Monopoles and Confinement

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Contents

1 Introduction

1.1 Dirac Approach to Magnetic Monopoles

The first great example of unification in physics is the one of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, when electromagnetic induction was discovered and it was established that magnetism is created by the motion of electric charge. The complete behaviour of the electromagnetic field was summarized by James Clerk Maxwell in 1864, in 4 elegant equations (we omit the factors of permittivity and permeability, dependent on the propagating medium):

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho, \qquad & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad & \nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{1}$$

where ρ and $\mathbf{j}$ are, respectively, the electric charge density and the electric current. There is a notorious asymmetry in these equations, due to the non-existence of magnetic charge density and magnetic current. This is equivalent to saying that there are no particles that are a source for the magnetic field. It is quite natural to ask if they exist, even though we never realised it.

The first to consider the theoretical possibility of a magnetic monopole was Paul Dirac in 1931. The monopole was then thought of as the end of a solenoid with infinitesimal radius and extended until infinity. The first difficulty that arises when we include a magnetic monopole in the electromagnetic theory is the fact that if $\nabla \cdot \mathbf{B} = \rho_m$, with ρ_m a non-zero magnetic charge density, the magnetic field cannot be expressed as a total rotational of the vector potential $\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$. Dirac found that in order to define both vector potential and magnetic charge in the theory, the vector potential had to be infinite in one direction, say $\hat{\mathbf{n}}$, so the magnetic field could be expressed as

$$\mathbf{B} = \nabla \times \mathbf{A} - 2\pi\delta^{(3)}(\hat{\mathbf{x}} - \hat{\mathbf{n}}). \tag{2}$$

This singularity in one direction is commonly called the *Dirac string*, and can be physically associated with the infinite solenoid from the monopole to infinity. We should note, however, that the vector potential is not uniquely defined, that is, it is possible to change the direction of the singularity by performing a gauge transformation of the field which leaves all the physics invariant. Therefore, the Dirac string is not a gauge invariant property and cannot be detected experimentally.

From the impossibility of observation of the Dirac string, Dirac showed [1] that if magnetic monopoles exist, the electric charge must be quantized, i.e, all electric charges must be integer multiples of a fundamental unit, e. The *Dirac quantization condition* reads

$$qg = 2\pi\hbar n, \tag{3}$$

where q and g are, respectively, any electric and magnetic charges occuring in Nature.

Electric charge quantization is actually observed in Nature, but no one has ever found another explanation for this phenomena. This argument makes many theoretical physicists strongly believe that magnetic monopoles must exist. A very well-known quote due to Dirac expresses the general feeling, “one would be very surprised if God hadn't made use of it”.

1.2 Magnetic Monopoles from QFT Concepts - Topological Defects

In Dirac's approach to magnetic monopoles nothing can be said about the mass of the monopole, but it is a natural question if we want to understand it as a new particle. A new formalism leading to a deeper understanding of magnetic monopoles was proposed by 't Hooft and Polyakov [2, 3] in 1974, when they found that a specific solution of a topological defect in quantum field theory could accurately describe a monopole. Furthermore, they found a general and very important feature about unification theories which are spontaneously broken to the one or more abelian subgroups $U(1)$ [4]:

Theorem 1.1. *Only if the underlying gauge group is compact, and has a compact covering group, must electric charges in the $U(1)$ gauge groups be quantized, and whenever the covering group of the underlying gauge group is compact, magnetic monopole solutions can be constructed.*

This result came in a time when physicists had already lost hope in the idea of magnetic monopoles, firstly because of Schwinger's failure to construct a consistent theory of monopoles after trying very hard for a long time [5], and secondly because the experimental results were very discouraging. 't Hooft and Polyakov's way of thinking about magnetic monopoles came as a fresh start in the field, and ever since all the implications have been largely studied.

A *topological defect* is a configuration of the fields whose symmetry was spontaneously broken due to the degeneracy of the potential minima, with the particularity that it doesn't assume the same vacuum expectation value (VEV) in the whole system. This phenomena is very common in condensed matter, as it is observed in liquid crystals, ferromagnets and many other systems. In quantum field theory, this idea can be systematically studied by observing whether there exist stationary and stable solutions to the field equations whose energy is finite but still assume different vacua in different points of space. As we will see, these topological defects are fundamentally different from the usual vacuum configuration (the trivial solution), where the VEV is the same everywhere. In QFT, most of the results come from perturbative expansions around the vacuum configuration, but topological defects cannot be obtained perturbatively, so they became a fascinating new field of study both for physicists and mathematicians.

Inspired on the two-dimensional topological defect, which simulates a non-dissipative wave or *soliton*, solutions of this type may also be called *topological solitons*. In this dissertation these terms will be used interchangeably.

1.3 Observation of Magnetic Monopoles

Up until today no one has ever observed a magnetic monopole. However, there are many difficulties in such an observation, since in all unification models where the properties of the monopole can be calculated in detail we find that it has a very large mass. In particular, some models estimate a mass of the order of 10^{16} GeV, which corresponds to 10^{-8} g, comparable to the mass of some bacteria [6].

The two main tools that humanity can use for detecting new particles are human-made particle accelerators, such as the Large Hadron Collider, and cosmic rays, which are essentially particles with a very large range of energies accelerated by natural processes in the universe. These hit the Earth constantly, and can be detected in space observatories.

Particle accelerators have a limited range of energy, therefore it is hardly surprising that no one has ever seen a monopole event. All that can be done with this technology is to impose a lower bound on the mass of the monopole [5]. Hence, the hope to detect magnetic monopoles lies only on cosmic rays.

The temperature of today's universe is too low to create monopoles. Nevertheless, the standard cosmological scenario predicts that monopoles were created in the early stages of our universe and therefore we should expect a small but existing flux of magnetic monopoles [5]. The fact that we do not observe them can be taken in two different ways: either our model of the universe's history is wrong and should be reformulated, or it is not a relevant piece of information, since there are many experimental difficulties in observing monopoles due to their slowness and scant number in cosmic rays. Magnetic monopoles continue to challenge experimentalists to find better ways to seek them.

If one day magnetic monopoles are indeed observed, it would provide evidence that there is a Grand Unified Theory (GUT) underlying the Standard Model, as it has been conjectured by many physicists in the past decades.

1.4 QCD and the Problem of Confinement

The defining characteristic of the strong interactions between quarks is that the interactions become arbitrarily weak at short distances (ultraviolet), a property that is generally called *asymptotic freedom*, and extremely strong at large distances (infrared), so the quarks stay bound to each other within hadron particles, called *permanent quark confinement*. This can be easily understood by a simple analogy with a rubber band connecting each pair of quark and antiquark particles. If we try to pull the two particles apart, the energy stored in the band will increase linearly with the separation, until it reaches sufficient energy, $2m_q$, that it becomes energetically more favorable to create another quark-antiquark pair and form another hadronic bound state. This procedure can be repeated, and the more energy we put in the system the more quark-antiquark pairs will be created, but never a single quark left loose. This implies that quarks will always appear confined in bound states. On the other hand, if we push the two particles close to each other, the rubber band will become soft, with no energy stored in it, and the particles will behave like there is no force at all connecting them besides the remaining Coulomb force due to their electrically charged character.

It is believed that this interaction is mediated by $SU(3)$ gauge fields, *gluons*, minimally coupled to the quark fields such that they can be described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \sum_f \bar{\psi}_f(iD_\mu\gamma^\mu - m_f)\psi_f, \tag{4}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu \tag{5}$$

with f_{abc} the stucture constants of the $su(3)$ Lie algebra, and the summation index f identifying the quark type, or its “flavour”. Note that quarks of different flavours have distinct masses. QCD has three types of charges, called *colours*, and one important characteristic of the strong force is that the gluons also carry colour, so they not only mediate the interaction, but also interact themselves.

The coupling constant g is a dimensionless parameter in the action, and, in the limit where the quark masses can be neglected, there is no intrinsic energy scale in QCD, thus the Lagrangian encodes the behaviour of the theory in different energy scales. However, we cannot understand both regimes, ultraviolet and infrared, with only one coupling constant g, equal in every energy scale. Therefore, we expect the effective coupling constant of the theory to be a function of the energy scale Λ, $\bar{g} = \bar{g}(\Lambda)$.

We can use perturbation theory to calculate scattering amplitudes in QCD. It is possible to compute the propagators and interaction vertices between quarks and gluons from the Lagrangian (4), and we find that the theory will contain divergences due to particles' self-energy. By renormalization schemes we can introduce a energy scale in the theory, and separate the divergent bits of the scattering amplitudes from the finite and physical part that now depends on the energy scale.

It is possible to find the dependence of the effective coupling constant on the scale of the theory and we see that it flows to zero in the ultraviolet limit and becomes large when we approach the infrared. However, when we approach the IR limit and the coupling constant increases, the calculation method fails, since perturbation theory only works for small couplings, and a non-perturbative proof that the coupling blows up in the IR limit does not exist.

There is still no satisfactory understanding of the dynamical mechanism behind quark confinement. However, it is believed that the mechanism should exist independently of the presence of quarks in the theory. A free Yang-Mills theory should encode, somehow, the confining mechanism. Note that, since the force that causes quark confinement depends only on their colour charge, any other particles with colour should also be confined, and only be observed in colourless combinations.

Since gluons also carry colour, this means that they are also confined, forming combinations called *glueballs*. These glueballs are massive, which means that there are no massless particles in QCD. This mass gap explains the short range observed for the strong force.

Topological defects in the Yang-Mills theory, such as *instantons*, allowed the discovery of a vacuum structure of QCD much more complex that one would naively expect. It became broadly accepted that the confining mechanism should be fundamentally based on this complex vacuum structure. Since then, confinement due to instantons has been proved in some non-abelian Yang-Mills models. QCD, however, remains unsolved.

In this dissertation we will prove confinement in the three-dimensional Georgi-Glashow model, based on the instanton solutions in Euclidean space-time which happen to have the same form as the 't Hooft-Polyakov solutions in four dimensions mentioned before.

1.5 Structure of the Dissertation

In this dissertation only the topological point of view of magnetic monopoles will be explored. An effort was made to present a self-contained discussion of topological defects, only with the strictly necessary details such that all calculations are well justified.

We begin by giving conditions on which classical field theories can accept soliton solutions and then studying three particular cases in detail. Firstly, two two-dimensional theories are solved with the goal of motivating what a topological defect is. Secondly, the soliton solutions on the Georgi-Glashow model in four-dimensional Minkowski space-time are studied. We find that, in this theory, the 't Hooft-Polyakov monopole emerges, and consequently we study the electromagnetic character of this solution.

We will then quantize the monopole solution and understand how the Hilbert space of a system containing magnetic monopoles looks like.

This is followed by the introduction of the concept of instantons, and the explanation of how they play the interesting role of introducing tunneling events in quantum field theory. We will find that instantons in d dimensions have the same form as static soliton solutions in $d+1$ dimensions, allowing us to understand the instantons in the three-dimensional Georgi-Glashow model as magnetic monopoles.

Finally we will see that these monopole/instanton events allow us to prove the existence of a mass gap in the theory and the confinement of electrically charged particles, following the proof given by Polyakov in his work: "Quark confinement and topology of gauge theories" [7].

Throughout this dissertation, except where explicitly stated, we will use natural units, with $\hbar = c = 1$.

2 Theories with Topological Solitons

We would like to find all the stable classical solutions to the equations of motion of a given theory, and check if any topological defect solution is allowed. However, it is not possible to know in general all the solutions to the equations of motion. More complex and higher-dimensional theories result in more complicated and non-linear partial differential equations to solve, which may not be analytically solvable.

It would be useful to find general theorems which rule out theories without soliton-like solutions, such as scaling arguments, or to find ways to categorize theories which are good candidates to have non-trivial solutions, such as homotopy classification of the theories via the boundary conditions of the fields.

2.1 Homotopy Classification of Field Theories

2.1.1 Brief Introduction to Homotopy Theory

For the discussion of topological solitons we will only be interested in topological spaces which are subspaces of $\mathbb{R}^m$, where continuous maps are defined as in usual analysis. Let $f, g : X \to Y$ be two continuous maps between two topological spaces X and Y.

f and g are said to be *homotopic*, $f \sim g$, if they can be continuously deformed into each other, that is, if we can define a continuous map $h : X \times [0, 1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

Homotopy is an equivalence relation of mappings since it is reflexive ($f \sim f$), symmetric ($f \sim g \Rightarrow g \sim f$) and transitive ($f \sim g$ and $g \sim h \Rightarrow f \sim h$). We can define $[f]$ as the set of all mappings $f' : X \to Y$ which are homotopic to f.

The homotopy classes of maps from S^n to a topological space Y form a group which is called the nth homotopy group of Y, $\pi_n(Y)$. In particular, for $n = 1$ the homotopy group $\pi_1(Y)$ is called the *fundamental group*, whose elements are the equivalence classes of closed paths in Y. It is straightforward to prove that these collections of homotopy classes have group structure, where for example in π_1 the group operation is given by going along one path after the other.

It is not easy to find the homotopy groups of topological spaces, but some relations are well known. The following are the most important concerning our analysis:

- $\pi_n(S^n) = \mathbb{Z}$,
- $\pi_k(S^n) = 0$ for $k < n$.

2.1.2 Application to Field Theories

For simplicity, and because the subjects studied in this dissertation depend only on static configurations of fields, only time-independent solutions of the field equations will be considered in this section.

The main ingredient needed for the homotopy classification of a field theory are the boundary conditions on spatial infinity. We have to impose some boundary conditions on the fields to ensure finiteness of energy, that is,

$$\int_{\infty} \mathcal{E}(\mathbf{x}) d^d x < \infty. \tag{6}$$

To obey this condition, the only relevant values of the field ϕ are the ones ϕ takes on spatial infinity, which can be identified with a $(d-1)$-sphere, S^{d-1}. Then, it becomes natural to define the maps $\phi^\infty : S^{d-1} \to \mathcal{M}$, where $\mathcal{M}$ is the manifold in which ϕ takes its values, as

$$\phi^\infty(\hat{\mathbf{x}}) \equiv \lim_{|\mathbf{x}|\to\infty} \phi(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^d, \ \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|} \in S^{d-1}. \tag{7}$$

To simplify, we can set the minimum of the potential at zero. Finiteness of energy requires that ϕ^∞ exists and takes value in the zeros of the potential of the theory, $V(\phi)$. Then, defining the *vacuum submanifold* of the theory, $\mathcal{V}$, as

$$\mathcal{V} \equiv \{\phi : V(\phi) = 0\} \subset \mathcal{M}, \tag{8}$$

the maps we need to classify are reduced to

$$\phi^\infty : S^{d-1} \to \mathcal{V}. \tag{9}$$

Note that, in a vacuum configuration, the field maps the whole space into a single point of $\mathcal{V}$, while in this case ϕ is only required to take value on $\mathcal{V}$ on spatial infinity, and ϕ^∞ can be any continuous map.

