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Pilot-wave Scalar Field Theory in the Schrödinger Representation

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Abstract

We consider the pilot-wave formulation of scalar field theory with a view to numerical modelling. We show that the standard $\mathbf{k}-$space representations of Bohm et al. (1987) unecessarily complicate the theory, and a simpler, more natural approach may be taken. We show this first by considering the functional-Wirtinger derivatives in the Bohm approach, and then concretely by the use of canonical transformations prior to quantisation. We then consider the volume element, and present an argument supporting it's representation in $\mathbf{k}-$space. We discuss the low energy limit of the theory, and suggest an attempt to construct a quantum mechanical sub-algebra from the algebra of field operators. Finally we consider the the success of our numerical approaches.

Contents

1 Introduction

Pilot-wave theory, known variously as de Broglie-Bohm theory, Bohmian mechanics, the causal or ontological interpretation is a group of formulations of quantum theory the first of which was presented by Louis de-Broglie in 1927, before being reformulated and re-presented by David Bohm in 1952. In their most basic form, that corresponding to elementary quantum mechanics, they are characterised by the interpretation of the continuity equation for the quantum probability density,

$$\frac{\partial|\psi|^2}{\partial t} + \nabla.\left(|\psi|^2\frac{\nabla S}{m}\right) = 0 \tag{1.1}$$

as a continuity equation for the normalised density ρ of an ensemble of system configurations,

$$\frac{\partial\rho}{\partial t} + \nabla.\left(\rho\dot{\mathbf{q}}\right) = 0, \tag{1.2}$$

each evolving such that if $\rho = |\psi|^2$ at some time, then they will also be so at any later time. S is the complex phase of the wavefunction; $\psi = |\psi|e^{iS}$.

Under this interpretation, $|\psi|^2$ each system state, often called the beable, (in this case the configuration space vector) $\mathbf{Q} \in \{\mathbf{q}\}$ necessarily moves

according to the canonical guidance equations,

$$\dot{\mathbf{Q}} = \frac{\nabla S(\mathbf{q}, t)}{m}\bigg|_{\mathbf{q}=\mathbf{Q}(t)}. \tag{1.3}$$

The relative advantages and disadvantages of a pilot-wave theory compared to a Copenhagen mechanics is a very interesting question, and one that we cannot do justice to here as we wish to move straight on to talking about field theories. We would emphasise how beneficial it can be to know more than one formulation of the same Physics. Personally I find the contrast between the two theories, and the possibility of equivalence of innequivalence of the theories captivating.

1.1 Pilot-wave field theory

Bosonic field theories were first treated by Bohm in 1952, in the first appendix of the second of his seminal papers, while treating the electromagnetic field in the Coulomb gauge. He took what seemed to be the obvious extension of a configuration vector beable to the field theoretic case; the field configuration became the beable. Despite the fame, or possibly infame, of these papers, (or maybe because of it) it seems likely that this was a part of the theory that took the back-seat a while. The forgetting a fleeting mention in Bohm & Hiley (1984), a good discussion on the Bosonic field didn't surface until Bohm et al. (1987). Though this seems to have solidified the field configuration as the beable of choice for Bosonic fields. It also brought into question the requirement for Lorentz covariance on the sub-quantum level; the ensemble covariant, trajectory not.

Fermionic fields, on the other hand, to did not receive pilot-wave treatment until Bell (1984) presented a lattice model where the beables were the

fermion numbers at each lattice point. Though this appears to be a notable break from tradition, it has seen a resurgence in popularity, with the recent development of two distinct continuous theories, Dürr et al. (2003) and Colin & Struyve (2007), the former stochastic, the latter deterministic. The anti-commuting field operators of Fermionic quantum field theories have historically proven a greater challenge to those looking to develop reasonable pilot-wave formulations.

1.2 Outline of dissertation

Although pilot-wave scalar field theory is now twenty three years old, there has been little computational investigation into the properties of the behaviour of individual pilot-wave fields. Indeed the only two papers we could find were Lam & Dewdney (1994*a*) and (1994*b*). This is as apposed to a significant amount of work put into standard pilot-wave theory. This was decided to be our focus, and it was kept deliberately open to allow for any interesting leads that presented themselves to be pointed out.

In sections 2.1-2.2 we will introduce the theory in it's canonical form; the Schrödinger representation in $\mathbf{x}-$space, and consider the possibilities of direct simulation in $\mathbf{x}-$space. As the project matured it became evident that the $\tilde{\phi}, \tilde{\phi}^*$ or it's real version, described in Bohm et al. (1987) and Holland (1993) were seeming to hinder calculations. These representations involve splitting $\mathbf{k}-$space in two, and doubling the number of operators defined over them, something that we shall show is unnecessary. In the first half of section 2.3 we give the motivation behind this thinking by showing an apparent choice of representation emerging through consideration of the functional derivatives. Toward the middle we will show concretely that the new representation is valid through considering the Fourier transform as a

canonical transform on the classical Hamiltonian. We will then derive the representations of Bohm et al. (1987) using the same methods. In the penultimate part of 2.3 we will consider the volume element, and whether one can extend a probabilistic interpretation to the $\mathbf{k}-$space wavefunctionals. We will then finish 2.3 by using our argument to develop guidance equations in the three representations.

Section 2.4 will then consider the how the theory should act in a low energy limit. We will suggest the possibility of a sub-algebra approximate to that of quantum mechanics that could be formed in the low energy limit, built from the field operators.

Section 3 will see us move on to numerical work. Though we have not had time to perform explicit calculations, we will show at least that a likely state in the $\phi(\mathbf{x})\left|0\right\rangle$ is a localised state. We will then explain the structure of the code that we developed, in the hope that it may aid future simulation.

2 Quantising the free massive scalar field for practical calculations

2.1 Conventions and the reality condition

One usually introduces quantum field theory by first treating the scalar field in the Fock representation. In the Fock representation entities such as $\left|0\right\rangle$ and $\hat{a}^{\dagger}$ are left in their vector language. This level of abstraction is unhelpful when introducing field beables into the theory. To ease the interpretation of the theory, pilot-wave quantum field theory is usually carried out in the Schrödinger representation where our field beables take a solid algebraic form. It is easy to mix up subtly differing expressions when working with pilot-wave field theory literature, and as such we will re-derive the basic

theory in several forms, highlighting the similarities and differences between each. Since we require eventually to return from our analysis to calculate actual field trajectories, we hope to keep the following discussion free for ambiguity, and to provide solid, calculable expressions wherever possible.

We begin with the expression for the standard convention for the Fourier transform we will use throughout. They are

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}.\mathbf{x}} \tilde{\phi}(\mathbf{k}), \quad \phi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} q_{\mathbf{k}} \tag{2.1}$$

$$\tilde{\phi}(\mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{k}.\mathbf{x}} \phi(\mathbf{x}), \quad q_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int_V d^3x\, e^{-i\mathbf{k}.\mathbf{x}} \phi(\mathbf{x}) \tag{2.2}$$

$$\int d^3x\, e^{i\mathbf{k}.\mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{k}), \quad \int_V d^3x\, e^{i\mathbf{k}.\mathbf{x}} = V\delta^{(3)}(\mathbf{k}). \tag{2.3}$$

It will be useful for the following discussion to use both continuum and box normalisation for the field, displayed on the left and right of (2.1),(2.2) and (2.3) respectively. The box normalisation makes the assumption of a periodic field $\phi(\mathbf{x}) = \phi(\mathbf{x} + L\mathbf{n}) \quad \forall \quad \mathbf{n} \in \mathbb{Z}^3$, where the box is defined as being of volume V and length L, it's exact position is arbitrary. The gain for making this assumption is realised in reducing the representation of the field from an uncountably to a countably infinite number of field variables i.e. discretising $\mathbf{k}-$space. We may then talk of a wavefunction instead of a wavefunctional, however one must be careful not to carry over all the standard results of quantum mechanics into $\mathbf{k}-$space unchanged. We shall discuss some of the differences as we proceed. As the goal of this project is the computational modelling of the pilot-wave theory of scalar fields, and discretisation is a necessary process when dealing with computers, we will also discuss discretising $\mathbf{x}-$space briefly. We will however remain using continuous variables for the majority of the time, as if nothing else,

it simplifies much of the notation.

We have chosen to use the complex exponential form of the transform throughout to be in keeping with standard QFT literature despite the fact that it is redundant for some of our analysis. To understand why we must consider the consequences of our reality condition. In Fourier transforming we double the degrees of freedom of the field from a continuum of real variables, $\phi(\mathbf{x})$, (one for each $\mathbf{x}$) to a continuum number of complex variables $\tilde{\phi}(\mathbf{k})$ (one for each $\mathbf{k}$). Remembering that our classical field is real then forces us to identify $\tilde{\phi}(\mathbf{k}) = \tilde{\phi}(-\mathbf{k})^* \Leftrightarrow q_{\mathbf{k}} = q^*_{-\mathbf{k}}$. Although it often doesn't make it explicit, standard QFT literature associates $\mathbb{R}^2$ with each complex plane and uses the field variables $\tilde{\phi}(\mathbf{k})$, $\tilde{\phi}(\mathbf{k})^*$ to span this space. When transformed into this basis the functional derivatives in the kinetic term of the Hamiltonian turn into Wirtinger-functional derivatives, behaving much like partial derivatives with respect to real variables. Then the reality condition, $\tilde{\phi}(\mathbf{k}) = \tilde{\phi}(-\mathbf{k})^*$ allows one to discard all the $\tilde{\phi}(\mathbf{k})^*$ giving the theory a kind of pseudo-real glaze.

Standard pilot-wave texts, e.g. Bohm et al. (1987), Holland (1993) apply the reality condition post-quantisation by restricting the theory to half of $\mathbf{k}-$space and imposing this on their field operators. One unfortunate consequence of this formulation is that expressions like $\sum_{\mathbf{k}} f(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\,|0\rangle$, or the continuous version $\int d^3k\, f(\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}\,|0\rangle$, no longer work in their usual form. They result in a theory that, though perfectly permissible, seems to emphasise the differences between itself and standard scalar field theory.

We show that by considering canonical transforms on the phase space prior to quantisation, we find a choice of $\mathbf{k}-$space representations, one of which more closely resembles the aesthetics of the Fock-representation. Unfortunately this was discovered too far into the time allowed for this dis-

sertation to make any impact on the computational work carried out. It seems however that were further simulations to be carried out on scalar field beables, this would be the representation in which to work.

