SOLUTIONS AND DYNAMICS OF HIGHER DIMENSIONAL BLACK HOLES

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Abstract

This essay will review black holes in higher dimensional flat space. Due to prediction made by the ADD theory and several unifying theories it is thought to be meaningful to examine general relativity in higher dimensions. Several black hole solutions have been found; static and stationary solution as well as the more exotic black rings and black strings, which disproves uniqueness in higher dimensions. The black strings have been proven to be unstable and the endstate of such strings violates the cosmic censorship conjecture. A branch of black strings is further found to able to change topology. Interestingly these have analogies in fluid dynamics.

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Introduction

The aim of this work is to give an review of black hole solutions and black hole dynamics in dimension $d>4$, for classical general relativity.

1.1 Justification

While General Relativity in higher dimensions has been a increasing area of study for the last couple of years, it might not be perfectly obvious why this is interesting. There is however a number of reasons justifying this as an important field and worth further study.

1.1.1 Scientific curiosity

From a purely theoretical point of view there is nothing stopping us at four dimensions when examining black holes and General Relativity. Rather, it turns out, the special characteristics of four-dimensional space-time restricts the possible solutions and is the cause of for example the uniqueness theorem of black holes. Uniqueness refers to the fact that for given black hole parameters (mass, charge and angular momentum) there only exists one possible unique black hole, rather than a variety. This was proven in four-dimensions, where the proof relies on the fact that the horizon topology of the black hole had to be S^2 [1].

In higher dimensions this is not necessarily true and it turns out that uniqueness sometimes breaks down for black holes. While uniqueness still holds for the static case in $d>4$, the Schwarzschild-Tangherlini black hole solution, there is no uniqueness theorem for the stationary cases [2]. There are even several counter-examples where you explicitly show non-uniqueness. The consequence is then that for the same mass, charge and angular momentum there exist several black hole solutions with different horizon topology. Phase transitions between the different phases is also allowed[3].

This implies that black holes in d>4 do not necessarily need to be stable. It has been found that dynamical instabilities due to tensor perturbations exist, which are dependent on the wavelength of the perturbation. This is called Gregory-Laflamme instability and was first described in 1993 [4].

The horizon topology of the black hole solutions are also considerably different and varied from that of a four-dimensional case, where only S^2 is allowed. The topology no longer needs to be spherical. One of the most straight forward examples of this is the black strings, where a black hole in q dimensions is extended by simply adding a spatial dimension. The new horizon will then have a geometry of $S^q \otimes \mathbb{R}$. This does not need to stop here, if $\mathbb{R}^p$ could be added making up a black brane, with horizon $S^p \otimes \mathbb{R}^p$.

One could also make a circle of the added dimensions to make a black string, with topology $S^q \otimes S^1$. This might at first glance be unstable, however with added rotation working against the gravitational pull, stable solutions are possible.

There is also the possibility of multi-black hole solutions, for example a stationary black hole surrounded by a black ring. This particular solution is called a black Saturn solution.

This richness and variety of solutions points to a great difference between the case of four-dimensional black holes and higher dimensional black holes. The physics of higher dimensional black holes is much more complicated with features such as non-uniqueness, non-spherical horizons and dynamical instabilities. It could then be valuable to compare $d>4$ with $d=4$ to see which principles of General Relativity hold generally and which are unique for $d=4$. This could be meaningful as it might give insight and deepen the understanding of the General relativity and black holes.

1.1.2 Unification

Other than scientific curiosity, it could also be of importance to investigate general relativity in higher dimensions.

It has been the suspicion of physicists for almost a century that if General Relativity is to be unified with quantum mechanics it is vital that higher dimensions are introduced. One of the first, and quite important, attempts to unify the forces was made by Theodor Kaluza in a paper in 1921 [5]. He managed to show that General Relativity in five dimensions, the product of four-dimensional Minkowski space and S^1, not only contains four-dimensional gravity but also the theory of electromagnetism. This theory was later extended by Oscar Klein who improved the mathematical framework [6]. Unfortunately, the theory contains inherent problems with the introduction of fermions, making it unphysical. However still to this day Kaluza-Klein theory is used as a toy theory.

There are still strong beliefs and hopes that all the fundamental forces might be unified in one great theory. More contemporarily, String Theory or M-theory is being put forward as this possible Theory of Everything. String theory seems to be able to describe both the quantum gravity, and further seems to be renormalisable. The theory prefers to exist in higher dimensions, usually either ten ("critical dimension") or eleven dimensions. This preference is brought about by the fact that quantum anomalies cancel in ten dimensions, which is required for the weakly coupled string theories [7]. To fully understand such a theory it would then be required to study General Relativity in higher dimensions [8].

An interesting extension of String Theory has also been proposed, by Juan Maldacena in 1997 [9], which relates superstring and M-theory with large N-limit of Conformal Field Theory(CFT). There is a conjectured equivalence between CFT and M-theory on Anti-de-Sitter space (AdS) [8]. Black holes in d dimensions can then be related to the properties of Quantum Field Theory in d-1 dimensions.

1.1.3 The Hierarchy Problem

One of the most gapping problems with present day physics, which future possible unified theories must explain, is the hierarchy problem of the standard model. A hierarchy problem occurs when fundamental parameters calculated are very different than those measured. In terms of the standard model problem is in simple words the question [10]; why is gravity 10^{33} times weaker than the weak force?

At the moment it looks like physics operates on two entirely different scales, the Planck scale m_{pl} and the electroweak scale m_{ew} [11]. Further when attempts are made to calculate quantum corrections to the Fermi's constant of the electroweak theory; it appears as if this constant is significantly larger than it should. Rather it should be of the order of the Newtons constant which rules gravity. Fermi's constant should not of the size it is unless there is some exact quantum cancellation, which is also problematic because it indicates fine tuning [12].

One solution to this problem would be supersymmetry, a theory where ordinary particles are each related to a supersymmetric partner. Another very appealing theory is that there exist higher dimensions, usually compact, imbedded in our space-time. This model is known as the ADD model, and the idea was first put forward by Arkani-Hamed et al. [12] in 1998.

If extra dimensions exist that could possibly explain why gravity is so much weaker. It would also unify the m_{pl} and m_{ew}, as the m_{pl} would not be a fundamental scale rather the effective four dimensional gravity. It would get its size as a consequence of the rather large size of the extra dimensions proposed [13] [11] [10].

This theory has some rather interesting consequences, the most important being that the scale of which quantum gravity effects could be seen is significantly earlier than previously believed, at scales of m_{ew} $\sim$TeV [12].

Several papers where quite quickly published pointing out that if quantum gravity effects where possible to achieve at orders of TeV, it was a real possibility of black holes being created at the Large Hadron Collider [14] [15].

The black holes produced in such a scenario are significantly smaller than the size of the extra dimensions. This in essence means that the black holes will behave as if they are living on flat space, making it meaningful to examine the properties of black holes in higher dimensional spacetime. Should this theory hold, miniature black holes might form, and then quickly disapear through Hawking Radiation, and thus be observed (for review of such black holes please see Kanti [16]). If this turned out to be the case, that would present an uniqe oppertunity to study a very exotic phenomena. This might also give further insight and data into the theory of Quantum Gravity.

1.2 Difference between d=4 and d>4

As previously mentioned the number and variety of black holes are far larger and more intricate than that of four dimensions. A very obvious reason for this would be the fact that higher dimensions get messier due to increased degrees of freedom. In four dimensions you have $4-1=3$ degrees of freedom of spatial dimensions. Add another dimension, and you get four spatial dimensions, which means two independent planes, each associated with one independent rotation. For d dimensions you then get $N = (\frac{d-1}{2})$ independent rotations, which can be written as $U(1)^N$. Each of these carry an angular momentum J_i, where $i = 1, ..., N$.

Even though part of the answer lies in the fact that the equations get complicated with more degrees of freedom, this is not the full explanation. Rather it has to do with why four dimensions are so special for general relativity. This is the smallest possible dimensions where the theory is not trivial. In three dimensions general relativity is topological and the Weyl tensor cancels perfectly, resulting in that objects do not interact. And while Einsteins field equations, which rule relativity, are non-trivial and highly non-linear there has been many extremely successful methods developed to handle them in four dimensions [17].

One example of a theory specific for four dimensions is the Newman-Penrose formalism. This formalism combines tetrad calculus and spinor calculus into one compact set of equations, making it possible to explicitly write out the Einstein equations. This makes dealing with Einsteins equations, which are non-linear, much easier. As it turns out it has been hard to extend this formalism into higher dimensions [18].

Further solutions with very high symmetry are usually studied. Spherically static using the Birkhoffs theorem and stationary solutions are the easiest and most common. In four dimensions this implies that there are two symmetries, time translation and axis rotation, which reduces the Lagrangian density specific form, and thus the theory into a sigma model.

There are some natural ways to extend this to higher dimensions, however these have some problems. If the axis symmetry is described by a $O(d-2)$ group, the axial orbits, $(d-3)$ spheres, are of non-zero curvature. The Lagrangian then gains a exponential term, which prevents straight-forward integration of the equations [17].

If commuting $U(1)$ symmetries are used, we get a $U(1)^{(d-3)}$ spatial symmetry which at first glance fixes the problem, and results in a integratable sigma model. However this is only possible for $d = 4, 5$. This is why there has been great success in expanding black hole theory to five dimensions but $d > 5$ has proven difficult [19].

Lastly a very big difference is that the potential of gravity is dimensionally dependent with

$$U_{GR} = -\frac{GM}{r^{d-3}} \tag{1.1}$$

The centrifugal potential on the other hand, which counteracts the gravity for rotating black hole, looks like

$$U_{centrifugal} = \frac{J^2}{M^2 r^2} \tag{1.2}$$

This is not dimensionally dependent as it is dependent on a rotation in a plane. This is particulaily interesting in the case of d=5, where (1.1) and (1.2)cancel eachother perfectly. This characteristic is important for the black rings, which first were thought to be unique to five dimensions [20]. This is however not thought to be the case anymore [21].

1.3 Outline

This review will be divided into 3 different sections excluding the introduction and conclusion.

General Relativity in $d > 4$

This section will talk about the generalisations necessary when extending general relativity from four dimensions to higher dimensions. Einstein's equation and the geodesic are stated. Further two different methods to derive the conserved charges. The concept of singularities is also discussed.

Black hole solutions

The higher dimensional analogies of Kerr and Schwarzschild black holes, the Tangherlini and Myers-Perry solutions are introduced. The similarities and differences between them and their lower-dimensional brothers are discussed.

The black ring solution mentioned before is presented. Its existence will break uniqueness in higher dimensions. Further, curiously there exist multi black hole solutions, for example the black Saturn solution or two rings rotating around each other.

The rod structure of black holes, which is a solution generating technique and black hole thermodynamics will be briefly discussed as well.

Dynamics

The different dynamics of black holes is examined. It was discovered in 1993 by Gregory and Laflamme [4] that black strings were not inherently stable, rather they had a wavelength-dependent instability which later came to be called the Gregory-Laflamme instability. The endstate of such instability was widely discussed, because it might possibly violate the cosmic protection conjecture. It was found that this was indeed the case.

The special case when the instability is stationary, in a so called non-uniform black string is presented, as well as the instability of ultra-spinning black holes.

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General Relativity in d>4

Corrections need to be made to General Relativity to generalize to d dimensions. In this section the basis framework for d-dimensional general relativity is given.

2.1 Basic Framework

The most common concepts of General Relativity are reviewed to establish conventions for future references.

2.1.1 Einstein's Field Equations

The basic idea of General Relativity is that space-time can be thought of as a smooth d-dimensional manifold $\mathcal{M}$ (where d=4 for conventional relativity) which contains a metric. This kind of manifold is called a Riemannian manifold. This manifold contains local Lorentzian covariance. A line element on such a manifold is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

for four dimensions $\mu,\nu=0,...,3$. However for arbitrary d dimensions we instead sum over $\mu,\nu=0,...,d-1$. The factor dx^μ is a oneform gradiant of some coordinate x^μ on $\mathcal{M}$. The line element further contains information about the structure of the spacetime. When $ds^2<0$, the spacetime is timelike,when $ds^2>0$ the spacetime is spacelike and $ds^2=0$ represents lightlike spacetime[22].

The metric $g_{\mu\nu}$ is central to relativity, as it defines different types of spacetimes and is used to raise and lower indicies. The signature of the metric used is $(-+ \ ... \ +)$[23].

To generalize the partial derivative, ∂_μ, in curved space the notion of covariant derivative, ∇_μ, is introduced. The covariant can take different forms depending on which spacetime you are considering. The Riemann tensor(or curvature tensor) is defined by commutating the covariant derivative and letting it act on a vector field V as follows [23]

$$[\nabla_\mu, \nabla_\nu]V_\rho = R_{\mu\nu\rho\sigma}V^\sigma \tag{2.2}$$

The Riemann tensor is in essence a measure of locally how close the metric is to that of Euclidian space. From explicitly writing out the Riemann tensor one can find several symmetric properties, namely[24]

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{\rho\sigma\mu\nu} \qquad\qquad R_{\mu[\nu\rho\sigma]} = 0 \tag{2.3}$$

The Riemann tensor also obeys the Bianchi identity [24]

$$\nabla_{[\mu}R_{\nu\rho]\sigma\tau} = 0 \tag{2.4}$$

It is often of value to consider the contraction of the Riemann tensor, the Ricci tensor, which is defined as[25]

$$R_{\mu\nu} \equiv g^{\rho\sigma}R_{\mu\rho\nu\sigma} \tag{2.5}$$

Similar to the Riemann tensor, the Ricci tensor is a represenation of how much the volume element of the geodisic in a manifold $\mathcal{M}$ differs from that of a sphere in Euclidian space . The Ricci tensor can be futher reduced to a Ricci scalar, which is defined as[25]

$$R \equiv g^{\mu\nu}R_{\mu\nu} \tag{2.6}$$

which is also a measure of curvature. By twice contracting (2.4) we find that

$$\begin{aligned} 0 &= g^{\tau\rho}g^{\sigma\mu}(\nabla_{mu}R_{\nu\rho\sigma\tau} + \nabla_\nu R_{\rho\mu\sigma\tau} + \nabla_\rho R_{\mu\nu\sigma\tau}) \\ &= \nabla^\sigma R_{\nu\sigma} - \nabla_\nu R + \nabla^\tau R_{\nu\tau} \end{aligned} \tag{2.7}$$

or

$$\nabla^{\sigma} R_{\nu\sigma} = \frac{1}{2}\nabla_{\nu} R \tag{2.8}$$

Then by defining a tensor, the Einstein tensor, as follows

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.9}$$

the Bianchi identity(2.7) is reduced to

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{2.10}$$

The geometry of spacetime is determined by the Einstein Equations, which in natural units look like

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.11}$$

$T_{\mu\nu}$ is the stress-energy tensor, which is the tensor generalisation for mass density[24]. This is the General Relativity equivalence to Newton's third law, which can be recovered in classical limits.

If the cosmological constant is added the equations become

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.12}$$

Please note that, as Durkee [23] points out, there are different conventions for how Λ is defined in higher dimensions. It is quite common to make the cosmological constant dimension dependent by replacing Λ with $\frac{2\Lambda}{d-2}$.

Further if flat vacuum space is assumed Einstein's equation reduces to

$$R_{\mu\nu} = 0 \tag{2.13}$$

Finally, as the Ricci tensor and the Ricci scalar are contractions they encrypt information about the trace of the Riemann tensor, not the full Riemann tensor. It is therefore useful to define a tensor that contains all the aspects of the Riemann tensor, the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{n-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu} \tag{2.14}$$

The complicated formations are so that all contractions of the tensor disappear, and so that the Weyl tensor keeps all the symmetric properties of the Riemann tensor defined in (2.3). An important property of the Weyl tensor is that it is invariant under conformal transformations [24].

2.1.2 The geodisic equation

Other than Einstein's field equations, a central feature in General Relativity is that of the geodesic. Geodesics are non-Euclidian spacetime equivalences to straight lines, and encode important information about the space and the movement of particles. A straight line can be defined as the shortest possible path length between two points, or more formally as the path where the tangent vector is parallel transported[24]. Studying the properties of solutions to the geodesic can be of importance as it can reveal structures and characteristics of a specific space.

