Entanglement in relativistic settings

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Chapter 1

Introduction

1.1 Introduction

At the beginning of the last century a revolution shook physics and led to the development of quantum mechanics, special relativity and subsequently general relativity. This last one is widely accepted as the theory of gravity and fits all observations so far. However, this theory includes ill-defined objects such as singularities (close to these, energies and distances reach the Planck scale). Such problems are expected to disappear in the quantized theory. But gravity reveals itself difficult to quantize. This remains one of the most important challenges of modern theoretical physics. On the other hand we have quantum mechanics, a theory with more controversial interpretations, but with even more tested results and by now also widely accepted.

The combination of both special relativity and quantum physics yields quantum field theory (QFT); one of the great scientific achievements of the twentieth century and the key to particle physics' confirmed predictions. If instead we consider general relativity we enter the domain of QFT in curved spacetime. Here many predictions are still waiting for experimental observation such as the famous Unruh or Hawking radiation[1].

In the early days of quantum physics the phenomenon of entanglement caught the attention of physicists, either because of its "strangeness"[2] or the apparent contradiction with other well established fields of physics, namely the Einstein-Podolsky-Rosen paradox[3]. Nowadays, quantum entanglement has become one of the most important resources of quantum information. A rapidly growing field in which many efforts are being made because of its remarkable applications. Many of these applications are considered impossible in the classical world; quantum communication, quantum cryptography, quantum simulation, to mention just a few. It is only recently that entanglement has been studied in a relativistic framework [4][5]. And shortly after in a general relativistic scenario[6][7].

Aim of the work

The aim of this work is to study how the entanglement is seen by different observers in relative motion, for both inertial and non-inertial cases. In the inertial context we first work with one-particle states as in [12][13]. However, we don't restrict ourself to initial maximally entangled states; we consider a wide variety of states by introducing preparation parameters. In the two-particle states section we extend the work in [14], considering new initial states (entangled in spin momentum) and studying different bipartitions (spin-spin and momentum-momentum), emphasizing the dependence not only on the preparation of the state but also on the Wigner rotation parameter. Although the "Alice and Bob work together" scenario has been naturally used in non-inertial cases, to the knowledge of the author, it has not been examined in the inertial case.

In the non-inertial case we mainly consider prepared single particle states. Again we work with more general states than in the literature by introducing preparation parameters. By setting specific values of these parameters we recover the results in [7] and [15]. Instead of using the "Alice and Bob work together" case, used in previous publications, we make Rob (the accelerated observer) work alone. Another new feature in this part is to consider entanglement between two different fields (boson-fermion system). And finally we look at the spin-momentum bipartition, not previously studied.

1.2 Structure of the thesis dissertation

This dissertation is divided into two parts:

Part I We start with a short introduction to the concept of entanglement and some entanglement measures which we will use throughout this work, namely entanglement entropy for pure states and negativity for mixed states. Next we go into relativistic quantum mechanics; presenting how a quantum mechanical particle state transforms under a Lorentz transformation and showing the Wigner rotation of the spin part of the state. Chapter 4 is the core of the first part, there we investigate the effects that the Lorentz transformation has on the entanglement. To do so, we consider two inertial observers, Alice and Bob. We make Alice prepare a set of states, both single and two-particle states, with a given amount of entanglement. Then we ask Bob to compute the entanglement for several bipartitions. Surprisingly, and as shown in previous literature, we will find that the entanglement of the majority of bipartitions is different for Alice and for Bob, i.e. is not Lorentz invariant. We show the dependence of entanglement for various settings on Δ. This parameter encodes the momentum of the particle(s) and the velocity of Bob with respect to Alice.

Part 2 Whereas in Part I we considered only special relativity, in this part we will work in a general relativistic framework. Therefore, we need to introduce some notion of QFT in curved spacetime. This is done in chapter 5 where we show the canonical quantization procedure for Minkowski and curved spacetimes. We then introduce Rob who uses Rindler coordinates, because he is constantly accelerated, and show how he describes the quantum field. Next we present the Bogoliubov coefficients that relates Alice's (inertial observer) and Rob's modes. After that we have the kernel of this part. Again we make Alice prepare settings[1] with some prescribed entanglement and then ask Rob to measure it. We observe an entanglement degradation between modes as seen in the literature. We also show the different behavior of entanglement for fermionic and bosonic fields in a combined system. Finally we look at a bipartition not yet studied: the spin-momentum bipartition.

[1] The settings are prepared such that there is no Wigner rotation, in this way we can study both effects separately.

Part I

Entanglement and Inertial observers

Chapter 2

Quantum Entanglement

In this chapter we will introduce the concept of non-classical correlation, and in particular of entanglement. Some entanglement measure will be presented for posterior use in this work. Many other sources can be found for a more detailed and complete study on entanglement see for example [8].

2.1 Definition of Entanglement

Quantum Entanglement is an essential quantum characteristic, inherent to all quantum systems that can be decomposed into two or more subsystems (composite quantum systems). Quantum entanglement give a notion about their structure, referring to its separability in terms of states of subsystems. In other words, entanglement measures the individuality of the subsystems, or the singleness of the quantum system under study.

2.1.1 Pure states

Consider two quantum systems A and B, and the Hilbert spaces associated to them $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. A general state for system A will be

$$|\varphi_A\rangle = \sum a_i |i_A\rangle$$

analogously for system B,

$$|\phi_B\rangle = \sum b_j |j_B\rangle$$

where the set $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are complete orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

The Hilbert space of the composite system is the tensor product of the two Hilbert spaces, $\mathcal{H}_A \otimes \mathcal{H}_B$ such that a general state of the composite system is given by

$$|\psi_{AB}\rangle = \sum c_{ij} |i_A\rangle \otimes |j_B\rangle$$

The state $|\psi\rangle_{AB}$ **is entangled if** it is not possible to write it as a tensor product of a state in A and a state in B, i.e.

$$|\psi_{AB}\rangle \neq |\varphi_A\rangle \otimes |\phi_B\rangle \leftrightarrow c_{ij} \neq a_i b_j$$

Otherwise it is said to be separable.

In terms of the density matrices formalism the state of the system is given by

$$\rho^{AB} = |\psi_{AB}\rangle \langle\psi_{AB}|$$

and the subsystem A and B are then described by their respective density matrices

$$\begin{aligned}\rho^A &= Tr_B\left(\rho^{AB}\right) = \sum_j \langle j_B|\, \rho^{AB}\, |j_B\rangle \\ \rho^B &= Tr_A\left(\rho^{AB}\right) = \sum_i \langle i_A|\, \rho^{AB}\, |i_A\rangle\end{aligned}$$

the state is said to be entangled if

$$\rho^A \neq |\varphi_A\rangle \langle\varphi_A| \ , \quad \rho^B \neq |\phi_B\rangle \langle\phi_B|$$

or in other words if the reduced density matrices are in a mixed state, i.e. they don't contain all the information of the system, something that ρ_{AB} does. In fact, if the reduce density matrices are maximally mixed (equal probability for all the possible states), the ρ_{AB} is said to be *maximally entangled*.

In terms of observables if the state is entangled the measurement performed on the subsystems are not independent of each others, they are said to be correlated. The relation for the expectation values of uncorrelated measurement $\langle O_A \otimes O_B\rangle_{\rho^{AB}} = \langle O_A\rangle_{\rho^A} \langle O_B\rangle_{\rho^B}$ don't hold for entangled states.

2.1.2 Mixed states

If from the beginning we have a lack of information about the actual state of the system and we only posses a statistical distribution of possible states, we will have to work with mixed states instead of dealing with pure states. In this case the mixedness of the reduced density matrices does not give conclusive information about the entanglement of the total system.

A state is said to be entangled, if it is not separable in the following sense

$$\rho^{AB} = \sum p_i \rho_i^A \otimes \rho_i^B$$

where p_i stands for the probability of the system to be in the state $\rho_i = \rho_i^A \otimes \rho_i^B$ and satisfies $\sum p_i = 1$ and $p_i \geq 0$. That is, the state is entangled if at least one of all the possible states in which the system can be found is not separable.

For mixed states correlations are in general not purely quantum mechanical. For example the state

$$\rho^{AB} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| \otimes |\downarrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\uparrow|\right)$$

is a separable mixed state but measurement will give totally correlated results.

2.2 Entanglement measures

At present there is no unique measure of entanglement. But all of them are positive functions of the state, $E(\rho)$, which satisfy the following axiom:

- Must be maximum for maximally entangled states (Bell states).
- Must be zero for separable states.
- Must be non-zero for all non-separable states.
- Must not grow under LOCC (Local operation + Classical Communication)

2.2.1 Pure states

For pure states there is a natural and well understood measurement of entanglement, the **entanglement entropy** defined as the *von Neumann entropy* of either of the two reduced density matrices

$$\begin{aligned} E(\rho) &= -Tr\left(\rho^A \log \rho^A\right) = -Tr\left(\rho^B \log \rho^B\right) \\ E(\rho) &= -\sum \lambda_i \log \lambda_i \end{aligned}$$

where λ_i are the eigenvalues of the reduced density matrix. Log is sometimes taken in base 2 such that it have an operational interpretation, or in base d, the dimension of the Hilbert space, so that it is bounded by 1(maximally entangled).

Another measure sometime used is the linear entropy defined as $E(\rho) = 1 - Tr\left(\rho_A^2\right)$ which is nothing else than the first order approximation of the entanglement entropy. $Tr\left(\rho^2\right)$ is called the purity of the state and is equal to one only if the state ρ is a pure state.

2.2.2 Mixed states

There are many different entanglement measures for mixed states like the entanglement of formation, the concurrence, entanglement cost, entanglement of distillation,etc. most of them difficult and complex to compute.

Here we will confine ourself to the negativity, a less computational demanding entanglement measure that will be used later on this work. Unfortunately negativity only accounts for distillable entanglement[1] in systems of more than dimension 2x3 (qubit-qutrit states), like some of ours later, ignoring bound entanglement[2]. Although we expect distillable entanglement to be enough to give qualitative idea of the entanglement behavior, suspicion remains.

Negativity, $\mathcal{N}$, is an entanglement monotone sensitive to distillable entanglement and is computed as follow: Starting with a general density matrix[3]

$$\rho_{AB} = \sum_{i,j,k,l} \rho_{ijkl} \left|i_A\right\rangle \left|j_B\right\rangle \left\langle k_A\right| \left\langle l_B\right|$$

[1] Entanglement that can be made maximum (Bell states) by means of LOCC.

[2] entangled state such that no pure entangled state can be obtained by LOCC.

[3] We make no difference between indices up or down. Neither do we between indices inside/outside bra and kets.

take the partial transpose defined as

$$\rho_{AB}^{pT_B} = \sum_{i,j,k,l} \rho_{ijkl} \left| i_A \right\rangle \left| l_B \right\rangle \left\langle k_A \right| \left\langle j_B \right|$$

compute it's eigenvalues (λ_i), and sum together all the negative ones.

$$\mathcal{N}_{AB} = \sum_{\lambda_i < 0} \lambda_i$$

negativity is zero for states with no distillable entanglement and its maximum depends on the dimension of the Hilbert space.

2.3 Mutual Information

Mutual information is a concept generalized from probability theory like the von Neumann entropy. It gives the amount information that two part of the system know from each other. It accounts for the correlations, quantum and classical, between the two variables. For a quantum bipartite system of density matrix ρ_{AB} the quantum mutual information is expressed

$$I_{AB} = S_A + S_B - S_{AB}$$

in terms of the Von Neumann entropies

$$\begin{aligned} S_{AB} &= -Tr\left(\rho^{AB} \log \rho^{AB}\right) \\ S_A &= -Tr\left(\rho^{A} \log \rho^{A}\right) \\ S_B &= -Tr\left(\rho^{B} \log \rho^{B}\right) \end{aligned}$$

Computing both entanglement and mutual information will give us an idea of how the information is distributed into quantum and classical. However, lately, it is being thought that entanglement isn't the only source of quantum correlation, and quantum discord is attracting more and more attention in quantum information.

Chapter 3

Relativistic Quantum mechanics

With time, group theory has become an essential tool to physics. Nowadays many concept are understood in terms of group theory. In particular, particles are considered as representations of the Poincaré group. This chapter is devoted to the study of the particle states and how they transform under Lorentz transformation. More can be found in relativistic quantum mechanics textbooks like [9].

3.1 Poincaré Group and its representations

The Poincaré group consist of Lorentz group plus the group of translations. The Lorentz group is defined as

$$\mathcal{L} := \left\{ \Lambda \in GL\left(4, \mathbb{R}\right); \Lambda^T \eta \Lambda = \eta \right\}$$

where η is the Minkowski metric $diag\left(- + + +\right)$. If we consider infinitesimal transformation we obtain the algebra satisfied by the generators,$M^{\mu\nu}$,

$$i\left[M^{\mu\nu}, M^{\rho\nu}\right] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\sigma\nu} M^{\rho\mu}$$

this together with the commutation relations

$$\begin{aligned} i\left[P^{\mu}, M^{\rho\sigma}\right] &= \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho} \\ \left[P^{\mu}, P^{\nu}\right] &= 0 \end{aligned}$$

form the Lie algebra of the Poincaré group.

The representation of the Poincaré group on the state vectors of infinite (because it's non-compact) dimensional Hilbert space is unitary (connected Lie group) and can be written as

$$\left|\psi\right\rangle' = U\left(\Lambda, a\right)\left|\psi\right\rangle$$

To find the irreducible representations we use the well-known Casimir operators

$$\begin{aligned} C_1 &= P^2 = P^\mu P_\mu \\ C_2 &= W^2 = W^\mu W_\mu \end{aligned}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma$ is the Pauli-Lubanski vector. But those alone don't classify completely the unitary, irreducible representations; since $sign(p^0)$ is unchanged by a Lorentz transformation. This translates into 6 distinct classes of irreps:

		Little Group
$p^2 = m^2 > 0, p^0 > 0$	massive	$SO(3)$
$p^2 = m^2 > 0, p^0 < 0$	unphysical	$SO(3)$
$p^2 = 0, p^0 > 0$	massless	$ISO(2)$
$p^2 = 0, p^0 < 0$	unphysical	$ISO(2)$
$p^\mu = 0$	vacuum	$SO(3,1)$
$p^2 < 0$	virtual particles	$SO(3)$

(3.1)

For a massive particle the Casimir operators are

$$\begin{aligned} C_1 &= P^2 = m^2 \\ C_2 &= W^2 = m^2 \mathbf{S}^2 \end{aligned}$$

where S is the spin operator. It is worth noting that since $\mathbf{S}^2$ is Lorentz invariant, statistics is frame independent.

3.1.1 Particle states and their Lorentz transformation

A base for the Hilbert space on which a unitary Poincaré transformation is realized can be constructed with the eigenstates of the complete set of commuting observables $\left\{P^2, \mathbf{S}^2, H, \mathbf{P}, S_z\right\}$, i.e. $\left\{\left|m, s; p^0, \mathbf{p}, \sigma\right\rangle\right\}$ where the states are labeled by their eigenvalues. We can simplify notation due to the invariance of the Casimir operators and set

$$\left|m, s; p^0, \mathbf{p}, \sigma\right\rangle \equiv |p, \sigma\rangle$$

such that the basis states are labelled by their four-momentum and the z-component of the spin (later simply called spin). This correspond to a basis of plane waves and, thus, transform under translations as

$$U(I, a)|p, \sigma\rangle = e^{-ipa}|p, \sigma\rangle$$

For Lorentz transformations it isn't that straight forward and we need the Wigner method or method of induced representations. A general Lorentz transformation takes the momentum $p^\mu \to \Lambda^\mu{}_\nu p^\nu$ and therefore $U(\Lambda)|p, \sigma\rangle$ must be a linear combination of all the states with momentum Λp, i.e.

$$U(\Lambda)|p, \sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(\Lambda, p)|\Lambda p, \sigma'\rangle$$

Since $U(\Lambda)$ is a representation it respect the group multiplication imposing conditions on the values of $D_{\sigma'\sigma}$. Those conditions are satisfied when we restrict $D_{\sigma'\sigma}(\Lambda, p)$ to $D_{\sigma'\sigma}(W, p)$ where W are Lorentz transformations that leave invariant a chosen standard momentum[1] (k^{μ}).

Consider the Lorentz transformation that takes k^{μ} to $p^{\mu} = L^{\mu}_{\ \nu}(p)\, k^{\nu}$ such that

$$|p,\sigma\rangle = U(L(p))\,|k,\sigma\rangle$$

then a Lorentz transformation, Λ, on $|p,\sigma\rangle$ is

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= U(\Lambda)\,U(L(p))\,|k,\sigma\rangle \\ &= U(I)\,U(\Lambda)\,U(L(p))\,|k,\sigma\rangle \\ &= U(L(\Lambda p))\,U\left(L^{-1}(\Lambda p)\right)U(\Lambda)\,U(L(p))\,|k,\sigma\rangle \\ &= U(L(\Lambda p))\,U\left(L^{-1}(\Lambda p)\,\Lambda L(p)\right)|k,\sigma\rangle \\ &= U(L(\Lambda p))\,U(W(\Lambda,p))\,|k,\sigma\rangle \end{aligned}$$

where $W(\Lambda, p) = L^{-1}(\Lambda p)\,\Lambda L(p)$ leaves the standard momentum k invariant and therefore only act on the spin degree of freedom of $|k,\sigma\rangle$. On the other hand $U(L(\Lambda p))$ by definition take k to Λp without touching the spin, then

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= U(W(\Lambda,p))\,|\Lambda p,\sigma\rangle \\ &= \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\,|\Lambda p,\sigma'\rangle \end{aligned} \tag{3.2}$$

$W(\Lambda, p)$ is an element of the little group. In other words, under a Lorentz transformation $U(\Lambda)$, the momentum label p goes to Λp, and the spin transform under the representation, $D_{\sigma'\sigma}$, of the little group, W.