Two field configurations are said to be topologically equivalent, or homotopic, if they can be deformed into each other without passing through forbidden configurations, i.e., configurations with infinite energy. This means that the image of the map ϕ^∞ should lay on the vacuum manifold during the entire deformation process. If two field configurations cannot be deformed into each other, they are said to be in different homotopy classes.

This idea is put into form in the following theorem [20]:

Theorem 2.1. *The connected components of the space of non-singular finite energy solutions are in one-to-one correspondence with the homotopy classes of mappings from S^{d-1} to $\mathcal{V}$, where d is the number of space dimensions.*

Our problem of identifying field theories with non-trivial solutions is therefore reduced to calculating the topological character of the ϕ^∞ mappings. As seen in the previous section, the homotopy class of a ϕ^∞ configuration is an element of $\pi_{d-1}(\mathcal{V})$.

Examples

- $\mathcal{V}$ is a single point, i.e., the symmetry is unbroken. The homotopy group is $\pi_{d-1}(\mathbb{1}) = 0$, and there is only one homotopy class. All solutions are topologically equivalent to the vacuum.

- If ϕ transforms in the fundamental representation of $SU(2)$, and the symmetry is spontaneously broken to $U(1)$, $\mathcal{V}$ is isomorphic to S^1. The classes of non-trivial solutions are related to $\pi_{d-1}(S^1)$, which means that in two spatial dimensions the theory has a non-trivial set of homotopy classes, isomorphic to $\mathbb{Z}$.

The presence of a gauge field does not change the topological classification of the theory, since the vacuum submanifold remains the same. However, they become very important in scaling arguments. One way to make sense of this is that on theories coupled to gauge fields, the asymptotic conditions imposed on the solutions to ensure finiteness of energy are on covariant derivatives instead of on regular ones. Thus, the gauge field can compensate the variation of the scalar field.

2.2 Scaling Arguments

The vacuum solution is spatially constant, so it has absolute minimal energy for all the fields. In order to find soliton configurations we should find other stationary points of the energy. Derrick [8] argued that a field configuration which is a stationary point of the energy should be invariant under spatial rescaling. Hence, theories for which the functional variation of the energy with respect to a spatial rescaling is different from zero, for any non-vacuum configurations, do not have stable soliton solutions.

Considering a spatial rescaling $\mathbf{x} \to \mu\mathbf{x}, \mu > 0$, and a finite-energy field configuration of any kind, $\Psi(\mathbf{x})$, let $\Psi^{(\mu)}(\mathbf{x})$ be the one-parameter family of fields rescaled by μ in the appropriate manner. The following theorem is formulated [21]:

Theorem 2.2 (Derrick's Theorem)**.** *If, for any finite-energy field configuration $\Psi(\mathbf{x})$ which is not the vacuum, the energy function $E(\mu) \equiv E(\Psi^{(\mu)}(\mathbf{x}))$ has no stationary point, then the only static solution of the field equations with finite energy is the vacuum.*

Remark 2.3. The boundary conditions are preserved by scaling invariance, therefore varying μ maintains the energy finite and the homotopy class of the mapping.

Using this theorem we can analyze the behavior of the most important theories:

Scalar Multiplet

$$\phi^{(\mu)}(\mathbf{x}) = \phi(\mu\mathbf{x}). \tag{10}$$

The gradient is given by

$$\nabla_{\mathbf{x}}\phi^{(\mu)}(\mathbf{x}) = \nabla_{\mathbf{x}}\phi(\mu\mathbf{x}) = \mu\nabla_{\mu\mathbf{x}}\phi(\mu\mathbf{x}) \tag{11}$$

and the energy can be generally written as

$$E(\phi) = \int d^d x f_{ab}(\phi)(\nabla\phi^a)(\nabla\phi^b) + V(\phi) \equiv E_1 + E_2, \tag{12}$$

so the energy depends on the scale like

$$\begin{aligned} E(\mu) &= \int d^dx \left(f_{ab}(\phi^{(\mu)})\nabla\phi^{(\mu)a}\nabla\phi^{(\mu)b} + V(\phi^{(\mu)})\right) \\ &= \int d^dx \left(\mu^2 f_{ab}(\phi(\mu\mathbf{x}))\nabla_{\mu\mathbf{x}}\phi^a(\mu\mathbf{x})\nabla_{\mu\mathbf{x}}\phi^b(\mu\mathbf{x}) + V(\phi(\mu\mathbf{x}))\right) \\ &= \int d^dx \mu^{-d}\left(\mu^2 f_{ab}(\phi)(\nabla\phi^a)(\nabla\phi^b) + V(\phi)\right) \\ &= \mu^{2-d}E_1 + \mu^{-d}E_2. \end{aligned} \tag{13}$$

Since E_1 and E_2 are two positive constants, the stationary points will depend crucially on the dimension of the theory:

$$\begin{aligned} \frac{dE}{d\mu} = (2-d)\mu^{1-d}E_1 - d\mu^{-1-d}E_2 = 0 \\ \Leftrightarrow (2-d)\mu^2 = \frac{E_2}{E_1}d. \end{aligned} \tag{14}$$

- $\mathbf{d = 1}$ There is a stationary point at $\mu = \sqrt{E_2/E_1}$, so it is possible to have non-trivial solutions, without any restrictions in the parameters of the theory.
- $\mathbf{d = 2}$ There is no solution for (14) unless the potential energy vanishes everywhere. In this case the energy is independent of the scale factor and there may exist nontrivial solutions.
- $\mathbf{d \geq 3}$ The left hand side of (14) becomes negative, and since both E_2 and E_1 are positive, there is no possible solution for real scalar factor μ. There are no scalar theories with soliton solutions.

Scalar Multiplet Coupled with a Yang-Mills Gauge Field

It is convenient to define

$$A^{(\mu)}(\mathbf{x}) = \mu A(\mu\mathbf{x}), \tag{15}$$

so the covariant derivative transforms like an ordinary derivative by scale transformation:

$$\begin{aligned} D_{\mathbf{x}}^{A^{(\mu)}}\phi^{(\mu)} &= (\partial_{\mathbf{x}}\phi^{(\mu)} + ieA^{(\mu)}\phi^{(\mu)})(\mathbf{x}) \\ &= \mu(\partial_{\mu\mathbf{x}}\phi + ieA\phi)(\mu\mathbf{x}) = \mu D_{\mu\mathbf{x}}^A\phi(\mu\mathbf{x}). \end{aligned} \tag{16}$$

The transformation of the field strength is given by

$$F^{(\mu)}(\mathbf{x}) = \mu^2 F(\mu\mathbf{x}), \tag{17}$$

since it involves one spatial derivative and one gauge field.

The analysis is analogous to the scalar theory, the usual energy is given by:

$$\begin{aligned} E[\phi, A] &= \int d^d x \left(|F|^2 + |D\phi|^2 + V(\phi)\right) \\ &= E_0 + E_1 + E_2 \end{aligned} \tag{18}$$

and with the rescaled fields the energy becomes

$$\begin{aligned} E(\mu) &= \int d^d x \left(\mu^4 |F|^2(\mu \mathbf{x}) + \mu^2 |D\phi|^2(\mu \mathbf{x}) + V(\phi)(\mu \mathbf{x})\right) \\ &= \mu^{4-d} E_0 + \mu^{2-d} E_1 + \mu^{-d} E_2. \end{aligned} \tag{19}$$

In this case, in one, two and three spatial dimensions we can find a stationary point for the energy, so Derrick's theorem does not rule out any of these theories. For more dimensions, there are no non-trivial solutions.

Pure Yang-Mills Field

In pure Yang-Mills gauge theory the only term in the energy is E_0, so $E(\mu) = \mu^{4-d} E_0$, which only may have soliton solutions in 4 dimensions, where the theory is scale-independent.

3 (1+1)d Solutions: the Kink

A two-dimensional scalar field theory, $\mu = t, x$, can be generally described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{20}$$

with a non-negative potential. Varying the action one finds the field equation

$$\partial_\mu \partial^\mu \phi + \frac{dV(\phi)}{d\phi} = 0. \tag{21}$$

We are interested in finding static solutions with finite energy, so we impose that $\partial_t \phi = 0$ and that the energy density

$$\mathcal{E} = \frac{1}{2}(\partial_x \phi)^2 + V(\phi) \tag{22}$$

vanishes at spatial infinity, i.e, the following conditions are satisfied:

$$\begin{aligned} &1. \quad \partial_x \phi^\infty = 0, \\ &2. \quad \phi^\infty \in \mathcal{V}. \end{aligned} \tag{23}$$

We also want classical stability of the solution, which means that a small perturbation around the solution does not grow exponentially with time. Expanding the field near the classical solution, we obtain

$$\phi(x,t) = \phi_c(x) + \psi_n(x) e^{i\omega_n t}, \tag{24}$$

where ψ_n are the normal modes for the oscillations. We can find these modes by solving the eigenvalue problem

$$\frac{\delta^2 S[\phi]}{\delta \phi^2}|_{\phi=\phi_c} \psi_n = \left[-\frac{d^2}{dx^2} + \frac{d^2 V(\phi)}{d\phi^2} \right] \psi_n = \omega_n^2 \psi_n, \tag{25}$$

since they are eigenfunctions of the second variational derivative of the action. The stability condition demands that the eigenvalues ω_n^2 are all non-negative.

Without yet imposing conditions in the potential we can see that equation (25) always has a zero frequency mode, $\psi_0 \propto \phi'$, corresponding to translational invariance. This is easily checked by considering the translated field $\phi_c(x + x_0) = \phi_c(x) + x_0 \phi_c'(x)$ and comparing with expression (24). In later sections we will find that the property of translational invariance becomes very important when we are quantizing the theory.

Two classic examples of kink solutions are ϕ^4 and sine-Gordon theories, both of which have discrete symmetry and degenerate vacua.

3.1 ϕ^4 Theory

The explicit form of the potential is given by

$$V(\phi) = \lambda(\phi^2 - v^2)^2, \tag{26}$$

which has degenerate vacua at

$$\phi = \pm v. \tag{27}$$

(21) then becomes

$$\partial^2\phi - 4\lambda(\phi^2 - v^2)\phi = 0. \tag{28}$$

Defining $\phi_{\pm}^{\infty}$ as the value of the field at $\pm\infty$, we can define the topological charge as

$$\nu = \frac{\phi_{+}^{\infty} - \phi_{-}^{\infty}}{2v} = \{0, 1, -1\}. \tag{29}$$

Configurations with $\nu = 0$ are in the same topological sector as the vacuum, and configurations with $\nu = \pm 1$ are respectively the kink and the anti-kink solutions.

The kink solution is given by [21]:

$$\phi(x) = v \tanh\left(\sqrt{2\lambda}v(x-a)\right), \tag{30}$$

where a is an arbitrary constant of integration related to the translational invariance. The energy density is then

$$\mathcal{E} = \frac{1}{2}\phi'^2 + \lambda(\phi^2 - v^2)^2 = 2\lambda v^4 \operatorname{sech}^4\left(\sqrt{2\lambda}v(x-a)\right), \tag{31}$$

and the total energy is

$$E = \int_{-\infty}^{\infty} \mathcal{E} dx = \frac{4}{3}v^3\sqrt{2\lambda}, \tag{32}$$

which corresponds to the mass of the kink.

As we can see in fig. 1, the energy density is a localized lump at $x = a$. Therefore, this solution can be interpreted as a massive particle in position a.

3.2 Sine-Gordon Theory

The analysis of the sine-Gordon theory is in everything analogous to the ϕ^4 theory, except that the potential

$$V(\phi) = m^2(1 - \cos\phi), \tag{33}$$

now has infinite degeneracy at values

$$\phi = 2\pi n, \qquad n \in \mathbb{Z}. \tag{34}$$

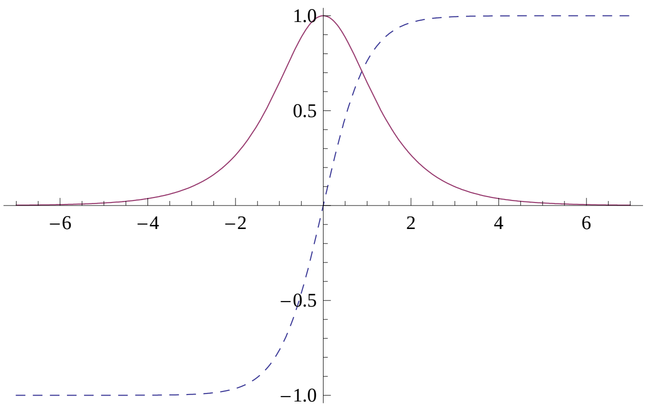

Figure 1: ϕ^4 kink solution for $\lambda = v = 1$. Dashed and full lines represent $\phi(x)$ and $\mathcal{E}(x)$ respectively.

In this case, we can identify the homotopy class of the solution by a single topological charge. Since the theory is invariant under the sum of a multiple of 2π, without loss of generality we can fix the value of ϕ_-^∞ to be the same for all ϕ. Hence, only the difference of the asymptotic values is important and we define the topological charge as

$$\nu = \frac{\phi_+^\infty - \phi_-^\infty}{2\pi}. \tag{35}$$

The field equation of this theory is

$$\partial^2\phi + m^2 \sin\phi = 0, \tag{36}$$

and the kink solution for this equation ($\nu = 1$) is given by [21]

$$\phi(x) = 4 \arctan e^{m(x-a)}, \tag{37}$$

where a is the arbitrary integration constant. The energy density of this solution is given by

$$\mathcal{E} = 4m^2 \operatorname{sech}^2[m(x-a)], \tag{38}$$

which reaches its maximum at $x = a$, motivating once again the interpretation of a as the particle location. The total energy, or the mass of the kink, is then

$$E = \int_{-\infty}^{\infty} \mathcal{E} dx = 8m. \tag{39}$$

4 A (3+1)d Solution: the 't Hooft-Polyakov Monopole

4.1 The Georgi-Glashow model

An example of an unification model with non-trivial solutions is the Georgi-Glashow model in 3+1 dimensions. It has $SU(2)$ local symmetry, and a triplet of scalar fields ϕ^a coupled with gauge fields A^a_μ, $a = 1, 2, 3$, both transforming in the adjoint representation of the group.