On a d-dimensional Riemannian manifold $\mathcal{M}$ the geodesic equation is

$$\nabla_\mu V = 0, \tag{2.15}$$

where V can be written as $V^\mu = \dot{x^\mu}$, in some coordinate system x^μ. A covenient way to find the geodesic is to start from the Lagrangian

$$L = \frac{1}{2}g_{\mu\nu}(x)\dot{x^\mu}\dot{x^\nu} \tag{2.16}$$

The Killing vector fields combined with the derivative of the Lagrangian, $\partial L/\partial \dot{x^\mu}$ defines conserved quantities or symmetries of a spacetime. These symmetries combined with the hidden symmetries associated with the Killing-Yano tensor turned out to render the geodesic integratable in the case of the Kerr metric[26]. It was futher proved that

Spacetime	4	5	6	7	8	9	10	11	≥12
Schwarzschild	$\odot$	$\odot$	$\bigoplus$	$\odot$	$-$	$\bigoplus$	$-$	$\bigoplus$	$-$
Schwarzschild-de Sitter	$\bigoplus$	$\odot$	$-$	$\odot$	$-$	$\bigoplus$	$-$	$\bigoplus$	$-$
Reissner-Nordrm	$\odot$	$\odot$	$-$	$\bigoplus$	$-$	$-$	$-$	$-$	$-$
Reissner-Nordrm-de Sitter	$\bigoplus$	$\odot$	$-$	$\bigoplus$	$-$	$-$	$-$	$-$	$-$

Table 2.1: Table is from Hackmann et al. [28]. The $\odot$ represent exact solutions by elliptic functions and the $\bigoplus$ solutions of hyperelliptic functions. The $-$ indicates, by known methods, unsolvable geodesic.

space-time in which the metric admitting a Killing-Yano tensor [27] makes the geodesic equation completely integratable.

Unfortunately this does not always generalize to higher dimensions; however several geodesics in higher dimensions have been solved exactly. Table 2.1 shows which spacetimes currently have solutions. Please see the article by Hackmann et al. [28] for in-depth calculations and discussions.

2.2 Conserved charge

2.2.1 Hamiltonian approach

Using a Hamiltonian approach, one can deduce the conserved charges in higher dimensional flat space (for full calculations please see Jamsin [29]). This approach requires that the spacetime is broken into space and time components. The Hamiltonain is given by

$$H[g_{ij},\pi^{ij}] = \int d^{D-1}(N(x)\mathcal{H}(x) + N^i\mathcal{H}_\rangle) \tag{2.17}$$

where π^{ij} is the conjugate momenta, $N = -(g^{00})^{-\frac{1}{2}}$, $N_i = g_{0i}$ and $\mathcal{H}$ the Hamiltonian density. By using canonical variables, the disturbance created by a Killing field $\xi = \xi^\perp \mathbf{n} + \xi^i \mathbf{e}_i$ is

$$Q[\xi] = Q_0 + \mathcal{I}[\xi] = \int d^{d-1}x(\xi^\perp(x)\mathcal{H} + \xi^i(x)\mathcal{H}_i(x)) + \mathcal{I}[\xi] \tag{2.18}$$

where $\mathcal{I}[\xi]$ is the boundary term. The term $\mathcal{I}[\xi]$ describes the conserved charge for some Killing field $\xi(x)$, when taking constraints into account. By taking an infinitesimal variation of $Q[\xi]$ and $Q_0[\xi]$, and requiring that they have defined derivatives, the following general expression is found for the conserved charge

$$\begin{aligned}\delta\mathcal{I}[\xi] &= \oint d^{d-2} s_l G^{ijkl}(\xi^{\perp}\delta g_{ij;k} - \xi^{perp}_{,k}\delta g_{ij}) \\ &\quad + \oint d^{d-2}(2\xi_k \delta\pi^{kl} + (2\epsilon^k \pi^{jl} - \xi\pi^{jk})\delta g_{jk}) \end{aligned} \tag{2.19}$$

where $G^{ijkl} = \frac{1}{2}\sqrt{g}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})$.

An explicit expression for the conserved charges can be found by looking at the asymptotic behaviour of the metric $g_{\mu\nu}$. For asymptotically flat space a general metric has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.20}$$

where $\eta_{\mu\nu}$ is the metric of Minkowski space and $h_{\mu\nu}$ is a perturbation. As the space is asymptotically flat, that is it approaches Minkowski space at infinity, the perturbation $h_{\mu\nu}\longrightarrow 0$ when $r\longrightarrow\infty$. By requiring that the solution should be valid at infinity and invariant under the Poincare group the RHS of (2.19) can be written as

$$\delta(\oint d^{d-2} s_r(\eta^{ik}\eta^{jr} - \eta^{ij}\eta^{kr})(\xi^{\perp}h_{ij;k} - \xi^{\perp}_{,k}h_{ij}) + 2\oint d^{d-2} s_r \pi^r_k \xi^k) \tag{2.21}$$

which reduces the equation to

$$\mathcal{I}[\xi] = \oint d^{d-2} s_r(\eta^{ik}\eta^{jr} - \eta^{ij}\eta^{kr})(\xi^{\perp}h_{ij;k} - \xi^{\perp}_{,k}h_{ij}) + 2\oint d^{d-2} s_r \pi^r_k \xi^k \tag{2.22}$$

Explicit calculations of the charge can be made. For example energy E, created by time translation symmetry, and angular momentum J_I, created by rotational symmetry, can for a stationary black hole be written as

$$E = \frac{(d-2)\Omega_{d-2}}{16\pi G}\mu \tag{2.23}$$

$$J_i = \frac{\Omega_{d-2}}{8\pi G}\mu a_i \tag{2.24}$$

2.2.2 Pertubation of Einstein's Equation

Using again an asymptotically flat metric (2.20), we may rewrite the Christoffel symbol as follows [22]

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\ &= \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu})\end{aligned} \tag{2.25}$$

Since we assume the perturbation $h \longrightarrow 0$, the calculations can be limited to the first order of Γ, all higher orders are disregarded. The Riemann tensor then becomes [24]

$$\begin{aligned}R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho\Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma\Gamma^{\lambda}_{\nu\rho} \\ &= \frac{1}{2}(\partial\rho\partial\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial\nu h_{\mu\rho} - \partial_{rho}\partial_\mu h_{\nu\sigma}\end{aligned} \tag{2.26}$$

Contracting, the Ricci tensor becomes

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \tag{2.27}$$

and the Ricci scalar becomes

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \tag{2.28}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ and $\Box$ is the d'Alembertian. These expressions can then be substituted back in the Einstein equation to give

$$\begin{aligned}G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h)\end{aligned} \tag{2.29}$$

By using the Lorentz gauge

$$\partial_\mu h^{\mu}{}_{\lambda} - \frac{1}{2}\partial_\lambda h = 0 \tag{2.30}$$

we can then simplify the expression further as follows [19]

$$-16\pi G T_{\mu\nu} = \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = \Box\overline{h}_{\mu\nu} \tag{2.31}$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$. As we have a first order pertubation, which is flat at infinity, the sources of mass M and angular momentum J are the same as pointlike sources, situated at the origin x^{μ}.

This can be solved using the Green function [30]

$$\overline{h}_{\mu\nu}(x^i) = \frac{16\pi G}{(d-3)\Gamma_{d-2}} \int \frac{T_{\mu\nu}}{|x^i - y^i|^{d-3}} d^{d-1}y^i \tag{2.32}$$

and by determining the input of the stress-energy tensor as

$$\int T_{00} d^{d-1}x = M \tag{2.33}$$

and

$$\int T_{0i} d^{d-1}x = 0 \tag{2.34}$$

The angular momentum tensor is defined as

$$J^{\mu\nu} = \int (x^{\mu}T^{\nu 0} - x^{\nu}T^{\mu 0}) d^{d-1}x \tag{2.35}$$

By integrating the Green equation (2.32) to first order, an expression for $\overline{h}_{\mu\nu}$ can be found, and subsequently

$$h_{00} = \frac{16\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}} \tag{2.36}$$

$$h_{ij} = \frac{16\pi G}{(d-3)(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}} delta_{ij} \tag{2.37}$$

$$h_{0i} = \frac{8\pi G}{\Omega_{d-2}} \frac{x^k J_{ki}}{r^{d-1}} \tag{2.38}$$

If angular momentum matrix J_{ij} can be put into a diagonal matrix form and polar coordinate coordinates are introduced, (2.38) can be written as [19]

$$h_{0\phi_a} = -\frac{8\pi g J_a}{\Omega_{d-2}} \frac{\mu_a^2}{rd-3} \tag{2.39}$$

where $\mu_a = \frac{r_a}{r}$. These results will yield identical results to the results from the Hamiltonian approach.

With these we can define the Newtonian gravitational potential energy

$$\Phi = -\frac{1}{2}h_{00} \tag{2.40}$$

and the Newtonian force

$$F = -\nabla\Phi = \frac{(d-3)8\pi GM}{(d-2)\Sigma_{d-2}r^{d-2}}\widehat{\mathbf{r}} \tag{2.41}$$

Further as we would like to compare properties of different black hole solutions it is meaningful to introduce scale-invariant variables for the physical parameters, namely the area of the horizon, A_h, and the angular momentum ,J_a. The following dimensionless quantities, a_H for surface area and j_a for spin, are suggested by Emparan et al. [31]

$$j_a^{d-3} = c_j \frac{J_a^{d-3}}{GM^{d-2}} \tag{2.42}$$

$$a_H^{d-3} = c_a \frac{A_h^{d-3}}{(GM)^{d-2}} \tag{2.43}$$

where the constants are

$$c_j = \frac{\Omega_{d-3}(d-2)^{(d-2)}}{2^{d+1}(d-3)^{\frac{d-3}{2}}} \tag{2.44}$$

$$c_a = \frac{\Omega_{d-3}(d-2)^{d-2}}{2(16\pi)^{d-2}}(\frac{d-4}{d-3})^{\frac{d-3}{2}} \tag{2.45}$$

2.3 Singularities

Before continuing to the different black hole solutions to the Einsteins equation and their properties, the actual concept of singularities will be discussed. A singularity can be defined as either a spacelike singularity, where matter collapse into on single point, or timelike, where infinite curvature exists but photons are still allowed to escape.

In a sense, this is an area where general relativity stands out from other theories. Often space and time is assumed before a theory is defined, as in electromagnetism. This means that for law of nature, such that Coloumbs law, there is no problem to define where the solutions blow up and create a singularity, which would be at $r = 0$ in this case [25].

In general relativity however, this is very much different as space-time is independent from the theory, but at the centre of it. A singularity in general relativity could instinctively be defined as where the metric or a curvature scalar is infinite or undefined. This is a problem as it is required for the manifold $\mathcal{M}$ and the metric $g_{\mu\nu}$ to be defined to define an occurrence in general relativity. This means that a singularity in general relativity is not necessarily a time or a place as it is the definition of spacetime itself that breaks down, it is not even a part of the manifold $\mathcal{M}$ [24].

There is also a problem with the precise definition of a singularity. One way would be to use the curvature, the Riemann tensor $R_{\mu\nu\rho\sigma}$, and find where it blows up as a measure. However this is inherently difficult because it is not always perfectly obvious where this blow up is due to an actual singularity or due to a misbehaving coordinate system. An extension of this thought would then be to examine the scalar contraction of the Riemann tensor, for example the Ricci scalar R or the Kretschmann scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. This however also has its drawbacks as these measures might become infinite at $r \longrightarrow \infty$, which is not desired. Further some solutions will have a singular curvature scalar even though they are well defined and there is no singularity [25].

A much better way of measuring singularities is to look at the geodesic around the singularity. If the geodesic is assigned affine values but cannot be extended, the geodesic is called incomplete. A singularity will then possess at least one incomplete geodesic [32].

Further, there is the question about how scientifically accurate the idea of a singularity is. The predictions of singularities are often based on perfectly symmetric models. If there is a small perturbation of these models, will there still be a singularity? The question was answered by Penrose and Hawkings, with their singularity theorem. The singularity theorem proved that singularities do exist and are a generic aspect of general relativity. For an exhaustive proof of this theorem please see chapter 9 of Wald [25].

– 3 –

Black Hole Solutions

Solution for $d = 4$		
	Static (J=0)	*Stationary (J≠0)*
No charge (Q=0)	Schwarzchild	Kerr
Charged (Q≠0)	Reissner-Nordstrom	Kerr-Newman

Table 3.1: The existing solutions for different parameters in four dimensions.

Solution for $d > 4$		
	Static (J=0)	*Stationary (J≠0)*
No charge (Q=0)	Schwarzchild-Tangherlini	Myers-Perry
Charged (Q≠0)	Tangherlini	–

Table 3.2: The equivalent to the solutions of table 3.1, in higher dimensions. Please note there exist higher dimensional solutions with non-spherical topology

Black holes have long been at the centre of the examination of general relativity. They have proven essential in the understanding of the theory and highlight the incompatibility of GR and quantum mechanics.

For the Einstein equation there exists four known exact black hole solutions in four dimensions. The black holes are described by eleven different parameters, as stated by the no-hair theorem; mass M, charge Q, three position variables x_i, three linear momentum variables P_i and three angular momentum variables J_i. Table 3.1 shows the known solutions in four dimensions.

Table 3.2 shows higher dimensional equivalents to the four dimension black holes. So far there has not been a proof of the Kerr-Newman solution in $d > 4$.

This section will discuss the solutions in $d > 4$ with $4d$ equivalents, as well as solutions unique to higher dimensions will introduced.

3.1 Schwarzschild-Tangherlini Solution

3.1.1 The metric

The spherically symmetric solution to Einstein solution, presented by Schwarzschild in 1969 [33], was one of the first exact solutions found. This solution was instrumental in proving general relativity, when it predicted small departures from Newtonian mechanics. This solution can be generalized to d dimensions.

The metric for spherically symmetric space can be written generally as [22]

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\Omega_{d-2}^2 \tag{3.1}$$

where $r^2 d\Omega_{d-2}^2$ is a line element on a unit (d-1)-sphere, where f and g are functions of r, $f(r)$ and $g(r)$.

If we assume a vacuum Einstein equation, so that $R_{\mu\nu} = 0$, it can be deduced that [30]

$$f = g^{-1} = \left(1 - \frac{C}{r^{d-3}}\right)^{\frac{1}{2}} \tag{3.2}$$

Using the linearized perturbation introduced in section 2.2.2. for a static source, and the gauge transform [19]

$$r \longrightarrow r - \frac{8\pi G}{(d-2)(d-3)\Omega_{d-2}} \frac{M}{r^{d-3}} \tag{3.3}$$

the integration constant, C, can be found in terms of the mass, M. The final metric is then [34]

$$ds^2 = -(1 - \frac{\mu}{r^{d-3}})dt^2 + \frac{dr^2}{(1 - \frac{\mu}{r^{d-3}})} + r^2 d\Omega_{d-2}^2 \tag{3.4}$$

This was found by Tangherlini in 1963 [34] and generalized the Schwarzschild black hole solutions to d dimensions. This solution is often called the Schwarzschild-Tangherlini solution.

The constant μ is introduced as a dimension dependent mass parameter which is defined as [19]

$$\mu = \frac{16\pi GM}{(d-2)\Omega_{(d-2)}} \tag{3.5}$$

First thing to be noticed is that this simplifies to the normal Schwarzschild metric when $d = 4$ [25]. Secondly even the higher dimensional version is very similar to the four-dimensional one. The only real difference is that the falloff, $\frac{1}{r}$, is replaced with the dimension dependent falloff $\frac{1}{r^{d-3}}$.

If the mass parameter $\mu < 0$ then we get a naked singularity. However this is not physical. If $\mu > 0$ we get a event horizon at

$$1 - \frac{\mu}{r_0^{d-3}} = 0 \longrightarrow r_0 = \mu^{\frac{1}{d-3}} \tag{3.6}$$

This is the same result as the classical calculations would give [35]. In Newtonian mechanics the horizon is where $K + \Phi = 0$. Using natural units so that the escape velocity c is $v = c = 1$, the kinetic energy becomes $K = \frac{1}{2}$. Combined with the knowledge from equation (2.41) we get

$$\Phi + K = -\frac{1}{2}h_{00} + \frac{1}{2} = 0 \longrightarrow h_{00} = 1 \tag{3.7}$$

Replacing for h_{00} from equation (2.36) we get

$$r_0 = \left(\frac{16\pi GM}{(d-2)\Omega_{d-2}}\right)^{\frac{1}{d-3}} \tag{3.8}$$

which corresponds to (3.6).

3.1.2 The motion of particle and light

For a Killing vector K_μ we know that [24]

$$K_\mu = \frac{dx^\mu}{d\lambda} = constant \tag{3.9}$$

Further for a metric (3.4) we know will be constant along a path so that [36]

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{3.10}$$

ϵ depends on what kind of particle is examined. For a massive particle we have $\epsilon = 1$. If we chose $\lambda = \tau$ we get

$$\epsilon = -g_{\mu\nu}V^\mu V^\nu = +1 \tag{3.11}$$

where V^μ is the velocity. If we have a lightlike particle we get $\epsilon = 0$ and for a spacelike geodesic we have $\epsilon = -1$ [25].

