Dissertation

Implications of f(R) modification to gravity

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Submitted in partial fulfilment of the requirements
for the degree of Master of Science of Imperial College London

2010

Abstract

During the last decades motivations for modified gravity have originated from both theoretical and observational level. The f(R) gravity is the simplest generalisation and hence has received increased attention. This dissertation is a review of certain implications of f(R) in the description of gravity. Starting from the field equations; after showing how they are derived in the metric formalism, an FLRW background is imposed and it is examined how inflation and Dark Energy dominated period are realised in the context of metric f(R) gravity. In each case, typical f(R) models are referred as examples. In addition there is a brief account of the standard cosmological picture for the above cosmological periods. In the end is displayed a summary of the viability conditions that f(R) theories should satisfy in order to be consistent.

Contents

Introduction

Modern cosmology would have never been a robust science without the appearance of Einstein's General Relativity. Before Einstein introduced his theory, describing the force of gravity, cosmology was more of a philosophical approach to the question, how the universe was created. The new theory not only enabled physicists to explain phenomena on the scale of the solar system, but it was actually the stepping stone that led them to the origins and the evolution of the whole universe.

Since 1915, when G.R. was introduced, many of its implications have been tested successfully through various experiments and astrophysical observations. It has also been the basis for numerous cosmological models, like the Big Bang which is now accepted as the theory best describing the beginning of everything from a spacetime singularity some 13,7 billion years ago. So all in all General Relativity is quite a successful theory and there is got to be a very good reason if someone wants to alter it.

Modifying Einstein's General relativity is not a new thing. The first person who suggested a slightly different version of G.R was Einstein himself. At the time Le-Maitre and Friedmann found exact solutions of Einstein's equations predicting an expanding universe most of the physicists, including Einstein, believed in a static universe. So Einstein modified the initial action of his theory, including the famous cosmological constant Λ in order to avoid the predicted expansion.

Other attempts for a different approach to the theory, came from scientists a few years right after its introduction. It was in 1919 and in 1923, when Weyl and Eddington first considered modifications of the theory by including higher order invariant terms in its action. Of course it was not an experimental or theoretical failure of the theory that motivated these efforts, rather than pure scientific will to obtain a better insight in the newly proposed theory. Though, as time passed, there were more and more indications that changing the action of general relativity would have its benefits.

The first important motivation came from the search for a unified theory of all fundamental forces. One of the steps towards an ultimate theory is to quantize gravity, in the same context of quantum fields as for the rest of the interactions. However this task is proved to be extremely subtle, with one of the main problems being the non renormalisability of general relativity. One way to get around this problem is to modify the standard G.R. action. In 1962, Utiyama and DeWitt showed that renormalization at one-loop demands that the Einstein-Hilbert action be supplemented by higher order curvature terms. Following that, Stelle showed that higher order actions are indeed renormalizable (but not unitary). More recent results show that when quantum corrections or string theory are taken into account, the effective low energy gravitational action admits higher order curvature invariants.

The latest motivations for modified gravities originate from contemporary astrophysical and cosmological observations. Two kinds of these type of observation are significant to us.

The first is the current matter-energy composition of the universe. By making accurate measurements of the cosmic microwave background fluctuations, WMAP is able to measure the basic parameters of the Big Bang model including the density and composition of the universe. We now know that this total density breaks down to: 4% *ordinary baryonic matter*, 20% *dark matter*, 76% *dark energy*.

Dark matter is an unknown form of matter, that remains unobserved in the laboratory. This elusive form of matter behaves the same way as ordinary matter under gravitation but it does not interact at all through electroweak or strong interactions. In addition to WMAP the existence of dark matter has been suggested as an explanation to the "missing mass" problem of galaxy clusters and individual galaxies.

The second observation has to do with dark energy and the growing amount of evidence that the universe is now undergoing an accelerated expansion phase. Data coming from type Ia supernovae observations suggest that this accelerated phase started when the universe was around 6 or 7 billion years old, when it is assumed that Dark Energy became dominant and took over the expansion, which was until then decelerating and driven by baryonic matter. So Dark energy is yet another unknown form of energy, with the unusual characteristic to act as a repulsive force, canceling the effect of gravity on large scales.

This late period acceleration is additional to the proposed early inflationary phase, which is used to solve many problems of standard cosmological model, like the horizon, the monopole and the flatness problem and as a mechanism to produce the initial perturbations that led to the current large scale structure. The difference between the two accelerated phases, is that inflation cannot be attributed to Dark Energy. That's because radiation and matter decelerating epochs must follow that of inflation, in order to achieve light element formulation through Big Bang Nucleosynthesis and the creation of the large scale structure amongst other observed features of our universe.

Taking all that into account it becomes apparent that deriving a picture for the evolution and the matter-energy content of the universe, from a theory that is in accordance with the observed, is at least challenging. The currently proposed cosmological model is the so called ΛCDM (Λ-Cold Dark Matter) or concordance model, supplemented with an inflationary phase, usually based on some scalar field. In this model Dark Energy is considered to be the same as with the energy of the vacuum, it has negative gravitational pressure and its effect is equivalent with a cosmological constant Λ. However this model is more of a phenomenological fit to the data, rather than a fundamental theory from which the observed universe is the unambiguous outcome. As such it does not explain the origin of inflation or the nature of Dark Matter and is inherited with the well known *cosmological constant problem* and the *coincidence problem*. The first one having to do with the magnitude problem, according to which the observed value of the cosmological constant is minuscule in comparison with the vacuum energy of the matter fields, about 120 orders of magnitude smaller. The later problem has to do with the strange coincidence, that the extremely small period of time when the energy density of the cosmological constant is comparable with that of matter, happens to be just now that we are here to observe it.

As long as the solution to this puzzles is not yet obtained in the context of standard General Relativity, attempts to a modified theory of gravity sound at least reasonable. So in this dissertation we are interested in examining how does f(R) modification succeeds on giving an answer to some of the aforementioned problems.

Chapter 1

The three types of f(R) gravity

So f(R) gravity is simply a generalisation of the Einstein-Hilbert action, whose Lagrangian is just the Ricci scalar $S_{EH} = \int d^4x\sqrt{-g}R$, to an action in which the Ricci scalar gets replaced by an arbitrary function f(R): $S = \int d^4x\sqrt{-g}f(R)$

There are three versions of f(R) gravities, depending on the formalism i.e. the variational principle we apply to the action in order to get the field equations. The standard way is to vary the action with respect to the metric, giving *the metric f(R)gravity.* The second way is to use the Palatini approach, which is to consider the connection independent of the metric and vary the action with respect to both the metric and the connection yielding the *Palatini f(R) gravity*, assuming of course that the matter action does not depend on the connection. We should note that these two methods lead to the same field equations in the case of the usual Einstein-Hilbert action. But in the context of f(R) their respective field equations differ. The third and most general way is the *metric-affine f(R)gravity*, in which we use the Palatini formalism abandoning the assumption that the matter action is independent of the connection.

1.1 Metric f(R) gravity

To get the field equations of f(R) gravity we have to vary with respect to the metric $g_{\mu\nu}$ the action (1.2)

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \psi) \tag{1.1}$$

Where $\kappa^2 = 8\pi G$ and we have added a matter action and ψ refers collectively to all matter fields.

1.1.1 Variation of the f(R) action in the metric formalism

We will derive the field equations varying the inverse metric

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \tag{1.2}$$

The variation on the action([1]) then becomes

$$\delta S = \int d^4x \delta(\sqrt{-g} f(R))$$

$$\delta S = \int d^4x \sqrt{-g} \delta(f(R)) + \int d^4x \delta(\sqrt{-g}) f(R) \tag{1.3}$$

We need to calculate the variation of $\sqrt{-g}$. To do that we use the identity $Tr(lnM) = ln(detM)$, where M is an invertible matrix and lnM is defined by $exp(lnM) = M$. Varying this identity and using the cyclic property of the trace, yields

$$Tr(M^{-1}\delta M) = \frac{1}{detM}\delta(detM) \tag{1.4}$$

Applying this to the inverse metric $M = g^{\mu\nu}$, and $detM = g^{-1} = detg^{\mu\nu}$ we get

$$\begin{aligned} Tr(g_{\mu\nu}\delta g^{\mu\nu}) &= g\delta(g^{-1}) \\ \delta(g^{-1}) &= \frac{1}{g} g_{\mu\nu} \delta g^{\mu\nu} \end{aligned}$$

Hence the variation of $\sqrt{-g}$ works out to be

$$\begin{aligned} \delta\sqrt{-g} = \delta[(-g^{-1})^{-1/2}] &= -\frac{1}{2}(-g^{-1})^{-3/2}\delta(-g) \\ &= -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \end{aligned} \tag{1.5}$$