For this theory we have the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{tr}(D_\mu\phi D^\mu\phi) - \frac{\lambda}{4}(|\phi|^2 - v^2)^2. \tag{40}$$

ϕ and A_μ take values in the Lie algebra, so we can write them in terms of the generators of $su(2)$:

$$\begin{aligned} A_\mu = A^a_\mu T^a, \qquad \phi = \phi^a T^a, \qquad F_{\mu\nu} = F^a_{\mu\nu}T^a,& \\ D_\mu\phi = \partial_\mu\phi + ie[A_\mu, \phi],& \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] = D_\mu A_\nu - D_\nu A_\mu - ie[A_\mu, A_\nu],& \end{aligned} \tag{41}$$

where the generators of $su(2)$, T^a, respect the commutation relation

$$[T^a, T^b] = i\varepsilon_{abc}T^c. \tag{42}$$

The equations of motion obtained by varying the action $S = \int d^4x\mathcal{L}$ are

$$\begin{aligned} D_\nu F^{a\mu\nu} &= -e\varepsilon_{abc}\phi^a D^\mu\phi^c, \\ D_\mu D^\mu\phi^a &= -\lambda\phi^a(\phi^b\phi^b - v^2), \end{aligned} \tag{43}$$

and the stress energy tensor, satisfying $\partial_\mu T^{\mu\nu} = 0$, is given by Noether's theorem

$$T_{\mu\nu} = F^a_{\mu\alpha}F^{a\nu\alpha} + D_\mu\phi^a D_\nu\phi^a - \frac{1}{2}g_{\mu\nu}D_\alpha\phi^a D^\alpha\phi^a - \frac{1}{4}g_{\mu\nu}F^a_{\alpha\beta}F^{a\alpha\beta} - g_{\mu\nu}\frac{\lambda}{4}(\phi^2 - v^2). \tag{44}$$

Hence, the total energy of the system is

$$\begin{aligned} E &= \int d^3x T_{00} \\ &= \int d^3x\left(F^a_{0\alpha}F^{a0\alpha} + D_0\phi^a D_0\phi^a + \frac{1}{2}D_\alpha\phi^a D^\alpha\phi^a + \frac{1}{4}F^a_{\alpha\beta}F^{a\alpha\beta} + \frac{\lambda}{4}(\phi^2 - v^2)\right) \\ &= \int d^3x\left(\frac{1}{4}F^a_{\alpha\beta}F^{a\alpha\beta} + \frac{1}{2}D_\alpha\phi^a D^\alpha\phi^a + \frac{\lambda}{4}(\phi^2 - v^2)^2\right) \\ &= \int d^3x\frac{1}{2}\left(E^a_n E^a_n + B^a_n B^a_n + D_n\phi^a D_n\phi^a + \frac{\lambda}{4}(\phi^2 - v^2)^2\right), \end{aligned} \tag{45}$$

where we made use of the definition of the non-abelian generalization of the Maxwell electromagnetic fields $E^a_n = F^a_{0n}$, $B^a_n = \frac{1}{2}\varepsilon_{nmk}F^a_{mk}$.

This energy is minimal if:

$$\begin{aligned} &1. \quad \phi^a\phi^a = v^2, \\ &2. \quad F^a_{mn} = 0, \\ &3. \quad D_n\phi^a = 0. \end{aligned} \tag{46}$$

Note that 2. implies that A^a_μ is a pure gauge.

Because ϕ has a non-zero expectation value, the $SU(2)$ gauge symmetry is spontaneously broken down to $U(1)$ by the Higgs mechanism. The *little group*, $U(1)$, corresponds to the unbroken generator. Supposing that ϕ assumes a given direction, the unbroken generator will correspond to the remaining rotational symmetry about the ϕ vector. Perturbative methods will give the particle spectrum of the theory which, in the general case where $\phi^a = \phi^a_v + \chi^a$ with ϕ_v respecting (46), consists of a massless vector boson

$$A^{em}_\mu = \frac{1}{v}\phi^a_v A^a_\mu \tag{47}$$

that we can identify as the photon, or the *electromagnetic projection* of the gauge potential, a massive scalar field $\chi = \frac{1}{v}\phi^a_v\phi^a$ with $M_H = v\sqrt{2\lambda}$ and two massive and charged vector bosons $W^\pm_\mu$ with $M_W = ev$. The charge is given by the minimal coupling of the photon via the covariant derivative

$$D_\mu = \partial_\mu + ieA^a_\mu T^a \equiv \partial_\mu + iQA^{em}_\mu, \tag{48}$$

where the electric charge operator Q is defined.

4.2 Existence of Topological Solitons in the Georgi-Glashow model

The Georgi-Glashow model allows for static soliton solutions. To see this we have to find solutions to the field equations (43) satisfying conditions (46) asymptotically, ensuring finite energy.

Given that the vacuum manifold of the theory is $\mathcal{V} = \{\boldsymbol{\phi} : |\boldsymbol{\phi}| = v\} \cong S^2$, we know by Theorem 2.1 that the space of non-trivial solutions is in one-to-one correspondence with $\pi_2(S^2) = \mathbb{Z}$. Therefore, there are infinitely many disjoint homotopy classes where the solutions may fall in. These classes are characterized by the *winding number*, $n = 0, \pm1, \pm2, \ldots$

The winding number measures, loosely speaking, how many times the vacuum submanifold S^2 is wrapped in the course of a single loop around the space boundary.

A solution with a given winding number cannot be continuously deformed into a solution with another winding number without passing through forbidden field configurations, i.e, configurations with infinite energy. In this sense, the solutions carry a conserved topological charge.

As in the examples we have seen in section 3, these non-trivial solutions have their energy distributions localized in space, as well as a world line, so they can be interpreted as particles. We can define the mass of the particle as the minimal energy of the solution, and associate the size of the particle's core with the spread of the energy distribution.

4.3 Electromagnetic Field Strength

For the vacuum configuration we understand the meaning of electromagnetic field strength $F_{\mu\nu} = \partial_\mu A^{em}_\nu - \partial_\nu A^{em}_\mu$, since the electromagnetic projection of the gauge field is constant in space-time and satisfies the Maxwell equations

$$D_\mu F^{\mu\nu} = J^\nu, \tag{49}$$

$$D_\mu {}^*F^{\mu\nu} = 0, \qquad {}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}. \tag{50}$$

But what about topologically non-trivial solutions? The big difference is that A^{em}_μ, which was a constant vector field given by $\phi^a_v A^a_\mu$, is now space-dependent: $A^{em}_\mu = \phi^a(x)A^a_\mu(x)$. The usual definition of field strength is no longer suitable, and thus we need to construct another definition.

First we note that since these particles are sources for the electromagnetic field, it only makes sense to talk about the electromagnetic field a particle creates far from the solution's core, where the internal structure of the particle is no longer relevant, as the fields have reached their asymptotic form. Secondly, we compute the dependence of the gauge field on ϕ, since we know they have to satisfy $D\phi = 0$ asymptotically:

$$\begin{aligned} \partial_\mu\phi^a - e\varepsilon_{abc}A^b_\mu\phi^c = 0 \\ \Leftrightarrow A^a_\mu = \frac{1}{v^2 e}\varepsilon_{abc}\phi^b\partial_\mu\phi^c + \frac{1}{v}\phi^a A^{em}_\mu, \end{aligned} \tag{51}$$

where $\phi^a\phi^a = v^2$ was also used. Finally, the new definition of electromagnetic field strength has to be gauge invariant, so the most simple way we can construct it is inspired on the Higgs mechanism for a randomly directed vacuum, (47):

$$\mathcal{F}_{\mu\nu} = \frac{1}{v}\phi^a F^a_{\mu\nu}. \tag{52}$$

Then, inserting (51) into $\mathcal{F}_{\mu\nu}$ we find

$$\mathcal{F}_{\mu\nu} = \partial_\mu A^{em}_\nu - \partial_\nu A^{em}_\mu + \frac{1}{v^3 e}\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c. \tag{53}$$

Analogously, we define the magnetic and electric fields as

$$\mathcal{B}_n = \frac{1}{v}\phi^a B^a_n = \frac{1}{2v}\phi^a\varepsilon_{nmk}F^a_{mk}, \tag{54}$$

$$\mathcal{E}_n = \frac{1}{v}\phi^a E^a_n = \frac{1}{v}\phi^a F^a_{0n}. \tag{55}$$

It is clear that in the vacuum, where the Higgs field is constant, (53) is reduced to the Maxwell field strength

$$\mathcal{F}^{vac}_{\mu\nu} = \partial_\mu A^{em}_\nu - \partial_\nu A^{em}_\mu. \tag{56}$$

However, in the general case the extra term does not vanish. This will result in a violation of the source-free Maxwell equation

$$\partial_\mu {}^*\mathcal{F}^{\mu\nu} = 0, \tag{57}$$

since the extra term on (53) will lead to a magnetic source

$$\begin{aligned} \partial_\mu {}^*\mathcal{F}^{\mu\nu} &= \partial_\mu(\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\mathcal{F}_{\sigma\rho}) \\ &= \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\varepsilon_{abc}\partial_\mu(\phi^a\partial_\sigma\phi^b\partial_\rho\phi^c) \equiv k^\nu. \end{aligned} \tag{58}$$

k^ν is immediately conserved by symmetry, $\partial_\nu k^\nu = 0$ by the contraction of the antisymmetric tensor with the symmetric partial derivatives. Therefore, we can define the magnetic charge of the solution as

$$\begin{aligned} g &= \int d^3x k^0 \\ &= \frac{1}{2ev^3}\int d^3x \varepsilon_{abc}\varepsilon_{mnk}\partial_m(\phi^a\partial_n\phi^b\partial_k\phi^c) \\ &= \frac{1}{2ev^3}\int_{S^\infty} d^2S_m\varepsilon_{abc}\varepsilon_{mnk}\phi^a\partial_n\phi^b\partial_k\phi^c \\ &= \frac{4\pi n}{e}, \end{aligned} \tag{59}$$

where this last step is done in detail in [9]. n is the winding number, so we find that the magnetic charge is associated to the homotopy class of the solution, that is, it is a topological invariant and should not change under small perturbations of the field.

4.4 't Hooft-Polyakov Ansatz

A field configuration with winding number 1 can be constructed if the Higgs field points in the radial direction at spatial infinity. This is the "hedgehog" configuration, term coined by Polyakov in his original paper [2]:

$$\phi^a \xrightarrow[r\to\infty]{} \frac{vr^a}{r}. \tag{60}$$

We can see this configuration in figure 2.

If we try to deform this solution into a vacuum configuration by performing a gauge transformation to the *unitary gauge*

$$\begin{aligned} \phi &\to g(x)\phi g^{-1}(x) = \phi_v, \\ A_\mu &\to g(x)A_\mu g^{-1}(x) + ie[\partial_\mu g(x)]g^{-1}(x), \end{aligned} \tag{61}$$

the result we will find for $g(x)$ will necessarily be singular in one direction. This means that the two configurations cannot be smoothly deformed into each other. The singularity plays the role of the Dirac string discussed in the introduction, allowing us to relate the Dirac monopole with

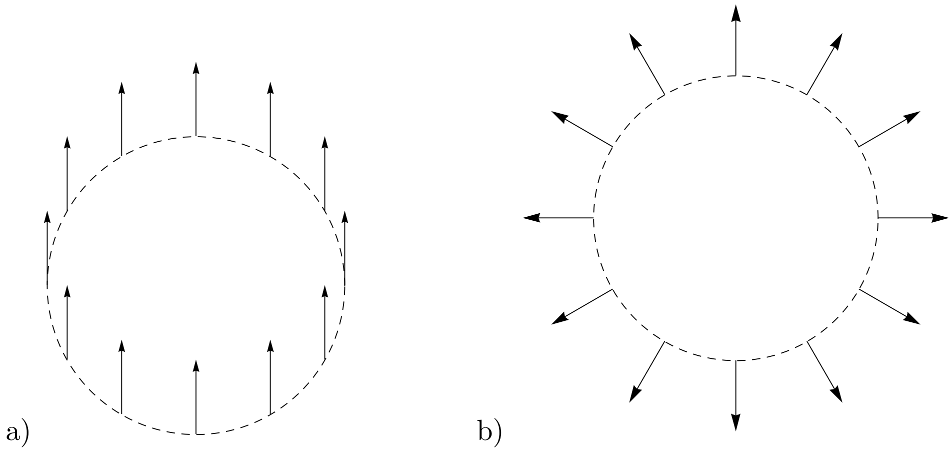

Figure 2: $\phi(x)$ configurations at spatial infinity S^∞, represented by the dashed circle. a) Vacuum configuration, the field is spontaneously broken to the same vacuum expectation value in all spatial infinity. b) Magnetic monopole configuration, the vacuum submanifold is spanned once ($n = 1$) in the course of a single loop around S^∞, known as the "hedgehog" configuration.

the 't Hooft-Polyakov monopole. Once again, the position of the string depends on the gauge chosen, so it cannot be a physical observable.

The condition of the vanishing covariant derivative, (46), gives the asymptotic behaviour of the gauge field:

$$\partial_n \left(\frac{r^a}{r} \right) - e\varepsilon_{abc} A_n^b \frac{r^c}{r} = 0, \tag{62}$$

with

$$\partial_n \left(\frac{r^a}{r} \right) = \frac{r^2 \delta_{an} - r_a r_n}{r^3} = -\varepsilon_{abc}\varepsilon_{bnk} \frac{r_c r_k}{r^3} \tag{63}$$

gives

$$A_k^a(r) \xrightarrow[r\to\infty]{} \frac{1}{e} \varepsilon_{ank} \frac{r_n}{r^2}. \tag{64}$$

We may notice that in this solution the spatial and isospin indices get mixed up. In section 5.4 we will see this fact leads to the conservation of the sum of the angular momentum and isospin, instead of simply angular momentum.

Also, looking only at the asymptotic behaviour of the fields, we see that the magnetic and electric fields take the form

$$\mathcal{B}_n = \frac{1}{2v} \varepsilon_{nmk} \phi^a F_{mk}^a = \frac{1}{2v} \varepsilon_{nmk} \phi^a (\partial_m A_k - \partial_k A_m - e\varepsilon_{abc} A_m^b A_k^c) \xrightarrow[r\to\infty]{} \frac{v r^a r^a r^n}{v e r^5} = \frac{r^n}{e r^3}, \tag{65}$$

$$\mathcal{E}_n = \frac{1}{v} \phi^a F_{0n}^a = \frac{1}{v} \phi^a (\partial_0 A_n - \partial_n A_0 - e\varepsilon_{abc} A_0^b A_n^c) = 0, \tag{66}$$

which corresponds to a point-like magnetic charge. This is why this solution is called the *'t*

Hooft-Polyakov magnetic monopole. If we wanted to rotate the field to the third direction, the result including the Dirac string would be

$$\mathcal{B}_n = \epsilon_{nmk}\partial_m A^3_k = \frac{r_n}{er^3} - 2\pi\delta_{n3}\theta(x_3)\delta(x_1)\delta(x_2), \tag{67}$$

as we anticipated in the introduction.

