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Quantum Field Theory and Phase Transitions

Symmetry Breaking and Unitary Inequivalence

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To my brother,

for his company during the cloudy

English summer

Outline

Assuming the reader is familiar with the relativistic formulation of QFT used in particle physics but not with that used in condensed matter, a chapter is focused on introducing the field formalism in condensed matter systems. Starting with the relativistic theory of a Klein-Gordon field, the non-relativistic limit is taken and particle and antiparticle solutions decoupled. To account for many-body aspects, we next add thermal degrees of freedom, foremost a *chemical potential*.

The third chapter is intended to review the concepts of *spontaneous symmetry breaking* and *phase transitions*, and to discuss the connection between them. Goldstone's theorem is enunciated and we expose the consequences for breaking the U(1) symmetry of the field theory of chapter 2. The language of *coherent states* is introduced next, to be used mainly through chapter 5.

Chapter 4 is spent working out some illustrative applications of the so far introduced concepts and formalism. The condensed matter field theory is put into practice to describe the phase transitions responsible for *superfluidity* and *superconductivity*. Emphasis is made on the symmetry breaking involved in these phenomena.

Finally, the fifth chapter is meant to be an analysis of the major issue of *inequivalent representations* in quantum field theories. The behaviour of example systems presented in the previous chapter is supposed to set the stage for this last one.

Contents

1

Introduction

The great success of quantum field theories in the description of particle physics has meant a big step towards understanding the elementary components of the universe. These theories regard elementary particles as quanta of underlying fields, and their interactions as being carried by virtual quanta of other fields which mediate the forces. A key feature of the field description is that fields have infinite degrees of freedom, and that's necessary to represent relativistic phenomena where particle number changes (pair creation and annihilation), necessary for a quantum treatment of interactions (emission and absorption of force mediators), etc... So, quantum field theories are many-particle theories, being also appropriate to describe condensed matter systems, where the number of constituent particles (and thus the number of degrees of freedom) is enormously huge. In typical condensed matter situations however, energies aren't high enough for new massive particles to be created and there's no presence of antimatter, thus particle number is conserved. The fields then represent two kinds of elementary excitations generally. They can describe the behaviour of the already present elementary particles submitted to a many-body situation, commonly dealt with by reducing it to a single particle problem on an effective mean-field which summarizes the many interactions. This excitations are called *quasiparticles*, because the natural properties of the elementary particles are

altered to account for the many-body physics, for example acquiring an effective mass or charge which are different to the intrinsic values. The other type of field quanta represent collective excitations. Being quantized excitations of fields, their treatment and many of their properties resemble those of single particles, and thus the name quasiparticles apply to them too. The terms particle, excitations or quanta will be used to refer to the same general concept through the following chapters. The context will provide a way around this abuse of terminology. Another abuse will be that of indistinctly using the terms vacuum and ground state, hoping again that the context makes things clear.

Both being described by the quantum field formalism, elementary particles and condensed matter share other similarities apart from the many degrees of freedom, like the central role of symmetries or the usual need for renormalization among other examples. Those links have been very relevant in order to establish or reinforce notions in those two fields of physics, which mutually contribute to each other as more connections are discovered and exploited. A famous topic is that of *spontaneous symmetry breaking*, devised by Nambu in his treatment of superconductivity and later applied to particle physics [35]. The related mechanism of particle mass generation introduced by Anderson in condensed matter [1], was later adapted to relativistic theories by Higgs, Kibble and other physicists, and became part of the standard model of particle physics. The description of phase transitions in condensed matter theories have been the basis to formulate relativistic versions [49], which are of great importance in cosmology. Furthermore, experiments in condensed matter can provide insight and confidence into cosmological theories [53][8]. An active area of more recent research where again reciprocal links are present is the study of topological order and defects.

The possibility of using quantum field theories to deal with phase transitions and symmetry breaking rests on their infinite degrees of freedom, but also lies in the availability of different representations for their algebraic structure [44]. Those

different representations are not equivalent to each other, differing in their physical content, and that can thus reflect the distinct behaviour of a system in different phases. Nevertheless, that turns out to be simultaneously a virtue and a complication. It is a complication in the sense that the choice of representation doesn't automatically come together with the theory, there's no universal prescription for it. For instance, whether field excitations consist of the first or the second kind of quasiparticles mentioned previously depends on the representation. We'll meet examples where a description centered around entities of the first type (those elementary particles with modified properties) requires, because of a phase transition taking place, to switch over to a different representation where the quanta are of the second type (collective excitations).

The non-uniqueness of the representation and thus of its corresponding elementary excitations has deep philosophical implications. Statistical mechanics has been successful in explaining the properties of ordinary matter from the microscopic physics of its components. However, the discovery of some rather bizarre phases, like *Bose-Einstein condensates*, has made the issue much more subtle. Condensates are understood as emergent phenomena which arise from microscopic behaviour in certain conditions, but can only be described through the above mentioned change to a representation which is disjoint from that of the elementary particles. Although the main perspective is to consider the building blocks of matter as fundamental, the impossibility of describing some particular states of matter in terms of them casts doubt on that point of view. And moreover, not only could collective excitations in condensates be considered as fundamental as quarks or leptons, but also could those be regarded as emergent [47].

2

Condensed Matter Limit of QFT

2.1 Motivation

The relativistic theory of noninteracting massive spinless particles is described by a quantum field operator $\widehat{\phi}(x)$ obeying the Klein-Gordon equation

$$(\Box + m^2)\widehat{\phi}(x) = 0 \tag{2.1}$$

where m is the mass, and the field is either real or complex depending on whether the particle is neutral or charged ($\widehat{\phi}(x) \in \mathbb{C}$ unless otherwise stated). Plane waves (eigenstates of four-momentum) are solutions to (2.1) provided the restriction for a free particle of mass m is satisfied. That is called the on-shell condition, which reads

$$k_0 = \pm E = \pm\sqrt{m^2 + |\mathbf{k}|^2} \tag{2.2}$$

Thus, the general solution to the Klein-Gordon equation can be written as a superposition of plane waves with all possible momenta which respect (2.2):

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{k}}} \left[\widehat{a}_\mathbf{k} e^{-ikx} + \widehat{b}^+_\mathbf{k} e^{ikx}\right] \tag{2.3}$$

The onshell condition reduces the integration over four-momenta to three-momenta, and we did explicitly split positive and negative energy modes ($k_0 = +\sqrt{m^2 + |\mathbf{k}|^2}$ from now on). We will always interpret $\widehat{\phi}$, $\widehat{a}$, $\widehat{b}$ as operators, stating explicitly that they aren't when otherwise. Thus we can drop the hats to easy the notation.

All this must match the physics described by the Schrödinger equation in the nonrelativistic limit, where $|k| \ll m$. That takes (2.2) to the nonrelativistic dispersion relation

$$E \approx m + \frac{|\mathbf{k}|^2}{2m} \tag{2.4}$$

with the difference between the total energy and the rest mass being small, which makes the mass term dominate the time dependency of each Fourier mode of the field ϕ. To soften the time dependence we can detach that term by the redefinition

$$\phi(x) = e^{-imt}\widetilde{\phi}(x) \tag{2.5}$$

We next take the time derivatives

$$\frac{\partial^2\phi(x)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\widetilde{\phi}(x)}{\partial t} - im\widetilde{\phi}(x)\right)e^{-imt} \tag{2.6a}$$

$$= \left(\frac{\partial^2\widetilde{\phi}(x)}{\partial t^2} - 2im\frac{\partial\widetilde{\phi}(x)}{\partial t} - m^2\widetilde{\phi}(x)\right)e^{-imt} \tag{2.6b}$$

and neglect the second order term in favor of the other two. Note those have the mass in them, and if we were not using natural units would have factors of c^2, therefore only those prevail in the non-relativistic limit ($c \to \infty$). Plugging that

into the Klein-Gordon equation yields

$$(\Box + m^2)\phi(x) = \frac{\partial^2\phi(x)}{\partial t^2} - \nabla^2\phi(x) + m^2\phi(x) \tag{2.7a}$$

$$\approx \left(-2im\frac{\partial\widetilde{\phi}(x)}{\partial t} - \nabla^2\widetilde{\phi}(x)\right)e^{-imt} = 0 \tag{2.7b}$$

which is first order in time. We did recover an equation with the form of Schrödinger's

$$i\frac{\partial\widetilde{\phi}(x)}{\partial t} + \frac{\nabla^2\widetilde{\phi}(x)}{2m} = 0 \tag{2.8}$$

Nevertheless, we should point out an important fact to avoid confusion. That is not Schrödinger's equation. The Schrödinger equation itself is a wave equation, and a postulate of Quantum Mechanics. The Klein-Gordon equation however, refers to a field operator instead of a wavefunction. In the non-relativistic limit, it turns to the form of the Schrödinger's equation, but will still be a relation between operators. The result doesn't come as surprise, as all we did is to modify the dispersion relation of the theory, in accordance with (2.4). The Lorentz covariance of the relativistic theory was broken by neglecting second order time derivatives, yielding an equation which is linear in time.

2.2 Taking the non-relativistic limit

2.2.1 Separating particles and antiparticles

The previous section was fine as a warm up, but more is to be done. Although the field $\widehat{\phi}$ should reproduce the low energy behaviour of our relativistic theory, it lacks several key features. Once out of the relativistic realm, pair creation is no longer possible. Furthermore, due to the matter-antimatter asymmetry of our universe, in

a typical condensed matter system there won't be anything but particles. We need a field description in which particles and antiparticles are conserved separately, and whose ground state can be defined as containing many particles but essentially no antimatter. By many particles we mean a macroscopically large number of them, where the thermodynamic limit can be taken.

We will deal with the large particle number in the next section, let's first find the fields which suitably describe particles and antiparticles independently. For that aim, the Klein-Gordon equation can be split into two equations which are first order in time derivatives [10]. That is done by considering the field and its time derivative as independent variables

$$\frac{\partial \phi(x)}{\partial t} \equiv \Pi^{+}(x) \tag{2.9a}$$

$$\frac{\partial \Pi^{+}(x)}{\partial t} = \left(\nabla^2 - m^2\right)\phi(x) \tag{2.9b}$$

The operator Π is the conjugate momentum of ϕ, with the usual equal time commutation relations being satisfied.

