Topology change in simplicial quantum gravity

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Abstract

This MSc thesis is a study of matrix models for two-dimensional quantum gravity and a higher-dimensional analogue thereof, group field theory. These are zero-dimensional field theories whose Feynman diagrammatic expansion yields a sum over topologies for a path-integral setting of quantum gravity. Care has been taken to provide detailed calculations and to point out subtleties and open issues.

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Chapter 1

Introduction

In this thesis, I will attempt to illustrate a particular viewpoint on the problem of quantizing gravity by using simplified toy models, namely gravity in two and three dimensions. The following brief introduction serves to argue for the necessity of a theory of quantum gravity and to motivate the particular approach presented in this work.

1.1 Why quantum gravity?

On a variety on theoretical and conceptual fronts, indications are that our understanding of both gravitation and quantum theory is left fundamentally incomplete if they remain two disparate theories in disparate mathematical frameworks. Despite the inability of current particle accelerators to access the relevant energy scales directly, a scheme for treating quantum effects and gravity in a unified manner is consequently desirable:

- Both quantum field theory and general relativity on their own are plagued by ultraviolet divergences and curvature singularities, respectively, both of which are sensitive to assumptions on short-distance physics. The fate of these mathematical pathologies is up to now unclear, much like the infinities that arose in early attempts to understand blackbody radiation before the advent of quantum theory in the early twentieth century.

- Black hole singularities seem to imply a loss of unitarity in the time evolution of quantum theory [Hawking, 1976]. This lead to the black hole information paradox, a resolution of which necessitates a theory of quantum gravity. If singularities are affected in such a framework, there similarly might be implications for cosmological singularities and the origin of the universe.

- A more mathematical argument has been given in [Thiemann, 2001]. The field equations of Einstein's theory state that

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where κ is a constant, $G_{\mu\nu}$ is a combination of the Riemann tensor and the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor for the matter distribution in spacetime. In quantum field theory however, the latter is an operator $\hat{T}_{\mu\nu}$ acting on a Hilbert space, which can not be equated to a classical field such as $G_{\mu\nu}$ in a straightforward way. A possibility is to replace $\hat{T}_{\mu\nu}$ with its expectation value $\langle\hat{T}_{\mu\nu}\rangle$. The computation of its value in field theory requires specifying a background metric g, while the subsequent solution of (1.1) will generally yield a metric $g' \neq g$. One might backfeed this solution to recompute $\langle\hat{T}_{\mu\nu}\rangle$. This iterative procedure however need not converge in general. A possible conclusion is that consistency can only be achieved by promoting g itself to an operator.

1.2 How quantum gravity?

Early attempts at quantizing gravity included the perturbative procedure that yielded the countless successes of quantum field theory. However, the perturbative nonrenormalizability as found in [Goroff and Sagnotti, 1986; 't Hooft and Veltman, 1974] limited the applicability of this approach to a treatment of phenomena well below the Planck scale. Yet, a conservative interpretation is that this does *not* imply a quantum field theory of gravity cannot be found, and renormalization might be achieved in a nonperturbative framework. This scenario has prominently been advocated in the context of the functional renormalization group [Gomis and Weinberg, 1996]. The focus of this thesis will be the complementary but related approach of lattice theory, namely putting the theory on a triangulation instead of a smooth manifold.

On a different note, it has long been speculated whether the topology of spacetime might in fact be dynamical as well: Wheeler coined the pictorial term "quantum foam" for quantum dynamics of the metric field at the Planck scale, subject to large topology-changing quantum fluctuations [A. and Wheeler, 1957]. A related unresolved question is whether the metric is in fact the appropriate dynamical variable. In particular, it has been argued that if a gauge connection is used instead, *topology change is unavoidable* [Horowitz, 1991]. The question whether toppology change does play a role being unanswered, it is desirable to solve this issue dynamically and not *a priori*. In that context, the approach to quantize gravity canonically in a Hamiltonian framework, requiring a fixed topology, is ill-suited, suggesting a path-integral approach to address the question of dynamical topology.

For these reasons, this thesis will focus on a *path-integral approach to lattice gravity*.

I close the introduction with a conceptual remark [Freidel and Louapre, 2003]. In practice, a sum over topologies in a path integral is hard to define: in a sum over all possible triangulations of three-manifolds, the gravity partition function is bounded from below [Ambjorn et al., 1991],

$$Z \geq \sum_{N_3} \lambda^{N_3} \mathcal{N}(N_3) \tag{1.2}$$

where λ is a constant, N_3 is the number of three-simplices in the triangulation and $\mathcal{N}(N_3)$ the number of possible triangulations for a given N_3. From combinatorial arguments, $\mathcal{N}(N_3) \propto N_3!$ if no topological restrictions are made. This factorial growth seems to obstruct any attempt to make a sum over topologies well-defined. In this thesis, I will attempt to illustrate a different take on this issue. Interpreting a triangulation not as a lattice regularization, but as a diagrammatic amplitude in a perturbative expansion offers a new perspective: It was first shown in [Dyson, 1952] that physical quantities in interacting quantum theory are non-analytic functions of the coupling constant. This is manifested in a vanishing radius of convergence for a series expansion that is at best asymptotic. Understanding this expansion as arising from a nonperturbatively defined quantity may allow standard tools from quantum field theory to be employed to give a well-defined meaning to its value. In the context of lattice gravity, this can be achieved by reinterpreting a sum over lattices as a sum over Feynman diagrams of an underlying zero-dimensional field theory. This particular viewpoint is the core theme of this work[1].

The thesis is structured as follows: in chapter 2, I will present a simple matrix model as a particular realization of the idea that a sum over topologies can arise as the perturbative expansion of an underlying path integral expression. Chapter 3 serves to set grounds for a realization of a similar scenario in three dimensions: I will present BF theory and argue that in three dimensions, it is equivalent to general relativity. I will outline a discretization procedure to arrive at the corresponding lattice theory and quantize it in a path integral framework. Chapter 4 will then put these ingredients together: I will first demonstrate how in two dimensions, the framework of group field theory is equivalent to the matrix models presented in the first chapter. Thereafter I will show that in three dimensions, it gives rise to a sum over topologies for lattice quantum gravity. In chapter 5, I will conlcude with pointing out open issues and further lines of development towards more realistic models.

[1]Let me emphasize that this viewpoint is by no means a very new one, and in particular not my own idea. A good exposition of the idea can be found in [Freidel and Louapre, 2003], and many other references I will give throughout the thesis.

Chapter 2

Matrix Models

The equivalence of lattice models of two-dimensional gravity and theories of dynamical matrices was first advocated in [David, 1985]. These types of models turned out solvable in the large N limit [Bessis et al., 1980; Brézin et al., 1978], with results for fixed topology in agreement with those obtained by the continuum theory, where computable. Later, a continuum limit was found nonperturbatively, including contributions from all topologies [Douglas and Shenker, 1990]. This sparked hope for a nonperturbative definition of string theory and quantum gravity in two dimensions. In the following sections, we will sketch the results crucial for gaining an intuition for an approach towards the three-dimensional case.

2.1 Gravity in two dimensions

Given a smooth, orientable two-manifold $\mathcal{M}$, the Einstein-Hilbert action in two dimensions with cosmological constant is given by

$$S_{EH}[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^2 x \sqrt{|g|}(R - 2\lambda), \tag{2.1}$$

where g is the metric tensor, $|g|$ the modulus of its determinant and R the curvature scalar. Here, κ is dimensionless and λ has dimensions of inverse area. In two dimensions, the identity

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \tag{2.2}$$

holds algebraically and the resulting set of classical solutions for gravity is consequently trivial if $\mathcal{M}$ is diffeomorphic to S^2. Upon quantization however, off-shell geometries contribute and large quantum fluctuations may change the genus of the surface. For higher genera the corresponding phase space is more complicated, rendering the corresponding quantum theory nontrivial. Using the fact that for a given topology, the area of $\mathcal{M}$ is given by $\Sigma = \int_{\mathcal{M}} \mathrm{d}^2x \sqrt{|g|}$, and the Euler characteristic of $\mathcal{M}$ by $4\pi\chi = \int_{\mathcal{M}} \mathrm{d}^2x \sqrt{|g|} R$, we can integrate the action to give

$$S_{EH}[g] = \frac{2\pi}{\kappa}\chi - \frac{\lambda}{\kappa}\Sigma. \tag{2.3}$$

A corresponding quantum theory including a sum over topologies can be defined by making sense of the symbolic functional integral expression

$$Z_{2D} = \sum_h \int \mathcal{D}g e^{-\frac{\lambda}{\kappa}\Sigma + \frac{2\pi}{\kappa}\chi}. \tag{2.4}$$

Here, h is the *genus* of the surface, related to the Euler characteristic by $\chi = 2 - 2h$. One such proposal, dating back to [Weingarten, 1982] is to replace[1] $\mathcal{M}$ with a triangulated surface Δ, and sum over connected triangulations of all genera h,

$$Z_{2D} = \sum_h \sum_{\Delta \text{ connected}} e^{-\frac{\lambda}{\kappa}\Sigma + \frac{2\pi}{\kappa}\chi}. \tag{2.5}$$

The goal of the next section is to present a tentative model that serves as a definition for the above sum (2.5) in terms of a perturbative expansion of a nonperturbatively defined quantity.

[1]For a concise review of the construction, see [Ambjorn, 1994].

2.2 Microscopic matrix action

Consider an ensemble of $N \times N$ hermitian Matrices $M = M^\dagger$, with dynamics governed by the $U(N)$-invariant action

$$S(M) = N\left(\frac{1}{2}\mathrm{tr}M^2 - g\mathrm{tr}M^3\right), \tag{2.6}$$

with g a coupling constant and tr the trace. Let us examine the diagrammatic expansion of the partition function

$$Z = \int [dM] e^{-S(M)}, \tag{2.7}$$

with the measure defined by integrating the independent components over $\mathbb{C}$,

$$[dM] = \prod_{i=1}^{N} dM^i{}_i \prod_{i<j} dM^i{}_j dM^{*i}{}_j. \tag{2.8}$$

Here, the index positions refer to the transformation behaviour of M under $U(N)$, which is given by the adjoint action

$$M^i_j \quad \to \quad M'^i{}_j = U(g)^i{}_k M^k{}_l U(g^{-1})^l{}_j, \quad g \in U(N). \tag{2.9}$$

In the last equation and in what follows, repeated indices will be summed over. The generating functional is

$$\begin{aligned} Z[J] &= \int [dM] e^{-N\mathrm{tr}\left(\frac{1}{2}M^2 + JM\right)} \\ &= \left(\frac{2\pi}{N}\right)^{N^2/2} e^{\frac{N}{2}\mathrm{tr}J^2}, \end{aligned} \tag{2.10}$$

where J is a hermitian $N \times N$ matrix serving as an external current. Denoting $Z[J]|_{J=0} = Z_0$, we can find the propagator by taking the second derivative with respect to M,

$$\langle M^i{}_j M^k{}_l \rangle = \frac{1}{Z_0} \frac{\partial^2 Z[J]}{\partial J^j{}_i \partial J^l{}_k}\bigg|_{J=0} = \frac{1}{N}\delta^i{}_l \delta^k{}_j. \tag{2.11}$$

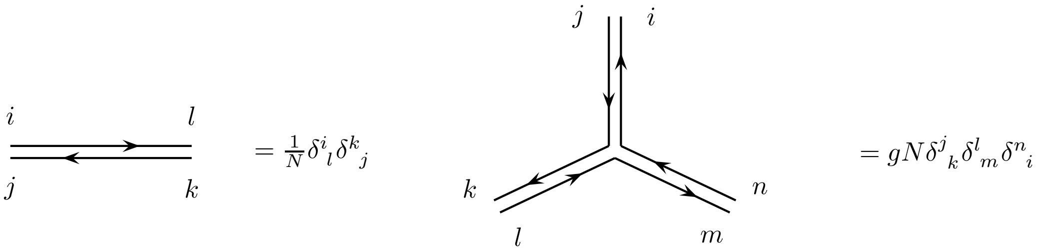


Figure 2.1: Feynman rules for the matrix model.

The vertex contribution can be read off from the action as

$$gN\delta^j{}_k \delta^l{}_m \delta^n{}_i. \tag{2.12}$$

The Feynman rules can conveniently be depicted graphically in terms of oriented double lines[1] as shown in Fig. 2.1. Vertices will thus contribute positive powers of gN, internal lines powers of N^{-1}, and each closed loop another factor of N from the resulting index contractions. The expansion of (2.7) will generate graphs without external legs only. In this expansion, the amplitude associated to a single graph Γ with V vertices, E edges (internal lines) and F faces (closed loops) is thus

$$A(\Gamma) = g^V N^V N^{-E} N^F = g^V N^{\chi(\Gamma)}, \tag{2.13}$$

where $\chi(\Gamma) = V - E + F$ is the Euler characteristic of Γ. The amplitude of each graph defines an amplitude for a triangulation Δ of a two-dimensional oriented[2] surface in the following way: associate to each vertex an equilateral triangle, with a double line passing through each edge. Join the triangles along edges that are connected with propagators. Using the fact that then, $\chi(\Gamma) = \chi(\Delta)$, we can associate (2.13) with an amplitude for a two-dimensional simplicial complex Δ

[1] This notation was first introduced by 't Hooft in his seminal paper to study the large N limit of $SU(N)$ Yang-Mills theory ['t Hooft, 1974].

[2] Had we used real symmetric matrices instead of hermitian matrices, upper and lower indices would be equivalent and consequently the lines in the Feynman diagrams unoriented, thus generating unoriented triangulations.