For massive particles where the standard momentum is $k^{\mu} = (m, 0, 0, 0)$ in its rest frame, the little group is nothing else than the rotation group $SU(2)$ and

$$D^{s}_{\sigma'\sigma}(W) = \langle s,\sigma'|\, e^{i\delta \mathbf{S}\cdot\mathbf{n}}\,|s,\sigma\rangle$$

So, if we are dealing with massive spin-$\frac{1}{2}$ particles we have the corresponding spin-$\frac{1}{2}$ representation of $SU(2)$

$$D_{\sigma'\sigma}(W) = I\cos\frac{\delta}{2} + i\left(\tilde{\boldsymbol{\sigma}}\cdot\widehat{n}\right)\sin\frac{\delta}{2} \tag{3.3}$$

where δ is the Wigner angle and $\boldsymbol{\sigma}_i$ are the Pauli matrices. This rotation is a consequence of the non-closeness of the boost generator algebra, i.e. two boost is equivalent to a boost and a rotation. That rotation is closely related to Thomas precession and is called the Wigner rotation.

[1] This is an underlying consequence of the unmixability of the different classes 3.1

Wigner rotation

To get the Wigner angle and the axis of rotation we need to know both $L(p)$ and Λ in the previous discussion. Consider for example that the particle has momentum p in the z axis from Alice reference frame and that Bob is boosted in the x direction with respect to Alice. In this case we have

$$L(p) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where α is the rapidity of Alice with respect to the rest frame of the particle, and β the one of Bob w.r.t. Alice. From the commutation of the boost generators $[K_i, K_j] = -ie\epsilon_{ijk}J_k$ we see the axis of rotation will be perpendicular to both boosts ($\vec{n} = \vec{v} \times \vec{w}$), in this case around the -y axis, so that the Wigner rotation will have the form

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta & 0 & \sin\delta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\delta & 0 & \cos\delta \end{pmatrix}$$

Now to get the angle we write

$$W(\Lambda, p) = L^{-1}(\Lambda p)\,\Lambda L(p) \Rightarrow L^{-1}(\Lambda p) = \Lambda L(p)\,W^{-1}(\Lambda, p)$$

$$L^{-1}(\Lambda p) = \begin{pmatrix} \cosh\alpha\cosh\beta & \sinh\beta\cos\delta - \cosh\beta\sinh\alpha\sin\delta & 0 & \sinh\beta\sin\delta + \cosh\beta\sinh\alpha\cos\delta \\ \cosh\alpha\sinh\beta & \cosh\beta\cos\delta - \sinh\alpha\sinh\beta\sin\delta & 0 & \cosh\beta\sin\delta + \sinh\alpha\sinh\beta\cos\delta \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & -\cosh\alpha\sin\delta & 0 & \cosh\alpha\cos\delta \end{pmatrix}$$

but since a boost matrix is symmetric we find

$$\begin{aligned} -\cosh\alpha\sin\delta &= \cosh\beta\sin\delta + \sinh\alpha\sinh\beta\cos\delta \\ \tan\delta &= \frac{\sinh\alpha\sinh\beta}{\cosh\alpha + \cosh\beta} \end{aligned}$$

from here we see δ range from $0 \le \delta \le \frac{\pi}{2}$. For more general formulas and various calculations I refer to [10].

Chapter 4

Entanglement and Inertial observers

This chapter is the main one of part I of this work. Here we will study what happens to entanglement when measured by different inertial perspectives for a variety of constructions, including two-particle and single-particle states. In all the cases the entanglement between two partition of the system will be computed.

4.1 Single-particle states entanglement

We start first with single particle states. A general one-particle state is written as

$$|\psi\rangle = \sum_{\sigma} \int d\mu\,(p)\, f_{\sigma}\,(p)\, |p,\sigma\rangle$$

where $d\mu\,(p) = \frac{1}{(2\pi)^3} \frac{d^3p}{2E_p}$ is the Lorentz-invariant measure introduced to normalize the basis states, such that

$$\sum_{\sigma} \int d\mu\,(p)\, \langle p,\sigma\, |p,\sigma\rangle = 1 \text{ and } \sum_{\sigma} \int d\mu\,(p)\, |f_{\sigma}\,(p)|^2 = 1$$

One of the first study of entanglement in relativistic frame was done precisely and surprisingly in the context of single particle states by Peres, Scudo and Terno in [4]. They considered a state with a Gaussian distribution $f_{\sigma}\,(p)$ and found that two observers related by a Lorentz boost will not agree on the entropy of the reduced spin state and therefore will see a different entanglement between spin and momentum. This is because the reduced spin and momentum density matrices don't transform covariantly.

To simplify calculations we will consider states which are assumed to be sufficiently peaked in momentum so that we can consider a discrete set of momentums and avoid the computational problems of the continuous.

Our system consist of a spin-$\frac{1}{2}$ particle that can only take two distinct values of momentum (p_1, p_2),such that the most general state we can work with, is

$$|\psi\rangle = A\,|p_1 \uparrow\rangle + B\,|p_1 \downarrow\rangle + C\,|p_2 \uparrow\rangle + D\,|p_2 \downarrow\rangle$$

The density matrix $\rho = |\psi\rangle\langle\psi|$ of the system is then

$$\rho = \begin{pmatrix} A^2 & AB^* & AC^* & AD^* \\ BA^* & B^2 & BC^* & BD^* \\ CA^* & CB^* & C^2 & CD^* \\ DA^* & DB^* & DC^* & D^2 \end{pmatrix} \begin{matrix} |p_1 \uparrow\rangle \\ |p_1 \downarrow\rangle \\ |p_2 \uparrow\rangle \\ |p_2 \downarrow\rangle \end{matrix}$$

We will confine ourself to the study of the following representative cases

1. A general separable state

$$|\psi\rangle = (\cos\chi\,|p_1\rangle + \sin\chi\,|p_2\rangle)(\cos\xi\,|\uparrow\rangle + \sin\xi\,|\downarrow\rangle) \tag{4.1}$$

   then we have $A = \cos\chi\cos\xi$, $B = \cos\chi\sin\xi$, $C = \sin\chi\cos\xi$, $D = \sin\chi\sin\xi$.

2. Here we have a state for which Alice's spin-momentum entanglement is parametrized by η

$$|\psi\rangle = \cos\eta\,|p_1 \uparrow\rangle + \sin\eta\,|p_2 \downarrow\rangle \tag{4.2}$$

   such that $A = \cos\eta$, $D = \sin\eta$.

4.1.1 Alice studies entanglement

We will consider the only bipartition that can be made for a single particle state with only two degrees of freedom, namely the spin-momentum bipartition. To measure the entanglement we use entanglement entropy introduced in 2.2.1. The momentum reduced density matrix is

$$\rho_{mom} = \begin{pmatrix} A^2 + B^2 & AC + BD \\ CA + DB & C^2 + D^2 \end{pmatrix} \begin{matrix} |p_1\rangle \\ |p_2\rangle \end{matrix}$$

which has eigenvalues $\lambda_\pm = \frac{1}{2}(1 \pm l)$, where $l = \sqrt{1 - 4\,(BC - AD)^2}$. Therefore the entanglement is given by

$$E = -\lambda_+ \log\lambda_+ - \lambda_- \log\lambda_-$$

Considering our two cases:

1. Alice sees a separable state, as expected

$$E = 0$$

2. Entanglement is parametrized by η,

$$E(\eta) = -\cos^2\eta\log\left(\cos^2\eta\right) - \sin^2\eta\log\left(\sin^2\eta\right)$$

4.1.2 Bob studies entanglement

Bob who's moving with respect to Alice decide to do the same computations. But the state he sees is $\rho = |\psi\rangle^{\Lambda} \langle\psi|^{\Lambda}$ where

$$|\psi\rangle^{\Lambda} = U(\Lambda, p)|\psi\rangle$$

The transformation is given by equation 3.2, then

$$\begin{aligned} U(\Lambda,p)|p_1 \uparrow\rangle &= \sum_{\sigma} D^{(1)}_{\sigma\uparrow} |p_1'\sigma\rangle \\ U(\Lambda,p)|p_1 \downarrow\rangle &= \sum_{\sigma_2'} D^{(1)}_{\sigma\downarrow} |p_1'\sigma\rangle \\ U(\Lambda,p)|p_2 \uparrow\rangle &= \sum_{\sigma} D^{(2)}_{\sigma\uparrow} |p_2'\sigma\rangle \\ U(\Lambda,p)|p_2 \downarrow\rangle &= \sum_{\sigma_2'} D^{(2)}_{\sigma\downarrow} |p_2'\sigma\rangle \end{aligned}$$

where $p'_{1(2)} = \Lambda p_{1(2)}$, and D is given by equation 3.3.

It has been shown that the entanglement is more affected by the Wigner rotation when spin and momentum are parallel to each other[1]. Also it is well known that the effect of the Wigner rotation are maximum when the boost axis is perpendicular to the momentum. For this, and without loss of generality, making the computations simpler we will consider both p_1 and p_2 in the z axis, and Bob moving in the x direction as in 3.1.1, obtaining

$$D^{(1)} = \begin{pmatrix} \cos\frac{\delta_1}{2} & \sin\frac{\delta_1}{2} \\ -\sin\frac{\delta_1}{2} & \cos\frac{\delta_1}{2} \end{pmatrix} \quad , \quad D^{(2)} = \begin{pmatrix} \cos\frac{\delta_2}{2} & \sin\frac{\delta_2}{2} \\ -\sin\frac{\delta_2}{2} & \cos\frac{\delta_2}{2} \end{pmatrix} \tag{4.3}$$

In general the Wigner angle range only between 0 and $\frac{\pi}{2}$. But since we haven't specified the direction of p_1, p_2 (the axis of rotation will depend whether they point in the $+z$ or $-z$ direction), this permits us to increase the range of $\delta_{1,2}$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Which is equivalent to allow rotation of $\delta \in [0, \frac{\pi}{2}]$ around $\pm y$-axis. Therefore

$$\Delta = \delta_1 - \delta_2 \in [-\pi, \pi]$$

Explicitly the states observed by Bob are

$$\begin{aligned} U(\Lambda,p)|p_1 \uparrow\rangle &= \cos\frac{\delta_1}{2}|p_1' \uparrow\rangle - \sin\frac{\delta_1}{2}|p_1' \downarrow\rangle \\ U(\Lambda,p)|p_1 \downarrow\rangle &= \sin\frac{\delta_1}{2}|p_1' \uparrow\rangle + \cos\frac{\delta_1}{2}|p_1' \downarrow\rangle \\ U(\Lambda,p)|p_2 \uparrow\rangle &= \cos\frac{\delta_2}{2}|p_2' \uparrow\rangle - \sin\frac{\delta_2}{2}|p_2' \downarrow\rangle \\ U(\Lambda,p)|p_2 \downarrow\rangle &= \sin\frac{\delta_2}{2}|p_2' \uparrow\rangle + \cos\frac{\delta_2}{2}|p_2' \downarrow\rangle \end{aligned}$$

[1] In fact, when spin and momentum are perpendicular the entanglement is not observer dependent in inertial frames.

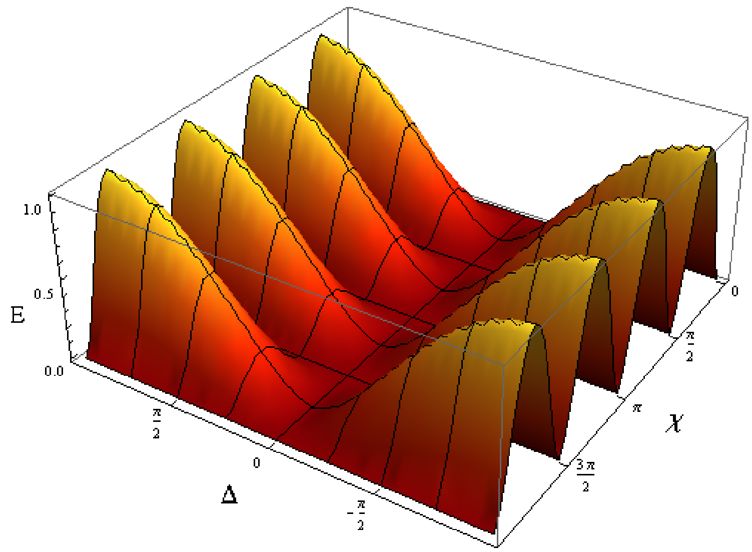

Figure 4.1: Spin-momentum entanglement entropy for preparation (4.1)

That give us the following transformations of the coefficients due to the Lorentz transformation

$$\begin{array}{c} A \longrightarrow A\cos\frac{\delta_1}{2} + B\sin\frac{\delta_1}{2} \\ B \longrightarrow -A\sin\frac{\delta_1}{2} + B\cos\frac{\delta_1}{2} \\ C \longrightarrow C\cos\frac{\delta_2}{2} + D\sin\frac{\delta_2}{2} \\ D \longrightarrow -C\sin\frac{\delta_2}{2} + D\cos\frac{\delta_2}{2} \end{array}$$

Repeating Alice's computation Bob obtain:

1. Entanglement different from zero (more than Alice)

$$E = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$$

where $\lambda_\pm = \frac{1}{2} \pm \frac{1}{4}\sqrt{3 + \cos 4\chi + 2\cos\Delta \sin^2 2\chi}$. It is interesting to see that for Alice separable states, the entanglement seen by Bob is totally independent of the spin state (ξ). Moreover the maximum of entanglement depend on how well is the state distributed in momentum (χ), in particular if the state has only one momentum (i.e. $\chi = 0, n\frac{\pi}{2}$), entanglement remains zero. For the equally distributed momentum ($\chi = \frac{\pi}{4}$) in the limit of the two modes moving in opposite direction, and Bob all at the speed of light ($\Delta \rightarrow \pm\pi$) entanglement approach a maximally entangled state (see figure 4.1).

2. Bob observes less entanglement than Alice

$$E = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$$

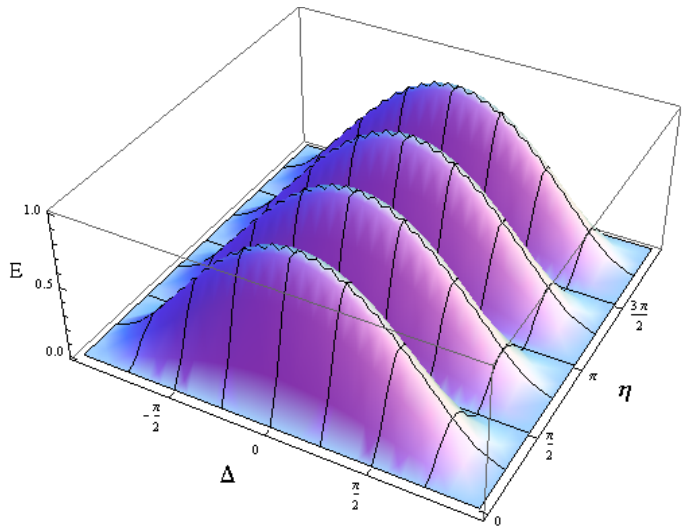

Figure 4.2: Spin-momentum entanglement entropy for preparation (4.2)

where $\lambda_{\pm} = \frac{1}{2} \pm \frac{\sqrt{2}}{8}\sqrt{6 - 2\cos\Delta + \cos(\Delta - 4\eta) + 2\cos 4\eta + \cos(\Delta + 4\eta)}$. Here we see the opposite effect; Bob's entanglement decreases from Alice's ($\delta_1 = \delta_2 = 0$) and tends to zero in the above mentioned limit (see figure 4.2). Despite the antiresemblance of the behavior it is worth to note that the increasing rate of entanglement for setting 1 is greater than the decreasing rate of setting 2 (see figure 4.3). So we can say that it is easier to create entanglement by boosting than to destroy it.

4.1.3 Alice and Bob work together

Instead of arguing with each other because of their disagreement, they look for a way of working together. They decide that Alice will be sensible only to the part of the state with momentum p_1, and Bob to the part with p_2. Therefore only the part associate to Bob will endure a Lorentz transformation getting

$$\begin{array}{c} A \to A \\ B \to B \end{array}, \quad \begin{array}{c} C \to C\cos\frac{\delta_2}{2} + D\sin\frac{\delta_2}{2} \\ D \to -C\sin\frac{\delta_2}{2} + D\cos\frac{\delta_2}{2} \end{array}$$

which is the same as obtained by Bob with $\delta_1 = 0$. Concluding that this share-out between Alice and Bob don't give anything new.