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{2.10}$$

Particle and antiparticle solutions, both stemming from the single field $\phi(x)$ and its hermitian conjugate, can be separated by a canonical transformation. We can rewrite the above system of PDE's using matrix notation

$$\frac{\partial}{\partial t}\begin{pmatrix} \phi \\ \Pi^{+} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \nabla^2 - m^2 & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \Pi^{+} \end{pmatrix} = M\begin{pmatrix} \phi \\ \Pi^{+} \end{pmatrix} \tag{2.11}$$

and try diagonalize M. Due to the absence of interactions, the fields are a combination of momentum eigenstates. So, it will be convenient to work in momentum

space to get rid of ∇ operators and have numbers instead. Thus, the field is written as

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{kx}}\phi(\mathbf{k},t) \tag{2.12}$$

Note the time component hasn't been Fourier transformed.

Let's first take a real scalar field φ as a simpler example, just to illustrate the procedure, which will be followed by the complex case. We have then

$$\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \Pi\end{pmatrix} = \begin{pmatrix}0 & 1\\ \nabla^2 - m^2 & 0\end{pmatrix}\begin{pmatrix}\varphi\\ \Pi\end{pmatrix} \tag{2.13}$$

In momentum space, derivatives read

$$\dot{\varphi}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{kx}}\dot{\varphi}(\mathbf{k},t) \tag{2.14a}$$

$$\nabla^2\varphi(\mathbf{x},t) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, |\mathbf{k}|^2\, e^{i\mathbf{kx}}\varphi(\mathbf{k},t) \tag{2.14b}$$

So, the equations of motion transform to

$$\frac{\partial\varphi(\mathbf{k},t)}{\partial t} \equiv \Pi(\mathbf{k},t) \tag{2.15a}$$

$$\frac{\partial\Pi(\mathbf{k},t)}{\partial t} = \left(-|\mathbf{k}|^2 - m^2\right)\varphi(\mathbf{k},t) \tag{2.15b}$$

The matrix to diagonalize is

$$M = \begin{pmatrix}0 & 1\\ -E_{\mathbf{k}}^2 & 0\end{pmatrix} \tag{2.16}$$

The eigenvalues are $\pm iE_{\mathbf{k}}$, and there exists a transformation that takes M to diag-

onal form

$$SMS^{-1} = \Lambda \tag{2.17}$$

That way, the equation of motion is rewritten

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\Pi} \end{pmatrix} = S^{-1}\Lambda S \begin{pmatrix} \varphi \\ \Pi \end{pmatrix} \tag{2.18}$$

and the fields in the new basis are $S\binom{\varphi}{\Pi}$. The transformation matrix is found to be

$$S = \begin{pmatrix} \sqrt{\frac{E_{\mathbf{k}}}{2}} & \frac{i}{\sqrt{2E_{\mathbf{k}}}} \\ \sqrt{\frac{E_{\mathbf{k}}}{2}} & \frac{-i}{\sqrt{2E_{\mathbf{k}}}} \end{pmatrix} \tag{2.19}$$

Therefore, we can now write down new fields which don't mix positive and negative energy solutions.

$$\psi(\mathbf{k},t) = \sqrt{\frac{E_{\mathbf{k}}}{2}}\varphi(\mathbf{k},t) + \frac{i}{\sqrt{2E_{\mathbf{k}}}}\Pi(\mathbf{k},t) \tag{2.20a}$$

$$\psi^{+}(\mathbf{k},t) = \sqrt{\frac{E_{\mathbf{k}}}{2}}\varphi(\mathbf{k},t) - \frac{i}{\sqrt{2E_{\mathbf{k}}}}\Pi(\mathbf{k},t) \tag{2.20b}$$

The system of PDE's was finally decoupled. It's interesting to see what commutation relation this fields satisfy. Fourier transforming the real field analogue of (2.10) yields

$$[\varphi(\mathbf{k},t), \Pi(\mathbf{q},t)] = (2\pi)^3 i\delta^{(3)}(\mathbf{k}+\mathbf{q}) \tag{2.21}$$

which makes it easy to find that the new equal time commutation relations are

$$[\psi(\mathbf{k},t), \psi^{+}(\mathbf{q},t)] = (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}) \tag{2.22}$$

That is nothing else but the canonical commutation relations for bosonic creation and annihilation operators. In fact, looking at what ψ is in terms of free fields one finds our old $a_{\mathbf{k}}$'s and $a^+_{\mathbf{k}}$'s. Consider the real field analogue of (2.3), and the corresponding expression for the conjugate momentum

$$\Pi(x) = -i \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{\frac{E_{\mathbf{k}}}{2}} \left[a_{\mathbf{k}} e^{-ikx} - a^+_{\mathbf{k}} e^{ikx} \right] \tag{2.23}$$

If φ and Π are taken back to position space in (2.20)

$$\psi(\mathbf{k}, t) = \int d^3\mathbf{x}\, e^{-i\mathbf{kx}} \left[\sqrt{\frac{E_{\mathbf{k}}}{2}} \varphi(\mathbf{x}, t) + \frac{i}{\sqrt{2E_{\mathbf{k}}}} \Pi(\mathbf{x}, t) \right] = e^{-iE_{\mathbf{k}}t} a_{\mathbf{k}} \tag{2.24}$$

So, the new fields in position space for the free case are

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a_{\mathbf{k}} e^{-ikx} \tag{2.25a}$$

$$\psi^+(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a^+_{\mathbf{k}} e^{ikx} \tag{2.25b}$$

That makes the decoupling more explicit.

It's now time to carry on with the complex scalar field. First of all, we'll make use of the decomposition of the field into two real fields

$$\phi = \frac{1}{\sqrt{2}} (\phi_R + i\phi_I) \tag{2.26}$$

and analogously for Π. This way, (2.9) turns into

$$\dot{\phi}_R + i\dot{\phi}_I = \Pi_R - i\Pi_I \tag{2.27a}$$

$$\dot{\Pi}_R - i\dot{\Pi}_I = (\nabla^2 - m^2)(\phi_R + i\phi_I) \tag{2.27b}$$

Or written in matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} \phi_R \\ i\phi_I \\ \Pi_R \\ i\Pi_I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \nabla^2 - m^2 & 0 & 0 & 0 \\ 0 & m^2 - \nabla^2 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_R \\ i\phi_I \\ \Pi_R \\ i\Pi_I \end{pmatrix} \tag{2.28}$$

Following the previous approach, we diagonalize the matrix and find the new fields. The eigenvalues are again $\pm iE_{\mathbf{k}}$, each with multiplicity 2. The matrix S is now

$$S = \begin{pmatrix} \sqrt{\frac{E_{\mathbf{k}}}{2}} & \sqrt{\frac{E_{\mathbf{k}}}{2}} & \frac{-i}{\sqrt{2E_{\mathbf{k}}}} & \frac{i}{\sqrt{2E_{\mathbf{k}}}} \\ \sqrt{\frac{E_{\mathbf{k}}}{2}} & -\sqrt{\frac{E_{\mathbf{k}}}{2}} & \frac{-i}{\sqrt{2E_{\mathbf{k}}}} & \frac{-i}{\sqrt{2E_{\mathbf{k}}}} \\ \sqrt{\frac{E_{\mathbf{k}}}{2}} & \sqrt{\frac{E_{\mathbf{k}}}{2}} & \frac{i}{\sqrt{2E_{\mathbf{k}}}} & \frac{-i}{\sqrt{2E_{\mathbf{k}}}} \\ \sqrt{\frac{E_{\mathbf{k}}}{2}} & -\sqrt{\frac{E_{\mathbf{k}}}{2}} & \frac{i}{\sqrt{2E_{\mathbf{k}}}} & \frac{i}{\sqrt{2E_{\mathbf{k}}}} \end{pmatrix} \tag{2.29}$$

and the new fields are the combinations

$$\psi(\mathbf{k},t) = \sqrt{\frac{E_{\mathbf{k}}}{2}}\phi(\mathbf{k},t) + \frac{i}{\sqrt{2E_{\mathbf{k}}}}\Pi^+(\mathbf{k},t) \tag{2.30a}$$

$$\bar{\psi}(\mathbf{k},t) = \sqrt{\frac{E_{\mathbf{k}}}{2}}\phi^+(\mathbf{k},t) + \frac{i}{\sqrt{2E_{\mathbf{k}}}}\Pi(\mathbf{k},t) \tag{2.30b}$$

and their hermitian conjugates. Again, the equal time commutation relations are

$$[\psi(\mathbf{x},t), \psi^+(\mathbf{y},t)] = [\bar{\psi}(\mathbf{x},t), \bar{\psi}^+(\mathbf{y},t)] = \delta^{(3)}(\mathbf{x}\text{-}\mathbf{y}) \tag{2.31}$$

with any other combinations being zero. The interpretation follows that of the real case, for ψ/ψ^+ and $\bar{\psi}/\bar{\psi}^+$ are responsible of particle and antiparticle creation/annihilation respectively. In terms of free fields it's made more evident

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a_{\mathbf{k}} e^{-ikx} \tag{2.32a}$$

$$\psi^+(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a^+_{\mathbf{k}} e^{ikx} \tag{2.32b}$$

$$\bar{\psi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, b_{\mathbf{k}} e^{-ikx} \tag{2.32c}$$

$$\bar{\psi}^+(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, b^+_{\mathbf{k}} e^{ikx} \tag{2.32d}$$

As a final remark, note the decoupled version of (2.28) again reproduces the form of Schrödinger's equation for the non-relativistic limit

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi \\ \psi^+ \\ \bar{\psi} \\ \bar{\psi}^+ \end{pmatrix} = \begin{pmatrix} -iE_{\mathbf{k}} & 0 & 0 & 0 \\ 0 & iE_{\mathbf{k}} & 0 & 0 \\ 0 & 0 & -iE_{\mathbf{k}} & 0 \\ 0 & 0 & 0 & iE_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \psi \\ \psi^+ \\ \bar{\psi} \\ \bar{\psi}^+ \end{pmatrix} \tag{2.33}$$

Performing a low momentum expansion of the energy and going back to position space

$$i\frac{\partial\psi(x)}{\partial t} \approx (m - \frac{\nabla^2}{2m})\psi(x) \tag{2.34a}$$

$$i\frac{\partial\psi^+(x)}{\partial t} \approx -(m - \frac{\nabla^2}{2m})\psi^+(x) \tag{2.34b}$$

and same for $\bar{\psi}(x)$,$\bar{\psi}^+(x)$. We must again emphasize the operator nature of the equation, and consequently the fact that it's not Schrödinger's equation.