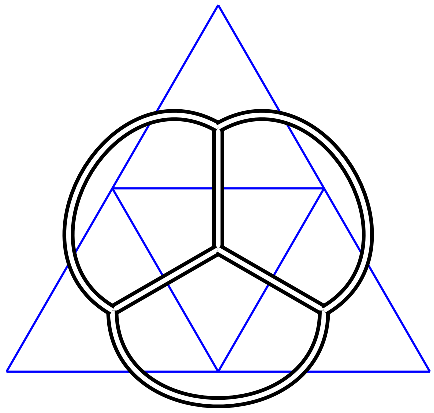


Figure 2.2: A Feynman diagram occuring in the expansion of (2.16) corresponding to a triangulation of S^2, proportional to g^4N^2. Labellings and orientations have been suppressed.

with T triangles:

$$A(\Delta) = g^T N^{\chi(\Delta)}. \tag{2.14}$$

An example is depicted in Fig. 2.2. Given $A(\Delta)$, upon inspection and comparison with (2.3), we may rename $N = e^{-2\pi/\kappa}$, $g = e^{\lambda\Sigma_0/\kappa}$, where Σ_0 denotes the area of a single triangle such that $T = \Sigma/\Sigma_0$. Then we have

$$A(\Delta) = \exp\left(\frac{\lambda}{\kappa}\Sigma - \frac{2\pi}{\kappa}\chi\right) = e^{-S_{EH}}. \tag{2.15}$$

Applying Wick's theorem, we see that at order n, the expansion of (2.7) generates all possible gluings of n triangles, weighted with the exponential of the Einstein-Hilbert action in two dimensions. To make contact with our initial goal of defining the path integral for two-dimensional quantum gravity, we can take the logoarithm of (2.7) to generate connected diagrams only, corresponding to triangulations of connected surfaces. We then arrive at the conclusion, that the *free energy* F of the matrix model provides a tentative definition of the *partition function* for two-dimensional gravity,

$$F = \log Z = \sum_h \sum_{\Delta \text{ connected}} e^{-S_{EH}}. \tag{2.16}$$

2.3 Continuum limt

Intuitively, to define a continuum limit, one needs the matrix model to approach a critical point for some value g_c of the coupling g, such that in the vicinity of g_c, the partition function function is dominated by graphs with a divergent number of vertices. This is reasonable because the average dual triangulation will correspondingly contain a divergent number of triangles, the areas of which can then be scaled to zero, keeping the physical area $\Sigma = n \cdot \Sigma_0$ fixed.

Another important remark is that while in the last section, we explicitly chose an interaction $V(M) = \lambda \mathrm{tr} M^3$, *a priori*, we need to allow for arbitrary potentials $V(M) = \sum_k \lambda_k \mathrm{tr} M^k$ to consistently deal with all possible quantum corrections. The crucial point however for finding a continuum limit by means of a critical point is that universality applies: while the critical values of the coupling constants λ_n depend on the form of $V(M)$, the critical exponents and the existence of phase transitions *do not*. A continuum limit will thus exist for any reasonable choice of $V(M)$ [Di Francesco et al., 1995].

Let me first demonstrate the existence of a simple continuum limit in which only surfaces of genus zero contribute, and thereafter discuss a more interesting limit for which all genera remain relevant. Writing $\chi(\Gamma) = 2 - 2h$, we can recast (2.13) explicitly in powers of *two* expansion parameters, g and $1/N^2$:

$$A(\Gamma) = N^2 g^V \left(\frac{1}{N^2}\right)^h, \tag{2.17}$$

where we can expect the expansion in h to be reliable for large N. The genus expansion of Z then reads

$$\begin{aligned} Z(N,g) &= \sum_h N^{2-2h} Z_h(g) \\ &= N^2 Z_0(g) + Z_1(g) + \frac{1}{N^2} Z_2(g) + \mathcal{O}\left(\frac{1}{N^4}\right). \end{aligned} \tag{2.18}$$

This expression clearly suggests that in large N-limit, the partition function is dominated by contributions of genus zero. Let us for the moment work in this regime. The genus zero contribution Z_0 can now be expanded in powers of g. Next, we make the following

Claim [Di Francesco et al., 1995, and references therein]. *The function $Z_0(g)$ diverges at a critical value g_c of the coupling constant g, with the singular piece having large-n behaviour*

$$Z_0(g) \propto \sum_n n^{\gamma-3} \left(\frac{g}{g_c}\right)^n \propto (g_c - g)^{2-\gamma}, \tag{2.19}$$

where γ is a critical exponent.

Consequently, near criticality, the expectation value of the area, being determined by the expectation value of the number of vertices $\langle n \rangle$, behaves as

$$\begin{aligned} \langle \Sigma \rangle = \Sigma_0 \langle n \rangle &\propto \sum_n n \cdot n^{\gamma-3} \left(\frac{g}{g_c}\right)^n \\ &\propto \Sigma_0 \frac{\partial}{\partial g} \log Z_0(g) \propto \frac{\Sigma_0}{g - g_c}. \end{aligned} \tag{2.20}$$

This allows to define a continuum limit of the matrix model action (2.6) by simultaneously taking

$$N \to \infty, \quad g \to g_c, \quad \Sigma_0 \to 0 \quad \text{with} \quad \Sigma = \frac{\Sigma_0}{g - g_c} \quad \text{fixed}. \tag{2.21}$$

Instead of taking the limit $N \to \infty$ first, and the limit $g \to g_c$ second, there is another possibility to obtain a different continuum theory by taking both limits *simultaneously* in a coordinated manner. To make this apparent, let us not yet take the large-N limit. We make another

Claim [Di Francesco et al., 1995, and references therein]. *The functions $Z_h(g)$, $h \geq 0$ diverge at the* same *critical value g_c as does $Z_0(g)$, scaling at large orders n as*

$$Z_h(g) \propto \sum_n n^{(\gamma-2)(1-h)-1} \left(\frac{g}{g_c}\right)^n \propto (g_c - g)^{(2-\gamma)(1-h)}. \tag{2.22}$$

Close to criticality, contributions from higher genera $h \geq 1$ are thus enhanced to contribute more significantly. At the same time, the divergent behaviour of $\langle n \rangle$

is preserved. In this regime, the genus expansion (2.18) can be rewritten

$$\begin{aligned} Z(N,g) &= \sum_h N^{2-2h} Z_h(g) \\ &\propto \sum_h N^{2-2h} f_h (g-g_c)^{(2-\gamma)(1-h)} \\ &\propto \sum_h f_h g_s^{2h-2}, \end{aligned} \tag{2.23}$$

where we defined $g_s = [N(g-g_c)^{(2-\gamma)/2}]^{-1}$. We can then take the limit

$$N \to \infty, \quad g \to g_c, \quad g_s = [N(g-g_c)^{(2-\gamma)/2}]^{-1} \quad \text{fixed.} \tag{2.24}$$

This is commonly referred to as the *double scaling limit*. Of course, still having Σ_0 as a free parameter at our disposal, we can take a corresponding continuum limit with fixed physical area just as before. This procedure gives a continuum theory to whose amplitude surfaces of all genera contribute significantly.

As a final remark, let us note that the continuum limit is intrinsically non-perturbative in g. This stems from the fact that in this limit, large powers of g dominate over lower order contributions. This is incompatible with the assumption of small g needed for perturbation theory, for which the critical point hence remains invisible.

Chapter 3

BF Theory

The first appearance of BF theories as a class of topological field theories and the relation to three-dimensional gravity dates back to [Horowitz, 1989; Schwarz, 1978]. This chapter will first give a review of the classical theory and demonstrate its equivalence to three-dimensional general relativity. Thereafter, I will discretize BF theory on a simplicial lattice. While certainly not unique, I shall demonstrate that the discretization procedure does not invoke any approximations. The simplicial lattice serves as a regularization for the path integral measure, and subsequent quantization will lead us to an expression for the partition function of a form first proposed by Ponzano and Regge [Ponzano and Regge, 1968]. The quantization procedure itself has appeared frequently in the literature [see e.g. Freidel and Krasnov, 1999], albeit sketchy and partially glossing over technical subtleties. The purpose of the last section in this chapter is thus to provide a more detailed exposition and comment on the relation between the initial proposal by Ponzano and Regge and the partition function for three-dimensional BF theory. This yields a crucial cornerstone to arrive at the generalized matrix models presented in the last chapter.

3.1 Classical Theory

Given a smooth, orientable manifold $\mathcal{M}$ with local coordinates $\{x^\mu\}$ and a Lie group G with Lie algebra $\mathfrak{g} \cong T_{\mathbb{1}}G$, BF theory in three dimensions is described by the set of $\mathfrak{g}$-valued one-form fields

$$A = A_\mu{}^i t^i \mathrm{d}x^\mu, \quad B = B_\mu{}^i t^i \mathrm{d}x^\mu, \tag{3.1}$$

with $\{t^i\}$ the basis of $\mathfrak{g}$. In the following, latin indices $i = 1, ...\dim(\mathfrak{g})$ will always refer to the Lie algebra, and greek indices $\mu = 1, 2, 3$ to $\mathcal{M}$. Using A as a gauge field, we can now introduce its curvature[1]:

$$F = \mathrm{d}A + A \wedge A, \tag{3.2}$$

Note that the second term in F vanishes only when G is abelian. The BF action is

$$S_{BF}[A, B] = \int_{\mathcal{M}} \mathrm{tr}(B \wedge F), \tag{3.3}$$

where tr is a nondegenerate bilinear form on $\mathfrak{g}$. The extremum conditions for the action are given by

$$0 = \frac{\delta S_{BF}}{\delta A} = \mathrm{d}_A B, \quad 0 = \frac{\delta S_{BF}}{\delta B} = F. \tag{3.4}$$

The first equation states that B is covariantly constant, and the second implies the flatness of the connection. Here, d_A is the gauge covariant derivative on forms,

$$\mathrm{d}_A \cdot = \begin{cases} \mathrm{d} + A \wedge \cdot & \text{acting on the fund. rep.} \\ \mathrm{d} + [A, \cdot] & \text{acting on the adjoint rep.} \end{cases} \tag{3.5}$$

The action has the following three symmetries:

1. Let $g \in G$. Then the *gauge transformation*

$$\begin{aligned} A \to A' &= gAg^{-1} + g\mathrm{d}g^{-1}, \\ B \to B' &= gBg^{-1} \end{aligned} \tag{3.6}$$

[1] Note that $\mathrm{d}_A A \neq F$, while $\mathrm{d}_A \delta A = \delta F$.

leaves S_{BF} invariant. The infinetesimal form is given by

$$\begin{aligned} \delta_X A &= \mathrm{d}_A X, \\ \delta_X B &= [B, X], \end{aligned} \tag{3.7}$$

with X a $\mathfrak{g}$-valued scalar and $[\cdot,\cdot]$ the commutator on $\mathfrak{g}$.

2. Let $\phi \in \mathfrak{g}$. Then, as a direct consequence of the Bianchi identity $\mathrm{d}_A F = 0$, the *translation*

$$\begin{aligned} \delta_\phi A &= 0, \\ \delta_\phi B &= \mathrm{d}_A \phi \end{aligned} \tag{3.8}$$

is a symmetry of S_{BF} up to a boundary term. Moreover, since $\mathrm{d}_A B = 0$, locally there is always a zero-form ϕ such that $B = \mathrm{d}_A \phi$ by Poincarés lemma. Together with the fact that all flat connections are locally equivalent, this implies that all solutions to (3.4) are locally the same up to gauge transformations. Theories with this property are referred to as *topological*, as locally, they have no physical degrees of freedom.

3. The action is manifestly invariant under *diffeomorphisms*. For the next discussion, an explicit form for infinitesimal diffeomorphisms is instructive: let ξ be an arbitrary vector field on $\mathcal{M}$. Then under a diffeomorphism generated by ξ, the fields transform as

$$\begin{aligned} \delta_\xi A &= \mathrm{d}(\imath_\xi A) + \imath_\xi(\mathrm{d}A), \\ \delta_\xi B &= \mathrm{d}(\imath_\xi B) + \imath_\xi(\mathrm{d}B), \end{aligned} \tag{3.9}$$

where $\imath$ denotes the interior product on forms.

One can now show that the above symmetries are not entirely independent. More specifically, consider the simultaneous transformations

$$\begin{aligned} \delta A &= \mathrm{d}_A X + \imath_\xi(\mathrm{d}_A A), \\ \delta B &= [B, X] + \mathrm{d}_A \phi + \imath_\xi(\mathrm{d}_A B). \end{aligned} \tag{3.10}$$

On-shell, these are simply a combination of a gauge transformation and a translation. Picking $X = \imath_\xi A$ and $\phi = \imath_\xi B$ everywhere, these take the form of a diffeomorphism generated by ξ if the equations of motion (3.4) are satisfied. This implies an *on-shell equivalence* between diffeomorphisms and local combinations of gauge transformations and translations.

