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Entanglement Criteria for Continuous-Variable States

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Abstract

Entanglement is an amazing feature of quantum mechanics which is a nontrivial non-local correlation between states. It is studied and exploited greatly in the field of quantum information. Traditionally, most quantum systems of interest in quantum information is described by discrete variable states, e.g. spin of particles. Thus, entanglement is studied to a considerable extent in such state. Nevertheless, in practice, there are a number of quantum systems that must be described by continuous variable states such as electromagnetic fields in quantum optics. Thus, it is also important to understand entanglement in the context of continuous variable states. In particular, to be able to study entanglement properly, it is very necessary to find a method to identify whether a state is entangled or separable and to quantify how much entangled a state is. This report is a review of several criteria and measures used to deal with entanglement in both Gaussian and non-Gaussian continuous variable states, including some prelimary backgrounds.

Contents

1 Introduction

1.1 General Overview and Outline

Quantum mechanics is accepted to be one of the two pillars supporting modern physics together with General Relativity. Not only do they change and extend our description of the world, but quantum mechanics is also a crucial underlying principle in many of our modern technologies, e.g. the integrated circuit (IC) and LASER. In addition, in the field of quantum information sciences, information processing and communication technology is enhanced greatly by employing many counter-intuitive properties of quantum systems such as superposition of states, uncertainty values of observables, non-local non-trivial correlation (entanglement), etc. These astounding properties enable a number of new possible breakthrough technologies.

Among all of the amazing quantum features, entanglement may be the one of the most interested. It was first studied and named by Erwin Schrödinger in 1935. An entangled system is composed of two or more subsystems whose overall state vector cannot be decomposed completely into a product of individual state vector of the subsystems. Thus, only the properties of the overall system can be completely determined by the overall state vector. In contrast, a subsystem cannot be described completely because of the lack of a state vector corresponding to it, i.e. there is no well-defined state asso-

ciated to the subsystem. This is the reason for the name entangled states - subsystems are not independent from but correlate to one another in a manner that preserve the behaviour of the overall system. Every manipulation of a subsystem will not only affect the manipulated subsystem, but also affect all the rest of the subsystems inevitably. For a bipartite entangled state, an outcome of a measurement on system 1 seems to be able to influence another outcome of the same measurement on system 2 simultaneously - even these two systems are space-like separated. This surprising property was investigated in the seminal paper of Einstein, Podolsky, and Rosen, the EPR paper [1], in 1935, in their attempt to point out the incompleteness of a quantum description of reality. This turned out to be the underlying idea leading to the famous Bell's inequality [2] invented to enable experimental discrimination between the classical correlation and quantum correlation. It was, finally, Alain Aspect who conducted an experiment in 1982 [3], following an adapted idea of Bell and verified the correctness of the quantum correlation.

Entanglement is not only interesting in the context of fundamental pure science, but also in its application. Entanglement is exploited greatly as a main resource in quantum information sciences to enhance the efficiency of information processing and enable some classical impossibility. For example, according to the E91 communication protocol proposed by Artur Ekert in 1991 [4], it is possible to develop a much more secure protocol than any of the classical possible methods. This is achieved by using the unique property of non-local correlation of entangled states to precisely check the presence of any eavesdropper. Another example is quantum teleportation which, again, employs the non-local correlation to enable the transfer of an unknown quantum state without sending the state through any medium.

Also, entangled states shared by two parties before hand enables the two parties to communicate two bits of information by transmitting only one bit by using the superdense coding protocol. Furthermore, there are a number of works which consider that entanglement is a crucial phenomenon in biological and chemical systems, e.g. the stability of DNA double helix [5]. Thus, it is a crucial task in quantum information to find a method to determine whether a state is entangled or not.

It is the main purpose of this dissertation to give a detailed survey of various methods used for determining whether a given bipartite system (two-particle or two-mode system) is entangled or separable, with the emphasis on the criteria for continuous variable systems. It starts later in this chapter with the introduction about some general basic definitions and mathematics. In chapter 2, a more detailed description about a discrete variable system, qubit, is introduced and several measures and criteria of entanglement used for qubit states will be explored with the emphasis on the Peres-Horodecki method. This is to provide more basic backgrounds before considering continuous variable systems. In chapter 3, a description of continuous variable (CV) systems are provided in the first part. Then, Wigner function is introduced as a proper mathematical tool for studying such states in phase space picture. This leads to a non-trivial simple type of states called Gaussian states, the states of minimum uncertainty that precisely characterize many systems in practice, such as LASER (coherent states) and thermal states. In chapter 4, a generalization of Peres-Horodecki criterion for variance (second statistical moment) are shown to be a necessary condition for the separability of general continuous variable states and also become a sufficient condition for Gaussian states. Moreover, several measures of entanglement for continuous states are described. In chapter 5, other criteria

proposed by Hillery, Zubairy, and Nha [6, 7] are discussed and shown to be sufficient conditions for entangled states. Several entangled states that are not recognised by the Peres-Horodecki method discussed in chapter 4, the non-Gaussian states, are shown to be recognized by this method. Finally, a conclusion and possible future work will be discussed briefly in chapter 6.

1.2 Basic Definitions and Mathematics

1.2.1 State Vector and Hilbert Space

The mathematical description of the simplest quantum system, i.e. a single particle (or single mode for continuous variable system), starts from the definition of a state vector, denoted in Dirac ket notation as $|\psi\rangle$, which is a unit-normed vector. This vector lives in a Hilbert space, which is by definition a complete complex inner product space [8, 9]. It can be expanded as a linear combination of basic vectors as follows,

$$|\psi\rangle = \sum_{i=1}^{n} \varphi_i |a_i\rangle$$

, where $|a_i\rangle$ represents one of the basic vectors in one of the basis of $\mathcal{H}$ and φ_i is the complex coefficient associating to each basic vector. It is assumed in this case and from now on until the section of continuous variable systems that $\mathcal{H}$ is completely spanned by finite n basic vectors.

Two different state vectors can be written as different linear combinations under the same basis but with different coefficients. These coefficients represent the proportion of each basic vector in the state vector, which can be found mathematically by performing an inner product,

$$\varphi_i = \langle a_i | \psi \rangle .$$

1.2.2 Density Matrix and Mixed State

Any quantum state which can be represented completely by only a state vector (in terms of any linear combination of a basis as above) is called a pure state. However, the information about which exact state a quantum system is in, is not typically available. In many cases, according to limited knowledge, it is only possible to identify a set of possible states a system can be in with corresponding probabilities. This is the concept of the statistical ensemble of the states. In contrast to a linear combination of states, which results in a single state and can be represented by a state vector, an ensemble is a mixture of states which cannot be described completely by one state vector. Thus, another method, called the density matrix, has to be employed instead.

A density matrix ρ represents an ensemble of states (which also includes a pure state, normally expressed by a state vector as described above) consisting of a set of states ψ_j, each with probability p_j, and defined in terms of Dirac notation as,

$$\rho = \sum_{j=1}^{m} p_j \left| \psi_j \right\rangle \left\langle \psi_j \right| = \sum_{j=1}^{m} p_j \rho_j$$

, where ρ_j is a short-hand notation of $\left| \psi_j \right\rangle \left\langle \psi_j \right|$. It is assumed here that ρ is an ensemble of m states.

A valid density operator must have the following properties [9]:

- $tr(\rho) = 1$: This follows directly from the definition of a density matrix and the fact that $\sum_{j=1}^{m} p_j = 1$.

- All eigenvalues of ρ are not negative numbers: This also follows from the fact that every probability p_j cannot be negative.

- $tr\left(\rho^2\right) \leq 1$: This follows from the fact that $0 \leq p_j \leq 1$. It can be exploited to determine whether a state is pure ($tr\left(\rho^2\right) = 1$) or mixed ($tr\left(\rho^2\right) < 1$).

The second property is very important since it provides a condition for checking whether a bipartite state is separable. The definition of a separable state and a bipartite state will be discussed later in this chapter while the discussion about the separable condition based on the second property will be presented in chapter 2 and 4.

1.2.3 Measurement

It is an assumption of quantum mechanics that a state contains all of the information about its corresponding system. However, a state is an abstract which cannot provide any physical realization by itself. In order to extract physical information, measurements have to be performed on the state. Measurements corresponding to measurable quantities can be represented mathematically as linear Hermitian operators which map states in a Hilbert space to other states in the same Hilbert space. Eigenstates of a Hermitian operator must be orthogonal to one another and form a complete set over the space that the operator is defined. This means that the eigenstates of a measurement operator forms an orthogonal basis of the corresponding Hilbert space.

Let M be a representation of a measurement corresponding to a measurable quantity. A state $|\psi\rangle$ being measured by M is expressed as $M|\psi\rangle$ (Similarly, for a mixed state ρ, it can be written as $\sum\limits_{j=1}^{m} p_j M|\psi_j\rangle\langle\psi_j|M^\dagger = M\rho M^\dagger = M\rho M$.). In a special circumstance where the state being measured is an eigenstate of the measurement operator, the state after being

measured will not change, i.e.

$$M \left| m \right\rangle = m \left| m \right\rangle$$

, where $\left| m \right\rangle$ is an eigenstate, with eigenvalue m. In other cases where the original state $\left| \psi \right\rangle$ is not an eigenstate, $\left| \psi \right\rangle$ can always be expressed as a linear combination of the basis formed by the eigenstates as follows,

$$\left| \psi \right\rangle = \sum_i \varphi_i \left| m_i \right\rangle.$$

After being measured, $\left| \psi \right\rangle$ will be forced to change into one of the eigenstates $\left| m_i \right\rangle$ in the sum above. One of the weirdness of quantum mechanics is that it is impossible to determine accurately which eigenstate the original state will collapse into before the measurement is performed, even in principle. It is only possible to determine the corresponding probability that the state will collapse into each eigenstate $\left| m_i \right\rangle$, which is equal to $\left| \varphi_i \right|^2$. Let $\left| m_f \right\rangle$ be the final state after measurement. The eigenvalue associated to this final state, m_f, can be interpreted as the outcome of the measurement and is called the observable.

1.2.4 Expectation Value and Variance

Due to the statistical behaviour of a quantum state, the average result of a set of measurement is called the expectation value and is denoted as $\left\langle M \right\rangle$. This can be calculated, again, by performing an inner product, generally written as [9],

$$\left\langle M \right\rangle = \left\langle \psi \left| M \right| \psi \right\rangle.$$

In terms of a density matrix ρ, this is equivalent to

$$\langle M \rangle = tr(\rho M)$$

, where on the right-hand side, the trace of the expression in the bracket is calculated.

After defining the statistical average of an outcome, it is natural to define variance, σ^2, of the outcome as follows,

$$\sigma_M{}^2 = \left\langle M - \left\langle M^2 \right\rangle \right\rangle = \left\langle M^2 \right\rangle - \langle M \rangle^2 .$$

The uncertainty principle is defined as the relationship between the variance of two operators. It can be shown using the definition of variance and the Cauchy-Schwarz inequality that, in general [8],

$$\sigma_A{}^2 \sigma_B{}^2 \geq (\frac{1}{2i} \langle [A, B] \rangle)^2$$

, where $[A, B]$ is the commutator between operator A and B. This is a very crucial relation which all of the physical states have to satisfy. The violation of this relation is exploited as the underlying principle for several separable criteria for continuous variable system described in chapter 4 and 5.

1.2.5 Creation and Annihilation Operators

Apart from the measuring operators, there exist some operators which are not Hermitian, i.e. the adjoint of them are not equal to themselves. These operators do not stand for any measureable quantities. However, such operators are also very important in quantum mechanics since they can manipulate states as well. Two remarkable operators of this type are annihilation

and creation operators, denoted as $a, a^\dagger$ respectively. They are the adjoints of each other and they are not equal.

Creation and annihilation operators arise from the analysis of a special system called harmonic oscillator. This type of system is characterized by a quadratic convex potential function. Because every analytic potential function at low energy can be well approximated by a quadratic potential, harmonic oscillator systems are extremely important. A remarkable characteristic property of this system is that it has discrete energy levels which are equally spaced [10]. Thus, to explore its properties, it is best to use the energy eigenstates as the basis. Also it is convenient to assign a non-negative integer to each energy eigenstate, starting from $|0\rangle$ for the ground state, $|1\rangle$ for the next higher energy eigenstate, and so on.

The origin of the names, creation and annihilation operators, becomes clearer by observing the actions of the operators on a state.

For a state $|n\rangle$ except the ground state ,

$$
\begin{aligned}
a\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,, \\
a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle
\end{aligned}
$$

The coefficients on the right-hand side are added to fulfil the normalization condition of the state.