The variation of f(R) is equal to:

[1]Without considering the matter action which will give the energy momentum tensor, and dropping for a while the κ^2 term

$$\delta f(R) = \frac{\partial f}{\partial R}\delta R = \frac{\partial f}{\partial R}(\delta g^{\mu\nu})R_{\mu\nu} + \frac{\partial f}{\partial R}g^{\mu\nu}\delta R_{\mu\nu}$$

So the variation of the action has three parts

$$\delta S = (\delta S1) + (\delta S2) + (\delta S3) \tag{1.6}$$

with

$$(\delta S1) = \int d^4x\sqrt{-g}\left[\frac{\partial f}{\partial R}R_{\mu\nu}\right]\delta g^{\mu\nu} \tag{1.7}$$

$$(\delta S2) = \int d^4x\sqrt{-g}\left[\frac{\partial f}{\partial R}g^{\mu\nu}\right]\delta R_{\mu\nu} \tag{1.8}$$

$$(\delta S3) = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}f(R)\right]\delta g^{\mu\nu} \tag{1.9}$$

Where the $(\delta S1)$ and $(\delta S3)$ terms are in the desired form, of a quantity multiplied by $(\delta g^{\mu\nu})$. To get the second term in the same form we need the variation of the Ricci tensor which is given by the Palatini equation

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda} \tag{1.10}$$

So the second term in the variation of the action becomes[2]

$$\begin{aligned}
(\delta S2) &= \int d^4x\sqrt{-g}F(R)g^{\mu\nu}[\nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}] \\
&= \int d^4x\sqrt{-g}F(R)[\nabla_\lambda g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}] \\
&= \int d^4x\sqrt{-g}F(R)[\nabla_\sigma g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - \nabla_\sigma g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda}] \\
&= \int d^4x\sqrt{-g}F(R)\nabla_\sigma[g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda}]
\end{aligned} \tag{1.11}$$

We can now insert the variation of $\delta\Gamma^\sigma_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$ as given in [5][3]

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\mu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})] \tag{1.12}$$

Using that in (1.11)

[2] From here and on $F(R) = \frac{\partial f}{\partial R}$

[3] Chapter 4.3, page 162, equation(4.64)

$$(\delta S2) = \int d^4x\sqrt{-g}(-\frac{1}{2}F(R))\nabla_\sigma[\delta^\lambda_\nu\nabla_\nu(\delta g^{\lambda\sigma}) + \delta^\lambda_\nu\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\alpha\beta}\nabla^\sigma(\delta g^{\alpha\beta})$$
$$- \delta^\sigma_\rho\nabla_\nu(\delta g^{\lambda\rho}) - g^{\mu\sigma}g_{\rho\lambda}\nabla_\mu(\delta g^{\lambda\rho}) + \delta^\sigma_\beta g_{\lambda\alpha}\nabla^\lambda(\delta g^{\alpha\beta})]$$

$$(\delta S2) = \int d^4x\sqrt{-g}(-\frac{1}{2}F(R))\nabla_\sigma[\nabla_\lambda(\delta g^{\lambda\sigma}) - \nabla_\lambda(\delta g^{\lambda\sigma}) + \nabla_\mu(\delta g^{\mu\sigma})$$
$$- g_{\alpha\beta}\nabla^\sigma(\delta g^{\alpha\beta}) - g^{\mu\sigma}g_{\rho\lambda}\nabla_\mu(\delta g^{\lambda\rho}) + g_{\lambda\alpha}\nabla^\lambda(\delta g^{\alpha\sigma})]$$

$$(\delta S2) = \int d^4x\sqrt{-g}(-\frac{1}{2}F(R))\nabla_\sigma[\nabla_\mu(\delta g^{\mu\sigma}) + \nabla_\alpha(\delta g^{\alpha\sigma})$$
$$- g_{\alpha\beta}\nabla^\sigma(\delta g^{\alpha\beta}) - g_{\rho\lambda}\nabla^\sigma(\delta g^{\lambda\rho})]$$

$$(\delta S2) = \int d^4x\sqrt{-g}(-\frac{1}{2}F(R))\nabla_\sigma[\nabla_\mu(\delta g^{\mu\sigma}) + \nabla_\mu(\delta g^{\mu\sigma})$$
$$- g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu})]$$

$$(\delta S2) = \int d^4x\sqrt{-g}(-\frac{1}{2}F(R))\nabla_\sigma[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\mu(\delta g^{\mu\sigma})]$$

Having found that, the total variation of the action reads out

$$\delta S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\{\left[\frac{\partial f}{\partial R}R_{\mu\nu}\right]\delta g^{\mu\nu}$$
$$+ (-\frac{1}{2}F(R))\nabla_\sigma[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\mu(\delta g^{\mu\sigma})]$$
$$+ [-\frac{1}{2}g_{\mu\nu}f(R)]\delta g^{\mu\nu}\} \tag{1.13}$$

1.1.2 Field equations of metric f(R) gravity

So from (1.13) the variation of the action (1.1) with respect to the metric gives the following field equations[4]

$$\frac{\delta S}{\delta g^{\mu\nu}} = F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa^2 T_{\mu\nu} \tag{1.14}$$

[4]Using the fact that F(R) is a scalar quantity and can go through the covariant derivative.

Where $\Box F = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$ and the energy momentum tensor of matter is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{1.15}$$

Which satisfies the continuity equation

$$\nabla^\mu T_{\mu\nu} = 0 \tag{1.16}$$

To find the trace of (1.14) we multiply with $g^{\mu\nu}$, leading to

$$F(R)R^\mu_\mu - \frac{1}{2}f(R)\delta^\mu_\mu - \nabla^\mu\nabla_\mu F(R) + \delta^\mu_\mu \Box F(R) = \kappa^2 T^\mu_\mu$$

Using the fact that $\delta^\mu_\mu = 4$ and that $\nabla^\mu\nabla_\mu \equiv \Box$ the trace of the field equations is

$$F(R)R + 3\Box F(R) - 2f(R) = \kappa^2 T \tag{1.17}$$

For $f(R) = R$ and $F(R) = 1$ we retrieve the Einstein field equations and their trace which is $R = -\kappa^2 T$, and thus in this case the Ricci scalar R is determined by the matter. Comparing the two cases we see that in the f(R)gravity, the term $\Box F(R)$ does not vanish, and we have an extra propagating degree of freedom $\phi \equiv F(R)$, called "scalaron", whose dynamics are determined by the trace equation (1.17).

A realisation of a universe undergoing an accelerated expansion is through the de Sitter solution, which corresponds to a vacuum solution of the equations where the Ricci scalar is constant i.e. $\Box F(R) = 0$. So for an f(R) model to give a de Sitter like solution it must satisfy the equation

$$F(R)R - 2f(R) = 0 \tag{1.18}$$

A model with $f(R) = aR^2$ satisfies the previous condition and gives rise to an exact de Sitter solution. Furthermore the first model to produce inflation was proposed by Starobinsky with $f(R) = R + aR^2$. In this model accelerated expansion stops when the term aR^2 becomes smaller than R. We will examine this model further in the following chapter.

1.2 Palatini f(R) gravity

As we said in the Palatini formalism we treat the metric $g_{\mu\nu}$ and the connection $\Gamma^{\alpha}_{\beta\gamma}$ as independent variables. In the case of standard G.R. the two variations lead to the same field equation. But in the case of f(R) the extra curvature terms will provide two separate equations. So varying the action (1.1) with respect to $g_{\mu\nu}$ we get a similar result with the metric formalism

$$F(R)R_{\mu\nu}(\Gamma) - \frac{1}{2}f(R)g^{\mu\nu} = \kappa T_{\mu\nu} \tag{1.19}$$

Where $R_{\mu\nu}(\Gamma)$ is the Ricci tensor corresponding to the connections $\Gamma^{\alpha}_{\beta\gamma}$ and it is in general different from the Ricci tensor calculated in terms of metric connections $R_{\mu\nu}(g)$. The trace of (1.19) is

$$F(R)R - 2f(R) = \kappa^2 T \tag{1.20}$$

And we immediately see that there is no term with $\Box F$ like in the metric formalism.

Now if we vary with respect to the connection $\Gamma^{\alpha}_{\beta\gamma}$ we get another field equation (see [6]) which is

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) =& \frac{\kappa^2 T_{\mu\nu}}{F} - \frac{FR(T) - f}{2F}g_{\mu\nu} + \frac{1}{F}(\nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F) \\ & - \frac{3}{2F^2}[\partial_\mu F\partial_\nu F - \frac{1}{2}g_{\mu\nu}(\nabla F)^2] \end{aligned} \tag{1.21}$$

Chapter 2

Cosmology within metric f(R) gravity

At this chapter we are interested in reproducing some basic features of the standard cosmological model in the context of f(R) gravities, which as we mentioned motivated the research for General Relativity alternatives. For that, we will focus on the metric f(R) gravity and see how the field equations give rise to the equivalent of the Friedmann equations. And following that we will show how inflation and late time acceleration are being achieved.