3.2 Relation with GR

The relation to general relativity in $2+1$ dimensions can be drawn as follows. A smooth $2+1$-dimensional manifold is equipped with a corresponding tangent space $T_p\mathcal{M}$ at every point $p \in \mathcal{M}$. Introducing a $\mathfrak{g}$-valued connection requires an additional abstract vector space V_p at every point, with isometry group G. However, upon choosing $G = SO(2,1)$, these two spaces are isomorphic. One can then introduce a $\mathfrak{so}(2,1) \cong \mathfrak{su}(1,1)$-valued one-form field $e = e^i_\mu t^i \mathrm{d}x^\mu$ with $i = 1,2,3$ and $t^i = -i\sigma^i/2$, with σ^i the three Pauli matrices. Then we have that $[t^i, t^j] = \epsilon^{ijk}\eta_{kl}t^l$, where ϵ^{ijk} is the usual totally antisymmetric tensor and $\eta_{kl} = \mathrm{diag}(-1,+1,+1)$. e then serves as an *explicit choice* for an isomorphism between these vector spaces at every point [Witten, 1988]. Let us now demonstrate how this isomorphism finds its place in BF theory. Choosing $G = SO(2,1)$, we make the identifications

$$A^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk}, \quad B^i = Ge^i, \tag{3.11}$$

with G a coupling constant of mass dimension $[G] = 1$ to make e dimensionless. The gauge transformations (3.6) are now the local frame rotations of the triad and the spin connection with a tangent space of Lorentzian signature[1]. Indeed, the appropriately rescaled action now takes the form

$$S[e,\omega] = \frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{ijk} e^i \wedge R^{jk}, \tag{3.12}$$

which is the usual Palatini-Kibble first-order action for gravity in three dimen-

[1] The Riemannian counterpart is obtained by replacing $\mathfrak{g} = \mathfrak{su}(2) \cong \mathfrak{so}(3)$.

sions. The field equations (3.4)

$$\mathrm{d}_\omega e = 0 \tag{3.13}$$

$$R = 0 \tag{3.14}$$

now acquire the interpretation of imposing vanishing torsion and spacetime curvature. General relativity in $2+1$ dimensions can thus be regarded as a special case of BF theory. To comment on the relation to the usual metric forumulation of gravity, let us identify the components of the metric and its inverse,

$$g = \eta_{ij} e^i \otimes e^j, \quad g^{-1} = \eta^{ij} e_i^{-1} \otimes e_j^{-1}, \tag{3.15}$$

or in components,

$$g_{\mu\nu} = e_\mu{}^i e_\nu{}^j \eta_{ij}, \quad g^{\mu\nu} = e^\mu{}_i e^\nu{}_j \eta^{ij} \tag{3.16}$$

where $e^\mu{}_i$ is the inverse triad. Equipped with an invertible triad, one can also define the Riemann curvature tensor,

$$R^\alpha{}_{\beta\mu\nu} = e^\alpha{}_i e_\beta{}^j R^i{}_{j\mu\nu}. \tag{3.17}$$

This allows to rewrite the action as

$$\begin{aligned} 16\pi G S[e,\omega] &= \frac{1}{4}\int_{\mathcal{M}} \epsilon_{ijk} e_\mu{}^i R^{jk}{}_{\nu\rho} \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\rho \\ &= \frac{1}{4}\int_{\mathcal{M}} \epsilon_{ijk} e_\mu{}^i e_\alpha{}^j e_\beta{}^k \epsilon^{\mu\nu\rho} R^{\alpha\beta}{}_{\nu\rho} \mathrm{d}^3x \\ &= \int_{\mathcal{M}} (\det e) \delta^\nu_\alpha \delta^\rho_\beta R^{\alpha\beta}{}_{\nu\rho} \mathrm{d}^3x \\ &= \int_{\mathcal{M}} \sqrt{|g|} R \mathrm{d}^3x, \end{aligned} \tag{3.18}$$

where g denotes the determinant of the metric and the identities

$$\det e = \sqrt{|g|}, \quad \epsilon_{\mu\alpha\beta}\epsilon^{\mu\nu\rho} = 4\delta^{[\nu}_\alpha \delta^{\rho]}_\beta \tag{3.19}$$

have been used. The last line in (3.18) is exactly the Einstein-Hilbert action in terms of the metric in three dimensions.

Let us however note that while BF theory and the first-order formulation of gravity are equivalent in three dimensions, the metric formulation is subtly different:

- The set of solutions in the first order formalism is larger than that of metric gravity: the equations of motion (3.13), (3.14) are also solved by non-invertible triads, for which

$$\mathrm{d}s^2 = \eta_{ij} e^i e^j = 0. \tag{3.20}$$

  These degenerate solutions have no analogue in the metric formulation of gravity as the inverse metric and the curvature tensor are ill-defined.

- A second difference becomes manifest in presence of a matter term S_{Matter} in the action. If S_{Matter} depends on ω, equation (3.13) acquires additional terms. This happens for kinetic terms for spinors, where the covariant derivative is of the form

$$\nabla_\mu \psi = \partial_\mu \psi + {\omega_\mu}^{ij} \gamma_i \gamma_j \psi, \tag{3.21}$$

  where the spinor indices have been supressed and the γ_i denote Dirac's matrices. This leads to torsion since then locally, $\mathrm{d}_\omega e \neq 0$. This is never the case for metric gravity.

For the purposes of this thesis let us simply note that up to now, the four-dimensional analogues of both formulations are compatible with experiments. The latter ultimately has to decide whether torsion plays a rôle in the dynamics of spacetime. Moreover, as we have seen, first-order gravity has a larger set of solutions and is thus slightly more general. With foresight to studying couplings to matter fields, let us furthermore note that coupling spinor fields to gravity requires a spin connection as present in the first-order formalism [Wald, 1984]. For these reasons, and for the fact that quantization methods for gauge theories are readily available, we choose to stick with the first-order formalism for gravity in the following, and in particular with its BF-theoretic incarnation. As a consequence of degenerate metrics being a solution, we have to deal with dynamical topology change in the quantum theory [Horowitz, 1991].

3.3 Discretization

The ultimate goal of this chapter is to provide a calculation of the path integral for BF theory, symbolically

$$\int \mathcal{D}A\mathcal{D}Be^{iS_{BF}[A,B]}. \tag{3.22}$$

To define this expression more rigorously, we need to make a choice of regularization and provide an appropriate definition of the measures $\mathcal{D}A$, $\mathcal{D}B$. Our choice will be to discretize $\mathcal{M}$ by replacing it with a simplicial complex corresponding to a triangulation of $\mathcal{M}$, and then consider the discretized fields as the dynamical data for the path integral in analogy with lattice gauge theory. One would then in principle need to specify a procedure for taking a continuum limit that coincides with the initial continuum action. Owing to the topological nature of three-dimensional gravity, this procedure will however turn out trivial. To carry out the discretization, we need to establish a few conventions regarding simplicial complexes and triangulations. In what follows, we will therefore recall a few definitions, and construct discrete analogues of basic operations on differential forms such as the wedge product, exterior derivative and Hodge dual. The treatment will largely follow the presentation given in [Sen et al., 2000] and [Thiemann, 2001]; the wedge variables introduced at the end of this section go back to [Reisenberger, 1997]. This machinery will allow us to derive the starting point for the path integral quantization given in the next section.

Definition 3.3.1. *A* p-simplex $\sigma^{(p)} = [v_0, ...v_p]$ *in* $\mathbb{R}^D$ *is the convex hull of* $p+1$ *vectors,*

$$\sigma^{(p)} := \left\{ \sum_{k=0}^{p} a_k v_k | a_k \geq 0, \sum_{k=0}^{p} a_k = 1 \right\}. \tag{3.23}$$

An orientation of $\sigma^{(p)}$ is induced by the order in which its vertices v_k appear in the list $[v_0, ...v_p]$. For a permutation $\pi \in S_{p+1}$, we say that $[v_0, ...v_p]$ and $[v_{\pi(0)}, ...v_{\pi(p)}]$ are equally oriented if π is even, and oppositely oriented if π is odd.

Definition 3.3.2. *The* boundary $\partial\sigma^{(p)}$ *of a p-simplex $\sigma^{(p)}$ is defined as the set of points for which $a_k = 0$, with $k = 0, ...p$. This defines a set of $p + 1$ different $(p - 1)$-simplices $\sigma_k^{(p-1)} = [v_0, ...v_{k-1}, v_{k+1}, ...v_p]$, the* faces *of $\sigma^{(p)}$.*

Definition 3.3.3. *The* barycentre *of a p-simplex $\sigma^{(p)} = [v_0, ...v_p]$ is defined as the point*

$$\hat{\sigma}^{(p)} := \frac{\sum_{k=0}^{p} v_k}{p+1}. \tag{3.24}$$

The faces share the orientation of $\sigma^{(p)}$ if k is even, and oppositely oriented if k is odd.

Definition 3.3.4. *A* simplicial complex *K is a collection of simplices $\sigma_i^{(p)}$, $p = 0, ...D$, $i = 1, ...N_p$, with the following properties:*

1. *All the subsimplices of each $\sigma_i^{(p)}$ also belong to K.*
2. *Two simplices $\sigma_i^{(p)}$, $\sigma_j^{(p)}$ intersect at most in a common subsimplex, which has opposite orientation in $\sigma_i^{(p)}$ and $\sigma_j^{(p)}$*

It can be shown that any differential manifold $\mathcal{M}$ admits a partition into a simplicial complex. The complex is then called a *triangulation* of $\mathcal{M}$, and its topology is inherited from the manifold. Such a triangulation will provide the lattice for our discretization of the dynamical variables.

Definition 3.3.5. *Let $K = \{\sigma_I^{(p)} | p = 0, ...D, i = 0, ...N_p\}$ be a simplicial complex.*

1. *A p-*chain *c is defined as the formal real linear combination,*

$$c = \sum_{i=1}^{N_p} c_i \sigma_i^{(p)}, \quad c_i \in \mathbb{R} \quad \forall \quad i. \tag{3.25}$$

The resulting vector space of all p-chains is denoted $C_p(K)$.

2. *We turn $C_p(K)$ into a Hilbert space, with an orthonormal basis provided by the p-simplices, by defining the inner product*

$$\langle \sigma_i^{(p)}, \sigma_j^{(p)} \rangle_K := \delta_{ij} \quad \forall \quad i, j = 1, ...N_p. \tag{3.26}$$

3. *The* boundary operator *on p-chains is defined by*

$$\begin{aligned} \partial : \quad & C_p(K) \longrightarrow C_{p-1}(K) \\ & \sigma_i^{(p)} \longmapsto \partial \sigma_i^{(p)} := \sum_{k=0}^{p} (-1)^k [v_0, ...v_{k-1}, v_{k+1}, ...v_p]. \end{aligned} \tag{3.27}$$

To proceed with a regularization of (3.22), we need to relate p-forms and p-chains and find an analogue of the wedge product. We are then sufficiently geared to discretize the form fields A, B on K. This is achieved by the following

Definition 3.3.6. *Let K be a simplicial complex and let $\Lambda^p(K)$ the space of p-forms defined on K.*

1. *The* Whitney map *is defined by*

$$\begin{aligned} W_K : \quad & C_p(K) \longrightarrow \Lambda^p(K) \\ & \sigma^{(p)} \longmapsto p! \sum_{k=0}^{p} (-1)^k a_k da_0 \wedge ...da_{k-1} \wedge da_{k+1} \wedge ...da_p, \end{aligned} \tag{3.28}$$

   where the a_k, $k = 0, ...p$ are the coefficients as defined in 3.3.1. Here, they are understood as local coordinates on $\sigma^{(p)}$.

2. *The* de Rham map *is defined by*

$$\begin{aligned} R_K : \quad & \Lambda^p(K) \longrightarrow C_p(K) \\ & \langle R_K(\omega), \sigma^{(p)} \rangle_K := \int_{\sigma^{(p)}} \omega. \end{aligned} \tag{3.29}$$

3. *The* wedge product *on p-chains is defined by*

$$\begin{aligned} \wedge_K : \quad & C_p(K) \times C_q(K) \longrightarrow C_{p+q}(K) \\ & \sigma^{(p)} \wedge_K \sigma^{(q)} := R_K\left(W_K(\sigma^{(p)}) \wedge W_K(\sigma^{(q)})\right). \end{aligned} \tag{3.30}$$

A variety of properties of the above operations will be useful in our derivation of a lattice version of BF theory, which we assemble in the following

Theorem 3.3.7. *The operations W_K, R_K and $\wedge_K$ obey the following relations:*

$$\begin{aligned} \sigma^{(p)} \wedge_K \sigma^{(q)} &= (-1)^{pq} \sigma^{(q)} \wedge_K \sigma^{(p)} \\ R_K \circ W_K &= \mathbb{1} \\ \int_{\sigma^{(p)}} W_K(\sigma'^{(p)}) &= \langle \sigma^{(p)}, \sigma'^{(p)} \rangle \end{aligned} \tag{3.31}$$

We refer the reader to [Whitney, 2005] for a proof. One more crucial piece for our discretization procedure is missing, namely what is called the *dual* complex $\star K$, which we define in what follows.

Definition 3.3.8. *Let $K = \{\sigma_i^{(p)} | p = 0, ...D, i = 1, ...N_p\}$ be a simplicial complex. For any $\sigma_{j_0}^{(p)} \in K$ consider all possible $(D-p)$-tuples of simplices $\sigma_{j_k}^{(p+k)}$ with $k = 1, ...D-p$ and $1 \leq j_k \leq N_{p+k}$, subject to the following condition:*

For all $l = 0, ...D-p-1$, the simplex $\sigma_{j_l}^{(p+l)}$ is a face of $\sigma_{j_l+1}^{(j+l+1)}$ with the induced orientation.

For each such $(D-p)$-tuple of simplices construct the $(D-p)$-simplex $[\hat{\sigma}_{j_0}^{(p)}, \hat{\sigma}_{j_1}^{(p+1)}, ...\hat{\sigma}_{j_{D-p}}^{(D)}]$, where we have used the barycentres of the respective simplices as defined in 3.3.3. The cell dual *to $\sigma_{j_0}^{(p)}$ is then defined by the map*

$$\begin{aligned} \star_K : \quad & C_p(K) \longrightarrow C_{D-p}(\star K) \\ & \sigma_{j_0}^{(p)} \longmapsto \star_K[\sigma_{j_0}^{(p)}] := \bigcup_{\substack{\sigma_{j_l}^{(p+l)} \subset \partial \sigma_{j_{p+l+1}}^{(p+l+1)} \\ l=0,...D-p-1}} [\hat{\sigma}_{j_0}^{(p)}, \hat{\sigma}_{j_1}^{(p+1)}, ...\hat{\sigma}_{j_{D-p}}^{(D)}], \end{aligned} \tag{3.32}$$

where the cell complex $\star K$ dual *to* K *is obtained by joining dual cells along common subcells.*

As the $(D-p)$-cells in $\star K$ are in one-to-one correspondence to the p-simplices of K, we can define a $\star_{\star K}$-operation on $\star K$ as the inverse of $\star_K$ on K. Note that in general, $\star K$ is *not* a simplicial complex, but a cell complex of more general type, composed of arbitrary polyhedra. Consequently, the operations we defined on simplicial complexes cannot be extended straightforwardly to $\star K$. To repair this, we need a notion of a *simplicial* complex in $\star K$. This is achieved by what is called *barycentric refinement*:

Definition 3.3.9. *Let* $\pi \in S_{p+1}$ *and let for each* $k = 0, ...p$

$$\hat{\sigma}(k)_\pi := \frac{\sum_{l=0}^{k} v_{\pi(l)}}{k+1} \tag{3.33}$$

be the barycentre of the k*-subsimplex* $[v_{\pi(1)}, ...v_{\pi(k)}]$. *Then*

1. *The* barycentric subdivision *of* $\sigma^{(p)}$ *is defined as the set of* $(p+1)!$ p*-simplices* $\sigma_\pi^{(p)} := [\hat{\sigma}_\pi^{(0)}, ...\hat{\sigma}_\pi^{(p)}]$.

