DISSERTATION

QUANTUM CORRECTION TO THE MASS OF THE 't HOOFT-POLYAKOV MONOPOLE

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Abstract

This dissertation is written in order to present some cases of calculations of the quantum corrections to the masses of low dimensional topological defects. The present document is focused on the cases of the *quantum kink* and especially the case of the *'t Hooft-Polyakov monopole*. In the first chapter we describe the semiclassical expansion in quantum mechanics and we apply this procedure in field theory, in the background of the trivial and kink solutions. Then we calculate the lowest order quantum correction to the mass of the kink in ϕ^4 in $1+1$ dimensions. In the second chapter we present the *Georgi-Glashow* model and its monopole solutions. In continuance we calculate the classical monopole mass at the *Bogomol'nyi-Prasad-Sommerfield (BPS)* limit. Finally at the end of this chapter we give the basic steps of the calculation of the one-loop correction to the monopole mass.

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Chapter 1

Quantization of static solutions

1.1 The semiclassical expansion

In this chapter we describe the semiclassical expansion [2, 13] by using the *correspondence principle*. This procedure relates quantum levels to classical orbits in a systematic approximation. In the following we generalize that procedure to field theory. Here we focus on the quantization of *static solutions*.

To begin with, we consider a non relativistic particle with unit mass, moving in one dimension under the influence of the potential $V(x)$. In *Classical Mechanics*, If we know the position x of the particle as a function of time then we can describe the particle. In order to obtain the position of the particle as function of time, we must solve the *Newton's equation*:

$$\frac{d^2x}{dt^2} = -\frac{dV}{dx} \tag{1.1}$$

In *quantum mechanics* particles are not described by giving their position x, but they are associated with a wavefunction which describes them. In particular, in order to determine their energy eigenstates we must solve the *Schrodinger's equation*:

$$H\Psi_n = (\frac{1}{2}\hat{p}^2 + V(x))\Psi_n(x) = E_n\Psi_n(x) \tag{1.2}$$

where $\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$ is the momentum operator. In the following, we give a relationship between the classical static solutions (obtained from Newton's equation) and the quantum energy eigenstates (determined by the Schrodinger's equation).

- The static solutions of the classical equation of motion are those which extremize the potential $V(x)$, i.e they are the solutions of the equation $\frac{dV}{dx} = 0$. Static solutions with $\left.\frac{d^2V}{dx^2}\right|_{x=sol} > 0$ represent *stable* static solutions, while static solutions with $\left.\frac{d^2V}{dx^2}\right|_{x=sol} < 0$ represent *unstable* static solutions.

The *"classical ground state"* represents the lowest energy solution and if it corresponds to the static solution $x(t) = a = const$, then its total energy is $E_0^{cl} = V(a)$.

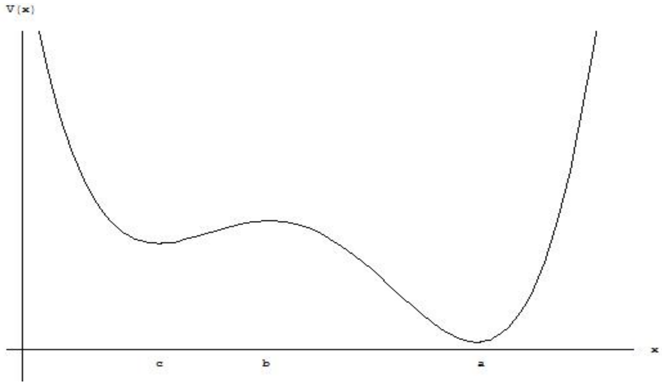

Figure 1.1: An example of a potential with two minima and one local maximum.

- In Quantum theory, the uncertainty principle does not allow the existence of such states. According to this principle the particle cannot have both zero momentum and a fixed position. Thus, even in the ground state (state with the lowest eigenvalue of energy) the particle will fluctuate around the classical static solution $x = a$. This fact gives to the ground state the amount of energy: $E_0 = E_0^{cl} + \Delta_0 = V(a) + \Delta_0$, where Δ_0 represents the *quantum correction* due to zero-point motion.

- If we consider an approximately harmonic potential in the vicinity of the classical static solution $x = a$, then we can perform a *"weak-coupling expansion"*. That is to take the Taylor expansion of the potential $V(x)$ around the classical static solution $x = a$. Let the expansion be [13]:

$$V(x) = V(a) + \frac{1}{2}\omega^2(x-a)^2 + \sum_{r=3}^{\infty} \frac{1}{r!}\lambda_r (x-a)^r \tag{1.3}$$

Then for those wavefunctions that satisfy the condition:

$$\lambda_r \left\langle (x-a)^r \right\rangle \ll \omega^2 \left\langle (x-a)^2 \right\rangle, \; r = 3, 4, ... \tag{1.4}$$

the effects of the anharmonic terms of $V(x)$ will be small. If we take λ_r to be sufficiently small, then the low-lying energy eigenstates whose spread is localized in the vicinity of the classical static solution $x = a$, will satisfy the previous condition. For these wavefunctions the potential will be dominantly that of a harmonic oscillator. The energies of these low-lying states [13] can be written as

$$E_n = V(a) + (n + \frac{1}{2})\hbar\omega + O(\lambda_r) \tag{1.5}$$

So the Quantum ground state in weak-coupling approximation becomes:

$$E_0 = V(a) + \frac{1}{2}\hbar\omega + O(\lambda_r) \tag{1.6}$$

- Equation (1.6) represent a relation between Quantum states and classical solutions. The left-hand side is the energy of the Quantum ground state. In right-hand side, the first term is the energy of the corresponding classical static solution. The second term represents the leading Quantum correction and with ω we denote the *classical stability frequency* of this solution.
  The $O(\lambda_r)$ corrections can be obtained by standard perturbation theory. These corrections will again involve only the constants λ_r and ω which give the derivatives of the potential at the classical solution $x = a$.

- In fact, not only the ground state, but the energies of a tower of low-lying states are similarly related to this classical solution through the relation (1.5).

- The full information about the state is contained in its wavefunction. Even though the ground state wavefunction $\Psi_0(x)$ has a spread in x, it will still be typically localized around the classical solution $x = a$. For instance, its position expectation value is given by:

$$\langle x \rangle \equiv \int x \left|\Psi_0(x)\right|^2 dx = a + ... \tag{1.7}$$

  where dots represent corrections due to the anharmonic constants λ_r and the quantity a above represents the classical solution.

- Now let us focus on the solution $x(t) = c = const$, which corresponds to an other stable classical static solution (local minimum of the potential $V(x)$). Classically this solution has energy $\widetilde{E}^{cl} = V(c)$ which is higher than $V(a)$. This solution is the analogue of the classical solitons in field theory. They are also static solutions but with higher energies than the classical vacua in the corresponding field theories.
  Even though $x = c$ is only a local minimum of $V(x)$, one can again attempt a *"weak-coupling"* approximation near it. Let the weak-coupling approximation be:

$$V(x) = V(c) + \frac{1}{2}\omega'^2(x-c)^2 + \sum_{r=3}^{\infty} \frac{\lambda'_r}{r!}(x-c)^r \tag{1.8}$$

- As before, if $\lambda_r^{'}$ are sufficiently small, then near the classical static solution $x = c$ the anharmonic effects will be small. One can try to construct a family of approximate harmonic oscillator states centered at $x = c$ with energies

$$\widetilde{E}_{n'} = V(c) + \hbar\omega^{'}(n^{'} + \frac{1}{2}) + O(\lambda_r^{'}) \tag{1.9}$$

  If the approximation was valid, the lowest of these states would have energy $\widetilde{E}_0 = V(c) + \frac{1}{2}\hbar\omega + O(\lambda_r^{'})$ and expectation value $\langle x \rangle = c + O(\lambda_r^{'})$.
  Thus we would again have an approximation to a set of energy eigenstates whose energy is related to $V(c)$ (the classical energy of a classical solution) and whose $\langle x \rangle$ is related to the solution $x = c$.