The representation we work in for the simulations will be in terms of real variables. This was decided early on as in this representation the connection with Harmonic oscillators more apparent. We use the decomposition

$$\begin{aligned} \tilde{\phi}(\mathbf{k}) &= \frac{1}{\sqrt{2}}\left(\tilde{\phi}_a(\mathbf{k}) - i\tilde{\phi}_b(\mathbf{k})\right), \\ \tilde{\phi}(-\mathbf{k}) &= \frac{1}{\sqrt{2}}\left(\tilde{\phi}_a(\mathbf{k}) + i\tilde{\phi}_b(\mathbf{k})\right), \end{aligned} \tag{2.4}$$

where the fields $\tilde{\phi}_a$ and $\tilde{\phi}_b$, and their conjugate momenta are defined only over a subset of $\{\mathbf{k}\}$ denoted $\{{}^{\mathbf{k}}/_{2}\}$. We may construct this subset by removing one by one, in any order, members whose negatives remain in the set $\mathbf{k}$ while retaining the 0 element. One may however wish to define it as something like

$$\{{}^{\mathbf{k}}/_{2}\} = \{\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) | \mathbf{k}_1 > 0,\, \mathbf{k}_2 < 0\} \cup \{\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) | \mathbf{k}_1 \geq 0,\, \mathbf{k}_2 \geq 0\} \tag{2.5}$$

in order to retain sensible integrals over this set. It will be useful to refer to arbitrary members of this set as being distinct from members of $\{\mathbf{k}\}$. We use the shorthand $+\mathbf{k} \in \{{}^{\mathbf{k}}/_{2}\}$, $-\mathbf{k} \notin \{{}^{\mathbf{k}}/_{2}\}$. We also use the primed notation of Bohm et al. (1987) to denote integrals and sums over $\{{}^{\mathbf{k}}/_{2}\}$,

$$\int' d^3k, \qquad \sum_{\mathbf{k}}'. \tag{2.6}$$

Finally, before we proceed to the process of quantisation, we note that proofs

requiring the identity (2.3) still work, but present themselves as

$$\int' d^3k \left(e^{i\mathbf{k}.\mathbf{x}} + e^{-i\mathbf{k}.\mathbf{x}}\right) = (2\pi)^3\delta^{(3)}(\mathbf{x}). \tag{2.7}$$

2.2 x−space representation

Canonical quantisation in x−space

The classical Hamiltonian for the free massive scalar field is

$$H(t) = \int d^3x \, \frac{1}{2}\Big(\pi(\mathbf{x},t)^2 + (\nabla\phi(\mathbf{x},t))^2 + m^2\phi(\mathbf{x},t)^2\Big). \tag{2.8}$$

To quantise this we turn our fields into the operators with the standard equal time commutation relations

$$\begin{aligned} \Big[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')\Big] &= i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \\ [\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{x}')] &= 0, \\ [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] &= 0, \end{aligned} \tag{2.9}$$

The lack of time-dependence of the field operators should alert the reader to the fact that we have chosen to keep in the Schrödinger picture in addition to the Schrödinger representation. We take our first step into the Schrödinger representation with a step analogous to that of standard quantum mechanics. We choose the field operators

$$\hat{\phi}(\mathbf{x}) = \phi(\mathbf{x}), \qquad \hat{\pi}(\mathbf{x}) = -i\frac{\delta}{\delta\phi(\mathbf{x})}, \tag{2.10}$$

which trivially satisfy the commutation relations. This may be done with any properly defined canonical variables. We recommend Struyve (2009) and Gitman & Tyutin (1990) for in depth discussion. These operators then

act upon a *wavefunctional*, $\psi[\phi;t)$, which satisfies the functional Schrödinger equation,

$$i\frac{\partial\psi[\phi;t)}{\partial t} = \hat{H}\psi[\phi;t) = \int d^3x\,\frac{1}{2}\left[-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + (\nabla\phi(\mathbf{x}))^2 + m^2\phi(\mathbf{x})^2\right]\psi[\phi;t). \tag{2.11}$$

Where the probability of finding the system in the state ϕ at time t is $|\psi[\phi;t)|^2$. By assuming a polar form of the wavefunctional, $\psi[\phi;t) = |\psi[\phi;t)|e^{iS[\phi;t)}$ and separating the real and imaginary parts of (2.11) we find

$$\frac{\partial S}{\partial t} + \frac{1}{2}\int d^3x\,\left[\left(\frac{\delta S}{\delta\phi(\mathbf{x})}\right)^2 + (\nabla\phi(\mathbf{x}))^2 - \frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta\phi(\mathbf{x})^2}\right] = 0, \tag{2.12}$$

$$\frac{\partial|\psi|^2}{\partial t} + \int d^3x\,\frac{\delta}{\delta\phi(\mathbf{x})}\left(|\psi|^2\frac{\delta S}{\delta\phi(\mathbf{x})}\right) = 0, \tag{2.13}$$

the quantum Hamilton-Jacobi and continuity equations respectively. The continuity equation leads us to conclude the guidance equation

$$\frac{\partial\Phi(\mathbf{x},t)}{\partial t} = \left.\frac{\delta S[\phi;t)}{\delta\phi(\mathbf{x})}\right|_{\phi=\Phi}, \tag{2.14}$$

for a field configuration $\Phi(\mathbf{x},t)$.

Discretising $\mathbf{x}-$space

We continue the discussion temporarily in one spatial dimension for ease of representation. Since the functional derivatives in (2.11) and (2.14) turn to partial derivatives under discretisation of $\mathbf{x}-$space,

$$\begin{aligned}\{\mathbf{x}\} &\to \{\mathbf{x}|\mathbf{x} = \epsilon\mathbf{n}\,\forall\,\mathbf{n}\in\mathbb{Z}\},\\ \phi(\mathbf{x}) &\to \phi_{\mathbf{x}},\\ \frac{\delta}{\delta\phi(\mathbf{x})} &\to \frac{\partial}{\partial\phi_{\mathbf{x}}},\end{aligned} \tag{2.15}$$

and we will end up discretising $\mathbf{k}-$space it is prudent to discuss this option. When we discretion $\mathbf{x}-$space the Hamiltonian reduces to a sum over harmonic oscillator Hamiltonians, each with an interaction term between adjacent oscillators corresponding to the $(\nabla\phi(x))^2$ in (2.11). This may be brought into an easier form to deal with via some guise of the finite difference derivative. If we choose a second order centre difference, e.g.

$$\left.\frac{\partial\phi}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}'} = \frac{\phi_{\mathbf{x}'+\epsilon} - \phi_{\mathbf{x}'-\epsilon}}{2\epsilon} + \mathcal{O}(\epsilon^2), \tag{2.16}$$

we may remove some of the dependence of the $(\nabla\phi(x))^2$ term from the spatial coordinate of other oscillators.

$$\begin{aligned}\hat{H} &= \frac{1}{2\epsilon}\sum_{\mathbf{x}}\left[-\frac{\partial^2}{\partial\phi_{\mathbf{x}}^2} + \left(\frac{\phi_{\mathbf{x}+\epsilon} - \phi_{\mathbf{x}-\epsilon}}{2\epsilon}\right)^2 + m^2\phi_{\mathbf{x}}^2\right], \\ &= \frac{1}{2\epsilon}\sum_{\mathbf{x}}\left[-\frac{\partial^2}{\partial\phi_{\mathbf{x}}^2} + \left(m^2 + \frac{1}{2\epsilon^2}\right)\phi_{\mathbf{x}}^2 - \frac{\phi_{\mathbf{x}+\epsilon}\phi_{\mathbf{x}-\epsilon}}{2\epsilon^2}\right].\end{aligned} \tag{2.17}$$

Unfortunately, as one might imagine, it is impossible to separate the cross-terms entirely.

Most of this work will focus on the development of methods in which to evolve beables corresponding to each Fourier mode in the expansions (2.1). Instead it may be easier for certain problems to use the $\mathbf{k}-$space solutions we will develop in subsequent sections to evolve the system directly in real space. For instance, since the free Hamiltonian Fourier transforms into a pure sum of HO Hamiltonians, we find that we are often left with wavefunctionals of the form $f(\phi(\mathbf{x}),\dots)\psi^0[\phi,t)$. The groundstate in this situation, for reasons we will elaborate on later doesn't contribute to the

motion of the field, and as such our guidance equations reduce to

$$\frac{\partial \Phi_{\mathbf{x}_i}}{\partial t} = Im\left(\frac{1}{f(\phi_{\mathbf{x}_i},\dots)}\frac{\partial f(\phi_{\mathbf{x}_1},\dots)}{\partial \phi_{\mathbf{x}_1}}\right)\Bigg|_{\phi_{\mathbf{x}_1}=\Phi_{\mathbf{x}_1}(t),\dots}. \tag{2.18}$$

Providing we have a $\mathbf{x}-$space expression for $f(\phi(x),\dots)$, it would then be possible to evolve the system directly in $\mathbf{x}-$space.

In standard quantum field theory one interprets the field operator $\hat{\phi}(\mathbf{x})$ as creating a particle at position $\mathbf{x}$. In the Schrödinger representation $\hat{\phi}(\mathbf{x})$ acts by simple multiplication and as such, by comparing this action with (2.18), we see that $\phi(\mathbf{k})$ produces a stationary field.

Due to the difficulty in Fourier transforming interaction terms, it seems that this may be the only way to proceed in an interacting theory.

2.3 k−space representations

Motivation

To motivate our approach to the quantisation we first follow the approach of Holland (1993), namely by attempting to Fourier transform the quantised theory before applying the reality condition. The first step in this program would be to transform the Hamiltonian

$$\hat{H} = \int d^3x\, \frac{1}{2}\left[-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + (\nabla\phi(\mathbf{x}))^2 + m^2\phi(\mathbf{x})^2\right]. \tag{2.19}$$

Although the second and third terms in the integrand transform easily, it is not immediately clear how the functional derivatives, corresponding to the conjugate momenta will transform. We would like to represent the conjugate momentum densities in a form resembling the *total*-functional

derivatives[†] of (2.10). We now have a complex field however, and we have no reason to expect that total-functional-complex derivatives with respect to the $\mathbf{k}-$space wavefunctional exist. In fact whether they exist or not will depend upon our choice of application of the reality condition. Through judicious application of this condition we will be able to ensure that our $\mathbf{k}-$space wavefunctional does indeed satisfy the Cauchy-Riemann equations. Before we go on however it is necessary to introduce some new notation.

Functional-Wirtinger derivatives

As we will be extending our discussion to functionals of a complex function, which may or may not be holomorphic, we must consider the derivatives we are using carefully. In treating normal functions f of a complex variable, $z = z_r + iz_i$, it is often useful to work solely in terms of the complex variables z and z^*. When doing this we may express partial derivatives with respect to z and z^* in terms of the partial derivatives with respect to the real and imaginary parts,

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial z_r} - i\frac{\partial f}{\partial z_i}\right), \\ \frac{\partial f}{\partial z^*} &= \frac{1}{2}\left(\frac{\partial f}{\partial z_r} + i\frac{\partial f}{\partial z_i}\right). \end{aligned} \tag{2.20}$$

These are called Wirtinger derivatives. When using the variables z and z^* one sometimes finds a volume element on the complex plane written

$$dz_r dz_i = \frac{1}{2} dz dz^*, \tag{2.21}$$

which may be motivated by evaluating the Jacobian determinant. Throughout, we'll use volume elements expressed in terms of the canonical variables

[†]we use the word *total*' in the sense that if $\{\mathbf{k}\}$ was a singleton and, there was no time dependence, then the derivative would be a total derivative

of the representation being used. These will be generalisations of (2.21), though if one feels uncomfortable using such expressions then one may always convert back into real coordinates.