For the ground state, it is defined that,

$$
a\,|0\rangle = 0.
$$

Though creation and annihilation operator themselves are not Hermitian, when combining together, it can become a Hermitian operator. For instance,

for the number operator defined as $N = a^\dagger a$, its adjoint is $N^\dagger = \left(a^\dagger a\right)^\dagger = a^\dagger a = N$. Thus, it is a Hermitian operator. This number operator acts on $|n\rangle$ as follows,

$$
\begin{aligned}
N\,|n\rangle &= a^\dagger a\,|n\rangle \\
&= a^\dagger \sqrt{n}\,|n-1\rangle \\
&= n\,|n\rangle .
\end{aligned}
$$

So the eigenvalue is the order of the corresponding energy level. This is the reason for the name of this operator.

Commutation relation is an important property between operators in quantum mechanics. For creation and annihilation operators, the commutation relation is as follows,

$$
[a, a^\dagger] = 1 \quad [a^\dagger, a] = -1
$$

1.2.6 Order of Operators

Because of the non-commutativity of operators in quantum mechanics, order of operators is very crucial because different order can lead to different results. For example, for creation and annihilation operators,

$$
\begin{aligned}
aa^\dagger\,|n\rangle &= a\sqrt{n+1}\,|n+1\rangle \\
&= (n+1)\,|n\rangle \; \neq a^\dagger a\,|n\rangle .
\end{aligned}
$$

This derivation can be shown in another way by employing the commutation relation between a and $a^\dagger$.

Suppose that b is an arbitrary operator and the combination of interest

is composed of one b and two $b^\dagger$. Normal order is defined as that all of the daggered operators are put to the left end and the undaggered operators are put to the right end. This is denoted as $: bb^{\dagger 2} := b^{\dagger 2} b$. The symbol $::$ means to arrange the expression in a normal order. Anti-normal order is defined as the reverse order of the normal order, i.e. all undaggered ones are on the left and all daggered ones are on the right. This is written as $\vdots b^{\dagger 2} b \vdots = bb^{\dagger 2}$. The symbol $\vdots\vdots$ signifies to arrange the expression in an anti-normal order. Symmetric order is defined as an expression of operators which mix together both normal and anti-normal order equally. This ordering is expressed as $S(bb^{\dagger 2}) = \frac{1}{3}(bb^{\dagger 2} + b^\dagger bb^\dagger b + b^{\dagger 2} b)$. [11]

1.2.7 Bipartite System

The mathematical treatment of quantum mechanics becomes exponentially complicated when it is applied to multiple particles (or multiple modes). This is because the state vector now lives in a more complicated space resulting from the tensor product of the corresponding Hilbert space, i.e. $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}2 \otimes \mathcal{H}_3 \otimes \ldots$ where each $\mathcal{H}_i$ is the Hilbert space associated to each single particle (or single mode). Due to this difficulty and complexity of many-particle system, most studies of interest have been done on a two-particle system, which is technically called a bipartite system.

The Hilbert space of bipartite states can be expressed as follows,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

A state associating to this new space can be written as a linear combination of a new basis constructed by a tensor product of the linear combination of

bases of each space. For instance, if bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ are

$$\{|a_1\rangle, |a_2\rangle, \ldots\}, \{|b_1\rangle, |b_2\rangle, \ldots\}$$

respectively, then the new basis will be

$$\{|a_1\rangle \otimes |b_1\rangle, |a_1\rangle \otimes |b_2\rangle, |a_2\rangle \otimes |b_1\rangle, |a_2\rangle \otimes |b_2\rangle, \ldots\}.$$

According to this basis, there is a set of states called separable states which can be expressed in a special form as follows,

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

, i.e. they can be written in a form being equivalent to tensor products between two state vectors, one living in the first Hilbert space and another one living in the second Hilbert space. For a bipartite system to be separable, it must be possible to write the density matrix in the following form [9],

$$\rho = \sum_{j=1}^{m} p_j \rho_1{}^j \otimes \rho_2{}^j$$

, where p_j is the probability of the states ρ_1^j and ρ_2^j to be found in this ensemble. The states that cannot be expressed in this form are called entangled states.

1.2.8 Reduced Density Matrix

A state corresponding to any multi-partite system encodes the overall information about that whole system. However, there are many situations in which only information of some parts of the system, e.g. the spin of only

one particle in a bipartite system, is required. This leads to the definition of a reduced density matrix which is the proper representation of a subsystem. A reduced density matrix is obtained by performing a partial trace over all but the subsystem in consideration. For simplicity, the following example will be shown for a bipartite state only.

The reduced density matrix ρ^A for subsystem A is [9]

$$\rho^A = tr_B(\rho^{AB})$$

, where ρ^{AB} is the density matrix of the bipartite system consisting of subsystems A and B whereas $tr_B(\rho^{AB})$ represents the partial trace over subsystem B.

The partial trace operation can be described as follows. First, every density matrix of bipartite system can be written as a linear combination in the following form,

$$\rho^{AB} = \sum_{i,j,k.l} c_{i,j} c_{k.l}^* \left(|a_i\rangle \otimes |b_j\rangle\right) \left(\langle a_k| \otimes \langle b_l|\right)$$

, where $\sum_{i,j} |c_{i,j}|^2 = 1$, $\sum_{k,l} |c_{k.l}^*|^2 = 1$, and $|a_i\rangle$ and $|b_j\rangle$ are elements of orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. [9].

Performing partial trace over system B as follows,

$$
\begin{aligned}
\rho^A = tr_B\left(\rho^{AB}\right) &= \sum_{i,j,k.l} c_{i,j} c_{k.l}^* \left(|a_i\rangle \langle a_k|\right) \otimes tr\left(|b_j\rangle \langle b_l|\right) \\
&= \sum_{i,j,k.l} c_{i,j} c_{k.l}^* \left(|a_i\rangle \langle a_k|\right) \otimes \langle b_l| b_j\rangle \\
&= \sum_{i,k.l} c_{i,l} c_{k.l}^* \left(|a_i\rangle \langle a_k|\right)
\end{aligned}
$$

The reduced density matrix shows a subtle relationship between entangle-

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ment and a mixed state. A reduced density matrix of a bipartite pure state will be a mixed state if and only if the former state is entangled. To examine this result, an example is shown in [9] using a Bell state (maximally entangled state),

$$
\begin{aligned}
\rho &= \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) \\
&= (|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)/2
\end{aligned}
$$

Performing partial trace over the second system,

$$
\begin{aligned}
\rho^1 &= tr_2(\rho) \\
&= \frac{1}{2}[|0\rangle\langle 0|tr(|0\rangle\langle 0|) + |0\rangle\langle 0|tr(|1\rangle\langle 1|) + \\
&\quad |1\rangle\langle 1|tr(|0\rangle\langle 0|) + |1\rangle\langle 1|tr(|1\rangle\langle 1|)] \\
&= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{I}{2}.
\end{aligned}
$$

The reduced density operator corresponding to subsystem 1 is a half of the identity matrix which is manifestly a maximally mixed state because it is composed of the pure states $|0\rangle$ and $|1\rangle$ equally. This result can be used as a method to check whether a pure state is entangled or not. It reflects the fact that the maximum knowledge available for an entangled state is the overall state of the system as suggested first by Schrödinger. The attempt to define a state to a subsystem (by performing partial trace) results in a mixed states that obviously represents the lack of definite information about the subsystem as has been discussed once above in the mixed state section.

1.2.9 Von Neumann Entropy

States and information or knowledge of systems has been discussed so far. However, it would be inconvenient and improper to continue the discussions if it is not clear how information is defined. To examine the concept of information properly, information needs to be quantified. This leads to the idea of entropy as the measure of information.

The concept of entropy was first discovered, quantified, and interpreted by many studies in the context of thermodynamics and statistical mechanics. Later, Claude Shannon, unintentionally, with a suggestion from John von Neumann, named a quantity in his mathematical communication theory as (information) entropy. These two quantities, from two unrelated fields, under the same name, were later shown to be surprisingly related. This is the beginning and the most important idea of the field of quantum information theory. It suggests that the physical degree of disorder of a system (statistical mechanics entropy) is directly related to its information (information entropy), according to the famous quote of Rolf Landauer: *Information is Physical!*

In fact, apart from these two entropies, there are several more entropies that have been defined. The discussion about entropies in quantum information can be long and interesting. However, only von Neumann entropy will be discussed here. Von Neumann entropy is a generalization of the thermodynamics entropy and classical information entropy to quantum mechanical systems. It is defined mathematically as,

$$S = -tr(\rho \log_2 \rho)$$

, where S is the von Neumann entropy and ρ is the density matrix corre-

sponding to the state under consideration. In addition, it is defined that if $\rho = 0$, then $S = 0\log_2 0 = 0$. In practice, ρ needs to be diagonalised first before taking logarithm, multiply, and taking trace. It can be shown that von Neumann entropy is always non-negative. The maximum value of the entropy can be obtained when all of the eigenvalues associated with the density matrix are the same, i.e. if, after being diagonalised,

$$\rho_{diag} = \begin{pmatrix} \frac{1}{n} & & \\ & \ddots & \\ & & \frac{1}{n} \end{pmatrix}$$

, where the entries $\frac{1}{n}$ follows from the property that $tr\rho = 1$ if the density matrix is an $n \times n$ matrix. The corresponding entropy is

$$\rho = -n \left[\frac{1}{n}\log_2 \left(\frac{1}{n} \right) \right] = \log_2 (n)$$

On the other hand, the minimum value of the entropy is zero which occurs if and only if the state is pure. This leads to the most notable basic property of von Neumann entropy, i.e. the entropy of an entangled pure state is zero while the entropy of any of its subsystems is positive (because the reduced density matrices corresponding to the subsystems represent mixed states as discussed before.) [9]. In general, it is possible for the overall entropy of a system to be less than the sum of the entropies associated with its individual subsystems, in the present of entanglement. This is called subadditivity property which occurs in any entangled pure multipartite state. Hence, von Neumann entropy can be employed to be a measure of entanglement for a pure bipartite state which would be discussed in more details in the following chapter.

2 Entanglement Criteria and Measures for Qubit

In this chapter, qubit which is the most simple quantum system will be discussed. Qubit is an example of a discrete variable (DV) system. The understanding of the concepts presenting in this chapter will help to understand similar concepts in continuous variable systems later.

2.1 Terminology

2.1.1 bit

A bit is an abstract entity encoding two possible states, denoted in general as 0 and 1. In computer sciences, information is represented in bits. Thus, bit can be stated as a fundamental unit of information.

2.1.2 Qubit

The word qubit is the combination of two words, quantum and bit. Hence, qubit is a fundamental unit of information in quantum information and quantum computation sciences. A qubit system can be implemented by any quantum systems whose Hilbert space can be spanned by only two orthogonal states, so it is the simplest discrete variable quantum system.

An example of this kind of system is the spin state of a spin-1/2 particle. Because of this simplicity, qubit systems have been studied to a considerable extent and a number of applications based on them have been proposed consistently. The most manifest difference between a classical bit and a qubit is that the state of a qubit can also be a combination between 0 and 1 (written in terms of state vectors), for example, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which results from the property of linear combination of quantum states. The factor $\dfrac{1}{\sqrt{2}}$ is the normalization factor. This possibility is the source of the advantage of a qubit over a classical bit.

2.2 Entanglement Criterion

2.2.1 Partial Trace Criterion for Pure State

Entanglement is an interesting and important phenomenon of a quantum system as being discussed in the previous chapter. It is crucial then to find methods to determine whether a pair of qubits is entangled or not. In the previous chapter, a simple method of partial trace has been shown to be able to identify entanglement of a pure state by observing the reduced density matrix. The state is entangled if the reduced density matrix represents a mixed state and is separable if the reduced density matrix remains a pure state. However, the partial trace method is limited to only pure states. To be able to cope with every qubit state, either pure or mixed one, another method called Peres-Horodecki criterion has to be employed.

2.2.2 Peres-Horodecki Criterion

Peres-Horodecki criterion (sometimes called Positive Partial Transpose: PPT) is a simple *separable condition* for a bipartite state. Unlike the partial trace

method, this criterion can be used for both pure and mixed states as well. It was first proposed as a necessary condition for every separable state by Asher Peres in 1996 [12]. By being a necessary condition, it means that every separable state must satisfy this condition, though some entangled states satisfy it as well. Later, Horodecki studied the criterion in more details and discovered that it was not only a necessary but also sufficient condition for separable states of 2×2 and 2×3 dimensions. Thus for a bipartite qubit state (2×2 dimensions), this criterion can be exploited to confirm exactly whether a state is entangled or separable.