- Such perturbation theory starting from a harmonic oscillator approximation, when applied to the local minimum $x = c$, treats the potential well near $x = c$ as if the other deeper well near $x = a$ did not exist. Actually, we know that wave packets built in the potential well around $x = c$ will tunnel into the well around $x = a$ and vice versa. Consequently the two subsets of energy levels will mix, but if the λ_r and $\lambda_r^{'}$ are all small the two minima $x = a$ and $x = c$ will be widely separated. Therefore the tunneling will be slow and the resultant change in energy eigenvalues due to tunneling will be small. To any finite order in the weak-coupling expansion, the set of levels $\widetilde{E}_{n'} = V(c) + \hbar\omega^{'}(n^{'} + \frac{1}{2}) + O(\lambda_r^{'})$ around $x = c$ can be considered separately from the set $E_n = V(a) + (n + \frac{1}{2})\hbar\omega + O(\lambda_r)$ around $x = a$.

- It is obvious that there will not be a set of levels built around the static solution $x = b = const$, which corresponds to an unstable classical static solution (local maximum in the potential V(x)), because it is unstable. The corresponding frequency would be imaginary, since $\omega^2 = \frac{d^2V}{dx^2}\Big|_{x=b} < 0$.

- When the harmonic constant ω vanishes, this whole procedure runs into trouble. No matter how small the λ_r may be, the weak-coupling condition (1.4) cannot be satisfied.
  So when the potential is independent of a coordinate, the corresponding harmonic frequency ω will vanish and the semiclassical expansion outlined above has to be modified.

- One can associate a tower of energy eigenstates with each static stable classical solution. It should be emphasized that the relationships

$$\left.\begin{array}{l} E_n = V(a) + (n + \frac{1}{2})\hbar\omega + O(\lambda_r) \\ \langle x \rangle = \int x \left|\Psi_0(x)\right|^2 dx = a + ... \end{array}\right\} \tag{1.10}$$

and

$$\left.\begin{array}{l} E_{n'} = V(c) + (n' + \frac{1}{2})\hbar\omega' + O(\lambda_r') \\ \langle x \rangle = c + O(\lambda_r') \end{array}\right\} \qquad (1.11)$$

are valid only in the weak-coupling expansion, where the anharmonic terms are treated as small. Note that anharmonic terms in $V(x)$ also lead to non-linear terms in the equation of motion $\frac{d^2x}{dt^2} = -\frac{dV}{dx}$. Correspondingly, quantization of static solitons in field theory will be valid only when the non-linear couplings are small. Nevertheless, the results may be non-perturbative. This is because the classical solution itself may be non-perturbative. Its properties form the leading terms, as for instance in the expansions (1.11). It is only the quantum corrections which are treated perturbatively.

1.2 The semiclassical expansion in field theory

In this paragraph we will apply the procedure that we discussed above to field theory, by giving an example. To do this we consider a system described by the Lagrangian:

$$L = \int d^3x \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} (\nabla \phi)^2 - U(\phi) \right\} \qquad (1.12)$$

where $U(\phi)$ is bounded from below. The kinetic energy is given by:

$$T[\phi] \equiv \int d^3x \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 \qquad (1.13)$$

and the potential energy is given by:

$$V[\phi] \equiv \int d^3x \left\{ \frac{1}{2} (\nabla \phi)^2 - U(\phi) \right\} \qquad (1.14)$$

So the Lagrangian can be written as $L[\phi] = T[\phi] - V[\phi]$.

The equations of motion for the field ϕ are obtained from the *Euler-Lagrange* equations:

$$\frac{\partial^2 \phi(\vec{x}, t)}{\partial t^2} = -\frac{\delta V[\phi]}{\delta \phi} \qquad (1.15)$$

Note that now the coordinates of the system is the field $\phi(\vec{x})$, so the potential energy is a function of the function $\phi(\vec{x})$, i.e a functional.

We observe that the equation of motion (1.15) is similar to Newton's equation (1.1), where now the static solutions of the Euler-Lagrange equation are the extrema of the potential in *field-space* satisfying the condition $\frac{\delta V[\phi]}{\delta \phi} = 0$. The stable solutions are given by the minima of the potential.

Now let $\phi(\vec{x}) = \phi_0(\vec{x})$ be one such minimum. Then we can perform a functional Taylor expansion of the potential V about ϕ_0. So we write[13]:

$$V[\phi] = V[\phi_0] + \int d^3x \frac{1}{2} \left\{ \eta(\vec{x}) \left[-\nabla^2 + \left. \frac{d^2U}{d\phi^2} \right|_{\phi_0(\vec{x})} \right] \eta(\vec{x}) + ... \right\} \tag{1.16}$$

where $\eta(\vec{x}) = \phi(\vec{x}) - \phi_0(\vec{x})$ and we have integrated by parts.

The operator $\left(-\nabla^2 + \left. \frac{d^2U}{d\phi^2} \right|_{\phi_0(\vec{x})} \right) \eta_i(\vec{x}) = \omega_i^2 \eta_i(\vec{x})$ is the generalization of the second derivative of the potential in the weak-coupling expansion (1.3) that we described in the previous paragraph. Here $\eta_i(\vec{x})$ are the orthonormal *"normal modes"* of fluctuations around $\phi_0(\vec{x})$.

Now one can construct a set of approximate harmonic-oscillator states, spread in field space around $\phi_0(\vec{x})$. The energies [13] of these states are:

$$E_{\{n_i\}} = V[\phi_0] + \hbar \sum_i \left(n_i + \frac{1}{2} \right) \omega_i + corrections \tag{1.17}$$

where n_i is the excitation number of the i-th normal mode.

In the previous formula:

- The first term is just the classical energy of the static solution $\phi_0(\vec{x})$.
- If any of the ω_i are equal to zero, then the simple ideas we have discussed so far need to be modified.
- These results are valid only in the weak-coupling approximation.
- If ϕ_0 is an absolute minimum, then it is the *"classical vacuum"* of the system. Because of the $(\nabla\phi)^2$ term in the expression of $V[\phi]$, such an absolute minimum would have to be space-independent.

1.3 Quantization of the kink solution

Let us apply now what we have discussed so far, to the case of a field theory with one scalar field $\phi(x,t)$ in (1+1) dimensions [2, 12, 13, 14] which is described by the Lagrangian:

$$L = \int dx \left[\frac{1}{2} \left(\frac{\partial\phi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial\phi}{\partial x} \right)^2 + \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4 - \frac{m^4}{4\lambda} \right] \tag{1.18}$$

In this system, the potential is:

$$V[\phi] = \int dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{\lambda}{4} \left(\phi^2 - \frac{m^2}{\lambda} \right)^2 \right] \tag{1.19}$$

and the equation of motion that a static classical solution satisfies is:

$$\frac{\partial V[\phi]}{\partial \phi(x)} = -\frac{\partial^2 \phi}{\partial x^2} - m^2 \phi + \lambda \phi^3 = 0 \tag{1.20}$$

This equation of motion has the following solutions [13]:

i. *Trivial solutions* $\phi(x,t) = \pm \frac{m}{\sqrt{\lambda}}$

ii. *Kink solutions* $\phi_k = \pm \frac{m}{\sqrt{\lambda}} \tanh \left[\frac{m(x-a)}{\sqrt{2}} \right]$

In the following we apply the semiclassical method [2, 13] to each of these static solutions.