There's an important remark concerning degrees of freedom. The relativistic

scalar field ϕ has two degrees of freedom per spacetime point (one when it's a real field). In our approach to the non-relativistic theory, each pair of fields ψ,ψ^+ and $\bar{\psi}$,$\bar{\psi}^+$ account for one degree of freedom, matching the two degrees of freedom of the relativistic case.

2.2.2 Large particle number

If we are to describe large amounts of particles, we must include thermal degrees of freedom [21]. These will account for those aspects of collective behaviour which can't be dealt with by direct application of the physics of low particle numbers. In the context of statistical mechanics, we'll work with a grand canonical ensemble, which is to say that both energy and particle number are allowed to fluctuate. Temperature and chemical potential enter as Lagrange multipliers, imposing the constraint that average energy and average number of particles be fixed. We won't be very concerned about temperature by now, because typical condensed matter systems have thermal energies which are very small in comparison with the energy scale of relativistic theories ($T \ll m$),[12]. Temperature is related to the average momentum of particles, which will be low enough to set $T \approx 0$. However, having finite densities is something unavoidable, and therefore a non-zero chemical potential will be our focus here. As it was mentioned, a finite chemical potential has to do with an overall conservation of particle number, though fluctuations are allowed.

According to its thermodynamic definition, the chemical potential is the variation of internal energy respect to particle number, with volume and entropy held fixed [22]. In the scheme of QFT, it is related to internal symmetries and their charges. Each conserved quantum number will give rise to its corresponding chemical potential [6]. Due to the fact that particles and antiparticles contribute oppositely to things like electric charge, lepton and barion number, etc... one associates them with a chemical potential which is equal in magnitude but has opposite sign.

For instance, in a relativistic theory of a charged scalar field there is a chemical potential related to charge conservation, which is equivalent to say the difference between particle and antiparticle number can't change. In our non-relativistic case, particles and antiparticles are conserved separately, the latter being pretty much absent. However, we haven't yet introduced that imbalance formally. That will be the main goal of this section, by incorporating the chemical potential in our description.

The density matrix for the grand canonical ensemble is

$$\rho = e^{-\beta(H-\mu N)} = e^{-\beta H_e} \tag{2.35}$$

where H_e is an effective Hamiltonian to include the particle number contribution to the energy. Following the same approach, we'll replace H by $H - \mu Q$ in our field theory, with Q being the charge operator for the corresponding conserved quantity. Going back to the example of a complex scalar field, H and Q read

$$H = \int d^3\mathbf{x} \left[\Pi^+\Pi + (\nabla\phi^+)(\nabla\phi) + m^2\phi^+\phi\right] \tag{2.36a}$$

$$Q = i\int d^3\mathbf{x} \left(\phi^+\Pi^+ - \phi\Pi\right) \tag{2.36b}$$

Note Q is the electric charge operator. Putting that together and integrating by parts the spatial derivatives, the effective Hamiltonian density is written as

$$\mathcal{H}_e = \Pi^+\Pi + \phi^+\nabla^2\phi + m^2\phi^+\phi - i\mu(\phi^+\Pi^+ - \phi\Pi) \tag{2.37}$$

The equations of motion can be derived from it

$$\dot{\phi} = \Pi^+ + i\mu\phi \tag{2.38a}$$

$$\dot{\Pi}^+ = (\nabla^2 - m^2)\phi + i\mu\Pi^+ \tag{2.38b}$$

Note that if we redefine the time derivative operator as

$$\widetilde{\partial}_t \equiv \partial_t - i\mu \tag{2.39}$$

then the equations of motion take the same form as in the previous section. One can then go straight to the decoupled equations in terms of ψ and undo the change to find

$$i\partial_t\psi(x) = i(\widetilde{\partial}_t + i\mu)\psi(x) = (E_{\mathbf{k}} - \mu)\psi(x) \tag{2.40}$$

The equations of motion for ϕ^+ and Π take the same form as those of ϕ and Π^+, except for an opposite sign in μ. Therefore

$$i\partial_t\bar{\psi}(x) = i(\widetilde{\partial}_t - i\mu)\bar{\psi}(x) = (E_{\mathbf{k}} + \mu)\bar{\psi}(x) \tag{2.41}$$

Before getting any further, let's discuss the value that μ should roughly take for the condensed matter regime. Bose-Einstein statistics for a system in equilibrium yield densities

$$n_\pm(\mathbf{k}) = e^{-\beta(E_{\mathbf{k}} \mp \mu)} \tag{2.42}$$

where the + and - subscripts refer to particles and antiparticles respectively. In the low energy limit, the condition $n_+(\mathbf{k}) \gg n_-(\mathbf{k})$ implies that μ is of the order of magnitude of the mass. That makes a crucial difference between the equations of

motion of ψ and $\bar{\psi}$, namely

$$i\dot{\psi}(x) \approx (m - \mu - \frac{\nabla^2}{2m})\psi(x) \tag{2.43a}$$

$$i\dot{\bar{\psi}}(x) \approx (m + \mu - \frac{\nabla^2}{2m})\bar{\psi}(x) \tag{2.43b}$$

Remember that we are studying the physics of the system at energies much smaller than the mass scale. Then, with $\mu \sim m$, the energy gap for ψ becomes small compared to the energy scale, while that of $\bar{\psi}$ stays at the same order. In our low energy regime, excitations of the particle fields are attainable but not those of the antiparticle fields. Consequently, $\bar{\psi}$ can be neglected because it won't add anything to the dynamics of the system, the field just stays in its ground state.

In the following, we'll adopt a different definition for chemical potential, since it will always come in the combination $\mu - m$:

$$\mu \equiv \mu - m \tag{2.44}$$

That new μ is the chemical potential usually found in the condensed matter literature. Forgetting about $\bar{\psi}$, the Lagrangian density for the system reads

$$\mathcal{L} = \psi^+ \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - \mu \right] \psi \tag{2.45}$$

Note $i\psi^+$ is the conjugate variable of ψ, as we implicitly considered in (2.22) (although the factor of i wasn't taken into account, note it makes (2.22) be consistent with (2.10)). The Hamiltonian of the system is given by

$$\mathcal{H} = \psi^+\dot{\psi} - \mathcal{L} = \frac{1}{2m}\nabla\psi^+\nabla\psi + \mu\psi^+\psi \tag{2.46}$$

where integration by parts was performed on the term with spatial derivatives. We

didn't write $\mathcal{H} = \psi^+\dot{\psi}+\psi\dot{\psi}^+-\mathcal{L}$, because we don't have a degree of freedom per field, the two fields together only account for one degree of freedom. This Hamiltonian (adding suitable interaction terms) will be used later in the example description of some condensed matter systems.

Fermions are handled in an analogous way to our treatment for bosons, and we won't deal with them except when discussing superconductivity. One has to define creation and annihilation operators satisfying anticommutation relations, so the fermionic version of (2.22) will be an equal time canonical anticommutation relation. Starting with Dirac's Hamiltonian requires a Foldy-Wouthuysen transformation to separate particles and antiparticles. Nevertheless, we won't pursue it here because the non-relativistic expression for (2.45) and (2.46) is the same as in the bosonic case, as should be expected. The non-relativistic fields won't carry spinor indices (will consider a different field for each spin), and their fermionic character is taken into account in the path integral formalism by integrating over Grassmann valued fields.

2.2.3 Elementary excitations of the many particle theory

By considering a finite density of particles to be present in the system, the energy gap for producing elementary excitations got drastically reduced. So, does it mean that putting together a large number of particles of mass m leads to creation of other much lighter particles of mass μ when exciting that system? Clearly not, the kinetic term involves m and not μ. And remember our current μ came from $m - \mu$, there is still the same mass gap to create a particle of mass m.

Now, the field quanta have to do with exciting the already present particles, including the fact of being surrounded by the rest. The equations of motion describe a boson in an effective potential which summarizes the presence of many other bosons. The chemical potential acts like that external potential, despite being a bit

confusing to have a thermodynamic quantity involved in the microscopic dynamics. Being a thermodynamic quantity, it fits in a phenomenological description of the system as a whole and ignores its microscopical details, just reflecting the relevant contribution coming from the internal degrees of freedom.

The main reason why we got this equations where elements from microscopic and macroscopic physics get mixed together traces back to (2.37). We said that the effective Hamiltonian includes the particle number contribution to energy, while we should really say that what it represents is the Helmholtz free energy. We are taking the statistical average over states which minimize the free energy, and that was merged with the Hamiltonian describing the microscopic dynamics. A clearer picture of all this is obtained by accommodating the thermodynamic aspects in boundary conditions, keeping the Hamiltonian untouched [12]. We'll stick to our effective Hamiltonian anyway, the trick does the job so that usual procedures can be carried on as if $\mu = 0$.

3

Spontaneous Symmetry Breaking

3.1 Broken symmetries and phase transitions

Physical systems are said to have symmetries under certain transformations when those leave the properties of the system unchanged. Transformations are assembled and classified by means of group theory, a system possessing a given symmetry meaning that its features are invariant respect to a certain group of transformations. That has deep implications for continuous transformations, namely that there are conserved quantities in the system, each of them related to a generator of the group in question (Noether's theorem).

Very often we deal with situations where, despite the laws of physics being symmetric at a fundamental level, nature doesn't exhibit that symmetry. That comes from the fact that physical states need not respect the underlying symmetries satisfied by physical laws. In those cases it's said the symmetry has been *spontaneously broken*. That is in contrast with explicitly broken symmetries, where the laws themselves aren't invariant. For instance, weak interactions violate parity invariance and therefore one must describe them in a way that includes that fact. Neither explicit symmetry breaking nor discrete symmetries like parity will be of our interest in the coming discussion.