We will require to use Wirtinger-functional derivatives, and so in order to distinguish between these and ordinary functional derivatives we will use the slashed notation $\not{\delta}$, keeping the regular δ for total functional derivatives. And so the $\mathbf{k}-$space Wirtinger-functional derivatives with respect to the wavefunctional are

$$\frac{\not{\delta}\psi}{\not{\delta}\tilde{\phi}(\mathbf{k})}, \quad \frac{\not{\delta}\psi}{\not{\delta}\tilde{\phi}(\mathbf{k})^*}. \tag{2.22}$$

We will however drop this notation after this section when there will be significantly less chance of confusion. Until then we emphasise that

$$\frac{\not{\delta}}{\not{\delta}\tilde{\phi}(\mathbf{k})} \neq \frac{\delta}{\delta\tilde{\phi}(\mathbf{k})}. \tag{2.23}$$

The choice of representation

Continuing, we note that since we have not yet applied our reality condition, we may consider for the moment the possibility that our $\mathbf{x}-$space field, $\phi(\mathbf{x})$ is complex. Then the wavefunctional must becomes $\psi = \psi[\phi_r, \phi_i] = \psi[\phi, \phi^*]$ and we can write the kinetic term in our Schrödinger equation (2.11) as

$$-\frac{1}{2}\int d^3x \, \frac{\not{\delta}^2\psi}{\not{\delta}\phi_r(\mathbf{x})^2}. \tag{2.24}$$

It is easy to see then that when we apply the reality condition, and $\psi = \psi[\phi_r, \phi_i] \to \psi[\phi_r]$, that this term reverts back to the expression of (2.11). Switching to our complex fields the term becomes

$$-\frac{1}{2}\int d^3x \left(\frac{\not{\delta}^2}{\not{\delta}\phi(\mathbf{x})^2} + \frac{\not{\delta}^2}{\not{\delta}\phi(\mathbf{x})\not{\delta}\phi(\mathbf{x})^*} + \frac{\not{\delta}^2}{\not{\delta}\phi(\mathbf{x})^{*2}} \right) \tag{2.25}$$

Here we may perform a little trick. Generally the chain rule does not exist to the functional derivative as it's domain and range exist within different spaces. Noting however that we may consider the wavefunction as a functional of either ϕ, ϕ^* or $\tilde{\phi}, \tilde{\phi}^*$ and each couple of fields as functionals of each other e.g.

$$\tilde{\phi}[\phi; \mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{k}.\mathbf{x}} \phi(\mathbf{x}), \tag{2.26}$$

we may construct the chain rule

$$\frac{\not{\delta}}{\not{\delta}\phi(\mathbf{x})} = \int d^3k \left(\frac{\not{\delta}\tilde{\phi}(\mathbf{k})}{\not{\delta}\phi(\mathbf{x})} \frac{\not{\delta}}{\not{\delta}\tilde{\phi}(\mathbf{k})} + \frac{\not{\delta}\tilde{\phi}(\mathbf{k})^*}{\not{\delta}\phi(\mathbf{x})} \frac{\not{\delta}}{\not{\delta}\tilde{\phi}(\mathbf{k})^*} \right). \tag{2.27}$$

From here it is simple to show that our Schrödinger kinetic term becomes

$$-\frac{1}{2} \int d^3k \left[\frac{\not{\delta}^2}{\not{\delta}\tilde{\phi}(\mathbf{k})\not{\delta}\tilde{\phi}(-\mathbf{k})} + 2\frac{\not{\delta}^2}{\not{\delta}\tilde{\phi}(\mathbf{k})\not{\delta}\tilde{\phi}(\mathbf{k})^*} + \frac{\not{\delta}^2}{\not{\delta}\tilde{\phi}(\mathbf{k})^*\not{\delta}\tilde{\phi}(-\mathbf{k})^*} \right] \psi[\tilde{\phi}, \tilde{\phi}^*]. \tag{2.28}$$

It is now that we choose to impose our reality condition[†], finding as we do so that we may make all but one of these terms disappear. To illustrate, if we specify that our wavefunction is a function of $\tilde{\phi}(\mathbf{k})^*$ only, then only the third term remains. If we choose to keep ψ as a function of both $\tilde{\phi}$ and $\tilde{\phi}^*$ but to reduce the set over which they are defined to $\{\mathbf{k}/2\}$, as we shall when using real fields, then only the second term remains, and the range of the integral halves. This is route taken implicitly by Holland (1993). In the next section we will promote the choice to keep the first term only. To do this we specify that $\psi = \psi[\tilde{\phi}]$, and we find the total Schrödinger equation

[†]Reminding ourselves again of what was discussed in section (2.1), i.e. that the wavefuctional cannot be a function of both $\tilde{\phi}(\mathbf{k})$ and $\tilde{\phi}(-\mathbf{k})^*$ as this would entail a probability distribution that would allow $\tilde{\phi}(\mathbf{k}) \neq \tilde{\phi}(-\mathbf{k})^*$.

to be

$$-i\frac{\partial\psi[\tilde{\phi},t)}{\partial t}=\frac{1}{2}\int d^3k\left[-\frac{\not{\delta}^2}{\not{\delta}\tilde{\phi}(\mathbf{k})\not{\delta}\tilde{\phi}(-\mathbf{k})}+\left(k^2+m^2\right)\tilde{\phi}(\mathbf{k})\tilde{\phi}(-\mathbf{k})\right]\psi[\tilde{\phi};t), \tag{2.29}$$

where we have remembered that ψ does indeed have time dependence. The other terms in the Hamiltonian, with a similar choice of forms, arise using the same method. We note here that since for a standard function of a complex variable the Cauchy Riemann equations may be written $\frac{\partial f}{\partial z^*}=0$. Our choice of application of the reality condition is the only one that ensures that $\frac{\not{\delta}\psi}{\not{\delta}\tilde{\phi}(\mathbf{k})^*}=0$, meaning a wavefunctional holomorphic in the function space over which it is defined.

On the choice of canonical variables

Although we've shown that the reality condition allows us some freedom as to the choice of the representation, we have not as of yet proved that the choices outlined above are legitimate representations of the theory. To prove this we prefer a different program of derivation to that described above. We prefer to start with the classical theory, then to make a canonical transformation into the variables we desire before quantising the theory. In addition to justifying the choice of variables, this method also allows us to make sense of how the volume element transforms.

Let us first define the properties of the canonical transform relevant to our discussion. We base our conventions and definitions upon those described in Goldstein et al. (2002). Let us consider for simplicity a Hamiltonian theory with N canonical variables q_i and their respective conjugate momenta p_i.

We wish to treat *restricted* canonical transformations of the form

$$\begin{aligned} q_i \to Q_i = Q_i(q,p)i \\ p_i \to P_i = P_i(q,p). \end{aligned} \tag{2.30}$$

In such transformations the Hamiltonian does not change form, and may simply be expressed in the new coordinates. Expressing the coordinates in their symplectic form,

$$\boldsymbol{\eta} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \boldsymbol{\zeta} = \begin{pmatrix} Q \\ P \end{pmatrix}, \tag{2.31}$$

the *symplectic condition* for a restricted canonical transformation is written

$$\mathbf{MJM}^T = \mathbf{J}, \tag{2.32}$$

where

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{2.33}$$

Here, $\mathbf{0}$ and $\mathbf{1}$ are N by N zero and identity matrices respectively. M is seen to be the Jacobian matrix of the transformation. The symplectic condition (2.32) may be written equivalently as

$$[\boldsymbol{\zeta}, \boldsymbol{\zeta}]_{\boldsymbol{\eta}} = \mathbf{J}, \tag{2.34}$$

where

$$\Big([\zeta, \zeta]_{\boldsymbol{\eta}}\Big)_{ij} = \frac{\partial \zeta_i}{\partial q_k}\frac{\partial \zeta_j}{\partial p_k} - \frac{\partial \zeta_i}{\partial p_k}\frac{\partial \zeta_j}{\partial q_k}, \tag{2.35}$$

the Poisson Bracket of ζ_i and ζ_j with respect to the the canonical variables

η, although it may be taken with respect to any set of canonical variables.

These two equivalent expressions, (2.32) and (2.34) represent a necessary and sufficient condition for the transformations (2.30) to be canonical, and as such provide us with a method of testing the legitimacy of our different Fourier representations. In the transition from the discrete to the continuous case the indices on our matrix expressions become continuous, however we will retain these expressions as we believe them to be instructive. In this case $1_{kk'} \to \delta(k-k')$, and so we choose to represent the continuous version of $\mathbf{J}$

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\delta} \\ -\boldsymbol{\delta} & \mathbf{0} \end{pmatrix}. \tag{2.36}$$

$\tilde{\phi}(\mathbf{k})$ representation

The standard $\mathbf{x}-$space Lagrangian is

$$L[\phi, \dot{\phi}] = \frac{1}{2}\int d^3x \, \left[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right]. \tag{2.37}$$

When we express this in terms of $\tilde{\phi}(\mathbf{k})$ with the transformation (2.2), it becomes

$$L[\tilde{\phi}, \dot{\tilde{\phi}}] = \frac{1}{2}\int d^3k \, \left[\dot{\tilde{\phi}}(\mathbf{k})\dot{\tilde{\phi}}(-\mathbf{k}) - (k^2+m^2)\tilde{\phi}(\mathbf{k})\tilde{\phi}(-\mathbf{k})\right]. \tag{2.38}$$

Dropping the slashed notation, the momentum field conjugate to $\tilde{\phi}$ is then

$$\pi_{\tilde{\phi}(\mathbf{k})} = \frac{\delta L}{\delta\dot{\tilde{\phi}}(\mathbf{k})} = \dot{\tilde{\phi}}(-\mathbf{k}), \tag{2.39}$$

and we may write the Hamiltonian

$$H[\tilde{\phi}, \pi_{\tilde{\phi}}] = \int d^3k\, \dot{\tilde{\phi}}(\mathbf{k})\pi_{\tilde{\phi}(\mathbf{k})} - L$$
$$= \frac{1}{2}\int d^3k \left[\pi_{\tilde{\phi}(\mathbf{k})}\pi_{\tilde{\phi}(-\mathbf{k})} + (k^2+m^2)\tilde{\phi}(\mathbf{k})\tilde{\phi}(-\mathbf{k})\right]. \tag{2.40}$$

The Poisson bracket

$$[\tilde{\phi}(\mathbf{k}), \pi_{\tilde{\phi}(\mathbf{k}')}]_{\phi\pi} = \int d^3x \left(\frac{\delta\tilde{\phi}(\mathbf{k})}{\delta\phi(\mathbf{x})}\frac{\delta\pi_{\tilde{\phi}(\mathbf{k}')}}{\delta\pi(\mathbf{x})} - \frac{\delta\tilde{\phi}(\mathbf{k})}{\delta\pi(\mathbf{x})}\frac{\delta\pi_{\tilde{\phi}(\mathbf{k}')}}{\delta\phi(\mathbf{x})}\right)$$
$$= \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{2.41}$$

may then be evaluated by noting that

$$\pi_{\tilde{\phi}(\mathbf{k})} = \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\mathbf{k}.\mathbf{x}}\pi(\mathbf{x}) \tag{2.42}$$

follows from (2.39). Writing $\boldsymbol{\zeta} = (\tilde{\boldsymbol{\phi}}, \boldsymbol{\pi}_{\tilde{\phi}})$ and $\boldsymbol{\eta} = (\boldsymbol{\phi}, \boldsymbol{\pi})$, we find

$$[\boldsymbol{\zeta}, \boldsymbol{\zeta}]_{\boldsymbol{\eta}} = \begin{pmatrix} [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}}, \boldsymbol{\pi}_{\tilde{\phi}}]_{\boldsymbol{\eta}} \\ [\boldsymbol{\pi}_{\tilde{\phi}}, \tilde{\boldsymbol{\phi}}]_{\boldsymbol{\eta}} & [\boldsymbol{\pi}_{\tilde{\phi}}, \boldsymbol{\pi}_{\tilde{\phi}}]_{\boldsymbol{\eta}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\delta} \\ -\boldsymbol{\delta} & \mathbf{0} \end{pmatrix}. \tag{2.43}$$

To quantise we associate the Poisson brackets equal to delta functions with commutators of the operators equal to imaginary delta functions. The only non-vanishing relations are

$$[\hat{\tilde{\phi}}(\mathbf{k}), \hat{\pi}_{\phi(\mathbf{k}')}] = i\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{2.44}$$

In the Schrödinger representation dictates field operators of the form

$$\hat{\tilde{\phi}}(\mathbf{k}) = \tilde{\phi}(\mathbf{k}), \qquad \hat{\pi}_{\phi(\mathbf{k}')} = -i\frac{\delta}{\delta\tilde{\phi}(\mathbf{k})}. \tag{2.45}$$

Our Hamiltonian operator is then

$$\hat{H} = \frac{1}{2}\int d^3k \left[\frac{-\delta^2}{\delta\tilde{\phi}(\mathbf{k})\delta\tilde{\phi}(-\mathbf{k})} + (k^2+m^2)\tilde{\phi}(\mathbf{k})\tilde{\phi}(-\mathbf{k})\right], \tag{2.46}$$

and we have a wavefunctional $\psi[\tilde{\phi}]$ that satisfies the Schrödinger equation $i\frac{\partial\psi}{\partial t} = \hat{H}\psi$. The groundstate solution is

$$\psi_0[\tilde{\psi}] = Ne^{-\frac{1}{2}\int d^3k\ E_{\mathbf{k}}\tilde{\phi}(\mathbf{k})\tilde{\phi}(-\mathbf{k})}, \tag{2.47}$$

from which it is simple to show that the groundstate energy is familiarly divergent; $E_0 = {}^1\!/_2 \int d^3k \frac{\delta\tilde{\phi}(\mathbf{k})}{\delta\tilde{\phi}(\mathbf{k})}$. The constant of proportionality N will be left untreated until we have have considered the volume element and we may apply a probabilistic interpretation to the $\mathbf{k}-$space wavefunctional.