1.3.1 The vacuum and its excitations

Consider now the trivial solution of the equation (1.20), which is $\phi_1(x,t) = \pm \frac{m}{\sqrt{\lambda}}$. We expand the potential around ϕ_1 and we take:

$$\begin{aligned}
V[\phi] = V[\phi_1 + \tilde{\phi}] &= \int dx \left\{ \frac{1}{2} \left(\frac{\partial(\phi_1 + \tilde{\phi})}{\partial x} \right)^2 - \frac{1}{2} m^2 (\phi_1 + \tilde{\phi})^2 + \frac{\lambda}{4} (\phi_1 + \tilde{\phi})^4 + \frac{m^4}{4\lambda} \right\} \\
&= \int dx \left\{ \frac{1}{2} \phi_1'^2 + \frac{1}{2} \tilde{\phi}'^2 + \phi_1' \tilde{\phi}' - \frac{1}{2} m^2 \phi_1^2 - \frac{1}{2} m^2 \tilde{\phi}^2 - m^2 \phi_1 \tilde{\phi} \right\} \\
&+ \int dx \left\{ \frac{\lambda}{4} \left[\phi_1^4 + 4\phi_1^3 \tilde{\phi} + 6 \phi_1^2 \tilde{\phi}^2 + 4 \phi_1 \tilde{\phi}^3 + \tilde{\phi}^4 \right] + \frac{m^4}{4\lambda} \right\} \\
&= V[\phi_1] + \int dx \left\{ \frac{1}{2} \tilde{\phi}'^2 - \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{3\lambda}{2} \phi_1^2 \tilde{\phi}^2 \right\} + \int dx \left\{ \lambda \phi_1 \tilde{\phi}^3 + \frac{\lambda}{4} \tilde{\phi}^4 \right\} \\
&+ \int dx \left\{ \phi_1' \tilde{\phi}' - m^2 \phi_1 \tilde{\phi} + \lambda \phi_1^3 \tilde{\phi} \right\}
\end{aligned}$$

The third integral equals to zero since ϕ_1 satisfies the equation of motion (1.20). The first integral becomes $\int dx \left\{ -\frac{1}{2} \tilde{\phi} \frac{\partial^2 \tilde{\phi}}{\partial x^2} - \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{3\lambda}{2} \phi_1^2 \tilde{\phi}^2 \right\}$ and the quantity $V[\phi_1]$ also equals to zero. By substitution of the expression of the static solution ϕ_1 we take the result:

$$V[\phi] = \int dx \frac{\tilde{\phi}}{2} \left(-\frac{\partial^2}{\partial x^2} + 2m^2 \right) \tilde{\phi} + m\sqrt{\lambda} \int \tilde{\phi}^3 dx + \frac{\lambda}{4} \int \tilde{\phi}^4 dx \tag{1.21}$$

where $\tilde{\phi}(x) \equiv \phi(x) - \phi_1(x) = \phi(x) - \frac{m}{\sqrt{\lambda}}$.

If the constant λ is sufficiently small, we can treat the cubic and the quartic terms in (1.21) by perturbation. In the lowest-order quadratic term, the second derivative of $V[\phi]$ at ϕ_1 is the operator $\left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)$ with:

i. eigenvalues: $k_n^2 + 2m^2$

ii. eigenfunctions: $e^{ik_n x}$

The allowed values of k_n are obtained in box-normalization [13, 14] by the condition:

$$k_n L = 2n\pi \tag{1.22}$$

where L is the length of the box. As $L \to \infty$ we perform the replacement

$$\sum_{k_n} \to \frac{L}{2\pi} \int dk \tag{1.23}$$

As we know the eigenvalues of the operator $\left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)$, i.e the frequencies, we can construct a tower of approximate harmonic oscillator states around ϕ_1. The lowest state of these states will have energy [13]:

$$E_{vac} = 0 + \frac{1}{2}\hbar \sum_n (k_n^2 + 2m^2)^{1/2} + O(\lambda) \tag{1.24}$$

where the zero represents the classical energy $V[\phi_1]$. This is the vacuum state of the system.

Higher excitations, analogous to (1.17), will have energies:

$$E_{\{N_n\}} = \hbar \sum_n (N_n + \frac{1}{2})(k_n^2 + 2m^2)^{1/2} + O(\lambda). \tag{1.25}$$

These correspond to the familiar quanta of the theory, where N_n of them have momentum $\hbar k_n$. We will call these quanta the *"mesons"* of this model. An identical set of vacuum and many-meson states can also be built around the solution $\phi_2 = -\frac{m}{\sqrt{\lambda}}$.

1.3.2 The quantum kink and its excitations

Having found the approximate harmonic oscillator energies for the states around the trivial static solution ϕ_1, we apply the same method [2, 12, 13, 14] to the kink solution $\phi_k(x) = \frac{m}{\sqrt{\lambda}} \tanh\left[\frac{mx}{\sqrt{2}}\right]$ with energy $V[\phi_k] = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}$. For simplicity we have considered the case where $a = 0$ in kink solution.

Now expand the potential around $\phi_k(x)$ and we obtain:

$$\begin{aligned} V[\phi] &= V[\phi_k] + \int dx \frac{1}{2}\eta(x)\left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_k^2\right)\eta(x) \\ &\quad +\lambda \int dx \left(\phi_k \eta^3 + \frac{1}{4}\eta^4\right) \end{aligned} \tag{1.26}$$

where $\eta(x) \equiv \phi(x) - \phi_k(k)$. The linear term in the above expansion is absent (in a similar way to the case to the trivial static solution) since the kink solution obeys the equation of motion (1.20). That is because the kink solution is an extremum of $V[\phi]$.

The second derivative of $V[\phi]$ at ϕ_k gives the operators $\left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_k^2\right)$ from which we obtain the eigenvalue problem [13]:

$$\left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_k^2\right)\eta_n(x) = \left[-\frac{\partial^2}{\partial x^2} - m^2 + 3m^2\tanh^2\left(\frac{mx}{\sqrt{2}}\right)\right]\eta_n(x) = \omega_n^2\eta_n(x) \tag{1.27}$$

If we change the variables to $z = \frac{mx}{\sqrt{2}}$ the above eigenvalue problem takes the form:

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + \left(3\tanh^2 z - 1\right)\right)\tilde{\eta}_n(z) = \frac{\omega_n^2}{m^2}\tilde{\eta}_n(z). \tag{1.28}$$

This has the form of a Schrodinger equation with potential $\left(3\tanh^2 z - 1\right)$. The eigenvalues and eigenfunctions of this Schrodinger equation are already known [8]. They are:

Discrete levels

i. $\omega_0^2 = 0$ with $\tilde{\eta}_0(z) = \frac{1}{\cosh^2 z}$

ii. $\omega_1^2 = \frac{3}{2}m^2$ with $\tilde{\eta}_1(z) = \frac{\sinh z}{\cosh^2 z}$

Continuum of levels

iii $\omega_q^2 = m^2\left(\frac{1}{2}q^2 + 2\right)$ with $\tilde{\eta}_q(z) = e^{iqz}\left(3\tanh^2 z - 1 - q^2 - 3iq\tanh z\right)$

The allowed values of q are fixed by periodic boundary conditions in a box of length L with $L \to \infty$, as we did for the allowed values of k_n in (1.22).

Now observe that the eigenfunctions of the continuum level $\tilde{\eta}_q(z)$ have an asymptotic behavior:

$$\tilde{\eta}_q(z) \xrightarrow{z\to\pm\infty} e^{i\left(qz\pm\frac{1}{2}\delta(q)\right)} \tag{1.29}$$

where

$$\delta(q) = -2\tan^{-1}\left(\frac{3q}{2-q^2}\right) \tag{1.30}$$

is just the phase shift of the scattering states of the associated Schrodinger problem (1.28). The periodic boundary conditions [2, 12, 13, 14] in the case of the kink require that:

$$q_n \frac{mL}{\sqrt{2}} + \delta(q_n) = 2n\pi \tag{1.31}$$

where n is a positive or negative integer. This condition fixes the allowed values of q_n. In the limit $L \to \infty$ we perform the replacement

$$\sum_{q_n} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \left(\frac{mL}{\sqrt{2}} + \frac{\partial}{\partial q}\left[\delta(q)\right]\right) \tag{1.32}$$

We should expect to construct a set of approximate harmonic oscillator states around the kink solution $\phi_k(x)$. Using the equation (1.17), we should expect the energies [13] of these states to be

$$\tilde{E}_{\{N_n\}} = V[\phi_k] + \hbar\sum_{n=0}^{\infty}\left(N_n + \frac{1}{2}\right)\omega_n + O(\lambda) \tag{1.33}$$

or

$$\tilde{E}_{\{N_n\}} = \frac{2\sqrt{2}m^3}{3\lambda} + \left(N_1 + \frac{1}{2}\right)\hbar\sqrt{\frac{3}{2}}m + m\hbar\sum_{q_n}\left(N_{q_n} + \frac{1}{2}\right)\left(\frac{1}{2}q_n^2 + 2\right)^{1/2} + O(\lambda) \tag{1.34}$$

However there is a difficulty: this analysis we described so far for the case of the kink solution does not hold for the $n = 0$ mode, since $\omega_0 = 0$. The treatment of such difficulties are not of the scope of the present discussion. Fortunately such difficulties do not occur to order λ^0 but they will arise only in the $O(\lambda)$ terms which are not explicitly shown in (1.33).