Spontaneous symmetry breaking is closely related to critical phenomena and phase transitions. The spontaneous breaking takes place when a system goes through a critical point, having several outcomes which are equivalent in light of the symmetry. However, one of them is picked out randomly and thus singled out respect to the rest, which breaks the symmetry. That arbitrary choice is caused by infinitesimal fluctuations around the state of the system, which has become unstable reaching the critical point. In that sense, it's normally not purely arbitrary but driven by any slight perturbation. Mathematically, a given element of the group has been made distinct from the others and the symmetry been broken, the selection not being prescribed by the theory but said to be spontaneous.

A change in the symmetry of a system implies a phase transition, though the opposite need not be true in general. In the specific case of a transition resulting in a less symmetrical phase we talk about symmetry breaking, and about restoring a symmetry when the system undergoes the inverse transition. Our interest will be in phase transitions which entail symmetry changes.

Another relevant concept to introduce is that of an *order parameter* [42]. The name stems from its attribute to characterize the usual emergence of order after the symmetry breaking phase transition. Such a parameter is normally zero above the phase transition and takes a finite value in the broken symmetry phase (often having 1 as an upper bound). Such parameters respond to the necessity of introducing additional variables to describe systems with broken symmetries.

Ferromagnetic materials provide a common example of spontaneous symmetry breaking, where the system is in a state with net magnetization and thus breaks rotational invariance. The breaking occurs in the transition to the ferromagnetic state, stage in which the system must pick a magnetization direction. The individual atomic magnetic dipole moments get aligned, with all possible alignment directions being a priori equivalent but not once we encounter the system oriented in a certain one. In practice, the orientation is usually imposed by an external magnetic field.

After the transition, the system still enjoys invariance respect to rotations with magnetization as an axis. Nevertheless, the group of those transformations (SO(2)) is smaller than the original one (SO(3)) and thus the magnetized phase is said to be less symmetric. The order parameter for this example is the net magnetization, which is zero prior to alignment (therefore reflecting the disordered state) and non-zero below the Curie point (ordered state).

3.2 Goldstone's Theorem

The least action principle lays at the core of most current physical theories. The symmetry arguments of the preceding section must therefore be manifest when writing down an action. Often, that reduces to the symmetries being present in the Lagrangian. Translating what we mentioned earlier to a more technical language, spontaneous breaking happens when a symmetry leaves the action invariant but not the ground state of the theory. The ground state is defined as that of minimum energy for some given physical conditions. That in turn means having a configuration of the fields which minimizes the expectation value of the Hamiltonian. The condition is satisfied for the fields being constant in spacetime (yielding zero for kinetic terms), together with minimizing the potential. Requiring the ground state to be invariant under the symmetry

$$U|0\rangle = e^{i\theta^\alpha T^\alpha}|0\rangle = |0\rangle \tag{3.1}$$

amounts to require $T|0\rangle = 0$, where T^α are the generators of the group and θ^α the parameters which specify the transformation. Recalling Noether's theorem, those generators are the quantum mechanical operators corresponding to conserved charges. So, for the ground state to be left invariant the charge operators must annihilate it. When that's not the case, the vacuum isn't uniquely defined. Say we

have a charge operator Q which acting on the vacuum

$$Q|0\rangle \neq 0 \tag{3.2}$$

It's said that Q is a broken generator. We can always redefine the energy values by performing a certain shift, so let's take $H|0\rangle = 0$. The charge conservation implies $[H, Q] = 0$, and together with $H|0\rangle = 0$ it leads to

$$H(Q|0\rangle) = [H, Q]|0\rangle = 0 \tag{3.3}$$

So, having a generator which doesn't annihilate the vacuum entails degeneracy, because the state $Q|0\rangle$ has the same energy as $|0\rangle$. Which is the true ground state is a matter of definition, any choice being equivalent usually. But is precisely that choice what spoils the symmetry.

The transformations (3.1) are said to be global when θ^α are taken to be constant, and local if their value is different for different points in spacetime. We will generally deal with the global ones, making the spacetime dependence explicit if otherwise.

We saw that when the vacuum is not annihilated by every generator of the group, the symmetry is broken. The general statement of Goldstone's theorem is that the breaking of k generators is accompanied by the appearance of k massless modes in the system (called *Goldstone modes*)[20]. The original relativistic formulation of the theorem is carried over to our nonrelativistic regime, ignoring possible subtleties which then arise [9][25].

Let's consider the non-relativistic bosonic system of previous chapter. The potential

$$V = \mu \left|\psi\right|^2 \tag{3.4}$$

has a minimum for $\langle\psi\rangle = 0$. A symmetry of (2.45) is that of rotating the fields in

the complex plane (i.e. U(1) transformations)

$$\psi' = e^{-i\theta}\psi \tag{3.5a}$$

$$(\psi^+)' = e^{i\theta}\psi^+ \tag{3.5b}$$

which become when infinitesimal

$$\delta\psi = -i\theta\psi \tag{3.6a}$$

$$\delta\psi^+ = i\theta\psi^+ \tag{3.6b}$$

The conserved charge is particle number

$$N = \int d^3\mathbf{x}\, j^0 = \int d^3\mathbf{x} \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\frac{\delta\psi}{\delta\theta} = \int d^3\mathbf{x}\, \psi^+\psi \tag{3.7}$$

and in the light of (2.32) it's easy to check that

$$N|0\rangle = \int d^3\mathbf{x}\, \psi^+\psi|0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} a^+_{\mathbf{k}} a_{\mathbf{k}}|0\rangle = 0 \tag{3.8}$$

So, the vacuum expectation value is zero and the conserved charge annihilates the vacuum, there's no broken symmetry.

Now take (2.45), flip the sign of μ and add some quartic interaction term

$$\mathcal{L} = i\psi^+\frac{\partial\psi}{\partial t} + \psi^+\frac{\nabla^2}{2m}\psi + \mu\psi^+\psi - \frac{g}{2}(\psi^+\psi)^2 \tag{3.9}$$

where the factor of 1/2 in the interaction term is unimportant, there's plenty of conventions. This Lagrangian provides an example of spontaneous breaking of the U(1) symmetry we had before. Our previous minimum of the potential is not a

minimum anymore. We must redefine the vacuum, but find that there's no unique choice. The same characteristic which makes the potential be symmetric leads to degenerate vacua. Namely, that the potential is a function of $|\psi|^2$, and thus if it's minimum is not at zero we can only get a condition over the modulus of ψ. The phase being unspecified means the potential's minimum forms a continuum, an S^1 manifold in particular. The system can pick any but just one concrete phase and that breaks the symmetry. The appearance of a Goldstone mode in a system like (3.9) will be shown in the next chapter [26][28].

3.3 Particle number uncertainty

A general result from many-particle quantum theory is that phase and particle number satisfy a Heisenberg uncertainty relation

$$\Delta\theta \, \Delta n \geq 1/2 \tag{3.10}$$

where capital delta delta refers to the uncertainty. It isn't a formal relation because there's no operator to associate with phase (albeit effort to define one [37][29][43]), same as there's no operator for time in quantum mechanics. However we still got the energy-time uncertainty relation, and similarly with phase and particle number.

For later convenience, we shall introduce the so called *coherent states* [32] due to Glauber, and brought to the condensed matter context by Anderson. They are defined as the eigenstates of creation and annihilation operators, which for one degree of freedom reads

$$a|\chi\rangle = \chi|\chi\rangle \tag{3.11}$$

Which means that either adding or removing particles from the system yields the state unchanged. These are in contrast with the Fock states $|n\rangle$ we're familiar with,

which are eigenstates of particle number. The coherent state can be written in the basis of Fock states

$$|\chi\rangle = e^{-\frac{|\chi|^2}{2}} \sum_{n=0}^{\infty} \frac{\chi^n}{\sqrt{n!}} |n\rangle \tag{3.12}$$

From (3.11), it's deduced that the average particle number is $\langle n\rangle = \langle\chi|a^+a|\chi\rangle = |\chi|^2$. Another property of coherent states is that they saturate the uncertainty relation, with (3.10) holding as an equality, $\Delta\theta\,\Delta n = 1/2$. This example system is known to represent a single particle on a harmonic potential. To avoid confusion respect to the common notion of particle, here we should rather speak of n as energy quanta rather than particle number.

The phase and particle number uncertainty relation applies to any number of degrees of freedom. Now notice a specific phase is set when the system described by (3.9) undergoes symmetry breaking, which automatically implies an infinite uncertainty in particle number (note the symmetry that just got broken had particle number as conserved charge). In that context, the field quanta are interpreted as physical particles. That's why a non-zero vacuum expectation value is often referred to as the presence of a condensate of particles. Some examples of that relation will be discussed in the following chapter.

4

Applications to Condensed Matter Systems

4.1 Bose-Einstein condensates

Bose-Einstein condensation is a phenomenon in which a large fraction of particles in a bosonic system occupy the lowest energy state (the ground state acquires a non-zero expectation value). Common requirements are that the bosons be weakly interacting and the temperature be near to absolute zero (thermal effects tend to drive particles to excited states, therefore destroying the condensate). There is a first order phase transition associated with Bose-Einstein condensation [4]. Though the ideal Bose gas exhibits this phenomenon, we will focus on the interacting case. The ideal gas can be described by (2.46), except that its chemical potential vanishes at condensation (together with taking the thermodynamic limit $V \to \infty$). The gapless mode automatically arises without need of further calculations.

4.1.1 Weakly interacting Bose Gas

We'll follow [31],[41] through this section. Consider a system of dilute cold particles of zero spin in a Bose-Einstein condensed phase. The behaviour of such a system is well described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\nabla\psi^{+}\nabla\psi - \mu\psi^{+}\psi + \frac{g}{2}(\psi^{+}\psi)^{2} \tag{4.1}$$

with the coupling constant g being small and where only two-particle pointlike interactions are considered (which is a valid approximation as far as the density is low enough and interactions aren't long ranged). Having a positive value for μ the system exhibits spontaneous symmetry breaking. The ground state expectation value is $|\langle 0|\psi|0\rangle|^2 = \mu/g$.