The creation and annihilation operators in this representation may be derived as usual by finding an operator $\hat{X}$ such that $[\hat{H},\hat{X}] = c\hat{X}$. Since it may be shown that

$$[\hat{H},\hat{\tilde{\phi}}(\mathbf{k})] = -i\hat{\pi}_{\tilde{\phi}(-\mathbf{k})}, \tag{2.48}$$

$$[\hat{H},\hat{\pi}_{\tilde{\phi}(\mathbf{k})}] = i(k^2+m^2)\hat{\tilde{\phi}}(-\mathbf{k}) \tag{2.49}$$

We may construct an operator, $a\hat{\tilde{\phi}}(\mathbf{k}) + b\hat{\pi}_{\tilde{\phi}(-\mathbf{k})}$, that satisfies the correct commutation relations with $c = E_{\mathbf{k}}$. As such we conclude that these operators are the creation and annihilation operators, and that the energy of the modes of the field ϕ are now discrete quantum numbers. The values of a and b are determined by considering normalisation; something we will not be ready to deal with until we have considered the transformation of the volume element.

Once the volume element is treated the final form of the operators is

found to be

$$a(\mathbf{k}) = \frac{1}{\sqrt{2E_{\mathbf{k}}}}\left(E_{\mathbf{k}}\tilde{\phi}(\mathbf{k}) + \frac{\delta}{\delta\tilde{\phi}(-\mathbf{k})}\right), \tag{2.50}$$

$$a(\mathbf{k})^{\dagger} = \frac{1}{\sqrt{2E_{\mathbf{k}}}}\left(E_{\mathbf{k}}\tilde{\phi}(-\mathbf{k}) - \frac{\delta}{\delta\tilde{\phi}(\mathbf{k})}\right), \tag{2.51}$$

so that upon rearranging and Fourier transforming we find that the $\mathbf{x}-$space field operators take their usual expansions,

$$\begin{aligned}\hat{\phi}(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_{\mathbf{k}}}}(\hat{a}(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}} + \hat{a}^{\dagger}(\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}}),\\ \hat{\pi}(\mathbf{x}) &= -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{E_{\mathbf{k}}}{2}}(\hat{a}(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}} - \hat{a}^{\dagger}(\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}}).\end{aligned} \tag{2.52}$$

$\tilde{\phi}(+\mathbf{k}), \tilde{\phi}(+\mathbf{k})^*$ representation

The Representation of Bohm et al. (1987) uses the variables $q_{\mathbf{k}}$ and $q_{\mathbf{k}}^*$ defined over a discretised $\{\mathbf{k}/2\}$. Here we present a continuum version of this theory using the same program of derivation as with the $\tilde{\phi}(\mathbf{k})$ representation.

If we are to retain the fields $\tilde{\phi}(+\mathbf{k}), \tilde{\phi}(+\mathbf{k})^*$ in the classical theory then we must choose modified forms of (2.1) and (2.2). These forms are

$$\tilde{\phi}(\mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}}e^{-i\mathbf{k}.\mathbf{x}}\phi(\mathbf{x}), \quad \tilde{\phi}(\mathbf{k})^* = \int \frac{d^3x}{(2\pi)^{3/2}}e^{i\mathbf{k}.\mathbf{x}}\phi(\mathbf{x}), \tag{2.53}$$

and their inverse

$$\phi(\mathbf{x}) = \int' \frac{d^3k}{(2\pi)^{3/2}}\left(e^{i\mathbf{k}.\mathbf{x}}\tilde{\phi}(\mathbf{k}) + e^{-i\mathbf{k}.\mathbf{x}}\tilde{\phi}(\mathbf{k})^*\right). \tag{2.54}$$

These may be derived easily by reducing $\{\mathbf{k}\}$ to $\{\mathbf{k}/2\}$ in (2.1) and (2.2), using the reality condition $\tilde{\phi}(-\mathbf{k}) = \tilde{\phi}(\mathbf{k})^*$. We emphasise that the fields are only defined over $\{\mathbf{k}/2\}$; one may not now talk of $\tilde{\phi}(-\mathbf{k})$. Using these

fields the Lagrangian (2.37) may be written

$$L[\tilde{\phi},\tilde{\phi}^*] = \int' d^3\mathbf{k}\left[\dot{\tilde{\phi}}\dot{\tilde{\phi}}^* - (k^2+m^2)\tilde{\phi}\tilde{\phi}^*\right], \tag{2.55}$$

meaning the conjugate momenta fields are

$$\pi_{\tilde{\phi}(\mathbf{k})} = \frac{\delta L}{\delta\dot{\tilde{\phi}}(\mathbf{k})} = \dot{\tilde{\phi}}(\mathbf{k})^*, \quad \pi_{\tilde{\phi}(\mathbf{k})^*} = \frac{\delta L}{\delta\dot{\tilde{\phi}}(\mathbf{k})^*} = \dot{\tilde{\phi}}(\mathbf{k}). \tag{2.56}$$

Adopting symplectic notation,

$$\boldsymbol{\eta} = \begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\pi}\end{pmatrix}, \quad \boldsymbol{\zeta} = \begin{pmatrix}\tilde{\boldsymbol{\phi}}\\ \tilde{\boldsymbol{\phi}}^*\\ \boldsymbol{\pi}_{\tilde{\phi}}\\ \boldsymbol{\pi}_{\tilde{\phi}^*}\end{pmatrix}, \tag{2.57}$$

The symplectic condition for a canonical transformation may be written

$$[\boldsymbol{\zeta},\boldsymbol{\zeta}]_{\boldsymbol{\eta}} = \begin{pmatrix} [\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\phi}}^*]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}},\boldsymbol{\pi}_{\tilde{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}},\boldsymbol{\pi}_{\tilde{\phi}^*}]_{\boldsymbol{\eta}} \\ [\tilde{\boldsymbol{\phi}}^*,\tilde{\boldsymbol{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}}^*,\tilde{\boldsymbol{\phi}}^*]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}}^*,\boldsymbol{\pi}_{\tilde{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\phi}}^*,\boldsymbol{\pi}_{\tilde{\phi}^*}]_{\boldsymbol{\eta}} \\ [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}},\tilde{\boldsymbol{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}},\tilde{\boldsymbol{\phi}}^*]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}},\boldsymbol{\pi}_{\tilde{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}},\boldsymbol{\pi}_{\tilde{\phi}^*}]_{\boldsymbol{\eta}} \\ [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}^*},\tilde{\boldsymbol{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}^*},\tilde{\boldsymbol{\phi}}^*]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}^*},\boldsymbol{\pi}_{\tilde{\phi}}]_{\boldsymbol{\eta}} & [\tilde{\boldsymbol{\pi}}_{\tilde{\phi}^*},\boldsymbol{\pi}_{\tilde{\phi}^*}]_{\boldsymbol{\eta}} \end{pmatrix} = \mathbf{J} \tag{2.58}$$

The non-zero Poisson brackets in this matrix are those in the top-right and bottom-left quadrants. One finds, for instance, that

$$[\tilde{\phi}(\mathbf{k}),\pi_{\tilde{\phi}(\mathbf{k}')}]_{\boldsymbol{\eta}} = \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{2.59}$$

$$[\tilde{\phi}(\mathbf{k}),\pi_{\tilde{\phi}(\mathbf{k}')^*}]_{\boldsymbol{\eta}} = \delta^{(3)}(\mathbf{k}+\mathbf{k}'). \tag{2.60}$$

Noting that in this representation $\delta^{(3)}(\mathbf{k}+\mathbf{k}') = 0$ as $\mathbf{k},\mathbf{k}' \in \{\mathbf{k}/2\}$, equation

(2.58) becomes

$$[\boldsymbol{\zeta},\boldsymbol{\zeta}]_{\boldsymbol{\eta}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{\delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\delta} \\ -\boldsymbol{\delta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\delta} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{2.61}$$

The Hamiltonian in the newly discovered canonical coordinates is

$$H[\pi_{\tilde{\phi}}, \pi_{\tilde{\phi}^*}, \tilde{\phi}, \tilde{\phi}^*] = \int' d^3k \left[\pi_{\tilde{\phi}}\pi_{\tilde{\phi}^*} + (k^2+m^2)\tilde{\phi}\tilde{\phi}^*\right]. \tag{2.62}$$

To quantise we turn the canonical coordinates into operators, which in the Schrödinger representation take the form

$$\begin{gathered} \tilde{\phi}(\mathbf{k}) \to \hat{\tilde{\phi}}(\mathbf{k}) = \tilde{\phi}(\mathbf{k}), \qquad \tilde{\phi}(\mathbf{k})^* \to \hat{\tilde{\phi}}(\mathbf{k})^* = \tilde{\phi}(\mathbf{k})^* \\ \pi_{\tilde{\phi}(\mathbf{k})} \to \hat{\pi}_{\tilde{\phi}(\mathbf{k})} = -i\tfrac{\delta}{\delta\tilde{\phi}(\mathbf{k})}, \quad \pi_{\tilde{\phi}(\mathbf{k})^*} \to \hat{\pi}_{\tilde{\phi}(\mathbf{k})^*} = -i\tfrac{\delta}{\delta\tilde{\phi}(\mathbf{k})^*}. \end{gathered} \tag{2.63}$$

The resultant Hamiltonian operator is

$$\hat{H} = \int' d^3k \left[-\frac{\delta^2}{\delta\tilde{\phi}\delta\tilde{\phi}^*} + (k^2+m^2)\tilde{\phi}\tilde{\phi}^*\right]. \tag{2.64}$$

It acts upon a wavefunctional $\psi[\tilde{\phi}, \tilde{\phi}^*; t)$ to form a functional Schrödinger equation $i\frac{\partial\psi}{\partial t} = \hat{H}\psi$. The groundstate wavefunctional is

$$\psi_0[\tilde{\psi}, \tilde{\psi}^*] = N_2 e^{-\int' d^3k \; E_{\mathbf{k}}\tilde{\phi}\tilde{\phi}^*}. \tag{2.65}$$

The $\tilde{\phi}, \tilde{\phi}^*$ is found to be very similar to the $\tilde{\phi}$ representation of the previous section. It does however have the unfortunate consequence of cluttering the theory. One must define all $\mathbf{k}-$space operators in their standard and

complex conjugate form, and take a sum of their Fourier transforms to retrieve the $\mathbf{x}-$space equivalents.

Real representation

The majority of the computational work of this project was carried out in the real representation before it was realised that said computational work would have been a great deal less tiresome in the $\tilde{\phi}$ representation. Although the real representation has some advantages, it suffers from the same problem as the $\tilde{\phi}, \tilde{\phi}^*$, namely the need to repeat many calculations.

The real representation's main advantage is in the fact that it is completely separable into a product of modal Harmonic oscillators. This allows one to construct a Hilbert space as a product of spaces isomorphic to those of harmonic oscillators. If one then uses the equivalent of a quantum mechanical position basis $|\mathbf{x}\rangle$ for each of these subspaces, we may use the position space representation for the HO raising and lowering operators as creation and annihilation operators.