4.1.4 Summary of results

In this section we have seen how entanglement between degrees of freedom of a particle is not a Lorentz invariant quantity. This comes from the dependence with the momentum of the spin state seen by Bob. If the state prepared by

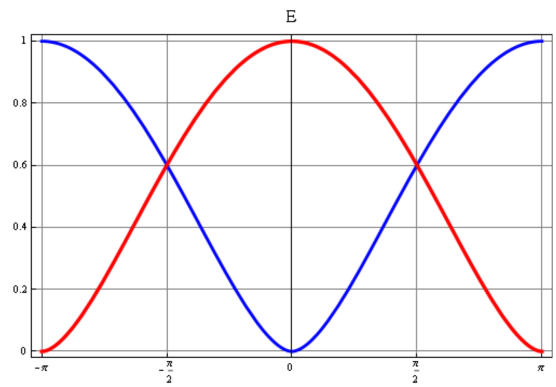

Figure 4.3: Spin-momentum entanglement for Alice prepared maximally entangled state (Red), and non-entangled state (Blue)

Alice has spin-momentum entanglement, Bob will see less entanglement than Alice. If Alice's state is not entangled, Bob will see it entangled. Furthermore, we have seen that entanglement is easier to increment by means of a Lorentz boost than to lessen; a detail that doesn't seem to have been noticed in previous works.

4.2 Two-particle states entanglement

The two-particle Hilbert space is given by $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where the states

$$|p_1, \sigma_1; p_2, \sigma_2\rangle = |p_1, \sigma_1\rangle \otimes |p_2, \sigma_2\rangle$$

are normalized as

$$\sum_{\sigma_1, \sigma_2} \int d\mu\left(p_1, p_2\right) \langle p_1, \sigma_1; p_2, \sigma_2 \,| p_1, \sigma_1; p_2, \sigma_2\rangle = 1$$

where $d\mu\left(p\right) = \frac{1}{(2\pi)^3} \frac{d^3 p}{2E_p}$ is the Lorentz-invariant measure. A general state is given by[2]

$$|\psi\rangle = \sum_{\sigma_1, \sigma_2} \int d\mu\left(p_1, p_2\right) f_{\sigma_1 \sigma_2}\left(p_1, p_2\right) |p_1, \sigma_1; p_2, \sigma_2\rangle$$

also normalized as

$$\sum_{\sigma_1, \sigma_2} \int d\mu\left(p_1, p_2\right) \left|f_{\sigma_1 \sigma_2}\left(p_1, p_2\right)\right|^2 = 1$$

[2]If we were considering indistinguishable fermions $f_{\sigma_1 \sigma_2}\left(p_1, p_2\right)$ would need to be anti-symmetric, so that it satisfies fermi statistics.

A study with independent momentum Gaussian $g(p)$ states can be found in [11] where they consider a Bell state with spread momentum, i.e. $f_{\sigma_1\sigma_2}(p_1,p_2) \propto \delta_{\sigma_1\sigma_2} g(p_1) g(p_2)$. This example shows how the entanglement is exchanged between spin and momentum degree of freedom under a Lorentz transformation, keeping the joint entanglement of the wave function invariant. This is because the global density matrix transforms covariantly, while the reduced spin and momentum density matrices don't. They also state and prove the following theorem

Theorem 1 *The entanglement between the spin and momentum parts of a pure state wave function, must be non-zero to allow the spin entanglement to increase under Lorentz transformation.*

Again and for the same reason like in the single particle section we consider only two momenta.

Our system consist of two spin-$\frac{1}{2}$ particles that can only take two distinct values of momentums (p_1, p_2 denoted by $1, 2$), that we additionally impose to be different for each particle. The more general state we can consider for this system is

$$\begin{aligned} |\psi\rangle &= A\,|1\uparrow;2\uparrow\rangle + B\,|1\uparrow;2\downarrow\rangle + C\,|1\downarrow;2\uparrow\rangle + D\,|1\downarrow;2\downarrow\rangle \\ &\quad +E\,|2\uparrow;1\uparrow\rangle + F\,|2\uparrow;1\downarrow\rangle + G\,|2\downarrow;1\uparrow\rangle + H\,|2\downarrow;1\downarrow\rangle \end{aligned} \tag{4.4}$$

with $A^2+B^2+C^2+D^2+E^2+F^2+G^2+H^2=1$.The density matrix $\rho = |\psi\rangle\langle\psi|$ is then

$$\rho = \begin{pmatrix} |A|^2 & AB^* & AC^* & AD^* & AE^* & AF^* & AG^* & AH^* \\ BA^* & |B|^2 & BC^* & BD^* & BE^* & BF^* & BG^* & BH^* \\ CA^* & CB^* & |C|^2 & CD^* & CE^* & CF^* & CG^* & CH^* \\ DA^* & DB^* & DC^* & |D|^2 & DE^* & DF^* & DG^* & DH^* \\ EA^* & EB^* & EC^* & ED^* & |E|^2 & EF^* & EG^* & EH^* \\ FA^* & FB^* & FC^* & FD^* & FE^* & |F|^2 & FG^* & FH^* \\ GA^* & GB^* & GC^* & GD^* & GE^* & GF^* & |G|^2 & GH^* \\ HA^* & HB^* & HC^* & HD^* & HE^* & HF^* & HG^* & |H|^2 \end{pmatrix} \begin{matrix} |1\uparrow;2\uparrow\rangle \\ |1\uparrow;2\downarrow\rangle \\ |1\downarrow;2\uparrow\rangle \\ |1\downarrow;2\downarrow\rangle \\ |2\uparrow;1\uparrow\rangle \\ |2\uparrow;1\downarrow\rangle \\ |2\downarrow;1\uparrow\rangle \\ |2\downarrow;1\downarrow\rangle \end{matrix} \tag{4.5}$$

In what follows we will consider this as the state seen by Alice. The state seen by Bob will be the Lorentz transformed of this one. This is our starting point, from which we will study the different entanglement settings below:

1. Separable in momentum and spin parts

$$\begin{aligned} |\psi\rangle &= |\psi_{mom}\rangle\,|\psi_{spin}\rangle = (\cos\chi\,|1;2\rangle + \sin\chi\,|2;1\rangle)\,(\cos\xi\,|\uparrow;\downarrow\rangle + \sin\xi\,|\downarrow;\uparrow\rangle) \\ &= \cos\chi\cos\xi\,|1\uparrow;2\downarrow\rangle + \cos\chi\sin\xi\,|1\downarrow;2\uparrow\rangle \\ &\quad + \sin\chi\cos\xi\,|2\uparrow;1\downarrow\rangle + \sin\chi\sin\xi\,|2\downarrow;1\uparrow\rangle \end{aligned} \tag{4.6}$$

comparing with 4.4 we have

$$B = \cos\chi\cos\xi, F = \sin\chi\cos\xi, C = \cos\chi\sin\xi, G = \sin\chi\sin\xi \tag{4.7}$$

with all others equal to zero.

2. Spin-momentum entangled

$$|\psi\rangle = \cos\eta\,|1\uparrow;2\downarrow\rangle + \sin\eta\,|2\downarrow;1\uparrow\rangle \tag{4.8}$$

such that

$$B = \cos\eta, G = \sin\eta \tag{4.9}$$

and all others zero.

Note that only setting 1 with $\chi = \frac{7\pi}{4}$ is compatible with indistinguishable fermions. That state is maximally entangled in momentum.

4.2.1 Alice studies entanglement

Several bipartitions of this system can be done allowing us to compute entanglement between different degrees of freedom. We will look at spin-spin, momentum-momentum, spin-momentum and particle1-particle2 entanglement. All of this will be done for Alice and in the next section for Bob's reference frames.

Spin-spin entanglement

When computing the spin-spin entanglement we ignore the momentum of the particles and therefore we have to trace them out of the density matrix, resulting in the reduced density matrix

$$\rho_{spin} = \begin{pmatrix} |A|^2+|E|^2 & AB^*+EF^* & AC^*+EG^* & AD^*+EH^* \\ BA^*+FE^* & |B|^2+|F|^2 & BC^*+FG^* & FH^*+BD^* \\ CA^*+GE^* & CB^*+GF^* & |C|^2+|G|^2 & GH^*+CD^* \\ DA^*+HE^* & HF^*+DB^* & HG^*+DC^* & |H|^2+|D|^2 \end{pmatrix} \begin{matrix} |\uparrow;\uparrow\rangle \\ |\uparrow;\downarrow\rangle \\ |\downarrow;\uparrow\rangle \\ |\downarrow;\downarrow\rangle \end{matrix} \tag{4.10}$$

Setting 1 Inserting the values 4.7 we get

$$\rho_{spin} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\xi & \frac{1}{2}\sin 2\xi & 0 \\ 0 & \frac{1}{2}\sin 2\xi & \sin^2\xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} |\uparrow;\uparrow\rangle \\ |\uparrow;\downarrow\rangle \\ |\downarrow;\uparrow\rangle \\ |\downarrow;\downarrow\rangle \end{matrix} \tag{4.11}$$

since the state is spin-momentum separable this density matrix is in a pure state. For this reason we use entanglement entropy defined in 2.2.1.

$$Tr_1\left(\rho_{spin}\right) = \begin{pmatrix} \cos^2\xi & 0 \\ 0 & \sin^2\xi \end{pmatrix}$$

Finally the entanglement entropy is

$$E\left(\rho_{spin}\right) = -\cos^2\xi\log\left(\cos^2\xi\right) - \sin^2\xi\log\left(\sin^2\xi\right) \tag{4.12}$$

It doesn't depend on χ as expected since the momentum and spin parts of the wave function are separable. Moreover maximally entangled states ($E\left(\rho_{spin}\right) = 1$) are reached for $\xi = n\frac{\pi}{4}$, which are the well-known Bell states.

Setting 2 This matrix is in general a mixed state (when there is spin-momentum entanglement), so we use the negativity to compute the entanglement.

$$\rho_{spin} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\eta & 0 & 0 \\ 0 & 0 & \sin^2\eta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} |\uparrow;\uparrow\rangle \\ |\uparrow;\downarrow\rangle \\ |\downarrow;\uparrow\rangle \\ |\downarrow;\downarrow\rangle \end{matrix} \tag{4.13}$$

$$\rho_{spin}^{pT} = \rho_{spin}$$

the partial transpose matrix has no negative eigenvalue, and therefore there is no spin-spin entanglement for this setting.

Momentum-momentum entanglement

To compute the momentum-momentum entanglement we overlook the spin, hence tracing over spin we get the reduced density matrix

$$\rho_{mom} = \begin{pmatrix} |A|^2 + |B|^2 + |C|^2 + |D|^2 & BF^* + CG^* + AE^* + DH^* \\ FB^* + GC^* + EA^* + HD^* & |F|^2 + |G|^2 + |H|^2 + |E|^2 \end{pmatrix} \begin{matrix} |1;2\rangle \\ |2;1\rangle \end{matrix} \tag{4.14}$$

Setting 1 For the same reason as for the spin-spin computation we use the entanglement entropy

$$\rho_{mom} = \begin{pmatrix} \cos^2\chi & \frac{1}{2}\sin 2\chi \\ \frac{1}{2}\sin 2\chi & \sin^2\chi \end{pmatrix} \begin{matrix} |1;2\rangle \\ |2;1\rangle \end{matrix} \tag{4.15}$$

$$Tr_1\left(\rho_{mom}\right) = \begin{pmatrix} \cos^2\chi & 0 \\ 0 & \sin^2\chi \end{pmatrix}$$

$$E\left(\rho_{mom}\right) = -\cos^2\chi \log\left(\cos^2\chi\right) - \sin^2\chi \log\left(\sin^2\chi\right) \tag{4.16}$$

this result was expected since it is the same kind of state as the spin states considered above.

Setting 2 Again we use negativity because this matrix is in general a mixed state (when there is spin-momentum entanglement).

$$\rho_{mom} = \begin{pmatrix} \cos^2\eta & 0 \\ 0 & \sin^2\eta \end{pmatrix} \begin{matrix} |1;2\rangle \\ |2;1\rangle \end{matrix} \tag{4.17}$$

$$\rho_{mom}^{pT} = \rho_{mom}$$

and again no negative eigenvalues, hence no momentum-momentum entanglement is found.

Spin-momentum entanglement

To compute it we can either use ρ_{spin} or ρ_{mom}. For simplicity we use ρ_{mom}.

Setting 1 States are separable. Even if 4.11 and 4.15 depend on different variables they have the same eigenvalues, 0 and 1. So that we find the required result

$$E(\rho) = -1 \log 1 - 0 \log 0 = 0$$

Setting 2 Now we find the entanglement parametrized by η:

$$E(\rho) = -\cos^2\eta \log\left(\cos^2\eta\right) - \sin^2\eta \log\left(\sin^2\eta\right) \tag{4.18}$$

Particle1-particle2 entanglement

Here we don't ignore any information of the system and hence don't trace any information out. We directly compute the entanglement entropy. The reduced density matrix for particle 1 is:

$$\rho_1 = \begin{pmatrix} |A|^2+|B|^2 & AC^*+BD^* & 0 & 0 \\ CA^*+DB^* & |C|^2+|D|^2 & 0 & 0 \\ 0 & 0 & |E|^2+|F|^2 & EG^*+FH^* \\ 0 & 0 & GE^*+HF^* & |G|^2+|H|^2 \end{pmatrix} \begin{matrix} |1\uparrow\rangle \\ |1\downarrow\rangle \\ |2\uparrow\rangle \\ |2\downarrow\rangle \end{matrix} \tag{4.19}$$

Setting 1 Inserting the values 4.7 into 4.19 :

$$\rho_1 = \begin{pmatrix} (\cos\chi\cos\xi)^2 & 0 & 0 & 0 \\ 0 & (\cos\chi\sin\xi)^2 & 0 & 0 \\ 0 & 0 & (\sin\chi\cos\xi)^2 & 0 \\ 0 & 0 & 0 & (\sin\chi\sin\xi)^2 \end{pmatrix} \begin{matrix} |1\uparrow\rangle \\ |1\downarrow\rangle \\ |2\uparrow\rangle \\ |2\downarrow\rangle \end{matrix} \tag{4.20}$$

is already diagonal, thus the entanglement entropy in a simplified form is

$$\begin{aligned} E(\rho) &= -\cos^2\chi \log\left(\cos^2\chi\right) - \sin^2\chi \log\left(\sin^2\chi\right) \\ &\quad -\cos^2\xi \log\left(\cos^2\xi\right) - \sin^2\xi \log\left(\sin^2\xi\right) \end{aligned} \tag{4.21}$$

Which is nothing else than the sum of spin and momentum entanglement, 4.12 and 4.16 , the maximum ($E(\rho) = \log 4$) is for Bell type states in spin and momentum ($\chi = \xi = \frac{n\pi}{4}$).

Setting 2 With the values 4.9 we get

$$\rho_1 = \begin{pmatrix} \cos^2\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^2\eta \end{pmatrix} \begin{matrix} |1\uparrow\rangle \\ |1\downarrow\rangle \\ |2\uparrow\rangle \\ |2\downarrow\rangle \end{matrix}$$

$$E(\rho) = -\cos^2\eta \log\left(\cos^2\eta\right) - \sin^2\eta \log\left(\sin^2\eta\right) \tag{4.22}$$

Before going to Bob's case here is a table summarizing the results of Alice.

Entanglement	spin	momentum	spin-mom.	Particle
Setting 1	$E(\xi)$	$E(\chi)$	0	$E(\xi)+E(\chi)$
Setting 2	0	0	$E(\eta)$	$E(\eta)$

4.2.2 Bob studies entanglement

Bob who's moving with respect to Alice decide to do the same computations. But the state he sees is $\rho = |\psi\rangle^{\Lambda}\langle\psi|^{\Lambda}$ where

$$|\psi\rangle^{\Lambda} = U(\Lambda,p)|\psi\rangle$$

considering our two settings we need to know how the states $|1\uparrow;2\downarrow\rangle$, $|1\downarrow;2\uparrow\rangle$, $|2\uparrow;1\downarrow\rangle$ and $|2\downarrow;1\uparrow\rangle$ transform under a Lorentz transformation. This is given by equation 3.2, then

$$\begin{aligned}
U(\Lambda,p)|1\uparrow;2\downarrow\rangle &= \sum_{\sigma_1',\sigma_2'} D_{\sigma_1'\uparrow}D_{\sigma_2'\downarrow}|\Lambda p_1,\sigma_1';\Lambda p_2,\sigma_2'\rangle\\
U(\Lambda,p)|1\downarrow;2\uparrow\rangle &= \sum_{\sigma_1',\sigma_2'} D_{\sigma_1'\downarrow}D_{\sigma_2'\uparrow}|\Lambda p_1,\sigma_1';\Lambda p_2,\sigma_2'\rangle\\
U(\Lambda,p)|2\uparrow;1\downarrow\rangle &= \sum_{\sigma_1',\sigma_2'} D_{\sigma_1'\uparrow}D_{\sigma_2'\downarrow}|\Lambda p_2,\sigma_1';\Lambda p_1,\sigma_2'\rangle\\
U(\Lambda,p)|2\downarrow;1\uparrow\rangle &= \sum_{\sigma_1',\sigma_2'} D_{\sigma_1'\downarrow}D_{\sigma_2'\uparrow}|\Lambda p_2,\sigma_1';\Lambda p_1,\sigma_2'\rangle
\end{aligned}$$

where D are given by equation 3.3. Again, like in the single particle section we will consider both p_1 and p_2 in the z axis, and Bob moving in the x direction, that is

$$D_1 = \begin{pmatrix} \cos\frac{\delta_1}{2} & \sin\frac{\delta_1}{2} \\ -\sin\frac{\delta_1}{2} & \cos\frac{\delta_1}{2} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \cos\frac{\delta_2}{2} & \sin\frac{\delta_2}{2} \\ -\sin\frac{\delta_2}{2} & \cos\frac{\delta_2}{2} \end{pmatrix} \tag{4.23}$$

with $\delta_{1,2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\delta_1 - \delta_2 = \Delta \in [-\pi,\pi]$. Before writing the states, let us clarify the notation: the two momentum will still be denoted by 1 and 2, now standing for Λp_1 and Λp_2 respectively. Explicitly the states are

$$\begin{aligned}
U(\Lambda,p)|1\uparrow;2\downarrow\rangle &= a_1|1\uparrow;2\uparrow\rangle + a_2|1\uparrow;2\downarrow\rangle + a_3|1\downarrow;2\uparrow\rangle + a_4|1\downarrow;2\downarrow\rangle\\
U(\Lambda,p)|1\downarrow;2\uparrow\rangle &= -a_4|1\uparrow;2\uparrow\rangle + a_3|1\uparrow;2\downarrow\rangle + a_2|1\downarrow;2\uparrow\rangle - a_1|1\downarrow;2\downarrow\rangle\\
U(\Lambda,p)|2\uparrow;1\downarrow\rangle &= -a_4|2\uparrow;1\uparrow\rangle + a_2|2\uparrow;1\downarrow\rangle + a_3|2\downarrow;1\uparrow\rangle - a_1|2\downarrow;1\downarrow\rangle\\
U(\Lambda,p)|2\downarrow;1\uparrow\rangle &= a_1|2\uparrow;1\uparrow\rangle + a_3|2\uparrow;1\downarrow\rangle + a_2|2\downarrow;1\uparrow\rangle + a_4|2\downarrow;1\downarrow\rangle
\end{aligned}$$

with

$$\begin{aligned}
a_1 &= \cos\frac{\delta_1}{2}\sin\frac{\delta_2}{2}, \quad a_2 = \cos\frac{\delta_1}{2}\cos\frac{\delta_2}{2}\\
a_3 &= -\sin\frac{\delta_1}{2}\sin\frac{\delta_2}{2}, \quad a_4 = -\sin\frac{\delta_1}{2}\cos\frac{\delta_2}{2}
\end{aligned}$$