Based on conditions (60) and (64), 't Hooft and Polyakov independently [2, 3] showed that the most general form of the fields can be expressed as

$$\begin{aligned} \phi^a &= \frac{r^a}{er^2}H(\xi), \\ A^a_n &= \varepsilon_{amn}\frac{r^m}{er^2}(1 - K(\xi)), \\ A^a_0 &= 0, \end{aligned} \tag{68}$$

with $\xi = ver$ and the following constraints to $H(\xi)$ and $K(\xi)$:

$$\begin{aligned} \lim_{\xi\to 0} K(\xi) = 1, \qquad \lim_{\xi\to 0} H(\xi) = 0, \\ \lim_{\xi\to\infty} K(\xi) = 0, \qquad \lim_{\xi\to\infty} H(\xi) = \xi. \end{aligned} \tag{69}$$

The equations of motion simplify to

$$\xi^2\frac{d^2K}{d\xi^2} = KH^2 + K(K^2-1) \qquad \xi^2\frac{d^2H}{d\xi^2} = 2K^2H + \frac{\lambda}{e^2}H(H^2 - \xi) \tag{70}$$

The exact solutions for these functions cannot be found in general and require detailed numerical calculation, but the existence of such solutions has been proved [10]. However, in the next section we will see that there is a limit where analytic solutions can be found, the BPS limit.

Nevertheless, the classical mass of the magnetic monopole has been calculated in general [11]. It only depends on the ratio

$$\frac{M_H}{M_W} = \frac{v\sqrt{2\lambda}}{ev} \propto \frac{\sqrt{\lambda}}{e} \tag{71}$$

and is given by

$$M = \frac{4\pi M_W}{e^2}\epsilon\left(\frac{M_H}{M_W}\right), \tag{72}$$

where ϵ is a monotonically increasing continuous function with limits $\epsilon(0) = 1$ and $\epsilon(\infty) = 1.787$. This mass, as we will see, corresponds to the BPS mass when $\frac{M_H}{M_W} \to 0$.

4.5 BPS Limit

The BPS limit, due to Bogomol'nyi, Prasad and Sommerfield [12, 13], gives a lower bound to the energy (45) of the monopole, which holds in the case where $\lambda \to 0^+$, i.e., the potential $V(\phi)$

vanishes but the conditions in (46) are still satisfied.

With the following calculation we can find this bound for the energy (note that $E_n = 0$, since the solution is static and $A_0 = 0$):

$$
\begin{aligned}
E &= \int d^3x \frac{1}{2}\left(B_n^a B_n^a + D_n\phi^a D_n\phi^a + \frac{\lambda}{4}(\phi^2 - v^2)^2\right) \\
&= \int d^3x \frac{1}{2}(B_n^a - D_n\phi^a)^2 + \int d^3x B_n^a D_n\phi^a
\end{aligned}
\tag{73}
$$

The equation reaches its minimum if the first term vanishes. In this case we have:

$$
B_n^a = D_n\phi^a \tag{74}
$$

so the energy satisfies the relation

$$
\begin{aligned}
E &\geq \int d^3x B_n^a D_n\phi^a \\
&= \int d^3x \left[B_n^a \partial_n\phi^a - e\varepsilon_{abc} B_n^a A_n^b \phi^c\right] \\
&= \int_{S^\infty} dS_n B_n^a \phi^a - \int d^3x \phi^a \left[\partial_n B_n^a - e\varepsilon_{abc} A_n^b B_n^c\right].
\end{aligned}
\tag{75}
$$

The second term vanishes due to the Bianchi identity (50), so this becomes

$$
E \geq \int_{S^\infty} dS B_n^a \phi^a = v \int_{S^\infty} dS \mathcal{B}_n = vg \tag{76}
$$

and the mass of the monopole, which corresponds to the minimum of its energy, is given by

$$
M = \frac{4\pi v}{e}, \tag{77}
$$

which is large in the weak-coupling regime, where by large we mean much greater than the corrections due to quantum fluctuations.

Also, in this limit it is possible to find an analytic solution for the equations of motion (70), since they are reduced to first order differential equations [12, 13],

$$
\xi \frac{dK}{d\xi} = -KH, \qquad \xi \frac{dH}{d\xi} = H + (1-K)^2, \tag{78}
$$

which can be analytically solved:

$$
K = \frac{\xi}{\sinh\xi}, \qquad H = \xi \coth\xi - 1. \tag{79}
$$

5 Quasiclassical Quantization

5.1 Quantum Meaning of Classical Solutions

Thus far, we have only talked about classical solutions to the field equations. However, in order to discuss phenomena such as quark confinement (which is a quantum effect), we need to understand how we can construct quantum field theories based on these solutions.

We have been associating the word “particle” to a localized function in space with some evolution in time, but this concept is not well-defined in classical field theory. It is only in quantum field theories that the concept of particle arises from the fields. The resemblance between the characteristics of classical solutions and particles should not induce us to think that there is a trivial relation between them. We should then look more carefully at the way we define particle in QFT.

A particle in QFT is an excitation of the fields, to which we can associate a state in the Hilbert space $\mathcal{H}$ of the theory. These states have the particularity of being eigenvectors both of the Hamiltonian and momentum operators, such that their eigenvalues obey the relation $E^2 - P^2 = M^2$ for a fixed constant M that we can associate with the particle's mass. So, at first sight we find that the concept of particle and classical solution of a field equation look a little disconnected from each other.

The particle states are postulated based on properties of quantum observables that we know the theory to have, like energy spectrum or form factor. The classical solution, ϕ^{cl}, yields this information at least to leading order. ϕ^{cl} is an extremum of the energy of the theory $U[\phi]$:

$$\frac{\delta U[\phi]}{\delta \phi}|_{\phi^{\mathrm{cl}}} = 0. \tag{80}$$

If we perform a small perturbation on the field, $\chi(x) = \phi(x) - \phi^{\mathrm{cl}}(x)$, the potential energy felt by this new field configuration can be obtained by a Taylor expansion around the classical solution,

$$U[\phi] = U[\phi^{\mathrm{cl}}] + \int d^3x \frac{1}{2}\chi(x)\left[-\nabla^2 + \frac{d^2V}{d\phi^2}|_{\phi^{\mathrm{cl}}}\right]\chi(x). \tag{81}$$

The eigenfunctions of the second variation of the potential energy are called *normal modes*, and form a complete set. To find the normal modes, we should solve the eigenvalue equation

$$\left[-\nabla^2 + \frac{d^2V}{d\phi^2}|_{\phi^{\mathrm{cl}}}\right]\chi_i(x) = \omega_i^2 \chi_i(x). \tag{82}$$

We can then write the perturbation as a linear combination of the normal modes. This means that $U[\phi]$ is reduced to a sum of harmonic oscillators with frequencies ω_i,

$$U[\phi] = U[\phi^{\mathrm{cl}}] + \int \frac{1}{2}\sum_i c_i^2 \omega_i^2 \chi_i^2(x) d^3x. \tag{83}$$

In quantum theory we know how to construct a set of approximate harmonic oscillator states spread around ϕ^{cl}, by quantizing the normal mode coefficients c_i. The energy of these states will be given by

$$E_{\{n_i\}} = U[\phi^{\mathrm{cl}}] + \hbar \sum_i (n_i + \frac{1}{2})\omega_i + \text{corrections} \tag{84}$$

where $\hbar$ was explicitly introduced for more clear recognition of the harmonic oscillator quantum states and the corrections are given by higher-order perturbations around the classical solution.

So we found that the energy is in first order related to the classical energy and the second order depends on the stability frequencies of the classical solution, ω_i.

5.2 Quantization of Soliton Solutions

The more convenient formalism to use when quantizing topological solitons is the path integral formalism, since it is more suitable to work with gauge freedom and the solutions we treat here usually involve gauge fields. This formalism depends fundamentally on the boundary conditions of the fields, because the partition function

$$\mathcal{Z} = \int_{\phi^{-\infty}, A_\mu^{-\infty}}^{\phi^{+\infty}, A_\mu^{+\infty}} \mathcal{D}\phi \mathcal{D}A_\mu e^{iS[\phi, A_\mu]} \tag{85}$$

is integrated over all possible configurations of the fields with some fixed boundary conditions $\phi^{\pm\infty}$, $A_\mu^{\pm\infty}$. The relation between the homotopy classes of the fields and the partition function now becomes clear, since the boundary conditions impose that the integration is made over a single homotopy class.

Usually, we are only considering perturbations around the vacuum configuration, so the whole theory is encoded in one partition function. However, if we consider theories with topologically non-trivial solutions, there is more than one homotopy class, so we will have a partition function corresponding to each one of them. Each partition function will describe one different sector in the Hilbert space. Thus, the topological charge behaves like a new quantum number of the theory, which has to be conserved, because there can be no decays of particles from one sector to another.

If we are quantizing a theory in the way described in section 5.1 around the vacuum configuration, the boundary conditions are fixed at zero and the particle spectrum is easy to compute, as it was already described in section 4.1. Finding the particle spectrum in the presence of a monopole ($n = 1$ sector of $\mathcal{H}$) will involve many other difficulties, since the classical field is no longer constant in space, and also we will have to consider the induced perturbations on the gauge field. However, the same principles are applied, since we may choose the unitary gauge and make $\phi \to \phi_v$ everywhere except on the Dirac string singularity. As we have seen before, this singularity cannot be observed, so we can do the calculations in this gauge, bearing in mind that the singularity exists and we have to be careful about it.

In order to calculate the partition function - and the possible physical quantities we can derive from it - we need to do the integration over the space of the fields, but since the normal modes span the entire space of configurations, we can simply integrate over the normal modes' coefficients. Since the fields are infinite-dimensional objects, we will have an infinite, but countable, number of integrations, which can be formally divergent. These divergences can be easily subtracted if the theory is renormalizable by adding the respective counterterms in the Lagrangian. However, if there are modes whose eigenvalue vanishes, the theory will have divergences related to the size of the symmetry group. As we will see in the next section, these divergences do not have a physical meaning and can be dealt with.

5.3 Zero Modes and Collective Coordinates

Eigenfunctions with zero frequency are called *zero modes* and are due to symmetries in the equations of motion. These symmetries do not impose symmetries in the solutions, but give us a way to find new solutions for the equations, since if we perform a transformation which leaves the equations of motion invariant, the solution may be transformed into a new one. In this way we can find a collection of solutions to the field equations which differ by a parameter in the argument. The solution can be labeled by this parameter - the *collective coordinate*.

In the Georgi-Glashow model, there are two main symmetries of the equations of motion: gauge freedom and translational invariance. Concretely, in the latter case, we find that all fields $\{\phi^{\mathrm{cl}}(x+a)\}$ are valid solutions. The corresponding collective coordinate will be the parameter a, corresponding to the position of the monopole core.

The gauge invariance of the theory can be treated in the usual way using the Faddeev-Popov method. These degrees of freedom can be removed by adding a gauge-fixing term in the action.

However, translational modes cannot be fixed in a similar way, and the integration over its collective coordinates - the position of the solutions - will not disappear from the integral. Therefore, in order to compute the integrations we should separate the degrees of freedom of the fields by its nature: Normal modes with non-zero eigenvalues, normal modes with zero eigenvalues due to gauge invariance which can be removed from the integration in the usual way, and three zero modes from the translational invariance of the action, one for each spatial dimension.

5.4 Energy Spectrum in the Monopole Background

For the Georgi-Glashow model considered in section 4, if we consider the perturbations in the fields

$$A_\mu = A_\mu^{\mathrm{cl}} + a_\mu, \qquad \phi = \phi^{\mathrm{cl}} + \chi, \tag{86}$$

we can Taylor expand the action around the classical monopole solution

$$\begin{aligned} S &= S^{\mathrm{cl}} + \frac{\delta S}{\delta A_\mu} a_\mu + \frac{\delta S}{\delta \phi}\chi + \frac{1}{2}\left[\frac{\delta^2 S}{\delta A_\mu^2} a_\mu^2 + \frac{\delta^2 S}{\delta \phi^2}\chi^2 + 2\frac{\delta^2 S}{\delta A_\mu^a \delta\phi^b} a_\mu^a \chi^b\right] + \mathcal{O}(a_\mu,\chi)^3 \\ &\equiv S^{\mathrm{cl}} + S^{\mathrm{II}} + \mathcal{O}(a_\mu,\chi)^3. \end{aligned} \tag{87}$$

The linear terms vanish, since the classical solution is an extremum of the action. The second variation appears as the coefficients of the quadratic terms in the perturbation. We find these terms by expanding the action[1]:

$$\begin{aligned}
S &= \operatorname{tr}\int d^4x\left[\frac{1}{2}F_{\mu\nu}F_{\mu\nu}+D_\mu\phi D_\mu\phi\right]+\int d^4x V(\phi) && (88)\\
&= \operatorname{tr}\int d^4x\Big[\frac{1}{2}F^{\rm cl}_{\mu\nu}F^{\rm cl}_{\mu\nu}+D_\mu\phi^{\rm cl}D_\mu\phi^{\rm cl}+\frac{1}{4}(D_\mu a_\nu-D_\nu a_\mu)^2+\frac{1}{2}ieF^{\rm cl}_{\mu\nu}[a_\mu,a_\nu]+\frac{1}{2}F^{\rm cl}_{\mu\nu}(D_\mu a_\nu-D_\nu a_\mu)\\
&\quad+\frac{1}{2}(D_\mu\phi^{\rm cl})^2+\frac{1}{2}(D_\mu\chi)^2-\frac{1}{2}e^2[a_\mu,\phi^{\rm cl}]^2+D_\mu\phi^{\rm cl}D_\mu\chi+ieD_\mu\chi[a_\mu,\phi^{\rm cl}]+ieD_\mu\phi^{\rm cl}[a_\mu,\chi]\\
&\quad+ieD_\mu\phi^{\rm cl}[a_\mu,\phi^{\rm cl}]\Big]+\int d^4x\left[V(\phi^{\rm cl})+V'(\phi^{\rm cl})\chi+\chi V''(\phi^{\rm cl})\chi\right], && (89)
\end{aligned}$$

where in (89) the covariant derivative is classical, $D_\mu X=\partial_\mu X+ie[A^{\rm cl}_\mu,X]$. Comparing the expressions (87), (88) and (89) we find

$$\begin{aligned}
S^{\rm II}=\operatorname{tr}\int d^4x\Big[&\frac{1}{4}(D_\mu a_\nu-D_\nu a_\mu)^2+\frac{1}{2}ieF^{\rm cl}_{\mu\nu}[a_\mu,a_\nu]+\frac{1}{2}(D_\mu\chi)^2-\frac{1}{2}e^2[a_\mu,\phi^{\rm cl}]^2\\
&+ieD_\mu\phi^{\rm cl}[a_\mu,\chi]+ieD_\mu\chi[a_\mu,\phi^{\rm cl}]\Big]+\int d^4x\chi V''(\phi^{\rm cl})\chi \qquad (90)
\end{aligned}$$

with

$$\chi V''(\phi^{\rm cl})\chi=-\lambda\left[({\phi_{\rm cl}}^2-v^2)\delta_{ab}+2\phi^a_{\rm cl}\phi^b_{\rm cl}\right]\chi^a\chi^b. \qquad (91)$$

Using integration by parts, we can write $S^{\rm II}$ in the following fashion:

$$\begin{aligned}
S^{\rm II} &= S^{\rm II}_{(a)}+S^{\rm II}_{(\chi)}+S^{\rm II}_{\rm mixed}, && (92)\\
S^{\rm II}_{(a)} &= \int d^4x\left[-D_\mu(D_\nu a^a_\mu-D_\mu a^a_\nu)a^a_\nu+ie[F^{\rm cl}_{\nu\mu},a_\mu]^a a^a_\nu+e^2[\phi^{\rm cl},[\phi^{\rm cl},a_\nu]]^a a^a_\nu\right],\\
S^{\rm II}_{(\chi)} &= \int d^4x\left[-(D_\mu D_\mu\chi^a)\chi^a+V''_{ab}\chi^a\chi^b\right],\\
S^{\rm II}_{\rm mixed} &= \int d^4x ie\left[[D_\nu\phi^{\rm cl},\chi]^a a^a_\nu+[D_\nu\chi,\phi^{\rm cl}]^a a^a_\nu\right]\\
&= \int d^4x ie\left[2[D_\nu\phi^{\rm cl},a_\nu]^a\chi^a+[\phi^{\rm cl},D_\nu a_\nu]^a\chi^a\right].
\end{aligned}$$

The detailed calculation of finding the normal modes of this theory will only be done in section 7.3 in order to find the integral measure of the monopole gas partition function. In this section we will only do a qualitative discussion of the results that can be found [25].