In addition to this, if one chooses the form of their canonical transforms carefully, one may use them to translate a usable volume element into $\mathbf{k}-$space, and the interpretation of the wavefunctional becomes simply that of an infinite dimensional HO. We shall discuss this further in the following section, and for the moment introduce our canonical transformations,

$$\begin{aligned} \phi_a(\mathbf{k}) &= \int \tfrac{d^3x}{(2\pi)^{3/2}}\sqrt{2}cos(\mathbf{k}.\mathbf{x})\phi, \quad \phi_b(\mathbf{k}) = \int \tfrac{d^3x}{(2\pi)^{3/2}}\sqrt{2}sin(\mathbf{k}.\mathbf{x})\phi, \\ \pi_{\phi_a(\mathbf{k})} &= \int \tfrac{d^3x}{(2\pi)^{3/2}}\sqrt{2}cos(\mathbf{k}.\mathbf{x})\pi, \quad \pi_{\phi_b(\mathbf{k})} = \int \tfrac{d^3x}{(2\pi)^{3/2}}\sqrt{2}sin(\mathbf{k}.\mathbf{x})\pi, \end{aligned} \tag{2.66}$$

which are again defined only over $\{\mathbf{k}/2\}$, and are related to the classical $\tilde{\phi}, \tilde{\phi}^*$, representation by $\phi_a(\mathbf{k}) = \frac{1}{\sqrt{2}}(\tilde{\phi}(\mathbf{k}) + \tilde{\phi}(\mathbf{k})^*)$, $\phi_b(\mathbf{k}) = \frac{i}{\sqrt{2}}(\tilde{\phi}(\mathbf{k}) - \tilde{\phi}(\mathbf{k})^*)$.

Their inverses are

$$\phi(\mathbf{x}) = \int^{'} \frac{d^3k}{(2\pi)^{3/2}} \left(\sqrt{2}cos(\mathbf{k}.\mathbf{x})\phi_a(\mathbf{k}) + \sqrt{2}sin(\mathbf{k}.\mathbf{x})\phi_b(\mathbf{k})\right), \tag{2.67}$$

$$\pi(\mathbf{x}) = \int^{'} \frac{d^3k}{(2\pi)^{3/2}} \left(\sqrt{2}cos(\mathbf{k}.\mathbf{x})\pi_{\phi_a(\mathbf{k})} + \sqrt{2}sin(\mathbf{k}.\mathbf{x})\pi_{\phi_b(\mathbf{k})}\right). \tag{2.68}$$

As we do not wish to subject the reader to another quantisation procedure, which in any case is very similar to that of the previous two sections we introduce the Hamiltonian in it's transformed and quantised form

$$\hat{H} = \frac{1}{2}\int^{'} d^3k \left[\left(-\frac{\delta^2}{\delta\phi_a(\mathbf{k})^2} + (k^2+m^2)\phi_a(\mathbf{k})\right) + (\phi_a \leftrightarrow \phi_b)\right], \tag{2.69}$$

and hence the groundstate in this representation is a simple product of HO groundstates

$$\psi_0[\phi_a, \phi_b] = \prod_{\mathbf{k}/2} \psi_0(\phi_a(\mathbf{k}))\psi_0(\phi_b(\mathbf{k})), \tag{2.70}$$

$$\psi_0(\phi_a(\mathbf{k})) = \left(\frac{E_\mathbf{k}}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}E_\mathbf{k}\phi_a(\mathbf{k})^2}. \tag{2.71}$$

The creation operators

$$\begin{aligned}
\hat{a}(\mathbf{k}) &= \sqrt{\frac{E_\mathbf{k}}{2}}\left(\hat{\tilde{\phi}}_a(\mathbf{k}) + \frac{i}{E_\mathbf{k}}\hat{\tilde{\pi}}_a(\mathbf{k})\right), \\
&= \sqrt{\frac{E_\mathbf{k}}{2}}\left(\tilde{\phi}_a(\mathbf{k}) + \frac{1}{E_\mathbf{k}}\frac{\delta}{\delta\tilde{\phi}_a(\mathbf{k})}\right), \\
\hat{b}(\mathbf{k}) &= \sqrt{\frac{E_\mathbf{k}}{2}}\left(\hat{\tilde{\phi}}_b(\mathbf{k}) + \frac{i}{E_\mathbf{k}}\hat{\tilde{\pi}}_b(\mathbf{k})\right);
\end{aligned} \tag{2.72}$$

the annihilation operators, naturally their adjoint. They satisfy the com-

mutators,

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \quad [\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{2.73}$$

with all others vanishing. The field operators are found to be

$$\hat{\phi}(\mathbf{x}) = \int' \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{E_\mathbf{k}}} \Big(cos(\mathbf{k}.\mathbf{x})(\hat{a}(\mathbf{k}) + \hat{a}^\dagger(\mathbf{k})) + sin(\mathbf{k}.\mathbf{x})(\hat{b}(\mathbf{k}) + \hat{b}^\dagger(\mathbf{k})) \Big) \tag{2.74}$$

$$\hat{\pi}(\mathbf{x}) = \int' \frac{d^3k}{(2\pi)^{3/2}} (-i)\sqrt{E_\mathbf{k}} \Big(cos(\mathbf{k}.\mathbf{x})(\hat{a}(\mathbf{k}) - \hat{a}^\dagger(\mathbf{k})) - sin(\mathbf{k}.\mathbf{x})(\hat{b}(\mathbf{k}) + \hat{b}^\dagger(\mathbf{k})) \Big) \tag{2.75}$$

An argument for the volume measure

We have thus far interpreted the wavefunctional $\psi[\phi; t)$ in line with Bohm's original formulation, as defining some sort of probability function. In order for one to interpret this concretely as a probability density we need a notion of volume within our space of functions. We achieve this by using a notional functional measure $\mathcal{D}\phi$, which we assume to be a continuum generalisation of the Lebesgue measure. One may naively represent this as $\prod_{\{\mathbf{x}\}} d\phi(\mathbf{x})$. However, as pointed out by Struyve (2009), such measures do not exist. We do not wish to go into detail about this problem, and instead refer the reader to Struyve (2009) for further discussion.

In order for us to have a probabilistic interpretation in our $\mathbf{k}-$space representations, it is necessary for us to be able to transform the volume element. Also since guidance equations are usually induced via interpreting the quantum probability as an ensemble probability, the volume element seems vital to the construction of a pilot-wave theory. For this reason we present an argument for it's transformation.

Since the Fourier transform is linear, it might seem that we can relate $\mathcal{D}\phi$ to it's $\mathbf{k}-$space via

$$\mathcal{D}\phi = |det(M)|\, \mathcal{D}\tilde{\phi}, \tag{2.76}$$

where M is the Jacobian Matrix of the transformation. As such it seems that we must evaluate a functional determinant. This is a highly non-trivial expression. We may get a taste of what evaluating this expression may involve through looking at a discrete case. As the Jacobian matrix must be square for us to take it's determinant, the only sensible scheme of discretisation would be one in which we discretise the sets $\{\mathbf{x}\}$ and $\{\mathbf{k}\}$ to have the same cardinality N. In this case the transforms may be represented as

$$\phi_{\mathbf{x}_i} = \frac{1}{\sqrt{N}} \sum_{\mathbf{j}} e^{i\mathbf{k}_j.\mathbf{x}_i} \tilde{\phi}_{\mathbf{k}_j}, \quad \tilde{\phi}_{\mathbf{k}_j} = \frac{1}{\sqrt{N}} \sum_{\mathbf{i}} e^{i\mathbf{k}_j.\mathbf{x}_i} \phi_{\mathbf{x}_i}. \tag{2.77}$$

Using the Einstein summation convention, the Jacobian would then be

$$\begin{aligned} J &= \epsilon^{i_1 \dots i_N} \frac{\partial \tilde{\phi}_{\mathbf{k}_{i_1}}}{\partial \phi_{\mathbf{x}_1}} \dots \frac{\partial \tilde{\phi}_{\mathbf{k}_{i_N}}}{\partial \phi_{\mathbf{x}_N}}, \\ &= \sqrt{N} \epsilon^{i_1 \dots i_N} e^{i(\mathbf{k}_{i_1}.\mathbf{x}_1 + \dots + \mathbf{k}_{i_N}.\mathbf{x}_N)}, \end{aligned} \tag{2.78}$$

where $\epsilon^{i_1 \dots i_N}$ is the totally antisymmetric Levi-Civita tensor. This clearly becomes ill defined in the continuous case. The problem becomes more unsettling if we remind ourselves that guidance equations are generally derived such that an ensemble of systems will statistically reproduce the quantum probability distribution.

Fortunately we may make an argument for the existence of a $\mathbf{k}-$space volume element, with the assumption of the $\mathbf{x}-$space element, by the use of one of Poisson's integral invariants. If one considers a discrete canonical transformation $\boldsymbol{\eta} \to \boldsymbol{\zeta}$, one may take the determinant of the symplectic

condition (2.32), to find that the Jacobian matrix for the transformation satisfies $det(\mathbf{M})^2 = 1$. As one takes the absolute value of $det(\mathbf{M})$ in transforming the phase space volume element,

$$d\boldsymbol{\eta} = |det(\mathbf{M})| d\boldsymbol{\zeta}, \tag{2.79}$$

one may conclude that for a canonical transformation the phase space volume element is invariant. This is the last of Poisson's integral invariants.

The transforms (2.67) and (2.68) have some useful properties. Firstly they are real and canonical, implying $det(\mathbf{M}) = \pm 1$. Secondly the transformations do not mix the fields and their canonical conjugates, and as such the matrix M takes block diagonal form. The determinant reduces to a product of determinants,

$$det(\mathbf{M}) = det \begin{pmatrix} \frac{\delta \boldsymbol{\phi}}{\delta \mathbf{a}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\delta \boldsymbol{\phi}}{\delta \mathbf{b}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\delta \boldsymbol{\pi}}{\delta \boldsymbol{\pi}_\mathbf{a}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\delta \boldsymbol{\pi}}{\delta \boldsymbol{\pi}_\mathbf{b}} \end{pmatrix} \tag{2.80}$$

$$= det\left(\frac{\delta \boldsymbol{\phi}}{\delta \mathbf{a}}\right) det\left(\frac{\delta \boldsymbol{\phi}}{\delta \mathbf{b}}\right) det\left(\frac{\delta \boldsymbol{\pi}}{\delta \boldsymbol{\pi}_\mathbf{a}}\right) det\left(\frac{\delta \boldsymbol{\pi}}{\delta \boldsymbol{\pi}_\mathbf{b}}\right). \tag{2.81}$$

Thirdly we note, from the form of (2.67) and (2.68), the equality of the first and third, and of the second and fourth factors in the product (2.81). We deduce

$$det\left(\frac{\delta \boldsymbol{\phi}}{\delta \mathbf{a}}\right) det\left(\frac{\delta \boldsymbol{\phi}}{\delta \mathbf{b}}\right) = \pm 1, \tag{2.82}$$

and finally, (2.76) becomes

$$\prod_{\mathbf{x}} d\phi(\mathbf{x}) = \prod_{\mathbf{k}}{}' da(\mathbf{k}) db(\mathbf{k}). \tag{2.83}$$

This may be extended, using the same kind of trick as (2.21), to the two complex representations we have described. The complex volume elements in these cases are

$$\prod_{\mathbf{x}} d\phi(\mathbf{x}) = \prod_{\mathbf{k}}{}' \frac{1}{2} d\phi(\mathbf{k}) d\phi(\mathbf{k})^* \tag{2.84}$$

$$= \prod_{\mathbf{k}} \frac{1}{2} d\phi(\mathbf{k}). \tag{2.85}$$

Probabilistic interpretation and guidance equations

If we take the imaginary part of the functional Schrödinger equation in the real representation we find the continuity equation,

$$\frac{\partial |\psi|^2}{\partial t} + \int d^3k \left[\frac{\delta}{\delta \phi_a} \left(|\psi|^2 \frac{\delta S}{\delta \phi_a} \right) + \frac{\delta}{\delta \phi_a} \left(|\psi|^2 \frac{\delta S}{\delta \phi_a} \right) \right] = 0, \tag{2.86}$$

which by the standard pilot-wave interpretation would imply the guidance equations,

$$\frac{\partial \tilde{\Phi}_a(\mathbf{k})}{\partial t} = \frac{\delta S}{\delta \tilde{\phi}_a(\mathbf{k})}\bigg|_{\tilde{\phi}=\tilde{\Phi}}, \qquad \frac{\partial \tilde{\Phi}_b(\mathbf{k})}{\partial t} = \frac{\delta S}{\delta \tilde{\phi}_b(\mathbf{k})}\bigg|_{\tilde{\phi}=\tilde{\Phi}}. \tag{2.87}$$