The description and the principle underlying the criterion can be explained as follows. First, for every physical state, the density matrix must be positive-semidefinite, i.e. all of its eigenvalues must not be negative. Next, consider a separable state,

$$\rho = \sum_{j=1}^{m} p_j \rho_{1j} \otimes \rho_{2j}.$$

To be valid, all of the density matrices must be positive-semidefinite. Let T be an operator performing transposition, i.e. $T(A) = A^T$, for an arbitrary square matrix A. Then to perform transposition on the first subsystem (partial transpose) is equivalent to operate $T \otimes I$ on the whole system (I is the identity operator). This gives,

$$\rho^{T_1} = (T \otimes I) \rho = \sum_{j=1}^{m} p_j T(\rho_{1j}) \otimes I(\rho_{2j}) = \sum_{j=1}^{m} p_j \rho_{1j}{}^T \otimes \rho_{2j}.$$

Because transposition does not change the eigenvalues of a matrix, $\rho_{1j}{}^T$ must still be positive-semidefinite. Thus, $\rho_{1j}{}^T$ is a valid density matrix representing a physical state. It implies directly that ρ^{T_1} is also a valid den-

sity matrix, representing a physical separable bipartite system and must be positive-semidefinite. This is the Peres' necessary condition: Every physical separable bipartite density matrix, when being applied by a partial transpose, must still be a physical density matrix, i.e. all of its eigenvalues must still not be negative. It was shown by Horodecki, as mentioned above, that this is also a sufficient condition for a qubit system which means any qubit states that do not satisfy this criterion are entangled.

To demonstrate the power of this criterion, it is best to consider a Werner state,

$$\rho = p \left| \Psi^+ \right\rangle \left\langle \Psi^+ \right| + (1 - p) \frac{I}{4}$$

where p is as before, a positive value represent probability, ranged from 0 to 1. This is a mixed state which is composed of one of the maximally entangled states: a Bell's states,

$$\begin{aligned}
\left| \Psi^+ \right\rangle &= \frac{1}{\sqrt{2}} (\left| 0 \right\rangle \left| 1 \right\rangle + \left| 1 \right\rangle \left| 0 \right\rangle) \\
\left| \Psi^+ \right\rangle \left\langle \Psi^+ \right| &= \frac{1}{2} [(\left| 0 \right\rangle \left\langle 0 \right| \otimes \left| 1 \right\rangle \left\langle 1 \right|) + (\left| 0 \right\rangle \left\langle 1 \right| \otimes \left| 1 \right\rangle \left\langle 0 \right|) + \\
&\quad (\left| 1 \right\rangle \left\langle 0 \right| \otimes \left| 0 \right\rangle \left\langle 1 \right|) + (\left| 1 \right\rangle \left\langle 1 \right| \otimes \left| 0 \right\rangle \left\langle 0 \right|)]
\end{aligned}$$

, and a maximally mixed state, the identity matrix I. This mixture is a function of p. It will become a pure state when $p = 0$ or 1. The details about entanglement measures and a proof to show that this particular state $\left| \Psi^+ \right\rangle$ is one of the maximally entangled states will be postponed until the next section. The task now is to find the critical proportion of the mixture p_{crit} which plays the role of the phase transition point for the whole system to transform from a separable state to an entangled state and vice versa. This is again going to show a subtle relationship between mixed states and

entanglement. To help to track the calculation easier, it is better to rewrite the density in the matrix form as follows,

$$|0\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle 0| \longrightarrow (1 \ 0)$$

$$|1\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle 1| \longrightarrow (0 \ 1)$$

$$|\Psi^+\rangle\langle\Psi^+| \longrightarrow \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\rho &= p\,|\Psi^+\rangle\langle\Psi^+| + (1-p)\frac{I}{4} \\
&= \frac{1}{2}p\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{(1-p)}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & 2p & 0 \\ 0 & 2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}.
\end{aligned}$$

$$(2.1)$$

This density matrix has to be positive-semidefinite, i.e. all of its eigenvalues must not be negative. Next, applying transpose to the first subsystem only

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(partial transpose). This will provide,

$$T(|\Psi^+\rangle\langle\Psi^+|) = \frac{1}{2}[(|0\rangle\langle 0| \otimes |1\rangle\langle 1|) + (|1\rangle\langle 0| \otimes |1\rangle 0)$$

$$+ (|0\rangle\langle 1| \otimes |0\rangle\langle 1|) + (|1\rangle\langle 1| \otimes |0\rangle\langle 0|)]$$

Hence,

$$\rho^{T_1} = \frac{1}{2}p\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{(1-p)}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 2p \\ 0 & p+1 & 0 & 0 \\ 0 & 0 & p+1 & 0 \\ 2p & 0 & 0 & 1-p \end{pmatrix}$$

By Peres-Horodecki criterion, ρ is separable if and only if both ρ and ρ^{T_1} are positive-semidefinite and it will be entangled if there exist, at least, one negative eigenvalue. Hence, the transition point must be the lowest positive value of p that makes one of the matrices to be singular, i.e. has a zero eigenvalue. This means the determinant of that matrix will vanish. Therefore, the transition point can be found by solving for the positive smallest solutions of two algebraic equations: $\det(\rho) = 0$ and $\det\left(\rho^{T_1}\right) = 0$. For this particular case,

$$\det(\rho) = 0$$

$$\left(\frac{1}{4}\right)^4 (1-p)\left[(p+1)^2(1-p) - (1-p)(2p)(2p)\right] = 0$$

$$(1-p)^2\left[(p+1)^2 - 4p^2\right] = 0$$

$$(1-p)^3 (3p+1) \;=\; 0$$

$$\det\left(\rho^{T_1}\right) \;=\; 0$$

$$\left(\frac{1}{4}\right)^4 \left[(1-p)^2 (p+1)(p+1) - (2p)^2(p+1)^2\right] \;=\; 0$$

$$(1-p)^2(p+1)^2 - (2p)^2(p+1)^2 \;=\; 0$$

$$(p+1)^3 (1-3p) \;=\; 0$$

The lowest positive value of p which is the transition point of the state from separable to entangled is $1/3$, i.e. $p_{crit} = 1/3$. Any state whose p is higher than p_{crit} will be entangled.

2.3 Entanglement Measures

After being able to identify entangled states, it is natural to quantify the degree of entanglement. There exist several measures, but what would be presented in this report are only the von Neumann entropy and negativity. The reasons for choosing these two measures are because von Neumann entropy is simple and powerful as the unique measure for bipartite pure states, whereas the negativity is closely related to the Peres-Horodecki criterion and will be shown later in chapter 4 to be generalized as a measure for continuous state.

2.3.1 Von Neumann Entanglement Measure for Pure State

Von Neumann entropy briefly discussed in the previous chapter is an entanglement measure for a bipartite pure state. According to this measure, the degree of entanglement of a subsystem with another subsystem is simply di-

rect proportional to its entropy computed from the corresponding reduced density matrix. It can be proved that the entropies of both subsystems are the same, so the amounts of entanglement associated with them are the same.

Because of this direct relation between entropy and entanglement, it can be shown now that the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)$ introduced before is one of the maximally entangled states.

$$\rho = |\Psi^+\rangle\langle\Psi^+| = \frac{1}{2}[(|0\rangle\langle 0| \otimes |1\rangle\langle 1|) + (|0\rangle\langle 1| \otimes |1\rangle\langle 0|)$$
$$+ (|1\rangle\langle 0| \otimes |0\rangle\langle 1|) + (|1\rangle\langle 1| \otimes |0\rangle\langle 0|)]$$

Finding the reduced density matrix of subsystem 1,

$$\begin{aligned} \rho^1 &= tr_2(\rho) \\ &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \end{aligned}$$

This is a diagonal matrix. Hence the diagonal entries are eigenvalues. Because both eigenvalues are equal, this density matrix possesses maximum entropy. By the definition of the measure, it is proved that the Bell state is the maximally entangled state as being claimed. Finding von Neumann entropy of the subsystem 1,

$$\begin{aligned} S_1 &= -tr(\rho^1 \log_2 \rho^1) \\ &= -\frac{1}{2}\log_2(\frac{1}{2}) - \frac{1}{2}\log_2(\frac{1}{2}) \\ &= \log_2 2 \\ &= 1 \end{aligned}$$

This is the magnitude of the entanglement of system 1 which, in this particular case, is the maximum value. The von Neumann entropy is zero for a separable state and increase to the maximum value of one for a maximally entangled state such as the Bell states. It can be proved that the magnitude of entanglement of system 2 always has the same value as that of system 1 for pure state.

Von Neumann entropy has been proved to be the only measure of entanglement for a bipartite pure state and it is an available measure for a general pure state. However, for a mixed state, the situation is more complicated. Thus, another measure is required.

2.3.2 Negativity

Not only be used for detecting entanglement of a qubit system, but the Peres-Horodecki criterion can also be extended to be a measure called negativity. To be rigorous, negativity is not exactly an entanglement measure, but entanglement monotone. It quantifies the amount of entanglement that is useful for various task in quantum information processing, e.g. teleportation. However, for qubit and Gaussian state, introduced in the next chapter, negativity can be thought of as a measure. This simple idea of negativity for qubit was proposed in [13]. The idea is based on the result after applying partial transpose to the system being examined. The Peres-Horodecki criterion states that all of the eigenvalues of the partial-transposed density matrix must not be negative for every separable state. For qubit systems, this was shown to be equivalent to the statement that there must be, at least, one negative eigenvalue for every entangled density matrix being partial transposed [14]. The idea of negativity is that the degree of entanglement reflects in the summation of the negative eigenvalues

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of the partial-transposed density matrix of the state. The description of the measure may be made clearer by expressing in terms of mathematics as follows [13],

$$\varepsilon(\rho) = 2\sum_i (-\lambda_i^-)$$

, where $\varepsilon(\rho)$ is the negativity and λ_i^- is the ith negative eigenvalue of ρ. Because, $tr\rho = 1$, it is manifest that $0 \leq \varepsilon(\rho) \leq 1$.

To demonstrate the use of this measure, the Bell state $|\Psi^+\rangle$ is employed again. Performing partial transpose on subsystem 1,

$$\rho^{T_1} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

, where ρ^{T_1} is the density matrix after being performed partial transpose on subsystem 1. This matrix has eigenvalues $-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$. Thus, the summation of negative eigenvalue is just $-\frac{1}{2}$ and the negativity $\varepsilon(\rho)$ is

$$\begin{aligned} \varepsilon(\rho) &= 2\sum_i (-\lambda_i^-) \\ &= 2 \times \frac{1}{2} \\ &= 1. \end{aligned}$$

It is noteworthy that both quantifications provide the same interpretation that the Bell state is the maximally entangled state.

Though negativity is first derived from the special case of Peres-Horodecki criterion, i.e. when the criterion is applied to qubit states and become both necessary and sufficient separability conditions, negativity was proved to

still be available for systems of higher dimension by [15]. In chapter 4, this measure is generalized to Gaussian continuous variable systems.

3 Continuous Variable System

This chapter is about the introduction to continuous variable (CV) system. Some basic terminologies and related mathematics will be presented. Phase space distribution and Gaussian/Non-Gaussian state will be discussed at the end. Entanglement of CV systems will be described in chapter 4 and 5.

3.1 Motivation

In his 1964 seminal paper, On the Einstein Podolsky Rosen paradox [1]. Bell examined correlation of spins between subsystems in a bipartite state and proposed his famous Bell's inequality to distinguish classical and quantum correlations of states. His work did not only initiate the verifiable study of the foundation of quantum mechanics, but also influence the use of qubits and other discrete variable (DV) systems as crucial tools in the field of quantum information. However, in the original EPR paper, Einstein, Podolsky, and Rosen did not proposed their thought experiment based on spins which are discrete variables. They investigated correlation of positions and momentums which are continuous variables.

A reason that made Bell adapt the EPR experiment might be that DV systems is simpler to handle mathematically. In addition, solely by DV, a number of novel fundamental understanding and applications are available, at least, in principle. However, DV systems are limited. The complete

description of many quantum systems, e.g. quantized electromagnetic field coupled to dissipative environment [11] and vibrational degree of freedom of trapped ions [16] are properly realized by continuous variable (CV) systems. Also, it is usually more difficult to experimentally deal with a DV system efficiently. This can be obviously impressed by comparing an experiment performing on a pair of photons to that performing on a pair of laser beams. The former experiment requires special equipments designed to be able to detect signals at very low intensity while the latter one does not. Hence, it is interesting and crucial, both practically and fundamentally, to extend the study about entanglement and other counter-intuitive quantum properties to continuous cases.