2. *The* barycentric refinement $B(K)$ *of* K *is defined as the set of barycentric subdivisions for all simplices in* K.

Obviously, $K \subset B(K)$. Moreover, since the p-cells of $\star K$ are unions of the p-simplices of $B(K)$, it follows that also $\star K \subset B(K)$. Since $B(K)$ is simplicial, all operations ∂, W_K, R_K, $\wedge_K$ and $\star_K$ on K can now be extended to $B(K)$ and thus also to $\star K$, because $C_p(\star K)$ is a subspace of $C_p(B(K))$. Furthermore, we can define an inner product $\langle \cdot, \cdot \rangle_{\star K}$ on $C_{D-p}(\star K)$ as $\langle \cdot, \cdot \rangle_{B(K)}$ on $C_p(B(K))$ by declaring dual cells to be orthonormal.

We can now state the following helpful

Theorem 3.3.10. *Let* K *be a triangulation of a differential manifold* $\mathcal{M}$ *and let* $x \in C_p(K)$, $y \in C_{D-p}(K)$. *Let furthermore* $E(x)$ *and* $E(y)$ *be the linear* *combinations of elements in $C_p(B(K))$ giving the sets x and y, respectively. Then*

$$\begin{aligned}\langle \star_K(x), y\rangle_{\star K} &= \frac{(D+1)!}{p!(D-p)!}\int_{\mathcal{M}} W_{B(K)}\left(E(x)\right)\wedge W_{B(K)}\left(E(y)\right)\\ \langle \star_{\star K}(y), x\rangle_K &= \frac{(D+1)!}{p!(D-p)!}\int_{\mathcal{M}} W_{B(K)}\left(E(y)\right)\wedge W_{B(K)}\left(E(x)\right)\end{aligned} \tag{3.34}$$

A proof was given in [Adams, 1996]. Now, all necessary ingredients to discretize the BF action (3.3) are at our disposal. The final result is given by the following theorem:

Theorem 3.3.11. *Let $\mathcal{M}$ be an orientable differential three-manifold and let K be a simplicial complex triangulating $\mathcal{M}$. Let the form fields B and F be defined as in section 3.1. Then*

$$\int_{\mathcal{M}} \mathrm{tr}(B\wedge F) = \sum_{\sigma^{(1)}\in C_1(B(K))}\int_{\star_K\sigma^{(1)}} \mathrm{tr}(F X_{\sigma^{(1)}}), \tag{3.35}$$

where $X_{\sigma^{(1)}} = \int_{\sigma^{(1)}} B$.

Proof. Using the second identity of theorem 3.3.7, we can rewrite

$$\begin{aligned}S_{BF}[A,B] &= \int_{\mathcal{M}} \mathrm{tr}(B\wedge F)\\ &= \int_{\mathcal{M}} \mathrm{tr}\left[W_{B(K)}\left(R_{B(K)}(B)\right)\wedge W_{B(K)}\left(R_{B(K)}(F)\right)\right].\end{aligned} \tag{3.36}$$

Next, using the skew-symmetry of the wedge product and invoking theorem 3.3.10, we find

$$S_{BF}[A,B] = \mathrm{tr}\langle \star_{\star K} R_{B(K)}(F), R_{B(K)}(B)\rangle_K. \tag{3.37}$$

Note that $\star_{\star K} R_{B(K)}(F) \in C_1\left(B(K)\right)$, since our manifold is three-dimensional and F is a two-form. We can thus insert a resolution of the identity in $C_1\left(B(K)\right)$

into (3.37), giving

$$S_{BF}[A,B] = \sum_{\sigma^{(1)} \in C_1(B(K))} \operatorname{tr}\left(\langle \star_{\star K} R_{B(K)}(F), \sigma^{(1)} \rangle_K \langle \sigma^{(1)}, R_{B(K)}(B) \rangle_K \right). \tag{3.38}$$

By definition of the de Rham map,

$$\langle \sigma^{(1)}, R_{B(K)}(B) \rangle_K = \int_{\sigma^{(1)}} B. \tag{3.39}$$

This defines an element of the Lie algebra for each $\sigma^{(1)} \in C_1(B(K))$,

$$\int_{\sigma^{(1)}} B =: X_{\sigma^{(1)}} \in \mathfrak{g}. \tag{3.40}$$

Using 3.3.10 and the skew-symmetry of the wedge product once more, we can rewrite the field-strength contribution

$$\begin{aligned} \langle \star_{\star K} R_{B(K)}(F), \sigma^{(1)} \rangle_K &= \int_{\mathcal{M}} W_{B(K)} \left(E(\sigma^{(1)}) \right) \wedge W_{B(K)} \left(R_{B(K)}(F) \right) \\ &= \langle R_{B(K)}(F), \star_K \sigma^{(1)} \rangle_{\star K} \\ &= \int_{\star_K \sigma^{(1)}} F, \end{aligned} \tag{3.41}$$

where in the last step we used the de Rham map. Putting everything together, we find that

$$S_{BF}[B,A] = \sum_{\sigma^{(1)} \in C_1(B(K))} \int_{\star_K \sigma^{(1)}} \operatorname{tr}(F X_{\sigma^{(1)}}), \tag{3.42}$$

which completes the proof. □

This defines the action for BF theory on a simplicial complex K. Remarkably, (3.42) is exact and in particular *independent of the chosen triangulation K of $\mathcal{M}$*. This is a particular feature of topological theories and can not be expected to hold for gravity in dimensions higher than 3.

Eventhough (3.42) can already be taken as a starting point for a path integral, I will in the last part of this section introduce a slight refinement of variables using the barycentric refinement $B(K)$. This will come in handy in section 3.4, where we will deal with a generating functional and couple the theory to an external current. First, let us define a set of two-cells called *wedges*:

Definition 3.3.12. *Let* $[v_0, ... v_3] = \sigma_v^{(3)}$ *be a tetrahedron in* $C_3(K)$. *Call its barycentre* $v = \hat{\sigma}^{(3)}$ *a* vertex. *Label its four boundary triangles as*

$$\begin{aligned}
\sigma_{0,v}^{(2)} &= [v_1, v_2, v_3], \\
\sigma_{1,v}^{(2)} &= [v_0, v_2, v_3], \\
\sigma_{2,v}^{(2)} &= [v_0, v_1, v_3], \\
\sigma_{3,v}^{(2)} &= [v_0, v_1, v_2],
\end{aligned} \tag{3.43}$$

with respective barycentres $b_i := \hat{\sigma}_{i,v}^{(2)}$. *Label the three boundary edges of each triangle as* $\sigma_{ij,v}^{(1)} := \sigma_{i,v}^{(2)} \cup \sigma_{j,v}^{(2)}$, *with barycentres* $b_{ij} := \hat{\sigma}_{ij,v}^{(1)}$. *Then the* wedge w_{ij}^v *is defined as*

$$w_{ij}^v := [v, b_i, b_{ij}] \cup [v, b_{ij}, b_j]. \tag{3.44}$$

The construction of a wedge is illustrated in Fig. 3.1. A wedge can be understood as a two-cell given by the union of two triangles belonging to the baryonic refinement $B(K)$. It is bounded by the loop

$$\partial w_{ij}^v = [v, b_i] \circ [b_i, b_{ij}] \circ [b_{ij}, v_j] \circ [b_j, v]. \tag{3.45}$$

To make use of these two-cells, we first decompose K into tetrahedra $\sigma_v^{(3)}$, one for each vertex v, and *thereafter* apply theorem 3.3.11:

$$\begin{aligned}
\int_{\mathcal{M}} \mathrm{tr}(B \wedge F) &= \sum_{\sigma_v^{(3)} \in C_3(K)} \int_{\sigma_v^{(3)}} \mathrm{tr}(B \wedge F) \\
&= \sum_{\sigma_v^{(3)} \in C_3(K)} \left[\sum_{\sigma_v^{(1)} \in C_1(\sigma_v^{(3)})} \mathrm{tr} \left(\int_{\star_{\sigma_v^{(3)}} \sigma_v^{(1)}} F \int_{\sigma_v^{(1)}} B \right) \right]
\end{aligned} \tag{3.46}$$

Now we note that the $\sigma_v^{(1)}$ are exactly the edges $\sigma_{ij,v}^{(1)}$, and their duals $\star_{\sigma_v^{(3)}} \sigma_v^{(1)}$ with

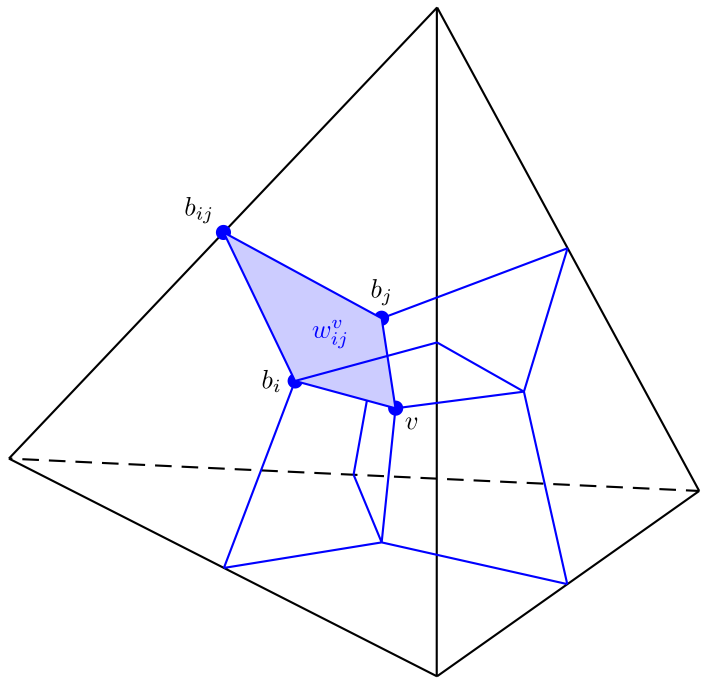


Figure 3.1: Definition of a wedge. The tetrahedron $\sigma_v^{(3)}$ is indicated in black, elements of the dual complex in blue.

respect to $\sigma_v^{(3)}$ are the wedges w^v_{ij} as defined in 3.3.12. Given their one-to-one correspondence, we can thus label all variables by the wedges alone, and sum over all wedges in the complex, writing in short-hand

$$S_{BF}[A,B] = \sum_w \int_w \mathrm{tr}(FX_w). \tag{3.47}$$

Again, this identity holds exactly, and is independent of the choice of K. This provides our starting point for the path integral quantization in the next section, and thanks to the exactness of the result, an otherwise required check that our choice of discretization approximates the initial continuum action becomes redundant.

3.4 Path integral quantization

We will now use the discretized form of the action to obtain a well-defined expression for the path integral of BF theory. For the sake of mathematical simplicity, in what follows we will deal with defining the path integral of BF theory for *Riemannian* signature, i.e. choose $\mathfrak{g} = \mathfrak{su}(2)$. At the classical level, it was irrelevant whether the corresponding gauge group was $SU(2)$ or $SO(3)$. This choice will however affect properties of the quantum theory. Historically, also with the motivation of coupling spinor fields to the theory, the choice $G = SU(2)$ was favored [Ponzano and Regge, 1968], which we shall stick with in the following. Note that a path integral for Riemannian signature is a different notion from the Euclidean path integral obtained from Wick rotation in quantum field theory. The latter defines a thermodynamic partition function, while in contrast the former remains a quantum-mechanical path integral, with a factor i in front of the action. Formally, we can see the importance of the factor i by noting that this procedure correctly implements the flatness constraint on the connection by means of a delta-functional,

$$\int \mathcal{D}A\mathcal{D}B e^{i\int_{\mathcal{M}} \mathrm{tr}(B\wedge F)} \propto \int \mathcal{D}A\delta(F). \tag{3.48}$$

The distribution $\delta(F)$ then enforces the condition that the holonomy of every contractible loop is trivial. To proceed, following [Freidel and Krasnov, 1999], we introduce a source two-form J and define the *generating functional*,

$$\begin{aligned} Z[J] &= \int \mathcal{D}A\mathcal{D}B e^{iS[J]}, \\ S[J] &= \int_{\mathcal{M}} [\mathrm{tr}(B\wedge F) + \mathrm{tr}(B\wedge J)], \end{aligned} \tag{3.49}$$

for which we now attempt to find a lattice counterpart using the results of section 3.3. Introducing a triangulation K of $\mathcal{M}$, the action in discrete variables reads

$$S[J] = \sum_w \int_w [\mathrm{tr}(FX_w) + \mathrm{tr}(JX_w)]\,. \tag{3.50}$$

Next, we need to specify a suitable integration measure. Since $X_w \in \mathfrak{su}(2)$,

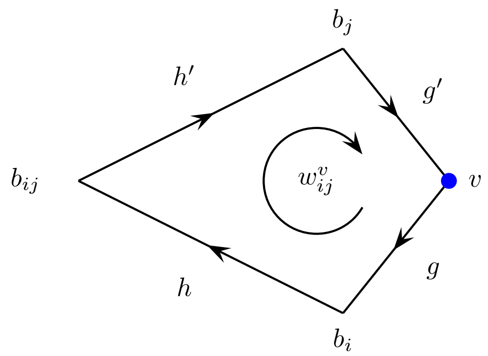


Figure 3.2: Holonomy of the connection around a wedge.