In the complex representation the one finds the continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \int d^3k \, \frac{\delta}{\delta \tilde{\phi}(\mathbf{k})} \left(|\psi|^2 \frac{\delta S}{\delta \tilde{\phi}(-\mathbf{k})} \right). \tag{2.88}$$

As such the guidance equations in the complex representation are

$$\frac{\partial\tilde{\Phi}(\mathbf{k})}{\partial t} = \frac{\delta S}{\delta\tilde{\phi}(-\mathbf{k})}\bigg|_{\tilde{\phi}=\tilde{\Phi}}. \tag{2.89}$$

In the complex ϕ, ϕ^* representation the continuity equation is

$$\frac{\partial|\psi|^2}{\partial t} + \int' d^3k \left[\frac{\delta}{\delta\tilde{\phi}}\left(|\psi|^2\frac{\delta S}{\delta\tilde{\phi}^*}\right) + \frac{\delta}{\delta\tilde{\phi}^*}\left(|\psi|^2\frac{\delta S}{\delta\tilde{\phi}}\right)\right] = 0. \tag{2.90}$$

As such, the guidance equations for the complex field are

$$\frac{\partial\tilde{\Phi}}{\partial t} = \frac{\delta S}{\delta\tilde{\phi}^*}\bigg|_{\tilde{\phi}=\tilde{\Phi}}, \quad \frac{\partial\tilde{\Phi}^*}{\partial t} = \frac{\delta S}{\delta\tilde{\phi}}\bigg|_{\tilde{\phi}=\tilde{\Phi}}, \tag{2.91}$$

as expected. Before we move one we would like to highlight a possible misinterpretation of these guidance equations. In using Wirtinger calculus we have rather blurred the distinction between real and complex variables. Since these variables now form our beables, this deserves clarification. In the complex $\tilde{\phi}$,$\tilde{\phi}^*$ representation we seem to have guidance equations for both the field $\tilde{\phi}$ and it's complex conjugate, and since we have been treating these fields as independent variables one could be tempted to proclaim these as independent beables. This of course does not make sense.

If they are not to be considered independent beables, then we must find that

$$\left(\frac{\partial\Phi^*}{\partial t}\right) = \left(\frac{\partial\Phi}{\partial t}\right)^*. \tag{2.92}$$

This follows logically from a property of the Wirtinger derivatives[†], namely

[†]See Bouboulis (2010) for a proof

that for a function $f = f(z, z^*)$,

$$\left(\frac{\partial f}{\partial z}\right)^* = \left(\frac{\partial f^*}{\partial z^*}\right), \tag{2.93}$$

Thus the second set of equations in (2.91) is equivalent to the first, and we may discard one set. A similar argument may be made for the beables $\tilde{\phi}(\mathbf{k})$ in the $\tilde{\phi}$ representation.

2.4 Particle states and the low energy limit

Part of the motivation of this project was in the question of how the standard results of non-relativistic quantum mechanics may be retrieved in a low energy quantum field theory. It is possible that an investigation into this may be more fruitful if considering a pilot-wave theory. The pilot-wave ontology allows us to consider the behaviour of a single quantum field, rather than the forced abstraction of a wavefunctional or Fock vector. Since the ontology of the pilot-wave formulation of quantum mechanics is that of point particles, a necessary requirement of a pilot-wave quantum field theory must be that it approximates this point particle ontology in some suitable non-relativistic limit. One imagines that this must involve stable-*localised*, possibly solitonic field excitations that, imbued with some kind of measurable function characterising position, would follow trajectories approximate to the trajectories described by a quantum mechanical guidance equation.

To this end we introduce some work of Valentini (1992) concerning localised particle states. We work for the moment in the $\tilde{\phi}$ representation.

If we define the positive and negative frequency field operators

$$\hat{\phi}(\mathbf{x}) = \hat{\phi}_+(\mathbf{x}) + \hat{\phi}_-(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} (\hat{a}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} + \hat{a}^\dagger(\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}}), \tag{2.94}$$

$$\hat{\pi}(\mathbf{x}) = \hat{\pi}_+(\mathbf{x}) + \hat{\pi}_-(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}(-i)\sqrt{\frac{E_\mathbf{k}}{2}}(\hat{a}(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}} - \hat{a}^\dagger(\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}}), \tag{2.95}$$

and normalise our Fock states[†]

$$|\mathbf{k}\rangle = \sqrt{2E_\mathbf{k}}\hat{a}^\dagger(\mathbf{k})\,|0\rangle\,, \tag{2.96}$$

such that

$$\langle\mathbf{k}|\mathbf{k}'\rangle = 2E_\mathbf{k}\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{2.97}$$

(where we have used $[\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$) we may define the operator,

$$\hat{X} = 2i\int d^3x\,\mathbf{x}\hat{\phi}_-(\mathbf{x})\hat{\pi}_+(\mathbf{x}). \tag{2.98}$$

This has been designed to mimic the action of the quantum mechanical position operator. The eigenkets of this operator are, in line with standard interpretations of quantum field theory, the those created by the $\mathbf{x}-$space field operator acting upon the vacuum;

$$\begin{aligned} |\mathbf{x}\rangle = \hat{\phi}(\mathbf{x})\,|0\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_\mathbf{k}}}\hat{a}^\dagger_\mathbf{k}e^{-i\mathbf{k}.\mathbf{x}}\,|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{2E_\mathbf{k}}e^{-i\mathbf{k}.\mathbf{x}}\,|\mathbf{k}\rangle\,. \end{aligned} \tag{2.99}$$

[†]Differing from Valentini (1992) we prefer to follow Peskin & Schroeder (1995) in using a Lorentz invariant definition of the single mode states. This makes only superficial difference to the formulation of Valentini.

One may verify that,

$$\begin{aligned}
\hat{X}\,|\mathbf{x}\rangle &= \int d^3y\,\mathbf{y}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3k''}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\sqrt{\frac{E_{\mathbf{k}'}}{E_{\mathbf{k}''}}}e^{i(-\mathbf{k}.\mathbf{y}+\mathbf{k}'.\mathbf{y}-\mathbf{k}''.\mathbf{x})}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}''}\,|0\rangle \\
&= \int d^3y\,\mathbf{y}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}e^{i(-\mathbf{k}.\mathbf{y}+\mathbf{k}'.\mathbf{y}-\mathbf{k}'.\mathbf{x})}\hat{a}^\dagger_{\mathbf{k}}\,|0\rangle \\
&= \mathbf{x}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}e^{-i\mathbf{k}.\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}}\,|0\rangle \\
&= \mathbf{x}\,|\mathbf{x}\rangle\,.
\end{aligned} \tag{2.100}$$

The normalisation of these states is

$$\begin{aligned}
\langle\mathbf{x}'|\mathbf{x}\rangle &= \langle 0|\,\hat{\phi}(\mathbf{x}')\hat{\phi}(\mathbf{x})\,|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}e^{-i\mathbf{k}.(\mathbf{x}-\mathbf{x}')} \\
&= D(\mathbf{x}'-\mathbf{x}),
\end{aligned} \tag{2.101}$$

the Feynman propagator for spacelike separations. As this goes as $e^{-m|\mathbf{x}'-\mathbf{x}|}$, for large $m/|\mathbf{x}'-\mathbf{x}|$, it may be possible to consider $|\mathbf{x}\rangle$ an approximation to a quantum mechanical position basis. Then a state vector would be represented

$$|\psi\rangle = \int d^3x\,|\mathbf{x}\rangle\,\psi_{QM}(\mathbf{x}), \tag{2.102}$$

where $\psi_{QM}(\mathbf{x})$ is a one particle wavefunction. If this were possible then the obvious extension would be to attempt to construct an operator with the eigenvalue $\mathbf{k}$. The total momentum operator,

$$\hat{\mathbf{P}} = -\int d^3x\hat{\pi}(\mathbf{x})\nabla\hat{\phi}(\mathbf{x}) = \int d^3k\,\mathbf{k}\,\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \tag{2.103}$$

suffices. The eigenkets of $\hat{\mathbf{P}}$ are simply the Fock states. If these operators were to mimic the results of quantum mechanics then their commutator

must at least approximate the correct solution, $[\hat{\mathbf{X}}, \hat{\mathbf{P}}] \simeq i\mathbf{1}$. The quantum mechanical Lie algebra would then correspond to a sub-algebra of that of the field theoretic algebra. As current forms of $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$ differ, this seems unlikely in the current formulation. There are however a couple of different forms of the total momentum operator, corresponding to different constructions in the classical theory. We may also construct a new *non-total* momentum operator. A quantum theory should by constructed from it's canonical variables, and as such the factor $\mathbf{x}$ in the integrand of (2.98) is a bit of an oddity. We note however that there is a factor $\mathbf{p}$ in the integrand of (2.103), corresponding a gradient operator. We wonder whether it would not be possible to formulate these operators purely in terms of canonical operators. We very much regret that time constraints have meant that we've not been able to investigate this possibility further.

If we attempt to extend these ideas to multiple particle states we find some results that may aid analysis. To illustrate; if we take the most obvious approach by defining a two particle state as

$$|\mathbf{x}, \mathbf{x}'\rangle = \hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{x}')\,|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\frac{1}{\sqrt{2E_{\mathbf{k}'}}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}'}e^{-i\mathbf{k}.\mathbf{x}}e^{-i\mathbf{k}'.\mathbf{x}'}\,|0\rangle + \frac{1}{2E_{\mathbf{k}}}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,|0\rangle\,,$$

we find an encouraging delta function preventing us creating two particles in one position. However if we act the position operator on this state we find

$$\hat{\mathbf{X}}\,|\mathbf{x}, \mathbf{x}'\rangle = (\mathbf{x}+\mathbf{x}')\,|\mathbf{x}, \mathbf{x}'\rangle\,. \tag{2.104}$$

As the commutator of the two field operators vanishes it is impossible to single out one of the particles upon which to act an operator, implying operators of this type would not be able to tell us the position of a single particle

in a multiple particle state. Similarly we find $\hat{P}\,|\mathbf{p},\mathbf{p}'\rangle = (\mathbf{p}+\mathbf{p}')\,|\mathbf{p},\mathbf{p}'\rangle$, and $|\mathbf{p},\mathbf{p}'\rangle = |\mathbf{p}',\mathbf{p}\rangle$. A quantum mechanical position operator should be formulated in terms of configuration space, and a momentum operator the momentum analogue of configuration space. The product of these spaces is of course phase space. These operators, in their current formulation, are unable to act in this way. Particle creation and annihilation might seem to be a great hurdle in constructing a working sub-algebra. Since, in a low energy limit there should not be particle creation/annihilation, and one assumes the dimensionality of phase space before constructing quantum mechanical position and momentum operators, it seems reasonable to assume the number of 'particles' when constructing $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$.

3 Computational simulation

We began our investigation into field trajectories with the intention of simulating trajectories of general solutions of (2.69). It was hoped that a program designed to do this could then be applied to the greatest number of specific problems. As already mentioned, the real representation in hindsight was probably not the wisest choice of representation in which to work. The Hilbert space of the real representation, when quantised so that it's canonical variables become operators, is larger than the Fock space in which the theory is required to operate. As such much of the programming involved duplication of processes for variables that, once the initial conditions had been specified, could have been reduced to simpler expressions. Much of the methodology however, is applicable to further undertakings in this area.