3.2 Basic Definitions and Mathematics

3.2.1 Linear Combination of Continuous Basis

Basic mathematical treatments for continuous variables are similar to those for discrete cases. A simple general rule is to change a linear combination from summation with respect to discrete basis to integration with respect to continuous basis instead. For instances, a state $|\psi\rangle$ can be expressed as a linear combination in position basis as,

$$|\psi\rangle = \int dx \varphi\left(x\right) |x\rangle$$

, where $|\varphi(x)|^2$ can be interpreted to be the probability of finding the system between the interval x and $x + dx$ which is called probability density. To be able to determine only a small interval, not an exact value, is a nature of continuous systems which distinguish them from discrete states. Given a

state $|\psi\rangle$, the probability density, corresponding to the interval x' to $x'+dx$, can be found by performing inner product as follows,

$$|\phi(x')|^2 = |<x'|\psi>|^2.$$

By following this general rule, the mathematics concerning expectation value, and variance are also similar to the discrete case [10]. Other definitions which are not related to the expansion into continuous basis do not change. This is not only restricted to the position basis. It is the same for any other continuous variables, e.g. momentum, as well.

To be rigorous, it must be emphasized that the position eigenstate, $|x\rangle$, and any other eigenstates which form continuous bases are not in Hilbert space. This follows from the fact that inner products of these states give Dirac delta functions which make these states cannot be normalized. Therefore, these states alone do not represent any physical entities, though they can be used as bases to span physical states.

3.2.2 Position, Momentum, Creation, and Annihilation Operators

Creation and annihilation operators have been discussed briefly in chapter 1. These operators are corresponding to a harmonic oscillator system which is really important in the analysis of any analytic potential and oscillation. In terms of position operator ($\hat{x}$) and momentum operator ($\hat{p}$), they can be expressed as follows [17, 10],

$$a \;\; = \;\; \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} + i\hat{p})$$

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$$a^\dagger \quad = \quad \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} - i\hat{p})$$

, where ω is the angular frequency corresponding to the oscillator.

The two equations above can be rearranged to be,

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger)$$

$$\hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}(a - a^\dagger).$$

The commutation relation between $\hat{x}$ and $\hat{p}$ follows directly from the commutation relation between a and $a^\dagger$.

$$[\hat{x}, \hat{p}] = i\hbar.$$

3.2.3 Quadratures

Two new operators can be defined from the definition of the annihilation operator [17],

$$\hat{X} \quad = Re(a) = \quad \sqrt{\frac{\omega}{\hbar}}\hat{x}$$
$$\hat{P} \quad = Im(a) = \quad \sqrt{\frac{1}{\hbar\omega}}\hat{p}.$$

By substituting $\hat{x}$ and $\hat{y}$ into the definitions of the new operators, they will become,

$$\hat{X} = \frac{1}{\sqrt{2}}(a + a^\dagger) \quad \hat{P} = \frac{1}{\sqrt{2}i}(a - a^\dagger).$$

The commutation relation between these operators is,

$$[\hat{X}, \hat{P}] = i.$$

The definition of quadratures in general is two objects which are 90 degree different in angle. In this context, the two operators, $\hat{X}$ and $\hat{Y}$ are 90 degree apart from each other on a complex plain, so they can be called quadrature operators. In addition, because they are only different from the position and momentum operators by constant factors, they will be referred to as $\hat{x}$ and $\hat{p}$ from now on. In quantum optics, it is these quadrature observables that are continuous.

3.2.4 Mode

Optical systems are important because most experiments in quantum information are conducted by means of optics. Because of that, optical systems have to be quantized. One of the most basic ideas of quantum optics is the quantization of electromagnetic wave.

An electromagnetic wave is a system of oscillating electromagnetic fields. Thus, due to the fact that every well-behaved oscillation can be expressed as a sum of sinusoidal waves (Fourier analysis), which are the characteristic motion of harmonic oscillator systems, it is sensible to represent the wave by a set of harmonic oscillators. Harmonic oscillator systems are categorised by their characteristic frequencies. Each distinct frequency is called mode. Hence, an electromagnetic wave can be decomposed into modes of oscillation [11].

Associate to each mode are a creation and an annihilation operators. Distinct modes have distinct operators which commute between distinct

modes. Let subscript k denote the mode with wave number k. A quantized single mode electromagnetic field can be expressed in terms of creation and annihilation operators as follows [17],

$$\hat{E}_k(\mathbf{r}, t) = E_0(a_k e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t+\phi)} + a_k^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t+\phi)})$$

, where E_0 and ϕ is a real constant. Rewritten the expression,

$$\hat{E}_k(\mathbf{r}, t) = E_0[(a_k e^{i\phi}+a_k^\dagger e^{-i\phi})\cos(\mathbf{k}\cdot\mathbf{r}-\omega_k t)+i(a_k e^{i\phi}-a_k^\dagger e^{-i\phi})\sin(\mathbf{k}\cdot\mathbf{r}-\omega_k t)]$$

The coefficients of the sine and cosine terms are $\frac{\pi}{2}$ radian apart. This can be shown explicitly by rotating the coefficient of the cosine term by $\frac{\pi}{2}$

$$(a_k e^{i(\phi+\frac{\pi}{2})} + a_k^\dagger e^{-i(\phi+\frac{\pi}{2})}) = i(a_k e^{i\phi} - a_k^\dagger e^{-i\phi})$$

Therefore, these are generalised quadrature operators which are continuous functions of phase angle, denoted as $\hat{x}_k^\phi$ and $\hat{p}_k^\phi$. They will return to be in the same form as the definition of quadrature operators defined before when $\phi = 0$. Finally, the field is written in terms of these generalized quadrature operators as,

$$\hat{E}_k(\mathbf{r}, t) \quad = \quad \sqrt{2}E_0[\hat{x}^\phi \cos(\mathbf{k}\cdot\mathbf{r} - \omega_k t) - \hat{p}^\phi \sin(\mathbf{k}\cdot\mathbf{r} - \omega_k t)].$$

The quadrature observables can be measured by a technique called *homodyne detection*. It can be seen that these variables are continuous because of the continuum of the phase angle ϕ.

3.3 Phase Space and Wigner Distribution

3.3.1 Phase Space Representation

Phase space is an abstract space used for representing states of a system in terms order pairs of positions and momenta. Classically, a state can be represented as a point in a phase space because both position and momentum of the state is allowed to have a precise value. However, this is not permitted in quantum mechanics because of the uncertainty principle. A pure quantum state must be represented by a smear that is characterized by its uncertainty instead of a point.

Because of the similarity between quadrature operators and position and momentum operators, it is convenient to employ phase space representation for studying the behaviours of continuous variable systems. This can be done by changing from the representation in terms of position and momentum order pairs to the representation in terms of order pairs of quadrature operators.

3.3.2 Wigner Distribution: A Quantum Phase Space Distribution

In classical mechanics, statistical values of an ensemble of states, e.g. the mean and variance of an observable, can be studied by its corresponding probability distribution in phase space. This is similar for an ensemble of quantum states. In quantum mechanics, an ensemble is equivalent to a mixed state which is described by a density matrix. Nevertheless, unlike a classical ensemble that has unique probability distribution, there are many possible phase space distributions corresponding to a density matrix [17]. This is a consequence of the non-commutative property of quantum oper-

ators, in particular creation and annihilation operators. Each distribution is suitable for a specific type of operators, though it is possible to use any distribution with any operator. Among all, three distributions are mostly exploited: Glauber-Sudarshan P-function, Wigner function, and Husimi Q-function. P-function and Q-function are appropriate distributions for operators that are expressed in terms of normal and anti-normal order of a and $a^\dagger$ respectively whereas Wigner distribution is proper for the symmetric ordered ones. Therefore, Wigner function is an appropriate distribution for the quadrature operators for they are symmetric-ordered [11, 17]. Since quadrature observables are the main continuous variables of interest in this dissertation, the details about Wigner function is discussed in more details in the following.

A general form of a Wigner representation of a single system density matrix is [18, 19],

$$W(x,p) = \frac{1}{\pi} \int dy (e^{2iyp} \langle x - y| \rho |x + y\rangle)$$

, where x, y and p are quadratures. The limit of the integral is all over the space. Several properties of a Wigner function are as follows [20, 17, 11],

$$\begin{aligned}
\int\int W(x,p)dxdy &= 1 \\
\int W(x,p)dx &= \langle p| \rho |p\rangle \\
\int W(x,p)dp &= \langle x| \rho |x\rangle.
\end{aligned}$$

The first equation above is the normalization condition. The second and the third equations show how one-dimensional distributions can be obtained from two-dimensional ones. These one-dimensional distributions are called

marginal distributions. The nth statistical moment of an operator $\hat{M}$ can be calculated by

$$\langle M^n \rangle = tr(\rho \hat{M}^n) = \int \int W(x,p) M^n(x,p) dx dp$$

, where $M(x,p)$ is a unique function corresponding to the operator $\hat{M}$, defined on phase space. Suppose that $S(\hat{x}^m, \hat{p}^n)$ is a symmetric ordered operator. An example for $m = 2$ and $n = 2$ is $S(\hat{x^2}, \hat{p^2}) = x^3 p^2 + pxp + p^2 x$ [11, 17]. There is a linear convention between a quantum operator and a classical one defined on phase space called Weyl correspondence [19] which can be computed by Wigner function as follows [17],

$$\left\langle S(\hat{x^m}, \hat{p^n}) \right\rangle = tr(\rho S(\hat{x^m}, \hat{p^n})) = \int \int W(x,p) x^m p^n dx dp.$$

Though it is a kind of distribution, Wigner function is not a proper probability distribution but a quasi-probability one. This is because it fails to satisfy the positive semidefinition which is a necessary property of a proper probability distribution. Nevertheless, due to its other elegant properties that fit well with the requirements of a quantum representation of phase space probability distribution for symmetric ordered operators, it is still exploited as a very efficient tool for providing phase space description of quantum ensembles [20, 17, 11]. As would be presented in the next chapter, this distribution plays an important role in the extension of Peres-Horodecki separable criterion from DV to CV systems.

3.4 Gaussian States

A Gaussian state is the most simple but non-trivial class of continuous variable systems [16]. It is characterized as a state whose Wigner function is a Gaussian function. Hence, only information about mean and variance are sufficient to describe a state. It is important and interesting not only because of its simple mathematics, but also because of its broad applications, for examples, many optical systems including LASER and thermal radiation can be well realized as Gaussian states. There are three important subclasses of Gaussian states: thermal state, coherent state, and squeezed state.

3.4.1 Thermal state

A thermal state arises in the situation when the system of interest is in thermodynamic equilibrium. This means that the only available information about the system is the average energy defined in terms of its temperature [11]. Thus it is a maximally mixed state constrained by energy, i.e. a quantum canonical ensemble, which cannot be represented by a single state vector. A thermal state is a good approximation of vacuum at non-zero temperature. In general, a density matrix of a thermal state at temperature T is in the form [11],

$$\rho = \frac{\exp(-\beta \hat{H})}{tr\left[\exp(-\beta \hat{H})\right]}$$

. , where $\hat{H}$ is a Hamiltonian operator, $\beta = \dfrac{1}{k_B T}$ is the thermodynamic beta, and k_B is the Boltzmann constant. For a single mode, this can be further manipulated to be,

$$\rho = \frac{\exp(-\beta \hbar \omega \hat{n})}{tr\left[\exp(-\beta \hbar \omega \hat{n})\right]}$$

, where ω is the frequency of the mode and $\hat{n}$ is the number operator.

Using this density matrix, the Wigner distribution for a single mode thermal state is [21],

$$W_{th}(x,p) = \frac{1 - e^{-\omega\beta}}{\pi(1 + e^{-\omega\beta})} \exp\left[\frac{-(1 - e^{-\omega\beta})}{1 + e^{-\omega\beta}}(x^2 + p^2)\right].$$

Obviously, this is a Gaussian function.

3.4.2 Coherent State

This is the most important state that represents LASER operating far above threshold and is also the basis for many other concepts in the field of quantum optics [11]. It was first applied to explain electromagnetic fields by Roy J. Glauber, a 2005 Nobel laureate in physics, in his 1963 work.

A Glauber displacement operator is defined as

$$\hat{D}(\alpha) = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}$$

A coherent state $|\alpha\rangle$ is created from a vacuum state $|0\rangle$ by Glauber displacement operator as follows,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle.$$

It is the eigenstate of the annihilation operator, i.e.