We want to know how the Hilbert space in the monopole spectrum looks like, and the spectrum of fluctuations of the fields χ and a_μ should give us this information. The main questions to be answered are which are the free particle states in the theory, if there exist bound states between the monopole and other particles, and which quantum numbers characterize this sector of $\mathcal{H}$.

We are only interested in the physics outside the monopole core, therefore we can assume

[1]Here, and in many occasions throughout the remainder of this dissertation, all the Lorentz indices will be kept below, in order not to clutter the notation. They are contracted in the usual way.

the asymptotic values of the fields, making $S^{\mathrm{II}}_{\mathrm{mixed}}$ vanish, and S^{II} become diagonal. Analysing the degrees of freedom of S^{II}, we see that there are three d.o.f coming from the Higgs field χ and four d.o.f. for every component of a_μ, which sum to a total of fifteen degrees of freedom. However, six of these d.o.f. are fixed when eliminating the gauge freedom, and only nine are left. The fact that we can do the calculations in the unitary gauge means that the 't Hooft-Polyakov monopole is the same as a Dirac monopole asymptotically, and we expect the fluctuations of the field to yield the same particles as in the vacuum sector. Thus, the degrees of freedom will be distributed as in the usual way: six d.o.f. for two massive spin-one bosons,

$$A^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(a^1_\mu \pm i a^2_\mu), \tag{93}$$

two d.o.f. for a massless photon, corresponding to the two possible polarizations, and one degree of freedom for the Higgs boson χ^3.

In terms of these physical variables we find that the eigenvalue equation gives the equations of motion of these fields

$$\begin{aligned} (D^2 + k^2)\chi^3 = 0, \qquad & k^2 = M_H^2 + \omega^2, \\ (D^2 + k^2 \mp 2e(\mathbf{S}\cdot\mathbf{B}))A^{\pm}_{\mu} = 0, \qquad & k^2 = M_V^2 + \omega^2, \end{aligned} \tag{94}$$

which is the Klein-Gordon equation with an interaction with the external magnetic field caused by the monopole. $\mathbf{S}$ stands for the spin operator, since the unit spin operator can be represented by a 3×3 matrix

$$(S^n)_{ij} = i\varepsilon_{nij} \tag{95}$$

with n, i and j spatial indices, we can see the similarity to the spin representation with the $SU(2)$ symmetry of the Higgs field. The isospin operator $\mathbf{T}$ is also represented by

$$(T^a)_{bc} = i\varepsilon_{abc}, \tag{96}$$

where, however, a, b and c are now Lie algebra indices. We had noted before, when we found the asymptotic values of the fields in the 't Hooft-Polyakov monopole, that these indices get mixed together. Given the standard angular momentum operator,

$$L_k = -i\varepsilon_{kmn} r_m \partial_n, \tag{97}$$

we can define the *generalized angular momentum* operator by

$$\mathcal{J} = \mathbf{T} + \mathbf{L} + \mathbf{S} = \mathbf{J} + \mathbf{S}, \tag{98}$$

which commutes with the Hamiltonian, in contrast with the usual angular momentum $\mathbf{L} + \mathbf{S}$. Equations (94) can be expressed by a single equation for the general boson, Φ, with charge q

and spin $\mathbf{S}\Phi = s\Phi$ ($s = 0$ for the Higgs boson and ± 1 for the vector bosons)

$$(D^2 + k^2 + 2q(\mathbf{S}\cdot\mathbf{B}))\Phi = 0, \qquad k^2 = M_\Phi^2 + \omega^2. \tag{99}$$

The modes which are also eigenfunctions of the charge operator (48) can be expanded by

$$\Phi(r) = \varphi(r)\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}}, \tag{100}$$

where $\varphi(r)$ is the radial dependence and $\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}}$ are the *monopole vector spherical harmonics*, which respect

$$\begin{aligned} \mathcal{J}^2\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}} &= \mathtt{j}(\mathtt{j}+1)\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}}, \qquad \mathcal{J}_3\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}} = \mathtt{m}\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}}, \\ \mathbf{J}^2\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}} &= j(j+1)\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}}, \qquad Q\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}} = q\mathbf{Y}^{(q)}_{\mathtt{j}j\mathtt{m}}. \end{aligned} \tag{101}$$

Besides the quantum number corresponding to total momentum, these are the quantum numbers which define a particle state in the monopole background. The conditions on these numbers and the characteristics of the respective solutions have been largely studied [14], but for the aims of this dissertation the only important property is that the free particle states carry these conserved quantum numbers.

The existence of bound states in this theory is also a broad subject, and it becomes even broader when we also consider fermions in the action. The number of bound states varies with the parameters and the quantum numbers of the states. However, we can say that in the case where $\mathtt{j} = q - 1$, there is an attractive potential that bounds the monopole with the charged vector boson, given by

$$V^{\pm}_{\text{centrifugal}} = -\frac{q^2 \mp 2s - j(j+1)}{r^2}. \tag{102}$$

For vector bosons $A^{\pm}_{\mu}$, with $q^2 = 1$ and $s = \mp 1$ in a spherically symmetric state ($\mathtt{j}$=0), the generalized momentum eigenvalue is $j = 1$ and $V_{\text{centrifugal}}$ is reduced to

$$V_{\text{centrifugal}} = -\frac{1}{r^2}. \tag{103}$$

This bound state, which is less energetic than the two particles separated, contains both electric and magnetic charge, and is usually called a *dyon* state.

6 Instantons

6.1 Instantons as Topological Defects

If we now consider time-dependent configurations of the fields, in d space and one time dimensions, the action is given by

$$S = \int d^d x dt \left[\frac{1}{2}(\partial_t \phi - \frac{1}{2}(\nabla \phi)^2 - V(\phi) \right]. \tag{104}$$

If we perform a Wick rotation to the time axis, $t \to \tau = it$, we get

$$iS[\phi] = -\int d^d x d\tau \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \right] \equiv -S_E[\phi], \tag{105}$$

where $S_E[\phi]$ is known as the *Euclidean action* or *pseudoenergy*, and the configurations with finite Euclidean action are called *instantons* or *pseudoparticles*.

We may notice that the Euclidean action corresponds to the energy of a time-independent field $\phi(\mathbf{x})$ in $(d+1)$ spatial dimensions, $\mathbf{x} \in \mathbb{R}^{(d+1)}$. Also, in the path integral formalism, configurations of the fields are weighted by

$$e^{iS} = e^{-S_E}, \tag{106}$$

which means that configurations with finite Euclidean action will dominate, just like the solutions with finite energy in the usual Minkowski space-time. For this reason, the same solutions will hold in both cases.

Hence, a soliton solution in $d+1$ dimensions in Minkowski space-time is formally equivalent to a instanton solution in d dimensions in Euclidean space.

6.2 A New Look at Tunneling in Quantum Mechanics

Instantons also allow us to have a different look at the phenomenon of tunnelling in quantum mechanics. First, let us review it as it is usually done. Considering the usual one-dimensional quantum mechanical system, the spatial dependence of the wave function $\psi(x,t)$ respects the time-independent Schrödinger equation:

$$\begin{aligned} H\psi(x) &= E\psi(x), \\ H &= \frac{1}{2}\left(-i\hbar \frac{\partial}{\partial x}\right)^2 + V(x) \\ \Rightarrow \hbar^2 \psi''(x) &= 2(V(x) - E)\psi(x). \end{aligned} \tag{107}$$

In the semiclassical limit, i.e., $\hbar \to 0$, it is possible to find an approximate solution to this differential equation by the WKB method [27]:

$$\psi(x) \approx \frac{1}{(2(V(x)-E))^{1/4}} \left[c_1 \exp\left\{ \frac{1}{\hbar} \int_a^x \sqrt{2(V(t)-E)} dt \right\} + c_2 \exp\left\{ -\frac{1}{\hbar} \int_a^x \sqrt{2(V(t)-E)} dt \right\} \right] \tag{108}$$

where the integration constants c_1 and c_2 are fixed by the boundary conditions, and a is a fixed integration point.

As we can see, in the classically forbidden zone, $V(x) - E > 0$, the wave equation decays exponentially:

$$\psi(x) \propto \exp\left\{ -\frac{1}{\hbar} \int_a^x \sqrt{2(V(t)-E)} dt \right\}. \tag{109}$$

Let A and B be the boundary points of the forbidden region. The amplitude of tunneling through the potential barrier is given by

$$Z \propto \int dt \exp\left\{ -\frac{1}{\hbar} \int_A^B \sqrt{2(V(x)-E)} dx \right\}. \tag{110}$$

We will find that the same result may be found by a completely different formalism, making use of instantons.

6.2.1 Instantons in Quantum Mechanics

Usual Quantum Mechanics can be thought of as a quantum field theory in $0+1$ dimensions, where the role of the field $\phi(t)$ is played by the position $x(t)$. The Euclidean action can be generally written as:

$$S_E[\phi] = \int \left(\frac{1}{2} (\partial_\tau \phi)^2 + V(\phi) \right) dt. \tag{111}$$

Note that this corresponds to the Minkowski action of the field in an inverted potential $-V(\phi)$.

We have seen that the solutions for this theory are the same as the static solutions in a $(1+1)d$ theory. In section 3 we found the form of these solutions for the ϕ^4 and sine-Gordon theories. Focusing on the ϕ^4, where the potential has the double well form, there exists a collection of possible solutions given by

$$\phi(\tau) = 2\lambda v^4 \operatorname{sech}^4(\sqrt{2\lambda}(\tau - \tau_0)), \tag{112}$$

which correspond to a tunneling trajectory, since they evolve from one minimum of the energy to another in an infinite time, as we can see in figure 3.

The amplitude of a state to evolve from a configuration $\phi(0)$ to $\phi(\tau')$ is given by the correlation function

$$\langle \phi(0)\phi(\tau') \rangle = \frac{\int \mathcal{D}\phi e^{-S_E[\phi]} \phi(0)\phi(\tau')}{\int \mathcal{D}\phi e^{-S_E[\phi]}}, \tag{113}$$

which can be calculated by the same techniques we explained in section 5.

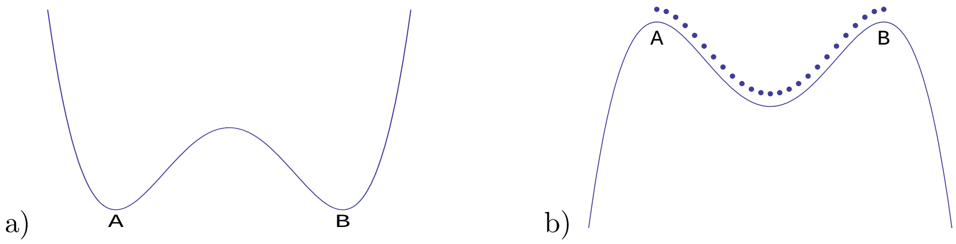

Figure 3: When we perform a Wick rotation in the action, the potential of the theory changes sign, a)→ b). Initially the fields assume one vacuua configuration A or B. By transforming the action the wells become hills, i.e, non-stable stationary points. Considering that the field assumes the stationary point A without any kinetic energy, it is possible that it reaches the stationary point B by the trajectory marked with the dotted line in an infinite time. This solution corresponds to the kink solution illustrated in fig. 1.

The pseudoenergy of each of the solutions (112) is

$$S_E^c = \frac{4}{3} v^3 \sqrt{2\lambda}. \tag{114}$$

At first sight, the contribution of these trajectories to the path integral (113) looks small. However, there are infinite trajectories, one for every value of τ_0, so the real contribution of the tunneling trajectories is given by

$$Z^{\text{tunneling}} = e^{-S_E^c} \int d\tau_0, \tag{115}$$

which becomes a very important contribution in large times τ'. This mechanism mimics the tunneling events understood for usual quantum mechanics and allows us to generalize the meaning of tunneling to quantum field theory.

7 Proof of Confinement in the (2+1)d Georgi-Glashow Model

QED can be formulated in two different ways, compact and non-compact, with identical continuum limit but different physical properties. The decision of which one is realized in Nature depends only on which of these properties we can actually see.

The main difference between the formulations is the existence of monopoles in the theory. There are various models where the abelian group has a compact formulation, but the two most important are *compact QED*, defined on the lattice, where the existence of monopoles is not discussed here but has been proved by Polyakov [15], and the Georgi-Glashow model, where the abelian group is embedded in SU(2), and the existence of monopoles is assured by the arguments given in this dissertation. Compactness of QED also implies a quantization of the electric flux, and consequently, quantization of the electric charge.

In theories where the abelian group appears in a direct product, for example in the Standard Model, where the gauge symmetry group is $SU(3) \times SU(2) \times U(1)$, monopoles do not exist. Furthermore, we cannot prove neither confinement nor charge quantization.

We will focus our analysis in the Georgi-Glashow model from now on.

7.1 Instantons in the (2+1)d Georgi-Glashow Model

The action of the Georgi-Glashow model in $3d$ Euclidean space-time is identical to the potential energy of the $(3+1)d$ Georgi-Glashow model in Minkowski space-time. Since we have only considered static solutions, the equivalence we have seen in section 6.1 tells us that this new model also contains these previously considered solutions. In particular, the 't Hooft-Polyakov monopole, which was a particle-like solution in four dimensions, is in this case a localized event and does not have a world line.

It is important to notice that the coupling constant in three dimensions has dimensions of mass. This fact introduces a characteristic scale in the theory, important to relate the phases of the theory with the length scale, something fundamental in the confinement phenomena.