2.1 Field equations in flat FLRW universe

As we know in the heart of the concordance model lies the cosmological principle that admits a universe which is flat, homogeneous and isotropic. This universe is described by FLRW (Friedmann-Lemaitre-Robertson-Walker) metric tensor with flat geometry.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\mathbf{x}^2$$

So the only non zero elements of the metric are

$$\begin{aligned} g_{00} &= -1 \\ g_{ii} &= a^2(t) \end{aligned}$$

And for the inverse metric g_{00} is the same and

$$g^{ii} = \frac{1}{a^2(t)} \tag{2.1}$$

As for the determinant g it is easy to be read off

$$g = a^6(t) \Rightarrow \sqrt{-g} = a^3(t) \tag{2.2}$$

So, introducing this metric into the field equations we will find the equivalent Friedmann equations for our modified gravity. To do so, we first need to calculate the components of the Ricci tensor through the Cristoffel symbols and since we are in the metric formalism these are given by the known relation

$$\Gamma^{\alpha}_{\mu\nu} = -\frac{1}{2} g^{\alpha\delta}(g_{\nu\delta,\mu} + g_{\mu\delta,\nu} - g_{\mu\nu,\delta})$$

Where the comma denotes partial derivatives.

In our particular case of flat (FLRW) geometry the only non-vanishing Cristoffel symbols are

$$\Gamma^{0}_{ii} = a\dot{a} \tag{2.3}$$

$$\Gamma^{i}_{0i} = \frac{\dot{a}}{a} \tag{2.4}$$

We can now proceed in the evaluation of the Ricci tensor. This is the only possible contraction between two indices of the Riemann tensor, that is compatible with its symmetries. So it follows up from the definition of the Rieamann tensor that in terms of the connection the Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\epsilon\alpha}\Gamma^{\epsilon}_{\mu\nu} - \Gamma^{\alpha}_{\nu\epsilon}\Gamma^{\epsilon}_{\mu\alpha}$$

Based on the Cristoffel symbols (2.3) the (00) component of the ricci tensor is

$$\begin{aligned} R_{00} &= \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{\epsilon\alpha}\Gamma^{\epsilon}_{00} - \Gamma^{\alpha}_{0\epsilon}\Gamma^{\epsilon}_{0\alpha} \\ &= -3\Gamma^{1}_{01,0} - 3(\Gamma^{1}_{01})^2 \\ &= -3\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \\ &= -3\frac{\ddot{a}a - \dot{a}^2}{a^2} - 3\left(\frac{\dot{a}}{a}\right)^2 \\ &= -3\frac{\ddot{a}}{a} \\ &= -3[\dot{H} + H^2] \end{aligned} \tag{2.5}$$

And the (ii) component is

$$
\begin{aligned}
R_{ii} &= \Gamma^{\alpha}_{ii,\alpha} - \Gamma^{\alpha}_{i\alpha,i} + \Gamma^{\alpha}_{\epsilon\alpha}\Gamma^{\epsilon}_{ii} - \Gamma^{\alpha}_{i\epsilon}\Gamma^{\epsilon}_{i\alpha} \\
&= \Gamma^{\alpha}_{ii,\alpha} + \Gamma^{\alpha}_{0\alpha}\Gamma^{0}_{ii} - \Gamma^{0}_{ii}\Gamma^{i}_{i0} - \Gamma^{i}_{i0}\Gamma^{0}_{ii} \\
&= \Gamma^{\alpha}_{ii,\alpha} + 3\Gamma^{1}_{01}\Gamma^{0}_{ii} - 2\Gamma^{0}_{ii}\Gamma^{i}_{i0} \\
&= \Gamma^{\alpha}_{ii,\alpha} + \Gamma^{i}_{0i}\Gamma^{0}_{ii} \\
&= \Gamma^{0}_{ii,0} + \Gamma^{i}_{0i}\Gamma^{0}_{ii} \\
&= \frac{\partial}{\partial t}(\dot{a}a) + \frac{\dot{a}}{a}\dot{a}a \\
&= \dot{a}^2 + a\ddot{a} + \dot{a}^2 \\
&= 2\dot{a}^2 + a\ddot{a} \qquad (2.6)
\end{aligned}
$$

The Ricci scalar, which is a contraction between the two indices of the Ricci tensor takes the form

$$
\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= g^{00}R_{00} + 3g^{ii}R_{ii} \\
&= (-1)\left(-3\frac{\ddot{a}}{a}\right) + 3\frac{1}{a^2}(2\dot{a}^2 + a\ddot{a}) \\
&= 6\left[\frac{\ddot{a}a + \dot{a}^2}{a^2}\right] \\
&= 6\left[\frac{\ddot{a}a - \dot{a}^2}{a^2} + 2\frac{\dot{a}^2}{a^2}\right] \\
&= 6[\dot{H} + 2H^2] \qquad (2.7)
\end{aligned}
$$

Using this we can rewrite the (00) component of the Ricci tensor as

$$
\begin{aligned}
R_{00} &= -3[\dot{H} + H^2] \\
&= -\frac{1}{2}6[\dot{H} + 2H^2 - H^2] \\
&= -\frac{1}{2}R + 3H^2 \qquad (2.8)
\end{aligned}
$$

To proceed in finding the modified Friedmann equations we need the expression for the energy-momentum tensor. The simplest case is to assume that the universe is filled with a perfect fluid with an energy-momentum tensor

$$
T^{\mu}_{\nu} = diag(-\rho_M, P_M, P_M, P_M) \qquad (2.9)
$$

$$
T_{\mu\nu} = g_{\alpha\mu}T^{\mu}_{\nu} \qquad (2.10)
$$

Before plugging these in the field equations it is useful first to see what the zero component of the conservation of energy equation (1.16) yields in this case

$$\begin{aligned}0 &= \nabla_\mu T^\mu_0 \\ &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \\ &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + P)\end{aligned} \tag{2.11}$$

The next step is to replace (2.8)into (1.14) which leads to

$$F(R)R_{00} - \frac{1}{2}f(R)g_{00} - \nabla_0\nabla_0 F(R) + g_{00}\Box F(R) = \kappa^2 g_{00} T^0_0 \tag{2.12}$$

Working out explicitly the fourth term of the LHS we have

$$\begin{aligned}g_{00}\Box F(R) &= (-1)(\frac{1}{\sqrt{-g}})\partial_0\big(\sqrt{-g}g^{00}\partial_0 F\big) \\ &= -\frac{1}{a^3}\frac{\partial}{\partial t}\left(-a^3\frac{\partial}{\partial t}F\right) \\ &= \frac{1}{a^3}3a^2\dot{a}\dot{F} + \frac{a^3}{a^3}\ddot{F} \\ &= 3H\dot{F} + \ddot{F}\end{aligned} \tag{2.13}$$

The third term with the covariant derivatives will just become $-\partial_0\partial_0 F$ since F(R) is a scalar quantity, and thus it will cancel the $\ddot{F}$ term coming from the last expression. So the remaining terms are

$$F(3H^2 - \frac{1}{2}R) + \frac{1}{2}f + 3H\dot{F} = \kappa^2 \rho_M$$

And the first modified Friedmann equations is

$$3FH^2 = (FR - f)/2 - 3H\dot{F} + \kappa^2\rho_M \tag{2.14}$$

For the second equation we take the (ii) part of (1.14):

$$F(R)R_{ii} - \frac{1}{2}f(R)g_{ii} - \nabla_i\nabla_i F(R) + g_{ii}\Box F(R) = \kappa^2 g_{ii} T^i_i \tag{2.15}$$