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$

, where $|\alpha\rangle$ denote the coherent state [11]. Because $a = \left(\dfrac{\hat{x} + i\hat{p}}{\sqrt{2}}\right)$, where $\hat{x}$ and $\hat{p}$ are quadrature operators, it is sensible to infer that $\alpha = \dfrac{x + ip}{\sqrt{2}}$.

The density matrix of a single mode coherent state is simple $|\alpha\rangle\langle\alpha|$. It

can be proved that the Wigner function for a single mode coherent state is in the form,

$$W_{co}(x, p) = \frac{4}{\pi} e^{-\frac{1}{2}|(x^2 + p^2)|}.$$

, which is manifestly a Gaussian function.

3.4.3 Squeezed State

Squeezed states arise purely from properties of quantum mechanics and play a very important role as a basis for many non-linear optical phenomena. The characteristic of this state is that the variance of one of the quadrature $\hat{x}$ (or $\hat{p}$) is less than $\frac{1}{2}$ which is the lowest bound in the cases of both vacuum and coherent states [11]. This brings about together with a compensation in the rise of the variance of the other operator $\hat{p}$ (or $\hat{x}$) such that the uncertainty principle is not violated (i.e. $\sigma_{\hat{x}}^2 \sigma_{\hat{p}}^2 \geq \frac{1}{4}$). According to the fact that the uncertainty of a system is equivalent to the area it possesses on phase space, the representation of a squeezed state forms an ellipse on phase space, in contrast to a vacuum and a coherent states which form circles. Hence, by performing a squeezing operator on a coherent state, the circle will be squeezed and become an ellipse of identical area.

The squeezed operator is [11]

$$
\begin{aligned}
\hat{S}^{\dagger}(\zeta) &= e^{-\frac{\zeta}{2}\hat{a}^{\dagger 2} + \frac{\zeta^*}{2}\hat{a}^2} \\
&= \exp\left[-\frac{1}{2}\hat{a}^{\dagger 2} \exp\left(i\varphi\right) \tanh r\right] \exp\left[-\frac{1}{2}\left(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}\right) \ln\left(\cosh r\right)\right] \\
&\quad \exp\left[\frac{1}{2}\hat{a}^2 \exp\left(-i\varphi\right) \tanh r\right]
\end{aligned}
$$

, where $\zeta = re^{i\varphi}$ and φ and r are real numbers characterizing the operator. This is essentially Glauber displacement operator with $\hat{a}^2$ instead of $\hat{a}$.

The single mode squeezed vacuum state $|\zeta\rangle$ is created from the vacuum state $|0\rangle$ by this squeezing operator as follows,

$$|\zeta\rangle = \hat{S}(\zeta)|0\rangle.$$

It can be proved that the Wigner function for a single mode vacuum squeezed state, with $\varphi = 0$, is in the form [11],

$$W_{sq}(x,p) = \frac{2}{\pi}\exp\left\{-2\left[\exp\left(-2r\right)p^2 + \exp\left(2r\right)x^2\right]\right\}.$$

, which is, again, a Gaussian function.

3.5 Non-Gaussian States

An example of states in this class is the number of particle state, i.e. Fock state. The characteristic of a non-Gaussian state is that its Wigner function is not Gaussian. Hence, only the first and the second statistical moments are not enough to describe it and the mathematics concerning this state is more difficult than a Gaussian one. Also, it is harder to deal with in the experimental aspects because it requires other techniques than the Gaussian operations which are more complicating or sensitive to conditions, such as photon counting. It is an advantage of CV system for enable easier implementation on real experiments compared to DV system. However, it is possible that the complication of non-Gaussian state might devalue this preference at some points. Anyway, there are ideas of using non-Gaussian states to enhance the efficiency of quantum technology. For example, in [22], it was shown that the possibility of long distance quantum communications based on continuous variable systems is confirmed by allowing non-Gaussian

operations in the protocols. Furthermore, it was recently discovered by [23] that non-Gaussian states can improve an efficiency and robustness of a quantum teleportation protocol significantly compared to Gaussian states. Hence, non-Gaussian systems are also interesting and important, though there are several drawbacks due to its complication. Currently, the knowledge about this kind of state is increasing because a great number of quantum researchers are putting their efforts into this field of research. Fruitful results are expected to be achieved in the near future.

4 Gaussian Entanglement

4.1 Overview

In the previous chapter, a brief introduction to continuous states and some of its mathematical treatments are presented. It has been shown how to represent a quantum ensemble in a phase space picture by a quasi-probability distribution called Wigner function. By that, states can be categorized by the forms of their corresponding Wigner functions as Gaussian and non-Gaussian states. The prestige status of Gaussian states is briefly described together with several important subclasses of the states.

In this chapter, a separability criterion proposed by R. Simon [18], which is basically a generalization of the Peres-Horodecki criterion to CV systems, and two entanglement measures: von Neumann entropy and negativity for CV systems [24] are explored. In addition, it will be shown that the simplicity of a Gaussian state provides an impressive result, i.e. it enhances the separable condition, from just a necessary one for a general CV state, to be also a sufficient condition in the case of a Gaussian state. Hence, it becomes also an entanglement criterion that can be employed to test exactly whether a Gaussian state is entangled or not.

Before the criterion and measures are presented, it is important to define what an entanglement is in CV systems, so that the following discussion

will be placed on a firm ground of understanding. Just like DV systems, a bipartite CV separable state is the state that can be expressed in terms of a summation of tensor products between two subsystems density matrices, i.e. it can be written as

$$\rho = \sum_j p_j \rho_1{}^j \otimes \rho_2{}^j.$$

An entangled state is the state that cannot be expressed in this form. The result of an entanglement is that the properties and behaviours of the subsystems correlate to each other in a non-trivial fashion. The only difference between DV and CV systems is that, while DV is described by a discrete basis, CV is described by a continuous basis instead such as the quadratures.

4.2 Peres-Horodecki Criterion for CV system

4.2.1 Partial Transpose as Reflection of Momentum

A two-mode CV system ρ can be expressed in terms of a Wigner function as [18],

$$
\begin{aligned}
W(x_1, p_1, x_2, p_2) &= \left(\frac{1}{\pi}\right)^2 \int \int dy_1 dy_2 (e^{2i(y_1 p_1 + y_2 p_2)}) \times \\
&\quad \sum_j p_j \left\langle x_1 - y_1 \right| \rho_1{}^j \left| x_1 + y_1 \right\rangle \otimes \left\langle x_2 - y_2 \right| \rho_2{}^j \left| x_2 + y_2 \right\rangle
\end{aligned}
$$

, where x_1, x_2, p_1 and p_2 are the quadrature variables defined in chapter 3. Performing transposition only on the subsystem 2 (partial transpose). It will become

$$
W^{PT}(x_1, p_1, x_2, p_2) = \left(\frac{1}{\pi}\right)^2 \int \int dy_1 dy_2 (e^{2i(y_1 p_1 + y_2 p_2)}) \times
$$

$$\sum_j p_j \langle x_1 - y_1 | \rho_1{}^j | x_1 + y_1 \rangle \otimes \langle x_2 + y_2 | \rho_2{}^j | x_2 - y_2 \rangle$$

,where the superscript PT denotes partial transpose. Redefine the variable y_2 to be $-y'_2$.

$$
\begin{aligned}
W^{PT}(x_1, p_1, x_2, p_2) &= \left(\frac{1}{\pi}\right)^2 \int \int dy_1 (-dy'_2)(e^{2i(y_1 p_1 - y'_2 p_2)}) \\
&\quad \sum_j p_j \langle x_1 - y_1 | \rho_1{}^j | x_1 + y_1 \rangle \otimes \langle x_2 - y'_2 | \rho_2{}^j | x_2 + y'_2 \rangle \\
&= \left(\frac{1}{\pi}\right)^2 \sum_j p_j \int dy_1 (e^{2i(y_1 p_1)}) \langle x_1 - y_1 | \rho_1{}^j | x_1 + y_1 \rangle \otimes \\
&\quad \int (-dy'_2) e^{(-p_2)y'_2} \langle x_2 - y'_2 | \rho_2{}^j | x_2 + y'_2 \rangle \\
&= W(x_1, p_1, x_2, -p_2).
\end{aligned}
$$

The equation on the first line is nearly in the same form with the original Wigner function, except for minus signs. On the second line, the minus sign in the exponential is grouped with p_2. Since $p_2 = \dfrac{dy'_2}{dt}$, it is obvious that $-p_2 = \dfrac{-dy'_2}{dt}$. Thus, the expression in the second and the third line are exactly equal. This shows an obvious equivalence between partial transpose operation and the reflection of a momentum direction.

According to Weyl correspondence, Wigner distribution relates quantum operators to their corresponding phase space variables [17] in which, for this situation, the operators and variables are the quadratures. For convenience, the quadratures are put together into a column vector. The quadrature

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operators and variables are expressed as follows,

$$\hat{\xi} = \begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \hat{x}_2 \\ \hat{p}_2 \end{pmatrix}, \quad \xi = \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix}$$

A matrix related to the commutators between the quadrature operators, Ω, can be written as [24, 18], Simon]

$$\Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $[\hat{\xi}_\alpha, \hat{\xi}_\beta] = i\Omega_{\alpha\beta}$ and α and $\beta = 1, 2, 3, 4$. The result of performing a partial transpose operation on the Wigner function can be seen to be equivalent to performing a operator Λ on ξ, where $\Lambda = \mathrm{diag}(1, 1, 1, -1)$.

4.2.2 Uncertainty Principle as Separable Criterion

According to quantum mechanics, every physical state is required to fulfil the uncertainty principle. In fact, Nha and Zubairy shows in [7] that the converse is also true, i.e. every state that satisfies the uncertainty principle is a physical state in quantum mechanics. In other words, it is the difference between the second statistical moment and the squared of the first moment that constrain a physical state. Therefore, the satisfaction of the uncertainty relation can be exploited to check whether a state is permitted or not, i.e. it can be used as a separable criterion in a similar way that the positivity of a density matrix does. In addition, because whether a state is mixed or pure, entangled or separable, does not depend on the first or the second statistical

moment but its difference in terms of covariance, the first moment can be set to be nil for convenience without any loss of generality.

To define the uncertainty principle for quadrature operators, a covariance matrix V must be defined as follows [18],

$$
\begin{aligned}
V_{\alpha\beta} &= \left\langle \left\{ \Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta \right\} \right\rangle \\
&= tr\left(\left\{ \Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta \right\} \hat{\rho} \right) \\
&= \int d^4\xi \, \Delta\xi_\alpha \Delta\xi_\beta W(\xi)
\end{aligned}
$$

, where

$$
\begin{aligned}
\Delta\hat{\xi}_\alpha &= \hat{\xi}_\alpha - \left\langle \hat{\xi} \right\rangle \\
\Delta\xi_\alpha &= \xi_\alpha - \langle \xi \rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\left\{ \Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta \right\} &= (\Delta\hat{\xi}_\alpha \Delta\hat{\xi}_\beta + \Delta\hat{\xi}_\beta \Delta\hat{\xi}_\alpha)/2 \\
&= \frac{1}{2}\left\langle \hat{\xi}_\alpha \hat{\xi}_\beta + \hat{\xi}_\beta \hat{\xi}_\alpha \right\rangle - \left\langle \hat{\xi}_\alpha \right\rangle \left\langle \hat{\xi}_\beta \right\rangle .
\end{aligned}
$$

This is a symmetric ordered function. Note that there is an equivalence between the average calculated by performing trace with a density matrix and that calculated by an integral of Wigner function according to Weyl correspondence.

The uncertainty principle in this context can be written as [24],

$$
V + \frac{i}{2}\Omega \geq 0 \tag{4.1}
$$

Every physical state needs to fulfil this relation. The proof is as follows,

$$
\begin{aligned}
V_{\alpha\beta} + \frac{i}{2}\Omega_{\alpha\beta} &= \frac{1}{2}\left\langle \hat{\xi}_\alpha \hat{\xi}_\beta + \hat{\xi}_\beta \hat{\xi}_\alpha \right\rangle - \left\langle \hat{\xi}_\alpha \right\rangle \left\langle \hat{\xi}_\beta \right\rangle + \frac{i}{2}\Omega_{\alpha\beta} \\
&= \frac{1}{2}\left\langle 2\hat{\xi}_\alpha \hat{\xi}_\beta - i\Omega_{\alpha\beta} \right\rangle - \left\langle \hat{\xi}_\alpha \right\rangle \left\langle \hat{\xi}_\beta \right\rangle + \frac{i}{2}\Omega_{\alpha\beta} \\
&= \left\langle \hat{\xi}_\alpha \hat{\xi}_\beta \right\rangle - \left\langle \hat{\xi}_\alpha \right\rangle \left\langle \hat{\xi}_\beta \right\rangle = \left\langle \hat{\xi}_\alpha \hat{\xi}_\beta \right\rangle
\end{aligned}
$$

, where on the second line, the commutation relation is employed. The conclusion on the last line is the result of choosing the first moment to be zero. Thus only the value of the second moment is involved. This is equivalent to $V + \frac{i}{2}\Omega = tr\left(\rho\hat{\xi}^2\right)$ which, from the positivity of ρ, must not be negative. This completes the proof. Moreover, because $tr\left(\Omega\right) = 0$, it means some of the eigenvalues of Ω are not positive. Hence, it can imply further that $V \geq 0$. This inequality is the starting point of the criterion for CV system.