That give us the following transformations of the coefficients due to the Lorentz transformation

$$\begin{array}{ll} A \rightarrow Ba_1 - Ca_4 & E \rightarrow Ga_1 - Fa_4 \\ B \rightarrow Ba_2 + Ca_3 & F \rightarrow Ga_3 + Fa_2 \\ C \rightarrow Ba_3 + Ca_2 & G \rightarrow Ga_2 + Fa_3 \\ D \rightarrow Ba_4 - Ca_1 & H \rightarrow Ga_4 - Fa_1 \end{array}$$

Bob is not as lucky as Alice because the entanglement becomes very cumbersome and extremely long to compute. Instead he decide to compute it for a limited but representative set of states seen by Alice:

$$\text{setting 1.1} \quad \longrightarrow \quad |\psi\rangle = \frac{1}{\sqrt{2}} |1;2\rangle (|\uparrow;\downarrow\rangle + |\downarrow;\uparrow\rangle) \tag{4.24}$$

$$\text{setting 1.2} \quad \longrightarrow \quad |\psi\rangle = \frac{1}{\sqrt{2}} (|1;2\rangle + |2;1\rangle) |\uparrow;\downarrow\rangle \tag{4.25}$$

$$\text{setting 1.3} \quad \longrightarrow \quad |\psi\rangle = \frac{1}{2} (|1;2\rangle + |2;1\rangle) (|\uparrow;\downarrow\rangle + |\downarrow;\uparrow\rangle) \tag{4.26}$$

$$\text{setting 2} \quad \longrightarrow \quad |\psi\rangle = \frac{1}{\sqrt{2}} (|1\uparrow;2\downarrow\rangle + |2\downarrow;1\uparrow\rangle) \tag{4.27}$$

The first three are included in setting 1 and the last one in setting 2. Their entanglement and their coefficient are shown below

State	s-s	m-m	s-m	Coefficients
4.24	1	0	0	$B = C = \frac{1}{\sqrt{2}}$
4.25	0	1	0	$B = F = \frac{1}{\sqrt{2}}$
4.26	1	1	0	$B = C = F = G = \frac{1}{2}$
4.27	0	0	1	$B = G = \frac{1}{\sqrt{2}}$

remind that our measure of entanglement[3] range from 0 to 1.

Spin-spin entanglement

Using 4.10 and inserting the corresponding new values of A-H into it, we get the following result for the various settings.

Setting 1.1(4.24) is included in the case with $\chi = 0, \pi$ for which we get a pure state with entanglement eq.4.12. The interesting $\chi = \pi/4$ case in which setting 1.2 (4.25) can be seen is displayed in figure 4.4 . States prepared by Alice with no spin-spin entanglement ($\xi = n\pi/2$) are generally not pure but negativity remains zero, while for $\xi = \pi/4, 5\pi/4$, the symmetric spin Bell states, purity and entanglement remains maximum(setting 1.3 correspond to this case). However, for the antisymmetric spin Bell states, $\xi = 3\pi/4, 7\pi/4$, entanglement is generally Δ dependent (except for $\chi = n\pi/2$) and entanglement can be made zero, not in the limit $\Delta \rightarrow \pi$, but when $\Delta \rightarrow \pi/2$ (see figure 4.5). But probably the

[3] Althought this is not true for negativity, we will rescale the maximum to 1, specially when the maximum correspond to a pure state with an entanglement entropy of 1.

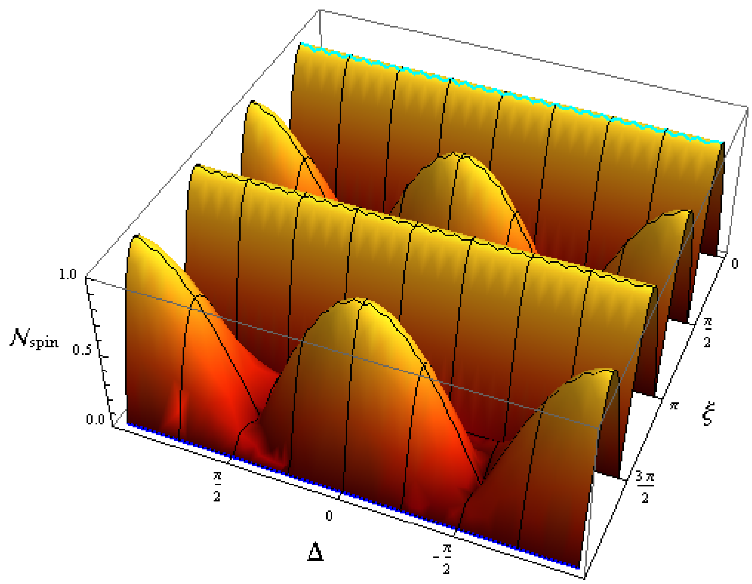

Figure 4.4: Negativity of the spin-spin bipartition for setting 1 with $\chi = \pi/4$. Includes setting 1.2 (Blue line) and setting 1.3 (Cyan line).

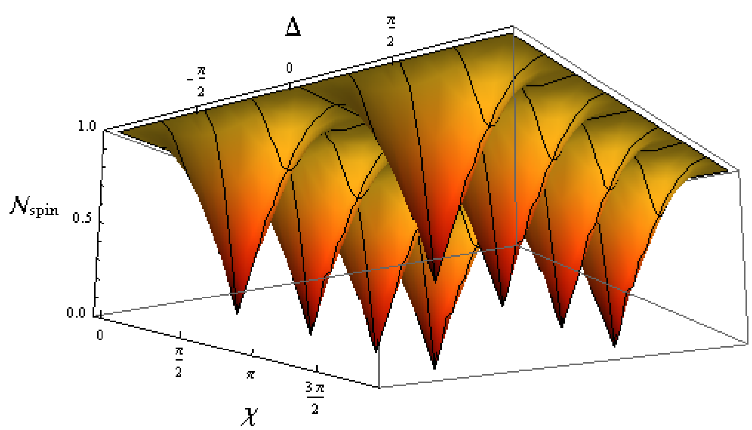

Figure 4.5: Negativity for an antisymmetric spin Bell state as a function of χ.

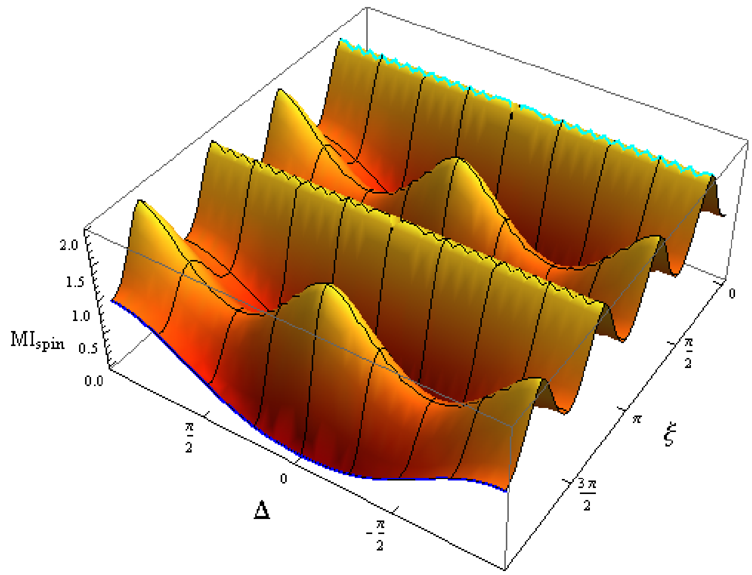

Figure 4.6: Mutual information between the two spin partition. Blue line for setting 1.2 and cyan for setting 1.3

more striking feature is that spin-spin entanglement is invariant for all states prepared by Alice with $\xi \in \{[0, \pi/2] \wedge [\pi, 3\pi/2]\}$.

For setting 2 (4.27) the state get in general mixed, but negativity remains zero.

Several cases have an invariant entanglement but we have to distinguish between those who remain pure and those who don't. The difference can clearly be seen computing the mutual information. For example, for (4.24) and (4.26) it is independent of δ_1, δ_2 while for (4.25) and (4.27) we have explicit dependence. Figure 4.6 is very similar to that of negativity except that for states prepared non-entangled by Alice, Bob will see an increase in mutual information. This could be in the form of classical correlations or even other types of quantum correlation like quantum discord. For setting 2 the opposite occurs, the mutual information decrease (see figure 4.7)

Momentum-momentum entanglement

Setting 1 The purity of the resulting state is

$$Tr\left(\rho_{mom}^2\right) = \cos^4\chi + \sin^4\chi + \frac{1}{8}\sin^2\chi\left(1 + \cos\Delta + 2\sin\frac{\Delta}{2}\sin 2\xi\right)^2$$

when the state is pure the entanglement entropy is the same as Alice's eq.4.16. In fact, 4.24 and 4.26's purity, entanglement and mutual information are inde-

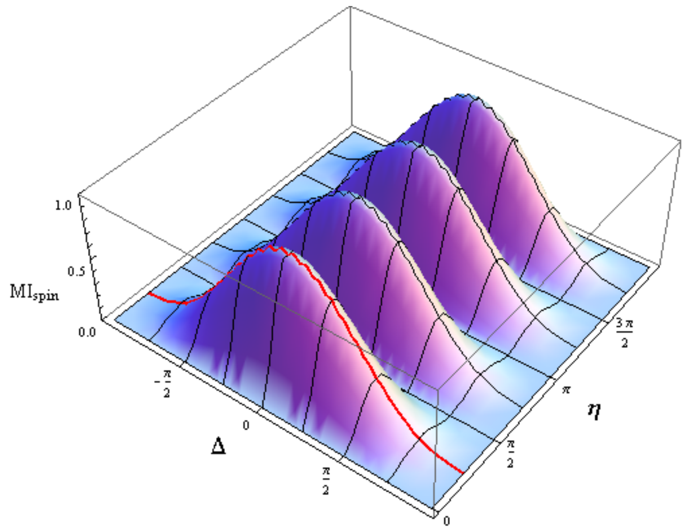

Figure 4.7: Mutual information for the spin-spin bipartition of setting 2.

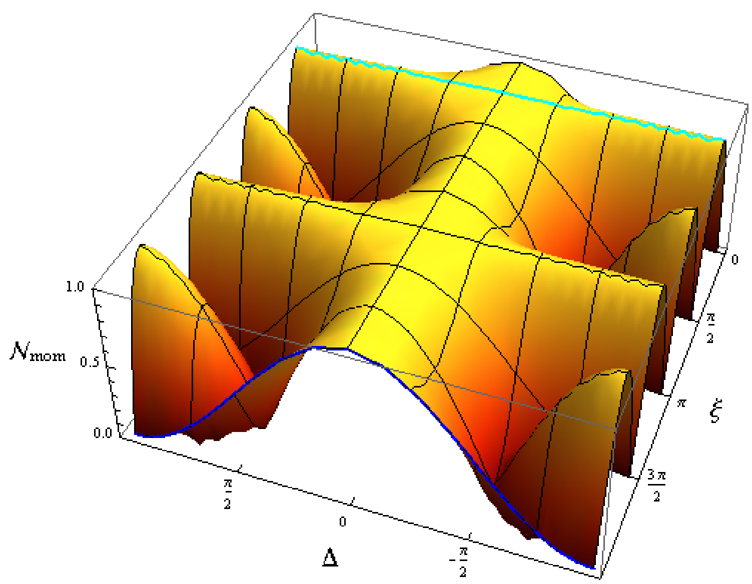

Figure 4.8: Negativity for the momentum-momentum bipartition of setting 1 with $\chi = \pi/4$,including 1.2 (Blue line) and 1.3 (Cyan line).

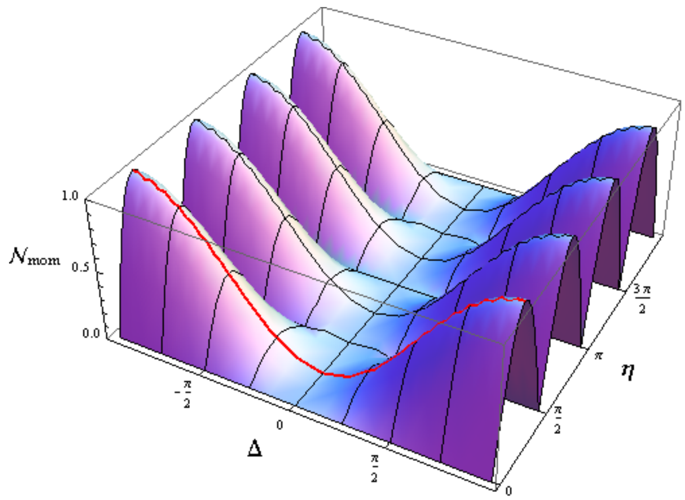

Figure 4.9: Negativity of the momentum-momentum bipartition for setting 2(Red line).

pendent of δ_1, δ_2. Negativity gives

$$\mathcal{N}_{mom} = \frac{1}{4}\left|\sin\chi\left(1+\cos\Delta+2\sin^2\frac{\Delta}{2}\sin 2\xi\right)\right|$$

which can be seen in figure 4.8 for $\chi = \pi/4$; in this case χ only change the maximum of the graph leaving it's form the same. It is interesting to note that now the only states that have invariant entanglement are the symmetric Bell states. The states with $\xi \in \{(0,\pi/2)\wedge(\pi,3\pi/2)\setminus\pi/4,5\pi/4\}$ will indeed have a Δ-dependent entanglement. In particular for 4.25 the state is pure for $\Delta = 0$ in which case entanglement entropy is the same as Alice. For all other values, negativity is

$$\mathcal{N}_{mom} = \cos^2\frac{\Delta}{2}$$

Thus, entanglement between the momenta of the particles tends to zero in the limit of $\Delta \to \pm\pi$, while for the antisymmetric Bell states entanglement decrease to zero for $\Delta \to \pm\pi/2$.

Setting 2 Purity is

$$Tr\left(\rho_{mom}^2\right) = \cos^4\eta+\sin^4\eta+\frac{1}{2}\sin^4\frac{\Delta}{2}\sin^2 2\eta$$

considering our state, 4.27, it will be pure in the limit $\Delta \to \pm\pi$, in which case entanglement is maximum. For non-pure reduced density matrices we get

$$\mathcal{N}_{mom} = \frac{1}{2}\left|\sin^2\frac{\Delta}{2}\sin 2\eta\right|$$

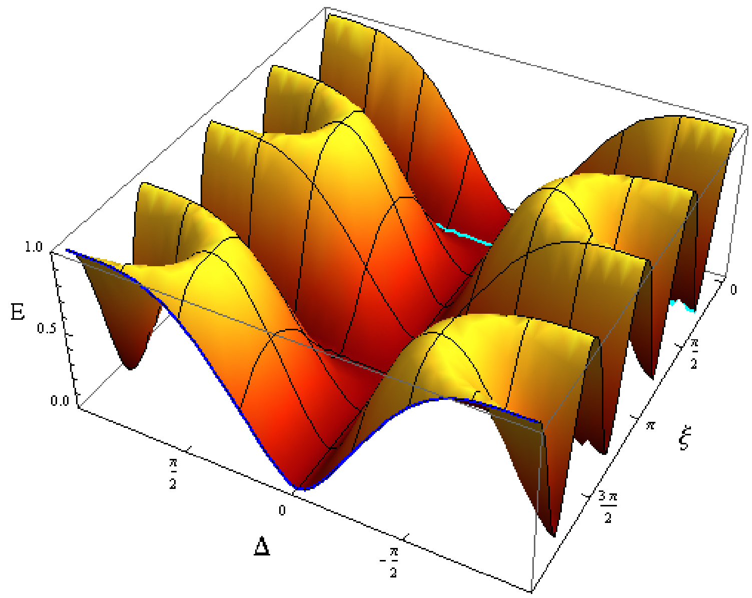

Figure 4.10: Entanglement entropy for spin-momentum bipartition of setting 1 including 1.2 (Blue line) and 1.3 (Cyan line)

which can be seen in figure 4.9.