As before the last term of the LHS is equal with

$$
\begin{aligned}
g_{ii}\Box F(R) &= (a^2)(\frac{1}{\sqrt{-g}})\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu F\big) \\
&= a^2(\frac{1}{\sqrt{-g}})\partial_0\big(\sqrt{-g}g^{00}\partial_0 F\big) + a^2(\frac{1}{\sqrt{-g}})\partial_i\big(\sqrt{-g}g^{ii}\partial_i F\big) \\
&= a^2\frac{1}{a^3}\partial_0\Big(a^3(-1)\dot{F}\Big) + a^2\frac{1}{a^3}\partial_i\left(a^3\frac{1}{a^2}\partial_i F\right) \\
&= -a^2\frac{1}{a^3}3a^2\dot{a}\dot{F} - a^2\frac{a^3}{a^3}\ddot{F} + \frac{1}{a}(\partial_i a)(\partial_i F) + \frac{a}{a}\partial_i\partial_i F \\
&= -3a^2H\dot{F} - a^2\ddot{F} + \frac{1}{a}\left(\frac{\partial t}{\partial \mathbf{x}}\frac{\partial}{\partial t}a\right)\left(\frac{\partial t}{\partial \mathbf{x}}\frac{\partial}{\partial t}F\right) + \partial_i\partial_i F \\
&= -3a^2H\dot{F} - a^2\ddot{F} + \frac{\dot{a}}{a}\left(\frac{\partial t}{\partial \mathbf{x}}\right)^2\dot{F} + \nabla_i\nabla_i F \\
&= -3a^2H\dot{F} - a^2\ddot{F} + a^2H\dot{F} + \nabla_i\nabla_i F \\
&= -2a^2H\dot{F} - a^2\ddot{F} + \nabla_i\nabla_i F
\end{aligned} \tag{2.16}
$$

Where I have used the fact that for flat FLRW metric $(\partial t/\partial \mathbf{x})^2 = a^2$. Now using the fact that the $\nabla_i\nabla_i$ cancels the term $-\nabla_i\nabla_i F(R)$ in the field equation, plugging (2.6) in (2.15) we have

$$
\begin{aligned}
(2\dot{a}^2 + a\ddot{a})F - \frac{a^2}{2}f - 2a^2H\dot{F} - a^2\ddot{F} &= \kappa^2 a^2 P_M \\
\left(\frac{2\dot{a}^2 + a\ddot{a}}{a^2}\right) - \frac{1}{2}f - 2H\dot{F} - \ddot{F} &= \kappa^2 P_M \\
\left(\frac{a\ddot{a} - \dot{a}^2}{a^2} + 3\frac{\dot{a}^2}{a^2}\right)F - \frac{1}{2}f - 2H\dot{F} - \ddot{F} &= \kappa^2 P_M
\end{aligned}
$$

We now replace the third term of the LHS from (2.14) with

$$
-f/2 = 3H^2F - \frac{1}{2}FR + 3H\dot{F} - \kappa^2\rho_M
$$

Leading to

$$
\begin{aligned}
\dot{H}F + 3H^2F + 3H^2F - \frac{1}{2}FR + 3H\dot{F} - 2H\dot{F} - \ddot{F} &= \kappa^2(\rho_M + P_M) \\
\dot{H}F + 6H^2F - \frac{1}{2}F6(2H^2 + \dot{H}) + H\dot{F} - \ddot{F} &= \kappa^2(\rho_M + P_M)
\end{aligned}
$$

Rearranging we get the second modified Friedmann equation

$$-2\dot{H}F = \ddot{F} - H\dot{F} + \kappa^2(\rho_M + P_M) \tag{2.17}$$

This equation along with

$$3FH^2 = (FR - f)/2 - 3H\dot{F} + \kappa^2\rho_M \tag{2.18}$$

Determine the background dynamics of a flat FLRW universe, governed by an f(R) gravity model.

For comparison the standard Friedmann equations that determine the evolution of the universe in the concordance model are derived from the Einstein equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, in the same way taking the (00) and (ii) components, since the Ricci tensor and scalar are the same cause we use the FLRW metric. So the resulting equations are

$$H^2 = \frac{\kappa^2}{3}\rho \tag{2.19}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho + 3P) \tag{2.20}$$

And in both cases we have the continuity equation (2.11)

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{2.21}$$

2.2 Inflation

In modern cosmology inflation is used to solve a number of cosmological problems such as the horizon problem, the flatness problem and the origin of the primordial density fluctuations. In the concordance model inflation is produced by a scalar field. The energy density of this field is the dominant ingredient of the very early universe, some time after the symmetry breaking of the electroweak force. To produce inflation we use the *slow roll approximation* which is to assume that the field is slowly rolling down to the minimum of its potential. In this case the energy density of the inflaton remains almost constant and its pressure is equal to $P \simeq -\rho$, which violates the strong energy condition i.e. $\rho + 3P < 0$ and we have accelerated expansion of the universe. However this picture is inherited with various inconsistencies. For example it leads to the production of certain topological defects and it requires significant fine tuning, since the initial value of the field should be carefully chosen for inflation to begin. As an alternative to this subtleties we will examine how inflation can be realised in f(R) gravity.

2.2.1 Inflation from scalar field

Before looking into the f(R) case, we will give an overview of the inflation theory in the the LCDM model.

The standard cosmological model gives a picture of the sequence of the various epochs of the early universe, implied by the Big Bang theory. The inflationary epoch, generated by a single scalar field ϕ called inflaton[1], starts right after the breaking of the $SU(3) \times SU(2) \times U(1)$ symmetry of the fundamental forces, when the universe was 10^{-36}sec old and lasted until 10^{-32}sec. At that period the universe's energy density is dominated by that of inflaton which drives the inflation and increases the volume of the universe by 10^{78} times, during this small fraction of cosmic time.

In order to have accelerated expansion (i.e. $\ddot{a} > 0$) and inflation we can see from (2.20) that the strong energy condition should be violated, instead of that we must have

$$(\rho + 3P) < 0 \tag{2.22}$$

In the perfect fluid approximation we have an equation of state $p = \omega\rho$ for each possible ingredient of the universe determined by the parameter of

[1] There are a number of theories that use more than one scalar fields to produce inflation, like the hybrid inflation that uses two scalar fields.

the state ω[2]. Therefore the acceleration condition becomes

$$\rho(1+3\omega) < 0 \Rightarrow \omega < -1/3 \tag{2.23}$$

In the following we will examine how to get exponential expansion from the inflaton.

The Lagrangian of a scalar field is with a potential $V(\phi)$

$$L = \frac{1}{2} g_{\mu\nu} \partial^{\mu}\phi \partial^{\nu}\phi - V(\phi) \tag{2.24}$$

Under a general coordinate translation $x^{\mu} \rightarrow x^{\mu} + \alpha^{\mu}$ the Lagrangian remains constant $\delta L = 0$ and from Noether's theorem we have the conserved current

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial L}{\partial(\partial_{\mu}\phi)} \partial^{\nu}\phi - g^{\mu\nu} L \\ &= \partial^{\mu}\phi \partial\nu\phi - g^{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \partial_{\alpha}\phi \partial_{\beta}\phi - V(\phi) \right] \end{aligned} \tag{2.25}$$

This equation defines the energy-momentum tensor for the inflaton field. Equating this with the energy-momentum tensor for a perfect fluid along with the assumption that ϕ is spatially homogeneous ignoring thus the gradients $(\nabla\phi)$, it will give us the energy density and pressure for the inflaton.

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.26}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.27}$$

We can get the equation of motion of the inflaton by replacing the above into (2.21).

For that we need the time derivative of ρ

$$\dot{\rho} = \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = \dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi}$$

and we also need the sum of ρ with P

$$\rho + P = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - V(\phi) = \dot{\phi}^2$$

[2]For example radiation has $\omega = 1/3$ and dust has $\omega = 0$

So the energy conservation equation (2.21) gives

$$\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi} + \dot{\phi}^2 = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{2.28}$$

This equation determines the evolution of the scalar field during inflation. In addition we have the two Friedmann equations taking the form

$$H^2 = \frac{1}{3\bar{m}_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{2.29}$$

$$\dot{H} + H^2 = -\frac{1}{3\bar{m}_p^2}\left(\dot{\phi}^2 - V(\phi)\right) \tag{2.30}$$

Which we wrote in this way, with $\dot{H}+H^2 = \frac{\ddot{a}}{a}$ and $\bar{m}_p^2 = \frac{1}{\kappa^2}$ being the reduced Planck mass, for convenience in our later discussion.

Now we can see from (2.22) that a primary condition for accelerated expansion, is the kinetic energy of the scalar field to be less than the potential energy $1/2\dot{\phi}^2 < V(\phi)$. It is therefore the shape of the potential and its dominance over kinetic term are significant in sustaining inflation, when these terms become comparable inflation stops.

A very useful approach on inflation, which is based exactly on that, is the slow roll approximation, in which inflation occurs if the evolution of the field is gradual that the potential dominates the kinetic energy, and the second derivative of ϕ is small enough to maintain this state for a sufficient period, i.e. we want

$$\dot{\phi}^2 \ll V(\phi)$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V'(\phi)|$$

Satisfying these conditions requires the smallness of two dimensionless quantities, known as slow roll parameters

$$\epsilon = \frac{1}{2}\bar{m}_p^2\left(\frac{V'}{V}\right)^2 \tag{2.31}$$

$$\eta = \bar{m}_p^2\frac{V''}{V} \tag{2.32}$$

So when $\epsilon, \eta \ll 1$ the field has the chance to roll down slowly and inflation can occur. These conditions are not sufficient, since one can choose initial conditions with $|1/2\dot{\phi}^2|$ so large that slow roll is not applicable. However most of the initial conditions are attracted to an inflationary phase if the slow roll parameters are small.