the discrete B-field will by integrated over the Lie algebra of $SU(2)$ for each wedge. To find a way to integrate over the connection, we will have to invoke an approximation, taking inspiration from ordinary lattice gauge theory. Consider a single wedge w^v_{ij}, and define the group elements

$$\begin{aligned} g &:= \mathcal{P}\exp\left(\int_{[v,b_i]} A\right), & h &:= \mathcal{P}\exp\left(\int_{[b_i,b_{ij}]} A\right), \\ h' &:= \mathcal{P}\exp\left(\int_{[b_{ij},b_j]} A\right), & g' &:= \mathcal{P}\exp\left(\int_{[b_j,v]} A\right), \end{aligned} \tag{3.51}$$

where $\mathcal{P}$ denotes path-ordering. Then, their product

$$ghh'g' := U_w \in SU(2), \tag{3.52}$$

gives the holonomy of the connection around the wedge[1], as illustrated in fig. 3.4. For U_w close to the identity [Göckeler and Schucker, 1989],

$$\begin{aligned} &U_w \approx \mathbb{1} + \int_w F, \\ &\mathrm{tr}(U_w X_w) \approx \mathrm{tr}(X_w + \int_w F X_w) = \mathrm{tr}(\int_w F X_w), \end{aligned} \tag{3.53}$$

where U_w was implicitly taken in the fundamental representation of $SU(2)$ and

[1]There is an ambiguity in choosing the base point of the holonomy. Here, it has been chosen so as to ensure the invariance of (3.54) under the discrete counterpart of gauge transformations (cf. [Freidel and Krasnov, 1999])

tr is now generally the trace on 2×2 complex matrices. For the second equation, we used the fact that elements of $\mathfrak{su}(2)$ are traceless. Using this approximation, and abbreviating $J_w := \int_w J$, the action becomes

$$S[J] = \sum_w [\mathrm{tr}(U_w X_w) + \mathrm{tr}(J_w X_w)]. \tag{3.54}$$

Note that at this point, the correspondence between (3.54) and the initial action is *not* exact anymore. The approximation (3.53) is however remniscient of the procedure in lattice gauge theory to define a lattice ation, and lattice refinements should improve this approximation. Moreover, off-shell configurations with holonomies U_w increasingly far from the identity should be subject to increasingly distructive interference in the path integral, since classically, only flat connections are solutions.

Using (3.54), our set of variables is a collection of group elements (g's and h's) corresponding to the connection, and a collection of Lie algebra elements corresponding to the B-field. For integrating functions of group elements, we can naturally employ the Haar measure dg on $L^2[SU(2)]$, and for the Lie algebra the usual Lesbegue measure on $\mathbb{R}^3$. We can thus finally *define* the path integral as:

$$Z[J] = \left[\prod \int_{SU(2)} dg \int_{SU(2)} dh\right] \left[\prod_w \int_{\mathfrak{su}(2)} dX_w\right] e^{i \sum_w [\mathrm{tr}(U_w X_w) + \mathrm{tr}(J_w X_w)]}. \tag{3.55}$$

with dg the Haar measure and dX_w the Lesbegue measure on $\mathfrak{su}(2)$, and the first product is understood to run over all g- and h-variables in $B(K)$. Before carrying out the integrals, a gauge-fixing procedure would be necessary to avoid an overcounting of physically equivalent configurations. I will at this point however work with (3.54) directly, without deriving a Faddeev-Popov determinant. As a result of the noncompactness of the translational symmetry (3.8), this will leave us with a divergent expression for the partition function. In this thesis, we will only be concerned with the formal expression to relate our results to the structure of group field theory later on. I refer the reader to [Freidel and Louapre, 2004] and [Freidel and Livine, 2003] for a careful treatment of symmetries and gauge-fixing, which in the end is indispensable to obtain a well-defined partition function.

We first perform the integration over the Lie algebra, given by a product of integrals of the form

$$\int_{\mathfrak{su}(2)} dX e^{i\mathrm{tr}(UX+JX)}. \tag{3.56}$$

To rewrite this as an explicit function of g's and h's, we use the identity

$$\int_{\mathfrak{su}(2)} dX e^{i\mathrm{tr}(gX)} = 4\pi[\delta_{SU(2)}(g) + \delta_{SU(2)}(-g)] \quad \forall g \in SU(2), \tag{3.57}$$

which is proven in Appendix B. We see that the support of the integrand is on $\mathbb{1}$ and $-\mathbb{1}$, effectively giving the delta function on $SO(3)$. The general approach in the literature [Freidel and Krasnov, 1999; Freidel and Louapre, 2004] has been to replace this expression by hand with a delta function on $SU(2)$,

$$\int_{\mathfrak{su}(2)} dX e^{i\mathrm{tr}(gX)} \to \delta_{SU(2)}(g). \tag{3.58}$$

This was mainly motivated by the fact that this choice yields the original Ponzano-Regge model [Ponzano and Regge, 1968]; let us however note that this is an *ad hoc* manipulation that does not derive from first principles. Using the Plancherel decomposition

$$\delta_{SU(2)}(g) = \sum_j (2j+1)\chi^j(g), \qquad j \in \frac{\mathbb{N}_0}{2}, \tag{3.59}$$

where $\chi^j(g)$ denotes the $SU(2)$ character for the j-representation, the partition function takes the form[1]

$$Z[J] = \left[\prod \int_{SU(2)} dg \int_{SU(2)} dh\right] \prod_w \sum_{j_w} (2j_w+1)\chi^{j_w}(U_w e^{J_w}). \tag{3.60}$$

Next, we will carry out the integration over the h's. Consider two adjacent wedges belonging to the *same* face of the dual complex, as shown in Fig. 3.3. The integral

[1] Had we proceeded with the original form (3.58), the sum over representations would include only $j \in \mathbb{N}_0$ – yielding representations of $SO(3)$ only.

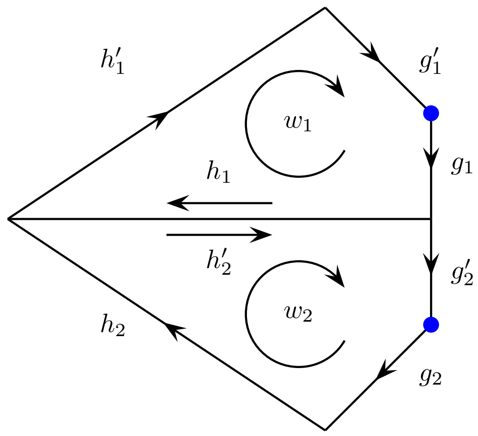


Figure 3.3: Two adjacent wedges. The blue dots indicate the holonomy base points.

over h_1 is of the form

$$...\int_{SU(2)} dh_1 \chi^{j_1}(g_1 h_1 h'_1 g'_1 e^{J_1}) \chi^{j_2}(g_2 h_2 h'_2 g'_2 e^{J_2})... \tag{3.61}$$

From Fig. 3.3, one can see that $h_1^{-1} = h'_2$, as both group elements are assigned to the same edge shared by the two wedges. The unitarity of the Wigner matrices then implies that $D^j_{ab}(h'_2) = \overline{D}^j_{ba}(h_2'^{-1}) = \overline{D}^j_{ba}(h_1)$. Expanding the characters into representation matrices, the integration thus takes the form

$$\begin{aligned} ...\int_{SU(2)} dh_1 D^{j_1}_{ab}(g_1) D^{j_1}_{bc}(h_1) D^{j_1}_{cd}(h'_1) D^{j_1}_{de}(g'_1) D^{j_1}_{ea}(e^{J_1}) \\ D^{j_2}_{fg}(g_2) D^{j_2}_{gh}(h_2) \overline{D}^{j_2}_{ih}(h_1) D^{j_2}_{ij}(g'_2) D^{j_2}_{jf}(e^{J_2})... \end{aligned} \tag{3.62}$$

Using Schur's orthogonality relation for the Wigner matrices (A.4) and keeping track of the resulting matrix contractions, the character expression after integration over h_1 is

$$...\frac{1}{2j_1+1} \chi^{j_1}(h'_1 g'_1 e^{J_1} g_1 g'_2 e^{J_2} g_2 h_2)... \tag{3.63}$$

In this expression, the group element $g_1 g'_2$ corresponds to the path-ordered exponential along the 1-simplex $[v_1, b_i] \circ [b_i, v_2] \equiv \star_K \sigma_i^{(2)}$ connecting the barycentres v_1, v_2 of the tetrahedra the two neighbouring wedges belong to, with $\sigma_i^{(2)}$ being their shared triangle. We can make a change of variables defining $g_1 g_2' := g_{\sigma_i^{(2)}}$ as the group element associated to $\star_K \sigma_i^{(2)}$. Repeating these steps for all wedges in the complex, we arrive at

$$\begin{aligned} Z[J] = \prod_{\sigma_i^{(2)}} \int_{SU(2)} dg_{\sigma_i^{(2)}} \prod_{\sigma^{(1)}} \sum_{j_{\sigma^{(1)}}} (2j_{\sigma^{(1)}} + 1) \\ \chi^{j_{\sigma^{(1)}}} \left(\exp(J^1_{\sigma^{(1)}}) g_{\sigma_1^{(2)}} ... \exp(J^{N(\sigma^{(1)})}_{\sigma^{(1)}}) g_{\sigma^{(2)}_{N(\sigma^{(1)})}} \right) \end{aligned} \tag{3.64}$$

Here, $\prod_{\sigma^{(1)}}$ indicates the product over all 1-simplices in K or, equivalently, all faces of the dual complex. $N(\sigma^{(1)})$ the number of edges composing the boundary of $\star_K \sigma^{(1)}$. At this stage, each edge $\sigma^{(1)} \in C_1(K)$ (equivalently, each face $\star_K \sigma^{(1)} \in C_2(\star K)$) is labelled by a representation label $j_{\sigma^{(1)}}$, and each face $\sigma^{(2)} \in C_2(K)$ (equivalently, each edge $\star_K \sigma^{(2)} \in C_1(\star K)$) by a group element $g_{\star_K \sigma^{(2)}}$ which we will now integrate over. Since every face in K is bounded by three edges, upon expanding the corresponding characters, each group element will appear three times in mutually uncontracted Wigner matrices. All g-integrals will thus be of the form

$$\begin{aligned} & \int_{SU(2)} dg \chi^{j_1}(...e^{J_1} g e^{J_1'}...) \chi^{j_2}(...e^{J_2} g e^{J_2'}...) \chi^{j_3}(...e^{J_3} g e^{J_3'}...) \\ = & \int_{SU(2)} dg D^{j_1}_{a_1 m_1}(...e^{J_1}) D^{j_1}_{m_1 m_1'}(g) D^{j_1}_{m_1' a_1}(e^{J_1'}...) \\ & \times D^{j_2}_{a_2 m_2}(...e^{J_2}) D^{j_2}_{m_2 m_2'}(g) D^{j_2}_{m_2' a_2}(e^{J_2'}...) \\ & \times D^{j_3}_{a_3 m_3}(...e^{J_3}) D^{j_3}_{m_3 m_3'}(g) D^{j_3}_{m_3' a_3}(e^{J_3'}...) \\ = & D^{j_1}_{a_1 m_1}(...e^{J_1}) D^{j_2}_{a_2 m_2}(...e^{J_2}) D^{j_3}_{a_3 m_3}(...e^{J_3}) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ & \times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} D^{j_1}_{a_1 m_1'}(e^{J_1'}...) D^{j_2}_{a_2 m_2'}(e^{J_2'}...) D^{j_3}_{a_3 m_3'}(e^{J_3'}...), \end{aligned} \tag{3.65}$$

where the primed quantities refer to one tetrahedron, and the unprimed to the other sharing the same face. The spins j_1, j_2 and j_3 label the respective edges bounding that face. In the last step we used the identity (A.10) relating the group integral of three Wigner matrices to the product of two Wigner $3j$-symbols. Finally, we can regroup the factors in the partition function into common factors for each tetrahedron. From (3.65), we see that for each tetrahedron, each of the four faces comes with a factor given by three Wigner matrices and a Wigner $3j$-symbol. Any two adjacent faces will however share one of these representation matrices and the corresponding current. We can thus introduce the function

$$\begin{aligned} A(\{j_i, J_i\}) &= \prod_{i=1}^{6} D^{j_i}_{m_i n_i}(e^{J_i}) \\ &\times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ n_1 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & n_5 & n_3 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ n_4 & n_2 & n_6 \end{pmatrix} \end{aligned} \tag{3.66}$$

to express the generating functional as a product over edges $\sigma^{(1)}$ and tetrahedra $\sigma^{(3)}$,

$$Z[J] = \prod_{\sigma^{(1)}} \sum_{j_{\sigma^{(1)}}} (2j_{\sigma^{(1)}} + 1) \prod_{\sigma^{(3)}} A(\{j_i^{\sigma^{(3)}}, J_i^{\sigma^{(3)}}\}). \tag{3.67}$$

This is the final expression for the generating functional. Interistingly enough, this is a purely combinatoric formula, containing only half-integers as a consequence of the compactness of $SU(2)$. From this, a variety of computations can be carried out, e.g. expectation values of observables or perturbation theory for interactions. Most straightforwardly, we can obtain the partition function for three-dimensinal BF theory, that is, the partition function for quantum gravity in three dimensions. Using the identities (A.3) and the reflection symmetry of the $3j$-symbold to invoke the definition (A.12) of the Wigner $6j$-symbol in terms of four $3j$-symbols. We then find

$$Z_{BF} = Z[J]|_{J=0} = \prod_{\sigma^{(1)}} \sum_{j_{\sigma^{(1)}}} (2j_{\sigma^{(1)}} + 1) \prod_{\sigma^{(3)}} \begin{Bmatrix} j_1^{\sigma^{(3)}} & j_2^{\sigma^{(3)}} & j_3^{\sigma^{(3)}} \\ j_4^{\sigma^{(3)}} & j_5^{\sigma^{(3)}} & j_6^{\sigma^{(3)}} \end{Bmatrix}. \tag{3.68}$$

It can be shown that the expression (3.68) depends on the topology only [Barrett and Naish-Guzman, 2009]. This means the above quantization procedure appropriately preserved the topological character of BF theory, and that upon regularization the partition function actually gives a *topological invariant* of K, and thus $\mathcal{M}$ [Turaev and Viro, 1992].