In the momentum-momentum bipartition mutual information behave in concordance with entanglement, unlike the previous studied spin-spin bipartition.

Spin-momentum entanglement

Here we don't trace any information out and therefore since ρ is a pure state, we can use entanglement entropy.

Setting 1 For 4.24 and 4.26 entanglement is invariant. In 4.25, blue line in figure 4.10, entanglement goes from zero to the maximum entanglement,1, reached in the limit $\Delta \to \pm\pi$.

$$E_B(\Delta) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$$

where $\lambda_\pm = \frac{1}{2}\left(1 \pm \cos^2 \frac{\Delta}{2}\right)$. The form of the graph is independent of χ, which only set the maxima. Again we can see an invariant behavior of the entanglement for the symmetric Bell states ($\xi = \pi/4, 5\pi/4$). Their neighborhood is like the momentum-momentum case, $\Delta-$dependent. In fact the two figure are closely related, if we turn one of them around, they look exactly the same[4].

[4] We can only compare them qualitatively because we are using two diferent measure of entanglement.

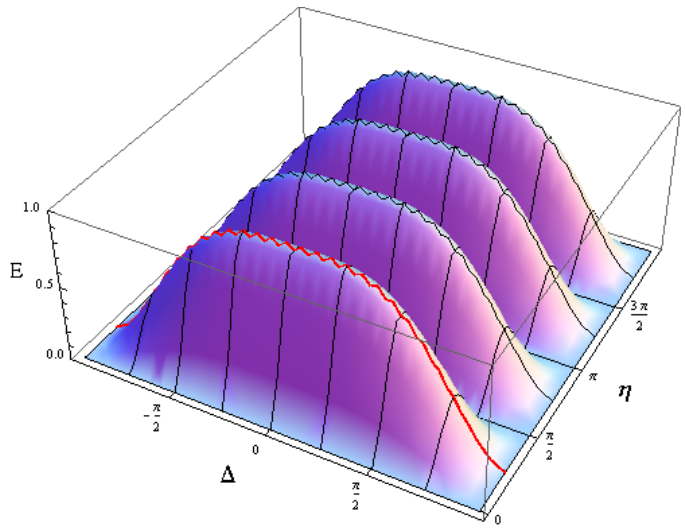

Figure 4.11: Entanglement entropy of setting 2 including state 4.27 (Red line)

Setting 2 Entanglement is then

$$E_B(\Delta) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$$

with $\lambda_\pm = \frac{1}{2}\left(1 \pm \sin^2 \frac{\Delta}{2}\right)$, plotted as a red line in figure 4.11.

We find again that entanglement is easier increased than degraded, but this time the difference is even stronger (see figure 4.12). We expect this effect to be emphasized for state of more particle.

Particle1-particle2 entanglement

Setting 1 As expected and found in all the previous works the entanglement of the particle-particle bipartition is the same as Alice's for all cases of $\chi, \xi, \delta_1, \delta_2$.

Setting 2 Again, Bob agrees with Alice for all values of $\eta, , \delta_1, \delta_2$.

4.2.3 Alice and Bob work together

Alice and Bob exchange their result and after arguing with each other for a while they decide to redo the computation together. We consider the two particle distinguishable such that the first correspond to Alice and the second to Bob[5]. Therefore Bob's particle will be Wigner rotated while Alice's won't. The state

[5] If the particle where indistinguishable another criterion, like momentum, should be taken in order to separate the state into Alice and Bob.

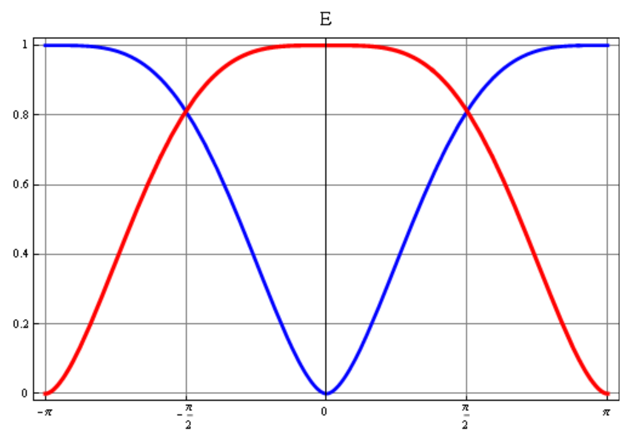

Figure 4.12: Spin-momentum entanglement for a two particle state, in a prepared maximally entangled state (Red) and non-entangled case (Blue)

under consideration here is $\rho = |\psi\rangle^{\Lambda} \langle\psi|^{\Lambda}$ where

$$|\psi\rangle^{\Lambda} = U(1) \otimes U(\Lambda, p) |\psi\rangle$$

considering our two settings we need to know how the states $|1\uparrow; 2\downarrow\rangle$, $|1\downarrow; 2\uparrow\rangle$, $|2\uparrow; 1\downarrow\rangle$, $|2\downarrow; 1\uparrow\rangle$ transform under a Lorentz transformation. This is given by equation 3.2, then

$$\begin{aligned}
U(1) \otimes U(\Lambda, p) |1\uparrow; 2\downarrow\rangle &= \sum_{\sigma_2'} D_{\sigma_2'\downarrow} |p_1\uparrow; \Lambda p_2, \sigma_2'\rangle \\
U(1) \otimes U(\Lambda, p) |1\downarrow; 2\uparrow\rangle &= \sum_{\sigma_2'} D_{\sigma_2'\uparrow} |p_1\downarrow; \Lambda p_2, \sigma_2'\rangle \\
U(1) \otimes U(\Lambda, p) |2\uparrow; 1\downarrow\rangle &= \sum_{\sigma_2'} D_{\sigma_2'\uparrow} |p_2\uparrow; \Lambda p_1, \sigma_2'\rangle \\
U(1) \otimes U(\Lambda, p) |2\downarrow; 1\uparrow\rangle &= \sum_{\sigma_2'} D_{\sigma_2'\downarrow} |p_2\downarrow; \Lambda p_1, \sigma_2'\rangle
\end{aligned}$$

where D's are given by 4.23. The first thing we note is the appearance of more than two different momenta that we will label $p_1 \to 1$, $p_2 \to 2$, $\Lambda p_1 \to 1'$, $\Lambda p_2 \to 2'$. This increase considerably the dimension of our system under study from $(2x2)^2$ to $(4x2)^2$, but hopefully most of the entries are cero. The resulting

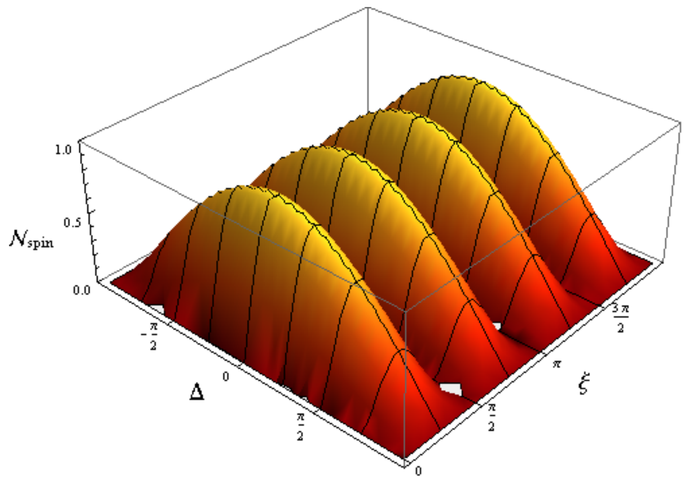

Figure 4.13: Negativity for the AB case for the spin-spin bipartition of setting 1, and fixed $\chi = \pi/4$.

states are

$$\begin{aligned}
U(1)\otimes U(\Lambda,p)\left|1\uparrow;2\downarrow\right\rangle &= \sin\frac{\delta_2}{2}\left|1\uparrow;2'\uparrow\right\rangle + \cos\frac{\delta_2}{2}\left|1\uparrow;2'\downarrow\right\rangle \\
U(1)\otimes U(\Lambda,p)\left|1\downarrow;2\uparrow\right\rangle &= \cos\frac{\delta_2}{2}\left|1\downarrow;2'\uparrow\right\rangle - \sin\frac{\delta_2}{2}\left|1\downarrow;2'\downarrow\right\rangle \\
U(1)\otimes U(\Lambda,p)\left|2\uparrow;1\downarrow\right\rangle &= \sin\frac{\delta_1}{2}\left|2\uparrow;1'\uparrow\right\rangle + \cos\frac{\delta_1}{2}\left|2\uparrow;1'\downarrow\right\rangle \\
U(1)\otimes U(\Lambda,p)\left|2\downarrow;1\uparrow\right\rangle &= \cos\frac{\delta_1}{2}\left|2\downarrow;1'\uparrow\right\rangle - \sin\frac{\delta_1}{2}\left|2\downarrow;1'\downarrow\right\rangle
\end{aligned}$$

Our density matrix is like 4.5 but now in the basis

$\{\left|1\uparrow;2'\uparrow\right\rangle, \left|1\uparrow;2'\downarrow\right\rangle, \left|1\downarrow;2'\uparrow\right\rangle, \left|1\downarrow;2'\downarrow\right\rangle, \left|2\uparrow;1'\uparrow\right\rangle, \left|2\uparrow;1'\downarrow\right\rangle, \left|2\downarrow;1'\uparrow\right\rangle, \left|2\downarrow;1'\downarrow\right\rangle\}$

so that we can use our expressions 4.10, 4.14 and 4.19. Our coefficient become

$$\begin{array}{cc}
A \to B\sin\frac{\delta_2}{2} & E \to F\sin\frac{\delta_1}{2} \\
B \to B\cos\frac{\delta_2}{2} & F \to F\cos\frac{\delta_1}{2} \\
C \to C\cos\frac{\delta_2}{2} & G \to G\cos\frac{\delta_1}{2} \\
D \to -C\sin\frac{\delta_2}{2} & H \to -G\sin\frac{\delta_1}{2}
\end{array}$$

Spin-spin entanglement

Introducing the new values into the reduced spin density matrix 4.10 we find a similar behavior to the study of Bob alone, but with two important differences. First of all, antisymmetric and symmetric spin Bell states have the same

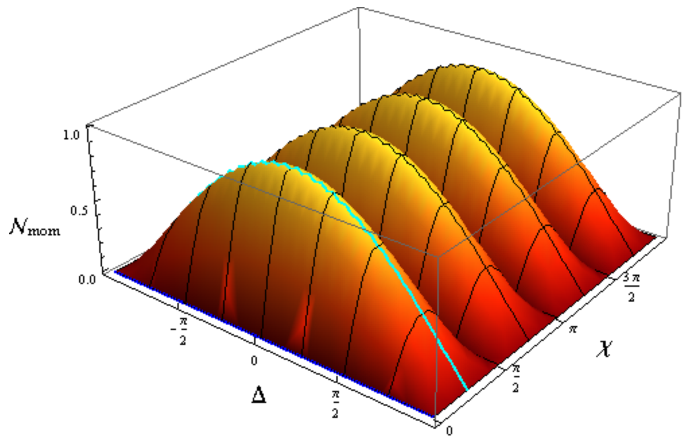

Figure 4.14: Negativity in AB case for momentum-momentum bipartition of setting 1, including setting 1.1 (Blue line) and 1.2/3 (Cyan line)

behavior, and second, entanglement descend to zero in the limit $\Delta \rightarrow \pm\pi$ (see figure 4.13). And in terms of mutual information nothing fancy occurs, i.e. it decreases always.

For setting 2 the state is generally mixed and negativity is cero for all values of Δ.

Momentum-momentum entanglement

Inserting our values in 4.14 we obtain a result independent of the spin preparation of the state,ξ. Negativity gives us the simple expresion

$$\mathcal{N}_{mom}^{AB} = \frac{1}{4}\sqrt{(1+\cos(\Delta))(1-\cos(4\chi))}$$

In preparation 2 the state is generally mixed and negativity remains zero for all values of Δ.

When Alice and Bob work together there is a total analogy between spin-spin and momentum-momentum bipartition, each one with it's corresponding parameter characterizing Alice's preparation of entanglement, ξ and χ respectively.

Spin-momentum entanglement

We find a relatively small expression, again independent of ξ, for the entanglement entropy

$$E^{AB}(\chi,\Delta) = -\frac{1}{4}(2-f(\chi,\Delta))\log\left[\frac{1}{4}(2-f(\chi,\Delta))\right] - \frac{1}{4}(2+f(\chi,\Delta))\log\left[\frac{1}{4}(2+f(\chi,\Delta))\right]$$

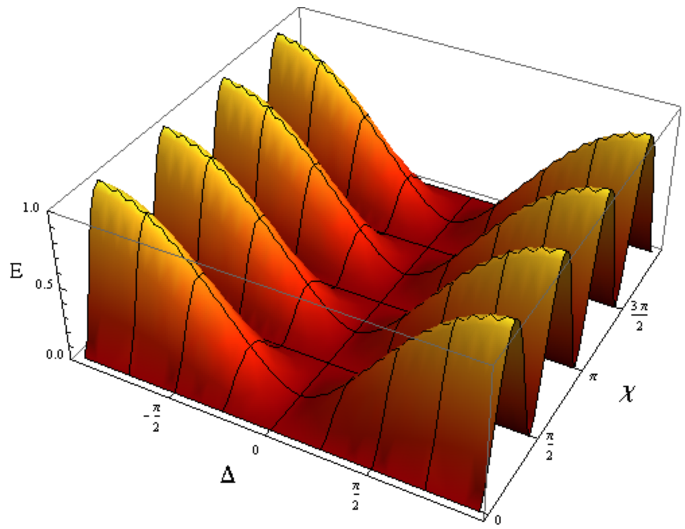

Figure 4.15: Entanglement entropy for the AB case of setting 1 in the spin-momentum bipartition.

where $f(\chi,\Delta)=\sqrt{3+\cos(4\chi)+2\cos\Delta\sin^2(2\chi)}$. This can be seen in figure 4.15.

For setting 2 entanglement entropy is independent of Δ.

Particle1-particle2 entanglement

Once more, entanglement is the same independently of Δ for all cases.

We present a table summarizing some of our qualitative results for the settings 1.1,1.2,1.3 and 2; "Ent. =" means entanglement remains equal, "Ent. $\nearrow$" means increase, and "Ent. $\searrow$" represent a decrease in entanglement.

Setting1	Alice	Bob	Alice-Bob
spin-spin	$E(\xi)$	Ent. = Ent. = Ent. =	Ent. = Ent. = Ent. $\searrow$
mom-mom	$E(\chi)$	Ent. = Ent. $\searrow$ Ent. =	Ent. = Ent. = Ent. $\searrow$
spin-mom	0	Ent. = Ent. $\nearrow$ Ent. =	Ent. = Ent. = Ent. $\nearrow$
part-part	$E(\xi)+E(\chi)$	Ent. =	Ent. =

Setting 2	Alice	Bob	Alice-Bob
spin-spin	0	Ent. $=$	Ent. $=$
mom-mom	0	Ent. $\nearrow$	Ent. $=$
spin-mom	$E(\eta)$	Ent. $\searrow$	Ent. $=$
part-part	$E(\eta)$	Ent. $=$	Ent. $=$

Notice how for Alice the particle-particle entanglement is the sum of all the other entanglements and how for Bob, if one increase the other decrease. We think this is due to the conservation of the part-part entanglement.

4.2.4 Summary of results

Alice and Bob always agree in the entanglement between the particles but that is not the case when they look at different bipartitions of the two particle system.

For instance, when Bob looks at the spin-spin bipartition he will always find equal or less entanglement than Alice. In fact, equal entanglement is found for a surprisingly large subset of states; not only for the Bell states, but also for a big neighborhood around the spin-symmetric Bell states. The other states with invariant entanglement are the non-entangled states, with separable spin and momentum parts, which show us an intriguing increase in mutual information with Δ. Being able to extinguish entanglement with Δ as small as $\pi/2$ is also an interesting fact.

When Bob studies the momentum-momentum bipartition, the entanglement can either decrease, remain equal or increase with respect to what Alice sees. Entanglement is invariant for the initially unentangled and for the spin symmetric Bell states, decreases for all other states separable in spin and momentum parts and increases for states with initial spin-momentum entanglement. However the increase and decrease are not equivalent, the increase reaches it's maximum in the limit $\Delta \to \pi$, while entanglement can be made zero for Δ as small as $\pi/2$.

The behavior of entanglement for the spin-momentum bipartition is very similar to the momentum-momentum one but with the increasing and decreasing states exchanged. They are the inverted image of each other. Again, as in the single particle section we see that this entanglement is easier to increase than to decrease, and we expect that the more particles are involved, the more emphasized the difference becomes.

When Alice and Bob work together, many interesting effects disappear. As a result, entanglement only grows in the spin-momentum bipartition, spin-spin and momentum-momentum are completely analogous to each other and extrema of entanglement are achieved only in the limit $\Delta \to \pi$.