Each Killing vector represents a conserved quantity; invariance under the time translation leads to conservation of energy and invariance under the $d-1$ component of angular rotation leads to conservation of angular momentum. As the direction of the angular momentum is conserved, the coordinate system can be rotated so that this direction always is the equatorial plane. This implies that $\theta = \frac{\pi}{2}$ [24].

The remaining Killing vectors represent the angular momentum and the energy. Energy arises from symmetry in time so that

$$K_\mu = \partial_t = \left(-\left(1 - \frac{\mu}{r^{d-3}}\right), 0, ...\right) \tag{3.12}$$

The magnitude of the angular momentum comes from

$$L = \partial_\phi = (0, ..., r^2 \sin^2\theta) \tag{3.13}$$

Since our choice of the direction of the angular momentum sets θ we get that $\sin\theta = \frac{\pi}{2}$ [25]. Using the form from (3.9) we get the two conserved quantities as follows

$$-\left(1 - \frac{\mu}{r^{d-3}}\right)\frac{dt}{d\lambda} = E \tag{3.14}$$

and

$$r^2\frac{d\phi}{d\lambda} = L \tag{3.15}$$

These quantities can be replaced back into the metric. We then obtain an expression for the energy in terms of the velocity and angular momentum of the test particle

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{\mu}{r^{d-3}}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0 \tag{3.16}$$

This can be written in a much more familiar form, similar to that of Newtonian mechanics

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \tag{3.17}$$

where the effective potential $V(r)$ is [24]

$$V(r) = \frac{1}{2}\left(1 - \frac{\mu}{r^{d-3}}\right)\left(\frac{L^2}{r^2} + \epsilon\right) \tag{3.18}$$

The form of the potential again sheds light on a difference between the four dimensional gravity and higher dimensional gravity. In four-dimensional gravity there exists a bounded circular orbit, outside of Schwarschild gravitational hole. Such orbits contain two turning points for the radial coordinate. These two points, r_1 and r_2, correspond to constant energy. The potential $V(r)$ has a minimum between these two turning points so that $r_1 > r_{min} > r_2$. At the minimum we have $E = V(r_{min})$, and the orbit of the test particle is circular [24].

A circular orbit is a necessary condition for bounded orbits. It is known that no such orbits exist in Newtonian gravity for $d > 4$. This can be shown to be true for General Relativity as well. By replacing $d-3$ with $k+1$, we can impose the condition $k \geq 1$ so that the dimension is always larger than 4. By replacing μ with r_0, and then r_0 and r with a new variable $y = \frac{r_0^2}{r^2}$ we get a new form for the potential [37]

$$V(y) = \left(1 - y^{\frac{1+k}{2}}\right)\left(1 + \frac{L^2}{r_0^2}y\right) \tag{3.19}$$

For $r_0 > r > \infty$ we have $0 > y > 1$. If we take the second derivative of the effective

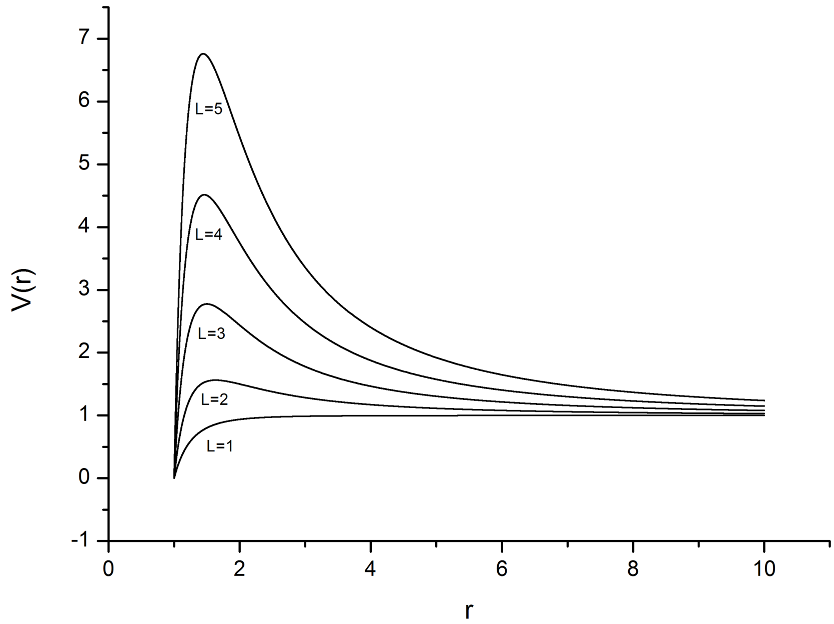

Figure 3.1: The effective potential for different values of L against the radial distance for d=5, with $r_0, \epsilon = 1$

potential we get

$$\frac{\partial(V(y))}{\partial y^2} = -\frac{k^2-1}{4} y^{\frac{k-3}{2}} - \frac{(k+3)(k+1)L^2}{4r_0^2} y^{\frac{k-1}{2}} \tag{3.20}$$

This function will never be zero for our set values of y. This means that for $y > 0$ there will only be one extremum. As $V(y) > 0$, the function has a maximum or is decreasing for all r, for $0 > y > 1$. As y is proportional to r^-2, the function V(r) will have the same properties. The consequence then is that there exists no circular orbit for $d > 4$ and thus no bounded orbits [37].

3.2 Myers-Perry Solutions

While the solution for static black holes and the solution for charged black holes were found early after the initial formulation of general relativity, the solution for a stationary black hole took longer. It was therefore a great success when the Kerr solution was found in 1963 [38]. In 1984 Myers and Perry managed to generalize the Kerr solution to arbitrary dimensions. The Kerr solution looks like [25]

$$ds^2_{Kerr} = -dt^2 + \frac{\rho}{\Delta}dr^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta d\phi - dt)^2 \tag{3.21}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{3.22}$$

and

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \tag{3.23}$$

3.2.1 The general Myers-Perry solution

The metric in the general solution is in the Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + hk_\mu k_\nu \tag{3.24}$$

where k_μ is a null vector with respect to the Minkowski space $\eta_{\mu\nu}$.

Due to the increased complexity when adding angular momentum, it is very hard to solve the Einstein equations. Myers and Perry instead approach the problem with making an ansatz for the metric in d dimensions as follows [30]

$$\begin{aligned} ds^2 \quad = \quad & -\alpha^2(r,\rho)(du + a\sin^2\theta d\phi)^2 + 2(du + a\sin^2\theta d\phi)(dr + a\sin^2\theta d\phi) \\ & +\rho^2(d\theta^2 + \sin^2\theta d\phi) + r^2\cos^2\theta d\Omega_{d-4} \end{aligned} \tag{3.25}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{3.26}$$

$$\alpha^2 = 1 - \frac{\mu}{r^{d-5}\rho^2} \tag{3.27}$$

This can be rewritten as following

$$
\begin{aligned}
ds^2 &= -dt^2 + \sin^2\theta(r^2+a^2)d\varphi^2 + \Delta(dt + a\sin^2\theta d\phi)^2 + \Psi dr^2 + \qquad (3.28)\\
&\quad \rho^2 d\theta^2 + r^2\cos^2\theta d\Omega_{d-4} \qquad (3.29)
\end{aligned}
$$

using the Boyer-Lindquist coordinate transforms, which are used for simplicity [39]. They are analogous to polar coordinates however instead of being spherical they are ellipsoidal, making the relationship between them and Cartesian space a little bit more complicated.The transforms are

$$
\begin{aligned}
dt &= du - \frac{(r^2+a^2)dr}{r^2+a^2-\mu r^{5-d}} \qquad (3.30)\\
d\varphi &= d\phi + \frac{adr}{r^2+a^2-\ mur^{5-d}} \qquad (3.31)
\end{aligned}
$$

The constants in equation (3.25) are

$$
\begin{aligned}
\Delta &= \frac{\mu}{r^{d-5}\rho^2} \qquad (3.32)\\
\Psi &= \frac{r^{d-5}}{r^{d-5}(r^2+a^2)-\mu} \qquad (3.33)
\end{aligned}
$$

This result is however not general; the case for $d = odd$ and $d = even$ is different due to the full angular momentum tensor which looks like

$$
\begin{pmatrix}
0 & J_1 & & & \\
J_1 & 0 & & & \\
 & & 0 & J_2 & \\
 & & J_2 & 0 & \\
 & & & & \ddots
\end{pmatrix} \qquad (3.34)
$$

For even d the last row and column of the matrix will disappear. The case for $d = odd$ and $d = even$ must then be analysed separately [30].

Odd d

For detailed derivations please see Robert C. Myers [30] By deducing expressions for $k_\mu dx^\mu$ and h metric can be written down for d dimensions where d=even as

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + 2\mu_i^2 a_i d\phi_i dr \\ &\quad + \frac{\mu r^2}{\Pi F}(dt + dr + a_i\mu_i^2 d\phi_i)^2 \end{aligned} \tag{3.35}$$

with

$$F = 1 - \frac{a_i^2\mu_i^2}{r^2 +_i^2} \tag{3.36}$$

and

$$\Pi = \prod_{i=1}^{\frac{N-2}{2}} (r^2 + a_i^2) \tag{3.37}$$

Please note that angular coordinate transforms are used as

$$x^i = (r^2 + a_i^2)^{\frac{1}{2}} \mu_i \cos[\phi_i - \arctan\frac{a_i}{r}] \tag{3.38}$$

$$y^i = (r^2 + a_i^2)^{\frac{1}{2}} \mu_i \sin[\phi_i - \arctan\frac{a_i}{r}] \tag{3.39}$$

The first part of the equation (3.35) is the metric for flat space while the rest is from the null vector field. The angle ϕ_i specifies the angle in each $x_i - y_i$ plane. The metric can be put into Boyer-Lindquist coordinates by the following transformations

$$dt = dt - \frac{\mu r^2}{\Pi - \mu r^2} dr \tag{3.40}$$

$$d\phi_i = d\phi_i + \frac{\Pi}{\Pi - \mu r^2}\frac{a_i dr}{r^2 + a_i^2} \tag{3.41}$$

to get the final form for the metric for odd d

$$ds^2 = dt^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r^2}{\Pi F}(dt + a_i\mu_i^{2d}\phi_i^2)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \tag{3.42}$$

Even d

By similar logic the metric for even d is

$$\begin{aligned} ds^2 = -dt^2 + dr^2 + r^2 d\alpha^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + 2\mu_i^2 a_i d\phi_i dr \\ + \frac{\mu r^2}{\Pi F}(dt + dr + a_i \mu_i^2 d\phi_i)^2 \end{aligned} \tag{3.43}$$

where Π and F are defined above. Again this can be transformed into Boyer-Linquist coordinates by the transforms

$$dt = dt - \frac{\mu r}{\Pi - \mu r} dr \tag{3.44}$$

$$d\phi_i = d\phi_i + \frac{\Pi}{\Pi - \mu r} \frac{a_i dr}{r^2 + a_i^2} \tag{3.45}$$

to obtain the metric for even d

$$ds^2 = -dt^2 + r^2 d\alpha^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r}{\Pi F}(dt + a_i \mu_i^2 d\phi_i^2)^2 + \frac{\Pi F}{\Pi - \mu r} dr^2 \tag{3.46}$$

The task to explicitly show that these are solutions to the Einstein equations is extremely tedious but is discussed in appendix A by Robert C. Myers [30].

3.2.2 The characteristics of the metric

As Emparan and Reall [19] suggests, simplifying the metric we can look at the rotation in a single plane, to easier see the physical properties of the solution. The Myers-Perry solution with rotation in only one direction looks like

$$\begin{aligned} ds^2_{MP} &= -dt^2 + \frac{\rho}{\Delta} dr^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{GM}{r^{d-5}\rho}(dt - a\sin^2\theta d\phi)^2 \\ &\quad + \sum d\theta^2 + r^2 \cos^2\theta d\Omega^2_{d-4} \end{aligned} \tag{3.47}$$

where

$$\rho = r^2 + a^2 \cos^2\theta \qquad \Delta = r^2 + a^2 - \frac{GM}{r^{d-5}} \tag{3.48}$$

This is very similar to the Kerr solution (3.21), which is obtained by taking d=4. Just like the Tangherlini solution [34] the most obvious difference is that the falloff now is $\frac{1}{r^{d-3}}$ instead of $\frac{1}{r}$. This means that again the competition between the centrifugal force and the gravitational force is strongly dimensionally dependent.

Further, there is obviously a horizon for the black hole where $g_{00}^{-1} = 0$, which means $\Delta(r) = 0$. This means that

$$r_0^2 + a^2 - \frac{\mu}{r_0^{d-5}} = 0 \tag{3.49}$$

will be the radius. This again is dimensionally dependent. Solving for $d = 4$ case we have

$$r_0 = GM - (G^2M^2 - a^2)^{\frac{1}{2}} \tag{3.50}$$

We have three possible solutions; one where $G^2M^2 > a^2$,$G^2M^2 = a^2$ and $G^2M^2 < a^2$. The extremal case is unstable and the third case corresponds to a naked singularity, which is deemed unphysical [25].

If we repeat the calculations for $d = 5$ we have

$$r_0 = (2GM - a^2)^{\frac{1}{2}} \tag{3.51}$$

This is rather similar to that of the four dimensional case. This will only have a solution up to the extremal case, which here corresponds to $2GM = a^2$. Solutions with $GM < a$ do not exist [19].

The surface area of the solutions is

$$A_H = r_0^{d-4}(r_0^2 + a^2)\Omega_{d-2} \tag{3.52}$$

Replacing for the extremal five dimensional case, we find that the area of the horizon is zero. Therefore this case must correspond to a naked ring singularity

For $d \geq 6$ the equation for $\Delta(r)$ (3.48) will always be positive for large r as the first term will dominate. For small r however the last term will dominate, making it negative. Therefore $\Delta(r)$ will have one positive root which is independent of the angular momentum for six dimensions or higher.

The angular momentum can then be made arbitrarily large, and such black holes are called ultra-spinning. For $j >> 1$ the black hole will flatten in along of the plane which it is rotating. The limit in which $j \to \infty$ the horizon topology will approach that of a black membrane. Emparan and Myers [40] do a in dept analysis of the shape of such ultra-spinning black holes.

Using the equation (2.36) the expression for mass can be deduced as follows

$$M = \frac{(d-2)\Omega_{d-2}}{16\pi G}\mu \tag{3.53}$$

Similarly the expression for angular momentum can be found in terms of mass from equation (2.39)

$$J = \frac{2}{d-2}Ma \tag{3.54}$$

These predictions agree with those made in section 2.2.1. Using the dimensionless variables suggested by] we find the following expression for the angular momentum J, expressed as j, and horizon area A_H expressed as a_h

$$j^{d-3} = \frac{\pi\Omega_{d-3}}{(d-3)^{\frac{d-3}{2}}\Omega_{d-2}}\frac{\nu^{5-d}}{1+\nu^2} \tag{3.55}$$

$$a_H^{d-3} = 8\pi\left(\frac{d-4}{d-3}\right)^{\frac{d-3}{2}}\frac{\Omega_{d-3}}{\Omega_{d-2}}\frac{\nu^2}{1+\nu^2} \tag{3.56}$$

The variable $\nu = \frac{r_0}{a}$ is used to furhter simplifying the equations.

The angular momentum j_H against the area a_H of the black hole, for different dimensions, can now be plotted for comparison. As seen in figure 3.2 the case of $d = 5$ is different from that of higher dimensions, which is a consequence of the existance of ultra-spinning black holes.