The usefulness of this approximation is that we can neglect the terms $1/2\dot{\phi}^2$ and $\ddot{\phi}$. Thus the energy density the pressure and the equation of motion of the inflaton along with the Friedmann equation become respectively:

$$\rho \simeq V(\phi) \tag{2.33}$$
$$P \simeq -V(\phi) \tag{2.34}$$
$$3H\dot{\phi} \simeq -V'(\phi) \tag{2.35}$$
$$H^2 \simeq \frac{1}{3\bar{m}_p^2}V(\phi) \tag{2.36}$$

Even though the smallness of ϵ and η is not a sufficient condition for slow roll to be applicable. We can show that the condition $\epsilon \ll 1$ is sufficient to produce accelerated expansion i.e. $\ddot{a}/a > 0$ or

$$\dot{H} + H^2 > 0 \Rightarrow -\frac{\dot{H}}{H^2} < 1 \tag{2.37}$$

To see that this inequality is equivalent with $\epsilon \ll 1$, we start differentiating (2.36) and using (2.35) and $V \simeq 3\bar{m}_p^2 V(\phi)H^2$ to get

$$\begin{aligned}
2H\dot{H} &= \frac{1}{3\bar{m}_p^2}\dot{\phi}V'(\phi) \\
2H\dot{H} &= -\frac{1}{3\bar{m}_p^2}\frac{V'}{3H}V' \\
2\dot{H} &= -\frac{(V')^2}{3\bar{m}_p^2 3H^2} \\
2\dot{H} &= -\frac{(V')^2}{3V} \\
-\frac{\dot{H}}{H^2} &= \frac{1}{2}\bar{m}_p^2\left(\frac{V'}{V}\right) = \epsilon
\end{aligned} \tag{2.38}$$

(2.39)

The equality of these two quantities shows that $\epsilon \ll 1$ is equivalent to accelerated expansion. A result that we will use in the next section as well.

As a simple example we can use the potential of a non interacting scalar field with mass m

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{2.40}$$

It easy to show that in this case the two slow roll parameters are the same

$$\epsilon = \eta = \frac{2\bar{m}_p^2}{\phi^2} \tag{2.41}$$

Thus we only have a unique condition for a valid slow roll determining the initial value of the field

$$\epsilon \ll 1 \Rightarrow \bar{m}_p^2 \ll \phi/2 \tag{2.42}$$

Under this condition the approximate motion and Friedmann equations (2.35),(2.36) become

$$3H\dot{\phi} = -m^2\phi \tag{2.43}$$

$$H^2 = \frac{1}{6\bar{m}_p^2} m^2 \phi^2 \tag{2.44}$$

This is a system of two differential equations that we can solve and calculate the evolution of the field and the scale factor during inflation.

Writing the Hubble parameter as $H = \frac{m\phi}{\sqrt{6}\bar{m}_p}$ and replacing in the equation of motion we have the evolution of the inflaton.

$$\dot{\phi} = \frac{\sqrt{6}\bar{m}_p}{3m}$$

$$\phi = \phi_i - \frac{\sqrt{6}\bar{m}_p}{3m} t \tag{2.45}$$

Plugging this in the expression for the Hubble parameter we find

$$\frac{\dot{a}}{a} = \frac{m}{\sqrt{6}\bar{m}_p} \left(\phi_i - \frac{\sqrt{6}\bar{m}_p}{3m} t \right)$$

$$a = a_i exp \left[\frac{m}{\sqrt{6}\bar{m}_p} \left(\phi_i t - \frac{\sqrt{6}\bar{m}_p}{3m} t^2 \right) \right] \tag{2.46}$$

Which of course shows that during inflation the scale factor grows exponentially.

Inflation will cease when the slow roll condition is no more satisfied i.e. when $\epsilon \sim 1$.

Finally, it is possible for someone to calculate the amount of inflation through the increase of the scale factor a. Using the fact that during inflation we have exponential growth of a this implies:

$$a(t) = a_i exp(Ht) \Rightarrow ln \left(\frac{a(t)}{a_i} \right) = Ht \tag{2.47}$$

As a measure for the amount of inflation we use the number of e-foldings from the start to the end of the inflationary period i.e. which in our case is

$$N \equiv \int ln \left(\frac{a(t_f)}{a(t_i)} \right) da = \int_{t_i}^{t_f} H dt \tag{2.48}$$

We can write this in terms of the initial and final values of ϕ

$$N = \int_{t_i}^{t_f} H \frac{dt}{d\phi} d\phi = \int_{t_i}^{t_f} \frac{H}{\dot{\phi}} d\phi = -\frac{1}{\bar{m}_p^2} \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi \tag{2.49}$$

Where for the ratio $H/\dot{\phi}$ we have divided (2.36) by (2.35). In any case under slow roll we have

$$N \simeq \frac{1}{\bar{m}_p^2} \int_{\phi_f}^{\phi_i} \frac{V}{V'} d\phi \tag{2.50}$$

For our example of $V = 1/2m^2\phi^2$ this gives

$$N = \frac{1}{\bar{m}_p^2} \int_{\phi_f}^{\phi_i} \frac{\phi}{2} = \frac{1}{2\bar{m}_p^2} (\phi_i^2 - \phi_f^2) \tag{2.51}$$

And since inflation stops when $\epsilon = 2\bar{m}_p^2/\phi^2$ is of the order of unity we can approximately say that the final value of inflaton field is

$$\phi_f^2 \simeq 2\bar{m}_p^2 \tag{2.52}$$

And so the number of e-foldings is determined by the initial value of the field

$$N \simeq \frac{1}{2\bar{m}_p^2} \phi_i^2 - 1 \tag{2.53}$$

For an inflationary model to be successful in terms of solving the horizon, flatness and monopole problems of the big bang model we require that $N \gtrsim 70$ and from this requirement we can determine a lower limit for the initial value of the field.

2.2.2 f(R) inflation: the general case

In contrast with the previous picture of a scalar field leading to inflation, in this case inflation arises as a consequence of the model.

We consider models of the form[3]

$$f(R) = R + bR^n, \tag{2.54}$$

[3]I will use the letter b for the coefficient rather than the usual a, so not to confuse it with the scale factor

with $(b > 0, n > 0)$

In this case the derivative of f with respect to R becomes

$$F = \frac{\partial}{\partial R} f(R) \tag{2.55}$$

$$= \frac{\partial}{\partial R}(R + bR^n) \tag{2.56}$$

$$= 1 + nbR^{n-1} \tag{2.57}$$

To see what form the field equation takes, we need the following product

$$FR = (R + nbR^n) \tag{2.58}$$

Along with the time derivative of F

$$\dot{F} = \frac{\partial}{\partial t} F = \frac{\partial}{\partial t}(1 + nbR^{n-1})$$

$$= n(n-1)bR^{n-2}\dot{R} \tag{2.59}$$

Replacing that in (2.14), the first modified Friedmann equation takes the form

$$3(1 + nbR^{n-1})H^2 = \frac{1}{2}(R + nbR^n - R - bR^n) - 3Hn(n-1)1 + nbR^{n-1}$$

$$3(1 + nbR^{n-1})H^2 = \frac{1}{2}(n-1)bR^n - 3Hn(n-1)1 + nbR^{n-1} \tag{2.60}$$

Cosmic acceleration can be realised in the regime $F = 1 + nbR^{n-1} \gg 1$. So dividing (2.60) with $3nbR^{n-1}$, under the approximation $F \simeq nbR^{n-1}$ gives

$$H^2 = \frac{1}{6}\left[\frac{(n-1)bR^n}{nbR^{n-1}}\right] - 3H\frac{n(n-1)bR^{n-2}\dot{R}}{3nbR^{n-1}}$$

$$= \frac{1}{6}\frac{n-1}{n}R - H\frac{\dot{R}}{R} \tag{2.61}$$

$$= \frac{n-1}{6n}\left(R - 6nH\frac{\dot{R}}{R}\right) \tag{2.62}$$

From (2.7) the time derivative of R is

$$\dot{R} = \frac{\partial}{\partial t}[6(2H^2 + \dot{H})]$$

$$= 6(4H\dot{H} + \ddot{H}) \tag{2.63}$$

So we have the quantity

$$
\begin{aligned}
\frac{\dot{R}}{R} &= \frac{6(4H\dot{H}+\ddot{H})}{6(2H^2+\dot{H})} \\
&= \frac{H\dot{H}\left(4+\frac{\ddot{H}}{H\dot{H}}\right)}{H^2\left(2+\frac{\dot{H}}{H^2}\right)} \\
&= \frac{4H\dot{H}}{2H^2} = 2\frac{\dot{H}}{H}
\end{aligned}
$$