Chapter 4

Group Field Theory

The last chapter has proceeded very much in the spirit of lattice gauge theory, which in a way is an orthogonal approach to evaluating physical quantities by a perturbative expansion in terms of Feynman diagrams. We are now arguing that in the context of quantum gravity, both notions might well be the same [Reisenberger and Rovelli, 2001]. This is the key conceptual step that allows us to interpret spacetime lattices as Feynman diagrams of an underlying field theory. The core structure of Feynman rules can be described as follows:

- Lines corresponding to *particles* are labelled by *irreducible representations* ρ of the Lorentz group $SL(2,\mathbb{C})$. *Antiparticles* are represented by lines labelled with the respective *dual representation* ρ^*. A set of n particles is then depicted by a set of n lines labelled with the tensor product $\rho_1 \otimes ... \otimes \rho_n$.

- An *interaction* with n ingoing and m outgoing particles is described by a vertex with n ingoing and m outgoing lines, labelled by an *intertwiner*

$$T: \quad \rho_1 \otimes ... \otimes \rho_n \longrightarrow \rho'_1 \otimes ... \otimes \rho'_m.$$

- An *amplitude* is then given by a *labelled graph*, in which the representation labels are mutually contracted as dictated by the vertex structure.

For instance, quantum electrodynamics is described by a spinor field ${\psi^a}_b$ in the fundamental representation ρ_ψ of $SL(2,\mathbb{C})$ and a gauge field A_μ in a four-dimensional real representation ρ_A. The three-point interaction is encoded in the structure of the gamma matrices

$$(\gamma^\mu)_a{}^b: \quad \rho_\psi \otimes \rho_\psi \longrightarrow \rho_A.$$

Now, the partition function of BF theory on a simplicial complex derived in the last chapter has the same structure, except that $SU(2)$ replaces $SL(2,\mathbb{C})$: the simplicial complex with its geometrical data is in essence a graph labelled by unitary representations j_i of $SU(2)$. The $3j$- and $6j$-symbols are intertwiners that map inbetween 3 and 6 representations, respectively, because they carry 3 and 6 spin labels. The evaluation of the partition function is fully determined by the representation labels and the contraction rules. In this light, a lattice refinement can also be depicted as going to a higher order in perturbation theory corresponding to a larger number of vertices. From this point of view, the model proposed falls into the general class of *spin foam models*, a term first coined by Baez in light of this interpretation [Baez, 1998].

This chapter will finally use the introduced concepts to present a potential generalization of matrix models to gravity in three dimensions, and it turns out that these proposals are exactly in line with the perspective outlined above. A major challenge is that in dimensions higher than two, it becomes notoriously hard to control topology and indeed as of now, the classification of topologies in $D \geq 3$ remains an open mathematical problem. Consequently, the naïve generalization of matrices to three-index tensors [Ambjorn et al., 1991; Godfrey and Gross, 1991; Sasakura, 1991] was soon realized to contain an insufficient number of free parameters to perform a topological expansion. As we have seen in the first chapter, the corroboration of the presence of a universal critical point allowing for a continuum limit rested crucially on the possibility to expand in genera at large N. To this end, Boulatov proposed a different type of model [Boulatov, 1992], which belongs to a general class of theories now commonly referred to as *group field theory* [see e.g. review in Freidel, 2005; Oriti, 2009]. Defining such a theory requires the choice of a group G, and different choices lead to different topological invariants as weights for the complexes. This choice plays the rôle of an additional parameter, endowing group field theory with a richer structure than tensor models. After a warm-up, this chapter will focus on the particular choice $G = SU(2)$ motivated from insights of the last chapter. The lattice amplitudes for gravity will then appear as the Feynman diagram amplitudes in the perturbative expansion of the corresponding group field theory.

4.1 Two dimensions

Before discussing a model for a three-dimensional generalization of matrix models, it is instructive to cast the matrix model presented in the first chapter into the formalism of group field theory. The generalization to higher dimensions then appears in a more straightforward way. To this purpose, let us consider a dynamical variable in the space of square integrable functions on two disjoint circles $\phi \in L^2[U(1) \times U(1)]$,

$$\phi : \quad U(1) \times U(1) \quad \longrightarrow \quad \mathbb{C}, \tag{4.1}$$

with the property that $\phi(g_1, g_2) = \phi^*(g_2, g_1) \quad \forall g_1, g_2 \in U(1)$. Let the dynamics be governed by the action

$$\begin{aligned} S[\phi] &= T[\phi] + V[\phi]; \\ T[\phi] &= \frac{1}{2} \int_{U(1)\times U(1)} dg_1 dg_2 |\phi(g_1, g_2)|^2, \\ V[\phi] &= \frac{\lambda}{3!} \int_{U(1)\times U(1)\times U(1)} dg_1 dg_2 dg_3 \phi(g_1, g_2)\phi(g_2, g_3)\phi(g_3, g_2). \end{aligned} \tag{4.2}$$

We can expand the field in fourier modes $\chi^m(g) = e^{img}$ on $U(1) \cong S^1$,

$$\phi(g_1, g_2) = M^i{}_j \overline{\chi}_i(g_1)\chi^j(g_2). \tag{4.3}$$

Here and in the following sections, we will always sum over repeated indices unless stated otherwise. The symmetry requirement on the field translates into the hermiticity of the Fourier coefficients, $M^i{}_j = M^*{}_j{}^i$. Using the orthogonality of the $U(1)$ characters,

$$\int_{U(1)} dg \overline{\chi}_i(g)\chi^j(g) = \delta_i^j, \tag{4.4}$$

one finds that

$$\begin{aligned} T[\phi] &= \frac{1}{2} M^j{}_k M^*{}_l{}^m \delta^l_j \delta^k_m, \\ V[\phi] &= \frac{\lambda}{3!} M^i{}_j M^k{}_l M^m{}_n \delta^n_i \delta^j_k \delta^l_m. \end{aligned} \tag{4.5}$$

The Fourier transformed action thus reads

$$S[\phi] = \frac{1}{2} \mathrm{tr} M^2 + \frac{\lambda}{3!} \mathrm{tr} M^3. \tag{4.6}$$

In fact, since there is an infinity of modes on the circle, giving rise to an infinite number of representations of $U(1)$, the matrices in (4.6) are of infinite size. This action thus corresponds to the $N \to \infty$ limit of the initial matrix model action (2.6) in chapter 2. We can use a cutoff on the mode expansion (4.3), letting m and n run to a finite value N. Indeed, renormalization group transformations for matrix models have been studied by integrating out matrix components, giving a change in scale $N+1 \to N$ [Brézin and Zinn-Justin, 1992]. This transformation acquires a natural interpretation in changing the cutoff scale for the Fourier modes of group field theory; the $N \to \infty$ limit simply corresponds to removing the cutoff.

To prepare for the three-dimensional model, let us now replace $U(1)$ by $SU(2)$ to arrive at a different toy model for two-dimensional random surfaces. As a dynamical variable, we take a field $P[\phi(g_1, g_2)]$, where P is the projector

$$P[\phi(g_1, g_2)] := \int_{SU(2)} dh [\phi(g_1 h, g_2 h) + \phi^*(g_2 h, g_1 h)] \tag{4.7}$$

projecting an arbitrary ϕ onto its hermitian right-invariant part, such that $P[\phi(g_1, g_2)] \in L^2[(SU(2) \times SU(2))/SU(2)]$, where $(SU(2) \times SU(2))/SU(2)$ is understood as a *right coset*. We start from accordingly modified kinetic and potential terms,

$$\begin{aligned} T[\phi] &= \frac{1}{2} \int_{SU(2)^{\times 2}} dg_1 dg_2 |P[\phi(g_1, g_2)]|^2, \\ V[\phi] &= \frac{\lambda}{3!} \int_{SU(2)^{\times 3}} dg_1 dg_2 dg_3 P[\phi(g_1, g_2)] P[\phi(g_2, g_3)] P[\phi(g_3, g_2)]. \end{aligned} \tag{4.8}$$

One can again expand ϕ into modes,

$$\phi(g_1, g_2) = \phi^{a_1 a_2 b_1 b_2}_{j_1 j_2} \overline{D}^{j_1}_{a_1 b_1}(g_1) R^{j_2}_{a_2 b_2}(g_2). \tag{4.9}$$

The Fourier transform on the coset space can be found by first expanding ϕ and then implementing the projecter P: using the orthogonality of the D-matrices (A.4), the projected field can then be written as

$$\begin{aligned} &P[\phi(g_1, g_2)] \\ &\quad = (\phi^{a_1 b_1 a_2 b_2}_{j_1 j_2} + \phi^{*a_2 b_2 a_1 b_1}_{j_2 j_1}) \overline{D}^{j_1}_{a_1 c_1}(g_1) D^{j_2}_{a_2 c_2}(g_2) \int_{SU(2)} dh \overline{D}^{j_1}_{c_1 a_1}(h) D^{j_2}_{c_2 a_2}(h) \\ &\quad = \sqrt{2j+1}\Phi^{ab}_j \overline{D}^j_{ac}(g_1) D^j_{bc}(g_2), \end{aligned} \tag{4.10}$$

where we defined

$$\Phi^{a_1 a_2}_{j_1} := \frac{1}{\sqrt{2j_1+1}} (\phi^{a_1 b_1 a_2 b_2}_{j_1 j_2} + \phi^{*a_2 b_2 a_1 b_1}_{j_2 j_1}) \delta^{j_1 j_2} \delta_{b_1 b_2} \tag{4.11}$$

and renamed indices. The components of the tensor Φ are the Fourier coefficients for the mode decomposition of elements of $L^2[(SU(2) \times SU(2))/SU(2)]$. They inherit the field symmetries, $\Phi = \Phi^\dagger$. The kinetic term can then be written as

$$\begin{aligned} S_K &= \frac{1}{2}\sqrt{2j+1}\sqrt{2k+1}\Phi^{ab}_j \Phi^{*cd}_k \\ &\times \int_{SU(2)^{\times 2}} dg_1 dg_2 \overline{D}^j_{am}(g_1) D^j_{bm}(g_2) D^k_{cn}(g_1) \overline{D}^k_{dn}(g_2) \\ &= \frac{1}{2}\Phi^{ab}_j \Phi^{*ab}_j. \end{aligned} \tag{4.12}$$

The potential becomes

$$\begin{aligned} S_I &= \frac{\lambda}{3!}\sqrt{2j_1+1}\sqrt{2j_2+1}\sqrt{2j_3+1}\int_{SU(2)^{\times 3}} dg_1 dg_2 dg_3 \\ &\times \overline{D}^{j_1}_{a_1 c_1}(g_1) D^{j_1}_{b_1 c_1}(g_2) \overline{D}^{j_2}_{a_2 c_2}(g_2) D^{j_2}_{b_2 c_2}(g_3) \overline{D}^{j_3}_{a_3 c_3}(g_3) D^{j_3}_{b_3 c_3}(g_1) \\ &= \frac{\lambda}{3!\sqrt{2j+1}} \Phi^{a_1 a_2}_j \Phi^{a_2 a_3}_j \Phi^{a_3 a_1}_j, \end{aligned} \tag{4.13}$$

giving the action

$$S = \sum_j \left(\frac{1}{2} \mathrm{tr}\Phi_j^2 + \frac{\lambda}{3!\sqrt{N}} \mathrm{tr}\Phi_j^3 \right) \tag{4.14}$$

for an infinite tower of noninteracting $N \times N$ hermitian matrix models labelled by j, with matrix size $N = 2j + 1$. After a rescaling $\sqrt{N} M_j := \Phi_j$, we recover the initial form of the matrix action (2.6) from the first chapter, $S = \sum_j N \left(\mathrm{tr} M_j^2/2 + \lambda \mathrm{tr} M_j^3/3! \right)$.