Recalling (2.32) and the identity

$$\int d^3\mathbf{x} e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}} = \delta^{(3)}(\mathbf{k}-\mathbf{q}) \tag{4.2}$$

the Hamiltonian can be rewritten in terms of creation and annihilation operators, which in normal ordered form reads

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\left(\frac{|\mathbf{k}|^2}{2m} - \mu\right)a^{+}_{\mathbf{k}}a_{\mathbf{k}} + \frac{g}{2}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}a^{+}_{\mathbf{k}}a^{+}_{\mathbf{q}}a_{\mathbf{p}}a_{\mathbf{k}+\mathbf{q}-\mathbf{p}}\right] \tag{4.3}$$

Supposing we were at $T = 0$, then most particles would be in the ground state (which for weak enough interactions can be regarded as the single particle ground state, $\mathbf{k} = 0$). Letting n be the total amount of bosons in the system and n_0 the occupation of the ground state, the condition

$$n - n_0 \ll n \tag{4.4}$$

allows for the following approximations, due to Bogoliubov [31]. The number operator $\hat{N}_0 = a_0^+ a_0$ has eigenvalue $n_0 \gg 1$, which means the commutator $[a_0, a_0^+] \ll n_0$ and thus can be neglected. The approximation consists in taking those operators as numbers

$$a_0^+ = a_0 = \sqrt{n_0} \tag{4.5}$$

and keeping terms in the Hamiltonian which are at most quadratic in $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^+$ for $\mathbf{k} \neq 0$. We're just making a mean-field approximation, but keeping the first corrections due to quantum fluctuations. The Hamiltonian now is

$$H = E_0 + \int\limits_{\mathbf{k}\neq 0} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2} \left[\left(\frac{|\mathbf{k}|^2}{2m} - \mu + 2gn_0 \right) (a_{\mathbf{k}}^+ a_{\mathbf{k}} + a_{-\mathbf{k}}^+ a_{-\mathbf{k}}) + gn_0 (a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+ + a_{\mathbf{k}} a_{-\mathbf{k}}) \right] \tag{4.6}$$

where in the first term we wrote $a_{\mathbf{k}}^+ a_{\mathbf{k}} = 1/2(a_{\mathbf{k}}^+ a_{\mathbf{k}} + a_{-\mathbf{k}}^+ a_{-\mathbf{k}})$, valid being inside of the integral and which will be convenient later. We would like to diagonalize the Hamiltonian, which is to say bring it to

$$H = E_0 + \int\limits_{\mathbf{k}\neq 0} \frac{d^3\mathbf{k}}{(2\pi)^3} E_{\mathbf{k}} \alpha_{\mathbf{k}}^+ \alpha_{\mathbf{k}} \tag{4.7}$$

That is done by means of a Bogoliubov transformation (See appendix of this chapter). Redefine

$$a_{\mathbf{k}} = u_{\mathbf{k}} \alpha_{\mathbf{k}} - v_{\mathbf{k}} \alpha_{-\mathbf{k}}^+ \tag{4.8a}$$

$$a_{-\mathbf{k}} = u_{\mathbf{k}} \alpha_{-\mathbf{k}} - v_{\mathbf{k}} \alpha_{\mathbf{k}}^+ \tag{4.8b}$$

and substitute into (4.6). We basically want to get rid of terms which only have

either creation or annihilation operators, thus arriving at the condition

$$2Au_{\mathbf{k}}v_{\mathbf{k}} = B(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) \tag{4.9}$$

where $A = \frac{|\mathbf{k}|^2}{2m} - \mu + 2gn_0$, and $B = gn_0$. Making use of the restriction $u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$ one finds

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = \frac{A}{\sqrt{A^2 - B^2}} \tag{4.10a}$$

$$2u_{\mathbf{k}}v_{\mathbf{k}} = \frac{B}{\sqrt{A^2 - B^2}} \tag{4.10b}$$

That gives the Hamiltonian the form in (4.7), with $E_{\mathbf{k}} = \sqrt{A^2 - B^2}$. In the mean field approximation $n_0 = |\langle 0|\psi|0\rangle|^2 = \mu/g$, thus $A = \frac{|\mathbf{k}|^2}{2m} + \mu$, and $B = \mu$, yielding

$$E_{\mathbf{k}} = \sqrt{\left(\frac{|\mathbf{k}|^2}{2m}\right)^2 + \mu\frac{|\mathbf{k}|^2}{m}} \tag{4.11}$$

Note the remarkable result, there's no energy gap in the spectrum anymore. For $|\mathbf{k}| \to 0$ the dispersion relation is

$$E_{\mathbf{k}} \approx \sqrt{\frac{\mu}{m}}|\mathbf{k}| \tag{4.12}$$

which describes a massless mode propagating at a speed of $v_s = \sqrt{\mu/m}$. That is nothing else but the Goldstone's boson associated with the symmetry breaking we pointed out earlier. It corresponds to oscillations of density in the condensate, which quanta are called phonons. These collective modes are the quantum analogue of sound, that's why the subscript was put in v_s.

In the description of a Bose-Einstein condensate, keeping track of the original constituent particles proves to be unsuccessful. One basically can't even discern any particle, they all behave coherently as a single entity. The thermal wavelength over-

comes the value for average interparticle spacing, making it impossible to identify or count particles. As we had mentioned before, the system acquires a definite phase and that renders the particle number all uncertain. This kind of behaviour is called quantum coherence.

4.1.2 The Landau criterion for superfluidity

The emergence of phonons lies at the heart of the superfluidity phenomenon. A superfluid set in uniform motion flows without resistance, it has zero viscosity. A normal fluid flowing through a pipe interacts with the walls and ends up dissipating its kinetic energy, which is converted into heat. A superfluid however, interacts with the walls too but for some reason maintains its momentum. Landau argued that the superfluid might not have allowed states to scatter into, arriving at his criterion for superflow. Let's discuss it in a classical context.

Consider a fluid of mass M moving along a pipe with velocity $\mathbf{v}$. Particles of the fluid can scatter against the wall if there are accessible final states, that is, lower energy states. Say the fluid is a condensate of bosons of mass m, so that in the fluid rest frame they all have $\mathbf{p} = 0$. A boson of momentum $\mathbf{p}$ has a momentum $\mathbf{p'} = \mathbf{p} - m\mathbf{v}$ in the pipe rest frame, so that for a total momentum $\mathbf{P}$ in the condensate frame we have $\mathbf{P'} = \mathbf{P} - M\mathbf{v}$ in the pipe frame. In the absence of an external potential, the internal potential energy of the fluid coming from particle interactions is the same in every frame. Knowing how momenta are related we find that $E^{'} = E - \mathbf{v}\mathbf{P} + \frac{1}{2}M|\mathbf{v}|^2$ for kinetic energies, and thus the change in energy for a transition seen from the pipe rest frame is $\Delta E^{'} = \Delta E - \mathbf{v}\Delta\mathbf{P}$. The relevant point is that one expects the condition

$$\Delta E - \mathbf{v}\Delta\mathbf{P} > 0 \tag{4.13}$$

to be fulfilled somehow. Final states would all be higher in energy ($\Delta E^{'} > 0$) and

therefore not accessible. Translated to the fluid rest frame, the condition (4.13) puts a restriction on the excitations of the condensate.

Imagine working with some fluid which lacks the phonon mode, and scattering a zero momentum particle. We get $\Delta\mathbf{P} = \mathbf{p}$ and $\Delta E = |\mathbf{p}|^2/2m$. However

$$\frac{|\mathbf{p}|^2}{2m} - \mathbf{vp} \not> 0 \tag{4.14}$$

the condition is not generally satisfied. Now take the case in which there is a phonon mode, which yields $\Delta E = v_s|\mathbf{p}|$. Landau's criterion

$$v_s|\mathbf{p}| - \mathbf{vp} > 0 \tag{4.15}$$

is satisfied as long as $|\mathbf{v}| < v_s$. The propagation speed of phonons in the fluid comprises a critical velocity for superflow, above which superfluidity ceases to exist.

The above treatment was however a bit oversimplified. The fluid does have allowed states to lose momentum to, though not enough of them to behave like a normal fluid. From a statistical point of view, the key difference lies in the density of states (that we'll denote by $n(E)$). At low energies, the density of states remains approximately constant for quadratically dispersing modes. On the other hand, $n(E) \propto E$ for linearly dispersing modes. There's far fewer possible states in the presence of gapless modes [51].

4.2 Superconductors

4.2.1 The origin of superconductivity

In the previous section we saw that superfluidity and Bose-Einstein condensation are intrinsically related phenomena. It is however not an exclusive phenomenon of bosonic systems. Under the appropriate conditions, fluids composed of fermionic

atoms do exhibit superfluidity as well. Fermion statistics prohibit condensation, but fermions can get around it by forming composite bosons (known as *Cooper pairs*). In fermionic superfluids, atoms pair together into diatomic molecules which are bosons and can thus condense.

The idea of fermion pairing came originally in the description of superconductivity by Bardeen, Cooper and Schrieffer [5]. Electrons in a superconductor are paired up into Cooper pairs, the metallic electron cloud constituting a Bose-Einstein condensate. The pair formation is a consequence of an effective attraction between electrons in the conduction band, which is mediated by phonons on the crystal lattice of the metal. Qualitatively speaking, nuclei in the lattice move slightly towards a bypassing electron due to coulombian forces. That produces a locally higher density of positive charge. The nuclei then oscillate a bit around their equilibrium positions before coming back to rest, and that oscillation perturbs the rest of the ion lattice. The propagating vibration is a phonon, carrying with it that higher density of positive charge that can affect a second electron (with opposite spin), effectively binding it to the first one. Due to the mass difference between electrons and nuclei, the distortion of the lattice caused by an electron is very small and only becomes important when there's no other sources of agitation for the nuclei (i.e. at very low temperatures). The correlation between electrons of a pair stays for long distances relative to the lattice spacing, so that many Cooper pairs overlap with each other. The situation is analogous to that of the weakly interacting bose gas previously considered. The condensate of electrons behaves collectively and coherently. Thus, even if the binding energy of a pair is small, breaking one affects the condensate as a whole. Collisions with the ion lattice don't provide enough energy to dismantle the condensate, and the superflow of electrons can go on.