Where we have used the approximations $|\frac{\dot{H}}{H^2} \ll 1|$ and $|\frac{\ddot{H}}{H\dot{H}} \ll 1|$ that are valid during inflation where the Hubble parameter H evolves slowly. In the end (2.14) can be written in the form

$$
\begin{aligned}
H^2 &= \frac{n-1}{6n}\left(6(2H^2+\dot{H}) - 6nH2\frac{\dot{H}}{H}\right) \\
H^2 &= \frac{n-1}{6n}\left(6(2H^2+\dot{H})2n\dot{H}\right) \\
H^2 &= \frac{n-1}{n}\left[2-(2n-1)\frac{\dot{H}}{H^2}\right] \\
\frac{n}{n-1} &= 2-(2n-1)\frac{\dot{H}}{H^2} \\
2-\frac{n}{n-1} &= (2n-1)\frac{\dot{H}}{H^2} \\
\frac{\dot{H}}{H^2} &= \frac{2(n-1)-n}{(n-1)(2n-1)} \\
&= \frac{n-2}{(n-1)(2n-1)}
\end{aligned}
$$

Or in other words

$$\frac{\dot{H}}{H^2} \simeq -\epsilon \tag{2.64}$$

Where $\epsilon = \frac{2-n}{(n-1)(2n-1)}$ is the equivalent slow roll parameter for our f(R) model. Integrating for positive ϵ

$$\frac{\frac{dH}{dt}}{H^2} = -\epsilon \Rightarrow \frac{dH}{H^2} = -\epsilon dt \Rightarrow \frac{1}{H} = \epsilon t \Rightarrow H = \frac{1}{\epsilon t} \tag{2.65}$$

Which means that the scale factor evolves as

$$\frac{\dot{a}}{a} = \frac{1}{\epsilon t} \Rightarrow \frac{da}{a} = \frac{1}{\epsilon t} dt \Rightarrow lna = \frac{1}{\epsilon} lnt = lnt^{\frac{1}{\epsilon}} \Rightarrow a \propto t^{\frac{1}{\epsilon}}$$

$$\Rightarrow a \propto t^{\frac{1}{\epsilon}} \tag{2.66}$$

Inflation is equivalent to a rapid expansion of the universe in a small cosmological period. So according to the previous result inflation occurs when $\epsilon < 1$, which means that

$$\frac{2-n}{(n-1)(2n-1)} < 1$$
$$2 - n < 2n^2 - 3n + 1$$
$$2n^2 - 2n - 1 > 0$$
$$2[n - \frac{1}{2}(1-\sqrt{3})][n - \frac{1}{2}(1+\sqrt{3})] > 0$$

$$n > \frac{1}{2}(1+\sqrt{3}) \tag{2.67}$$

For $n = 2$ we have $\epsilon = 0$ so that the Hubble parameter is constant in the regime $F \gg 1$. Models with $n > 2$ lead to super inflation with $\dot{H} > 0$. So the standard inflation with decreasing H could be achieved in models with $\frac{1}{2}(1+\sqrt{3}) < n < 2$.

2.2.3 Starobinsky's inflation

As a simple example we will examine the Starobinsky model. As we said this was the first model of modified gravity, that was used as an origin of inflation other than the usual scalar field option. The model is determined by the function

$$f(R) = R + \frac{R^2}{6M^2} \tag{2.68}$$

So we have a case with $n = 2$ and $b = \frac{1}{6M^2}$, where M is a constant with dimensions of mass.

$$R^n = R^2 = [6(2H^2 + \dot{H})]^2 = 36(4H^4 + \dot{H}^2 + 4H^2\dot{H}) \tag{2.69}$$

$$\frac{\partial}{\partial t}[6(2H^2 + \dot{H})] = 6(4H\dot{H} + \ddot{H}) \tag{2.70}$$

Replacing in (2.60)

$$
\begin{aligned}
3(1+2\frac{1}{6M^2}R)H^2 &= \frac{1}{2}\frac{1}{6M^2}R^2 - 6\frac{1}{6M^2}H\dot{R} \\
3(1+2\frac{1}{3M^2}6(2H^2+\dot{H}))H^2 &= \frac{1}{12M^2}36(4H^4+\dot{H}^2+4H^2\dot{H}) - \frac{1}{M^2}H6(4H\dot{H}+\ddot{H}) \\
3M^2H^2+12H^4+6\dot{H}H^2 &= 12H^4+3\dot{H}^2+12\dot{H}H^2-24\dot{H}H^2-6H\ddot{H} \\
3M^2H^2+6H\ddot{H}+18H^2\dot{H}-3\dot{H}^2 = 0
\end{aligned}
$$

Dividing the last line with $6H$ we get

$$
\begin{aligned}
\ddot{H}+\frac{1}{2}M^2H+3H\dot{H}-\frac{1}{2}\frac{\dot{H}^2}{H} &= 0 \\
\ddot{H}-\frac{1}{2}\frac{\dot{H}^2}{H}+\frac{1}{2}M^2H &= -3H\dot{H} \qquad (2.71)
\end{aligned}
$$

As we said during inflation the Hubble parameter evolves very slowly, which means that the first two terms in (2.71) are negligible, allowing us to write $\dot{H} \simeq -M^2/6$, solving this one we get

$$
\begin{aligned}
\int_{H_i}^{H} dH &\simeq -\int_{i}^{t}(M^2/6)dt \\
H &\simeq H_i - (M^2/6)(t-t_i) \qquad (2.72) \\
\int_{a_i}^{a}\frac{da}{a} &\simeq \int_{t_i}^{t}H_i dt - \int_{0}^{t-t_i}(M^2/6)(t-t_i)d(t-t_i) \\
a &\simeq a_i exp[H_i(t-t_i)-(M^2/12)(t-t_i)^2] \qquad (2.73)
\end{aligned}
$$

With H_i and a_i being the Hubble parameter and the scale factor at the beginning of inflation ($t = t_i$) and

$$R \simeq 12H^2 - M^2 \qquad (2.74)$$

Inflation continues as long as the slow rolling condition is satisfied

$$
\begin{aligned}
\epsilon = -\frac{\dot{H}}{H^2} \simeq \frac{M^2}{6H^2} &\lesssim 1 \\
\Rightarrow H^2 &\gtrsim M^2 \qquad (2.75)
\end{aligned}
$$

At time $(t = t_f)$ we have the slow roll parameter becoming of the order of unity $\epsilon \simeq 1$ and inflation stops. So at the end of inflation from (2.64) we see

$$\frac{\dot{H}_f}{H^2} \simeq 1 \Rightarrow H_f \simeq M/\sqrt{6} \tag{2.76}$$

Plugging this into (2.72) we have that

$$\begin{aligned} M/\sqrt{6} &\simeq H_i - (M^2/6)(t_f - t_i) \\ (M^2/6)t_f &\simeq H_i + (M^2/6)t_i - M/\sqrt{6} \\ t_f &\simeq \frac{6H_i}{M^2} + t_i \end{aligned} \tag{2.77}$$

Where we have discarded a term $\sqrt{6}/M$ since the latest measurements from WMAP constrain the mass scale to be very small $M \simeq 13$Gev

And from (2.74) at this epoch the Ricci scalar decreases to $R \simeq M^2$.

Using the result from (2.77) the number of e-foldings in our model is equal to

$$N \simeq H_i \frac{6H_i}{M^2} - \frac{M^2}{12}\left(\frac{6H_i}{M^2}\right)^2 \simeq \frac{3H_i^2}{M^2} \simeq \frac{1}{2\epsilon(t_f)} \tag{2.78}$$

Where in the last equality we used $\epsilon \simeq M^2/6H^2$. We already mentioned that we need at least $N \gtrsim 70$, to solve the horizon and monopole problems. Which means that $\epsilon(t_f) \lesssim 7 \times 10^{-3}$.

2.3 Generating Dark Energy in metric f(R)

We already mentioned that the existence of Dark Energy in the universe is supported by two of the most significant modern cosmological observational facts. The first one is the composition of the universe, which shows that Dark Energy constitutes 72% of the total energy density. And the second one is the strong indications coming mainly from the type Ia Supernovae (SN Ia) observations that the universe is currently undergoing an accelerated expansion phase, which has followed that of the matter epoch.