Part II

Entanglement and non-Inertial observers

Chapter 5

Introduction to QFT in curved spacetimes

This chapter is a very brief introduction to QFT in curved spacetime presented via the standard canonical quantization. Only the minimum necessary for understanding the next chapters will be presented. More about this extensive topic can be found in many textbooks, for instance [16].

5.1 Scalar field quantization

The equation for scalar fields is the Klein-Gordon equation

$$\left(\Box - m^2\right)\Phi = 0 \tag{5.1}$$

who's general solution can be expressed as a sum of negative and positive frequency solutions, $u_i(x,t)$ and $u_i^*(x,t)$,

$$\Phi(x,t) = \sum_i \alpha_i u_i(x,t) + \alpha_i^* u_i^*(x,t)$$

5.1.1 In Minkowski spacetime

In Minkowski spacetime it is easy to separate between positive and negative frequency because this spacetime admits a *global* timelike Killing vector, ∂_t (a pointer in the direction of time). Therefore, positive frequency solution[1] satisfy

$$\partial_t u_k(x,t) = -i\omega_k u_k(x,t) \tag{5.2}$$

[1] Any inertial observer will agree with that because Lorentz transformation leave the sign of t invariant.

and negative ones are $u_k^*(x,t)$. We can choose those solutions to form an orthonormal basis of solution so that their scalar product satisfy

$$\begin{aligned}(u_j,u_k) &= -i\int d^3x\,(u_j\partial_t u_k^* - u_k^*\partial_t u_j) = \delta_{jk} = -\left(u_j^*,u_k^*\right) \\ (u_j,u_k^*) &= 0\end{aligned} \tag{5.3}$$

this allow us to construct a Fock space using the standard canonical field quantization scheme:

- Promote the classical Klein-Gordon field and it's canonical conjugate momentum to quantum operators satisfying the equal time commutation relations

$$\begin{aligned}\left[\Phi(x,t),\Pi(x',t)\right] &= i\delta(x-x') \\ \left[\Phi(x,t),\Phi(x',t)\right] &= \left[\Pi(x,t),\Pi(x',t)\right] = 0\end{aligned} \tag{5.4}$$

- Replace the complex amplitudes α_i and α_i^* by annihilation and creation operators a_i and $a_i^\dagger$ who inherit commutation relations from 5.4,

$$\begin{aligned}[a_i,a_j^\dagger] &= (u_i,u_j) = \delta_{ij} \\ [a_i,a_j] &= [a_i^\dagger,a_j^\dagger] = 0\end{aligned}$$

- Use a_i to define the vacuum state of the field ($a_i|0\rangle = 0$) and the creator operator, $a_i^\dagger$, to build the Fock space

$$\left|n_{i_1}^1,n_{i_2}^2,\ldots,n_{i_k}^k\right\rangle = \frac{1}{\sqrt{n^1!n^2!\cdots n^3!}}(a_{i_1}^\dagger)^{n^1}(a_{i_2}^\dagger)^{n^2}\cdots(a_{i_k}^\dagger)^{n^k}|0\rangle$$

Note that this quantization procedure is equivalent for all inertial observers since a Poincaré transformation only relabel annihilation (creation) operators between themselves without mixing with the creation (annihilation) ones, making the vacuum state a Poincaré invariant.

Alice learns quantum fields

So, if Alice is an inertial observer in Minkowski spacetime, her Fock space is build from solutions of 5.2. And therefore a one mode state is

$$|1_\omega\rangle_M = a_{\omega,M}^\dagger|0\rangle_M \equiv u_\omega^M \propto \frac{1}{\sqrt{\omega_k}}e^{-i\omega_k t}$$

where M stand for Minkowskian modes. Note that in this part of the thesis we are labeling the modes with their frequency instead of their momentum, the relation is $\omega_k = \sqrt{k^2+m^2}$. However we will limit ourself to massless fields ($\omega_k = |k|$) in order to work with Bogoliubov coefficients of a simpler form.

The field expanded in term of Alice's modes take the form

$$\phi = \sum_i\left(a_{\omega_i,M}u_{\omega_i}^M + a_{\omega_i,M}^\dagger u_{\omega_i}^{M*}\right) \tag{5.5}$$

5.1.2 In curved Spacetime

In curved spacetime we generalize Klein-Gordon equation 5.1 by means of the covariant derivative[2] $\partial_\mu \to \nabla_\mu$.

$$\left(\nabla_\mu \nabla^\mu - m^2\right) \Phi = 0$$

and now orthonormal conditions 5.3 have to be defined using a Cauchy hypersurface Σ

$$(u_j, u_k) = -i \int_\Sigma d\Sigma n^\mu \left(u_j \partial_\mu u_k^* - u_k^* \partial_\mu u_j\right)$$

where $d\Sigma$ is the volume element and n^μ is a future oriented timelike unit vector orthogonal to Σ.

Non-stationary spacetime don't have *global* timelike Killing vector making it impossible to separate between positive and negative frequency solutions globally. Hence, there is no natural way of defining the vacuum and Fock space. Interpretation in term of particles become a fuss. Only when the spacetime possess asymptotically stationary region could that interpretation be recovered.

For stationary spacetime we have a timelike Killing vector field ξ^μ with which we can generalize what we did for Minkowski. Such that positive frequency modes $u_k(x,t)$ will satisfy

$$\xi^\mu \nabla_\mu u_j = -i\omega_j u_j$$

where $\omega_j > 0$. With this, the quantization procedure is totally analogous.

5.2 Accelerated observer: Introducing Rob

Up to now we have only considered inertial observers, Alice and Bob. Now we will also consider an observer with constant acceleration that will be named Rob. Since Alice and Rob are not related by a Poincaré transformation they will generally not have the same Fock space, and in particular don't agree on the vacuum state of the field. This is because they have different proper coordinates; Alice uses Minkowskian coordinates (t, x) while Rob uses Rindler coordinates (τ, ξ) both describing the flat Minkowski spacetime.

5.2.1 Rindler coordinates

The Rindler coordinates are related to the Minkowskian coordinate through

$$t = \xi \sinh(a\tau) \quad , \quad x = \xi \cosh(a\tau)$$

where each constant ξ correspond to the trajectory of an observer accelerated with acceleration a with -lightspeed in the infinite past and passing through $x = \xi$ at time $t = \tau = 0$. Rob is one of those observers, see figure 5.1. The

[2] For fields with spin degree of freedom covariant derivative is defined in a different way.

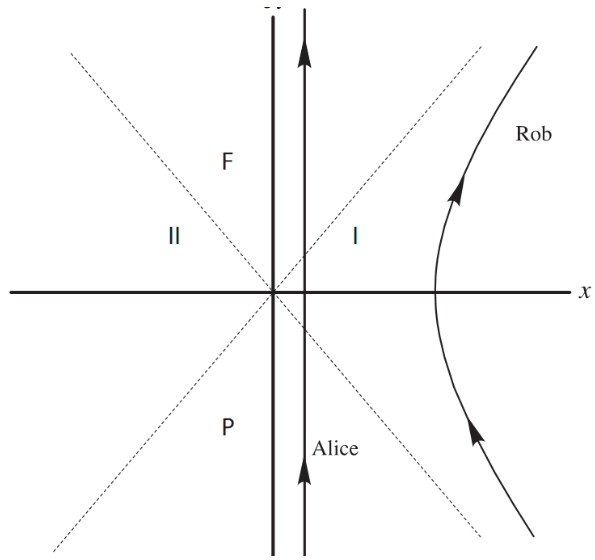

Figure 5.1: Alice and Rob's trajectories in the Minkowski spacetime.

coordinates (τ, ξ) take values in the $(-\infty, +\infty)$ but they don't cover the whole Minkowski spacetime. Three more sets of Rindler coordinates are needed to cover it all. The other set that interest us to complete the Cauchy surface needed for the quantization is the one associated to region II, i.e.

$$t = -\xi \sinh{(a\tau)} \quad , \quad x = -\xi \cosh{(a\tau)}$$

Both regions are globally causally disconnected. In fact, observers in both regions will perceive a horizon at proper distance a^{-1} in the opposite direction of acceleration.

5.2.2 Rob learns quantum fields

Since Rob need two causally disconnected region to complete the needed Cauchy surface, he will have to build two Fock spaces, one for each region. Thus having a vacuum and excited states for each region; single excitation are

$$\begin{aligned} |1_\omega\rangle_I &= a^\dagger_{\omega,I}\,|0\rangle_I \equiv u^I_\omega \propto \frac{1}{\sqrt{\omega}}e^{-i\omega\tau} \\ |1_\omega\rangle_{\mathrm{II}} &= a^\dagger_{\omega,\mathrm{II}}\,|0\rangle_{\mathrm{II}} \equiv u^{\mathrm{II}}_\omega \propto \frac{1}{\sqrt{\omega}}e^{i\omega\tau} \end{aligned}$$

the different sign comes from the relation between the two regions,i.e. they are spacetime reflection of each other. The field expanded in terms of these

Rindler modes is expressed:

$$\phi = \sum_i \left(a_{\omega_i, I} u^I_{\omega_i} + a^\dagger_{\omega_i, I} u^{I*}_{\omega_i} + a_{\omega_i, \mathrm{II}} u^{\mathrm{II}}_{\omega_i} + a^\dagger_{\omega_i, \mathrm{II}} u^{\mathrm{II}*}_{\omega_i} \right) \tag{5.6}$$

5.3 Bogoliubov transformation

Now since 5.5 and 5.6 is the same object in two different basis, we can after some algebra and using 5.3 relate both basis finding

$$u^M_{\omega_j} = \sum_i \left[\alpha^I_{ji} u^I_{\omega_i} + \beta^I_{ji} u^{I*}_{\omega_i} + \alpha^{\mathrm{II}}_{ji} u^{\mathrm{II}}_{\omega_i} + \beta^{\mathrm{II}}_{ji} u^{\mathrm{II}*}_{\omega_i} \right]$$

where we have defined the Bogoliubov coefficients in terms of the scalar products

$$\alpha^{I(\mathrm{II})}_{ij} = \left(u^M_{\omega_i}, u^{I(\mathrm{II})}_{\omega_j} \right) \quad , \quad \beta^{I(\mathrm{II})}_{ij} = -\left(u^M_{\omega_i}, u^{I(\mathrm{II})*}_{\omega_j} \right)$$

Now, to get the relation between ladder operators we compute $a_{\omega_i, M} = \left(\phi, u^M_{\omega_i}\right)$,

$$a_{\omega_i, M} = \sum_j \left[\alpha^{I*}_{ij} a_{\omega_j, I} - \beta^{I*}_{ij} a^\dagger_{\omega_j, I} + \alpha^{\mathrm{II}*}_{ij} a_{\omega_j, \mathrm{II}} - \beta^{\mathrm{II}*}_{ij} a^\dagger_{\omega_j, \mathrm{II}} \right]$$

analogously for $a^\dagger_{\omega_i, M}$. They are related by linear combination with ladder operator of Rindler space. Note that frequencies are mixed; i.e. a single frequency Minkowski mode correspond to a multichromatic mode when transformed into Rindler basis. In other words, the Bogoliubov coefficients, α_{ij} and β_{ij}, are not diagonal. However, we will be working in an approximation which indeed has diagonal coefficients and hence don't mix frequencies.

5.3.1 Single-mode approximation

The single mode approximation has been widely used in relativistic quantum information, in fact it wasn't until 2010 in [17] when the first work without this approximation was done. The assumption is that the transformation only involved one frequency, i.e. $\omega_i = \omega_j$. Such an assumption looks very arbitrary, but it was shown in [18] that it is equivalent to find an orthonormal basis $\{\psi^U_{\omega_i}, \psi'^U_{\omega_i}\}$ related to Minkowski modes

$$\psi^U_{\omega_j} = \sum_i C_{ij} u^M_{\omega_i} \quad , \quad \psi'^U_{\omega_j} = \sum_i C'_{ij} u^M_{\omega_i}$$

such that the Bogoliubov coefficient relating this basis and the Rindler basis are diagonal

$$\begin{aligned}
\alpha^{I}_{ij} &= \left(\psi^{U}_{\omega_j}, u^{I}_{\omega_i}\right) = \cosh r_{s,\omega_i}\delta_{ij} = \alpha'^{\text{II}}_{ij} \\
\beta^{I}_{ij} &= -\left(\psi^{U}_{\omega_j}, u^{I}_{\omega_i}\right) = 0 = \beta'^{\text{II}}_{ij} \\
\alpha^{\text{II}}_{ij} &= \left(\psi^{U}_{\omega_j}, u^{\text{II}}_{\omega_i}\right) = 0 = \alpha'^{I}_{ij} \\
\beta^{\text{II}}_{ij} &= -\left(\psi^{U}_{\omega_j}, u^{\text{II}*}_{\omega_i}\right) = \sinh r_{s,\omega_i}\delta_{ij} = \beta'^{I}_{ij}
\end{aligned}$$

where r_s (s stands for scalar) is defined as

$$\tanh r_{s,\omega_i} = \exp\left(-\frac{\pi\omega_i}{a}\right)$$

Those modes $\{\psi^{U}_{\omega_i}, \psi'^{U}_{\omega_i}\}$ are a specific choice of the so-called Unruh modes. This choice give us the following relation between ladder operators

$$a_{\omega_i,U} = \cosh r_{s,\omega_i} a_{\omega_i,I} - \sinh r_{s,\omega_i} a^{\dagger}_{\omega_i,\text{II}} \tag{5.7}$$

and the same for $a'_{\omega_i,U}$ exchanging the label I with II.

5.3.2 Results for spin $\frac{1}{2}$ fields

Everything we have shown can be done analogously for the massless Dirac field. Positive (particle) and negative (antiparticle) solution of the Dirac equation in Minkowskian coordinates, $u^{+}_{\omega,\sigma,M}$ and $u^{-}_{\omega,\sigma,M}$, are used to expand the field

$$\phi = \sum_{\sigma}\int d^3k\left(c_{\omega,\sigma,M}u^{+}_{\omega,\sigma,M} + d^{\dagger}_{\omega,s,M}u^{-}_{\omega,\sigma,M}\right)$$

where σ represent the spin degree of freedom and takes the values $\sigma = \{\uparrow,\downarrow\}$. The Fock space is constructed from a pair (one for each, particles and antiparticles) of creation and annihilation operators, $c^{(\dagger)}_{\omega,\sigma,\Sigma}$, $d^{(\dagger)}_{\omega,s,\Sigma}$, which instead of commutation relation satisfy usual anticommutation relation for the ladder operators of fermionic fields

$$\{c_{\omega,\sigma,\Sigma}, c^{\dagger}_{\omega',\sigma',\Sigma'}\} = \delta_{\omega\omega'}\delta_{\sigma\sigma'}\delta_{\Sigma\Sigma'} \quad , \quad \{d_{\omega,\sigma,\Sigma}, d^{\dagger}_{\omega',\sigma',\Sigma'}\} = \delta_{\omega\omega'}\delta_{\sigma\sigma'}\delta_{\Sigma\Sigma'}$$

where Σ stands for M,U,I or II. Then mode states are

$$\begin{aligned}
c^{\dagger}_{\omega,\sigma,\Sigma}\left|0\right\rangle_{\Sigma} &= \left|\sigma_{\omega}\right\rangle \\
c^{\dagger}_{\omega,\sigma,\Sigma}c^{\dagger}_{\omega,\sigma',\Sigma}\left|0\right\rangle_{\Sigma} &= \left|\sigma\sigma'_{\omega}\right\rangle\delta_{\sigma,-\sigma'}
\end{aligned}$$

therefore only allowing the following states for each mode $\{\left|0_{\omega}\right\rangle^{\pm}_{\Sigma}, \left|\uparrow_{\omega}\right\rangle^{\pm}_{\Sigma}, \left|\downarrow_{\omega}\right\rangle^{\pm}_{\Sigma}, \left|\uparrow\downarrow_{\omega}\right\rangle^{\pm}_{\Sigma}\}$.

Again we can work with the Minkowskian Unruh modes which have the next diagonal relation with the Rindler modes

$$\begin{aligned} c_{\omega_i,\sigma,U} &= \cos r_{d,i} c_{\omega_i,\sigma,I} - \sin r_{d,i} d^{\dagger}_{\omega_i,-\sigma,\text{II}} \\ d^{\dagger}_{\omega_i,\sigma,U} &= \cos r_{d,i} d^{\dagger}_{\omega_i,\sigma,\text{II}} + \sin r_{d,i} c_{\omega_i,-\sigma,I} \end{aligned} \tag{5.8}$$

the same for $c'_{\omega_i,\sigma,U}$ and $d'^{\dagger}_{\omega_i,\sigma,U}$ interchanging labels I and II. Now r_d (Dirac) is given by

$$\tan r_{d,i} = \exp\left(\frac{-\pi\omega_i}{a}\right)$$

Notice how the relation (spacetime reflection of each other) between region I and region II, makes c_U annihilate a particle in I but create an antiparticle in II.

Now that Alice and Rob have the necessary tools to write single-particle and two-particle states and their relation, they are ready to study entanglement for some settings to be discussed in the next chapters.

Chapter 6

Entanglement and non-Inertial observers

This is the main chapter of this part. Here we study how is entanglement seen by different non-inertial (accelerated) observers. For it we consider different settings and observers combinations and fields.