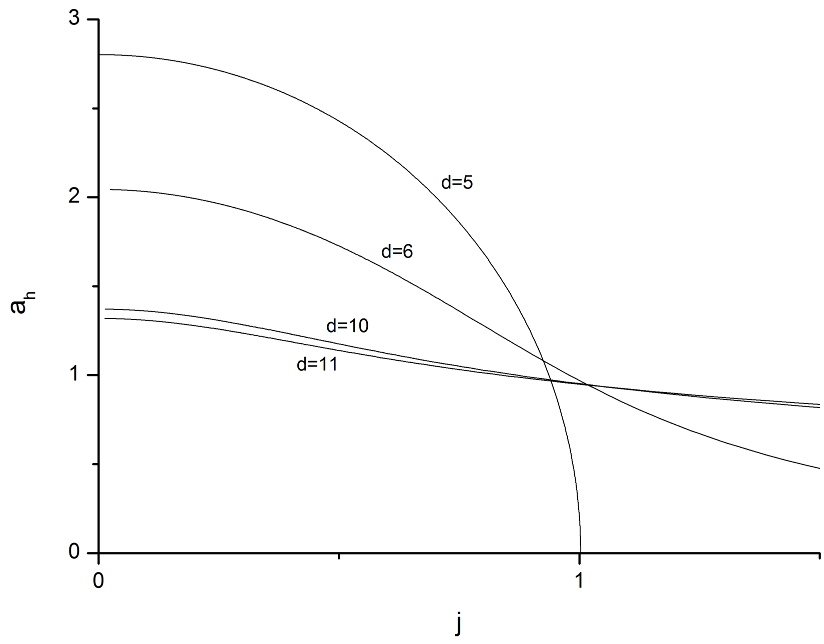

Figure 3.2: The dimensionless area a_H vs the dimensionless angular momentum j for d=5,6,10,11

3.2.3 Motion of particles and light

Just as in the case with the Schwarzschild-Tangherlini solution, the Myers-Perry solution does not have bounded orbits. This proof is limited to five dimensions. This can be shown using the Hamiltonian-Jacobi method which allows separation of variables. By using the Hamiltonian to define [37]

$$-\frac{\partial S}{\partial \lambda} = H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} \tag{3.57}$$

where S is the action. The full expression for S can be deduced to show the two partial derivatives

$$\frac{\partial S_\theta}{\partial \theta} = \omega_\theta \sqrt{\Theta} \tag{3.58}$$

$$\frac{\partial S_r}{\partial r} = \omega_\theta \sqrt{X} \tag{3.59}$$

where $\Theta = \Theta(\theta)$ and $X = X(r)$. The circular orbits in the equatorial plane for a Myers-Perry metric would then have the property

$$\mathcal{X} = 0 \tag{3.60}$$

$$\frac{\partial \mathcal{X}}{\partial r} = 0 \tag{3.61}$$

where $\mathcal{X} = 4\Delta^2 X$ and $\Delta = (x + a^2)(x + b^2) - r_0^2 x$. Again by looking at the second derivative of the $\mathcal{X}(r)$ function the nature of the extremum point can be deduced. The second derivative is

$$\frac{\partial^2 \mathcal{X}}{\partial x^2} = 2[(E^2 - m^2)(3r + a^2 + 2b^2) - \theta^2 + r_0^2 m^2] = 0 \tag{3.62}$$

which clearly only has one solution for r, making it impossible for it to be a minimum. The Myers-Perry black hole does not have a circular orbit and thus no bound orbits in five dimensions [37].

3.3 Black Strings and Branes

Uniqueness is proven in four dimensions and prevents any other horizion topology than that of a sphere. However this is not applicable in higher dimensons and you could easily construct a new category of black holes, black strings. Black strings are the product of a black hole solution and a new spacetime direction and look like [19]

$$ds^2_{d+1} = ds^2_{bh} + dx^i dx^i \tag{3.63}$$

where d is the dimension of the black hole metric, ds^2_{bh}. The full dimension of the solution is then d+1. It has horizon topology $S^{d-2} \otimes \mathbb{R}$.

Another set of solutions where there is no restriction on the dimension of $\mathbb{R}$. We could add a arbitrary number of new spacetime directions. The solutions will then look like [4]

$$ds^2_{d+q} = ds^2_{bh} + \sum_{i=1}^{q} dx^i dx^i \tag{3.64}$$

The horizon toplogy is $S^{d-2} \otimes \mathbb{R}^q$.

Both these solutions are valid as they are the product of two Ricci flat manifolds, which makes a Ricci flat manifold. The black branes and black strings turn out to be important as they can be pertubated and decay [19].

This is a non-unique object, with a cylindrical event horizon. The mass of the black string will be infinite for an arbitrary dimension. However if the extended space z is compact and finite, with a length L, this problem can be avoided [41].

This black string in five dimensions still looks like an ordinary black hole in four dimensions. There is however new physic involved as the added space is a new degree of freedom and it appears at energies of order L^{-1}. The fact that the extra space dimension is of finite size means that the extra dimension in practice introduces a period in one direction.

For a large extra dimension, $L >> \mu$, the solution will approach the five dimensional Schwarzschild solution [7]. However if the mass is bigger it is hard to find exact solutions and the black hole is often examined numerically. The black hole can be too big to fit in the extra dimension, creating string-like solutions called non-uniform black strings.

3.3.1 Caged black hole

For space with extra dimensions which has a periodicity L, it should be possible to construct a small black hole which is so small that it does not feel the presence of the extra dimension. However if the black hole gains mass and grows it should start to notice the effects of the extra dimension and due to that become deformed [2].

It is not possible to theoretically describe such a black hole, rather it has to be examined numerically [42], [43]. Further such a black hole might be examined through pertubative expansion [44].

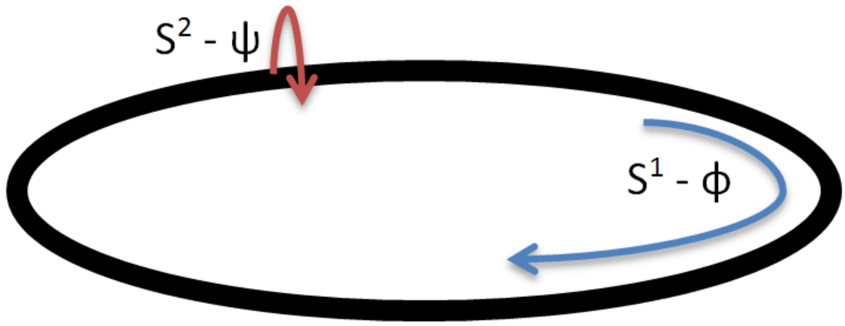

Figure 3.3: A black ring with surface topology $S^2 \otimes S^1$.

3.4 Black Rings

Imagine the black string, $S^{d-3} \otimes \mathbb{R}$, being curved into a ring with the horizon topology $S^{d-3} \otimes S^1$. These solutions are called black rings. Intuitively these solutions should be unstable, however if the ring is given an angular momentum the centripetal force will work against the gravitational force. In the case of five dimensions the centripetal force cancels perfectly with gravity. The black rings were found in five dimensions in 2002 by Emparan and Reall [45]. While the solutions so far are limited to five dimensions, it is thought that this could be extended to higher dimensions.

3.4.1 Solution in five dimensions

The original metric for a black ring was found by Emparan and Reall [45] in 2001 and was rotating in the S^1 direction as seen in figure 3.3. However the solution was slightly clumpsy. Since then a more elegant solution has been found. One such is given by [46]

$$\begin{aligned} ds^2 &= -\frac{F(y)}{F(x)}\left(dt - R\frac{1+y}{F(y)}d\psi\right)^2 \\ &\quad + \frac{R}{(x-y)^2}F(x)\left(-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right) \end{aligned} \tag{3.65}$$

where the functions F(ϵ) and G(ϵ) are

$$F(\epsilon) = 1 + \lambda\epsilon \tag{3.66}$$

$$G(\epsilon) = (1 - \epsilon^2)(1 + \nu\epsilon) \tag{3.67}$$

and

$$C = \sqrt{\lambda(\lambda - \nu)\frac{1+\lambda}{1-\lambda}} \tag{3.68}$$

The coordinates in the case of a rotating black ring are described in terms of torioidal coordinates,(t, x, y, ϕ, ψ) [47]. The limits for (x, y) are

$$-\infty \leq y < -1 \qquad -1 \leq x \leq 1 \tag{3.69}$$

The ring is rotating in the ψ direction and λ and ν are dimensionless parameters in the range $0 < \nu \leq \lambda < 1$. The parameter ν measures the ratio of the radius of the S^1 and the S^2. When $\lambda, \nu \longrightarrow 0$ flat spacetime is recovered, which in torodial coordinates looks like

$$ds^2 = \frac{R^2}{(x-y)^2}\left((y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right) \tag{3.70}$$

This is not obviously flat space, however it can be transformed to the more familiar form for flat space if the following transformations are used [20]

$$y = -\frac{R^2 + r_1^2 + r_2^2}{\sqrt{(r-1^2 + r_2^2 + R^2)^2 - 4R^2 r_2^2}} \tag{3.71}$$

$$x = \frac{R - r_1^2 - r_2^2}{\sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4R^2 r_2^2}} \tag{3.72}$$

The remaining metric after the transforms looks like

$$ds^2 = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2 \tag{3.73}$$

which clearly is flat space for two radial and two angular coordinates. Further as the limits for (r_1, r_2) is $0 \leq (r_1, r_2) < \infty$, the limits in (3.69) can be deduced from the functions (3.71) and (3.72).

An obvious problem with the metric is that there are apparently too many variables. As it is required for the angular momentum and gravity to be balanced, the radius of the black ring is expected to be fixed. Therefore we should find two variables; measures of mass and angular momentum. Instead we have three parameters in λ, ν and R [45].

Another problem with the metric in the current form is that the orbits of ψ and ϕ do not close smoothly, creating conical singularities [20]. To avoid this, the angle can be set with periodicity. Firstly we have singularities at $x, y = 1$ which is avoided if

$$\Delta\psi = \Delta\phi = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu} \tag{3.74}$$

Further to avoid conical singularity at $x = +1$ we must set

$$\Delta\phi = 2\pi\frac{\sqrt{1+\lambda}}{1+\mu} \tag{3.75}$$

Combining (3.74) and(3.75) fixes the value for λ in terms of ν as

$$\lambda = \frac{2\mu}{1+\nu^2} \tag{3.76}$$

This actually fixes the problem with too many variables as well, leaving us with only two, ν and R.

From R and ν all the physical parameters can be deduced (see [48] for details);

$$\begin{aligned} M &= \frac{3\pi R}{4G}\frac{\lambda}{1-\nu} && (3.77) \\ J &= \frac{\pi R^3}{2G}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1+\nu)^2} && (3.78) \\ & && (3.79) \end{aligned}$$

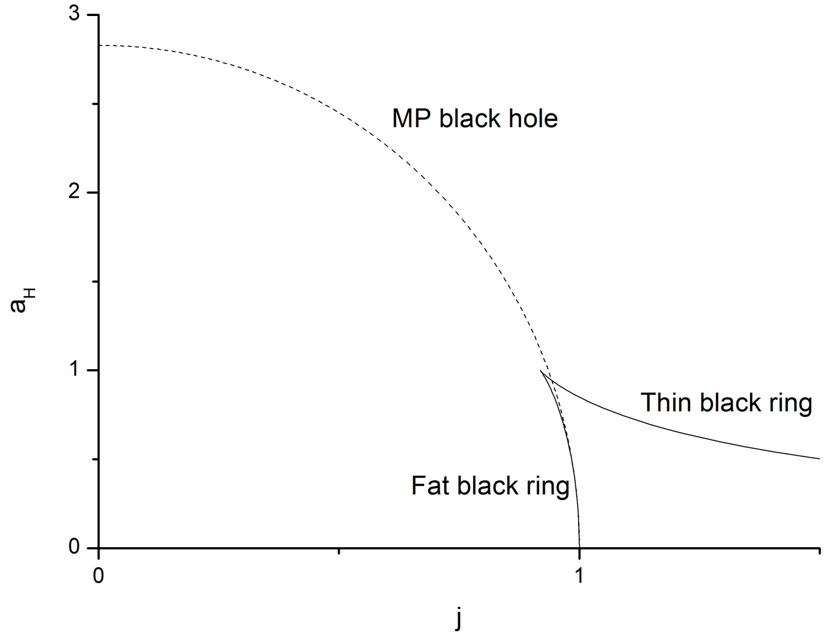

Figure 3.4: The dimensionless a_H vs j for a Myers-Perry black hole in five dimensions, and for the two branches of black rings.

3.4.2 Non-uniqueness of black holes

Using the dimensionless variables introduced in section 2.2.2, we find that

$$a_H = 2\sqrt{\nu(1-\nu)} \tag{3.80}$$

$$j = \sqrt{\frac{(1+\nu)^3}{8\nu}} \tag{3.81}$$

By looking at the derivative with respect to ν we find

$$\frac{da_H}{d\nu} = \frac{1-2\nu}{\sqrt{-(\nu-1)\nu}} = 0 \text{ at } \nu = \frac{1}{2} \tag{3.82}$$

$$\frac{dj}{d\nu} = \frac{1}{\sqrt{32}}\frac{(\nu+1)^2(2\nu-1)}{\nu^2}\sqrt{\frac{\nu}{(\nu+1)^3}} = 0 \text{ at } \nu = \frac{1}{2} \tag{3.83}$$

As can be seen in figure 3.4 there obviouslyis a turning point at $\nu = \frac{1}{2}$. For a_H; $\nu = \frac{1}{2}$ is a maximum at $a_H = 1$. For the j; $\nu = \frac{1}{2}$ is a minimum at $j = \sqrt{\frac{27}{32}}$ [19].

The point $\nu = \frac{1}{2}$is a meeting point for two branches of solutions. The thin black ring solutions extend from $\nu = 0$ to $\nu = \frac{1}{2}$. For this branch as $\nu \longrightarrow 0$, $j \longrightarrow \infty$ and $a_H \longrightarrow 0$.

The fat black ring solutions go from $\nu = \frac{1}{2}$ to $\nu = 1$ and have a smaller area than the thin black rings. The range for $\nu \longrightarrow 1$ is $j \longrightarrow 1$ and $a_H \longrightarrow 0$. This solution meets the Myers-Perry black hole in the naked singularity at $j = 1, a_H = 0$.

As seen in figure 3.4 for the values $\sqrt{\frac{27}{32}} \leq j < 1$ there exist three possible horizon topologies for the black hole with the same angular momentum; the Myers-Perry black hole, the fat black ring and the thin black ring. This clearly violates the uniqueness theorem, and is a definite proof that uniqueness to not generalised to higher dimensions [46].

3.4.3 Two angular momentum

The first black ring solution discovered by Emparan and Reall [45] was described with one angular momentum, along the S^1 ring. Pretty soon after that paper was published, Mishima and Iguchi [49] and Figueras [50] described solutions with angular momentum along the S^2 but not along the S^1. These solutions turned out to be unstable as the rotation no longer works against the gravity and thus have conical singularities.

Pomeransky and Sen'kov [51] presented a solution where the ring had rotation in both the S^1 direction and the S^2 direction. This solution was found working with the Weyl solution described in section 3.6 (for details please see Pomeransky and Sen'kov [51]). The metric found turned out to be rather compact and looks like

$$\begin{aligned} ds^2 \;=\; & -\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 - 2\frac{J(x,y)}{H(y,x)}d\psi d\phi - \frac{F(y,x)}{H(y,x)}d\phi^2 \\ & -\frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right) \end{aligned} \tag{3.84}$$

The coordinates are essentially the same as used before [20] . The variable Ω is given by

$$\Omega = -\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda}}{H(y,x)}\Bigg((1-x^2)y\sqrt{\nu}\psi + \frac{(1+y)}{(1-\lambda+\nu)} \times(1+\lambda-\nu+x^2y\nu(1-\lambda-\nu)+2\nu x(1-y))d\phi\Bigg) \tag{3.85}$$

The functions G, H, J and F are defined as [51]

$$G(x) = (1-x^2)(1+\lambda x+\nu x^2) \tag{3.86}$$

$$H(x,y) = 1+\lambda^2-\nu^2-2\lambda\nu(1-x^2)y+2x\lambda(1-y^2\nu^2)+x^2y^2\nu(1-\lambda^2-\nu^2) \tag{3.87}$$

$$J(x,y) = \frac{2k^2(1-x^2)(1-y^2)\lambda\sqrt{\nu}}{(x-y)(1-\nu)^2}(1+\lambda^2-\nu^2+2(x-y)\lambda\nu-xy\nu(1-\lambda^2-\nu^2) \tag{3.88}$$

$$\begin{aligned} F(x,y) &= \frac{2k^2}{(x-y)^2(1-\nu)^2}(G(x)(1-y^2)\Bigg(\Bigg[(1-\nu)^2-\lambda^2\Bigg](1+\nu)+y\lambda(1-\lambda^2+2\nu-3\nu^2)\Bigg) \\ &+ G(y)(2\lambda^2+x\lambda\Bigg((1-\nu)^2+\lambda^2\Bigg)+x^2\Bigg((1-\nu)^2-\lambda^2\Bigg)(1-\nu)+x^\lambda(1-\lambda^2-3\nu^2+2\nu^3) \end{aligned} \tag{3.89}$$

Similar to the case of one angular momentum, the number of variables is reduced to avoid conical singularities .