In order to have accelerated expansion (i.e. $\ddot{a} > 0$) we sawe from the second Friedmann equation of classic G.R.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.79}$$

We need some form of energy-matter approximated as a perfect fluid with an equation of state, determined by the parameter ω,that would violate the strong energy condition.

The simplest candidate for Dark Energy is the Cosmological Constant Λ, which is considered to be equivalent with the vacuum energy of particle physics. And the reason is exactly that its equation of state is $\omega = -1$ which obviously violates the above condition and leads to accelerated growth of the scale factor.

However there exist the problems of coincidence and magnitude attached to this picture. Specifically for the magnitude problem if the origin of the cosmological constant is a vacuum energy, then a huge disagreement arises from the comparison between the calculated energy scale of the vacuum relative to the Dark Energy density today. The zero point energy of some field of mass m with momentum k and frequency ω is given by $E = \omega/2 = \sqrt{k^2 + m^2}/2$. Summing over the zero-point energies of this field up to a cut-off scale $k_{max}(\gg m)$, we obtain the vacuum energy density

$$\rho_{vac} = \int_0^{k_{max}} \frac{d^3k}{(2\pi)^3} \frac{1}{2}\sqrt{k^2 + m^2} \tag{2.80}$$

Since the integral is dominated by the mode with large $k(\gg m)$, we find that

$$\rho_{vac} \approx \int_0^{k_{max}} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} k = \frac{k_{max}^4}{16\pi^2} \tag{2.81}$$

Taking the cutoff scale k_{max} to be the Planck mass m_{pl}, the vacuum energy density can be estimated as $\rho_{vac} \simeq 10^{74}\text{GeV}^4$. This is about 10^121 times larger than the observed value $\rho_{DE}^{(0)} \simeq 10^{-47}\text{GeV}^4$

So, as we said this particular discrepancy, along with the *coincidence* problem, have strongly motivated the search for other explanations about the origin of Dark Energy.

2.3.1 The effective parameter of state

The way to realise Dark Energy in f(R) models is to define an effective state parameter and to seek the kind of models that their resulting ω_{eff} satisfies (2.23).

$$\omega_{eff} < -\frac{1}{3}$$

At first we can write the metric field equations (1.14) in a more standarised form, with the Einstein tensor on the left hand side, as:

$$G_{\mu\nu} \equiv R_{\mu\nu} - 1/2 g_{\mu\nu} R = \kappa^2 \left(T_{\mu\nu}^{(M)} + T_{\mu\nu}^{(curv)} \right) \tag{2.82}$$

Where $G_{\mu\nu} \equiv R_{\mu\nu} - 1/2 g_{\mu\nu} R$ and

$$\kappa^2 T_{\mu\nu}^{(curv)} = g_{\mu\nu}(f-R)/2 + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F + (1-F) R_{\mu\nu} \tag{2.83}$$

Since $\nabla^\mu G_{\mu\nu} = 0$ and $\nabla^\mu T_{\mu\nu}^{(M)} = 0$, it follows that

$$\nabla^\mu T_{\mu\nu}^{(curv)} = 0 \tag{2.84}$$

Thus the continuity equation holds equally well for this effective energy-momentum tensor $T_{\mu\nu}^{(curv)}$.

So what we do in this way is to separate the usual matter tensor, coming from the variation of the matter action, from the rest of the field equations which we treat as an effective energy tensor which essentially is an additional curvature source determined by each f(R) model and can be used to describe the dark energy presence in the universe.Since we use for our other sources the perfect fluid approximation, we can do the same for this curvature source, i.e. $T_\nu^{\mu(curv)} = diag(-\rho_{curv}, P_{curv}, P_{curv})$. Then we can define the effective parameter of state as

$$\omega_{eff} = \frac{P_{curv}}{\rho_{curv}} \tag{2.85}$$

The continuity equation in this case leads to

$$\dot{\rho}_{curv} + 3H(\rho_{curv} + 3P_{curv}) = 0 \tag{2.86}$$

Inserting the effective equation of state for the geometric fluid then we have

$$\frac{\dot{\rho}_{curv}}{\rho_{curv}} = -3H(1+\omega_{curv}) \tag{2.87}$$

We can deduce the effective energy density either by (2.83) or from the first modified Friedmann equation (2.14) after rewriting it as

$$H^2 = \frac{1}{3f'}[\kappa^2\rho_M + (f'R-f)/2 - 3H\dot{F}] \tag{2.88}$$

So if we compare this with (2.19) we see that it is in the form

$$H^2 = \frac{\kappa^2}{3f'}\rho_M + \frac{\kappa^2}{3}\rho_{eff}$$

Where the effective energy density is equal

$$\begin{aligned}\rho_{eff} &= \frac{1}{f'}\left[(f'R-f)/2 - 3H\dot{f}'\right] \\ &= \frac{1}{f'}\left[(f'R-f)/2 - 3Hf''\dot{R}\right]\end{aligned} \tag{2.89}$$

There are to ways to get the effective state parameter. The first is from the equation of state $\omega_{eff} = \rho_{curv}/P_{curv}$, using (2.83) to find P_{curv}. Or we can use (2.87)

$$\omega_{eff} = -1 - \frac{1}{3H}\frac{\dot{\rho}_{curv}}{\rho_{curv}} \tag{2.90}$$

Following the last way, we have to take the time derivative of ρ_{curv} which is equal to

$$\begin{aligned}\dot{\rho}_{curv} &= \frac{d}{dt}\left[\frac{1}{f'}\left[(f'R-f)/2 - 3Hf''\dot{R}\right]\right] \\ &= -\frac{1}{f'^2}\left(\frac{1}{2}(f''\dot{R}R + f'\dot{R} - f'\dot{R}) - 3\dot{H}f''\dot{R} - 3Hf'''\dot{R}^2 - 3Hf''\ddot{R}\right) \\ &= -\frac{1}{f'^2}\left(\frac{1}{2}f''\dot{R}6(\dot{H}+2H^2) - 3\dot{H}f''\dot{R} - 3Hf'''\dot{R}^2 - 3Hf''\ddot{R}\right) \\ &= -\frac{1}{f'^2}(6H^2f''\dot{R} - 3Hf'''\dot{R}^2 - 3Hf''\ddot{R})\end{aligned} \tag{2.91}$$

Dividing the last with $3H$ yields

$$\dot{\rho}_{curv} = \frac{1}{f'^2}(\ddot{R}f'' + \dot{R}(f'''\dot{R} - 2Hf'')) \tag{2.92}$$

Substituting this result in (2.90) then we see that the effective state parameter is

$$\omega_{eff} = -1 + \frac{1}{f'} \frac{\ddot{R}f'' + \dot{R}(f'''\dot{R} - 2Hf'')}{(f'R - f)/2 - 3Hf''\dot{R}} \tag{2.93}$$

There is a number of f(R) models that give an ω_{eff} that satisfies the condition (2.23) and produce late time accelerated expansion, as stated in [14].

An example is models with $f(R) = f_0 R^n$, where n is a real number. If we assume that the scale factor grows as a generic power law $a = a_0(t/t_0)^\alpha$ then ω_{eff} as a function of n works out to be

$$\omega = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \tag{2.94}$$

For $n \neq 1$, which is the limit where f(R) reduce to the usual Einstein gravity, and the parameter α is given in terms of n from

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2} \tag{2.95}$$

Obviously there are many values of n that give the desired ω_{eff}. For instance, $n = 2$ which is the Starobinsky model for inflation studied in the previous section, gives $\omega_{eff} = -1$ and $\alpha = \infty$ as expected.

Power-law solutions for $a(t)$ can be found in this family of models, that provide a satisfactory fit to SNeIa data and a good agreement with the estimated age of the Universe in the range $1.366 < n < 1.376$

An additional example is models of the form $f(R) = R - \mu^{2(n+1)}/R^n$, where μ is a suitable parameter. Again if we assume power law solutions for the scale factor then the effective state parameter in this case is

$$\omega_{eff} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)} \tag{2.96}$$

The most typical model of these is $f(R) = R - \mu^4/R$, which has $\omega_{eff} = -2/3$

The difference between these two examples is that in the later we have terms inversely proportional to R for positive n, contrary to the case of R^n. This is important because in the late time acceleration, we want the extra curvature term to be negligible at early times and dominate at the late stages of cosmic evolution. Consequently models with terms $1/R^n$ are more suitable for Dark energy, since these terms are negligible relative to R, at the early universe when curvature is bigger and become important when $R \to 0$ at late times.

2.3.2 Dark Energy using dynamical equations

There is a more general method addressing the problem of Dark Energy, introduced in detail in [1] and discussed in chapter 4 of [6], which we are going to mention in order to give a complete overview of the case.