From relations 5.7 we can easily compute the vacuum and first excited state of Minkowski in terms of Rindler modes

$$|0_i\rangle_U = \frac{1}{\cosh r_i}\sum_{n=0}^{\infty}\tanh^n r_i\,|n_i\rangle_I\,|n_i\rangle_{\text{II}} \tag{6.1}$$

$$|1_i\rangle_U = \frac{1}{\cosh^2 r_i}\sum_{n=0}^{\infty}\tanh^n r_i\sqrt{n+1}\,|(n+1)_i\rangle_I\,|(n+1)_i\rangle_{\text{II}} \tag{6.2}$$

analogously with 5.8 we get for the spin-$\frac{1}{2}$ field

$$\begin{aligned}|0_i\rangle_U &= \cos^2 r_i\,|0_i\rangle_I\,|0_i\rangle_{\text{II}} + \sin r_i\cos r_i\,|{\uparrow_i}\rangle_I\,|{\downarrow_i}\rangle_{\text{II}} \\ &\quad + \sin r_i\cos r_i\,|{\downarrow_i}\rangle_I\,|{\uparrow_i}\rangle_{\text{II}} + \sin^2 r_i\,|p_i\rangle_I\,|p_i\rangle_{\text{II}}\end{aligned} \tag{6.3}$$

$$|{\uparrow_i}\rangle_U = \cos r_i\,|{\uparrow_i}\rangle_I\,|0_i\rangle_{\text{II}} + \sin r_i\,|p_i\rangle_I\,|{\uparrow_i}\rangle_{\text{II}} \tag{6.4}$$

$$|{\downarrow_i}\rangle_U = \cos r_i\,|{\downarrow_i}\rangle_I\,|0_i\rangle_{\text{II}} - \sin r_i\,|p_i\rangle_I\,|{\downarrow_i}\rangle_{\text{II}} \tag{6.5}$$

where $|p_i\rangle \equiv |{\uparrow_i\downarrow_i}\rangle$, p stands for pair.

It is important to remember here that since we are dealing with anticommuting operators the order of the operators, and hence, the order of the particles in the notation are important, for example $|p_i\rangle \equiv -|{\downarrow_i\uparrow_i}\rangle$. In fact, negativity depends on the ordering convention, even of the out-traced modes. This is discussed in [22]. They conclude that there isn't really a problem because the field is always measured by means of a physical detector[1]. The ordering has to be chosen in concordance with the interaction between the field and the detector;

[1] A physical system that interacts with the field and then is measured.

such that all anticommutation signs disappear from the computation. Although there is no proposed physical detector for fermions, we know a priori that a detector in region I will need to have all the region II operators rightmost.

In all this part of the work we consider the acceleration to be in the same direction of the spin in order to avoid Wigner rotation in contrast to what we did in part I of this work. In this way we study both effect separately. Trajectories in curved spacetime also cause Wigner rotation, this can be studied in the tetrad formalism as done for example in [24], [25].

6.1 Alice prepares the states

We will consider a set of states prepared by Alice, many of them already considered in the literature and for which we reproduce the result but also some new cases.

First of all and as an example we consider a setting with very simple transformation rules

$$|\psi\rangle = \cos\xi \, |\uparrow_1\rangle_U \, |\downarrow_2\rangle_U + \sin\xi \, |\downarrow_1\rangle_U \, |\uparrow_2\rangle_U \tag{6.6}$$

this is a two particle state similar to 4.6 with $\chi = 0$. We say similar because here the two particle are undistinguishable and the sates are antisymmetric by construction.

Single-particle states Next example will be a single particle state like in section 4.1

$$|\psi\rangle = A \, |\uparrow_1\rangle_U \, |0_2\rangle_U + B \, |\downarrow_1\rangle_U \, |0_2\rangle_U + C \, |0_1\rangle_U \, |\uparrow_2\rangle_U + D \, |0_1\rangle_U \, |\downarrow_2\rangle_U$$

with

1. $A = \cos\eta$ and $D = \sin\eta$, i.e. equivalent to 4.2

$$|\psi\rangle = \cos\eta \, |\uparrow_1\rangle_U \, |0_2\rangle_U + \sin\eta \, |0_1\rangle_U \, |\downarrow_2\rangle_U$$

2. $A = \cos\chi\cos\xi$, $B = \cos\chi\sin\xi$, $C = \sin\chi\cos\xi$, $D = \sin\chi\sin\xi$ which is analog to 4.1.

It is important to note that those states won't be seen as single-particle by Rob. He will see more particles than the prepared by Alice. This is a general feature of QFT in curved spacetime and the reason for the Unruh and Hawking radiation.

Although the state considered are in analogy with those of section 4.1, we are not measuring the same entanglement. Here we measure the entanglement between the two modes. Consider for example the entanglement measured by Alice for setting 2: Negativity gives

$$\mathcal{N} = \frac{1}{2} \, |\sin 2\chi|$$

which is surprisingly independent of ξ.

In order to show one of the striking differences between bosonic and fermionic fields for non-inertial observers Alice prepares the next setting. It consist of two independent fields: one bosonic, the other a fermionic, such that the mode "1" of the scalar field is entangled with mode "2" of the Dirac field. The state looks like

$$|\psi\rangle = \cos\eta \left|0_1^b\right\rangle_U \left|\uparrow_2^f\right\rangle_U + \sin\eta \left|1_1^b\right\rangle_U \left|\downarrow_2^f\right\rangle_U \tag{6.7}$$

6.2 Rob studies entanglement

Although most of the previous works consider our previously named "Alice and Bob work together" scenario, i.e. partition corresponding to Alice don't transform, here we will consider that both modes transform. But since both modes transform with different parameters (r_1, r_2) we can easily recover previous result by setting $r_1 = 0$ ($a_1 = 0$ or $\omega_1 = \infty$).

6.2.1 First example (6.6)

Even if it's very long we will explicitly write down the computation with all its steps for this case, which is the simpler, omitting the details for the following settings. Now we start with

$$|\psi\rangle = \cos\xi \left|\uparrow_1\right\rangle_U \left|\downarrow_2\right\rangle_U + \sin\xi \left|\downarrow_1\right\rangle_U \left|\uparrow_2\right\rangle_U$$

writing in terms of Rindler modes

$$\begin{aligned}
\left|\uparrow_1\right\rangle_U \left|\downarrow_2\right\rangle_U &= \left(\cos r_1 \left|\uparrow_1\right\rangle_I \left|0_1\right\rangle_{\mathrm{II}} + \sin r_1 \left|p_1\right\rangle_I \left|\uparrow_1\right\rangle_{\mathrm{II}}\right)\left(\cos r_2 \left|\downarrow_2\right\rangle_I \left|0_2\right\rangle_{\mathrm{II}} - \sin r_2 \left|p_2\right\rangle_I \left|\downarrow_2\right\rangle_{\mathrm{II}}\right) \\
&= \cos r_2 \cos r_1 \left|\uparrow_1\right\rangle_I \left|0_1\right\rangle_{\mathrm{II}} \left|\downarrow_2\right\rangle_I \left|0_2\right\rangle_{\mathrm{II}} - \sin r_2 \cos r_1 \left|\uparrow_1\right\rangle_I \left|0_1\right\rangle_{\mathrm{II}} \left|p_2\right\rangle_I \left|\downarrow_2\right\rangle_{\mathrm{II}} \\
&\quad + \cos r_2 \sin r_1 \left|p_1\right\rangle_I \left|\uparrow_1\right\rangle_{\mathrm{II}} \left|\downarrow_2\right\rangle_I \left|0_2\right\rangle_{\mathrm{II}} - \sin r_2 \sin r_1 \left|p_1\right\rangle_I \left|\uparrow_1\right\rangle_{\mathrm{II}} \left|p_2\right\rangle_I \left|\downarrow_2\right\rangle_{\mathrm{II}}
\end{aligned}$$

ordering all region II operators rightmost and simplifying notation like $\left|\uparrow_1\right\rangle_I \left|\downarrow_2\right\rangle_I \equiv \left|\uparrow_1\downarrow_2\right\rangle_I$

$$\begin{aligned}
\left|\uparrow_1\right\rangle_U \left|\downarrow_2\right\rangle_U &= \cos r_2 \cos r_1 \left|\uparrow_1\downarrow_2\right\rangle_I \left|0_1 0_2\right\rangle_{\mathrm{II}} - \sin r_2 \cos r_1 \left|\uparrow_1\ p_2\right\rangle_I \left|0_1\ \downarrow_2\right\rangle_{\mathrm{II}} \\
&\quad - \cos r_2 \sin r_1 \left|p_1\ \downarrow_2\right\rangle_I \left|\uparrow_1\ 0_2\right\rangle_{\mathrm{II}} - \sin r_2 \sin r_1 \left|p_1 p_2\right\rangle_I \left|\uparrow_1\downarrow_2\right\rangle_{\mathrm{II}}
\end{aligned}$$

analogously (exchanging $\uparrow$ and $\downarrow$ and transferring the minus sign from $\sin r_2$ to $\sin r_1$)

$$\begin{aligned}
\left|\downarrow_1\right\rangle_U \left|\uparrow_2\right\rangle_U &= \cos r_2 \cos r_1 \left|\downarrow_1\uparrow_2\right\rangle_I \left|0_1 0_2\right\rangle_{\mathrm{II}} + \sin r_2 \cos r_1 \left|\downarrow_1\ p_2\right\rangle_I \left|0_1\ \uparrow_2\right\rangle_{\mathrm{II}} \\
&\quad + \cos r_2 \sin r_1 \left|p_1\ \uparrow_2\right\rangle_I \left|\downarrow_1\ 0_2\right\rangle_{\mathrm{II}} - \sin r_2 \sin r_1 \left|p_1 p_2\right\rangle_I \left|\downarrow_1\uparrow_2\right\rangle_{\mathrm{II}}
\end{aligned}$$

then the state in Rindler modes is

$$\begin{aligned}
|\psi\rangle &= A \left|\uparrow_1\downarrow_2\right\rangle_I \left|0_1 0_2\right\rangle_{\mathrm{II}} + B \left|\uparrow_1\ p_2\right\rangle_I \left|0_1\ \downarrow_2\right\rangle_{\mathrm{II}} + C \left|p_1\ \downarrow_2\right\rangle_I \left|\uparrow_1\ 0_2\right\rangle_{\mathrm{II}} + D \left|p_1 p_2\right\rangle_I \left|\uparrow_1\downarrow_2\right\rangle_{\mathrm{II}} \\
&\quad + E \left|\downarrow_1\uparrow_2\right\rangle_I \left|0_1 0_2\right\rangle_{\mathrm{II}} + F \left|\downarrow_1\ p_2\right\rangle_I \left|0_1\ \uparrow_2\right\rangle_{\mathrm{II}} + G \left|p_1\ \uparrow_2\right\rangle_I \left|\downarrow_1\ 0_2\right\rangle_{\mathrm{II}} + H \left|p_1 p_2\right\rangle_I \left|\downarrow_1\uparrow_2\right\rangle_{\mathrm{II}}
\end{aligned}$$

with

$$\begin{array}{ll} A = \cos\xi\cos r_2\cos r_1 & E = \sin\xi\cos r_2\cos r_1 \\ B = -\cos\xi\sin r_2\cos r_1 & F = -\sin\xi\sin r_2\cos r_1 \\ C = -\cos\xi\cos r_2\sin r_1 & G = -\sin\xi\cos r_2\sin r_1 \\ D = -\cos\xi\sin r_2\sin r_1 & H = -\sin\xi\sin r_2\sin r_1 \end{array}$$

The density matrix looks like 4.5 but with the corresponding basis. Tracing over region II we get the reduced density matrix that Rob will observe for 6.6

$$\rho_I = \begin{pmatrix} A^2 & 0 & AE & 0 & 0 & 0 & 0 \\ 0 & B^2 & 0 & 0 & 0 & 0 & 0 \\ AE & 0 & E^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D^2+H^2 \end{pmatrix} \begin{array}{l} |\uparrow_1\downarrow_2\rangle_I \\ |\uparrow_1 p_2\rangle_I \\ |\downarrow_1\uparrow_2\rangle_I \\ |\downarrow_1 p_2\rangle_I \\ |p_1\uparrow_2\rangle_I \\ |p_1\downarrow_2\rangle_I \\ |p_1 p_2\rangle_I \end{array}$$

The partial transpose of this matrix is

$$\rho_I^{pT} = \begin{pmatrix} 0 & AE & 0 \\ AE & 0 & 0 \\ 0 & 0 & \rho_I(diag) \end{pmatrix}$$

where $\rho_I(diag)$ is the 7x7 ρ_I matrix without the non-diagonal elements. Therefore the only negative eigenvalue gives the negativity

$$\mathcal{N}_{spin} = AE = \frac{1}{2}\left|\sin 2\xi\cos^2 r_2\cos^2 r_1\right|$$

This result is plotted in figure 6.1. The preparation parameter ξ only set the maximum of entanglement;for all its values, entanglement will be degraded in the same way. As expected, due to the symmetry of the state under exchange of the two modes, it is symmetric in r_1 and in r_2. It is interesting to see that even in the limit of both infinite acceleration ($r_1, r_2 \longrightarrow \pi/4$) entanglement don't vanishes. This is a property of fermionic fields and seems to be caused by fermi statistics. If we set $r_1 = 0$ and $\xi = \pi/4$ we obtain the same result as in [15].

6.2.2 Single particle states

Setting 1

Now we do the same with the state

$$|\psi\rangle = \cos\eta\,|\uparrow_1\rangle_U|0_2\rangle_U + \sin\eta\,|0_1\rangle_U|\downarrow_2\rangle_U$$

The algebra is much longer because we have to deal with all the elements of the basis of the modes I and II, that we present here for completeness

$$\begin{array}{cccccccc} |0_1 0_2\rangle_I & |0_1\uparrow_2\rangle_I & |0_1\downarrow_2\rangle_I & |0_1 p_2\rangle_I & |\uparrow_1 0_2\rangle_I & |\uparrow_1\uparrow_2\rangle_I & |\uparrow_1\downarrow_2\rangle_I & |\uparrow_1 p_2\rangle_I \\ |\downarrow_1 0_2\rangle_I & |\downarrow_1\uparrow_2\rangle_I & |\downarrow_1\downarrow_2\rangle_I & |\downarrow_1 p_2\rangle_I & |p_1 0_2\rangle_I & |p_1\uparrow_2\rangle_I & |p_1\downarrow_2\rangle_I & |p_1 p_2\rangle_I \end{array}$$

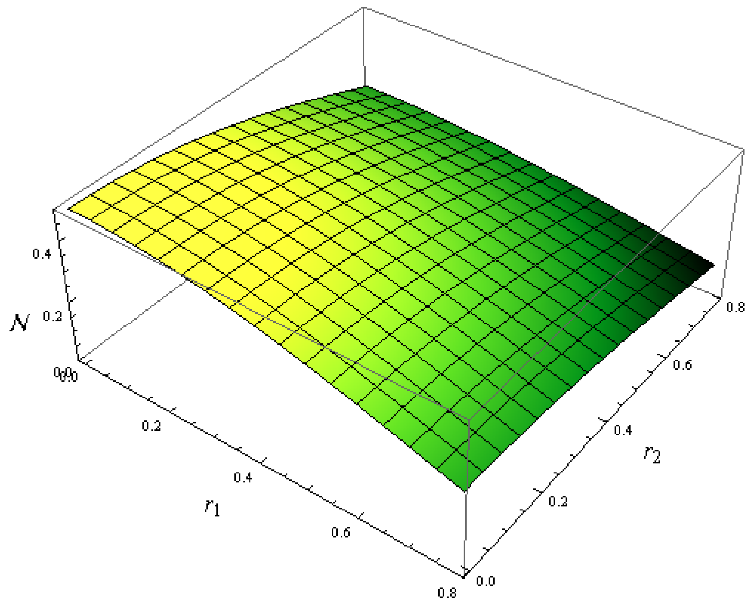

Figure 6.1: Negativity for the state 6.6 with $\chi = \pi/4$ as a function of r_1 and r_2.

that makes us work with general 32x32 matrices[2] and once we trace out region II we end up with 16x16 matrices. Note that we work with up to 4 particle states.

The result for this setting with $\chi = \pi/4$ looks like the previous example shown in figure 6.1. The $r_1 \leftrightarrow r_2$ symmetry is particular for this choice of parameter. Now the preparation parameter, η, plays an active role in the form of the negativity, breaking the symmetry between r_1 and r_2. If we choose η, such that there is more weight on one of the basis state, e.g. $|\uparrow_1\rangle_U |0_2\rangle_U$, i.e. $\cos\eta > \sin\eta$, the negativity will decrease more and faster with the parameter associated with the mode in the vacuum, e.g. r_2; see figure 6.2. So, we can make the relative decrease of entanglement (difference in color) as large as we want (for a fixed acceleration) by taking the state closer and closer to separable, see figure 6.3. Therefore the better we can prepare and control quasi-separable states the less acceleration we need to observe an appreciable difference in entanglement. Something very similar is discussed in [23], but with a really different setting[3]; in fact there the negativity increase! The experimental feasibility presented there is completely analogous for both cases (including values of acceleration needed).

As a last comment we would get exactly the same considering the state

$$|\psi\rangle = \cos\chi\, |\uparrow_1\rangle_U\, |0_2\rangle_U + \sin\chi\, |0_1\rangle_U\, |\uparrow_2\rangle_U \tag{6.8}$$

[2] Most of them with many zeros. That makes the computation possible.