It is simple to extend this relation to other Hermitian operators. This is done by employing the fact that the quadratures form a complete basis on phase space, so every Hermitian operator of a two-mode state can be defined uniquely by a matrix of four real numbers. For example an operator $\chi(d)$ can be expressed as $\chi(d) = d^T\hat{\xi} = d_1\hat{x}_1 + d_2\hat{p}_2 + d_3\hat{x}_1 + d_4\hat{p}_2$, where d is the matrix containing the four real number $d_i, i = 1, 2, 3, 4$. [18] The corresponding uncertainty relation can be obtained by the same procedure as that for ξ.

Similar to the Peres-Horodecki for DV systems, by performing partial transpose, another inequality is obtained. As it is shown above that the effect of partial transpose operation on a Wigner function is equivalent to operating $\Lambda = \text{diag}(1,1,1,-1)$ to ξ, the partial transpose will change the

former inequality $V + \frac{i}{2}\Omega \geq 0$ to be

$$\Lambda V \Lambda + \frac{i}{2}\Omega \geq 0. \qquad (4.2)$$

By the Peres-Horodecki criterion, this inequality must also be fulfilled by every separable state.

4.2.3 Invariant Form of Separable Condition

Two subsystems in a bipartite state are typically separated apart and operations normally perform locally to one of the subsystems. This local operation together with classical communication denoted as LOCC is a very important operation in quantum information. Thus, a practical separable criterion should take this local operation into account too. To develop the two inequalities to satisfy this task, the covariance matrix is examined.

A covariance matrix of a two-mode state is a 4×4 skew-symmetric matrix, the symmetry of which is described by a symplectic group, $Sp(4, R)$. A symplectic group is a group of operations that preserves a skew-symmetric matrix. $Sp(4, R)$ means symplectic group of 4×4 dimensional matrix with only real entry. The covariance matrix is a skew-symmetric because of the symplectic relation between quadrature operators. In this case, the elements of the group are corresponding to any unitary operators acting on the system. By imposing the condition of local operation, the operator can be divided into two blocks with respect to each system [18]. Let S_{local} be an element of the group, it is expressed in a matrix-form as,

$$S_{local} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

, where $S_1, S_2 \in Sp(2, R)$. The local operation constraint reduces the former symmetry group to be one of its subgroup, $Sp(2, R) \otimes Sp(2, R)$.

This form of the symmetry operator suggests that the form of the covariance matrix of a separable state, i.e. the state that is preserved under local operation, should be

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

, where A, B and C are 2×2 matrices. The unitary operator in the form described above operate on each block of the covariance matrix as follows,

$$A \to S_1 A S_1^T \quad B \to S_2 B S_2^T \quad C \to S_1 C S_2^T.$$

This operation must preserve several entities such as $\det A, \det B, \det C$, and $tr A J C J B J C^T J$, where J is the skew-symmetric matrix that has been presented before. These entities altogether can form another inequality that is equivalent to the separable inequality presented, with an additional property of manifestly invariance under local operation.

$$\det A \det B + \left(\frac{1}{4} - \det C \right)^2 - tr \left(A J C J B J C^T J \right) \geq \frac{1}{4} \left(\det A + \det B \right)$$

$$(4.3)$$

This equivalence between the two inequalities (4.1) and (4.3) can be proved directly by substituting terms calculated from a specific matrix written in

the form,

$$
V_0 = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix} .
\tag{4.4}
$$

, into both (4.1) and (4.3) inequalities.

However, it was proved by Duan [25] that it is always possible to transform any covariance matrix into the above form by a proper local transformation. Hence, every covariance matrix is proved to satisfy the new inequality above.

Next, the equivalence inequalities to (4.2) followed from (4.3) will be derived. Partial transpose operation is equivalent to the similarity transformation by the matrix Λ, i.e. $\tilde{V} = \Lambda V \Lambda$, where $\tilde{V}$ is the transformed covariance matrix. By considering how each block of V, i.e. A, B and C, is transformed, it appears that only block C is changed, i.e.

$$
c_2 \to -c_2 \qquad \det C \to -\det C
$$

It is easy to verify straightforwardly that the trace term will not change. Finally, this leads to the change of (4.3) to be

$$
\det A \det B + \left(\frac{1}{4} + \det C \right)^2 - tr\left(AJCJBJC^T J \right) \geq \frac{1}{4} \left(\det A + \det B \right)
\tag{4.5}
$$

It is different from (4.3) only in the sign in front of the $\det C$ term. This inequality is equivalent to (4.2). A separable state must satisfy both (4.3) and (4.5).

It is even more convenient to express the inequalities as follows,

$$\det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - tr\left(AJCJBJC^T J\right) \geq \frac{1}{4}\left(\det A + \det B\right).$$

(4.6)

The absolute sign enables the combination of (4.3) and (4.5) into one. Thus, this is the final form of the Peres-Horodecki criterion for continuous state proposed by Simon [18].

Before continuing to the next section, it is best to summarise again that (4.1) and (4.3) are the conditions in which every physical or permitted state must satisfies while the satisfaction of (4.2) and (4.5) are only necessary for separable states. It should be emphasised that, in general, the inequalities (4.2) and (4.5) are only necessary conditions for a CV separable state, but not sufficient ones. Therefore, they are not entanglement necessary condition.

4.2.4 Gaussian States: From Separability to Entanglement Criterion

Nevertheless, this is not the case for Gaussian states. Due to the simplicity of the states, the Peres-Horodecki separability criterion becomes also a sufficient condition. This means it can be employed to detect exactly whether a Gaussian state is entangled or separable. A Gaussian state is simple because the complete description of the state can be made by only two information, the first and the second statistical moments. Furthermore, it has been discussed that it is the variance that constrain a physical state and it is possible to assign a zero mean value to the state without any loss of generality. Therefore, the separability of a Gaussian state can be completely

described by the knowledge of its second moment solely. This becomes more manifest when considering the Wigner function of a general Gaussian state, $W(\xi) = \left(4\pi^2\sqrt{\det V}\right)^{-1}\exp\left(-\frac{1}{2}\xi^T V^{-1}\xi\right)$, which is characterized only by V. According to [18], a Gaussian state is separable if $V \geq \frac{1}{2}$. Thus, the remaining task is to show mathematically that the Gaussian state that fulfils the separable inequality (4.6) is separable by checking its covariance matrices.

Comparing between (4.3) and (4.5). (4.3) is the condition that every physical state has to obey while only separable states follow (4.5). Hence, a conclusion that can be drawn immediately is a state with $\det C \geq 0$ always fulfils both condition simultaneously. In other word, it always fulfil (4.6). Hence, first, this class of states is required to check whether all of them are separable.

Every physical state can provide a variance matrix V_0 in the form of (4.4). Two local squeezing operators which are elements of $Sp(2,R) \otimes Sp(2,R)$ can be formed as $S_{local1} = \text{diag}(x, x^{-1}, x^{-1}, x)$ and $S_{local2} = \text{diag}(y, y^{-1}, y^{-1}, y)$. After being operated by the two local squeezing operators, the covariance matrix will become [18],

$$V_0' = S_{local1}S_{local2}V_0S_{local2}S_{local1} = \begin{pmatrix} y^2x^2a & 0 & y^2c_1 & 0 \\ 0 & y^{-2}x^{-2}a & 0 & y^{-2}c_2 \\ y^2c_1 & 0 & y^2x^{-2}b & 0 \\ 0 & y^{-2}c_2 & 0 & y^{-2}x^{-2}b \end{pmatrix}$$

Suppose first that $\det C > 0$. It is possible to choose $x = [(c_1a + c_2b)/(c_2a + c_1b)]^{1/4}$. By performing rotations with an equal amount of angle on $x_1 - x_2$ and $p_1 - p_2$ planes, the covariance matrix can be manipulated to be diagonal, denoted as $V_0'' = \text{diag}(\kappa_+, \kappa_+', \kappa_-, \kappa_-')$, where $\kappa_\pm = \frac{1}{2}[y^2x^2a +$

$x^{-2}b \pm [(x^2 a - x^{-2}b)^2 + 4c_1^2]^{1/2}]$ and $\kappa'_\pm = \frac{1}{2}y^{-2}\{x^{-2}a + x^2 b \pm [(x^{-2}a - x^2 b)^2 + 4c_2^2]^{1/2}\}$. Consider $V_0'' + \frac{i}{2}\Omega \geq 0$ which is the condition that every state has to fulfil. By choosing an appropriate y, it is possible to set all the eigenvalues of V_0'' to be greater than $\frac{1}{2}$. Hence, $V_0'' \geq \frac{1}{2}$. This means that the state corresponding to V_0'' is separable. Note that rotations of the two planes by an equal angle are just the change of basis. It does not affect the uncertainty relation, so the state corresponding to V_0' is also separable. Finally, because V_0' and V_0 are related by local canonical transformations, the conclution is V_0 is separable for $\det C > 0$.

For $\det C = 0$ case, suppose that $c_1 \geq 0$ and $c_2 = 0$. It is possible to transform V_0 using a diagonal matrix $\text{diag}\left(\sqrt{2a}, 1/\sqrt{2a}, \sqrt{2b}, 1/\sqrt{2b}\right)$. This will result in,

$$V_0' = \begin{pmatrix} 2a^2 & 0 & 2abc_1 & 0 \\ 0 & 1/2 & 0 & 0 \\ 2abc_1 & 0 & 2b^2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

, where V_0' is the transformed V_0. The inequality associated to this covariance matrix, $V_0' + \frac{i}{2}\Omega \geq 0$, implies that $V_0' \geq \frac{1}{2}$. Hence the state associated to this covariance is separable and the original state is proved to be separable too.

Eventually, it has been shown that, the states with $\det C \geq 0$ must be separable. This is a sufficient condition. Combining together this result with the inequality (4.6), the Peres-Horodecki separability criterion for CV states become also an entanglement criterion. It can be employed to check exactly which Gaussian state is separable or entangled. Explicitly, for $\det C \geq 0$, the states are confirmed to be separable automatically. For $\det C < 0$, there are two possibility. If the states violate (4.6), then they are definitely

entangled. On the other hand, if they satisfy the condition, then so do their mirror reflection counterparts with $\det C > 0$. Thus, their counterparts are separable states. This means, in return, the original states are separable since the mirror reflection of a separable state is also separable. The last note for this section is that the sufficient condition arises because Gaussian state can be completely described by the second moment.

4.3 CV Entanglement Measures

There exists several entanglement measures for CV system. In this section, only two of them will be presented: von Neumann entropy (or entropy of entanglement) and negativity. These are just generalisations of the measures from DV state. As usual, the measures for Gaussian states are based on variances.

Recall that, in thermal equilibrium, a density matrix can be expressed as [11],

$$\rho = \frac{exp(-\beta \hat{H})}{tr[exp(-\beta \hat{H})]}$$

Represent in Fock state using number basis, the density matrix for one mode can be recasted as,

$$\rho = (1 - exp(-\beta \hbar \omega)) \sum_{n=0}^{\infty} exp(-\beta \hbar \omega n)$$

The term before summation is computed using the convergence of a geometrical series.