4.2 Three dimensions

For a model of three-dimensional gravity, we again choose $G = SU(2)$, but now a field

$$\phi : \quad SU(2)^{\times 3} \quad \longrightarrow \quad \mathbb{C}. \tag{4.15}$$

Let us demand the reality condition $\phi(g_1, g_2, g_3) = \phi^*(g_1, g_2, g_3)$, yielding unoriented lines in the diagrammatic expansion. Again, we will also demand $SU(2)$-*right-invariance:*

$$\phi(g_1, g_2, g_3) = \phi(g_1 h, g_2 h, g_3 h) \quad \forall h \in SU(2) \tag{4.16}$$

and furthermore, for a permutation $\pi \in S_3$ with signature $|\pi|$,

$$\phi(g_1, g_2, g_3) = (-1)^{|\pi|} \phi(g_{\pi(1)}, g_{\pi(2)}, g_{\pi(3)}). \tag{4.17}$$

Equation (4.16) implies that $\phi \in L^2[SU(2)^{\times 3}/SU(2)]$. For notational simplicity, we will in the following assume that ϕ posssesses the required symmetries. Yet again, these can always be explicity imposed on an arbitrary field ϕ' through the projector

$$\begin{aligned} \phi(g_1, g_2, g_3) &= P[\phi'(g_1, g_2, g_3)] \\ &:= \frac{1}{|S_3|} \sum_{\pi \in S_3} (-1)^{|\pi|} \int_{SU(2)} dh \phi'(g_{\pi(1)} h, g_{\pi(2)} h, g_{\pi(3)} h), \end{aligned} \tag{4.18}$$

where $|S_3| = 6$ is the order of S_3. We will consider an action of ϕ^4-type defined by

$$\begin{aligned}
S[\phi] &= T[\phi] + V[\phi], \\
T[\phi] &= \frac{1}{2}\left[\prod_{i=1}^{3}\int_{SU(2)} dg_i\right]\phi(g_1,g_2,g_3)^2, \\
V[\phi] &= \frac{\lambda}{4!}\left[\prod_{i=1}^{6}\int_{SU(2)} dg_i\right]\phi(g_1,g_2,g_3)\phi(g_5,g_4,g_3)\phi(g_5,g_1,g_6)\phi(g_2,g_4,g_6).
\end{aligned} \tag{4.19}$$

Our goal is to examine the structure of the perturbative expansion of the quantity

$$Z = \int \mathcal{D}\phi e^{-S[\phi]}, \tag{4.20}$$

where the integration measure remains to be specified. For this purpose, we will expand the field in Fourier modes and derive the Feynman rules. The mode expansion on the coset space $SU(2)^{\times 3}/SU(2)$ can again be found by first using the Peter-Weyl theorem to expand the field and then implementing the invariance property. Leaving the permutation symmetry of the field implicit for notational simplicity, we find

$$\begin{aligned}
\phi(g_1,g_2,g_3) &= P[\phi'(g_1,g_2,g_3)] \\
&= \phi'^{m_1k_1m_2k_2m_3k_3}_{j_1j_2j_3} D^{j_1}_{m_1n_1}(g_1)D^{j_2}_{m_2n_2}(g_2)D^{j_3}_{m_3n_3}(g_3) \\
&\times \int_{SU(2)} dh D^{j_1}_{n_1k_1}(h)D^{j_2}_{n_2k_2}(h)D^{j_3}_{n_3k_3}(h) \\
&= \sqrt{d_{j_1}d_{j_2}d_{j_3}}\,\Phi^{m_1m_2m_3}_{j_1j_2j_3} D^{j_1}_{m_1n_1}(g_1)D^{j_2}_{m_2n_2}(g_2)D^{j_3}_{m_3n_3}(g_3) \\
&\times \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}.
\end{aligned} \tag{4.21}$$

Here we used the integral representation (A.10) of the product of two $3j$-symbols, abbreviated $d_j = (2j+1)$ and defined the coset space Fourier coefficients as

$$\Phi^{m_1 m_2 m_3}_{j_1 j_2 j_3} := \frac{1}{\sqrt{d_{j_1} d_{j_2} d_{j_3}}} \phi'^{m_1 k_1 m_2 k_2 m_3 k_3}_{j_1 j_2 j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}. \tag{4.22}$$

The symmetries of the group field induce symmetries on the Fourier coefficients[1],

$$\begin{aligned} \Phi^{*m_1 m_2 m_3}_{j_1 j_2 j_3} &= (-1)^{\sum_i (j_i + m_i)} \Phi^{-m_1 - m_2 - m_3}_{j_1 j_2 j_3}, \\ \Phi^{m_{\pi(1)} m_{\pi(2)} m_{\pi(3)}}_{j_{\pi(1)} j_{\pi(2)} j_{\pi(3)}} &= (-1)^{|\pi|} \Phi^{m_1 m_2 m_3}_{j_1 j_2 j_3}. \end{aligned} \tag{4.23}$$

We can now again use the orthogonality relation for the representation matrices (A.4) and the 3j-symbols (A.11) to find the Fourier transformed kinetic term:

$$\begin{aligned} T[\phi] = & \sqrt{d_{j_1} d_{j'_1} ... d_{j_3} d_{j'_3}} \Phi^{m_1 m_2 m_3}_{j_1 j_2 j_3} \Phi^{m'_1 m'_2 m'_3}_{j'_1 j'_2 j'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} j'_1 & j'_2 & j'_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix} \\ & \times \left[\prod_{i=1}^{3} \int_{SU(2)} dg_i D^{j_i}_{m_i n_i}(g_i) \overline{D}^{j'_i}_{m'_i n'_i}(g_i) \right] \\ & = |\Phi^{m_1 m_2 m_3}_{j_1 j_2 j_3}|^2 \end{aligned} \tag{4.24}$$

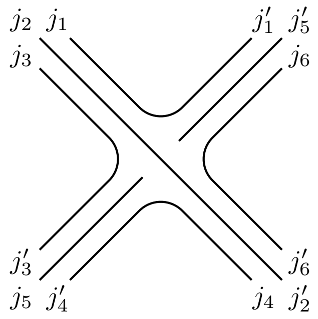
Expanding the potential term in Fourier modes and exploiting the permutation (A.8) and reflection (A.9) symmetries of the 3j-symbol gives

$$V[\phi] = \frac{\lambda}{4!} (-1)^{\sum_i (j_i + m_i)} \Phi^{-m_1 - m_2 - m_3}_{j_1 j_2 j_3} \Phi^{m_3 - m_5 m_4}_{j_3 j_5 j_4} \Phi^{-m_4 m_2 m_6}_{j_4 j_2 j_6} \Phi^{-m_6 m_5 m_1}_{j_6 j_5 j_1} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{4.25}$$

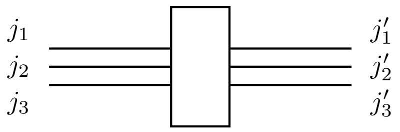
A natural integration measure for the partition function is thus

$$\mathcal{D}\phi = \prod_{\{j_1, j_2, _3\}} \prod_{\{-j \leq m \leq j\}} d\Phi^{m_1 m_2 m_3}_{j_1 j_2 j_3}. \tag{4.26}$$

[1] In fact, the Fourier coefficients obey all symmetries of the 3j-symbol, except for the condition $m_1 + m_2 + m_3 = 0$ [Boulatov, 1992]



$$= \tfrac{\lambda}{4!} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \textstyle\prod_{i=1}^{6} \delta_{j_i,j_i'} \delta^{m_i,m_i'}$$



$$= \tfrac{1}{3!} \textstyle\sum_{\pi \in S_3} \prod_{i=1}^{3} \delta_{j_i,\pi(j_i')} \delta^{m_i,\pi(m_i')}$$

Figure 4.1: Feynman rules for 3D group field theory. The magnetic labels m_i have been suppressed; the box indicates the sum over permutations.

We can thus obtain the propagator for Φ,

$$\langle \Phi^{m_1 m_2 m_3}_{j_1 j_2 j_3} \Phi^{m_1' m_2' m_3'}_{j_1' j_2' j_3'} \rangle = \frac{1}{|S_3|} \sum_{\pi \in S_3} \prod_{i=1}^{3} \delta_{j_i,\pi(j_i')} \delta^{m_i,\pi(m_i')}. \tag{4.27}$$

Due to the abscence of derivatives in the kinetic term, this propagator is independent of the labels j, m of the attached fields. We see however that the factor $\delta_{j_i,\pi(j_i')} \delta^{m_i,\pi(m_i')}$ forces conservation of these labels along the lines of propagation, analogous to momentum conservation in ordinary quantum field theory. The vertex contribution is

$$\frac{\lambda}{4!} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \prod_{i=1}^{3} \delta_{j_i,j_i'} \delta^{m_i,m_i'}. \tag{4.28}$$

Equipped with these Feynman rules, let us now study the amplitudes of graphs occuring in the expansion of (4.20). Only graphs without external legs will contribute. Symbolically, the amplitude $A(\Gamma)$ of such a graph Γ with n vertices v is thus given by

$$A(\Gamma) = \frac{(-\lambda)^n}{\mathrm{sym}_\Gamma} \sum_{\{j_i, m_i\}} \prod_{\text{vertices}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}, \tag{4.29}$$

where we have reintroduced an explicit sum over j_i and m_i, and sym_Γ denotes a symmetry factor. As this amplitude is independent of the magnetic indices m_i, the sum over the latter simply yields a factor $2j_l + 1$ for each single line l forming a loop, with j_l the representation label on that line. As a consequence,

$$A(\Gamma) = \frac{(-\lambda)^n}{\mathrm{sym}_\Gamma} \prod_l \sum_{j_l} (2j_l + 1) \prod_{\text{vertices}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{4.30}$$

We see already that this amplitude is strikingly similar to the Ponzano-Regge amplitude derived in the last chapter. To arrive at the simplicial interpretation of these amplitudes, let us associate a 1-simplex $\sigma_l^{(1)}$ of length j_l to each single line. The field $\Phi_{j_1 j_2 j_3}^{m_1 m_2 m_3}$ can then be associated to a two-simplex with edge lengths j_1, j_2, j_3, and boundary $\sigma_1^{(1)} \circ \sigma_2^{(1)} \circ \sigma_3^{(1)}$. This is consistent because the field Φ will identically vanish unless the *triangle inequality* (A.7)

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2 \tag{4.31}$$

is satisfied. This can bee seen from the Fourier expansion (4.22), which is proportional to a $3j$-symbol, which vanishes if (4.31) is violated. It can then be seen that the combinatorial structure of the vertex is such that four triangles $\sigma_i^{(2)}$, $i = 1...4$ are joined to form a 3-simplex $\sigma^{(3)}$ with edge lengths $\{j_i | i = 1...6\}$ and $\partial\sigma^{(3)} = \sigma_1^{(2)} \circ \sigma_2^{(2)} \circ \sigma_3^{(2)} \circ \sigma_4^{(2)}$, as illustrated in Fig. 4.2. Moreover, the vertex ampliute given by the $6j$-symbol vanishes for any edge configurations that fail to match to form a closed 3-simplex, as conjectured initially in [Ponzano and Regge, 1968] and finally proven in [Roberts, 1999]. The propagator identifies faces of 3-simplices, with the matching condition for the shared triangles implemented by the conservation of the spin labels j_i along the lines of propagation. We are thus guaranteed that each Feynman diagram corresponds to a three-dimensional simplicial complex. We can thus associate the amplitude $A(\Gamma)$ with an amplitude $A(K)$ for a complex K,

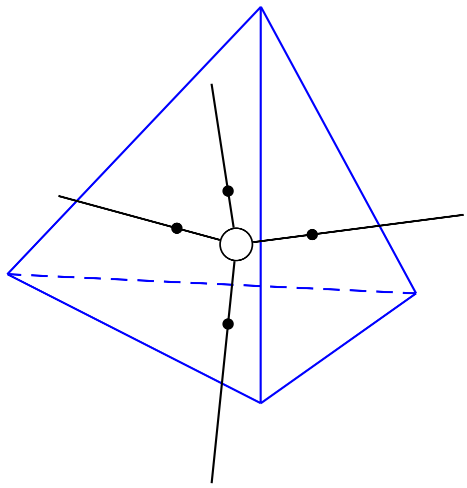


Figure 4.2: Group field theory vertex and its dual tetrahedron.

$$A(\Gamma) \equiv A(K) = \frac{(-\lambda)^n}{\mathrm{sym}_K} \prod_{\sigma^{(1)}} \sum_{\sigma^{(1)}} (2j_{\sigma^{(1)}} + 1) \prod_{\sigma^{(3)}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{4.32}$$

This is exactly the non-regularized form of the Ponzano-Regge amplitude as derived in chapter 3. As anticipated, the lattice amplitude for BF theory we derived in chapter 3 appears as a Feynman diagram amplitude in the perturbative expansion of group field theory. To obtain the sum over connected complexes, we may take the logarithm to obtain the free energy for group field theory,

$$F = \log\left(\int \mathcal{D}\phi e^{-S[\phi]}\right) = \sum_{K \text{ connected}} A(K). \tag{4.33}$$

Chapter 5

Conclusion and Outlook

In this thesis, I have outlined a particular viewpoint on the quantization of gravity and attempted to illustrate this by means of simple toy models in two and three dimensions. In the introduction 1, I have argued that topology change can not be ruled out to matter at the quantum level, and that in this light, a nonperturbative path integral approach is best suited. I have furthermore tried to make plausible that a sum over topologies in this framework can be defined by interpreting lattices as Feynman diagrams of an underlying zero-dimensional field theory. In chapter 2, I have presented a simple matrix model as a particular realization of this idea for two-dimensional gravity and sketched how a continuum limit is achieved. I went on in chapter 3 to introduce BF theory as a theory of gravity in three dimensions, discussed its discretization on a simplicial lattice and the quantization of the resulting lattice theory. At the beginning of chapter 4, I used these results to reinterpret the lattice amplitudes as amplitudes for Feynman diagrams. In the following sections, I presented the framework of group field theory: I demonstrated the equivalence to matrix models in two dimensions, and then gave a description of a model that generalizes the latter to a tensor-type theory of three-dimensional gravity. The Feynman diagram amplitudes of this theory turned out to be the lattice amplitudes derived in chapter 3, implementing explicitly the perspective announced in the introduction.

What remains is a long list of outstanding issues that have to be adressed to assess the viability of this general approach to yield a reasonable quantum theory for gravity in our universe. In what follows, I will attempt to give an outlook by briefly commenting on the most pressing of these open questions – by far not being exhaustive.

5.1 Renormalization and continuum limit

A first crucial cornerstone to establish the viability of the model described in the last chapter is to define the partition function in an unambiguous way. This was achieved by the proof that the asymptotic series arising in the perturbative expansion of the theory is *uniquely Borel summable* [Freidel and Louapre, 2003]. For this proof however, an additional interaction term of the form

$$\phi(g_1, g_2, g_3)\phi(g_3, g_5, g_4)\phi(g_4, g_5, g_6)\phi(g_6, g_2, g_1)$$

had to be included, which is of different combinatoric structure. This relates to an important open issue, namely the study of arbitrary interaction terms as opposed to the simple ϕ^4 interaction presented herein. That issue is most naturally adressed in a framework of renormalization for group field theory. The reader is referred to [Rivasseau, 2010] and [Gurau and Ryan, 2011] for a general conceptual outline of this very young field.