We interpret these energy states in the following way:

i. The lowest-energy state in (1.33), which corresponds to $N_n = 0$ will be interpreted as the state of the quantum kink particle at rest. The $N_n = 0$ state in (1.33), is not the vacuum state and will be interpreted instead as the quantum kink.

ii. The next higher energy level in (1.33) to order λ^0, arises when the $n = 1$ mode is excited once, i.e when $N_1 = 1$. This state has energy

$$\tilde{E}_1 \equiv \tilde{E}_{\{N_1=1;N_{q_n}=0\}} = \tilde{E}_0 + \sqrt{\frac{3}{2}} m\hbar + O(\lambda).$$

This state may be interpreted as a discrete excited state of the kink particle. Higher excitations of this mode ($N_1 > 1$) give higher excited states of the kink.

iii. The $N_q \neq 0$ states in (1.34) can be thought of as the scattering states of the mesons of this theory in the presence of the kink particle.

1.4 Quantum corrections to the kink mass

Here we will evaluate the mass of the quantum kink particle. So far we have associated the kink particle with the lowest energy level in the set (1.33). This energy is given by:

$$\tilde{E}_0 = \frac{2\sqrt{2}m^3}{3\lambda} + \frac{1}{2}\hbar m\sqrt{\frac{3}{2}} + \frac{1}{2}\sum_{q_n} \hbar m \left(\frac{1}{2}q_n^2 + 2\right)^{1/2} + O(\lambda). \tag{1.35}$$

This expression is formally divergent. The infinite series over $\sum_{q_n}$ becomes in the continuum limit (1.32) a quadratically divergent integral. In fact this is not a problem for us, since the energy of the vacuum (1.24) is also quadratically divergent. What matters physically is the difference in energy between any given state and the vacuum state. This difference is given by subtracting (1.24) from (1.35):

$$\tilde{E}_0 - E_{vac} = \frac{2\sqrt{2}m^3}{2\lambda} + \frac{1}{2}\sqrt{\frac{3}{2}}\hbar m + \frac{1}{2}\sum_{n} \left\{ m\left(\frac{1}{2}q_n^2 + 2\right)^{1/2} - \left(k_n^2 + 2m^2\right)^{1/2} \right\} + O(\lambda) \tag{1.36}$$

If we put the system in a finite box of length L then k_n and q_n are related by the periodic boundary conditions (1.22) and (1.31) which give

$$2n\pi = k_n L = \frac{q_n m L}{\sqrt{2}} + \delta(q_n) \tag{1.37}$$

So the first term in the sum of the expression (1.36) becomes

$$m\left(\frac{q_n^2}{2}+2\right)^{1/2} = \left[\left(k_n-\frac{\delta_n}{L}\right)^2+2m^2\right]^{1/2} = \left[k_n^2+2m^2-2\frac{k_n\delta_n}{L}+\frac{\delta_n^2}{L^2}\right]^{1/2}$$
$$= \left(k_n^2+2m^2\right)^{1/2}-\frac{k_n\delta_n}{L}\left(k_n^2+2m^2\right)^{-1/2}+O(1/L^2)$$

Thus the whole term in the sum of the expression (1.36) takes the form:

$$\left[\left(k_n-\frac{\delta_n}{L}\right)+2m^2\right]^{1/2}-\left(k_n^2+2m^2\right)^{1/2} = -\frac{k_n\delta_n}{L}\left(k_n^2+2m^2\right)^{-1/2}+O(1/L^2) \quad (1.38)$$

Going to the limit $L\to\infty$ and using the replacement (1.23) we obtain the expression:

$$\tilde{E}_0-E_{vac}=\frac{2\sqrt{2}m^3}{3\lambda}+\frac{1}{2}\sqrt{\frac{3}{2}}\hbar m-\frac{\hbar}{4\pi}\int_{-\infty}^{\infty}dk\frac{k\delta(k)}{\sqrt{k^2+2m^2}}+O(\lambda). \quad (1.39)$$

In the limit $L\to\infty$ using the relation between k_n and q_n in (1.37) we can express the phase shift given in the equation (1.30) in terms of k. Then we take:

$$\delta(k)=-2\tan^{-1}\left(\frac{3\sqrt{2}mk}{2m^2-2k^2}\right) \quad (1.40)$$

Integrating by parts the integral in (1.39) we take

$$\begin{aligned}\tilde{E}_0-E_{vac}&=\frac{2\sqrt{2}m^3}{3\lambda}+\frac{1}{2}\sqrt{\frac{3}{2}}\hbar m-\frac{\hbar}{4\pi}\left[\delta(k)\sqrt{k^2+2m^2}\right]_{-\infty}^{\infty}\\&+\frac{\hbar}{4\pi}\int_{-\infty}^{\infty}dk\sqrt{k^2+2m^2}\frac{d\delta(k)}{dk}+O(\lambda)\end{aligned} \quad (1.41)$$

The term $\left[\delta(k)\sqrt{k^2+2m^2}\right]_{-\infty}^{\infty}$ is computed at the following lines. We take first the limit of this expression at $k\to\infty$ and we have:

$$\lim_{k\to\infty}\left\{\sqrt{k^2+m^2}\delta(k)\right\}=\lim_{k\to\infty}\frac{\delta(k)}{\frac{1}{\sqrt{k^2+2m^2}}}=\lim_{k\to\infty}\frac{\frac{\delta(k)}{dk}}{-k\left(k^2+2m^2\right)^{-3/2}}=$$
$$12\sqrt{2}m\lim_{k\to\infty}\frac{\frac{m^2+k^2}{4m^4+4k^4+10k^2+m^2}}{\frac{k}{(k^2+2m^2)^{3/2}}}=12\sqrt{2}m\lim_{k\to\infty}\frac{\left(m^2+k^2\right)\left(k^2+2m^2\right)^{3/2}}{k\left(4m^4+4k^4+10k^2m^2\right)}=$$
$$12\sqrt{2}m\lim_{k\to\infty}\frac{k^2\left(k^2+2m^2\right)^{3/2}}{4k^5}=3\sqrt{2}\lim_{k\to\infty}\frac{\left(k^2+2m^2\right)^{3/2}}{k^3}=3\sqrt{2}\lim_{k\to\infty}\frac{\left(k^2+2m^2\right)^{1/2}}{k}=3\sqrt{2}m$$

where we have used the *Del'Hospital's* rule in the steps of computing the limit. Also we have used the fact that

$$\frac{d\delta(k)}{dk} = -12\sqrt{2}m\frac{m^2+k^2}{4m^4+4k^4+10k^2m^2}$$

In a similar way we find that

$$\lim_{k\to-\infty}\left\{\sqrt{k^2+m^2}\delta(k)\right\} = -3\sqrt{2}m$$

Thus the equation (1.41) takes the form:

$$\begin{aligned}\tilde{E}_0 - E_{vac} &= \frac{2\sqrt{2}m^3}{3\lambda} + \frac{1}{2}\sqrt{\frac{3}{2}}\hbar m - \frac{3\hbar m}{\pi\sqrt{2}} \\ &- \frac{3m\hbar}{\sqrt{2}\pi}\int_{-\infty}^{\infty} dk\frac{k^2+m^2}{(2k^2+m^2)\sqrt{k^2+2m^2}} + O(\lambda)\end{aligned} \tag{1.42}$$

Note that although the quadratic divergence in $\tilde{E}_0$ has been removed by subtracting out the E_{vac}, still there is a logarithmic divergence in the difference $\tilde{E}_0 - E_{vac}$. This divergence cannot be removed by adding a divergent constant to the Lagrangian, since $\tilde{E}_0 - E_{vac}$ is the difference between two energy levels. In fact this divergence can be removed by normal-ordering the Hamiltonian. In fact, the removal of the logarithmic divergence will be realized by adding suitable *"counter terms"* [13, 14] to the Hamiltonian. For a ϕ^4 theory in $1+1$ dimensions, the only divergences that occur are known to be removable by just normal ordering. The Hamiltonian of our theory is:

$$H = \int dx\left\{\frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}m^2\phi^2\frac{\lambda}{4}\phi^4 + \frac{m^4}{4\lambda}\right\} \tag{1.43}$$

In the quantized theory, operators such as $\phi^2(x,t)$, $\phi^4(x,t)$ etc. are formally divergent and therefore so is the Hamiltonian. Consequently the energy levels calculated from this Hamiltonian will also be divergent. This is the reason to the divergence in $\tilde{E}_0 - E_{vac}$.