The one angular momentum with rotation along the S^1 direction can be recovered by taking $\nu \to 0$, renaming $\lambda \to \nu$ and substitute $R^2 = 2k^2(1+\lambda)^2$. Further flat space is recovered when $\lambda = 0$ [19].

The solutions with rotation only along the S^2 cannot be recovered as a balance condition is inherent to this solution [20].

There are restrictions on what angular momentum is allowed for the $J(S^1)$ if the angular momentum for S^2, $J(S^2)$ is fixed. Non-zero $J(S^2)$ means that this bit of the black ring behaves as a Kerr black hole, and thus follow the limits for angular momentum on the Kerr black hole. As the size of S^2 cannot become arbitrarily small, which means that that there cannot be an arbitrarily large $J(S^1)$ [52].

Another feature of this solution is that for large values of $J(S^2)$, the fat ring branch will disappear. For details please see Elvang and Rodriguez [52].

3.5 Multiple Black Hole Solutions

Other than the black hole solutions introduced, several concentric multi-black hole solutions have been found.

3.5.1 Di-ring

Figure 3.5: The di-ring solution, figure from Godazgar [53].

This is described as the simplest of the multiple black ring solutions. This is two black rings in five dimensions, with horizon topology $S^2 \otimes S^1$, with different radius rotated along S^1 in the same plane, as seen in figure 3.5.

Iguchi and Mishima [54] first derived these solutions in 2007. The same year Evslin and Krishnan [55] derived di-rings using an inverse scattering method. For detailed derivations please see these papers.

Iguchi and Mishima [54] suggests that for the same mass and angular momentum, black di-rings can literally take an infinite number of parameters. This means that the solution is continuously non-unique.

3.5.2 Bicycle black ring

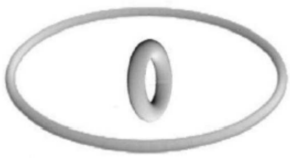

Figure 3.6: The bicycle black ring solution, figure from Godazgar [53].

The bicycle black rings are solutions with two black rings. This is a similar solution to the black di-ring solution; however they rotate around perpendicular plane opposed to the same plane in the black di-ring as seen in figure 3.6. This system was first described by Elvang and Rodriguez [52] in 2008

The way this system is set up, the rotation along S^1 for the rings affects the rotation along S^2 for the other ring due to frame dragging. This means that even if there is only initial momenta along S^1 the angular velocity will be non-zero along both S^1 and S^2.

The particular solution introduced by Elvang and Rodriguez [52] only contains four independent parameters; mass for both rings and angular momenta along S^1. It should however be expected that there exists a general solution with six parameters; 2 masses, 2 angular momenta along S^1 and two angular momenta along S^2.

3.5.3 Black Saturn

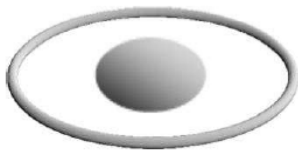

Figure 3.7: The black saturn solution, figure from Godazgar [53].

A black Saturn solution is a Myers-Perry black hole surrounded by a black ring. The black hole and the black ring will interact due to gravitational effects and the interaction will create quite curious phenomena. It was found by Elvang and Figueras [56] in 2007.

The black Saturn have stable solutions for both co-rotation of the individual black holes and counter-rotation. The different black holes will interact due to frame dragging. If the black rings and the black hole are counter-rotating it is possible for the angular velocity of the Myers-Perry black hole to cancel. This means that the solution will be stable with an rotating black ring and a non-rotating black hole.

Like the previous solutions there will be a continuous non-uniqueness however in the case of black Saturn the non-uniqueness is 2-fold. This relationship is further complicated if both thin and fat black rings are admitted.

3.6 Rod Structure of Black Holes

3.6.1 The Weyl solution for d=4

Several solution-creating methods where discovered after the introduction of general relativity. One of the most important one is the Weyl solution which assumes axisymmetry, stationary and (d-2) Killing vectors.

This can be derived in four dimensions as follows. By assuming that a solution is stationary, that the solution contains isometries σ_t with timelike orbits, and axisymmetry, that the solution further contains isometries $\varkappa_\phi$ with closed spacelike orbits, the general solution can be simplified. A further condition is that the actions of the symmetries commute, i.e. that the rotation commutes with the time translation. A metric for such a system will be independent of the time coordinate, t, and the angular coordinate, ϕ, and look like Wald [25]

$$ds^2 = \Sigma_{\mu,\nu} g_{\mu\nu}(x_2, x^3) dx^\mu dx^\nu \tag{3.90}$$

Here the metric, $g_{\mu\nu}$, will be 10 unknown function of two variables. However by choosing an appropriate coordinate system together with the assumption that the tangent spaces of the killing fields associated with the isometries are integrable will limit the options for the functions contained in $g_{\mu\nu}$ greatly. By tweaking the coordinates and implementing restraints due to being in vacuum equation 2.13 and assuming two commuting killing vectors we get the very much simplified version of the metric [57]

$$ds^2 = -e^{2U}(dt - wd\phi) + e^{-2U}[r^2 d\phi^2 + e^{2\gamma}(dr^2 + dz^2) \tag{3.91}$$

U will be a function of (r, z) axisymmetric solution to Laplaces equation in flat $(d-1)$ dimensional space. The metric for such space will be

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 \tag{3.92}$$

This is a powerful solution generating technique for four-dimensional general relativity and was first discovered by Weyl in 1917. What is peculiar is that since the function U is harmonic it can be thought of as a potential produced by a source. For example, the Schwarzschild solution is retained if U is set to be a zero-radius rod with a density of $\frac{1}{2}$ per length [19].

3.6.2 Weyl solution for d>4

This technique has been generalized to higher dimensions for spaces which contain $(d-2)$ killing vector fields. As it turns out the higher-dimensional solution will also have harmonic functions corresponding to thin rods. Many of the multi-black hole solutions were derived using this method. The metric in arbitrary dimensions will look like

$$ds^3 = \Sigma_{i=1}^{d-2} \epsilon_i e^{2U_i} (dx^i)^2 + g_{ab} dy^a dy^b \tag{3.93}$$

where the local coordinate transformation

$$g_{ab} dy^a dy^b = e^{2C} dZ dZ \tag{3.94}$$

is assumed to always be possible [19].

This form for a d-dimensional metric with d-2 commuting killing vectors can be used to solve the Einstein equation. This becomes the equation

$$\partial Z \partial Z \exp\left(\Sigma_j U_j\right) = 0 \tag{3.95}$$

which has the general solution

$$\sum_j U_j = log(w(Z) + \overline{w}(Z)) \tag{3.96}$$

Taking the example of flat space, the two solutions will be [57]

$$U_1 = \frac{1}{2} \log[a \mp z + \sqrt{(a \mp z)^2 + r^2}] + constant \tag{3.97}$$

$$U_2 = \frac{1}{2} \log[-a \pm z + \sqrt{(-a \pm z)^2 + r^2}] + constant \tag{3.98}$$

where U_1 represent the solution that extend from $-\infty \to a_1$, while U_2 extends from $a_1 \to \infty$. These two choices can be represented graphically as seen in figure 1. The rods are all along the z-axis and have zero thickness and represent an interval along the z-axis where the Killing vector of the space has a fixed point, for spacelike killing vectors, or becomes null, for timelike killing vectors [58].

Figure 3.8: This figure shows the rod structure of flat space, where the rods represent the places in space where Killing vectors have fixed points. U_i is the sources for time and space potential sources

3.6.3 Rod structure of black holes

Dimensions	Commuting Killing vectors	Difference
4	2	2
5	3	2
6	3	3
7	4	3
8	4	4
9	5	4

Table 3.3: The number of commuting Killing vectors for different dimensions. The Weyl method only works if the difference is 2, that is for d=4,5

The technique requires that there exists (d-2) commuting Killing vector fields. However the maximum amount of commuting Killing vector fields in higher dimensions are

$$1 + \frac{d-2}{2} \tag{3.99}$$

As seen in table 3.3, as the solutions require that that there is $(d-2)$ commuting killing vectors, this method is not useful for higher than five dimensions. However the rod structure then clearly has a great potential not only for finding solutions of the Einstein equation but for visualizing black holes. The rods in general represent fixed points in Killing vector fields [57].

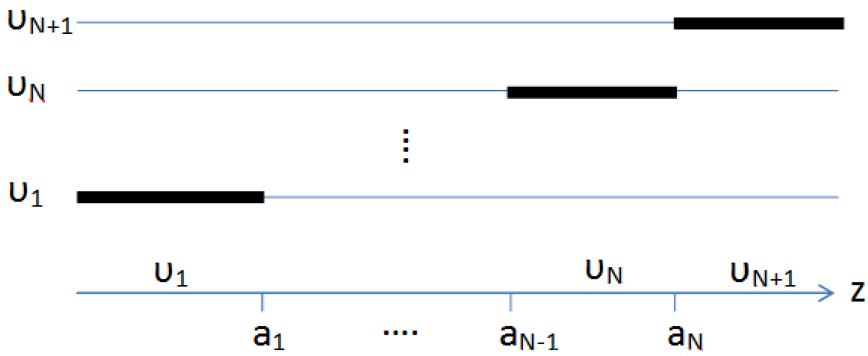

Figure 3.9: This figure shows how the most common way to visualise rod structure works. The z-axis is devided into rods; they are represented by 'rods' on the line depending on which Killing vector field it involves.

The normal way to visualise rods is to divide the z-axis for into sections $[-\infty, a_1, .., a_k, +\infty]$. Depending on which Killing vector the rod represent, different lines is used, as seen in figure 3.9 [59].

The general structures for rods is that if the rod is that they are intervals along the z-axis where the action of a specific Killing vector has fixed points, if the Killing vector is spacelike, or where the Killing vector becomes null, for timelike. That is for finite time Killing vectors, it represent an event horizon in spacetime while for semi-infinite the rod corresponds to an accelerated horizon. The rod sources for spatial coordinates represent a fixed point in the orbits of the specific Killing vector in question. If the rod extends to infinity, the fixed point for that particular obit will also extend to infinity, which usually represents an axis of rotational symmetry.

Emparan and Reall [57] created a classification for rod-structures. Class 0 is the class where there are no finite rods, only infinite rods, or semi-infinite rods. Flat space belongs to this class, as seen in figure 3.8.

Figure 3.10: The rod structure of a four-dimensional Schwarzchild black hole.

Class 1 is if there is one finite rod. The Schwarzschild black holes belong to this class. For four dimensional Schwarzschild black holes there is a finite rod for the time Killing vector, which represent the horizon at (a_1, a_2). Semi-finite rods exist for the same angular Killing vector, goring from $(-\infty, a_1)$ and from $(a_2, +\infty)$. This can be seen in figure 3.10. A five dimensional black hole again has the only one finite rod where the timelike Killing vector is null, representing the horizon. There is however semi-infinite rods for two different angular Killing vectors. Myers-Perry Black hole will have the same rod structure as Schwarzschild in five dimensions.

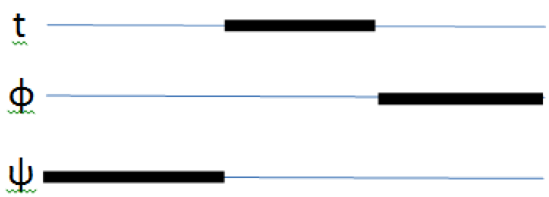

Figure 3.11: The rod structure for a five-dimensional Schwarzchild-Tangherlini black hole or a Myers-Perry black hole.

Class 2 is where there exist two finite rods and two semi-infinite rods. A black ring can be drawn using this method. Here one of the rods represent the time Killing vector, that is the horizon, while the other is a fixed set of orbits for one of the angular Killing vectors. This can be seen in figure 3.11.

The multiple black hole solutions can further be put into a rod structure. This can be seen in figure 3.12.

3.7 Black Hole Thermodynamics

By analysing the quantum mechanical effects of black holes, Stephen Hawking reached a quite stunning conclusion in his paper from 1975; the thermodynamical rules could be applied to black holes. This is extraordinary as this approach connects quantum effects with classical physics. Black hole thermodynamics will not be examined in detail; however the most important results will be represented.

For classical gravity the laws for uncharged black holes are

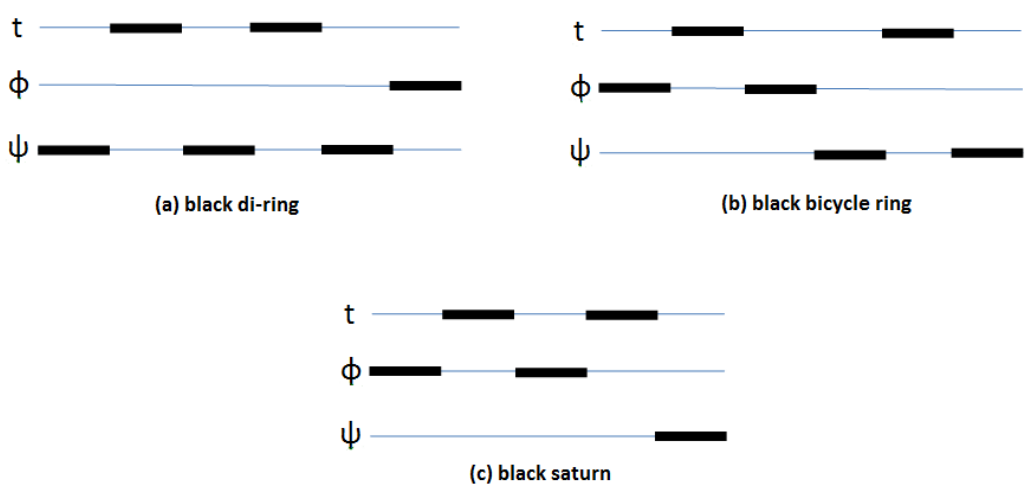

Figure 3.12: (a) Rod structure of black di-rings, (b) the rod structure of black bicycle rings, (c) the rod structure of black saturns.

- *The zeroth law* Classically if system A is in thermal equilibrium with system B and system B is in thermal equilibrium with system C, then system A is in thermal equilibrium with system C [60]. For black holes the analogy is that the surface gravity κ will be constant over the horizon [24].

- *The first law* The infinitesimal transfer of heat into a system and work done on the system is equal to the change in internal energy or [60]

$$dU = dQ + dW \tag{3.100}$$

  The black hole version of this law is

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ \tag{3.101}$$

  for an uncharged black hole. M is the mass, κ is the surface gravity, A is the horizon area, Ω the angular velocity and J the angular velocity [24].

- *The second law* The entropy of all system must stay the same of increase, $\Delta S \geq 0$ [60]. The equivalence is then that for any allowed physical process the area of the horizon of the black hole can never decrease, $\Delta A \geq 0$ [24].

- *The third law* For thermodynamics, this law states that it is impossible to cool a system to absolute zero [60]. The analogy for black holes is that it is impossible to decrease the surface gravity κ to zero [24].

These laws can be extended to higher dimensions, where the constants change however the general meaning of each law will stay the same.

– 4 –

Dynamics

In this section the dynamics of black strings and non-uniform black strings is discussed, as well as the fate of their fluid analogies.

4.1 Thermodynamical argument for instability

A heuristic argument involving black hole thermodynamics can be used to argue that instabilities are innate to black strings. The terms of thermodynamics the most stable configuration is the one with largest entropy. This corresponds to the second law of thermodynamics introduced in section 3.9, where in black hole terms this means that the horizon area can never shrink. If there exist two states with different entropy one will inherently be preferred over the other, that is the black hole with largest event horizon [60].

Black hole thermodynamics connects the horizon area of the black hole with the entropy, $S = \frac{A}{4}$ where A is the black hole horizon area [3].

By adding an extra dimension we have changed the Planck mass, which is dimension dependent. This means that the gravitational constant G is changed as [41]

$$G_5 = G_4 L = L \tag{4.1}$$

where $G_4 = 1$ in natural units. The scale is then renormalized by the size of the extra dimension, L. By relating the area of a black hole, A, with the entropy as $S = \frac{A}{4}$, we can compare the entropy for a black string with dimension size L and a caged black hole. This will then be [41]

$$S_{BH} = \frac{\pi^2 r_5^3}{2L} \tag{4.2}$$

$$S_{BS} = \pi r_+^2 \tag{4.3}$$

where r_i is the radius of the horizon for the black hole and the black string respectively. Solving for black hole mass we get

$$M_{d=5} = \frac{3\pi r_5^2}{8L} \tag{4.4}$$

Entropy can then be re-written for the same mass as [4]

$$S_{BH} = 4\pi M^2 \frac{8L}{27\pi M} \tag{4.5}$$

$$S_{BS} = 4\pi M^2 \tag{4.6}$$

Here we clearly have different size of the two entropies. For a large L, the black holes will be preferred over black strings. This hints to the fact that the black strings will decay into black holes. If there is an unstabililty for the black string, it would be quite logical to come to the conclusion that black strings should decay into black holes. This was the thoughts of Gregory and Laflamme [4] when they introduced the work that showed that black strings are indeed unstable.