The average number of a boson (e.g. photon) is calculated by

$$\bar{n} = tr(\rho \hat{n})$$

$$\begin{aligned}
&= (1 - exp(-\beta\hbar\omega)) \sum_{n=0}^{\infty} (n) exp(-\beta\hbar\omega n) \\
&= (1 - exp(-\beta\hbar\omega)) \left(-\frac{1}{\hbar\omega}\frac{d}{d\beta} \right) \sum_{n=0}^{\infty} exp(-\beta\hbar\omega n) \\
&= [exp(\beta\omega\hbar) - 1]^{-1}
\end{aligned}$$

Thus,

$$exp(-\beta\hbar\omega) = \frac{\bar{n}}{1+\bar{n}}$$

Substitute this back, the density matrix is represented in terms of the average $\bar{n}$ as,

$$\rho = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle\langle n|$$

Consider the covariance matrix of a bipartite Gaussian state, V. Williamson theorem confirms that it can always be transformed into a diagonal matrix, $diag(\nu_1, \nu_1, \nu_2, \nu_2)$ [18, 24]. These diagonal elements are corresponding to the terms, $\hat{x}_1^2, \hat{p}_1^2, \hat{x}_2^2$ and $\hat{p}_2^2$, respectively. The quadrature operators can be represented in terms of creation and annihilation operators as $\hat{x}_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right)$ and $\hat{p}_i = \frac{1}{\sqrt{2}i}\left(a_i - a_i^\dagger\right)$, $i = 1, 2$. The variances corresponding to these operators in number basis have the forms [24],

$$\begin{aligned}
\nu_{x_i} = \nu_{p_i} &= \langle n|\frac{1}{2}\left(a_i^\dagger a_i + a_i a_i^\dagger\right)|n\rangle \\
&= \langle n|\frac{1}{2}\left(2a_i^\dagger a_i + 1\right)|n\rangle \\
&= \bar{n} + \frac{1}{2}
\end{aligned}$$

Thus, $\bar{n}_i = \nu_i - \frac{1}{2}$. This suggests that every density operator of a Gaussian bipartite state can be written in the form a tensor product between two

single-mode density operators as follows,

$$\rho^{\otimes} = \underset{i}{\otimes} \frac{2}{2\nu_i + 1} \sum_{n=0}^{\infty} \frac{(2\nu_i - 1)^n}{(2\nu_i + 1)^n} |n\rangle\langle n|$$

Finally, the density matrix is written in terms of variance as required.

4.3.1 Entropy of Entanglement

Similar to its DV counterpart, the entropy of entanglement or von Neumann entropy is the unique measure of entanglement for a bipartite pure state. Follow from the definition of the von Neumann entropy, it is shown in [26, 17, 24, 27] that

$$S_i = \left(\frac{2\nu_1 + 1}{2}\right) \log_2 \left(\frac{2\nu_1 + 1}{2}\right) - \left(\frac{2\nu_1 - 1}{2}\right) \log_2 \left(\frac{2\nu_1 - 1}{2}\right)$$

, where S_i is the entropy of the subsystem i, for $i = 1, 2$. Nevertheless, for pure state, $S_1 = S_2$. The range of value for this particular measure is from 0 for a separable state to 1 for a maximally entangle state.

4.3.2 Negativity

The idea of negativity is discussed in chapter 2. Briefly, the original state is partial-transposed. The negativity is equal to the summation of the negative eigenvalues of the partial-transposed density matrix.

A partial-transposed density matrix can be represented in the form,

$$\rho^{\otimes PT} = \underset{i}{\otimes} \frac{2}{2\tilde{\nu}_i + 1} \sum_{n=0}^{\infty} \frac{(2\tilde{\nu}_i - 1)^n}{(2\tilde{\nu}_i + 1)^n} |n\rangle\langle n|.$$

From the definition of negativity in chapter 2,

$$c = 2\sum_i (-\lambda_i^-).$$

This is equivalent to

$$\varepsilon(\rho) = tr|\tilde{\rho}| - 1.$$

Substitute the density matrix in the form of variance into the above equation. Consider the term,

$$tr|\tilde{\rho}| = \frac{2}{2\tilde{\nu}_i + 1} \sum_{n=0}^{\infty} \frac{(2\tilde{\nu}_i - 1)^n}{(2\tilde{\nu}_i + 1)^n}$$

For $\tilde{\nu}_i \geq \frac{1}{2}$, this term is equal to 1. For $\tilde{\nu}_i < \frac{1}{2}$, this term is equal to $\frac{1}{\tilde{\nu}_i}$. Therefore, the negativity is equal to

$$\varepsilon(\rho) = max[0, \frac{1}{2\nu} - 1].$$

The state is separable if $\varepsilon(\rho) = 0$. One of the advantages of negativity is its simplicity and wide availability. Thus, it is greatly used in the study of Gaussian state.

5 Non-Gaussian Entanglement

5.1 Overview

The previous chapter is devoted for Gaussian states. It has been shown that Gaussian states are special because of its simplicity, nice features, and broad applications. However, as discussed briefly in chapter 3, non-Gaussian entanglement is also crucial to study because it provides promising results leading to the improvement of several important communication protocols. It is claimed in [17] that the true power of quantum mechanics can only be exploited via non-Gaussian states and non-Gaussian interactions. An example of non-Gaussian state is the number of photon state in quantum optics.

A guiding idea to develop entanglement criteria for non-Gaussian states is hinted in the Peres-Horodecki criterion for Gaussian state. As the criterion for Gaussian states is expressed in terms of variance, an interesting question to ask is if it is also possible and convenient to formulate an entanglement condition for non-Gaussian states in terms of variance.

This question has positive answers. In this chapter, entanglement criteria for non-Gaussian states from two papers are reviewed. The first one is proposed by Mark Hillery and M. Suhail Zubairy in 2006 [6] and the other one is published in 2008 by Hyunchul Nha and M. Suhail Zubairy [7]. For

convenience to refer to these two works later on, the former will be denoted as H-Z whereas the latter will be refered to as N-Z. The underlying principle of both criteria are uncertainty principles. In fact, the differences between these criteria are just the choices of operators and uncertainty relations being used. Different choices provide different conditions and require different experimental techniques to measure.

The main ideas and methods of both H-Z and N-Z are very similar. They constructed a particular form of operators and study the uncertainty relations of these operators to seek for necessary conditions for separability which is equivalent to sufficient conditions for entanglement.

The entanglement measures for non-Gaussian states will not be presented in this chapter because they are too complicated. Generally, it is difficult to extend the simple existing measures from discrete variable case to continuous case except for Gaussian states, the simplicity of which enables that generalisation to be simple to deal with.

5.2 Hillery and Zubairy Criteria

H-Z construct two sets of operators. The first set satisties $su(2)$ algebra, so they are spin-like operators. The other set satisfies $su(1,1)$ algebra which is isomorphic to $sp(2,R)$ algebra [28]. The importance of these two groups in quantum optics were shown in [29]. Briefly, $su(2)$ is related to the interactions between atom and photon whereas $su(1,1)$ is related to the non-linear parametric generation and conversion of two photons. H-Z entanglement criteria are based on the comparing between two inequalities, one is derived for every physical states to fulfil while the other one is a necessary condition that every separable state needs to satisfies. Thus, a sufficient condition for

a state to be entangled is concluded to be the satisfaction of the former condition but violate the latter one. The criteria for both groups of operators are separately descibed below. This is essentially what is described in [6] with some additional explanations in details.

5.2.1 su(2) operators

The spin-like operators are defined as $L_1 = ab^\dagger + a^\dagger b$, $L_2 = i(ab^\dagger - a^\dagger b)$, and $L_3 = a^\dagger a + b^\dagger b$, where the operators a and b are annihilation operators associating to different modes. These operators obey the $su(2)$ commutation relation, i.e. $[J_k, J_m] = i\epsilon_{kmn} J_n$, where $J_i = L_i/2$. The variances of the operators can be expressed as

$$(\Delta L_i)^2 = \left\langle (L_i)^2 \right\rangle - \langle L_i \rangle^2$$

for $i = 1, 2$. By using the definition of L_i, it is straightforward to derive the following,

$$(\Delta L_1)^2 + (\Delta L_2)^2 = 2\left[\langle (N_a + 1)N_b \rangle + \langle N_a(N_b + 1) \rangle - 2\left| \left\langle ab^\dagger \right\rangle \right|^2 \right]$$

, where $N_a = a^\dagger a$ and $N_b = b^\dagger b$. For separable state, this equation becomes,

$$(\Delta L_1)^2 + (\Delta L_2)^2 = 2\left[\langle (N_a + 1) \rangle \langle N_b \rangle + \langle N_a \rangle \langle (N_b + 1) \rangle - 2\left| \langle a \rangle \left\langle b^\dagger \right\rangle \right|^2 \right]$$

$$(5.1)$$

According to Cauchy-Schwarz inequality:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Set $y = 1$ and $x = a$, this will give $|\langle a \rangle|^2 \leq \langle a, a \rangle = \langle a^\dagger a \rangle = \langle N_a \rangle$. This is also true for b, i.e. $|\langle b \rangle|^2 \leq \langle N_b \rangle$. Hence,

$$(\Delta L_1)^2 + (\Delta L_2)^2 \geq 2 \left(\langle N_a \rangle + \langle N_b \rangle \right) \tag{5.2}$$

This is a separability condition, but just for product state. To extend it to cover all of the separable states, recall that a general separable state can be expressed as the sum of product states. The variance of an observable B for a separable state $\rho = \sum_m p_m \rho_m$ where, p_m is the probability associated to each density matrix of a product state ρ_m is

$$
\begin{aligned}
(\Delta B)^2 &= tr\left(\rho B^2\right) - [tr\left(\rho B\right)]^2 \\
&= \sum_m p_m tr\left(\rho_m B^2\right) - \left(\sum_m p_m tr\left(\rho_m B\right)\right)^2 \\
&= \sum_m p_m tr\left(\rho_m B^2\right) - \left(\sum_m p_m tr\left(\rho_m B\right)\right)^2 \\
&\quad + \sum_m p_m[tr(\rho_m B)]^2 - \sum_m p_m[tr(\rho_m B)]^2 \\
&= \sum_m p_m \left(\Delta B_m\right)^2_+ \sum_m p_m[tr(\rho_m B)]^2 \\
&\quad - \left(\sum_m p_m tr(\rho_m B)\right)^2 \\
&\geq \sum_m p_m \left(\Delta B_m\right)^2
\end{aligned}
$$

The inequality will become an equation if the expectation value of B with respect to each state ρ_m is vanished. This shows that the variance of any observable calculated with respect to the overall system density matrix of a separable state can never less than the summation of variance of individual subsystems.

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Another inequality can be found by considering, the commutators

$$
\begin{aligned}
(\Delta L_1)(\Delta L_2) &\geq \frac{1}{2i}\left|\langle [L_1, L_2] \rangle\right| \\
\frac{1}{4}[L_1, L_2] &= i\frac{L_3}{2} \\
\frac{1}{2i}[L_1, L_2] &= L_3 = a^\dagger a - b^\dagger b = N_a - N_b
\end{aligned}
$$

Therefore, $(\Delta L_1)(\Delta L_2) \geq |\langle N_a - N_b \rangle|$. This leads further to the relation,

$$
\begin{aligned}
(\Delta L_1 - \Delta L_2)^2 &\geq 0 \\
(\Delta L_1)^2 + (\Delta L_2)^2 &\geq (\Delta L_1)(\Delta L_2) + (\Delta L_2)(\Delta L_1) \\
&\geq 2\left|\langle N_a - N_b \rangle\right| \tag{5.3}
\end{aligned}
$$

This relation, in contrast with (5.2) that is a condition for separable state, (5.3) is a condition for every physical state. It is obvious that there exists some states which satisty only (5.3) but fail to satisfy (5.2). Those states are confirmed to be entangled. Thus, these relations together are necessary condition for separable states and sufficient for entangled states.

By further examining (5.1), the inequality (5.2) will be violated if $\langle N_a N_b \rangle < \left|\langle ab^\dagger \rangle\right|$. This indicates that

$$
\langle N_a N_b \rangle < \left|\langle ab^\dagger \rangle\right|^2 \tag{5.4}
$$

is an entanglement condition. However, Cauchy-Schwarz inequality states that every physical state must obey

$$
\left|\langle ab^\dagger \rangle\right|^2 \leq \langle N_a (N_b + 1) \rangle \tag{5.5}
$$

It is, again, obvious that there exists some states that satisfy both (5.5) and

(5.4) because (5.4) is obviously weaker than (5.5). Hence, (5.4) combines the essences of both (5.2) and (5.3) in determining an entangled state within one condition. The satisfaction of (5.4) is sufficient to confirm that the state in consideration is entangled.