To remove the logarithmic divergence we replace [13] the original Hamiltonian by the normal-ordered one. In fact the normal-ordered Hamiltonian can be written as

$$: H := H - \int_{-\infty}^{\infty} dx\left(\frac{1}{2}\delta m^2\phi^2 + D\right) \tag{1.44}$$

where the constants δm^2 and D can be evaluated in perturbation theory. For δm^2 the contribution to order λ comes from the Feynman diagram:

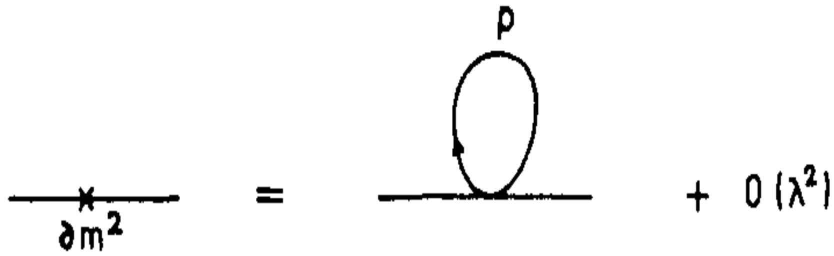

Figure 1.2: The one loop Feynman diagram that contributes to order λ.

Thus for the counter-term δm^2 [13, 14] we have

$$\begin{aligned} \delta m^2 &= -3i\lambda\hbar \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+2m^2} = \frac{3\lambda\hbar}{4\pi} \int_{-m\Lambda}^{m\Lambda} \frac{dp}{\sqrt{p^2+2m^2}} \\ &= \frac{3\lambda\hbar}{4\pi} \int_{-\Lambda}^{\Lambda} \frac{dp}{\sqrt{p^2+2}} \end{aligned} \tag{1.45}$$

where $m\Lambda$ is the momentum cut-off. We will not calculate the constant D since the kink mass involves the difference between two energy levels. Thus in our calculation, constant D cancels.

The replacement of the Hamiltonian by the normal ordered Hamiltonian implies the change of the potential (1.19) by the amount $\Delta V = -\int dx \left(\frac{1}{2}\delta m^2 \phi^2(x) + D\right)$. This implies the change in the classical energies of the static solutions $\phi_k(x)$ and $\phi_1(x)$. Finally this affect the quantum energy levels $\tilde{E}_0$ and E_{vac}, since the classical energies are their leading terms. As a result, we have to add to equation (1.42) the amount

$$
\begin{aligned}
\Delta\tilde{E}_0 - \Delta E_{vac} &= -\frac{1}{2}\int_{-\infty}^{\infty} dx\delta m^2\left(\phi_k^2 - \phi_1^2\right) = \frac{1}{2}\delta m^2\int_{-\infty}^{\infty}\frac{m^2}{\lambda}\left(1-\tanh^2\frac{mx}{\sqrt{2}}\right)\\
&= \frac{m^2}{2\lambda}\delta m^2\int_{-\infty}^{\infty}dx\frac{1}{\cosh^2\frac{mx}{\sqrt{2}}} = \frac{\sqrt{2}m}{2\lambda}\delta m^2\int_{-\infty}^{\infty}du\frac{1}{\cosh^2 u}\\
&= \frac{2\sqrt{2}m}{\lambda}\delta m^2\int_{-\infty}^{\infty}du\frac{1}{\left(e^u+e^{-u}\right)^2} = \frac{2\sqrt{2}m}{\lambda}\delta m^2\int_{-\infty}^{\infty}du\frac{e^{2u}}{\left(e^{2u}+1\right)^2}\\
&= \frac{\sqrt{2}m}{\lambda}\delta m^2\int_0^{\infty}dz\frac{1}{\left(z+1\right)^2} = -\frac{\sqrt{2}m}{\lambda}\delta m^2\left\{\left.\frac{1}{z+1}\right|_0^{\infty}\right\} = \frac{\sqrt{2}m}{\lambda}\delta m^2
\end{aligned}
\tag{1.46}
$$

where in the last step we used the substitution $z = e^{2u}$.

Before we continue we must insert the momentum cut-off in the integral of the expression (1.42). Thus we consider the integral

$$
I = -\frac{3m\hbar}{\sqrt{2}\pi}\int_{-\infty}^{\infty}dk\frac{k^2+m^2}{\left(2k^2+m^2\right)\sqrt{k^2+2m^2}}
\tag{1.47}
$$

When we insert the momentum cut-off in this integral this becomes:

$$
\begin{aligned}
I &= -\frac{3m\hbar}{\sqrt{2}\pi}\int_{-m\Lambda}^{m\Lambda}dk\frac{k^2+m^2}{\left(2k^2+m^2\right)\sqrt{k^2+2m^2}}\\
&= -\frac{3m\hbar}{\sqrt{2}\pi}\int_{-\Lambda}^{\Lambda}dp\frac{p^2+1}{\left(2p^2+1\right)\sqrt{p^2+2}}
\end{aligned}
\tag{1.48}
$$

where in the last equality we used the substitution $p = k/m$. Then the expression (1.42) takes the form:

$$
\begin{aligned}
\tilde{E}_0 - E_{vac} &= \frac{2\sqrt{2}m^3}{3\lambda} + \frac{1}{2}\sqrt{\frac{3}{2}}\hbar m - \frac{3\hbar m}{\pi\sqrt{2}}\\
&- \frac{3m\hbar}{\sqrt{2}\pi}\int_{-\Lambda}^{\Lambda}dp\frac{p^2+1}{\left(2p^2+1\right)\sqrt{p^2+2}} + O(\lambda)\\
&= \frac{2\sqrt{2}m^3}{3\lambda} + \frac{1}{2}\sqrt{\frac{3}{2}}\hbar m - \frac{3\hbar m}{\pi\sqrt{2}}\\
&- \frac{3\hbar m}{2\sqrt{2}\pi}\int_{-\Lambda}^{\Lambda}dp\left\{\frac{1}{\sqrt{p^2+2}} + \frac{1}{\left(2p^2+1\right)\sqrt{p^2+2}}\right\}
\end{aligned}
\tag{1.49}
$$

Now insert the expression for the counter-term into the previous result and we get for the renormalized kink mass the following expression:

$$M \equiv \left(\tilde{E}_0 + \Delta\tilde{E}_0\right) - (E_{vac} + \Delta E_{vac}) = \frac{2\sqrt{2}m^3}{3\lambda} + m\hbar\left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{3}{\pi\sqrt{2}}\right)$$
$$- \frac{3\sqrt{2}m\hbar}{4\pi}\int_{-\Lambda}^{\Lambda} dp \frac{1}{(2p^2+1)\sqrt{p^2+2}} + O(\lambda\hbar^2) \tag{1.50}$$

where we have introduced the cut-off into the integral. Note that the logarithmic divergences cancel. Then we can take the cut-off to infinity. All we have to do now is to compute the integral.