4.2 Perturbation of metric

The instabilities can also explicitly be show by perturbing the metric, using the form (3.64) introduced in section 3.3. Using the perturbation $g_ab \longrightarrow g_{ab} + h_{ab}$, where $h_{ab} <<$ 1, we get a Ricci tensor [61]

$$R_{ab} \longrightarrow R_{ab} - \frac{1}{2}\nabla_L h_a b \tag{4.7}$$

where ∇_L is the curved space wave function for two massless particles as [41]

$$\nabla_L h_{ab} = \Box + 2R_{abcd}h^{cd} - 2R^c_{(a}h_{b)c} - 2\nabla_{(a}\nabla^c h_{b)c} + \nabla_a \nabla_b h \tag{4.8}$$

In empty spacetime from (2.13) we then have the condition

$$\nabla_L h_{ab} = 0 \tag{4.9}$$

Further we have the de Donder gauge transform

$$\nabla_a h^a_b = 0 = h \tag{4.10}$$

which reduced equation (4.8) to

$$\nabla_L h_{ab} \longrightarrow \Box h_{ab} + 2R_{abcd}h^{cd} = 0 \tag{4.11}$$

One important property of General Relativity is that it is invariant under general coordinate transforms, generated by ξ^a. The metric transforms as

$$g_{ab} \longrightarrow g_{ab} + 2\xi_{(a;b)} \tag{4.12}$$

which means that the pure gauge perturbation is

$$h_{ab} = 2\xi_{(a;b)} \tag{4.13}$$

The ξ^a is divergence free and harmonic, which means that this perturbation satisfies both (4.11) and (4.10). The degrees of freedom is $\frac{(N-2)(N+1)}{2}$ to the Lichnerowicz, $(N-1)$ is pure gauge while the rest is in fact physical [4].

There are however a issues with the boundary conditions of this problem; the horizon is a singularity in Schwarzschild coordinates which makes it hard to define $h_{ab} << 1$ close to the horizon . As the singularity in the Schwarzschild coordinates is a coordinate singularity this whole problem can be avoided by transforming into Kruskal coordinates

[41].

It is now possible to find the perturbation using separation of the variables as well as the symmetries present. The symmetries in this case is rotational symmetries, time symmetry and is invariant along z. Considering the spherical symmetries it can be deduced that there will be no off-diagonal inputs for the angles. This puts the perturbation into the form [4]

$$h_{ab} = e^{\Omega t} e^{i\mu_i x^i} \begin{pmatrix} h_{tt} & h_{tr} & 0 & 0 & ... & h_{tz} \\ h_{tr} & h_{rr} & 0 & 0 & ... & h_{rz} \\ 0 & 0 & h_{\theta\theta} & 0 & ... & 0 \\ 0 & 0 & 0 & h_{\theta\theta}\sin^2\theta & ... & 0 \\ .. & .. & .. & .. & \ddots & ... \\ h_{tz} & h_{rz} & .. & .. & .. & h_{zz} \end{pmatrix} \tag{4.14}$$

The time-symmetry leads to an oscillating behaviour described by $e^{i\mu_i z}$, where $\mu = \mu(k)$ and the z-symmetry leads to a growing instability $e^{\Omega t}$, where Ω is the inverse decay time. As proven by Gregory and Laflamme [4] for any unstable mode the inputs for h_{zz} and for $h_{z\mu}$ must vanish for any unstable solution. This can be show explicitly. For simplicity the d=5 case is examined. Them we can write $h_{zz} = e^{i\mu z} e^{\Omega t} h$ and hence obtain the equation [41]

$$h + \left(\frac{2r - r_+}{r - r_+}\right)\frac{h}{r} - (\mu^2 r(r - r_+) + \Omega^2 r^2)\frac{h}{(r - r_+)^2} = 0 \tag{4.15}$$

Looking at the two limits we obtain the behaviour

$$h \longrightarrow e^{\pm\sqrt{\Omega^2 + \mu r}} \text{ as } r \longrightarrow \infty \tag{4.16}$$

$$h \longrightarrow (r - r_+)^{\pm\Omega r_+} \text{ as } r \longrightarrow r_+ \tag{4.17}$$

Looking at the limit behaviour any solution must disappear at the horizon and at ∞, and hence must have a turning point between the two values. This solution however does not allow for such a turning point, resulting in h_{zz} must disappear. With similar

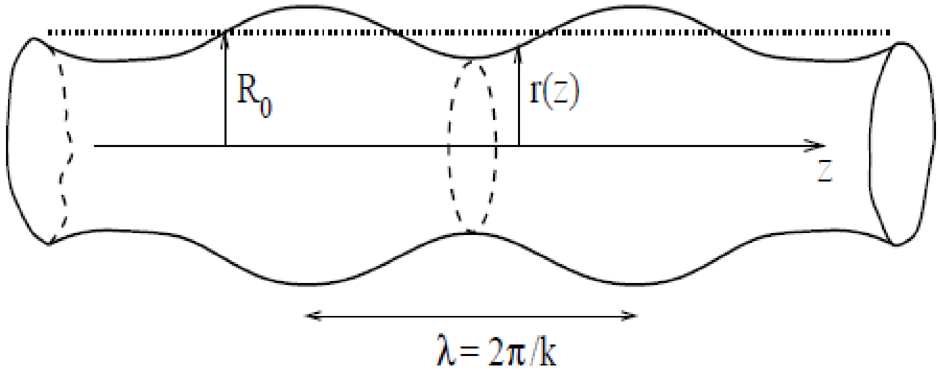

Figure 4.1: This figure shows the black string after some arbitrary time of instability. The λ is the wavelenght of the instability, the r(z) the changed radius and R_0 the original radius.

argument it can be deduced that $h_{z\mu}$ also must disappear [41].

The result then is a matrix which contain four components and can be described as $h_{\mu\nu} = e^{i\mu_i z}e^{\Omega t}h_{\mu\nu}(r)$. With gauge condition the equation reduced to the second order differential equation as

$$
\begin{aligned}
0 \;=\; & \Bigg[-\Omega^2 - \mu^2 V + \frac{(D-3)^2(\frac{r_+}{r})^{2(D-3)}}{4r^2}\Bigg]H^{tr} - \Bigg[\mu^2[(D-2) - 2\Big(\frac{r_+}{r}\Big)^{D-3} + (4-D)\Big(\frac{r_+}{r}\Big)^{2(D-3)]} \\
+ \; & \frac{\Omega^2[(D-2)+(2D-7)\big(\frac{r_+}{r}\big)^{D-3}]}{rV} + \frac{3(D-3)^2\big(\frac{r_+}{r}\big)^{2(D-3)}[(D-2)-\big(\frac{r_+}{r}\big)^{D-3}]}{4r^2V}\Bigg]H^{tr} \\
+ \; & \Bigg[(\mu^2 + \frac{\Omega^2}{V})^2 + \frac{\Omega^2(4(D-2) - 8(D-2)\big(\frac{r_+}{r}\big)^{D-3} - (53-34D+5D^2)\big(\frac{r_+}{r}\big)^{2(D-3)}}{4r^2V^2} \\
+ \; & \frac{\mu^2[4(D-2) - 4(3D-7)\big(\frac{r_+}{r}\big)^{D-3} + (D^2+2D-11)\big(\frac{r_+}{r}\big)^{2(D-3)}}{4r^2V} \\
+ \; & \frac{(D-3)^2\big(\frac{r_+}{r}\big)^{2(D-3)}[(D-2)(2D-5) - (D-1)(D-2)\big(\frac{r_+}{r}\big)^{D-3} + \big(\frac{r_+}{r}\big)^{2(D-3)}]}{4r^4V^2}\Bigg]H^{tr} \qquad (4.18)
\end{aligned}
$$

This was the form of the perturbation found by Gregory and Laflamme [4]. Please notice that the D used in this context is not the dimension rather the dimension of the sphere, i.e. for dimension d the surface topology is $S^D \otimes R^{d-(D+1)}$ [3].

4.2.1 Significance of the Instability

The equations determining the perturbation must be numerically solved. A subtly however is that there will not exist satisfying solutions to all values of μ and Ω. There will however for every particular μ be a specific wavelength Ω_μ that yields a solution

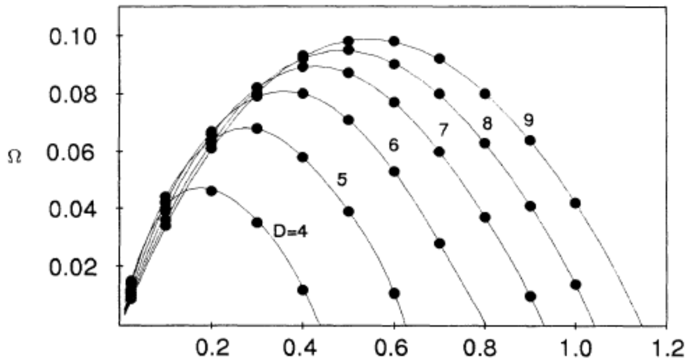

Figure 4.2: This is the calculations from Gregory and Laflamme [4], showing the values for μ and Ω for which there is a solution for different dimensions.

As seen in figure 4.2, for each Ω, there is a specific μ for which Ω goes to zero. This cutoff value will be of importance later, and is usually called critical mode for Gregory-Laflamme instabilities.

It has been proven that there exists a perturbation which results in an instability that is physical, however what is the consequence of having this present? As shown this instability will grow with time and change the shape of the event horizon [7].

By using Kruskal coordinates, it can be explored what happens in details what happens to the horizon under the effect of the instability by looking at the geodesic.

Again looking at the five dimensional case, for unperturbed space, the null geodesic is described as $R = \pm T + R_0$. $R = T$ describes the future event horizon. The geodesic is [41]

$$\begin{aligned}\left(\frac{dR}{dT}\right)^2 &= 1 + \frac{1}{\mathcal{U}}(h_{TT} + 2h_{TR}\dot{R} + h_{RR}\dot{R}^2) \\ &= 1 + \epsilon \cos mz (R+T)^{2r_+\Omega - 2}\left(1 + \frac{dR}{dT}\right) \qquad (4.19)\end{aligned}$$

where $\epsilon << 1$, representing the magnitude of the initial perturbation. The horizon is shifted to, in Kruskal coordinates

$$R = T + \epsilon \cos mzT^{2r_+\omega-1} \tag{4.20}$$

which are, in more recognizable Schwarzschild coordinates look like

$$r = r_+ + \epsilon T^{2\Omega} \cos mz \tag{4.21}$$

The horizon gained a oscillatory dependence on z, and will fluctuate as seen in figure 4.3. The consequence is clearly that the shape of the horizon no longer is cylindrical in shape; rather it will get larger in some places while it gets smaller in some placer. The total horizon area will in this process grow which is consistent with the thermodynamical arguments introduced, as area is a measure of the entropy [62].

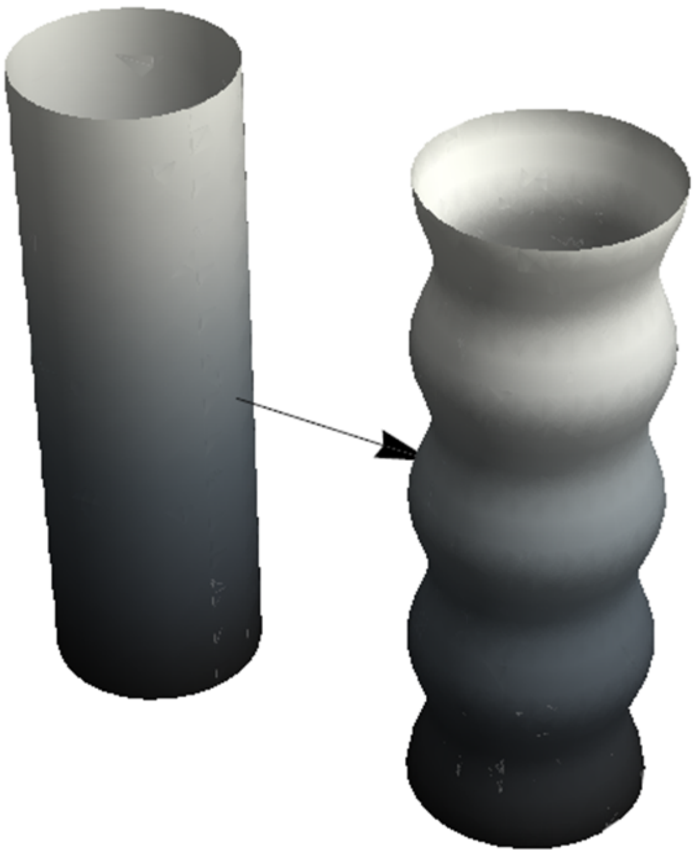

Figure 4.3: This is a 3d image of the inital string and then the string some time, from Gregory [41].

The logical endpoint of these ripples would be for them to pinch off, creating smaller black holes which turned out to quite controversial [63]. These small black holes would then be caged within the larger extra dimension. Further it would seem like at the exact point when the cylinder pinches off, a naked singularity will be created. This starts the discussion on whether or not naked singularities can exist. The cosmic censorship conjecture states that naked singularities do not exist (other than Big Bang); a singularity is always behind a horizon [25]. The final answer of what the endstate for Gregory-Laflamme unstable black strings where would not be discovered for almost two decades after Gregory and Laflamme [4] initial paper on the topic.

4.3 The final state of the Gregory Laflamme instability

Gregory and Laflamme [4] to suggest that for black holes with $L > r_0$, black holes have larger entropy than that of the black string. This should suggest that the black strings in theory should be distorted into black holes. The same instability exists for black branes, which then would suffer a similar fate.

4.3.1 The cosmic censorship conjecture in four dimensions

In the earlier days of General Relativity there were discussions concerning the nature of black holes and singularities. The main concern was that the exact nature of a singularity was unknown, as the physics of such phenomena has not yet been properly discovered [64]. In these regions classical physics breaks down and have to, in the future, be replaced with quantum gravity. This could possibly result in spacetime failing to be asymptotically predictable, making it impossible to tell the outcomes. However if the singularities was behind an event horizon, causality would be regained. The horizon would hide the misbehaving singularity, making the effect of the unknown physics impossible to spot by an outside observer [65].

This led to Roger Penrose conjecturing that naked singularities do not exist in 1969 [66], in what he called the cosmic censorship conjecture. It stated that an outside observer could never observe a singularity from future null infinity. He revisited the definition of the conjecture in 1979, which resulted in a newer version of the conjecture. Wald [25] stated in physical terms the conjecture as following;

All physical reasonable spacetimes are globally hyperbolic, i.e. apart from a possible initial singularity (such as big bang singularity) no singularity is ever visible to any observer.

While there does not exist any evidence either for or against the conjecture, there have been a peculiarly persistent lack of counterevidence [67]. There have exhaustive attempts to try to find naked singularities in four dimensions, both theoretically and numerically, which have always been unsuccessful. No naked singularities where could be created, as long as the system had physical initial parameters [25]. This has led to strong suspicions that the conjecture is true. The final state of the instability of long wavelengths have been studied in detail as it could possibly shed light on the comic censorship conjecture in higher dimensions.

4.3.2 The cosmic censorship conjecture in $d > 4$

If the cosmic censorship conjecture is assumed to be correct, the black strings should collapse to a new black string state, never bisecting the horizon. This fate of the black string was argued by Horowitz and Maeda [63], who stated theoretically that naked singularities should be impossible to achieve.

They conjectured that it would be impossible for the horizon to pinch off in a finite affine time, drawing the conclusion that a black string must settle down into a new non-uniform black string state.

The proof argues that for the horizon to pinch off the parameter θ in the metric must approach zero in a finite time. This is proven to impossible for a time parameter λ.

It was highlighted by Horowitz and Maeda [63] of the article that there was a loophole in the proof, namely that it was possible for decay of the horizon in the form

$$\mathcal{L} = e^{-(\ln \lambda)^{\alpha}} \tag{4.22}$$

with the variable $0 < \alpha < \frac{1}{2}$. It was argued that for the time parameter this decay would be very slow, resulting in basically was a new static form of a black string and was disregarded as unnatural.