It is possible to generalise inequalities to cover the cases of $a^m(b^\dagger)^n$ in which for $m = n = 1$ would provide the same results as that are already obtained. Consider a pure product state,

$$\left|\left\langle a^m b^{\dagger n}\right\rangle\right|^2 = |\langle a^m\rangle|^2 \, |\langle b^n\rangle|^2 \le \left\langle (a^\dagger)^m a^m\right\rangle \left\langle (b^\dagger)^n b^n\right\rangle = \left\langle (a^\dagger)^m a^m (b^\dagger)^n b^n\right\rangle$$

The inequality is the consequence of Cauchy-Schwarz inequality. To find the condition for a general separable state, it is convenient to define $A = a^m$ and $B = b^n$.

$$
\begin{aligned}
\left|\left\langle AB^\dagger\right\rangle\right| &= \sum_k \left|tr(p_k \rho_k AB^\dagger)\right| \\
&\le \sum_k p_k \left|tr(\rho_k AB^\dagger)\right| \\
&\le \sum_k p_k \left(\left\langle A^\dagger AB^\dagger B\right\rangle_k\right)^{1/2} \\
&\le \left(\sum_k p_k\right)^{1/2} \left(\sum_k p_k \left\langle A^\dagger AB^\dagger B\right\rangle_k\right)^{1/2} \\
&\le \left(\sum_k p_k \left\langle A^\dagger AB^\dagger B\right\rangle_k\right)^{1/2}
\end{aligned}
$$

This proves the validity of the inequality $\left|\left\langle a^m b^{\dagger n}\right\rangle\right|^2 \le \left\langle (a^\dagger)^m a^m (b^\dagger)^n b^n\right\rangle$ for a general separable state.

5.2.2 su(1,1) operators

The operators which satisfy this algebra are expressed in the form, $K_1 = ab + a^\dagger b^\dagger$, $K_2 = i(a^\dagger b^\dagger - ab)$ and $K_3 = a^\dagger a - b^\dagger b$. The mathematical method to find separability condition for these operators is slightly more difficult than that of the $su(2)$ operators. Define a new operator which is a function of a phase ϕ as

$$K(\phi) = e^{i\phi} a^\dagger b^\dagger + e^{-i\phi} ab.$$

This operator becomes K_1 when $\phi = 0$ and become K_2 when $\phi = \dfrac{\pi}{2}$. It is proved in [6] that states are separable if $K(\phi) \geq 1$ and entangled if $K(\phi) < 1$.

It is also possible to find other inequalities corresponding to this class of operators. Consider an example associating to a product state,

$$|\langle ab \rangle| = |\langle a \rangle \langle b \rangle| \leq [\langle a^\dagger a \rangle \langle b^\dagger b \rangle]^{1/2}$$

Therefore,

$$|\langle ab \rangle| \leq [\langle N_a \rangle \langle N_b \rangle]^{1/2} \tag{5.6}$$

As usual, this inequality for product states is still valid for general separable states, i.e.

$$|\langle a^m b^n \rangle| \leq [\langle \left(a^\dagger\right)^m a^m \rangle \langle \left(b^\dagger\right)^n b^n \rangle]^{1/2}. \tag{5.7}$$

Separable states are required to fulfil this condition.

In general, for $m = n = 1$,

$$|\langle ab \rangle| \leq [\langle N_a + 1 \rangle \langle N_b + 1 \rangle]^{1/2}. \tag{5.8}$$

As always, this is the condition for every physical state to fulfil. There exist states that fulfil (5.8) but fail to fulfil (5.6). Those states are confirmed to be entangled. (5.6) together with (5.8) provide another sufficient condition for entanglement. In other words, the satisfaction of $[\langle N_a \rangle \langle N_b \rangle]^{1/2} < |\langle ab \rangle|$ confirms that the state in consideration is entangled.

To manifestly see the power of this criterion, consider a two-mode squeezed vacuum state

$$|\psi\rangle = \left(1 - x^2\right)^{1/2} \sum_{n=0}^{\infty} x^n |n\rangle_a |n\rangle_b$$

, where $0 \leq x \leq 1$. To check the state using (5.6), relevant terms are computed as follows.

$$
\begin{aligned}
|\langle ab \rangle| &= |\langle \psi | ab | \psi \rangle| \\
&= \left| \left(1 - x^2\right) \sum_n^{\infty} x^{2n} n \right| \\
&= \frac{x}{1 - x^2}
\end{aligned}
$$

The second line comes from the action of annihilation operators on the state, i.e. $a |n\rangle_a = \sqrt{n} |n - 1\rangle_a$ and $b |n\rangle_b = \sqrt{n} |n - 1\rangle_b$. Next, the term $\langle N_a \rangle$ is calculated by

$$
\begin{aligned}
\langle N \rangle_a &= \langle \psi | N_a | \psi \rangle \\
&= \left(1 - x^2\right) \sum_n^{\infty} n x^{2n} \\
&= \frac{x^2}{1 - x^2}
\end{aligned}
$$

The term $\langle N_b \rangle$ is equal to this value too. Substitute these values into the inequality (5.6), the inequality will be invalid because $0 \leq x \leq 1$. This disagreement indicates that this state is entangled.

So far, it has been shown that a simple idea of comparing between two inequalities, one to be obeyed by every physical state and another one to be satisfied by every separable state, leads to several criteria both in terms of $su(2)$ and $su(1,1)$ operators. It is discovered that, finally, the two conditions, in each case, can be summarised into one sufficient condition of entanglement. These conditions have an impact in practice because crucial operations in quantum optics can be catagorised mathematically into these two groups.

Next, another criterion invented by Nha and Zubairy will be presented. It will be shown to be a generalised criterion such that the $su(2)$ criterion of H-Z is a special case.

5.3 Nha and Zubairy Criteria

Similar to the previous criteria, in this work, particular operators are constructed and two inequalities are generated and compare to obtain a sufficient condition of entanglement. However, instead of mainly employing Cauchy-Schwarz inequality to generate the inequality relation, N-Z used Sch*ö*dinger-Robertson inequality which is a stronger uncertainty relation compared to the traditional Heisenberg one. The general form of Sch*ö*dinger-Robertson inequality for two operators, A and B, is

$$\left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq \frac{1}{4} \left| [A,B] \right|^2 + \frac{1}{4} \left\langle \Delta A \Delta B \right\rangle_S^2$$

, where $\langle \Delta A \Delta B \rangle_S \equiv \langle \Delta A \Delta B + \Delta B \Delta A \rangle$. Note that this inequality becomes the Heisenberg uncertainty if the last term is vanished.

This is one inequality. Another inequality is obtained via partial-transpose operation, just as being done in the Peres-Horodecki method. For a bipartite continuous variable state, this is equivalent to the reflection of the second component of the quadrature operator of the second subsystem. The fulfilment of both inequalities simultaneously is required if the state under consideration is separable, though some entangled states may also satisfy them in the same way. The violation of one of the inequality confirms that a state is entangled. This is then become a sufficient condition for entanglement.

In fact, N-Z started their paper by showing first that the satisfaction of uncertainty relations is a sufficient condition for a permitted state in quantum mechanics. Then the criterion was derived for discrete variable states. For continuous states, two sets of operators were constructed. Two quadrature opeators, $X_i^{(m)} \equiv a_i^{\dagger m} + a_i^m$ and $Y_i^{(m)} \equiv -i \left(a_i^{\dagger m} - a_i^m \right)$, for $i = 1, 2$, are defined for the first set. These operators can be combined together to become Hermitian operators: $H_1 = X_1^{(m)} + X_2^{(n)}$ and $H_2 = Y_1^{(m)} + Y_2^{(n)}$. A separable condition in the form of Schrödinger-Robertson inequality is straighforwardly expressed as,

$$\Delta^2 H_1 \Delta^2 \tilde{H}_2 \geq \left\langle C_1^{(m)} + C_2^{(n)} \right\rangle^2 + \left\langle \Delta H_1 \Delta \tilde{H}_2 \right\rangle_S^2$$

, where $C_i^{(m)} \equiv [a_i^m, a_i^{\dagger m}]$ and $\tilde{H}_2 \equiv Y_l^{(m)} - Y_2^{(n)}$. This form of $\tilde{H}_2$ is equivalent to the reflection of momentum on phase space which is the result of partial-transpose operation acting on the density matrix.

Another set of operators are defined as $X_{mn} \equiv a_1^{\dagger m} a_2^n + a_1^m a_2^{\dagger n}$ and $Y_{mn} \equiv$

$-i\left(a_1^{\dagger m}a_2^n - a_1^m a_2^{\dagger n}\right)$ which, again, provide Hermitian operators as follows, $H_1 = a_1^{\dagger m}a_2^{\dagger n} + a_1^m a_2^n$ and $H_2 = -i\left(a_1^{\dagger m}a_2^{\dagger n} - a_1^m a_2^n\right)$. These operators provide an inequality in the form,

$$\left(\Delta^2 X_{mn} + \left\langle C_1^{(m)}C_2^{(n)}\right\rangle\right)\left(\Delta^2 Y_{mn} + \left\langle C_1^{(m)}C_2^{(n)}\right\rangle\right) \geq \left\langle[a_1^m a_2^n, a_1^{\dagger m}a_2^{\dagger n}]\right\rangle^2$$
$$+ \left\langle\Delta X_{mn}\Delta Y_{mn}\right\rangle_S^2$$

These two inequalities have an amazing feature that they can be shown to be a generalisation of several existing criteria [25, 30, 31, 32, 7] in a sense that those other criteria are special cases of this one. It can be proved that even H-Z criteria presented in the previous section are also a special case of these two inequalities. An experimental test to verify these criteria is seemed to be possible according to an experimental scheme proposed by Shchukin and Vogel [33]. Therefore, N-Z criterion is very promising.

5.4 Discussion

Before ending this chapter, there are two points that should be emphasised. Firstly, the uncertainty principle has been exploited as the crucial underlying principle of every criterion discussed, even the Peres-Horodecki one in the previous chapter. It has also been proved in [7] that the fulfilment of the uncertainty principle is both a necessary and sufficient condition for a valid quantum physical state. Secondly, the criteria that have been discussed here act as sufficient conditions, but not necessary ones. This means that there may exist some entangled states which fulfil the separable inequalities, thus cannot be detected by these methods. To ensure that an entangled state is recognised, a necessary and sufficient condition is required. However, such condition has not been discovered and it might not exist. This is a very dif-

ficult problem in which many researchers in quantum information are trying to solve. In practice, because there is no perfect single criterion, many criteria may have to be employed together to increase the efficiency of the test. For example, the advantage of Peres-Horodecki method, introduced in chapter 4, is that it is not only a sufficient but also a necessary condition of entanglement of Gaussian states. However, it cannot detect some entangled states in Fock basis. On the other hand, Hillery-Zubairy condition has an advantage of being able to detect entangled Fock states, but it fails to be a necessary condition. In the situation that requires higher accuracy, various criteria must be exploited together.

6 Final Remark

Several entanglement criteria and measures has been discussed both for discrete variable systems and continuous variable systems. Start from the most basic definitions and formalisms in quantum mechanics, the argument has been gradually develop step-by-step, from Hilbert space, state vectors and density matrices to the definition of variance and entropy, from pure and mixed states to separable and entangle states, and from the detials about DV states to Gaussian states and non-Gaussian states in CV systems. The Peres-Horodecki separability criterion, first proposed and proved to be a necessary and sufficient condition for separability of a class of low dimensional DV states, is later proved to provide a necessary condition for separability of general CV states. Specifically for Gaussian states, the Peres-Horodecki becomes also a sufficient condition for separable states which means that it can be used to detect precisely whether a Gaussian state is separable or entangled. Moreover, the criterion is shown to provide density matrices with negative eigenvalues which can be exploited as an entanglement measure for DV and Gaussian states, called negativity. For non-Gaussian state, several criteria are presented following from the results of the works of Hillery and Zubairy in 2006 and Nha and Zubairy in 2008.

Entanglement is a unique property which distinguish the quantum description of the world from the traditional classical ones. In the context of

pure sciences, entanglement is an exciting property which is highly counter-intuitive but true. The understanding of the transition from a separable state which is classical to an entangled state which is purely quantum is absolutely important for the more comprehensible understanding of quantum mechanics. Also, the existence of the upper limit of entanglement points to something which is much more fundamental than what is known todays. In terms of application, it is this non-trivial non-local correlation of states that enables the possibility to develop breakthrough technologies. Hence, inventions of entanglement criteria and measures are really important as necessary tools to study this amazing phenomenon efficiently. Nevertheless, until now, there is no perfect entanglement criterion and measure for non-Gaussian CV state. Thus, it is a very crucial task to develop a better criterion and measure for this particular system in the near future. It has been described in the text that the generalised entanglement criterion proposed by Nha and Zubairy [7] is seemed to be very powerful and promising. It may enable to unify existing entanglement criterion into one. Therefore, it is a good starting point for a future work to continue.

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