To simplify the result, first we compute the antiderivative. To do this, we set $p = \sqrt{2}\tan u$ and the integral becomes:

$$\int dp \frac{1}{(2p^2+1)\sqrt{p^2+2}} = \int du \frac{1}{4\tan^2 u + 1}\frac{1}{\cos u} = \int du \frac{\cos u}{4\sin^2 u + \cos^2 u} =$$
$$\int du \frac{\cos u}{3\sin^2 u + 1} = \int dy \frac{1}{1+3y^2} = \frac{1}{3}\int dy \frac{1}{y^2+\frac{1}{3}} = \frac{\sqrt{3}}{3}\tan^{-1}\left(\sqrt{3}y\right) + const =$$
$$\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\sin u\right) + const = \frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{3}p}{\sqrt{p^2+4}}\right\} + const$$

where in the intermediate steps we have set $y = \sin u$. Now we have that the integral we want to calculate is:

$$\int_{-\Lambda}^{\Lambda} dp \frac{1}{(2p^2+1)\sqrt{p^2+2}} = \frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{3}p}{\sqrt{p^2+4}}\right\}\Bigg|_{-\Lambda}^{\Lambda}$$

Taking the limit $\Lambda \to \infty$ this integral becomes:

$$\int_{-\infty}^{\infty} dp \frac{1}{(2p^2+1)\sqrt{p^2+2}} = \frac{2\pi}{3\sqrt{3}}$$

Having done with the integral, we substitute the result to the equation (1.50) and we find that the renormalized mass of the kink particle[2, 13] is:

$$M = \frac{2\sqrt{2}}{3\lambda}m^3 + m\hbar\left(\frac{1}{6}\sqrt{\frac{3}{2}} - \frac{3}{\pi\sqrt{2}}\right) + O(\lambda\hbar^2) \tag{1.51}$$

This expression [2, 12, 13, 14] gives the mass of the kink particle. Let us now make some observations about the result.

i. The first term in the mass of the quantum kink particle is the energy of the classical static kink solution. The next term represents the leading correction due to quantum fluctuations.

ii. The leading term, the energy of the classical kink is singular as $\lambda \to 0$. Thus the result is non-perturbative.

iii. This result is valid only in the weak-coupling approximation. The non-perturbative nature of the result is brought about by the classical contribution. The quantum corrections are being treated perturbatively in powers of λ.

Chapter 2

Quantum one-loop correction to the 't Hooft-Polyakov monopole mass

2.1 The model and its monopole solutions

In this chapter we continue our discussion to the calculation of the quantum correction [5] of the 't Hooft-Polyakov monopole mass, which is the aim of this dissertation. Before we begin our discussion it is remarkable to refer that the classical [13, 18] monopole mass can be analytically computed only in the BPS limit. However when one reaches the BPS limit the quantum correction turns out to be increasing.

The model we use here consists of scalar fields $\phi^a(\vec{x},t)$ and vector fields $A^a_\mu(\vec{x},t)$ in (3+1) dimensions [5, 11, 13, 16]. The index $a = 1, 2, 3$ is an internal space index, which will transform according to local $SU(2)$ transformations. For any given a, ϕ^a is a scalar and A^a_μ $(\mu = 1, 2, 3)$ is a vector under Lorentz transformations. The system we deal with is described by the action

$$S = \int d^4x \left\{ -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2}\left(D_\mu\phi^a\right)^2 - \frac{1}{4}\lambda\left(\phi^2 - F^2\right)^2 \right\} \tag{2.1}$$

The Lagrangian density of this theory is

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2}\left(D_\mu\phi^a\right)^2 - \frac{1}{4}\lambda\left(\phi^2 - F^2\right)^2 \tag{2.2}$$

Here the field tensor $F^a_{\mu\nu}$ is defined by

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu \tag{2.3}$$

and the covariant derivative $D_\mu\phi^a$ by

$$D_\mu \phi^a \equiv \partial_\mu \phi^a + g\epsilon^{abc} A^b_\mu \phi^c. \tag{2.4}$$

The covariant derivatives of other triplets of fields such as A^a_μ and $F^a_{\mu\nu}$ are defined in a similar way. The real constants $g, \lambda > 0$ and F are parameters of the model.

Now we can calculate the equations of motion using the *Euler-Lagrange* equations for the *Higgs* fields ϕ^a and the *gauge* fields A^a_μ.

- The equation of motion that comes from the *Higgs* fields are given by:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^a\right)} - \frac{\partial \mathcal{L}}{\partial \phi^a} = 0 \Rightarrow D_\mu D^\mu \phi^a = -\lambda \left(\phi^b \phi^b\right) \phi^a + \lambda F^2 \phi^a \tag{2.5}$$

- The equation of motion that comes from the *gauge* fields are given by:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A^a_\nu\right)} - \frac{\partial \mathcal{L}}{\partial A^a_\nu} = 0 \Rightarrow D_\mu F^{a\mu\nu} = g\epsilon^{abc} \left(D^\nu \phi^b\right) \phi^c \tag{2.6}$$

At this point of our discussion we will find the classical vacuum solutions[5, 11, 13, 16]. This will help us to find the boundary conditions that any finite-energy configuration must satisfy. We will restrict ourselves to static solutions which satisfy $A^a_0(\vec{x}) = 0$ for all $\vec{x}, a$. Such solutions provide us an expression for the conserved energy of the system which is

$$E = \int d^3x \left\{ \frac{1}{4} F^a_{ij} F^{aij} + \frac{1}{2} D_i \phi^a D^i \phi^a + \frac{1}{4} \lambda \left(\phi^a \phi^a - F^2\right)^2 \right\}. \tag{2.7}$$

Then the energy has a minimum (which is zero) when

$$A^a_i(\vec{x}) = 0 \tag{2.8}$$

$$\phi^a(\vec{x})\phi^a(\vec{x}) = F^2 \tag{2.9}$$

$$D_i \phi^a = 0 \tag{2.10}$$

The condition [13] for finite E is that the fields approach some configuration with zero energy at spatial infinity sufficiently fast. We see from the expression (2.7) that this condition for the *Higgs* fields is

$$r^{3/2} D_i \phi^a \to 0 \tag{2.11}$$

$$\phi^a \phi^a \to 0 \tag{2.12}$$

as $r \equiv |\vec{x}|$. Also if we consider the expression (2.4) for the covariant derivative in terms of spherical polar coordinates (r, θ, ϕ), then the $\theta-$component $D_\theta \phi^a$ is given by

$$D_\theta \phi^a = \frac{1}{r}\frac{\partial \phi^a}{\partial \theta} + g\epsilon^{abc} A_\theta^b \phi^c. \tag{2.13}$$

As long as the *Higgs* field falls fast enough in order to satisfy the asymptotic form (2.11), $\partial\phi^a/\partial\theta$ need not vanish as $r \to \infty$. For a non zero $\partial\phi^a/\partial\theta$ as $r \to \infty$, the $\theta-$component of the *gauge* field A_θ^b must be matched with $\partial\phi^a/\partial\theta$ in such a way that $D_\theta\phi^a$ goes to zero as $r \to \infty$. This implies that the $\theta-$component of the *gauge* field, A_θ^b, falls off to zero as fast as $1/r$. A similar statement holds for the azimuthal components of $(grad\phi^a)$ and A_μ^b. We must note that the part which involves $F_{ij}^a F^{aij}$ in the expression (2.7) will decrease as $1/r^4$ and will be integrable.

It is shown by *Gerard 't Hooft* [16] that there is a monopole solution in this model which satisfies the equations of motion (2.5) and (2.6) and which has the appropriate asymptotic form that we discussed above. This monopole solution is given by the following equations [5, 11, 13, 16, 18]:

$$\left.\begin{array}{l} A_0^a = 0 \\ A_i^a = \epsilon_{aik} n_k W(r) \\ \phi^a = n_a \phi(r) \end{array}\right\} \tag{2.14}$$

where $n_a = \frac{x^a}{r}$ and in the limit $r \to \infty$ the functions $W(r)$ and $\phi(r)$ have the asymptotic behavior $W(r) = \frac{1}{gr}$ and $\phi(r) = F - \frac{1}{gr}$.

At this point it would be useful to define the *scalar boson mass* μ and the *the vector field mass m*. To do this we consider small fluctuations around the vacuum. Thus we consider a small fluctuation χ of the scalar field $\vec{\phi}$ around the trivial vacuum $\left|\vec{\phi}\right| = F$, where only the third isotopic component of the *Higgs* field is non-vanishing, i.e we take:

$$\vec{\phi} = (0, 0, F + \chi) \tag{2.15}$$

Substitution of this expansion into the Lagrangian of our theory yields, up to terms of second order

$$D_\mu \phi^a D^\mu \phi^a \approx \partial_\mu \chi \partial^\mu \chi + g^2 F^2 \left[\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2 \right] \tag{2.16}$$

and

$$V(\phi) \approx \frac{\lambda}{2} F^2 \chi^2 \tag{2.17}$$

Further analysis shows that the perturbative spectrum consists of a massless photon A_μ^3 corresponding to the unbroken $U(1)$ subgroup, massive vector fields $A_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \pm A_\mu^2\right)$ with mass $m = gF$, and neutral scalars having a mass $\mu = F\sqrt{2\lambda}$.