The search for such possible states however turned out to be problematic. There was several examples of just such stable, non-uniform black states found [68] [69]. However it was shown that they all had a entropy which was smaller than that of a uniform black string. Lehner and Pretorius [3] showed that it was impossible for these solutions to be endstates for the uniform black string, at least in $d < 13$.

Further as it turned out this proof was wrong, as shown by Marolf [70] and Garfinkle et al. [61]. As it turns out the form of the decay 4.22 is actually exactly how the decay of the horizon will look like.

Further the affine time parameter used, λ was not the time for a static observer, rather it was the time at the horizon. As one might expect, this time will behave strangely as the horizon size of the horizon approaches zero.

As it turns out, it is possible for the horizon to pinch off in finite advanced time, which is preferred over the affine time parameter used by Marolf [70]. λ will approach infinity in a finite time for an outside observer, making it possible for the horizon to decay.

The behaviour of the affine time parameter λ compared to the advanced time of a static outside observer was examined in a numerical analysis by Garfinkle et al. [61]. As seen in figure 4.4 the parameter s, which is a measure of λ as $\lambda = e^s$, will quickly grow very large very quickly compare to the advanced time of an outside observer, t.

The options of a black string evolving into smaller black holes are still a possibility.

4.4 Dynamics of instabilities on black strings

4.4.1 Using linear perturbation approximation

The findings of the endstates comes from numerically solve instability of the black strings. As stated before, it has been conjectured that the end-state of a system should

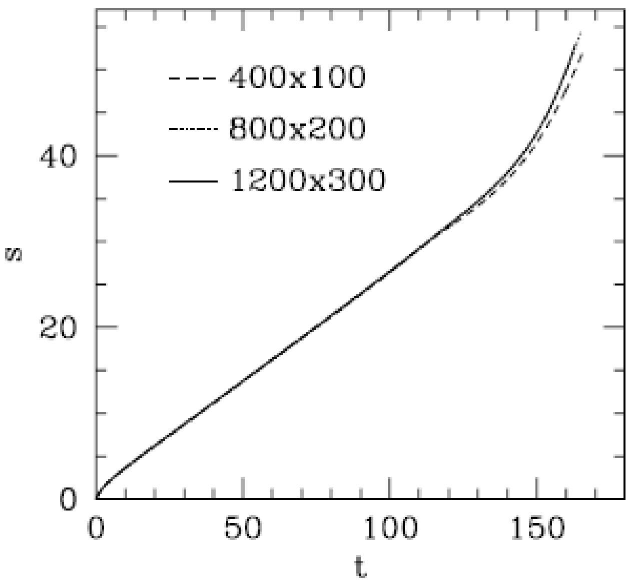

Figure 4.4: This figure is taken from Garfinkle et al. [61] and shows the affine time s, where $\lambda = e^s$, vs the advanced time t. The different curves come from three different simulations of identical initial data but different resolution.

result in the horizon of the black string being pinched off at even intervals, creating a naked singularity. This conjecture comes partly from thermodynamical consideration; the black string has smaller entropy than that of a spherical black hole [41].

This is quite extraordinary as it is a system without un-physical initial parameters that reaches naked singularities, that is evolving to a state where quantum gravity would be needed to exactly solve the system Wald [25].

Initial trials to find the end-states where made using a linear perturbation. This was done by amongst others Choptuik et al. [62] in 2003. They numerically solved the full equations of motions, and then simulate it. The results found by Gregory and Laflamme [4] was recovered. The strings developed uncertainties for $L > L_c$ but where stable for $L < L_c$.

Further they found that the black string evolved into a state of consecutive S^3 black holes connected by a very thin black string. Result was achieved as seen in figure 4.5 ; the black strings decayed towards black holes which still where connected by a very thin string.

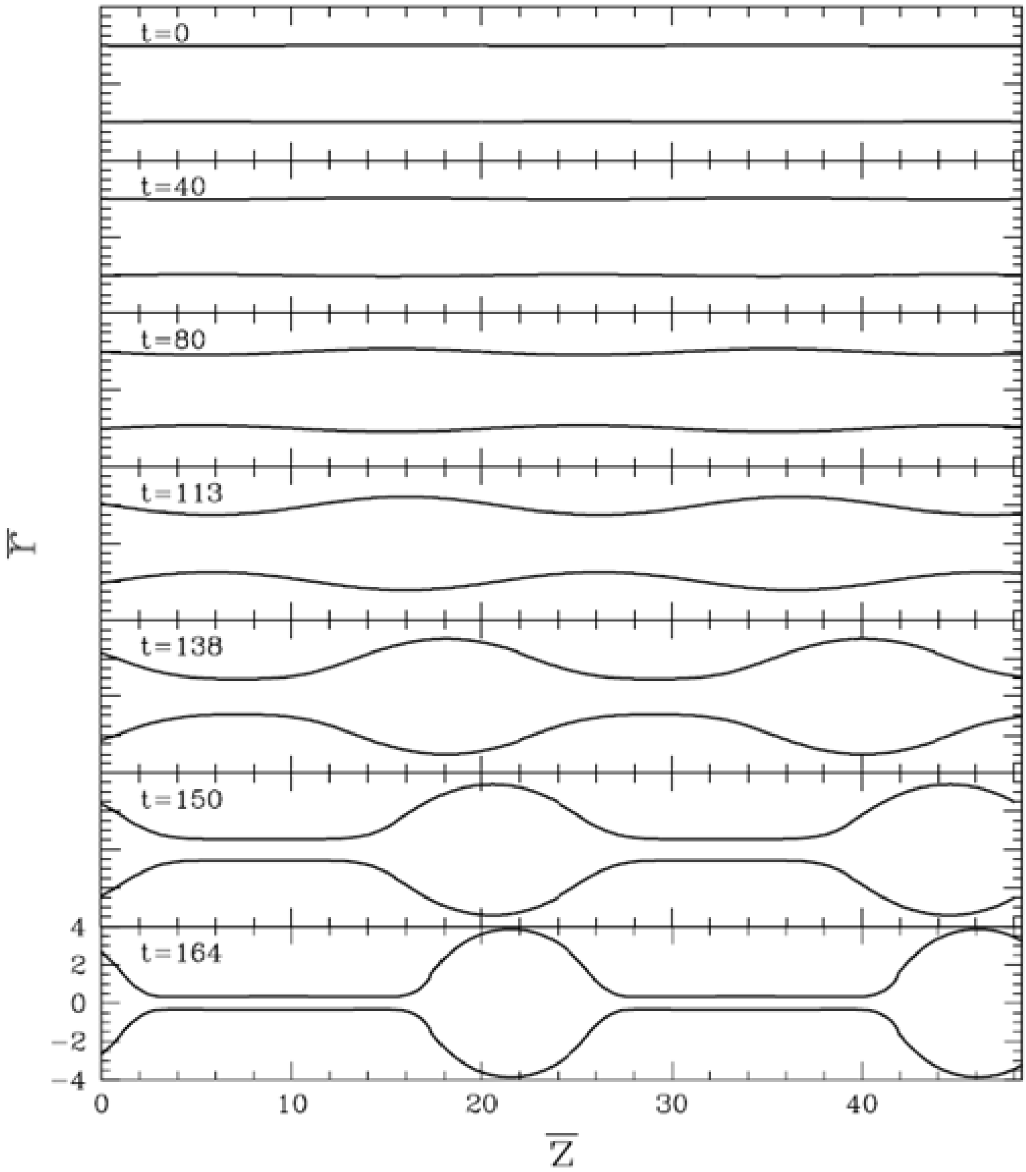

Figure 4.5: This figure is taken from Choptuik et al. [62] and shows the apparent horizon at different timesteps for the coordinates r and z. The dimensions of θ and ϕ have been suppressed for simplicity.

Two possible scenarios was suggested by Choptuik et al. [62]; either the black string is pinched off in finite time or that there was a new, non-uniform state.

However it turned out to problematic to go further as the code for these simulations consistently seemed to crash before the real end-state was reached. This was thought by the authors to be due to a large gradient being developed in one of the functions used, which is theorised to be due to the coordinate choices made.

4.4.2 Higher Order Perturbation Approximations

A quite logical step would then be that the black string connecting the black hole would also be subjected to the Gregory-Laflamme instability, creating a cascade effect. This was exactly what was found by Lehner and Pretorius [71]. They managed to come up with a method of numerical analysis that allowed for evolution further than Choptuik et al. [62].

The initial data used by Lehner and Pretorius [71] was the same as that by Choptuik et al. [62]. The boundary values where found by looking at the intrinsic metric and curvature of the space, which at $t = 0$ can be used to define Hamiltonian and momentum constraints.

The coordinates used where of the form $(\bar{t}, \overline{x}, \overline{y}, \overline{z}, \overline{w}) \equiv = (t, r\cos\phi\sin\theta, rsin\phi\sin\theta, r\cos\theta, z)$, where ϕ and θ is angular coordinates of the two-sphere and $\overline{w} = z$ is the direction of the string.

The perturbation introduced has the form

$$\gamma_\Omega = 1 + A\sin\left(w\frac{2\pi q}{L}\right)e^{\frac{-(r-r_0)^2}{\delta_r^2}} \tag{4.23}$$

and is a perturbation along the z and r direction. A determines the size of the perturbation, q describes the spatial frequency in the z-direction and r_0 and $delta_r$ control the perturbation in the r-direction. The variables describes are set at constant values [3].

Cartesian Harmonic gauge was implemented, which has the form

$$\nabla^{\nu}\nabla_{\nu}x^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\nu}(\sqrt{-g}g^{\nu\epsilon}\partial_{epsilon}x^{\mu}) \equiv 0 \tag{4.24}$$

This makes it possible to explicitly write the metric in symmetric hyperbolic evolution equations.

By writing the constraints as

$$C_{\mu} \equiv g_{\mu\nu}\nabla^{\epsilon}\nabla_{\epsilon}x^{\nu} = 0 \tag{4.25}$$

the time derivative can be related back to the Hamiltonian and momentum constraints. As it turns out, this was hard to solve numerically as truncation errors resulted in a non-zero C_{μ}. The solution used by Lehner and Pretorius [3] was to introduce a *constraint dampening* to the Einstein equations. This introduces two new parameters, κ and ρ which damps unwanted zero-wavelengths which grew along z.

As Gregory and Laflamme [4] showed that there only exist a instability in the z-direction, it is possible to reduce the number of dimensions to make the code more efficient by enforce the existing $SO(3)$ symmetry. This is called the *cartoon method* [72] [73] and makes it possible to only evolve at $\overline{y} = \overline{z} = 0$ slices of spacetime.

As the dynamics of the black string is found by looking at the horizon, it is the horizon that should be described by the simulation. However the real horizon is hard to find in numerical simulations, making it more appropriate to study the apparent horizon, which is the outermost surface where outward null expansion is zero and inward null expansion is less than zero [3]. It has been shown that the apparent horizon is more or less indistinguishable from the event horizon [74].

4.4.3 Violating the cosmic censorship

Lehner and Pretorius [71] used Runge-Kutta of fourth order to evolve the black strings, while monitoring several quantities.

The first quite obvious result of evolving a black string over time is that the horizon

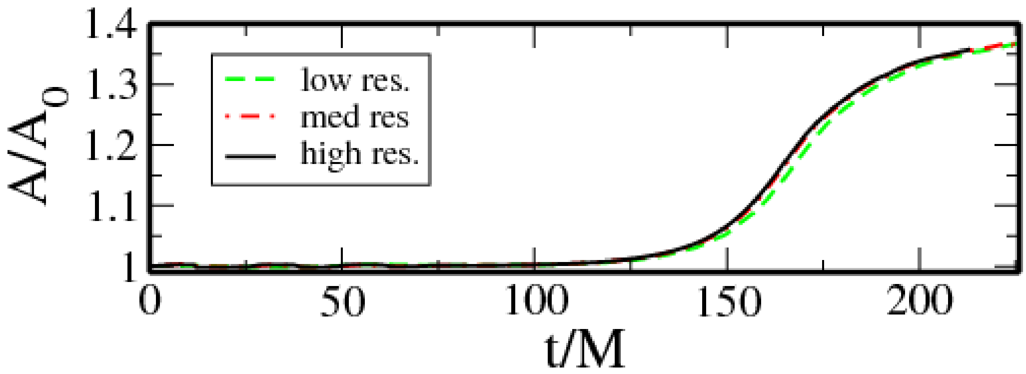

Figure 4.6: This figure is taken from Lehner and Pretorius [71] and shows the surface area of the black string vs time, in normalised units, for three different resolutions.

surface area, which is analogous to the entropy, increases over time as expected, as seen in figure 4.6. As discussed in the paper, the result reached with the lowest resolution, that is a final area of $1.374A_0$, is exactly the same area of what a five-dimensional black holes of the same mass would have [3].

Further the horizon of the black string can be examined. As seen in figure 4.7, the apparent event horizon takes the same shape as seen in the work of Choptuik et al. [62]; spherical black holes connected by thin black strings.

However as Lehner and Pretorius [71] is able to further evolve the strings, the cascade effect mentioned before becomes apparent. The strings separating the black holes are inherently unstable and will decay towards smaller black holes, again separated by black strings. This pattern repeats it turns out, making it possible to approximate the endstate of the string by looking at the generations of ever smaller black holes. A new generation is defined as when the black hole has a radius r, which is 1.5 times larger than that of the surrounding string. A time t_i which describes the time it takes for a new generation to be created from connecting string segments can be deduced. This time seems to grow smaller the higher the number of generations gets [3].

As it takes a finite time for a new generation of ever smaller black holes to be created by instabilities in the strings, it can be approximated when the radius of the connecting string segments reaches zero. As it turns out the time t_{n+1} for a generation $n+1$ seems

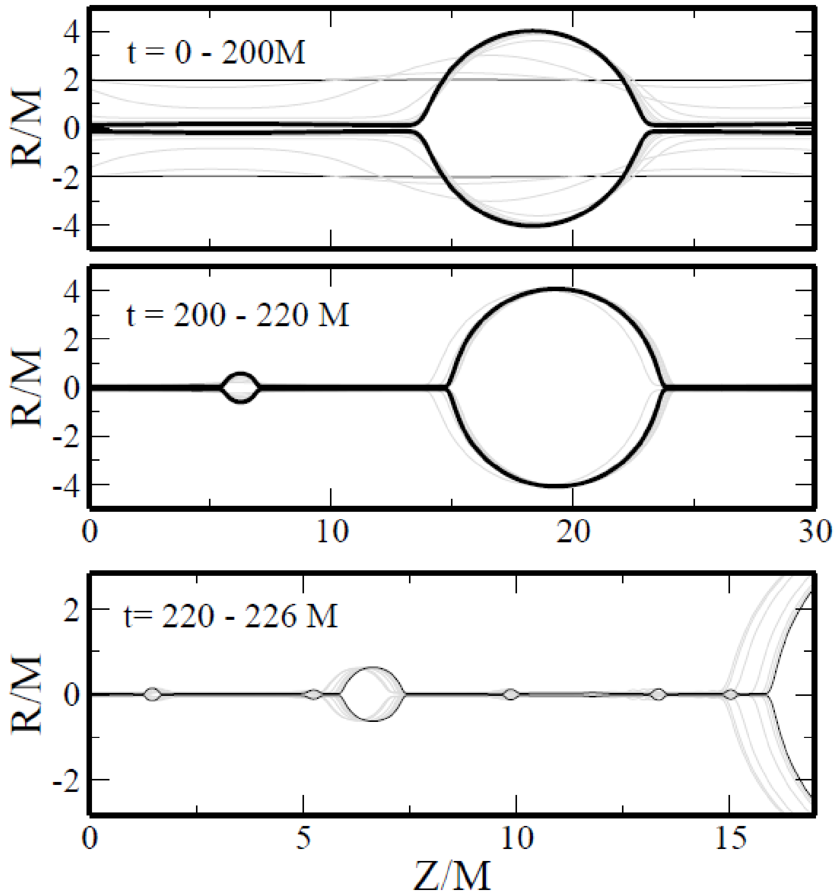

Figure 4.7: This figure is taken from Lehner and Pretorius [71] and shows the cascade effect for black holes at three different times.

to be dependent on the former generation as $t_{n+1} = \frac{t_n}{X}$, where X is some number. For the parameters used by], $X = 4$. The total time to reach the endstate is then finite as

$$t_{final} \propto \sum_{i=0}^{\infty} t_1 X^i = \frac{t_1}{1 - X} \tag{4.26}$$

as this is simply a geometric series [3].