The Euclidean action for this theory is

$$S_E = \int d^3x \frac{1}{2}\,\mathrm{tr}(F_{mn}F_{mn}) + \mathrm{tr}(D_m\phi D_m\phi) + \frac{\lambda}{4}(\phi^2 - v^2)^2. \tag{116}$$

From this, we can deduce that the dimensions of the fields and coupling constants are as follows:

$$[\phi] = [A] = [e] = [v] = [mass]^{\frac{1}{2}}, \qquad [\lambda] = [mass]^1. \tag{117}$$

7.2 Partition Function of the Monopole Gas

We know [24] that the field strength caused by a gas of N monopoles ($n_a = 1$) and antimonopoles ($n_a = -1$) fixed at the positions $\{x_a\}$, sufficiently far from each other, is given by the superposition of the fields of each pseudoparticle (65)

$$\mathcal{B}_n = \frac{1}{e}\sum_{a=1}^{N} n_a \frac{(x-x_a)_n}{|x-x_a|^3} \tag{118}$$

and the action of the system can be written as

$$S \approx NS^{(1)} + S^{(2)}, \tag{119}$$

where the first contribution $S^{(1)}$ comes from the self pseudoenergy of a single monopole (72) and the second contribution $S^{(2)}$ comes from the interactions between them.

$$S = N\frac{4\pi M_W}{e^2}\epsilon\left(\frac{\lambda}{e^2}\right) + \frac{\pi}{2e^2}\sum_{a\neq b}\frac{n_a n_b}{|x_a - x_b|} \tag{120}$$

with n_a the winding number of the solution.

To calculate the partition function

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}A_\mu \exp\{-S\} \tag{121}$$

we need to execute two fundamental steps. First, we need to remove the non-physical infinities caused by gauge invariance. This is done by expanding the field in its normal modes as it was outlined in section 5.1. Recovering the expansion in (83) for coefficients ξ_n and normal modes $\chi^{(n)}$ and $a^{(n)}$, we obtain

$$\mathcal{Z} = \int (\text{measure}) \times \exp\{-S_{cl}\}\prod_n \exp\left\{-\int d^3x\xi_n^2\omega_n^2(\chi_{(n)}^2 + a_{(n)}^2)\right\} d\xi_n. \tag{122}$$

After calculating the integration measure, we will see that $\mathcal{Z}$ can indeed be written in the form

$$\mathcal{Z} = \sum_{N,\{n_a\}}\frac{\zeta^N}{N!}\int d^3R_1....d^3R_N \exp\left\{-\frac{\pi}{2e^2}\sum_{a\neq b}\frac{n_a n_b}{|R_a - R_b|}\right\}. \tag{123}$$

Then, we need to calculate this integral and evaluate the physical meaning of this function, by means of correlation functions of physical quantities in this theory. We will find that there exists a mass gap, i.e., no massless particles exist in the theory.

Finally, to complete the proof of confinement in this model, we will compute the interaction between particles, showing that there is a confining force which is linear in the particle separation.

7.3 Calculation of the Integral Measure

Throughout this section, all factors of ie will be omitted. Since ie is a dimensional constant, we can easily add the necessary factors to the final expression by dimensional analysis. To this effect, note that all fields α in this section have dimensions $[\alpha] = [mass]^{-\frac{1}{2}}$.

To find the normal modes of S^{II} in (92), we want to solve the eigenvalue problem:

$$D_\mu(D_\nu a_\mu^{(n)} - D_\mu a_\nu^{(n)}) - [F_{\nu\mu}^{\mathrm{cl}}, a_\mu^{(n)}] + [\phi^{\mathrm{cl}}, [\phi^{\mathrm{cl}}, a_\nu^{(n)}]] + [D_\nu\phi^{\mathrm{cl}}, \chi^{(n)}] - [D_\nu\chi^{(n)}, \phi^{\mathrm{cl}}] = -\omega_n^2 a_\nu^{(n)}, \tag{124}$$

$$(D_\mu D_\mu \chi^{(n)}) - V''\chi^{(n)} - 2[D_\nu\phi^{\mathrm{cl}}, a_\nu^{(n)}] - [\phi^{\mathrm{cl}}, D_\nu a_\nu^{(n)}] = -\omega_n^2\chi^{(n)}, \tag{125}$$

where we must remember that V'' is an operator and by $V''\chi^{(n)}$ we mean $V''_{ab}\chi_b^{(n)}$. First, we note that if the fields can be expressed in the form

$$a_\mu^{(0)} = D_\mu\alpha(x), \qquad \chi^{(0)} = [\phi_{\mathrm{cl}}, \alpha], \tag{126}$$

they are normal modes with zero eigenvalue. Substituting on equation (124), using $(D_\mu D_\nu - D_\nu D_\mu)\alpha = [F_{\mu\nu}^{\mathrm{cl}}, \alpha]$, we get

$$\begin{aligned} 0 &= D_\mu(D_\nu D_\mu\alpha - D_\mu D_\nu\alpha) - [F_{\nu\mu}^{\mathrm{cl}}, D_\mu\alpha] + [\phi^{\mathrm{cl}}, [\phi^{\mathrm{cl}}, D_\nu\alpha]] + [D_\nu[\phi^{\mathrm{cl}}, \alpha], \phi^{\mathrm{cl}}] + [D_\nu\phi^{\mathrm{cl}}, [\phi^{\mathrm{cl}}, \alpha]] \\ &= [D_\mu F_{\nu\mu}^{\mathrm{cl}}, \alpha] + [[D_\nu\phi^{\mathrm{cl}}, \alpha], \phi^{\mathrm{cl}}] + [D_\nu\phi^{\mathrm{cl}}, [\phi^{\mathrm{cl}}, \alpha]] \\ &= [D_\mu F_{\nu\mu}^{\mathrm{cl}} - [\phi^{\mathrm{cl}}, D_\nu\phi^{\mathrm{cl}}], \alpha], \end{aligned} \tag{127}$$

where we used the Jacobi identity on the last line. This commutator vanishes due to the equations of motion in (43). On (125), we obtain:

$$\begin{aligned} 0 &= D_\mu D_\mu[\phi^{\mathrm{cl}}, \alpha] - V''(\phi^{\mathrm{cl}})[\phi^{\mathrm{cl}}, \alpha] - 2[D_\mu\phi^{\mathrm{cl}}, D_\mu\alpha] - [\phi^{\mathrm{cl}}, D_\mu D_\mu\alpha] \\ &= [D_\mu D_\mu\phi^{\mathrm{cl}}, \alpha] - V''(\phi^{\mathrm{cl}})[\phi^{\mathrm{cl}}, \alpha] \\ &= [D_\mu D_\mu\phi^{\mathrm{cl}}, \alpha]^a + \lambda(\phi_{\mathrm{cl}}^2 - v^2)[\phi^{\mathrm{cl}}, \alpha]^a + 2\lambda\phi_{\mathrm{cl}}^a\phi_{\mathrm{cl}}^b[\phi_{\mathrm{cl}}, \alpha]^b \\ &= [D_\mu D_\mu\phi_{\mathrm{cl}} + \lambda(\phi_{\mathrm{cl}}^2 - v^2)\phi_{\mathrm{cl}}, \alpha]^a + 2\lambda\phi_{\mathrm{cl}}^a\varepsilon_{bcd}\phi_{\mathrm{cl}}^b\phi_{\mathrm{cl}}^c\alpha^d, \end{aligned} \tag{128}$$

where we used the expression of V'' in (91). The first term vanishes by the equations of motion, and the second one by contraction of ε_{bcd} with $\phi_{\mathrm{cl}}^b\phi_{\mathrm{cl}}^c$. The only other zero-eigenvalue modes are related to the translational collective coordinates.

Since the normal modes should be orthogonal to each other,

$$\begin{aligned} 0 &= \mathrm{tr}\int d^3x \left[a_\mu^{(n)}a_\mu^{(0)} + \chi^{(n)}\chi^{(0)}\right] \\ &= \mathrm{tr}\int d^3x \left[a_\mu^{(n)}D_\mu\alpha + \chi^{(n)}[\phi^{\mathrm{cl}}, \alpha]\right] \\ &= \mathrm{tr}\int d^3x \left[-(D_\mu a_\mu^{(n)})\alpha - [\phi^{\mathrm{cl}}, \chi^{(n)}]\alpha\right], \end{aligned} \tag{129}$$

and because $\alpha(x)$ is an arbitrary function, we find that the non-zero modes must satisfy the condition

$$D_\mu^{A^{\mathrm{cl}}} a_\mu^{(n)} + [\phi^{\mathrm{cl}}, \chi^{(n)}] = 0. \tag{130}$$

To eliminate the zero modes due to gauge freedom, we can choose the gauge such that the fields satisfy the condition (130),

$$D_\mu^{A^{\mathrm{cl}}} A_\mu + [\phi^{\mathrm{cl}}, \phi] = 0. \tag{131}$$

To include this gauge condition in the integral, we use the standard procedure introduced by Faddeev and Popov,

$$1 = \int \mathcal{D}G\delta(G) = \int \mathcal{D}\alpha\delta(G[A_\mu^\alpha]) \det\left(\frac{\delta G[A_\mu^\alpha]}{\delta\alpha}\right), \tag{132}$$

where $G[A_\mu^\alpha]$ is the gauge condition (131) and the determinant of an operator is defined as the product of its eigenvalues. The variation can be calculated as follows: The action of a gauge transformation on the fields is given by

$$\begin{aligned} A'_\mu &= g(x)A_\mu g^{-1}(x) + g(x)\partial_\mu g^{-1}(x), & (133)\\ \phi' &= g(x)\phi g^{-1}(x), & (134) \end{aligned}$$

where $g(x)$ is an element of the group and can be written as an exponential of elements of the Lie algebra

$$g(x) = e^{\alpha(x)} = e^{\alpha(x)^a T^a} \approx 1 + \alpha, \tag{135}$$

with T^a the generators of $SU(2)$. Then we have

$$\begin{aligned} \delta A_\mu &= \partial_\mu\alpha + [A_\mu, \alpha] = D_\mu^A \alpha, & (136)\\ \delta\phi &= [\phi, \alpha], & (137) \end{aligned}$$

and the variation of the gauge condition with respect to α has the simple form

$$\frac{\delta(D_\mu^{A^{\mathrm{cl}}} A_\mu + [\phi^{\mathrm{cl}}, \phi])}{\delta\alpha} = D_\mu^{A^{\mathrm{cl}}} D_\mu^A + [\phi^{\mathrm{cl}}, [\phi, \tag{138}$$

So we can rewrite the identity (132) as

$$1 = \int \mathcal{D}\alpha\delta\left(D_\mu^{A^{\mathrm{cl}}} A_\mu + [\phi^{\mathrm{cl}}, \phi]\right) \det\left(D_\mu^{A^{\mathrm{cl}}} D_\mu^A + [\phi^{\mathrm{cl}}, [\phi,\right) \tag{139}$$

What we want to do now is to reduce the functional integral $\int \mathcal{D}\phi\mathcal{D}A_\mu$ to regular integrals. Since the normal modes form a complete set of the solutions of the fields we expand them in a linear combination of their normal modes

$$\begin{aligned} A_\mu &= A_\mu^{\mathrm{cl}} + \sum_n \xi_n a_\mu^{(n)} + D_\mu^{A^{\mathrm{cl}}} \alpha, & (140)\\ \phi &= \phi^{\mathrm{cl}} + \sum_n \xi_n \chi^{(n)} + [\alpha, \phi^{\mathrm{cl}}], & (141) \end{aligned}$$

where $a_\mu^{(n)}$ and $\phi^{(n)}$ correspond to the non-zero modes only. Then we can rewrite the measure as a function of the coefficients of the modes only. However, in order to do this we must find the Jacobian of the coordinate substitution. We see that the unit of length in the A_μ and ϕ manifold is given by

$$(\delta l)^2 = \mathrm{tr}\int d^3x \left[(\delta A_\mu)^2 + (\delta\phi)^2\right] = \sum_n (\delta\xi_n)^2 + (D_\mu^{A^{\rm cl}}\delta\alpha)^2 + [\delta\alpha, \phi^{\rm cl}]^2. \tag{142}$$

We can then define a metric as

$$g_{ab} = D_\mu^{A^{\rm cl}(a)} D_\mu^{A^{\rm cl}(b)}\delta_{ab} + \phi_{\rm cl}^2\delta_{ab} - \phi_{\rm cl}^a\phi_{\rm cl}^b, \tag{143}$$

where an index (a) on a differential operator means that it acts on an object with index a. If the coefficients ξ_n are normalized, the measure becomes

$$\int \mathcal{D}\phi\mathcal{D}A_\mu = \int \prod_n d\xi_n \mathcal{D}\alpha\sqrt{g}. \tag{144}$$

Including the Faddeev-Popov determinant we find

$$\begin{aligned} \int \mathcal{D}\phi\mathcal{D}A_\mu &= \int \prod_n d\xi_n \mathcal{D}\alpha\delta\left(D_\mu^{A^{\rm cl}}A_\mu + [\phi^{\rm cl}, \phi]\right)\det\left(D_\mu^{A^{\rm cl}}D_\mu^A + [\phi^{\rm cl}, [\phi, \right)\sqrt{g} \\ &= \int \prod_n d\xi_n \det\left(D_\mu^{A^{\rm cl}}D_\mu^A + [\phi^{\rm cl}, [\phi, \right)\sqrt{g}, \end{aligned} \tag{145}$$

where the delta function eliminated the integration over the zero modes, $\mathcal{D}\alpha$, and the gauge degrees of freedom completely disappear from the integral. However, there are still the zero modes related to the symmetries of the classical solution. We should change the integration over these modes to an integration over the respective collective coordinates.

Calculation of the translational collective coordinates

The zero modes related to the translational symmetry of the action are given by

$$\begin{aligned} a_\mu^{(\lambda,0)} &= C^{-\frac{1}{2}}\partial_\lambda A_\mu^{\rm cl} \\ \chi^{(\lambda,0)} &= C^{-\frac{1}{2}}\partial_\lambda\phi^{\rm cl} \end{aligned} \tag{146}$$

with C a normalization constant. On the one hand, these zero modes are not eliminated from the integral when fixing the gauge (131). The usual way to deal with this is to associate a collective coordinate to these degrees of freedom, as explained in section 5.3. On the other hand, in this form these modes do not yet satisfy condition (130) either.