[3] A different choice of Unruh modes (outside the single mode approximation)

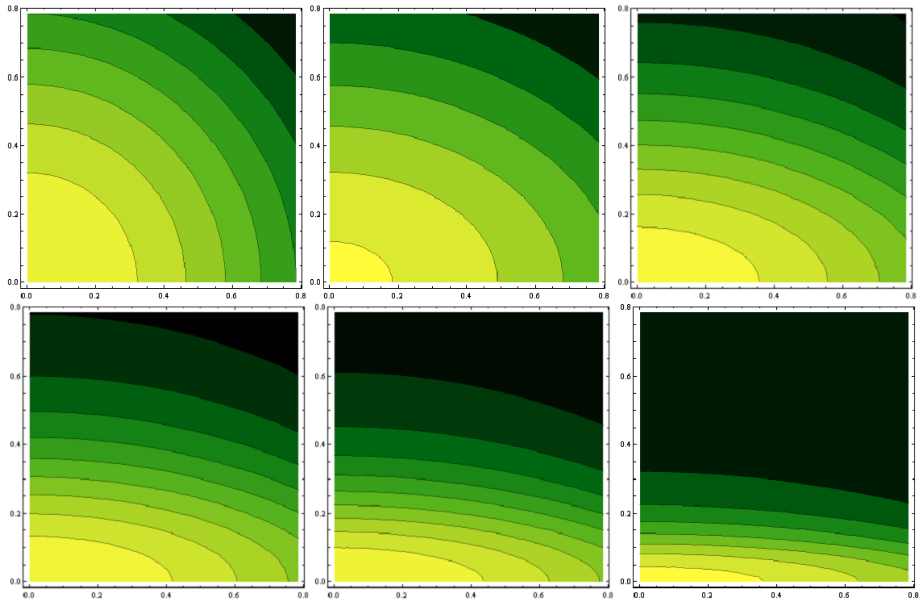

Figure 6.2: Density plot of negativity for various values of $\eta = [\pi/4, 0.4, 0.2, 0.1, 0.05, 0.015]$. The first one is like figure 6.1. r_2 and r_1 are the vertical and horizontal axis respectively.

which is included in the next study with $\xi = 0$.

Setting 2

Following the same process, which for this setting is very long and tedious, we get a disappointing resembling behavior to the previous setting, where χ plays the role of η. The principal difference is the role of the parameter, ξ, which only becomes noticeable for large values of acceleration, but still very small.

6.2.3 Boson-Fermion system

Same work again: rewriting the state (6.7) in terms of Rindler modes using (6.1), (6.2),(6.4) and (6.5); tracing out the modes in region II; partial transpose the reduced density matrix; and computing the negativity. We only have to be careful with the reordering of the modes considering a graded algebra, having anticommutation relation only between two fermionic operators.

We find the result shown in figure 6.4. Setting $r_1 = 0$ (and $\eta = \pi/4$) we recover the result found in [15], i.e. entanglement survives in the infinite acceleration limit. Setting $r_2 = 0$ (and $\eta = \pi/4$) we find the same as [7]; entanglement tends to zero for infinite acceleration, see figure 6.5.

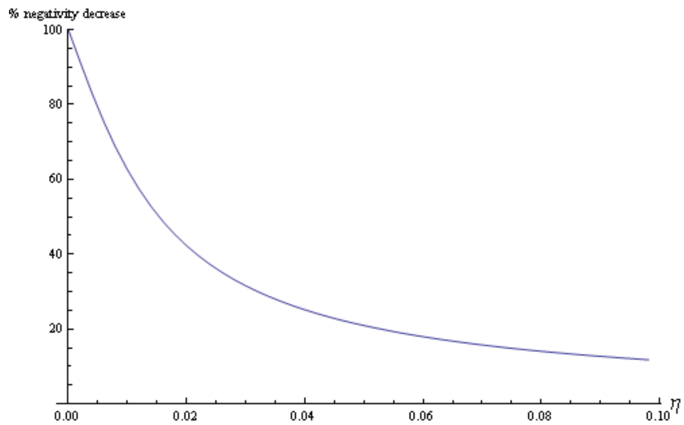

Figure 6.3: Percent in negativity decrease for a fixed acceleration, $r_2 = 0.15$ ($r_1 = 0$), for different preparation of the state close to a separable state

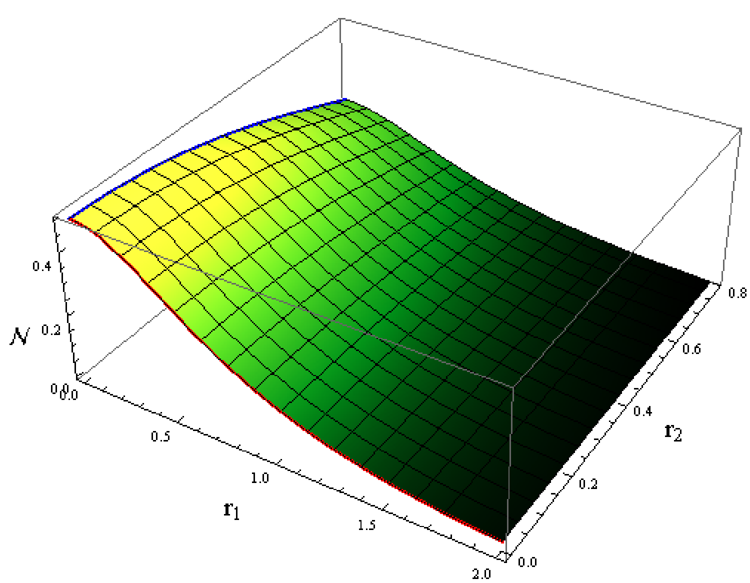

Figure 6.4: Negativity for the boson-fermion system with $\eta = \pi/4$.

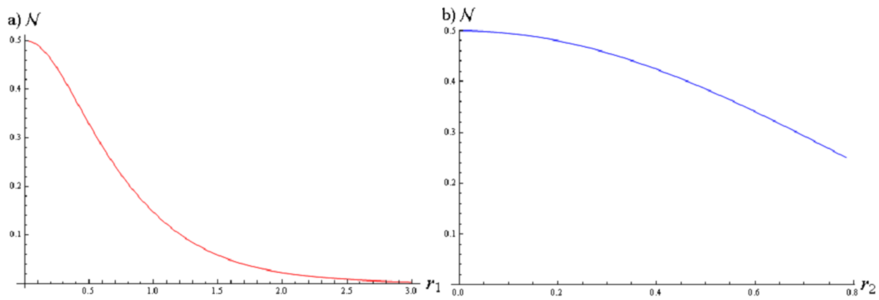

Figure 6.5: Bosonic(a) and fermionic (b) behavior of entanglement.

At the beginning this difference was thought to be because of the disparity in the dimension of Hilbert spaces. But results of several setting, bosonic and fermionic, with variety of Hilbert space dimensions, points that this is solely due to statistics.

6.3 Spin-Momentum entanglement

Until now we have computed the entanglement between the two modes. We will now consider a different bipartition, the spin-momentum bipartition, which to knowledge of the author haven't been considered in previous literature. We expect to find a change in this entanglement because as in part I, the transformation depend on the momentum (frequency of the mode) of the particle and we don't expect that tracing over region II counteract the expected change in entanglement.

The basis we are using don't have this two degrees of freedom separated. In order to compute the entanglement of this bipartition we need a basis in which we have in one side the total momentum and in the other the total spin. We are speaking of total momentum and spin because although we start with a single particle state we end up with more particles. From the standard addition of angular momentum the change of basis to momentum (occupation number) - total spin is given by

$$\begin{aligned}
&|00\rangle = |00\rangle|S\rangle\,, \quad |0\uparrow\rangle = |01\rangle|D_+\rangle\,, \quad |0\downarrow\rangle = |10\rangle|D_-\rangle\,, \quad |0p\rangle = |02\rangle|S\rangle \\
&|p0\rangle = |20\rangle|S\rangle\,, \quad |p\uparrow\rangle = |21\rangle|D_+\rangle\,, \quad |p\downarrow\rangle = |21\rangle|D_-\rangle\,, \quad |pp\rangle = |22\rangle|S\rangle \\
&|\uparrow 0\rangle = |10\rangle|D_+\rangle\,, \quad |\uparrow\uparrow\rangle = |11\rangle|T_+\rangle\,, \quad |\uparrow p\rangle = |12\rangle|D_+\rangle \\
&|\downarrow 0\rangle = |10\rangle|D_-\rangle\,, \quad |\downarrow\downarrow\rangle = |11\rangle|T_-\rangle\,, \quad |\downarrow p\rangle = |12\rangle|D_-\rangle \\
&|\uparrow\downarrow\rangle = \tfrac{1}{\sqrt{2}}\left[|11\rangle|T_0\rangle + |11\rangle|S\rangle\right], \quad |\downarrow\uparrow\rangle = \tfrac{1}{\sqrt{2}}\left[|11\rangle|T_0\rangle - |11\rangle|S\rangle\right]
\end{aligned}$$

where $|T_{\pm,0}\rangle$ stands for the three states of the triplet, $|D_\pm\rangle$ for the two states of the doublet, and $|S\rangle$ for the singlet. Now we are ready to change basis to

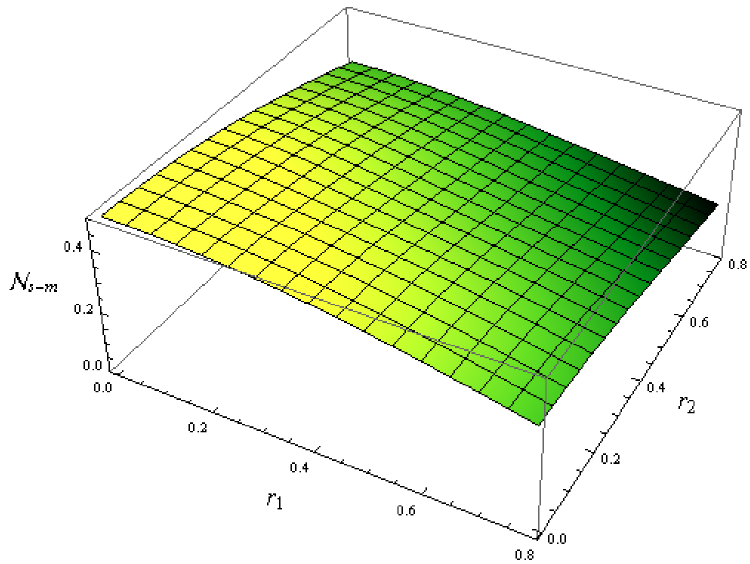

Figure 6.6: Spin-momentum negativity for setting 1 with $\eta = \pi/4$.

the reduced density matrix of region I, and then compute the negativity for the spin-momentum bipartition seen by Rob.

The state considered in the first example is initially not entangled in spin-momentum and remains unentangled for Rob. This is totally analogous to what happens in part I. The symmetry between 1 and 2, and up and down, prevent the entanglement to be changed.

Setting 1

In this case the state is prepared with spin-momentum entanglement and we observe it's degradation due to the acceleration, see figure 6.6. The parameter η set the initial maximum point for entanglement, but depending of the choice of η we also find a light tilt, producing an asymmetry between 1 and 2. This isn't really a surprise because the transformation is different for up and down.

Setting 2 Alt

Instead of considering the general setting 2 we take for simplicity the particular case 6.8. In contrast to what happens for the mode-mode entanglement, we now not find the same result as setting 1. In fact, the behavior is the complete opposite. The state being prepared separable in spin-momentum has an increase in his entanglement with the acceleration, see figure 6.7.

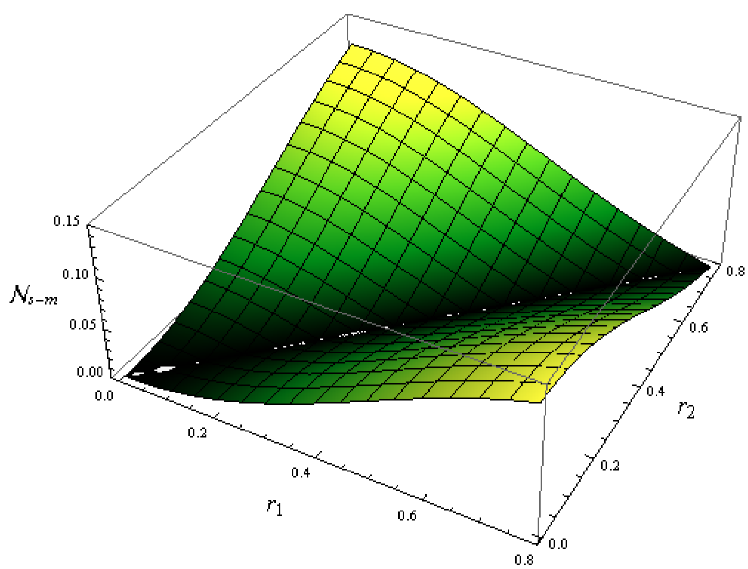

Figure 6.7: Negativity for the spin-momentum bipartition of state 6.8 with $\chi = \pi/4$.

Chapter 7

Conclusions and Future work

7.1 Part I

In this section we list the main conclusions already presented in subsections 4.1.4 and 4.2.4. We then make some comments and give proposals for future work.

Conclusions

In previous works:

- Entanglement between degrees of freedom of a single particle is not Lorentz invariant.
- Alice and Bob always agree in the entanglement between particles but not for other bipartitions.

Additional conclusions in our work:

- Spin-momentum entanglement is more rapidly increased than decreased and this effect seems to be exacerbated with more particles.
- There are whole regions of the Hilbert space which have invariant entanglement for the spin-spin bipartition.
- Mutual information doesn't always behave in concordance with entanglement.
- "Alice and Bob work together" makes the spin-spin and momentum-momentum entanglement analogous.

Comments and further work

Although we are presenting non-invariant quantitites, none of the results are in contradiction with the principle of relativity since there is no possible argument to fix a prefered reference frame. If we give a measurement device to both Alice and Bob, they will in principle agree about the outcome of the measurement. However, in order for this to happen, the device must also be described relativisticaly, i.e. Bob will see Alice's device Lorentz transfomed. The transformation of quantum measurement devices is a problem that deserves further investigation. In terms of operators there has been a long discussion about the form of a relativistic spin operator [19][20], and more recently in [21] where an operator is defined by studying the transformation properties of a Stern-Gerlach apparatus.

We have seen that entanglement for bipartitions other than for the particle-particle case is in general not Lorentz invariant. In fact, many other physical quantities are not Lorentz invariant, such as energy or momentum. These are part of the four-momentum $p^\mu = \left(p^0, p^i\right)$ which transform as a Lorentz vector; it is $p^\mu p_\mu = m^2$ which is invariant. In our context, the entanglement between particles would play the role of the invariant quantity (m). For example, in section 4.2.1 the particle-particle entanglement is the sum of the entanglements of spin-spin, momentum-momentum, and spin-momentum bipartitions. These similarities lead us to raise a natural question: are the entanglements of distinct bipartitions part of a larger object that transform under a certain representation of the Lorentz group? The main difficulty when trying to answer this question is the lack of a well-defined entanglement measure for mixed states. In fact, trying to answer this question could give some hints into that problem and lead to a new or refined measure of entanglement for mixed states. If we want to go deeper into this problem, our work, or similar, would need to be repeated with other entanglement measures. One should look for a relationship between the particle entanglements and the entanglements of all other bipartitions; the relationship could be a simple sum, or most likely a generalized sum.

Extentions to this work This work can be extended in diverse ways. The first and most obvious is to consider states that we haven't yet considered, e.g. prepared states with spin-momentum and spin-spin (or momentum-momentum) entanglement for which we expect theorem 1 to apply. In particular it would be interesting to study the whole range of possible states. By doing this, we could create a map of the Hilbert space telling us which regions have invariant entanglement for each bipartition.

Another option to extend this work would be to study not only the correlations due to entanglement but all the correlations, classical cases and quantum discord. As mentioned before, this should be performed especially for the spin-spin bipartition of setting 1.1, in order to understand the discrepancy in behavior between entanglement and mutual information.

One more option is to consider different particles, i.e. particles with different spins and likewise for massless particles.

Finally, more particles could be added to the setting. Especially to check whether the difference between increasing and decreasing rate of the spin-momentum entanglement is further emphasized. If this is the case, experimental verification of this effect could be much easier to perform.

7.2 Part II

Conclusions and further work

In this work we have considered more general states, using preparation parameters, but in our result they don't seem to play an important role. Actually, the behavior is driven by the initial entanglement almost independently of the form of the state. However, as well as in previous literature, we computed the entanglement between modes. This is the reason for our last computation, which, this time, indeed shows an important dependence with the detailed preparation of the state. This opens the door to many further computation of entanglement between different degrees of freedom and it's just a matter of time before we see numerous publications of this.

Considering two particle states is the natural extension to this work, if we want to compare the results with Part I we should consider undistinguishable particles. This is done by considering two fields as in our boson-fermion system.

We believe that the mode-mode entanglement plays the role of particle-particle entanglement in Part I of this work, however is no longer invariant due to the presence of the gravitational field[1]. Here it is worth noting that we have been working without any kind of interaction with gravity, this should be replicated in further works, for example for minimal coupling.

The different behavior[2] of entanglement for bosons and fermions leads us to wonder what would happen in more elaborate systems where that distinction isn't so clear. For example the case of anyons, or even in theories like supersymmetry where fermions and bosons transform into each other.

In conclusion, this is a new field which we believe deserves much more research, especially in the areas highlighted above.

[1] Although we are, in reality, working in flat spacetime, the use of Rindler coordinates in this context is equivalent to having a gravitational field.

[2] It was recently shown in [26][27] and discussed in [28] that fermionic entanglement can also vanish when we start with a mixed state.

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