A finite time to reach an endstate, which has the property that the radius of strings goes to zero, is an explicit violation of the cosmic censorship for generic initial data. The conclusion drawn is then that the cosmic censorship conjecture do not seem to hold in higher dimensions and confirms the suspicions presented by] almost two decades earlier.

4.5 Non-uniform black strings

Several investigation were made into the critical mode for Gregory-Laflamme instabilities, that is when $\Omega = 0$ and $k = k_c$ in equation as seen in figure 4.8 **??**.

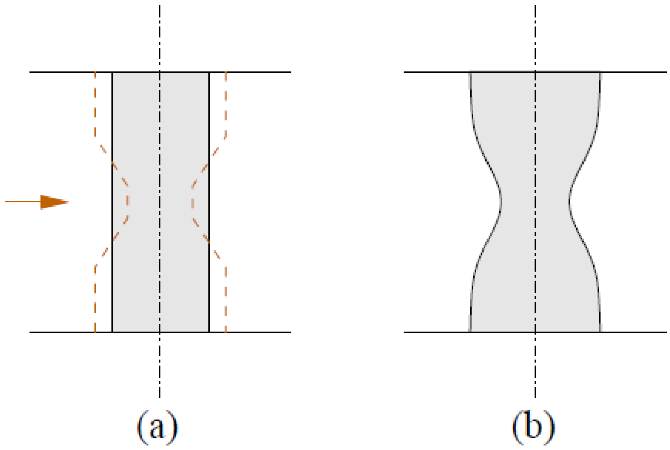

Figure 4.8: (a) shows Gregory-Laflamme instability for the critical mode of a black string. (b) showes the shape of a non-uniform balck string. This figure is Kol [7]

This sets the equation to

$$(\nabla_L)^{\rho\omega}_{\mu\nu}(P_{c,\rho\omega}e^{ik_c z/r_0}) = 0 \tag{4.27}$$

This is the threshold mode for the Gregory-Laflamme instability that depends directly upon the string dimension, z. This branch of solutions is called non-uniform black strings. This branch of solutions has a topology $S^1 \otimes S^{d-2}$, where the S^1 is a circle due to the Kaluza-Klein compactification. The non-uniform black strings meets the uniform string branch at $\mu = \mu_{GL}$ and has a local SO(d-2)symmetry.

For each dimension a critical value of μ_{GL} can be found, corresponding to where $k = k_{GL}$ and $\Omega = 0$ [4]. By looking at a number $\gamma(\mu = \mu_{GL}$ further information about the behaviour of γ; if $\gamma > 0$ the non-uniform black string will increase in mass, if $\gamma < 0$ the mass will decrease which results in different entropies [75].

This leads to a vastly different behaviour depending on the sign of γ. As found by

Sorkin [76] γ is always positive for $d < 14$ while it seems to be negative for $d \geq 14$. This means that there is a critical dimension where there the system changes behaviour, $d = 14$.

4.5.1 Phase diagram for non-uniform black hole

The question of what happens with caged black holes as they increase in size has been asked. The most popular scenario as to date is that the caged black hole will develop into a non-uniform black string [44]. This however means that the topology would have to change, which at the meeting point of the two solutions impossible to explain using currently known physics [7].

For caged black holes is that $n \to 0$ as $\mu \to 0$ [77]. This indicates that the solution will approach that of a Schwarzschild solution as the mass becomes very small [75]. Metrics for a caged black hole was found by an ansatz in [44]. More references. There have also been numerical analyses of the caged black holes for different dimension (see [69] [2] [78]. Similarly there have been several numerical analyses of the non-uniform strings (see [79] [43]).

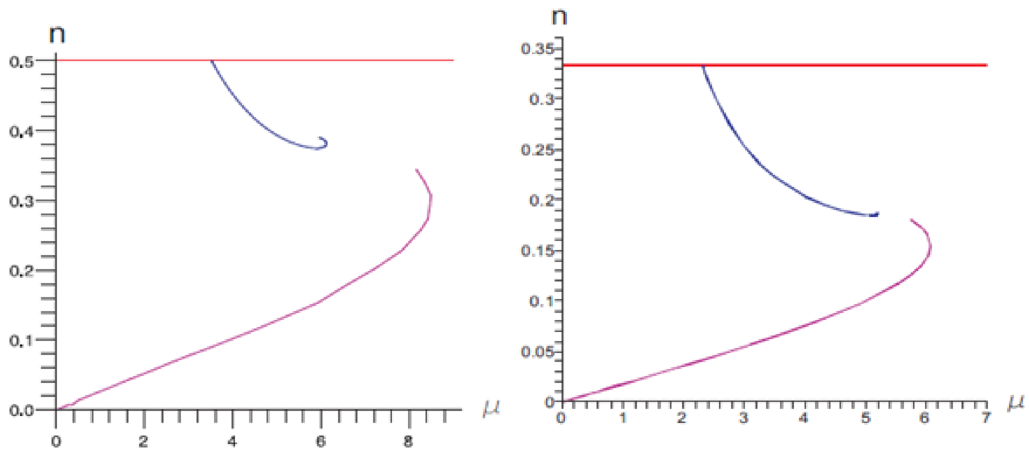

Figure 4.9: This figure is from [2] with data from Wiseman [69], Sorkin [76] and Kudoh and Wiseman [80]. It shows the phase diagram for a non-uniform black string for five dimensions (left) and six dimensions (right). The red line is a stable black string, the purple line is a caged black hole and the blue lne a non-uniform string.

By plotting these results in a phase diagram (μ, n), the notion that the two branches meet is implied. This can be seen in figure 4.9 where the blue branch is a non-uniform string while the purple branch is a caged black hole, in five and six dimensions. The red line is uniform black string branch.

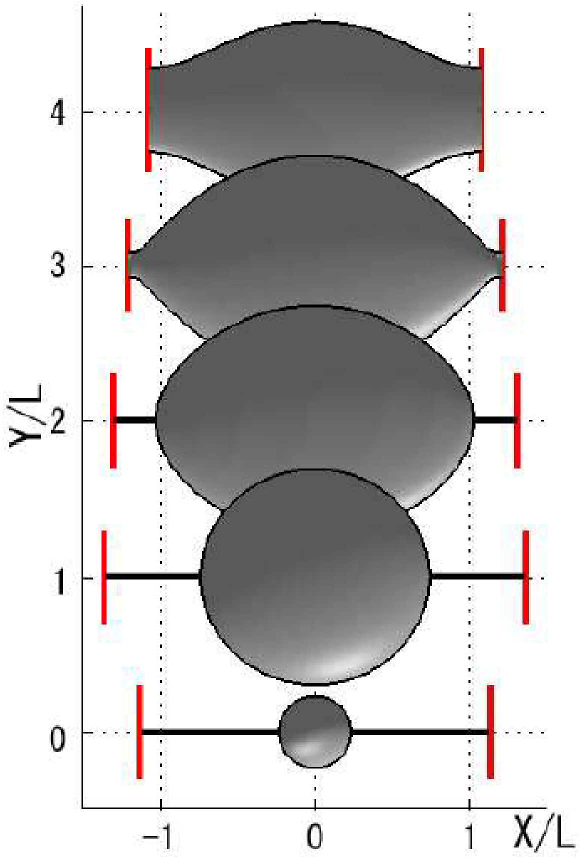

Figure 4.10: This figure is from Kol:2004ww and shows the apperant horizon for a six dimensional black hole and non-uniform black string.

This can further be seen in figure 4.10, where the distortion of caged black holes clearly approaches that of a non-uniform black string.

4.6 Fluid analogy of black strings

Just as black holes seem to have an eerie connection to thermodynamics, it was similarly found to have a connection to fluid dynamics. The first hint of this was the introduction of a toy model by Kip S. Thorne [81] called the membrane paradigm, with the goal to create a more accessible theory for physicist with a nonrelativistic background. This theory suggested that the horizon of a black hole could be described by the Navier-

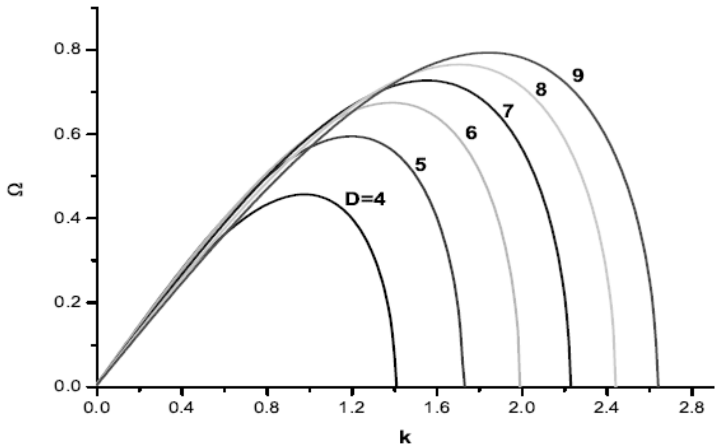

Figure 4.11: This figure is taken from Cardoso and Dias [83] and shows the growth rate Ω vs the wavenumber k for a cylinder experiencing Rayleigh-Plateau instability. The surface-tension and density is the same as that of a black string.

Stokes equation. These findings dates back to the 70s when it was found that external gravitational forces deforms the horizon just as if it were viscous [82].

This analogy between the two fields turns out to be important for the dynamics of black strings. As Cardoso and Dias [83], there is similarities between the Gregory-Laflamme instabilities and the classical instability of a membrane, the Rayleigh-Plateau instability. A system ruled by the Rayleigh-Plateau instability does in fact pinch off. The two instabilities are not perfectly equal; they have different relationship between the growth rate Ω and the wavelength k. The shape of the relationship however is similar, as can be seen by comparing figure 4.11 and figure 4.2. Further both instabilities disappear for modes other than the s-modes.

Further the endpoint of a Rayleigh-Plateau instability is the cylinder pinching off and drops forms [84]. A beam of fluid will have the satellite formation shown in black strings; between large drops smaller and smaller drops will be created. The sizes of these drops are dependent on the viscosity of the fluid; the lower viscosity the more drops. A picture of such phenomena can be seen in figure 4.12, which shows an extended fluid breaking up. This behaviour is similar to that of the black string (for idept review see [84]).

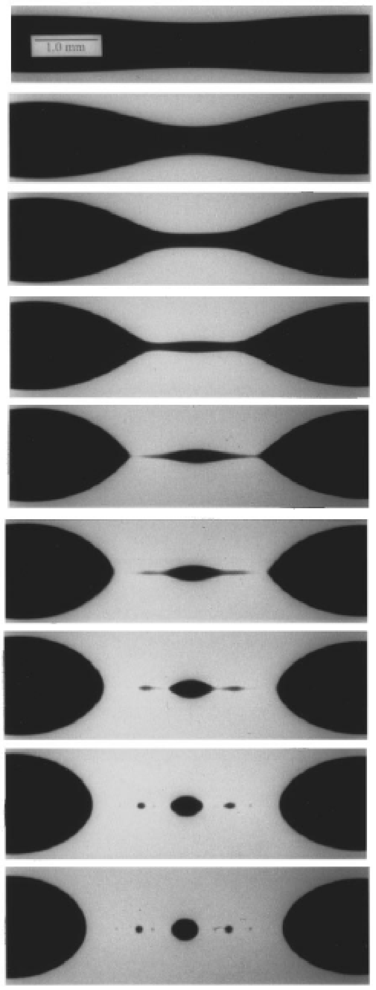

Figure 4.12: This figure is take from Eggers [84] and shows an extended fluid suspended in another fluid, at different times.

4.7 Beyond the classical regim

It would be highly likely that the final stages of black strings being pinched off would be governed by quantum gravity. The result reviewed here are all assuming classical physics. This classical examination has shown a cascading behaviour, similar to that of a fluid, along the horizon of the black string [3].

When the horizon grows smaller and smaller the classical gravity would no longer be warranted. The effects of quantum mechanics would have been considered. This is usually thought to happen at scales smaller than the Planck length that is when the radius of the black string r $l_p = 1.62 \times 10^{-35}$ [2].

Another possible problem which could hinder the use of classical physics is if the timescale of Hawking radiation τ_H becomes smaller than that of the instability τ_{GL}. It can be show however that this do not happen until after the Planck time scale is reached [2].

– 5 –

Conclusion

Higher dimensional general relativity has been an area of increasing interest lately, due to several unifying theories demanding more than four dimensions. Flat space black hole is further of interest as it is a possibly that micro-black hole may be created in the Large Hadron Collider, made possible by the ADD theory.

It has not however been perfectly straight forward to generalize all the concepts from the familiar four-dimensional general relativity, such as uniqueness, to higher dimensions. While uniqueness is an inherited part of classical gravity this partly due to the special features of four-dimensional space. In higher dimensions there is nothing that limits the horizon topologies to be spherical, making it possible to have a number of different black holes.

Except the higher dimensional equivalences to Schwarzschild and Kerr black holes, the Tangherlini and Myers-Perry black hole, one of the first unique to higher space solutions found was the black string. The existence of black strings explicitly violates uniqueness in five dimensions. It is further possible to find multiple black hole solutions which consist of several black rings or black rings and black holes.

One of the most interesting new branches of black holes however is the black string, or more generally the black membrane. This is a black hole crossed with a spatial dimension, with the topology $S^p \otimes R$ for string and $S^p \otimes R^q$ for branes. As shown by Gregory and Laflamme [4] it is possible to evolve this kind of black hole solutions using a instability called the Gregory-Laflamme instability.

As shown by Lehner and Pretorius [71] this instability on a string will end up violating another well-established concept in four dimensions; the cosmic censorship conjecture. This conjecture prohibits the existence of naked singularities. The endstate of Gregory-Laflamme evolved black string however seems to be several smaller black holes, which means that the horizon of the string must pinch off. At the bifurcation of the horizon a naked singularity would be created.

Other quite surprising results come from the study of a specific kind of black strings, non-uniform black strings. These are strings with a specific wavelength which creates strings with constant perturbation. It seems like the non-uniform black string can change its topology if evolved, from a non-uniform black string to a black hole.

To fully understand these results a theory describing quantum gravity is needed. However until a theory of quantum gravity is (hopefully) discovered investigating these phenomena gives important insights to the structure and frame work for such a possible theory. Examining higher dimensional black holes is also important to be able to analyse the result from the Large Hadron Collider for micro-black holes, for the affirmation or repudiation of the ADD theory.

5.1 Future work and open questions

There are several open questions which could be asked and possible future work

- It would be highly desirable to be able to find and classify all higher dimensional black hole solutions in flat space. This could be problematic due to the amount of possible black holes in higher dimensions, unless effective solutions generating techniques are found.

- It would be interesting to try to generalize the Kerr-Newman black hole in higher dimensions. There have been higher dimensional black holes found with charge, but usually they are constructed to have gauge charge which does not properly correspond to the Kerr-Newman case [85]. While Kerr-Newman might exist in higher dimensions it might be considerably more complicated than in the four dimensional case. This remains an unsolved problem for higher dimensional black holes.

- For Gregory-Laflamme instabilities it would be very interesting to extend the study to other black hole branches for example black strings which are perturbed by unstable modes. As previously mentioned an ultra-spinning black hole behaves in its limits as a black membrane. This kind of object would also be interesting to examine, as well as to see if initial charge of the black strings changes the evolution.

- It would also, of course, always be of interest to improve the simulation to higher order, for more accurate results. While it has be simulated for a long time period by Lehner and Pretorius [71], it would still be interesting to investigate the topology of the string close to when and where a possible naked singularity could appear.

- Further as hinted by Sorkin [76], the behaviour of black strings is dimension dependent. Black strings seem to have a critical dimension, $d = 14$; below that dimension black holes have higher entropy however above it seems like the string have higher entropy. This allows for the question of how dimension dependent the phase diagrams of black strings and non-uniform black strings are. It would be interesting to examine the dynamics of black strings around the critical dimension by simulating the behaviour and to try to find the endstate for $d > 13$.

- Though not touched upon in this essay, it would be interesting to examine the Gregory-Laflamme instability in (anti)-de-Sitter space, where the curvature is non-zero.

- The apparent violation of the cosmic censorship conjecture discussed in section implies that naked singularities are formed in unstable black strings. It would be interesting to try to find a solid mathematical construction of the cosmic censorship conjecture, to in-depth study the consequences of Gregory-Laflamme instability.

- Lastly the fluid analogy of the black strings implies a deeper relationship between fluid dynamics and black hole dynamics, as suggested by the membrane paradigm. This would be useful to study further as it might be possible simulate

Gregory-Laflamme instabilities as a Rayleigh-Plateau instabilities, and apply fluid dynamical concept on black holes.

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