4.2.2 Superconductivity as a symmetry breaking phase transition

The work done in [41] guides the following derivation. Let's start with the effective Lagrangian

$$\mathcal{L} = \sum_{\alpha=\uparrow,\downarrow} \psi_\alpha^+ \left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \right) \psi_\alpha - g\psi_\uparrow^+ \psi_\downarrow^+ \psi_\downarrow \psi_\uparrow \tag{4.16}$$

which describes an ideal fermion gas, plus an attractive interaction term (which means $g < 0$). The arrows refer to the electron's spin. It does not refer to the underlying lattice at all, neither to phonons mediating electron interactions, but this effective theory is accurate enough for our purpose. The fields here are Grassmann fields, but we'll soon be back to ordinary bosonic fields when describing Cooper pairs. Note the global U(1) symmetry of the Lagrangian.

Having two fields (spin up and spin down) it will be neater to work in matrix notation. Define

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^+ \end{pmatrix} \tag{4.17a}$$

$$\Psi^+ = \begin{pmatrix} \psi_\uparrow^+ & \psi_\downarrow \end{pmatrix} \tag{4.17b}$$

The partition function

$$Z = \int \mathcal{D}\Psi^+ \mathcal{D}\Psi e^{i\int d^4x \mathcal{L}} \tag{4.18}$$

cannot be evaluated directly because is not gaussian. Furthermore, we want to describe things in terms of fermion pairs, and actually we'll kill two birds with one stone through a Hubbard-Stratonovich transformation. By making use of the

identity

$$|a-b|^2 = |a|^2 + |b|^2 - a^*b - b^*a \tag{4.19}$$

and taking $a = \Phi$, $b = g\psi_\downarrow\psi_\uparrow$ (where Φ is not yet specified) we rewrite the interaction term as

$$g|\psi_\downarrow\psi_\uparrow|^2 = \frac{1}{g}|\Phi - g\psi_\downarrow\psi_\uparrow|^2 - \frac{1}{g}|\Phi|^2 + \Phi^+\psi_\downarrow\psi_\uparrow + \Phi\psi_\uparrow^+\psi_\downarrow^+ \tag{4.20}$$

Choosing $\Phi = g\psi_\downarrow\psi_\uparrow$ the first term on the right hand side cancels, so that we only have quadratic terms on ψ. More than just doing a mathematical trick, we are essentially defining a field describing pairs of electrons with opposite spins! And with the coupling constant included, in order to reflect that without the attractive interaction there are no Cooper pairs. Connecting with chapter 3, Φ would be the order parameter for the system in question ($\langle\Phi\rangle = 0$ in the normal conducting state, and becomes nonzero after the transition to superconductor). Obviously, we can't just plug (4.20) in and done. Because the interaction term is inside a functional integral and Φ depends on the fields to be integrated, an extra integration over Φ is necessary to be consistent. Putting all the ψ terms into matrix form, the partition function reads

$$Z = \int \mathcal{D}\Psi^+\mathcal{D}\Psi \int \mathcal{D}\Phi^+\mathcal{D}\Phi\, e^{iS[\Psi,\Psi^+,\Phi,\Phi^+]} \tag{4.21a}$$

$$S[\Psi,\Psi^+,\Phi,\Phi^+] = \int d^4x \left[\Psi^+ \begin{pmatrix} i\partial_t + \nabla^2/2m + \mu & -\Phi \\ -\Phi^+ & i\partial_t - \nabla^2/2m - \mu \end{pmatrix} \Psi + \frac{1}{g}|\Phi|^2\right] \tag{4.21b}$$

We can now integrate out the fermionic degrees of freedom. Remember the gaussian integral over Grassmann numbers

$$\int \left(\prod_i d\theta_i^* d\theta_i \right) \exp[-\theta_i^* M_{ij} \theta_j] = \det(\mathbf{M}) \tag{4.22}$$

The result is therefore

$$Z = \int \mathcal{D}\Phi^+ \mathcal{D}\Phi \begin{vmatrix} i\partial_t + \nabla^2/2m + \mu & -\Phi \\ -\Phi^+ & i\partial_t - \nabla^2/2m - \mu \end{vmatrix} e^{\frac{i}{g}\int d^4x\, |\Phi|^2} \tag{4.23}$$

To evaluate the determinant we'll make use of the identity

$$\det(\mathbf{M}) = \exp[\mathrm{tr}\{\ln(\mathbf{M})\}] \tag{4.24}$$

However, that trace applied to our expression shouldn't be confused with summing diagonal terms of the matrix appearing in (4.21). Working over a discrete set of variables, like that of (4.22), would indeed lead to the common notion of trace. The matrix it acts upon is an operator acting on a vector of variables. In our case, we need the trace of an operator acting on a space of functions, which will be called a functional trace (and denoted Tr). Note the possible confusion, happens that we have a matrix with operator elements, but hadn't we written things in matrix form we would still have a functional trace acting on some operator. The details on how to define Tr are put in an appendix. The result is

$$Z = \int \mathcal{D}\Phi^+ \mathcal{D}\Phi\, e^{iS[\Phi,\Phi^+]} \tag{4.25a}$$

$$S[\Phi,\Phi^+] = \int d^4x \left[\frac{i}{g}|\Phi|^2 + \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left\{ \ln \begin{pmatrix} k^0 - |\mathbf{k}|^2/2m + \mu & -\Phi \\ -\Phi^+ & k^0 + |\mathbf{k}|^2/2m - \mu \end{pmatrix} \right\} \right] \tag{4.25b}$$

Now, if we take a further approximation replacing Φ by its ground state average, and extremize the resulting expression for the action

$$\frac{\delta S[\langle\Phi\rangle, \langle\Phi^+\rangle]}{\delta\langle\Phi^+\rangle} = 0 \tag{4.26}$$

we obtain the corresponding equation of motion for $\langle\Phi\rangle$, which will indicate symmetry breaking if the result is not $\langle\Phi\rangle = 0$. Make use of

$$\delta \operatorname{tr}\{\ln(\mathbf{M})\} = \operatorname{tr}\{\delta \ln(\mathbf{M})\} = \operatorname{tr}\{\mathbf{M}^{-1}\delta\mathbf{M}\} \tag{4.27}$$

where in our case

$$\mathbf{M}^{-1} = \frac{1}{(k^0)^2 - (|\mathbf{k}|^2/2m - \mu)^2 - |\langle\Phi\rangle|^2} \begin{pmatrix} k^0 + |\mathbf{k}|^2/2m - \mu & \langle\Phi\rangle \\ \langle\Phi^+\rangle & k^0 - |\mathbf{k}|^2/2m + \mu \end{pmatrix} \tag{4.28a}$$

$$\delta\mathbf{M} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \delta\langle\Phi^+\rangle \tag{4.28b}$$

As $\langle\Phi\rangle$ is spacetime independent, the integral over x gives some uninteresting constant factor, and the result reads

$$\frac{i}{g}\langle\Phi\rangle + \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^0)^2 - (|\mathbf{k}|^2/2m - \mu)^2 - |\langle\Phi\rangle|^2 + i\epsilon} \operatorname{tr} \begin{pmatrix} -\langle\Phi\rangle & 0 \\ k^0 - |\mathbf{k}|^2/2m + \mu & 0 \end{pmatrix} = 0 \tag{4.29}$$

Where we put $i\epsilon$ to avoid the integral hitting a pole.

The last expression is the BCS gap equation, which we can rewrite as

$$\frac{i}{g} - \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^0)^2 - (|\mathbf{k}|^2/2m - \mu)^2 - |\langle\Phi\rangle|^2 + i\epsilon} = 0 \tag{4.30}$$

Note that the $\mathbf{M}$ we were working with is just the Fourier transformed matrix of that involved in the action for Ψ, prior to integrating out the fermionic degrees of freedom. Its inverse is the fermion propagator, the pole of which gives the dispersion relation

$$E = \sqrt{(|\mathbf{k}|^2/2m - \mu)^2 + |\langle\Phi\rangle|^2} \tag{4.31}$$

It tells us that a minimum energy $|\langle\Phi\rangle|$ is required for exciting an electron, the energy necessary to break the bound state that a Cooper pair means. If $|\langle\Phi\rangle| = 0$ there is no symmetry breaking and we recover the dispersion relation for electrons in a many-body effective potential.

Contour integration methods allow the integral over k^0 in (4.30) to be performed, yielding

$$\frac{1}{g} + \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{E} = 0 \tag{4.32}$$

We're supposing the attraction between electrons to be weak, $|\langle\Phi\rangle|$ is small and therefore $1/E$ must peak around $|\mathbf{k}|^2/2m - \mu = 0$. We can approximate

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \approx n(0) \int_{-\omega_D}^{\omega_D} d\mathcal{E} \tag{4.33}$$

where $\mathcal{E} = |\mathbf{k}|^2/2m - \mu$, and $n(0)$ is the density of states for $\mathcal{E} = 0$. The cutoff ω_D is the Debye frequency, which is the maximum energy phonons can carry. Then

$$\frac{1}{2} n(0) \int_{-\omega_D}^{\omega_D} d\mathcal{E} \frac{1}{E} = n(0) \ln \left(\frac{2\omega_D}{|\langle\Phi\rangle|} \right) \tag{4.34}$$

which means that

$$|\langle\Phi\rangle| = 2\omega_D e^{-g\,n(0)} \tag{4.35}$$

The Cooper pair field has a nonvanishing vacuum expectation value, and the U(1) symmetry is broken. Remember $g < 0$, so that the bigger its magnitude is, the stronger the interaction between electrons and thus the higher the departure from $|\langle\Phi\rangle| = 0$. Similarly to the case considered in the previous section, a linearly dispersing mode appears, this time with a propagation speed of $v_s = \sqrt{2\mu/3m}$. Whatever the spin orientation is, the symmetry transformations

$$\psi_\alpha^{'} = e^{-i\theta}\psi_\alpha \tag{4.36a}$$

$$(\psi_\alpha^{+})^{'} = e^{i\theta}\psi_\alpha^{+} \tag{4.36b}$$

that the Lagrangian satisfied, are not a symmetry of the Cooper pair ground state when it's nonzero.

$$\Phi^{'} = g\psi_\downarrow^{'}\psi_\uparrow^{'} = e^{-2i\theta}\Phi \tag{4.37}$$

Again, we found some condition on the modulus of the field's vacuum expectation value (4.35), the phase being unrestricted. Once more, the broken symmetry is related to specifying some value for that phase and the consequent particle number indeterminacy.