2.2 The classical mass of the monopole at the BPS limit

Here we will derive an expression which gives the lower bound [13, 18] of the energy of a static configuration. This expression first derived by *Bogomol'nyi* (1976) and relates the energy of a static configuration to its topological index. In order to find the classical mass of the monopole we have to use the *Bogomol'nyi condition* [13, 18] that we will refer later.

To begin with we consider the limit $\lambda \to 0$. In this limit the energy for a static solution with $A_0^a = 0$, which is given in the expression (2.7), takes the form:

$$\begin{aligned} E &= \int d^3x \left\{ \frac{1}{4} F_{ij}^a F^{aij} + \frac{1}{2} D_k \phi^a D^k \phi^a \right\} \\ &= \int d^3x \sum_{i,j,a} \frac{1}{4} \left(F_{ij}^a - \epsilon_{ijk} D_k \phi^a \right)^2 + \int d^3x \frac{1}{2} \epsilon_{ijk} F_{ij}^a D_k \phi^a \end{aligned} \tag{2.18}$$

Now we can write the second term in the second line of the above expression as

$$\int d^3x \frac{1}{2} \epsilon_{ijk} F_{ij}^a D_k \phi^a = \int d^3x \partial_k \left(\frac{1}{2} \epsilon_{ijk} F_{ij}^a \phi^a \right) \tag{2.19}$$

where we have used the identity $D_\mu \tilde{F}^{a\mu\nu} = 0$ where $\tilde{F}^{a\mu\nu}$ is the dual field $\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a$. Then using the *Gauss theorem* we finally obtain

$$\int d^3x \frac{1}{2} \epsilon_{ijk} F_{ij}^a D_k \phi^a = \oint_{S^2} d\sigma_k \left(\frac{1}{2} \epsilon_{ijk} F_{ij}^a \phi^a \right) \tag{2.20}$$

Before we continue we must mention an other gauge invariant definition of the electromagnetic field tensor $F_{\mu\nu}$. This definition of the field tensor presented by *Gerard 't Hooft*[16] has the following form:

$$F_{\mu\nu} \equiv \hat{\phi}^a F_{\mu\nu}^a - \frac{1}{g} \epsilon^{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c \tag{2.21}$$

where $\hat{\phi}^a = \frac{\phi^a}{|\phi|}$.

As we discussed above, for any finite-energy configuration it must be: $D_\mu \phi^a \to 0$ and $\hat{\phi}^a \to \phi^a / F$ at the limit $r \to \infty$. Thus the magnetic field obtained from the field tensor (2.21) has the following asymptotic behavior[13]:

$$B_k = \frac{1}{2} \epsilon_{ijk} F_{ij} \to \frac{1}{2F} \epsilon_{ijk} F_{ij}^a \phi^a \tag{2.22}$$

as $r \to \infty$.

Using this result, the equation (2.20) takes the form:

$$\int d^3x \frac{1}{2}\epsilon_{ijk}F^a_{ij}D_k\phi^a = F\oint_{S^2} d\sigma_k B_k = 4\pi m F = 4\pi\frac{Q}{g}F \tag{2.23}$$

where Q is called *homotopy index*. This is a topological index and any further discussion about it is out of the scope of this dissertation. We have to mention that in this case the *homotopy index* has the value $Q = 1$ [13]. Also m is the *monopole charge* [13] which is related to the *homotopy index* by the relation $m = Q/g$.

Then we obtain the following inequality [13, 18] which gives the lower bound for the energy of the static solutions:

$$E = \frac{4\pi QF}{g} + \int d^3x \sum_{i,j,a} \frac{1}{4}\left(F^a_{ij} - \epsilon_{ijk}D_k\phi^a\right)^2 \geq \frac{4\pi QF}{g}. \tag{2.24}$$

For any value of the *homotopy index*, the energy is minimized if and only if the fields satisfy the *Bogomol'nyi condition* [13, 18]:

$$F^a_{ij} = \epsilon_{ijk}D_k\phi^a. \tag{2.25}$$

If a field configuration satisfies the *Bogomol'nyi condition* then it minimizes the static energy and is therefore a static classical solution. Accordingly the inequality in (2.24) tells us that the monopole will have a classical mass [5, 11, 13, 18] $4\pi F/g$.

2.3 One-loop correction to the monopole mass

In this final paragraph we present the calculation of the quantum correction [5] to the monopole mass. In fact we give the basic steps of the procedure we use here.

Before we continue we observe that our theory, described by the action (2.1), is gauge invariant under $SO(3)$ gauge transformations. It is known that in gauge invariant theories there is a large set of field configurations that make the functional integral $\int \mathcal{D}Ae^{\imath S[A]}$ being badly divergent. Where with A we mean the gauge field of a gauge invariant theory. This difficulty is due to gauge invariance.

To see this difficulty in the present theory, we consider the *Yang-Mills* term of our Lagrangian which is $\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$. Now we express this term in the following way:

$$\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \\
&= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) \\
&- \frac{g}{2}\epsilon^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)A^{b\mu}A^{c\nu} - \frac{g^2}{4}\epsilon^{abc}\epsilon^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}
\end{aligned} \tag{2.26}$$

From the above expression we see that in the limit $g \to 0$ we cannot compute the propagators of the gauge fields, since when we try to compute these propagators we see that the quantity $-\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)$ lead us to the 4×4 matrix $-k^2\eta_{\mu\nu} + k_\mu k_\nu$ which is singular. The solution of this problem is to fix the gauge. To do this we follow the *Faddeev-Popov* method. Following this method first we vary the action around the monopole solution (that minimizes the energy) and then we see that a convenient gauge-fixing function is the $G^a = D_\mu \alpha^a_\mu + g\epsilon_{abc}\phi^b\chi^c$, where α^a_μ and χ^c are fluctuations of the gauge fields and scalar fields respectively.

Now, after we have introduced the *ghost terms* in the original Lagrangian, during the *Faddeev-Popov* procedure, we collect the terms that give the second variation of the action. Then we take for the second variation [5] of the action the result:

$$\begin{aligned}\delta^2 S &= \int d^4x \left\{ \frac{1}{2}\left(D_\mu \alpha^a_\nu\right)^2 + \frac{1}{2}g^2\left(\epsilon_{abc}\phi^b\alpha^c_\mu\right)^2 + \frac{1}{2}\left(D_\mu\chi^a\right)^2 \right.\\ &\quad + \frac{1}{2}g^2\left(\epsilon_{abc}\phi^b\chi^c\right)^2 + \delta^2 V + \left|D_\mu\psi^a\right|^2 + g^2\left|\epsilon_{abc}\phi^b\psi^c\right|^2 \\ &\quad \left. + g\alpha^a_\mu\alpha^b_\nu\epsilon_{abc}F^c_{\mu\nu} + 2g\epsilon_{abc}\alpha^a_\mu\chi^b D_\mu\phi^c \right\} \qquad (2.27)\end{aligned}$$

where $\delta^2 V$ is the second variation of the potential and $\psi^a, a = 1, 2, 3$ are the *ghost fields*. The non-diagonal terms on the third line of the above expression, together with the term $\delta^2 V$ can be ignored since they don't have any sensible contribution to the singular part of the quantum correction [5] which then becomes:

$$\delta M = \frac{3}{2}\ln\det\frac{H}{H^{(0)}} \qquad (2.28)$$

where the operators H and $H^{(0)}$ are defined by the expression:

$$-\left(D^2_k\right)_{ab} + g^2\left(\phi^2\delta_{ab} - \phi^a\phi^b\right) \qquad (2.29)$$

on the background of the monopole solution and of the trivial solution, correspondingly. We note that the coefficient 3 in the expression (2.28) for the quantum correction, comes from the counting of the degrees of freedom. In this model the number of the degrees of freedom is 3.