Moreover, whether or not group field theory may serve as a model of quantum gravity crucially depends on whether a suitable continuum limit for the theory can be defined at all. As we have seen, in the case of matrix models, the existence of a continuum limit could be established beyond much doubt thanks to the *exact solvability* of these models. There is however no reason to expect that one is as fortunate in higher dimensions. An investigation of the phase diagram of group field theory thus requires an approximation scheme of some sort. More specifically, one needs to determine whether a critical point exists that corresponds to a second-order phase transition, as was the case for matrix model. For this purpose, it is helpful to realize that second-order phase transitions correspond to *infrared fixed points of the renormalization group flow*, as has first been clarified by Wilson [Wilson, 1971]. In this light, a study of the behaviour of group field theory under renormalization is also a natural framework to adress the question of a continuum limit.

5.2 Further developments

To make this topic tractable in a thesis of this scope, it was necessary to restrict the treatment to simplified toy models of gravity. To arrive at physically more realistic theories, the following four generalizations have to be adressed:

1. *Higher dimensions.* BF theory generalizes straightforwardly to arbitrary dimensions D by promoting B to a $D-2$-form and choosing the gauge group $SO(D)$. The discretization procedure outlined in section 3.3, and the quantization procedure given in section 3.4 then go through in the same fashion. A group field theory corresponding to BF theory in $D = 4$ was first given in [Ooguri, 1992].

   It is however only for $D = 3$ that the equivalence to first-order general relativity holds, and BF theory remains purely topological for any D. It was Plebański who observed first that general relativity in $D = 4$ can be written as a *constrained* BF theory [Plebański, 1977]. This lead to a large amount of work going into spinfoam models, with the strategy to quantize BF theory in $D = 4$ and impose the constraints *after* quantization [for a review, see e.g. Livine, 2011]. Let us however note that this procedure is not generally agreed upon and moreover has been argued to be inconsistent with the rules of Dirac quantization [Alexandrov and Roche, 2011]. What is more, even classically Plebański's action principle is not strictly equivalent to first-order gravity, having unphysical topological sectors that also solve the constraints [Livine, 2011].

   A possible way to avoid these issues might be given by action principles that are of the form of a topological field theory with a symmetry breaking term [MacDowell and Mansouri, 1977]. The breaking of topological symmetry turns gauge degrees of freedom into physical ones, giving rise to an non-topological gravity action. In that case, the additional term in the action can be treated as a honest interaction, with an expansion that has been shown to be rapidly converging [Smolin and Starodubtsev, 2003]. One might attempt to proceed by identifying a suitable group field theory corresponding to these action principles.

In any case, since gravity is topological only in $D \leq 3$, one cannot expect independence of the choice of discretization in $D > 3$. A great virtue of the group field theory framework is then that the sum over triangulations implements an averaging procedure, in principle giving triangulation independence when all orders are included.

2. *Lorentzian signature.* Switching to a model of Lorentzian spacetime amounts to replacing the gauge group $SO(D)$ with $SO(D-1,1)$. The noncompactness of the Lorentz group then gives rise to variety of technical issues, and in particular a careful gauge fixing procedure is required to avoid divergent integrals over the volume of the gauge group. For a review on how this can be dealt with to give convergent expressions, see e.g. [Baez and Barrett, 2001].

3. *Cosmological constant.* Introducing a nonzero cosmological constant in BF theory corresponds to adding a term $\propto \Lambda(B\wedge B\wedge B)$ in $D=3$, or $\propto \Lambda(B\wedge B)$ in $D=4$ to the action. As a result, the initial gauge group gets replaced by what is called its "quantum deformed" version [Baez, 1996]. In fact, having only a finite number of representations, these quantum groups serve as a regularization by cutting off otherwise infinite sums over representation labels. As a consequence, the partition function for BF theory turns out finite and in $D=3$ yields Turaev-Viro invariant [Turaev and Viro, 1992], a topological invariant for three-manifolds. Note that quite obviously, BF theory only allows for a cosmological constant term in $D=3$ and $D=4$.

4. *Matter.* The literature contains a great variety of proposals on how to include matter couplings in the framework of lattice BF theory. One approach has been to introduce the relevant degrees of freedom at the classical level and keep track of these in the discretization procedure. In particular, point particles [Freidel and Louapre, 2004] and gauge fields [Speziale, 2007] have been coupled to the Ponzano-Regge model along these lines.

   A phenomenologically more removed, but mathematically more straightforward way is to consider supergroups as gauge groups. As a first instance, in $D=3$, $SU(2)$ was replaced by the supergroup OSp(1|2) [Livine and

Oeckl, 2004], and the distinction between matter and gravitational degrees of freedom was done after quantization. One might anticipate that a corresponding group field theory[1] could be constructed by considering fields $\phi \in L^2[\mathrm{OSp}(1|2)^{\times 3}/\mathrm{OSp}(1|2)]$.

On a more speculative note, it has also been argued that matter degrees of freedom could emerge from a non-supersymmetric group field theory directly [Di Mare and Oriti, 2010].

[1] The author wonders whether this might allow for a new class of candidates for worldvolume theories of extended objects in string theory.

Appendix A

Recoupling theory of $SU(2)$

Herein, I will give a condensed and by far not exhaustive treatment of elementary objects of recoupling theory. The focus will be on the particular relations needed for the purposes of this thesis.

The unitary irreducible representations ρ_j of $SU(2)$ are labelled by half-integers $j \in \mathbb{N}_0/2$ called "spins" and are of respective dimensions $2j+1$. Introducting a basis of ρ_j in Dirac notation $\{|jm\rangle | m = -j, ...j\}$, the elements of the Wigner representation matrices are given by

$$D^j_{mn}(g) = \langle jm|g|jn\rangle. \tag{A.1}$$

The Wigner matrices are unitary

$$D^j_{ab}(g^{-1}) = \overline{D}^j_{ba}(g), \tag{A.2}$$

and furthermore obey

$$\overline{D}^j_{ab}(g) = (-1)^{(a-b)} D^j_{-a-b}(g). \tag{A.3}$$

They are orthogonal with respect to the Haar measure dg,

$$\int_{SU(2)} dg \overline{D}^j_{ab}(g) D^k_{cd}(g) = \frac{1}{2j+1}\delta^{jk}\delta_{ac}\delta_{bd} \tag{A.4}$$

and thus form a basis of $L^2[SU(2)]$. We can furthermore denote a basis on $\rho_{j_1} \otimes \rho_{j_2}$

by $\{|j_1 m_1 j_2 m_2\rangle | m_i = -j_i, ... j_i\}$. The $3j$-symbol is then defined in terms of the Clebsch-Gordan coefficient $\langle j_1 m_1 j_2 m_2 | j_3 m_3\rangle$ as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} := \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle. \tag{A.5}$$

Given three representations labelled j_1, j_2, j_3, the $3j$-symbol is, up to normalization the unique intertwiner

$$\{3j\}: \quad \rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3} \longrightarrow \mathbb{C}. \tag{A.6}$$

The $3j$-symbol is nonzero only if the following conditions are satisfied:

$$\begin{aligned} & m_1 + m_2 + m_3 = 0, \\ & j_1 + j_2 + j_3 \in \mathbb{N}, \\ & |m_i| \leq j_i \quad \forall i, \\ & |j_1 - j_2| \leq j_3 \leq j_1 + j_2. \end{aligned} \tag{A.7}$$

A permutation $\pi \in S_3$ acts on the columns of the $3j$-symbol as follows:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{cases} \begin{pmatrix} j_{\pi(1)} & j_{\pi(2)} & j_{\pi(3)} \\ m_{\pi(1)} & m_{\pi(2)} & m_{\pi(3)} \end{pmatrix} & \text{for } \pi \text{ even,} \\ (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_{\pi(1)} & j_{\pi(2)} & j_{\pi(3)} \\ m_{\pi(1)} & m_{\pi(2)} & m_{\pi(3)} \end{pmatrix} & \text{for } \pi \text{ odd.} \end{cases} \tag{A.8}$$

A sign swap in the magnetic indices gives a phase,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{A.9}$$

A triple integral of Wigner matrices is proportional to a product of two $3j$-

symbols,

$$\int_{SU(2)} dg D^{j_1}_{a_1 b_1}(g) D^{j_2}_{a_2 b_2}(g) D^{j_3}_{a_3 b_3}(g) = \begin{pmatrix} j_1 & j_2 & j_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ b_1 & b_2 & b_3 \end{pmatrix}. \tag{A.10}$$

The $3j$-symbols moreover obey the orthogonality relation

$$\sum_{\{a_i\}} \begin{pmatrix} j_1 & j_2 & j_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ a_1 & a_2 & a_3 \end{pmatrix} = 1, \tag{A.11}$$

where in this equation, only the explicit sum is carried out, and other repeated indices are *not* summed over.

The $6j$-symbol is defined by a contraction of four $3j$-symbols amoung their magnetic indices m_i,

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = & \sum_{\{m_i\}} (-1)^{j_4+j_5+j_6+m_4+m_5+m_6} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ & \times \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & m_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & -m_5 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -m_4 & m_2 & m_6 \end{pmatrix}, \end{aligned} \tag{A.12}$$

where again all summations are explicit. As a consequence of the symmetries of the $3j$-symbol, the $6j$-symbol is nonzero only if the tuples $(j_1, j_{2,3})$, (j_1, j_5, j_6), (j_4, j_5, j_3) and (j_4, j_2, j_6) simultaneously obey the triangle inequality. The $6j$-symbol admits a natural action of the tetrahedral symmetry group S_4 which is given by the permutation of the four vertices of a tetrahedron with its six edges labelled by the six spins. This translates into the invariance of the $6j$-symbol under any permutation of two of its columns,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_4 & j_6 & j_5 \end{Bmatrix} = \begin{Bmatrix} j_3 & j_2 & j_1 \\ j_6 & j_5 & j_4 \end{Bmatrix}, \tag{A.13}$$

and the invariance under the exchange of upper and lower arguments in any two columns:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1 & j_2 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_5 & j_6 \\ j_4 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_2 & j_6 \\ j_1 & j_5 & j_3 \end{Bmatrix}. \tag{A.14}$$

Appendix B

Delta function identity

Here, I restate and prove the identity used in section 3.4 relating the delta function on the Lie algebra of $SU(2)$ to the delta function on the group itself:

Theorem B.0.1. *Let* $X \in \mathfrak{su}(2)$ *and* $g \in SU(2)$. *Let* tr *be the trace on* 2×2 *complex matrices. Then*

$$\int_{\mathfrak{su}(2)} dX e^{i\mathrm{tr}(gX)} = 4\pi[\delta_{SU(2)}(g) + \delta_{SU(2)}(-g)], \tag{B.1}$$

where $\delta_{SU(2)}(g)$ *is defined with respect to the Haar measure* dg *on* $L^2[SU(2)]$,

$$\int_{SU(2)} dg \delta_{SU(2)}(g) f(g) = f(\mathbb{1}) \qquad \forall f \in L^2[SU(2)]. \tag{B.2}$$

Proof. We closely follow the proof as presented in [Freidel and Louapre, 2004]. Expressing the $\mathfrak{su}(2)$ generators in terms of the Pauli matrices as $t^i = -i\sigma^i/2$, $i = 1, 2, 3$, we can write $X = -iX^i\sigma^i/2$. We parametrize the group element as

$$g = \cos\theta \mathbb{1} + in^i\sigma^i \sin\theta \tag{B.3}$$

with $n^i n^i = 1$. Using $\mathrm{tr}(\sigma^i\sigma^j) = 2\delta^{ij}$, we find

$$\mathrm{tr}(gX) = X^i n^i \sin\theta. \tag{B.4}$$

Using this, we evaluate the integral over $\mathfrak{su}(2)$,

$$\begin{aligned}\int_{\mathfrak{su}(2)} dX e^{i\mathrm{tr}(gX)} &= \int_{\mathbb{R}^3} d^3X e^{iX^i n^i \sin\theta} \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{n}\sin\theta) \\ &= \frac{2\pi^2}{|\sin\theta|^2}\delta(|\sin\theta|),\end{aligned} \tag{B.5}$$

where the boldface indicates that $\mathbf{n} \in \mathbb{R}^3$, and in the last step we used the identity $\delta^{(3)}(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|^2}\delta(|\mathbf{x}|)$. The composition of a function $f(x)$ with the delta function is defined by a sum over the roots[1] x_i of $f(x)$,

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i|}. \tag{B.6}$$

We can thus rewrite

$$\begin{aligned}\frac{2\pi^2}{|\sin\theta|^2}\delta(|\sin\theta|) &= \frac{2\pi^2}{|\sin\theta|^2}\sum_{n\in\mathbb{Z}} \frac{\delta(\theta - \pi n)}{|\cos\theta|} \\ &= \frac{2\pi^2}{|\sin\theta|^2}\sum_{n\in\mathbb{Z}} [\delta(\theta - 2\pi n) + \delta(\theta - \pi(2n+1))].\end{aligned} \tag{B.7}$$

Using the parametrization (B.3), the normalized Haar measure on $L^2[SU(2)] \cong L^2(S^3)$ takes the form

$$dg = \frac{2}{\pi} d\theta (\sin\theta)^2 d\Sigma \tag{B.8}$$

where $d\Sigma$ is the normalized measure on S^2. From (B.2), the delta function on $SU(2)$ is thus

$$\begin{aligned}\delta(g) &= \frac{\pi}{2(\sin\theta)^2}\sum_{n\in\mathbb{Z}} \delta(\theta - 2\pi n), \\ \delta(-g) &= \frac{\pi}{2(\sin\theta)^2}\sum_{n\in\mathbb{Z}} \delta[\theta - \pi(2n+1)].\end{aligned} \tag{B.9}$$

[1]We assume the roots of $f(x)$ to be simple.

Comparing this expression with (B.7), we find

$$\int_{\mathfrak{su}(2)} dX e^{i\mathrm{tr}(gX)} = 4\pi[\delta_{SU(2)}(g) + \delta_{SU(2)}(-g)], \tag{B.10}$$

which proves (B.1). $\square$

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