3.1 A computable expression for field trajectories

When moving from continuum to box normalisation of $\mathbf{k}-$space as specified by the transforms (2.1) and (2.1), we make the associations

$$\tilde{\phi}(\mathbf{k}) \leftrightarrow q_{\mathbf{k}}, \quad \int \frac{d^3k}{(2\pi)^3} \leftrightarrow \frac{1}{V}\sum_{\mathbf{k}}, \quad \frac{\delta}{\delta\tilde{\phi}(\mathbf{k})} \leftrightarrow \frac{\partial}{\partial q_{\mathbf{k}}}. \tag{3.1}$$

We make the further association[†] $q_{\mathbf{k}} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}} - ib_{\mathbf{k}})$, to find the real representation. The Hamiltonian becomes

$$i\frac{\partial\psi}{\partial t} = \sum_{\mathbf{k}}{}' \frac{1}{2}\left[-\left(\frac{\partial^2}{\partial a_{\mathbf{k}}^2} + \frac{\partial^2}{\partial b_{\mathbf{k}}^2}\right) + E_k^2(a_{\mathbf{k}}^2 + b_{\mathbf{k}}^2)\right]\psi. \tag{3.2}$$

Although in general the solutions of this equation are not product wavefunctions, and we shall be constructing ones that are not, the groundstate plainly is. We write it

$$\psi^0(\mathbf{a},\mathbf{b}) = e^{i\theta}\prod_{\mathbf{k}}{}' \Psi^0(a_{\mathbf{k}})\Psi^0(b_{\mathbf{k}}), \tag{3.3}$$

$$\Psi^0(a_{\mathbf{k}}) = \left(\frac{E_{\mathbf{k}}}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}E_{\mathbf{k}}a_{\mathbf{k}}^2} e^{-\frac{1}{2}iE_{\mathbf{k}}t} \tag{3.4}$$

a simple product of modal harmonic oscillator groundstates. From this we may construct any state of the free system by acting a number of creation operators,

$$\hat{a}_{\mathbf{k}}^\dagger = \sqrt{\frac{E_{\mathbf{k}}}{2}}\left(a_{\mathbf{k}} - \frac{1}{E_{\mathbf{k}}}\frac{\partial}{\partial a_{\mathbf{k}}}\right), \tag{3.5}$$

$$\hat{b}_{\mathbf{k}}^\dagger = \sqrt{\frac{E_{\mathbf{k}}}{2}}\left(b_{\mathbf{k}} - \frac{1}{E_{\mathbf{k}}}\frac{\partial}{\partial b_{\mathbf{k}}}\right), \tag{3.6}$$

[†]Note that we have retained hats on all operators to avoid confusion with field variables

on a number of groundstates and taking a superposition with some relative phases. Since we are dealing with harmonic oscillators the creation operators act such that

$$\psi^n(a_{\mathbf{k}}) = \psi^0(a_{\mathbf{k}})\sqrt{\frac{1}{2^n n!}}H_n(\sqrt{E_{\mathbf{k}}}a_{\mathbf{k}})e^{-iE_{\mathbf{k}}t} = \psi^0(a_{\mathbf{k}})h^n(a_{\mathbf{k}}), \tag{3.7}$$

where $H_n(\sqrt{E_{\mathbf{k}}}a_{\mathbf{k}})$ are the Hermite polynomials in $\sqrt{E_{\mathbf{k}}}a_{\mathbf{k}}$. In the case where our wavefunction is a product, the phases in (3.7) will sum to produce an overall phase in (3.3). In any wavefunction with an overall phase, the guidance equations,

$$\frac{\partial\alpha_{\mathbf{k}}}{\partial t} = Im\left(\frac{1}{\psi}\frac{\partial\psi}{\partial a_{\mathbf{k}}}\right)\Big|_{a_{\mathbf{k}}=\alpha_{\mathbf{k}}(t)}, \quad \frac{\partial\beta_{\mathbf{k}}}{\partial a_{\boldsymbol{\kappa}}} = Im\left(\frac{1}{\psi}\frac{\partial\psi}{\partial b_{\mathbf{k}}}\right)\Big|_{b_{\mathbf{k}}=\beta_{\mathbf{k}}(t)} \tag{3.8}$$

may be seen to vanish. This is analogous to the standard quantum mechanical case.

Taking a superposition of (3.3), an arbitrary wavefunction for our field may be represented

$$\psi(\mathbf{a},\mathbf{b}) = \psi^0(\mathbf{a},\mathbf{b})\sum_i c^i e^{i\theta^i}\left[\prod_{\mathbf{k}/2} h^{n^i_{\mathbf{k}}}(a_{\mathbf{k}})h^{m^i_{\mathbf{k}}}(b_{\mathbf{k}})\right] \tag{3.9}$$

Since the ψ^0 and it's derivative are real any contribution that the groundstate wavefunction may have made drops out of the guidance equations leaving our final result for an arbitrary field

$$\frac{\partial\alpha_{\boldsymbol{\kappa}}}{\partial t} = Im\left(\frac{\sum_i c^i e^{i\theta^i}\left[\prod_{\mathbf{k}/2} h^{n^i_{\mathbf{k}}}(a_{\mathbf{k}})h^{m^i_{\mathbf{k}}}(b_{\mathbf{k}})\right]\frac{\partial}{\partial a_{\boldsymbol{\kappa}}}\left(\ln h^{n^i_{\boldsymbol{\kappa}}}(a_{\boldsymbol{\kappa}})\right)}{\sum_i c^i e^{i\theta^i}\left[\prod_{\mathbf{k}/2} h^{n^i_{\mathbf{k}}}(a_{\mathbf{k}})h^{m^i_{\mathbf{k}}}(b_{\mathbf{k}})\right]}\right)\Big|_{\mathbf{a}=\boldsymbol{\alpha}(t)\,\mathbf{b}=\boldsymbol{\beta}(t)}. \tag{3.10}$$

Although this isn't the prettiest of equations it is general, exact and quite

computable. More so the bulk of this equation is polynomial and modern computers are very efficient in calculating polynomials, as apposed to exponents or logarithms where they must resort to power series.

The simplest non-trivial case is the superposition of a groundstate and a state with one excited mode in the first excited state, $n^1_{\mathbf{k}} = 1$. The corresponding polynomial is then $h^1(a_{\mathbf{k}}) = \sqrt{2E_{\mathbf{k}}}a_{\mathbf{k}}e^{-iE_{\mathbf{k}}t}$ and ,with all other $h^0 = 1$, the velocity of the system in the $\mathbf{k}^{th}$ direction is

$$\frac{\partial \alpha_{\mathbf{k}}}{\partial t} = Im\left(\frac{c^2 e^{i\theta^2}\sqrt{2E_{\mathbf{k}}}e^{-iE_{\mathbf{k}}}}{c^1 e^{i\theta^1} + c^2 e^{i\theta^2}a_{\mathbf{k}}\sqrt{2E_{\mathbf{k}}}e^{-iE_{\mathbf{k}}}}\right)\Bigg|_{\mathbf{a}=\boldsymbol{\alpha}(t)\,\mathbf{b}=\boldsymbol{\beta}(t)}. \tag{3.11}$$

The velocity in all other directions will be zero.

3.2 Position eigenstates

We may retrieve the wavefunction of the localised field excitations of 2.4 by taking the inner product of the state with the basis of the representation we are working in. In the current case we have the basis $\{|\mathbf{a}, \mathbf{b}\rangle\}$, where $\mathbf{a} = (a_{\mathbf{k}_1}, a_{\mathbf{k}_2}, \dots)$, $\mathbf{b} = (b_{\mathbf{k}_1}, b_{\mathbf{k}_2}, \dots)$ normalised in the usual way,

$$\langle \mathbf{a}, \mathbf{b}|\mathbf{a}', \mathbf{b}'\rangle = \delta^{(3)}(a_{\mathbf{k}_1} - a'_{\mathbf{k}_1})\delta^{(3)}(b_{\mathbf{k}_1} - b'_{\mathbf{k}_1})\dots \tag{3.12}$$

The groundstate in this basis is

$$|0\rangle = \int d\mathbf{a}d\mathbf{b}\,\psi^0(\mathbf{a}, \mathbf{b})\,|\mathbf{a}, \mathbf{b}\rangle\,. \tag{3.13}$$

The discrete version of the single particle state is then

$$|\mathbf{x}\rangle \equiv \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}{}' \frac{1}{\sqrt{E_{\mathbf{k}}}}\left(\hat{a}^\dagger_{\mathbf{k}} cos(\mathbf{k}.\mathbf{x}) + \hat{b}^\dagger_{\mathbf{k}} sin(\mathbf{k}.\mathbf{x})\right)\int d\mathbf{a}d\mathbf{b}\,\psi^0(\mathbf{a}, \mathbf{b})\,|\mathbf{a}, \mathbf{b}\rangle\,. \tag{3.14}$$

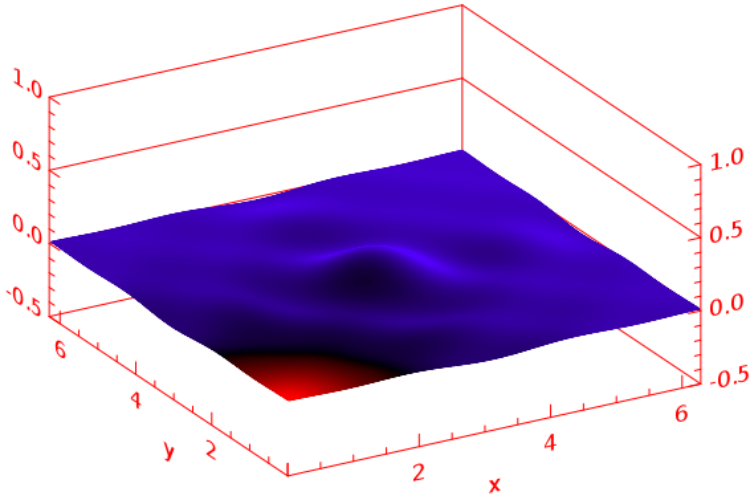

Figure 0.1: A likely $\phi(\mathbf{x})$ in $|\mathbf{x} = (\pi, \pi)\rangle$ calculated on a 7 by 7 grid of $\mathbf{k}$ points, with $m = 20$

making the wavefunction

$$\psi^x(\mathbf{a}, \mathbf{b}) \equiv \langle \mathbf{a}, \mathbf{b} | \mathbf{x} \rangle \tag{3.15}$$

$$= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}}{}' \sqrt{2} \left(cos(\mathbf{k}.\mathbf{x}) a(\mathbf{k}) + sin(\mathbf{k}.\mathbf{x}) b(\mathbf{k}) \right) \psi^0(\mathbf{a}, \mathbf{b}). \tag{3.16}$$

This, in the $\tilde{\phi}$ representation, takes the form $\psi^x[\tilde{\phi}] = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}.\mathbf{x}} \tilde{\phi}(\mathbf{x}) \psi^0[\phi]$. One may verify these equations by using the reverse canonical transformations (2.1) and (2.67).

Plotting a likely $|\mathbf{x}\rangle$

If the assertion made in section to 2.4, namely that a pilot-wave field theory must produce localised fields in the low energy limit is true, then a likely field configuration of $|\mathbf{x}\rangle$ must be localised. A simple verification of the plausibility of this was carried out. A basic Monte-Carlo random stepper algorithm was written to find the maxima of the probability amplitude corresponding to (3.16). Assuming a two dimensional $\mathbf{x}-$space for ease of plotting, the sum was taken over grids of $\mathbf{k}$ of varying sizes. It was found that with the computational resources available to us, convergence to a field configuration was not possible. This impeded for possibility of varying m. Nevertheless, as shown in figures 0.1, 0.2 and 0.3, the resultant field configurations are clearly localised.

3.3 On constructing a code to calculate a general field trajectory

As we are not presenting results there is no benefit to be gained by detailing the finer parameters of the code generated. Instead we wish to outline the structure of the code we wrote as we believe, due to it's generality, it may be useful to others considering simulating similar trajectories. The code has been made available at http://nicku.co.uk with documentation and instructions for compilation under a GPL license.