Now we substitute the monopole solution (2.14) in the expression of the operator H and we obtain the following expression for this operator [5]:

$$\begin{aligned}H_{ab} &= \left(-\partial^2_0 - \vec{\nabla} + g^2W^2(r) + g^2\phi^2(r)\right)\delta_{ab} + 2\vec{T}_{ab}\cdot\vec{L}\frac{W(r)}{r} \\ &\quad + \left(g^2W^2(r) - g^2\phi^2(r)\right)n_a n_b \qquad (2.30)\end{aligned}$$

where with $\vec{L}$ we denote the angular-momentum operator and with $\vec{T}$ we denote the generators of the $SO(3)$ group which are $T^c_{ab} = \imath\epsilon_{abc}$.

Then we demand the operator $\vec{J} = \vec{L} + \vec{T}$ to commute with H. We observe that the spectrum of the operator H consists of triplets with fixed $\vec{J}$ and J_3 with $L = J-1, J, J+1$, where J_3 is the third component of the operator $\vec{J}$. Now the operator H is reduced to

$$\begin{aligned} H &= -\partial_r^2 + \omega^2 + g^2 W^2 + g^2\phi^2 \\ &+ \frac{1}{r^2} diag\left(J\left(J-1\right), J\left(J+1\right), \left(J+1\right)\left(J+2\right)\right) \\ &+ \frac{gW}{r} diag\left(2\left(J-1\right), -2, -2\left(J+2\right)\right) \\ &+ g^2\left(W^2-\phi^2\right)\begin{pmatrix} J & 0 & -\sqrt{J\left(J+1\right)} \\ 0 & 0 & 0 \\ -\sqrt{J\left(J+1\right)} & 0 & J+1 \end{pmatrix}\frac{1}{2J+1} \end{aligned} \tag{2.31}$$

Our next step is to substitute the expression of the asymptotic behavior of the function $W(r)$, where $W(r) = \frac{1}{gr}$ at $r \to \infty$. Then we see that the matrix expression (2.31) for the operator H can be diagonalized to give the following two operators [5]:

$$-\partial_r^2 + \omega^2 + \frac{J\left(J+1\right)-1}{r^2} + g^2\phi^2 \tag{2.32}$$

and

$$-\partial_r^2 + \omega^2 + \frac{J\left(J+1\right)}{r^2} + g^2\phi^2 \tag{2.33}$$

where these two expressions are valid at the limit $r \to \infty$.

Now since the spectrum of the operator (2.33) has not any singularity at the limit $\lambda \to \infty$, we take into account only the operator (2.32), which has a singularity at this limit. Taking into account the operator (2.32) the quantum correction (2.28) becomes [5]:

$$\begin{aligned} \delta M &= 3\int \frac{d\omega}{2\pi}\left(\sum_{J=1}^{\infty}\left(2J+1\right)\ln det\left(-\partial_r^2+p^2\right)\right. \\ &\left. - \sum_{L=0}^{\infty}\left(2L+1\right)\ln det\left(-\partial_r^2+\tilde{p}^2\right)\right) \end{aligned} \tag{2.34}$$

where

$$p = \sqrt{\frac{J\left(J+1\right)-1}{r^2} + \omega^2 + g^2\phi^2} \tag{2.35}$$

and

$$\tilde{p} = \sqrt{\frac{L\,(L+1)}{r^2} + \omega^2 + m^2} \tag{2.36}$$

with m to be the mass of the vector field, which is $m = gF$.

In order to compute the determinants in the equation (2.34) we use the relation [4]:

$$\frac{det\,(-\partial_r^2 + V_1(r))}{det\,(-\partial_r^2 + V_2(r))} = \frac{N_1(\infty)}{N_2(\infty)}, \tag{2.37}$$

where the $N_1(r)$ and $N_2(r)$ are the solutions of the equations

$$\begin{array}{l} (-\partial_r^2 + V_1(r))\,N_1(r) = 0 \\ (-\partial_r^2 + V_2(r))\,N_2(r) = 0 \end{array} \tag{2.38}$$

having identical regular behavior at the limit $r \to \infty$. To find the functions $N_1(r)$ and $N_2(r)$ for the operators from (2.34) it is correct to use the quassiclassical expansion [6] including the second-order terms. This is valid because we are interested only in terms singular at the limit $\lambda \to \infty$. By using the formula of the quassiclassical expansion, the expression (2.34) becomes [5]:

$$\begin{aligned} \delta M &= 3\int \frac{d\omega}{2\pi}\int_0^\infty dr \left\{ \sum_{J=1}^{\infty} (2J+1)\left[p + \frac{1}{8p^3}\left(\frac{dp}{dr}\right)^2\right] \right. \\ &\left. - \sum_{L=0}^{\infty} (2L+1)\left[\tilde{p} + \frac{1}{8\tilde{p}^3}\left(\frac{d\tilde{p}}{dr}\right)^2\right] \right\} \end{aligned} \tag{2.39}$$

We can see from (2.39) that δM includes quadratic and logarithmic ultraviolet divergences. To regularize these divergences we use the *Pauli-Villars* procedure [5], i.e we modify formula (2.28) (and all the following formulas up to (2.39) correspondingly) to:

$$\delta M = \frac{3}{2}\left(\ln det \frac{H}{H^{(0)}} - \sum_i C_i \frac{\ln\,(H + M_i^2)}{H^{(0)} + M_i}\right) \tag{2.40}$$

with conditions $\sum_i C_i = 1$ and $\sum_i C_i M_i^2 = 0$.

Following this procedure, when we perform the calculations we end up to the following formula for the quantum correction δM [5]:

$$
\begin{aligned}
\delta M &= \frac{3}{4\pi}\left(m^2 \ln m^2 - \sum_i C_i\left(M_i^2+m^2\right)\ln\left(M_i^2+m^2\right)\right)\int drr^2\left(g^2\phi^2-m^2\right) \\
&+\frac{3}{4\pi}\left(\ln m^2-\sum_i C_i \ln M_i^2\right)\int drr^2\left(g^2\phi^2-m^2\right)^2 \\
&+\frac{1}{8\pi m^2}\int drr^2\left(g^2\phi^2-m^2\right)^3+O(m) \qquad (2.41)
\end{aligned}
$$

To renormalize this formula we must subtract the perturbative counterterms from it. In our model there are four types of possible counterterms: $\left(F^a_{\mu\nu}\right)^2$, $\left(D_\mu\phi^a\right)^2$, ϕ^2 and ϕ^4. The first two counterterms are not singular at the limit $\lambda \to 0$, thus we can neglect them. The second two counterterms can be rewritten as:

$$\delta\mathcal{L}_{c.t.} = a\left(g^2\phi^2-m^2\right)+b\left(g^2\phi^2-m^2\right)^2 \qquad (2.42)$$

We define the constants a and b in terms of an effective potential:

$$
\begin{aligned}
\frac{d}{d\phi}\left(\delta\mathcal{L}_{c.t.}+\delta V_{eff}\right)\Big|_{\phi=F} &= 0, \\
\frac{d^2}{d\phi^2}\left(\delta\mathcal{L}_{c.t.}+\delta V_{eff}\right)\Big|_{\phi=F} &= 0, \qquad (2.43)
\end{aligned}
$$

where δV_{eff} [5] is a one-loop correction to the effective potential:

$$\delta V_{eff} = \frac{3}{32\pi^2}\left(g^4\phi^4\ln g^2\phi^2-\sum_i C_i\left(M_i^2+g^2\phi^2\right)^2\ln\left(M_i^2+g^2\phi^2\right)\right) \qquad (2.44)$$

Now obtaining from equations (2.42), (2.43) and (2.44) the constants a and b, we perform the integration of (2.42) over d^3x and we add it to (2.41). Then we finally obtain the following result for the quantum correction δM [5]:

$$
\begin{aligned}
\delta M &= \frac{1}{8\pi m^2}\int_0^\infty drr^2\left(g^2\phi^2-m^2\right)^3 \\
&= -\frac{m}{2\pi}\ln\frac{m^2}{\mu^2}+O(m) \qquad (2.45)
\end{aligned}
$$

where $\mu = F\sqrt{2\lambda}$ is the scalar boson mass and we have taken into account the asymptotic behavior of $\phi(r)$.

So, we see that the one-loop quantum correction to the monopole mass is negative and increases in absolute value when one reaches the *BPS* limit.

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