We still have the freedom to fix $\alpha(x)$. If we choose

$$\alpha^{(\lambda)}(x) = -A_\lambda^{\rm cl}, \tag{147}$$

the translational modes (146) become

$$\begin{aligned} a_\mu^{(\lambda,0)} &= C'^{-\frac{1}{2}}(\partial_\lambda A_\mu^{\rm cl} - D_\mu A_\lambda^{\rm cl}) = N'^{-\frac{1}{2}} F_{\lambda\mu}^{\rm cl}, \\ \chi^{(\lambda,0)} &= C'^{-\frac{1}{2}}(\partial_\lambda \phi^{\rm cl} + [A_\lambda^{\rm cl}, \phi^{\rm cl}]) = C'^{-\frac{1}{2}} D_\lambda^{A^{\rm cl}} \phi^{\rm cl}, \\ C' &= \int d^3x \left[(F_{\lambda\mu}^{\rm cl})^2 + (D_\lambda^{A^{\rm cl}} \phi^{\rm cl})^2 \right] \end{aligned} \tag{148}$$

and now they satisfy the condition (130)

$$D_\mu^{A^{\rm cl}} a_\mu^{(\lambda,0)} + [\phi^{\rm cl}, \chi^{(\lambda,0)}] = D_\mu^{A^{\rm cl}} F_{\lambda\mu}^{\rm cl} + [\phi^{\rm cl}, D_\lambda^{A^{\rm cl}} \phi^{\rm cl}] = 0 \tag{149}$$

by the equations of motion.

Now we want to expand the translated vector field and find the coefficient of the zero mode, ξ_0. Let R_λ be the centre of the translated solution (the collective coordinate). Then the shifted field is given by

$$\begin{aligned} A_\mu^{\rm cl}(x+R) - D_\mu(R_\sigma A_\sigma) &\approx A_\mu^{\rm cl}(x) + R_\sigma \partial_\sigma A_\mu^{\rm cl}(x) - R_\sigma D_\mu^{A^{\rm cl}} A_\sigma^{\rm cl} \\ &= A_\mu^{\rm cl}(x) + R_\sigma D_\sigma^{A^{\rm cl}} A_\mu^{\rm cl}(x) - R_\sigma D_\mu^{A^{\rm cl}} A_\sigma^{\rm cl} - R_\sigma [A_\mu^{\rm cl}, A_\sigma^{\rm cl}] \\ &= A_\mu^{\rm cl}(x) + R_\sigma F_{\sigma\mu}^{\rm cl} \end{aligned} \tag{150}$$

and we find that

$$\xi_0^{(\lambda)} \approx C'^{\frac{1}{2}} R_\lambda \Rightarrow d^3\xi_0 = C'^{\frac{3}{2}} d^3R. \tag{151}$$

We are now ready to write the full measure as integrations over the centres of the monopoles and coefficients of the non-zero oscillation modes:

$$\int \mathcal{D}\phi \mathcal{D}A_\mu = \sum_{N,\{n_a\}} \int \frac{1}{N!} C'^{\frac{3N}{2}} d^3R_1 \dots d^3R_N \prod_{n\neq 0} d\xi_n \det\left(D_\mu^{A^{\rm cl}} D_\mu^A + [\phi^{\rm cl}, [\phi, \right) \sqrt{g}. \tag{152}$$

In the one-loop approximation, [7]

$$\det\left(D_\mu^{A^{\rm cl}} D_\mu^A + [\phi^{\rm cl}, [\phi, \right) \approx \det\left(D_\mu^{A^{\rm cl}} D_\mu^{A^{\rm cl}} + [\phi^{\rm cl}, [\phi^{\rm cl}, \right). \tag{153}$$

The measure is then independent of ξ_n, and on (122) we can do the Gaussian integral on the ξ_n (the normal modes are normalised so that the integral on x gives 1):

$$\int d\xi_n \exp\left\{ - \int d^3x \xi_n^2 \omega_n^2 (\chi_{(n)}^2 + a_{(n)}^2) \right\} = \frac{\sqrt{\pi}}{\omega_n}. \tag{154}$$

The partition function can then be written

$$\mathcal{Z} = \sum_{N,\{n_a\}} \int \frac{1}{N!} C'^{\frac{3N}{2}} d^3R_1 \dots d^3R_N M_W^{3N} \exp\{-S_{cl}\} \det\left(D_\mu^{A^{\rm cl}} D_\mu^{A^{\rm cl}} + [\phi^{\rm cl}, [\phi^{\rm cl}, \right) \sqrt{g} \prod_{n\neq 0} \frac{\sqrt{\pi}}{\omega_n}, \tag{155}$$

where $S_{\rm cl}$ is given by equation (120), and the other factors are independent of the positions R_i. These factors can be evaluated for one monopole,

$$C'^{\frac{3}{2}} M_W^3 \det \left(D_\mu^{A^{\rm cl}} D_\mu^{A^{\rm cl}} + [\phi^{\rm cl}, [\phi^{\rm cl}, \right) \sqrt{g} \prod_{n\neq 0} \frac{\sqrt{\pi}}{\omega_n} = \frac{M_W^{7/2}}{e} \alpha \left(\frac{\lambda}{e^2} \right), \tag{156}$$

where α is a function that could be calculated. For N monopoles, this factor simply appears N times. Finally, substituting the classical action, we find

$$\mathcal{Z} = \sum_{N,\{n_a\}} \int d^3R_1 \ldots d^3R_N \frac{1}{N!} \left[\frac{M_W^{7/2}}{e} \alpha \left(\frac{\lambda}{e^2} \right) e^{-\frac{4\pi M_W}{e^2} \epsilon \left(\frac{\lambda}{e^2} \right)} \right]^N \exp \left\{ -\frac{\pi}{2e^2} \sum_{a\neq b} \frac{n_a n_b}{|R_a - R_b|} \right\}, \tag{157}$$

which corresponds to (123) with

$$\zeta = \frac{M_W^{7/2}}{e} \alpha \left(\frac{\lambda}{e^2} \right) e^{-\frac{4\pi M_W}{e^2} \epsilon \left(\frac{\lambda}{e^2} \right)}. \tag{158}$$

The partition function becomes, as anticipated,

$$\mathcal{Z} = \sum_{N,\{n_a\}} \frac{\zeta^N}{N!} \int d^3R_1 \ldots . d^3R_N \exp \left\{ -\frac{\pi}{2e^2} \sum_{a\neq b} \frac{n_a n_b}{|R_a - R_b|} \right\}. \tag{159}$$

7.4 Calculation of the Mass Gap

With the partition function of the monopole gas, we should be able to find correlation functions for various physical observables. However, the form of the integral (159) is not yet suitable for it. We would like the partition function to be written as a functional integral, so we could find, by the usual way in path integral formalism, a diagrammatic expansion.

In order to do this we can use the method explained in Zinn-Justin's book [28] based on the following identity:

$$\int \mathcal{D}\varphi \exp \left\{ -\frac{1}{2} \int d^3x [(\nabla\varphi)^2] + i \sum_i c_i \varphi(x_i) \right\} = \exp \left\{ -\frac{1}{2} \sum_{i,j} c_i c_j \Delta(x_i - x_j) \right\} \tag{160}$$

for a massless boson field with propagator

$$\Delta(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{p^2} e^{ip\cdot x} = \frac{1}{2\pi^2|x|} \int \frac{\sin(p)dp}{p} = \frac{1}{4\pi|x|}. \tag{161}$$

Then, if we rewrite the partition function as a functional integral over the auxiliary field $\varphi(x)$,

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\varphi e^{-\frac{2\pi^2}{e^2}\int(\nabla\varphi)^2 d^3x} \sum_N \sum_{\{n_a=\pm 1\}} \frac{\zeta^N}{N!} \int d^3R_1 ... d^3R_N e^{i\frac{4\pi^2}{e^2}\sum n_a \varphi(R_a)} \\
&= \int \mathcal{D}\varphi e^{-\frac{e^2}{8\pi^2}\int(\nabla\varphi)^2 d^3x} \sum_N \frac{\zeta^N}{N!} \left(\int d^3R (e^{i\varphi(R)} + e^{-i\varphi(R)}) \right)^N \\
&= \int \mathcal{D}\varphi \exp\left\{ -\frac{e^2}{8\pi^2} \int \left[(\nabla\varphi)^2 - \frac{16\pi^2\zeta}{e^2}\cos(\varphi) \right] d^3x \right\}, \qquad (162)
\end{aligned}$$

we find an equivalence of our model to a massless boson in a sine-Gordon potential. The field φ can be regarded as the "dual photon field", since it results from the action of the monopole gas and encodes the magnetic field structure. We can still work this integral a little bit further, writing the action as

$$S_\varphi = \frac{1}{2}\left(\frac{e}{2\pi}\right)^2 \int d^3r \left[(\nabla\varphi)^2 + \frac{8\pi^2\zeta}{e^2}\varphi^2 + \mathcal{O}(\varphi^4) \right] \qquad (163)$$

and neglecting higher order terms, since they are much smaller than the quadratic term [7]. This theory then turns out to be a free bosonic theory with a massive field φ whose mass M is given by

$$M^2 = \frac{8\pi^2\zeta}{e^2}. \qquad (164)$$

We know that the two-point correlation function $\langle\varphi(k)\varphi(-k)\rangle$ is reduced to the Feynman propagator

$$\langle\varphi(k)\varphi(-k)\rangle = D_F(k) = \left(\frac{2\pi}{e}\right)^2 (2\pi)^3 \frac{1}{k^2+M^2}. \qquad (165)$$

But we will need to calculate other correlation functions, namely correlation functions of the electromagnetic field, so it is useful to define the generating functional

$$\mathcal{G}[\eta] = \mathcal{Z} e^{-\int \eta(x)\rho(x) d^3x}, \qquad (166)$$

introducing an external source $\eta(x)$ coupled to the monopoles with density

$$\rho(x) = \sum_a n_a \delta(x - x_a). \qquad (167)$$

We can find the form of $\mathcal{G}[\eta]$ by the same method used in (162):

$$\begin{aligned}
\mathcal{G}[\eta] &= \int \mathcal{D}\varphi e^{-\frac{e^2}{8\pi^2}\int(\nabla\varphi)^2 d^3x} \sum_N \sum_{\{n_a=\pm 1\}} \frac{\zeta^N}{N!} \int d^3R_1 ... d^3R_N e^{i\sum n_a(\varphi(R_a)+\eta(R_a))} \\
&= \int \mathcal{D}\varphi \exp\left\{ -\frac{e^2}{8\pi^2} \int \left[(\nabla\varphi)^2 - 2M^2\cos(\varphi+\eta) \right] d^3x \right\} \\
&= \int \mathcal{D}\varphi \exp\left\{ -\frac{1}{2}\left(\frac{e}{2\pi}\right)^2 \int \left[(\nabla\varphi - \nabla\eta)^2 - 2M^2\cos(\varphi) \right] d^3x \right\}, \qquad (168)
\end{aligned}$$

where in the last step we made the substitution $\varphi \to \varphi - \eta$.

The function $\mathcal{G}[\eta]$ allows us to calculate the correlation functions of the monopole densities by taking the functional derivative with respect to the source $\eta(x)$

$$\langle \rho(x_1)...\rho(x_n)\rangle = \frac{\mathcal{Z}\rho(x_1)...\rho(x_n)}{\mathcal{Z}} = \frac{1}{\mathcal{Z}}\frac{\delta}{\delta\eta(x_1)}...\frac{\delta}{\delta\eta(x_n)}\mathcal{G}[\eta]|_{\eta=0}. \tag{169}$$

We can also work in momentum space. We then need the Fourier transform of the generating function, which is given by

$$\mathcal{G}[\eta] = \int \mathcal{D}\varphi \exp\left\{-\frac{1}{2}\left(\frac{e}{2\pi}\right)^2\left[\int \frac{d^3k}{(2\pi)^3}k^2(\varphi-\eta)(k)(\varphi-\eta)(-k)\right] + 2\zeta\cos\varphi(0)\right\} \tag{170}$$

and the correlation functions are calculated in the same way.

We need to calculate $\langle\rho(-k)\rho(k)\rangle$. Starting by

$$\begin{aligned}\langle\rho(k)\rangle &= \frac{1}{\mathcal{Z}}\frac{\delta}{\delta\eta(k)}\mathcal{G}[\eta]|_{\eta=0}\\ &= \frac{1}{\mathcal{Z}}\left[\mathcal{G}[\eta]\int\mathcal{D}\varphi\frac{e^2}{(2\pi)^5}k^2(\varphi-\eta)(-k)\right]|_{\eta=0}\\ &= \frac{e^2}{(2\pi)^5}k^2\langle\varphi(-k)\rangle,\end{aligned} \tag{171}$$

we can then see

$$\begin{aligned}\langle\rho(-k)\rho(k)\rangle &= \frac{1}{\mathcal{Z}}\frac{\delta}{\delta\eta(-k)}\frac{\delta}{\delta\eta(k)}\mathcal{G}[\eta]|_{\eta=0}\\ &= \frac{1}{\mathcal{Z}}\left[\frac{\delta}{\delta\eta(-k)}\mathcal{G}[\eta]\int\mathcal{D}\varphi\frac{e^2}{(2\pi)^5}k^2(\varphi-\eta)(-k)\right]|_{\eta=0}\\ &= \frac{1}{\mathcal{Z}}\left[\left(-\frac{e^2}{(2\pi)^5}k^2+\int\mathcal{D}\varphi\left(\frac{e^2}{(2\pi)^5}k^2\right)^2((\varphi-\eta)(-k)(\varphi-\eta)(k))\right)\mathcal{G}[\eta]\right]|_{\eta=0}\\ &= -\frac{e^2}{(2\pi)^5}k^2+\left(\frac{e^2}{(2\pi)^5}\right)^2k^4\langle\varphi(-k)\varphi(k)\rangle\\ &= -\frac{e^2}{(2\pi)^5}k^2+\left(\frac{e^2}{(2\pi)^5}\right)^2\left(\frac{e^2}{(2\pi)^5}\right)^{-1}\frac{k^4}{k^2+M^2}\\ &= -\frac{e^2}{(2\pi)^5}\left[k^2-\frac{k^4}{k^2+M^2}\right].\end{aligned} \tag{172}$$

Now, we can write the magnetic field (118) as a function of the monopole density as

$$\mathcal{B}_n(x) = \frac{1}{e}\int d^3y\frac{(x-y)_n}{|x-y|^3}\rho(y), \tag{173}$$

or, by performing a Fourier transform

$$\mathcal{B}_n(k) = \frac{i4\pi k_n}{ek^2}\rho(k). \tag{174}$$

The correlation functions of the field $\mathcal{B}$ are then given by

$$\langle \mathcal{B}_n(-k)\mathcal{B}_m(k)\rangle = \langle \mathcal{B}_n(-k)\mathcal{B}_m(k)\rangle^{(0)} - \left(\frac{4\pi}{e}\right)^2 \frac{k_n k_m}{k^4} \langle \rho(-k)\rho(k)\rangle \tag{175}$$

where the first term corresponds to the bare Green function of the $\mathcal{B}$-field, i.e., the propagator of the massless photon, which has the form [24]

$$\langle \mathcal{B}_n(-k)\mathcal{B}_m(k)\rangle^{(0)} = \frac{4}{(2\pi)^3}\left[\delta_{nm} - \frac{k_n k_m}{k^2}\right]. \tag{176}$$

Inserting the correlation function (172) and the Green function (176) in (175) we find

$$\begin{aligned} \langle \mathcal{B}_n(-k)\mathcal{B}_m(k)\rangle &= \frac{4}{(2\pi)^3}\left[\delta_{nm} - \frac{k_n k_m}{k^2}\right] - \left(\frac{4\pi}{e}\right)^2 \left(-\frac{e^2}{(2\pi)^5}\right) \frac{k_n k_m}{k^4}\left[k^2 - \frac{k^4}{k^2+M^2}\right] \\ &= \frac{4}{(2\pi)^3}\left[\delta_{nm} - \frac{k_n k_m}{k^2}\right] + \frac{4}{(2\pi)^3}\frac{k_n k_m}{k^2} - \frac{4}{(2\pi)^3}\frac{k_n k_m}{k^2+M^2} \\ &= \frac{4}{(2\pi)^3}\left[\delta_{nm} - \frac{k_n k_m}{k^2+M^2}\right]. \end{aligned} \tag{177}$$

This highlights an interesting feature - the photon acquires a small mass M. Therefore, there are no massless particles in the theory.

