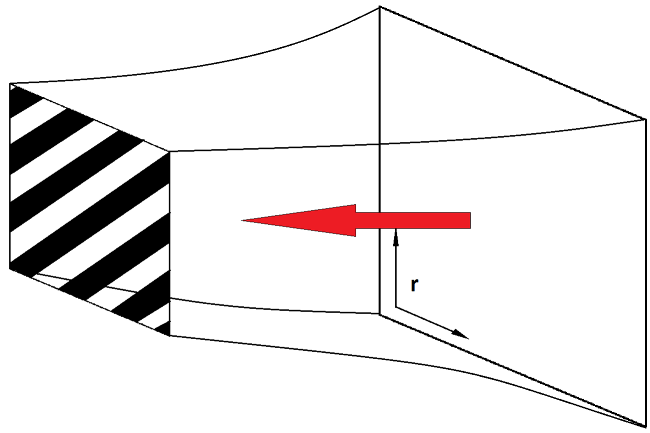


Figure 3.3: This figure illustrates the waves falling through the black hole horizon, depicted by the striped region at $u = R$, which describes dissipation of the CFT at $T \neq 0$.

The use of black branes in holography describes much of the CFT physics accurately. For example waves that propagate in the $D + 1$ dimensional space-time will get damped as they lose energy crossing the black hole horizon. This damping is related by the above identity to the transport coefficients corresponding to the conserved charge of the CFT.

The second point is that the action considered thus far has been that of the ABJM model [1], which describes a large class of CFTs, however it does not describe that of the B-H model given in a previous section. In order to consider a broader class of dual models methods similar to those of effective field theory will be used [20]. One way is to add terms to the action with a higher number of derivatives. The terms must preserve the underlying symmetries including parity, i.e. they cannot include the totally antisymmetric ϵ tensor. The next logical terms to consider are of fourth order in derivatives, since the action contains terms up to second order. Both of these constraints imposed together allow only 15 terms to be constructed using the degrees of freedom to hand, [19]. This can be whittled down further by using integration by parts, Riemann tensor identities such as $R_{[abc]d} = 0$ and Bianci-like identities such as $\nabla_{[a}F_{bc]} = 0$ so that only eight terms remain

$$\alpha_1 R^2 \ ; \ \alpha_2 R_{ab}R^{ab} \quad ; \quad \alpha_3 (F^2)^2 \ ; \ \alpha_4 F^4 \ ; \ \alpha_5 \nabla^a F_{ab} \nabla^c F^b_c \ ;$$
$$\alpha_6 R_{abcd} F^{ab} F^{cd} \quad ; \quad \alpha_7 R^{ab} F_{ac} R^c_b \ ; \ \alpha_8 R F^2 \tag{3.7.8}$$

where $F^4 = F^a_b F^b_c F^c_d F^d_a$, $F^2 = F_{ab}F^{ab}$, and $R^2 = R_{ab}R^{ab}$.

These interactions might appear in string theory as quantum or higher order α' corrections to the second order derivative SUGRA action [10, 4]. So these terms would probably be part of a perturbative expansion such that higher order terms are suppressed by e.g. the string scale over the curvature scale. In the dual CFT they would represent corrections suppressed by inverse powers of the 't Hooft coupling and/or number of colors N_c.

By focussing only on terms that affect charge transport and preserve the underlying symmetries we can neglect the first two terms as these depend only upon curvature terms which will add corrections to the stress-energy tensor of the CFT, but not the conductivity. Because terms α_3 and α_4 include four field strength terms they will not contribute to the conductivity which can at most involve only two field strength terms, at least at zero density. The fifth term proportional to α_5 contains two field strength terms and so in principle could modify the conductivity, however because of the additional derivatives it will lead to higher derivative terms in the equations of motion of the theory which, as is explained in [21] will induce non-unitary properties in the dual CFT. The last two terms are also second order in F and so appear to modify the conductivity, however as is discussed in [20, section 7] their contribution is relatively trivial in that they renormalize the coefficient of the Maxwell term e. Hence only the term α_6, which includes the Riemann curvature tensor and two field strength terms, will modify the conductivity in question, and so is the only term that needs to be added to the action. The addition of this term into the action also has the property of breaking the self-duality of the Einstein-Maxwell theory, [20, section 6].

Modifying the previous action to include this additional interaction term gives

$$\mathcal{S}_{EM} \to \mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4e^2}F_{ab}F^{ab} + \frac{\gamma L^2}{e^2}C_{abcd}F^{ab}F^{cd}\right]. \tag{3.7.9}$$

where C_{abcd} is the Weyl curvature tensor, the traceless part of the Riemann curvature tensor. Thus the term added is not quite the term proportional to $alpha_6$, but a linear combination of the terms $alpha_{6,7,8}$ which as stated before will renormalize the coupling e. This action is hopefully dual to a class of models that realize a CFT description of which the B-H model is one. The two free parameters in the action are γ and e. As some reference shows by considering the $\omega \to \infty$ limit of the correlators in the CFT it can be shown that e is related to Σ_∞ and γ is dual to a 3-point correlator of the current J_μ and the stress energy tensor [20].

The method used to apply the gauge-gravity duality to the B-H model is to use effective field theory to justify the use of the effective action as the $D + 1$ gravity theory. Field theory methods can then be used to compute dynamics of the CFT at zero-temperature which can then be compared with those of the gravity theory to match parameters in both theories. Then the case in which $T \neq 0$ is considered in the $\omega \to \infty$, i.e. the long time limit, and compared with the gravity dual in which the black hole solution is used to describe zero-temperature physics. This approach is similar to the implementation of the Boltzmann equations used in the solvable phases of the B-H model. This suggests that the gravity theory replaces the Boltzmann equation for CFTs.