Considering non relativistic matter and radiation, whose energy densities ρ_m and ρ_r satisfy

$$\dot{\rho}_m + 3\rho_m = 0 \tag{2.97}$$

$$\dot{\rho}_r + 4H\rho_r = 0 \tag{2.98}$$

The starting point of this method is the two modified Friedmann equations (2.14) and (2.17)

$$3FH^2 = (FR - f)/2 - 3H\dot{F} + \kappa^2(\rho_m + \rho_r) \tag{2.99}$$

$$-2\dot{H}F = \ddot{F} - H\dot{F} + \kappa^2[\rho_m + 4/3\rho_r] \tag{2.100}$$

For a general f(R) model we introduce the following dynamical variables

$$x_1 \equiv -\tfrac{\dot{F}}{HF}, \quad x_2 \equiv -\tfrac{f}{6FH^2}, \quad x_3 \equiv \tfrac{R}{6H^2}, \quad x_4 \equiv \tfrac{\kappa^2\rho_r}{3FH^2}$$

Also here it turns out to be convenient to work with the density parameter

$$\Omega_X \equiv \frac{\kappa_X^2}{3FH^2} \tag{2.101}$$

Where $X = m, rad$ or DE

It is then obvious from (2.99) that we have

$$\Omega_M \equiv \frac{\kappa^2}{3FH^2} = 1 - x_1 - x_2 - x_3 - x_4 \tag{2.102}$$

Together with the density parameters for radiation and Dark Energy.

$$\Omega_r \equiv x_4, \quad \Omega_{DE} \equiv x_1 + x_2 + x_3$$

Using the form of the equations of motion and the above density parameters, it is straightforward to derive the following system of equations concerning the previously defined dynamical parameters.

$$\frac{dx_1}{dN} = -1 - x_3 - 3x_2 + x_1^2 - x_1x_3 + x_4$$

$$\frac{dx_2}{dN} = \frac{x_1x_3}{m} - x_2(2x_3 - 4 - x_1)$$

$$\frac{dx_3}{dN} = \frac{x_1x_3}{m} - 2x_3(x_3 - 2)$$

$$\frac{dx_4}{dN} = -2x_3x_4 + x_1x_4$$

Where $N = lna$ is the number of e-foldings, and

$$m \equiv \frac{dlnF}{dlnR} = \frac{Rf''(R)}{f'(R)}$$
$$r \equiv -\frac{dlnf}{dlnR} = -\frac{Rf'(R)}{f} = \frac{x_3}{x_2}$$

Additionally we have in this case the effective equation of state defined as

$$\omega_{eff} \equiv -1 - 2\dot{H}^2/(3H^2) = -(2x_3 - 1)/3 \tag{2.103}$$

It is possible now to find the values of the variables x_1, x_2, x_3 and x_4, that model dependent.

The fixed points of the system correspond to models of f(R) that describe universes with different properties and some of them have properties, like an effective equation of state, that match with the real universe we live in.

So, in the absence of radiation ($x_4 = 0$) the fixed points of the above dynamical system are

$$\begin{array}{lll} P1 : (x_1, x_2, x_3) = (0, -1, 2) & \Omega_m = 0 & \omega_{eff} = -1 \\ P2 : (x_1, x_2, x_3) = (-1, 0, 0) & \Omega_m = 2 & \omega_{eff} = 1/3 \\ P3 : (x_1, x_2, x_3) = (1, 0, 0) & \Omega_m = 0 & \omega_{eff} = 1/3 \\ P4 : (x_1, x_2, x_3) = (-4, 5, 0) & \Omega_m = 0 & \omega_{eff} = 1/3 \\ P5 : (x_1, x_2, x_3) = (\frac{3m}{1+m}, -\frac{1+4m}{2(1+m)^2}, \frac{1+4m}{2(1+m)}) & \Omega_m = 1 - \frac{m(7+10m)}{2(1+m)^2} & \omega_{eff} = -\frac{m}{1+m} \\ P6 : (x_1, x_2, x_3) = (\frac{2(1-m)}{1+2m}, \frac{1-4m}{m(1+2m)}, -\frac{(1-4m)(1+m)}{m(1+2m)}) & \Omega_m = 0 & \omega_{eff} = \frac{2-5m-6m^2}{3m(1+2m)} \end{array}$$

Examining theses six points we have the interesting remarks

- The matter-dominated epoch ($\Omega_m \simeq 1$ and $\omega_{eff} < 0$) can be realized only by the point P5 for m close to 0.
- Either the point P1 or P6 can be responsible for the late-time cosmic acceleration. The former is a de Sitter point ($\omega_{eff} = -1$) with $R = -2$, in which case the condition (1.18) is satisfied.
- The point P6 can give rise to the accelerated expansion ($\omega_{eff} < -1/3$) provided that $m > (\sqrt{3}-1)/2$, or $-1/2 < m < 0$, or $m < -(1+\sqrt{3})/2$.

Discussion

All in all, what we have shown is simplified cases in which metric f(R) gravity can produce the desired characteristics of the universe for which the concordance model explanations are inherited with various inconsistencies and motivate the search for other theories. We have seen in particular how inflation and late time acceleration, attributed to dark energy, can be realised in various f(R) models.

However apart from providing alternatives for these inconsistencies, a modified theory of gravity should reproduce the well established theoretical results and provide a satisfactory fit to the observations where General Relativity has been proven triumphant. Among others, it should pass the tests imposed by Solar System and terrestrial experiments on relativistic gravity, it should give a coherent picture of the universe with an inflationary phase followed by radiation and matter era, where we have the formulation of the light elements and the evolution of the initial perturbations to the Large Scale structure, leading to a period of late time acceleration. Finally it should also be consistent with other fundamental physical theories, such as quantum field theories that describe the rest of the natural forces.

All this means that a viable f(R) model is subject of certain constraints. A summary of these constrains as given in [8] is

1. possess the correct cosmological dynamics
2. not suffer from instabilities, which is equivalent with $f_{,R}$ and $f_{,RR}$, and ghosts
3. have the correct Newtonian and post-Newtonian limit
4. give rise to cosmological perturbations compatible with the data from cosmic microwave background and large scale structure surveys
5. have a well posed Cauchy problem

So concerning the above constraints the current state of the various f(R) models is:

- **Metric f(R) gravity**: there are models that pass all the theoretical and observational constrains. The most popular of these are given in Table (3.1). The biggest problem is whether curvature singularities exist for relativistic strong field stars.

Table 2.1: Explicit forms of f(R) in (i) Hu-Sawicki, (ii) Starobinsky, (iii) Tsujikawa, and (iv) the exponential gravity models.

model	f(R)	constant parameters
i)	$R - \frac{c_1 R_{HS}(R/R_{HS})^p}{c_2(R/RHS)^p+1}$	$c_1, c_2, p(>0)$,$R_{HS}(>0)$
ii)	$R + \lambda R_S \left[\left(1+\frac{R^2}{R_S^2}\right)^{-n} - 1 \right]$	$\lambda(>0), n(>0), R_S$
iii)	$R - \mu R_T tanh \left(\frac{R}{R_T}\right)$	$\mu(>0), R_T(>0)$
iv)	$R - \beta R_E \left(1 - e^{-R/R_E}\right)$	β, R_E

- **Palatini f(R) gravity**: these theories failed for various reasons; they contain a nondynamical scalar field, the Cauchy problem is ill-posed, and discontinuities in the matter distribution generate curvature singularities.

- **Metric-affine gravity**: this class of theories is not yet sufficiently developed to assess whether it is viable according to the criteria listed here, and its cosmological consequences are unexplored.

Finally, a new way to test a theory of gravity on very Large scales, comes from the Chandra observation of galaxy clusters. A modified gravitational force should also affect the rate at which the initial density perturbations can grow during the evolution of the universe and become massive clusters of galaxies, providing a sensitive test of the theory. Recent studies on theoretical analysis of Chandra's cluster data[4] strongly support that Einstein's gravity, which until now was only tested from laboratory to Solar System scales, is also valid on scales larger that 130 million light years. In one of these studies they used Einstein's theory to calculate the number of massive clusters that have formed under the force of gravity during the last five billion years, which is in complete agreement with the observations.

Results like these strengthen the position of General Relativity, against opponent theories and place even tighter constrains, for modifications such as f(R) gravities, which they must pass if they are to be consider realistic.

In any case, even if f(R) is proven an unsatisfactory alternative is not at all ill-motivated, for it is certain that investigating f(R) models, will have provide a deeper insight in several aspects of Einstein's theory and its generalisations, and secure our view of General Relativity as the unique fundamental theory of gravity.

[4]See for example [12] and [11] for detail.

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