The outline problem was: solve equation (3.17) for a general wavefunction (3.9) constructed from a arbitrary number of product wavefunctions, each with an arbitrary number of excitations in an arbitrary number of modes. The actual algorithm was a Runge-Kutta-Fehlberg, as described in Press et al. (1992).

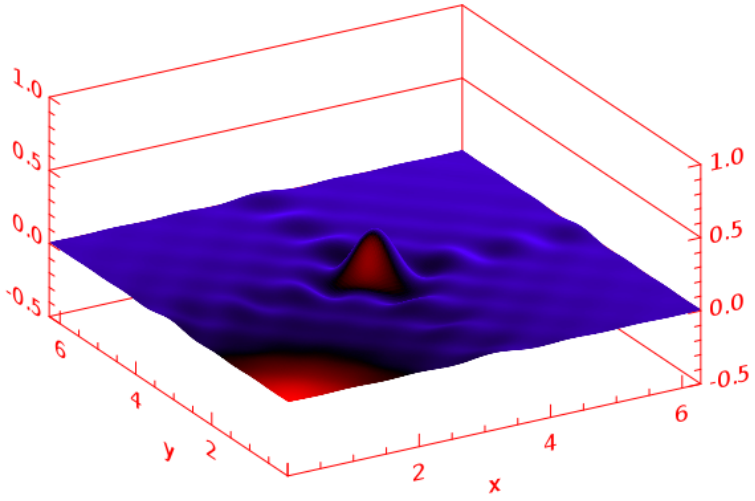

Figure 0.2: A likely $\phi(\mathbf{x})$ in $|\mathbf{x} = (\pi, \pi)\rangle$ calculated on a 9 by 9 grid of $\mathbf{k}$ points, with $m = 20$

The equation for a free field trajectory,

$$\frac{\partial \alpha_{\boldsymbol{\kappa}}}{\partial t} = Im\left(\frac{\sum_i c^i e^{i\theta^i}\left[\prod_{\mathbf{k}/2} h^{n^i_{\mathbf{k}}}(a_{\mathbf{k}}) h^{m^i_{\mathbf{k}}}(b_{\mathbf{k}})\right]\frac{\partial}{\partial a_{\boldsymbol{\kappa}}}\left(\ln h^{n^i_{\boldsymbol{\kappa}}}(a_{\boldsymbol{\kappa}})\right)}{\sum_i c^i e^{i\theta^i}\left[\prod_{\mathbf{k}/2} h^{n^i_{\mathbf{k}}}(a_{\mathbf{k}}) h^{m^i_{\mathbf{k}}}(b_{\mathbf{k}})\right]}\right)\Bigg|_{\mathbf{a}=\boldsymbol{\alpha}(t)\,\mathbf{b}=\boldsymbol{\beta}(t)}, \tag{3.17}$$

requires a code with great flexibility. The greatest problems to surmount were, we think

- Firstly, and most importantly, (3.17) implies that there will only be motion in the $\mathbf{k}-$space directions in which there excitations in the superposed wavefunctions, so the trajectory must be calculated in an arbitrarily dimensional space.

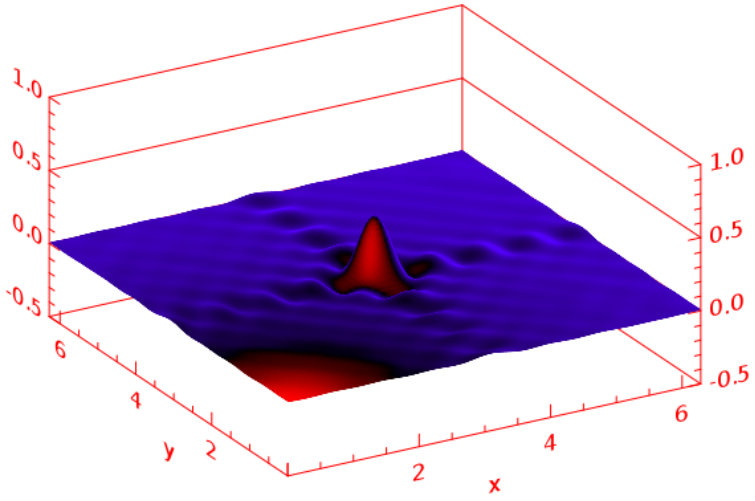

Figure 0.3: A likely $\phi(\mathbf{x})$ in $|\mathbf{x} = (\pi, \pi)\rangle$ calculated on a 17 by 17 grid of $\mathbf{k}$ points, with $m = 20$

- Secondly, there may be an arbitrary number of superposed fields each with an arbitrary number of excited oscillators.

- Finally, though products must necessarily be calculated over the Hermite polynomials of each respective product wavefunction in the superposition, the $\mathbf{k}-$sub-configuration space of each wavefunction may overlap.

These three points, when combined suggested a program with large derived data types, the first of which (named *wvfn*) was that containing all

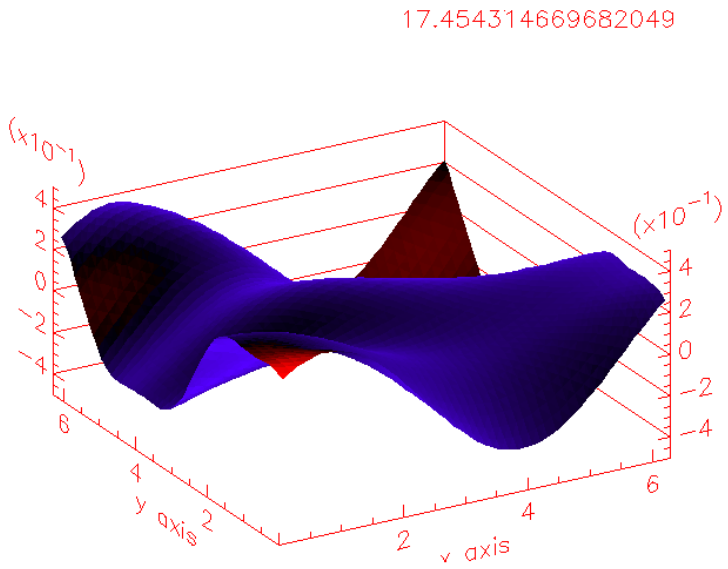

Figure 0.4: The program produces a real-time video as it run of the field trajectory. This is a screenshot of some typical output.

the information required to calculate

$$c^i e^{i\theta^i} \left[\prod_{\mathbf{k}/2} h^{n^i_{\mathbf{k}}}(a_{\mathbf{k}}) h^{m^i_{\mathbf{k}}}(b_{\mathbf{k}}) \right]. \tag{3.18}$$

for a product wavefunction. There was required to be an arbitrary number of these, so a linked listed structure with each *wvfn* pointing to the next was the most appropriate choice. Since the configuration space of the system was different from the configuration space of each product wavefunction, the actual position of the system in $\mathbf{k}-$space was kept in terms of a basis consisting of all excited modes in a separate derived type *ppos*. The modes in each *wvfn*, then point to the position vectors in *ppos* so that we need not update positions in *wvfn*; as the trajectory progresses we only need update the positions in the *ppos*. Finally, as we calculate each velocity vector in the basis contained in *ppos*, we need to be able to know where the relevant

excited oscillators in the *wvfn* are. To this end we used an idea found in Metcalf & Reid (1999). Arrays of pointers are not included in the Fortran standard, though the same effect may be achieved by using a derived type. The elements of the vectors *ppos* are then able to contain pointers to the excitations to be summed over, in the same way that the excitations point to the positions.

4 Summary and conclusions

In this dissertation we began by wishing to make numerical simulations of the low energy limit of scalar field theory. We did not manage to achieve this within the available time, though we have developed a code that can do this. We showed that the standard $\mathbf{k}-$space representations used for the theory complicate it unnecessarily, and showed that a simple one is valid. We also made a plausible argument for the volume element in $\mathbf{k}-$ space, something that is surely vital to a pilot-wave theory.

We very much regret that we have not had the time to try to develop a low energy sub-algebra of the field operator algebra. The non-relativistic limit is something that We feel should be an integral part of a quantum field theory. For a pilot wave theory, where one may simulate the trajectories of point particles of known wavefunctions with ease, it could be very profitable to compare the difference between low energy field trajectories and point particles.

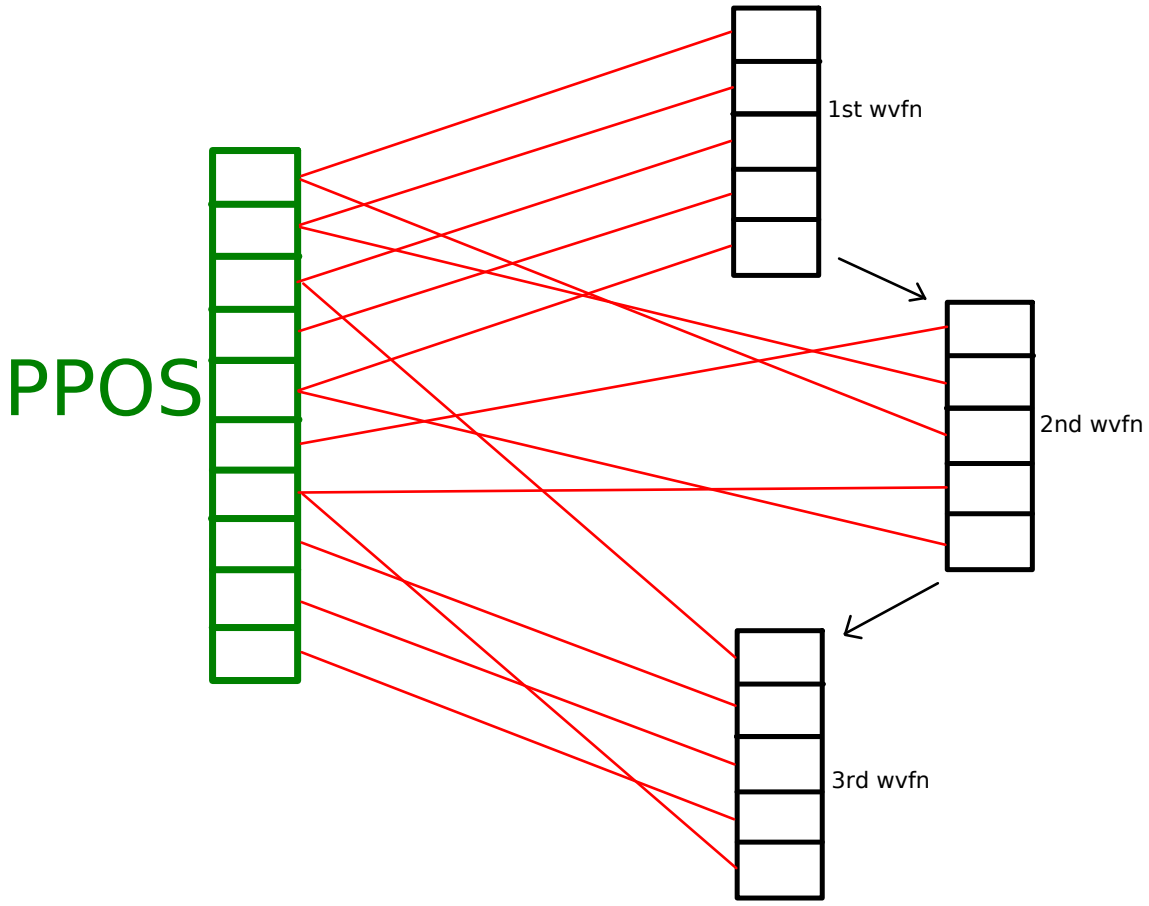

Figure 0.5: This represents the underlying structure of the code. The need to iterate over both the configuration space coordinates and over product wavefunction excitations is achieved through ordering in *ppos* and the *wvfn* respectively. The pointers, represented in red link each oscillator excitation in the *wvfn* to its $\mathbf{k}-$space basis vector and coordinate in *ppos* and vice versa. The iterations may then descend down the arrays, and obtain all information they require.

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