4.A The Bogoliubov transformation

Consider a system of bosons described in terms of creation and annihilation operators $a_{\mathbf{k}}$,$a^+_{\mathbf{k}}$. These satisfy

$$[a_{\mathbf{k}}, a^+_{\mathbf{q}}] = (2\pi)^3\delta^{(3)}(\mathbf{k\text{-}q}) \tag{4.38}$$

One can define some new operators via a canonical transformation [38]. Say we introduce

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}}a_{\mathbf{k}} + v_{\mathbf{k}}a^+_{-\mathbf{k}} \tag{4.39}$$

where u and v are some coefficients which we can assume to be real and depend only on $|\mathbf{k}|$. For the transformation to be canonical

$$[\alpha_{\mathbf{k}}, \alpha^+_{\mathbf{q}}] = (2\pi)^3\delta^{(3)}(\mathbf{k\text{-}q}) \tag{4.40a}$$

$$[\alpha_{\mathbf{k}}, \alpha_{\mathbf{q}}] = [\alpha^+_{\mathbf{k}}, \alpha^+_{\mathbf{q}}] = 0 \tag{4.40b}$$

the coefficients must satisfy

$$u^2_{\mathbf{k}} - v^2_{\mathbf{k}} = 1 \tag{4.41}$$

The transformation can be inverted to give

$$a_{\mathbf{k}} = u_{\mathbf{k}}\alpha_{\mathbf{k}} + v_{\mathbf{k}}\alpha^+_{\mathbf{k}} \tag{4.42}$$

4.B The functional trace

Proceeding through analogy, if for an ordinary diagonal matrix ($\mathbf{M}_{ij} = \lambda_i \delta_{ij}$) one has

$$\mathrm{tr}\{\ln(\mathbf{M})\} = \sum_i \ln(\lambda_i)\delta_{ij}\Big|_{i=j} \tag{4.43}$$

we can then define for an operator $\hat{\mathbf{M}}$

$$\mathrm{Tr}\{\ln(\hat{\mathbf{M}})\} = \int d^4x \, \ln\Big[\hat{\mathbf{M}}\delta^{(4)}(x - x^{'})\Big]\Big|_{x=x^{'}} \tag{4.44}$$

It is convenient to make use of the Fourier transform (4.2), and for $\hat{\mathbf{M}}$ only operating on the x variable we write

$$\mathrm{Tr}\{\ln(\hat{\mathbf{M}})\} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \, e^{ikx^{'}} \ln(\hat{\mathbf{M}}) e^{-ikx}\Big|_{x=x^{'}} \tag{4.45}$$

That form will be useful because in field theory the operator usually contains space-time derivatives, which have plane waves as their eigenfunctions. Therefore $\hat{\mathbf{M}}$ is replaced by its eigenvalues and setting $x = x^{'}$ the exponentials cancel

$$\mathrm{Tr}\{\ln(\hat{\mathbf{M}})\} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \, \ln(\mathbf{M}(k)) \tag{4.46}$$

For the case in which $\hat{\mathbf{M}}$ is written as a matrix of operator elements

$$\mathrm{Tr}\{\ln(\hat{\mathbf{M}})\} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \;\; \mathrm{tr}\{ln(\mathbf{M}(k))\} \tag{4.47}$$

5

Inequivalent Representations and Multiple Vacua

5.1 Finite degrees of freedom

5.1.1 Canonical commutation relations

We'll follow [44] in this chapter. In the same way as one can choose a reference frame out of infinite possibilities, generally one always has to define a specific vacuum. It is a common practice to define excited states (and the whole spectrum of the theory) by applying creation operators repeatedly on the vacuum. Thus, by choosing some vacuum we're consequently choosing the state space, although to build that space we make use of some particular set of creation and annihilation operators. Usually, turns out that the choice of vacuum implies a set of operators (through the requirement that all annihilation operators must annihilate it) and conversely, the definition of those operators entails a choice of state space.

Let's put that in a more formal way. Consider we're dealing with only one degree of freedom, and say we have a candidate vacuum which isn't annihilated by our annihilation operator $a|0\rangle \neq 0$. There exists some other operator α so that

$\alpha|0\rangle = 0$. Then, the vacuum we had is the actual vacuum respect to α, so let's call it $|0\rangle_\alpha$. Both operators can be related by a transformation

$$a = S\alpha S^{-1} \tag{5.1}$$

and that leads to

$$\alpha|0\rangle = S^{-1}aS|0\rangle = 0 \tag{5.2}$$

For that to hold, a must annihilate $S|0\rangle$, and consequently $S|0\rangle = |0\rangle_a$. It makes no difference to pick $|0\rangle_\alpha$ or $|0\rangle_a$ as long as it's accompanied by the correct operator. Therefore, our concern about the different vacua can be reduced to studying their corresponding operators.

An implicit condition on creation and annihilation operators is that they satisfy their characteristic canonical commutation relation. The term canonical commutation refers commonly to the relation (2.10), though as far as we are concerned that implies (4.38) and viceversa and therefore receiving the same name.

The state space we have been mentioning is a vector space, and more specifically a Hilbert space $\mathscr{H}_S$. An operator is a map between state vectors, the set of operators thus constituting a conjugate space to the state Hilbert space, which is again a Hilbert space itself, $\mathscr{H}_O$. The canonical commutation relations define an algebra over $\mathscr{H}_O$, so that a given choice for the set of creation and annihilation operators is a representation of the algebra. In the following, we'll commit some abuse of language talking about $\mathscr{H}$ to mean a representation over $\mathscr{H}$.

5.1.2 Unitary equivalence

In Quantum Mechanics, von Neumann's theorem assures the Hilbert space is unique up to unitary equivalence [40]. The theorem proves the existence of a unitary version

of (5.1) involving any two operators, as far as the total number of them is finite. The corresponding version of (4.38) for that case is

$$[a_i, a_j^+] = \delta_{ij} \tag{5.3}$$

where i,j take discrete values. In the context of quantum mechanics unitary transformations preserve probabilities and expectation values in general, so the physical equivalence follows. That is not the case when there's infinite degrees of freedom. The Fock spaces of quantum field theory need not be unitarily equivalent to each other, and the physics they bring might differ substantially. From now on, when talking about equivalence or inequivalence, it will implicitly mean unitary equivalence or inequivalence.

We can see the mentioned equivalence making use of the coherent states introduced in chapter 3. Consider again a system with only one degree of freedom. Say we have operators a and α with respective vacua $|0\rangle_a$ and $|0\rangle_\alpha$. The transformation

$$\alpha = a + \theta \tag{5.4}$$

is the Bogoliubov transformation for coherent states, where θ is a complex constant parameter. One can see that

$$a|0\rangle_\alpha = -\theta|0\rangle_\alpha \tag{5.5}$$

Therefore $|0\rangle_\alpha$ is a coherent state respect to a, with eigenvalue $-\theta$. The translation (5.4) can be written as

$$\alpha = U(\theta) a U(\theta)^{-1} \tag{5.6a}$$

$$U(\theta) = e^{\theta^* a - \theta a^+} \tag{5.6b}$$

Which implies the following relation between vacua

$$|0\rangle_\alpha = U(\theta)|0\rangle_a \tag{5.7}$$

The Baker-Campbell-Hausdorff formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}\dots \tag{5.8}$$

used on $U(\theta)$ (where further nested commutators vanish and suffices to include the terms shown above) yields

$$U(\theta) = e^{-\frac{1}{2}|\theta|^2}e^{-\theta a^+}e^{\theta^* a} \tag{5.9}$$

And putting that in the relation between vacua

$$|0\rangle_\alpha = e^{-\frac{1}{2}|\theta|^2}e^{-\theta a^+}|0\rangle_a = e^{-\frac{1}{2}|\theta|^2}\sum_{n=0}^{\infty}\frac{(-\theta)^n}{\sqrt{n!}}|n\rangle_a \tag{5.10}$$

If $|0\rangle_\alpha$ can be expressed as a superposition of states from the Hilbert space of a, $\mathscr{H}(a)$, then any state in $\mathscr{H}(\alpha)$ can. The collection of state spaces can be parameterized by θ, and related to each other by the unitary transformation (5.6). This can be extended to the case of several operators a_i, α_i provided the number of them is finite. Any state in a given space can be written as a well defined sum of states from another space, thus being equivalent.

We shall make a couple of remarks to avoid confusion. In quantum mechanics, if one has a system of two distinguishable particles, the Hilbert space of the first particle is considered different to that of the second. Nevertheless, the total Hilbert space (expressed as the tensor product of the individual spaces) is equivalent to any other that might arise if choosing another space for any or both particles. Equally, in the scalar Quantum Field Theory considered at the start of chapter 2, for two fields

each of which satisfies the Klein-Gordon equation but with distinct masses, the state spaces belong to different Fock spaces. All this might indeed sound obvious, as they are not the same physical system. We won't expect finding a state for a particle of mass m_1 in the state space of a particle of mass $m_2 \neq m_1$. Considerations about equivalence are meant to refer to representations of a given specific system, and not inbetween different systems. But that's where the striking fact comes in, as even when considering two identical systems we often face the problem of inequivalence (for systems with infinite degrees of freedom). One can argue about those being really identical, because typically there's phase transitions involved, and as we saw the behaviour can change drastically from a phase to another.