This results in a form for the conductivity $\sigma(\omega)$ related to the two-point function of the current in the CFT. The results depends on both of the free parameters e and γ, however, by considering the ratio $\sigma(\omega)/\sigma_\infty$ the e dependence factors out. By imposing causality on the gravity side as shown in [20, section 5] the value of γ is constrained so that $|\gamma| < /12$. The results are shown in this figure

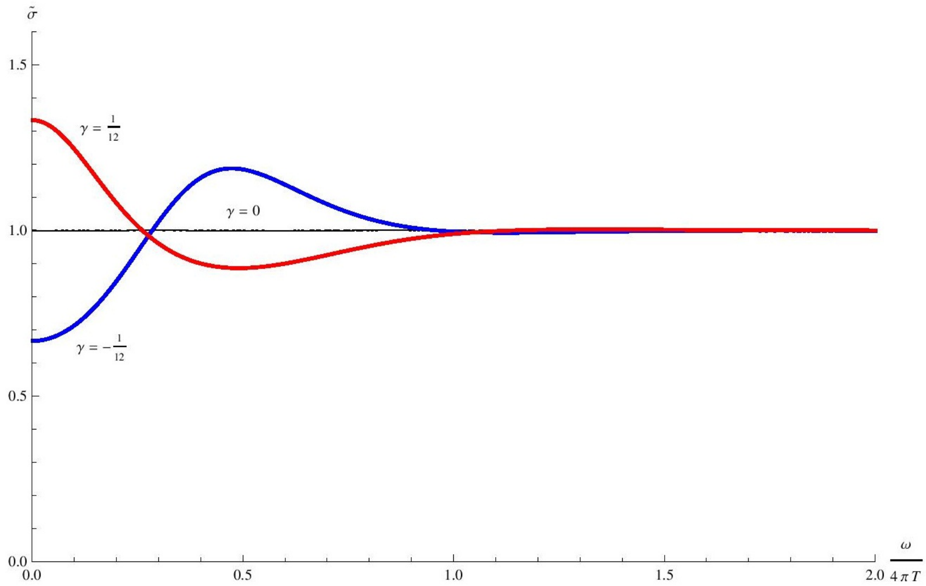


Figure 3.4: The frequency dependent conductivity obtained by the method described above where the extremal values of γ have been chosen. The conductivity is in its dimensionless form such that $\tilde{\sigma} = 1/e^4\sigma$ where the charge e is defined in 3.7.1. This figure was taken from [20].

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Looking at the form of this function for allowed values of γ and ω one notices how little the form of the function varies from $\gamma = 0$ to its extremal values. It also corresponds very well with the conductivity forms obtained using the Boltzmann equation by extrapolating from both quantum phases to the critical point: the $\gamma > 0$ part resembles the particle-like Boltzmann approach and the $\gamma < 0$ part resembles the vortex-like approach. Thus γ appears to determine how close the behavior of the CFT favors particle-like behavior, or vortex-like behavior. The value of this parameter has not been ascertained yet in the case of the CFT of the B-H model, but it can be obtained by using the method detailed above.

Chapter 4

Conclusion

This dissertation has reviewed the AdS/CFT correspondence and demonstrated an example of how it can be used to describe certain condensed matter systems under conditions such that traditional methods do not work. The usual methods for computing the dynamical properties of the B-H model for different values of its physical parameters involve Boltzmann statistics, and assume that all excitations can be described by particle excitations that exist over a long time scale. This assumption results in a contradiction when this picture is extrapolated from the insulating phase of the B-H model and the superfluid phase towards the critical point between the two phases: the two pictures do not give the same result.

Holographic methods however, offer an alternative approach to the problem due to the system exhibiting conformal invariance at the quantum critical point, thus allowing the AdS/CFT correspondence to be applied. The problem is not only solvable using this method, its solution also complements the results in the two quantum phases. Despite this work the frequency dependent conductivity for the B-H model is still not well understood at the quantum critical point. As shown in figure 3.7 the dimensionless conductivity is determined as a function of frequency and of the parameter γ dual to physical observables in the QFT, however the value of γ is still undetermined.

This dissertation has been concerned solely with bosonic systems mainly due to their simplicity in order that the concepts can be easily introduced. A natural question to ask is if there are any more systems for which these methods are applicable, especially fermionic systems. All the systems that can be modeled by the B-H model come under the class of conformal quantum matter systems which are all incompressible as was shown in the phases of the B-H model 3.2. There are also systems with non-zero compressibility that appear to favor the use of holographic methods, notably a set of materials called ‘strange metals’ whose description in terms the fractionalized Fermi liquid mirrors that of a charged black hole in anti-deSitter space [27].

The simplest example of a compressible state from the gravity theory point of view is described by the Reissner Nordström solution to Einstein’s field equations [36, pages 313, and 317-318]. It turns out that even this simple example has many properties in common with the fractionalized Fermi liquid described above [31, 32]. Going beyond this simple case is topic of ongoing research [11, 13, 9, 14] by including additional interactions between the matter fields and the geometry.

It is clear that the AdS/CFT correspondence is going to continue to deliver interesting new physics which will help tackle longstanding problems not just in high energy theory but in many other areas such as condensed matter theory, heavy ion collisions. Further work in this field will address important questions not just for condensed matter systems but also for quantum gravity, and in particular string theory. It is possible that string theory is the theory of everything and a valid quantum theory of gravity, however AdS/CFT shows that it could be a tool for better understanding quantum systems. Considering the work that has been done in the last 15 years alone one can hope to see the resolution to many of these problems very soon.

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