7.5 Calculation of the Confining Condition

In the previous section we found that there is a lower bound on the energy of a photon propagating in a monopole plasma, which leads to the appearance of a non-vanishing mass in its propagator. In this section we will see what the monopole gas imposes on the vacuum energy of two separated electric charges.

If we consider the creation of a negative and a positive charged particles in Euclidean spacetime, allowing them to be statically separated by a distance R for an Euclidean time T and then annihilate, as it is illustrated in Fig. 2, we see that the world lines of both these particles form a closed loop C with a current density j_m. The interaction of the current density with the gauge field can be expressed as an interaction term in the action, given by [26]

$$S_{int} = \int d^3x j_m A_m^{em} \tag{178}$$

and since the electric current is entirely due to the two charges in the loop, this integral can be reduced to

$$S_{int} = e \oint_C dx_m A_m^{em}. \tag{179}$$

We can define an auxiliary quantity $W = \exp(-S_{int})$, which is called the *Wilson loop* and has the good properties of being gauge and Lorentz invariant. The expectation value of the Wilson loop, $\langle W\rangle$, then yields the additional potential energy of the two static charges separated by the

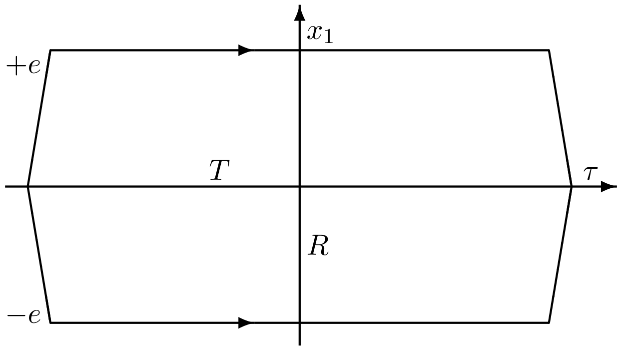

Figure 4: World lines of the charged particles.

distance R, $\Delta E(R)$ [16]:

$$\lim_{T\to\infty} \langle W \rangle = \lim_{T\to\infty} \frac{\exp[-ET - \Delta E(R)T]}{\exp[-ET]} \tag{180}$$

$$\Rightarrow \Delta E(R) = \lim_{T\to\infty} -\frac{1}{T} \log \langle W \rangle \,. \tag{181}$$

To find whether the two charges experience a confining force between them in this model, we need to calculate $\langle W \rangle$.

Firstly we can use Stokes' theorem to reduce $\langle W \rangle$ to

$$\langle W \rangle = \left\langle \exp\left\{ -e \oint_C dx_m A_m^{em} \right\} \right\rangle = \left\langle \exp\left\{ -e \int_S dS_m \mathcal{B}_m \right\} \right\rangle , \tag{182}$$

where the integration is taken on the surface S, enclosed in the contour C. This integral is equal to the magnetic flux through S. Without loss of generality, we can take the loop to lie on the $x_1 - x_2$ plane ($x_2 = \tau$), so the flux is equal to the integral of the third component of the magnetic field on the surface.

In the previous section we found a way to calculate the expectation value of quantities in the form

$$\left\langle \exp\left\{ - \int \eta(x)\rho(x) d^3x \right\} \right\rangle , \tag{183}$$

in order to define the generating functional (168). Since $\mathcal{B}$ can be written in terms of ρ as we have seen in (173), if we fix the function η to be

$$\eta(x) = \int_S dS_m^{(y)} \frac{(x-y)_m}{|x-y|^3} , \tag{184}$$

we find that

$$e \int_S \mathcal{B}_m dS_m = \int \eta(x)\rho(x) d^3x \tag{185}$$

and we can express the expectation value of the Wilson loop as

$$\langle W \rangle = \left\langle \exp\left\{ - \int \eta(x)\rho(x) d^3x \right\} \right\rangle . \tag{186}$$

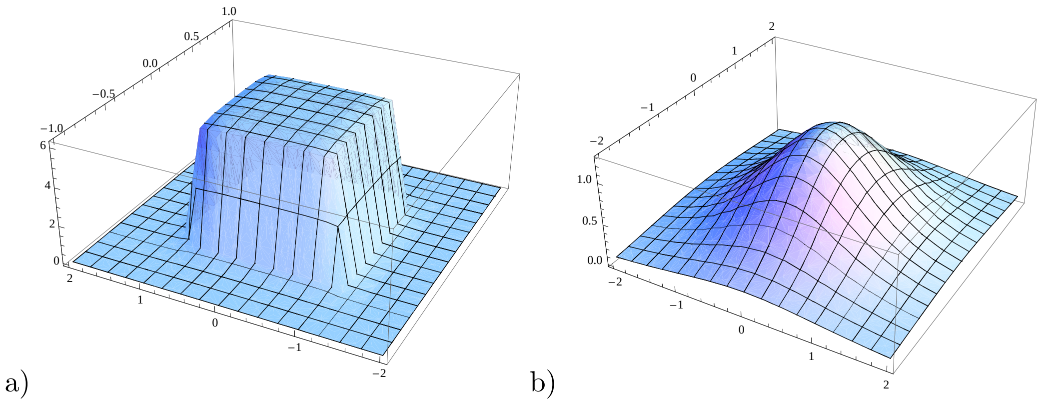

Figure 5: Plot of the function $\eta(x)$ for a loop with $R = 1$ and $T = 2$ with variables x_1, x_2 and two fixed values of x_3. a) $x_3 = 0.01$, where we can see that close to the loop $\eta(x)$ approaches a top-hat function with the value 2π inside the loop and zero otherwise. b) $x_3 = 1$, where the function is already spread, and trending smoothly to zero.

From (185), we see that $\eta(x)$ measures the contribution made by a monopole in position x to the magnetic flux through the surface S. This physical interpretation of the function $\eta(x)$ allows us to infer the behaviour of the integral (184).

With the modified Maxwell equations discussed in section 4.3, the Gauss law for a magnetic monopole takes the form

$$\oint \mathcal{B} \cdot dS = g = \frac{4\pi}{e} \tag{187}$$

so if we consider a monopole arbitrarily close to the loop ($x_3 \to 0$ and x_1, x_2 inside the contour), the contribution to the flux through the surface should be half of the total magnetic flux created by the monopole. Similarly, if $x_3 \to 0$ and we are outside the contour, the contribution should be zero. We then expect that

$$\lim_{x_3 \to 0} |\eta(x)| = 2\pi\theta(x_1, x_2), \tag{188}$$

where $\theta(x_1, x_2)$ is 1 if x_1, x_2 are inside the contour and 0 if they are outside. Furthermore, this step should become smoother as x_3 increases, and $\eta(x)$ goes to zero. Performing a numerical integration, we see that this is in fact true (fig. 5).

Thus, we found that the interaction term, $\int dx \eta(x)\rho(x)$, is of the order of unity and consequently much larger than the potential of the theory, $2M^2 \cos\varphi$. This means that the approximation we did in the previous section, (163), is no longer valid, and terms in higher powers of φ should be taken into account. Furthermore, the quantum corrections to the potential can be suppressed and the only contribution to the partition function can be written as

$$\langle W \rangle = \mathcal{G}[\eta] = \exp\left\{ -\frac{1}{2}\left(\frac{e}{2\pi}\right)^2 \int d^3x \left[(\nabla\varphi^{\text{cl}} - \nabla\eta)^2 - 2M^2\cos(\varphi^{\text{cl}}) \right] \right\} \tag{189}$$

where φ^{cl} is given by the equation of motion

$$\nabla^2(\varphi^{\text{cl}} - \eta) = M^2 \sin(\varphi^{\text{cl}}). \tag{190}$$

The Laplacian of $\eta(x)$ is given by

$$\nabla^2\eta(x) = 4\pi\delta'(x_3)\theta(x_1, x_2). \tag{191}$$

Far from the contour C this equation can be approximated to only depend on the coordinate x_3. Note that we are interested in the behaviour of the potential for large R, and we took the limit $T \to \infty$, so both are big enough that we can consider that this is the case. Then

$$\nabla^2\varphi^{\text{cl}} = \varphi''_{\text{cl}}(x_3) = M^2\sin(\varphi^{\text{cl}}), \tag{192}$$

which is the sine-Gordon model studied in section 3.2, and as we have seen, is analytically solvable. Then, the exact solution for φ^{cl} should be of the form

$$\varphi^{\text{cl}}(x_3 \neq 0) = 4\arctan e^{\pm Mx_3} + 2\pi n \tag{193}$$

since the theory is invariant under the sum of 2π. To find the suitable solution for (190) we should consider the boundary condition on the surface. The Laplacian of $\eta(x)$ imposes a jump of $+4\pi$ in the solution at $x_3 = 0$, hence a possible solution of φ^{cl} is

$$\varphi^{\text{cl}}(x_3) = \begin{cases} 4\arctan(e^{-Mx_3}), & x_3 > 0 \\ 4\arctan(e^{-Mx_3}) - 4\pi, & x_3 < 0. \end{cases} \tag{194}$$

However, this solution is only valid when x_1 and x_2 lay inside the loop, and outside the loop the field φ^{cl} vanishes. Therefore, the integration should only be within these limits. The Wilson loop becomes

$$\begin{aligned} &\langle W\rangle \approx \\ &\approx \exp\left\{-\frac{1}{2}\left(\frac{e}{2\pi}\right)^2 \int_{-R/2}^{R/2} dx_1 \int_{-T/2}^{T/2} dx_2 \int dx_3 \left[(\partial^2_{x_3}\varphi^{\text{cl}} - \partial^2_{x_3}\eta)\varphi^{\text{cl}}(x_3) + 2M^2[1 - \cos(\varphi^{\text{cl}}(x_3))]\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\left(\frac{e}{2\pi}\right)^2 RT \int dx_3 \left[M^2\sin(\varphi^{\text{cl}})\varphi^{\text{cl}}(x_3) + 2M^2[1 - \cos(\varphi^{\text{cl}}(x_3))]\right]\right\} \\ &= \exp\{-\gamma RT\}, \end{aligned} \tag{195}$$

where γ is a number given by

$$\gamma = \frac{1}{2}\left(\frac{e}{2\pi}\right)^2 \int dx_3 \left[M^2\sin(\varphi^{\text{cl}})\varphi^{\text{cl}}(x_3) + 2M^2[1 - \cos(\varphi^{\text{cl}}(x_3))]\right] = \frac{e^2M}{\pi}. \tag{196}$$

From (181), we then obtain $\Delta E(R) = \gamma R$, so we have proved that the energy grows linearly with the distance and therefore there is a confining force between the particles.

7.6 The Phases of the Model

The calculation done in this section is not always true, since it depends crucially on the three dimensionful parameters in the theory e, v and λ, and we used a specific limit to do the approximations in our calculation. We may ask what would have happened if we had considered other limits.

Two dimensionless ratios of these quantities are enough to define the system. It is convenient to choose them to be $\frac{e}{v}$ and $\frac{M_W}{M_H}$, since these are ratios which appear often in the calculations.

Three phases are expected in this theory, an unbroken phase, a confining phase and a Coulomb phase [28]:

- *Symmetric phase:* The symmetry is not spontaneously broken, the states consist only on neutral states, singlets of the group transformation.
- *Coulomb phase:* The symmetry is spontaneously broken and the particle states contain massive and charged particles which interact by a Coulomb potential.
- *Confining phase:* The limits discussed above are respected and the result is satisfied, there is a confining force between charged particles which grow linearly with its separation.

The ratio $\frac{e}{v}$ will determine whether the Higgs mechanism indeed happens. If it is much smaller than one, the symmetric configuration is highly unstable and we can safely consider that we are in either the Coulomb or confining phases of the theory. The distinction between this two phases will depend on the second ratio $\frac{M_W}{M_H}$.

8 Conclusions

With this calculation we have proved that there are no massless particles in the broken phase of the Georgi-Glashow model, and that the charge is confined by a linear force imposed by the background of monopole particles.

Polyakov's confinement mechanism cannot be straightforwardly applied to QCD since there are two fundamental differences between the $2 + 1$d Georgi-Glashow model and QCD. The first one is that despite the fact that there are instanton solutions in free QCD [17], and in particular a monopole solution with unit topological charge, the electromagnetic field tensor, $\mathcal{F}^{\mu\nu}$, falls off like $1/r^4$, instead of the Couloumb force in the Georgi-Glashow model which falls off like $1/r^2$ [18]. The second one consists of the scale (or conformal) invariance of free QCD, which is not present in the Georgi-Glashow model since there is a Higgs field introduced in the theory fixing the energy scale.

Unfortunately, due to the limited amount of time to do the dissertation, it was not possible to explore this fascinating field further than the late seventies, and recent progresses were not included in this discussion.

However, it is evident that the journey to find the right nature of the confinement mechanism is still running strong, and new significant breakthroughs are slowly albeit constantly happening. Since the proof of confinement in an $N = 2$ super-Yang-Mills theory, by Seiberg and Witten in 1994 [19], the route of supersymmetric non-abelian gauge theories has been the most followed. Despite the fact that the proof for $d = 4$ free Yang-Mills theory has not yet been found, some limits are already understood.

Actually, solving confinement in four-dimensional pure Yang-Mills is such a difficult and important problem that it has been considered by the Clay Mathematics Institute to be one of the seven millennium problems, a prize of one million dollars being awarded to the person who can finally find a solution.

Surprisingly, I have not been able to find a detailed review of Polyakov's calculations, and breaking into the missing steps and shortened explanations revealed to be an extremely challenging work. I believe it might be of interest to anyone who wants to understand this proof to have access to a more detailed version of it, like the one I have tried to offer.

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