5.2 Infinite degrees of freedom

The problem of many vacua brings additional difficulty when considering systems with infinite degrees of freedom. In particular, the unitary equivalence theorem won't hold in general when the number of degrees of freedom is not finite.

There are still equivalent representations, those of possibly greater relevance being the ones spanned by symmetry transformations. A familiar example are the spacetime symmetry transformations, for instance the translation

$$\phi(\mathbf{x}', t') = e^{i[H(t'-t)-\mathbf{P}(\mathbf{x}'-\mathbf{x})]}\phi(\mathbf{x}, t)e^{-i[H(t'-t)-\mathbf{P}(\mathbf{x}'-\mathbf{x})]} \tag{5.11}$$

The field operators at each side of the expression are different operators, but we denote both with the same symbol and let the spacetime label do the rest. We allow that abuse of notation because of the existing equivalence. Equally, note that even though the operators $a_{\mathbf{k}}, a^{+}_{\mathbf{k}}$ for different $\mathbf{k}$'s are independent, we name them all the same except for a subscript. Invariance under Lorentz boosts is behind that notation. Symmetry transformations are meant to leave the vacuum invariant, and

state spaces which share the vacuum are automatically equivalent.

To put a different case, which is also familiar, we'll consider an example of unitarily equivalent spaces that don't share the vacuum. For instance, the usual procedure when one has $H|0\rangle \neq 0$ is to perform an energy shift to have zero energy for the ground state, which amounts to redefine the Hamiltonian as

$$H^{'} = H - \langle 0|H|0\rangle \tag{5.12}$$

so that $H^{'}|0\rangle = 0$. We can naively think of a transformation generated by an operator R satisfying $R|0\rangle \neq 0$ and $[R, H] \neq 0$, which relates the two Hamiltonians

$$H^{'} = e^{i\theta R} H e^{-i\theta R} \tag{5.13}$$

Considering an infinitesimal transformation ($\theta \ll 1$)

$$H^{'} \approx H + i\theta[R, H] \tag{5.14}$$

we obtain the condition that $[R, H] = i/\theta\langle 0|H|0\rangle$ for the transformation to be an energy shift. But as we saw at the beginning of the chapter, we can either redefine operators or states. That is, $H^{'}|0\rangle = 0$ means that

$$H(e^{-i\theta R}|0\rangle) = 0 \tag{5.15}$$

so we could well have chosen to transform $|0\rangle$ to $e^{-i\theta R}|0\rangle$ and kept the Hamiltonian. The physical equivalence under overall energy shifts doesn't come as a surprise. All we're saying with this examples is that we can relate states from equivalent state spaces, something we knew already. In the following section we'll make use of coherent states once more to discuss the inequivalence explicitly.

5.2.1 Unitary inequivalence

Consider a system described by a set of creation and annihilation operators $a_{\mathbf{k}}$,$a_{\mathbf{k}}^{+}$ satisfying (4.38) and whose corresponding vacuum is denoted as $|0\rangle_a$. The Bogoliubov transformation of the finite case, which related the operators to some other α,α^{+} will relate inequivalent Fock spaces this time. Say

$$\alpha_{\mathbf{k}} = a_{\mathbf{k}} + \theta_{\mathbf{k}} \tag{5.16}$$

and the corresponding vacuum for α is denoted $|0\rangle_\alpha$. Again, that vacuum is a coherent state for a

$$a_{\mathbf{k}}|0\rangle_\alpha = -\theta_{\mathbf{k}}|0\rangle_\alpha \tag{5.17}$$

The operator in (5.7) for this case is

$$U(\theta) = \exp\left\{\int \frac{d^3\mathbf{k}}{(2\pi)^3}(\theta_{\mathbf{k}}^{*}a_{\mathbf{k}} - \theta_{\mathbf{k}}a_{\mathbf{k}}^{+})\right\} \tag{5.18}$$

And (5.10) takes the form

$$|0\rangle_\alpha = \exp\left\{-\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}|\theta_{\mathbf{k}}|^2 - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\theta_{\mathbf{k}}a_{\mathbf{k}}^{+}\right\}|0\rangle_a \tag{5.19}$$

A reasonable requirement on vacua is that they be translationally invariant. Applying that to $|0\rangle_\alpha$, we require $\theta_{\mathbf{k}}$ not to depend on position. It's spatial dependence is obtained by Fourier transformation, and imposing that to equal a constant θ

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\theta_{\mathbf{k}}e^{i\mathbf{kx}} = \theta \tag{5.20}$$

one obtains by inverting the transformation

$$\theta_{\mathbf{k}} = \int d^3\mathbf{x} e^{-i\mathbf{k}\mathbf{x}}\theta = \theta\delta^{(3)}(\mathbf{k}) \tag{5.21}$$

That implies

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}|\theta_{\mathbf{k}}|^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\theta^2\delta^{(3)}(\mathbf{k})\delta^{(3)}(\mathbf{k}) = \theta^2\delta^{(3)}(0) \tag{5.22}$$

and then

$$\exp\left\{-\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}|\theta_{\mathbf{k}}|^2\right\} = 0 \tag{5.23}$$

which means (5.19) is not a well defined relation, because it's coefficient is zero. We can't write the ground state of the Fock space $\mathscr{F}(\alpha)$ as a superposition of states of $\mathscr{F}(a)$, and the same is true for any other state of $\mathscr{F}(\alpha)$ (because the space is built from $|0\rangle_\alpha$). To put it another way, on the grounds of (5.19) the inner product of any vector from $\mathscr{F}(a)$ with $|0\rangle_\alpha$ vanishes ($\mathscr{F}(a)$ is supposed to contain finite norm vectors only). The two spaces are inequivalent, and in contrast with the finite case, now θ parameterizes an infinite set of inequivalent Fock spaces.

Recall that after (3.12) we related the average particle number of a coherent state to it's eigenvalue respect to the annihilation operator. If we now evaluate the expectation value of the number operator $N(a)$ in the vacuum of α we find

$${}_\alpha\langle 0|N(a)|0\rangle_\alpha = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,{}_\alpha\langle 0|a^+_{\mathbf{k}}a_{\mathbf{k}}|0\rangle_\alpha = \int \frac{d^3\mathbf{k}}{(2\pi)^3}|\theta_{\mathbf{k}}|^2 \tag{5.24}$$

which was shown to diverge. The vacuum $|0\rangle_\alpha$ contains an indefinite amount of the excitations that $a_{\mathbf{k}}$,$a^+_{\mathbf{k}}$ are responsible for.

So far we have met some examples of this sort of paradox arising in condensed matter systems undergoing phase transitions. However, that is not an exclusive characteristic of condensed matter, there are phase transitions in relativistic QFT

as well, most notably those of QCD and the Electroweak transition. It isn't exclusive of symmetry breaking phenomena either. In relativistic QFT it becomes necessary to perform radiative corrections and renormalization procedures, which is nothing else but a change between inequivalent state spaces (and which amounts to a transformation of the Bogoliubov kind). For instance, the removal of infrared divergences in QED is done by taking into account the cloud of infinite soft photons that surround every electrically charged particle. Equally, effects like vacuum polarization lead to a description of photons carrying a cloud of charged particles. Turns out that to formulate an interacting field theory in a consistent way, the correct state space is that of dressed particles or infraparticles, name referring to the bare particles together with the mentioned clouds. That space is inequivalent to the space of bare particles alone, we again face an indefinite amount of particles present in the clouds. In theories of curved spacetime, when two observers lack a mutual global coordinate system their representations of quantum fields will be inequivalent, and that leads to a discrepancy of particle content in the vacuum between them [45].

Another example of inequivalence is that between a free field theory and an interacting one, summarized in Haag's theorem. It rules out the possibility of rigorously defining an interaction picture, because such a picture is based on relating states of the interacting theory to those of the free theory, concretely by unitary transformations generated by the free Hamiltonian.

As a final example, we'll note that the inequivalence shows up even in the simplest case. In the canonical quantization of the Klein-Gordon field one runs into the problem of an infinite energy vacuum, coming from the individual zero-point energies of an infinite set of harmonic oscillators. That is solved by defining a renormalized Hamiltonian which is obtained substracting the vacuum energy. That is another example of inequivalence, for if one considers (5.12) with an infinite vacuum expectation value for H, then there's no operator R which can generate the transformation (5.12) unitarily.

5.2.2 Construction of Quantum Field Theories

The issue of unitary inequivalence in quantum field theories is a central aspect to take into account for their construction. Familiar procedures are canonical quantization, the path integral quantization, etc... The canonical quantization refers to the construction of a quantum theory starting from its classical counterpart. The canonical variables and their conjugate momenta are promoted to operators, and required to satisfy canonical commutation relations. Generally, some subtleties arise on the way, like in the last example of the previous section. To get to the correct state space one had to renormalize, something which amounts to prescribe a normal ordering for creation and annihilation operators inside the Hamiltonian (displacing all $a_{\mathbf{k}}$'s to the right of $a^{+}_{\mathbf{k}}$'s, we used it in (4.3) without further justification).

However, that seemingly *ad-hoc* consideration is due to our lack of a universal way to deal with the above mentioned. Different quantization procedures might or might not be suitable for given situations, and the same applies to operator ordering [7]. The root of that difficulty to find a physical description of a system originates to big extent in the existence of inequivalent representations, and in the absence of a general way to pick the right one. Recent efforts to tackle this matter have lead to more algebraic [24] and geometrical [13] perspectives of field theory.

6

Conclusion

This thesis is intended to be a modest analysis of the similarities and differences between quantum field theories of high energy particle physics and condensed matter physics. The approach assumes the reader is familiar with the first, therefore being centered on the condensed matter part. We've seen how to formulate a field theory of condensed matter as an effective, low-energy, finite density version of the known relativistic theory, enabling the background formalism to be used throughout.

Of special interest is the interplay between methods and concepts in high energy and condensed matter physics, and to our purpose, those of spontaneous breaking of symmetries and phase transitions. We've thus dedicated a chapter to revisiting these notions, setting the stage for a field description of example cases of phase transitions in condensed matter. Those examples try to highlight the relevance of shifting between field descriptions in terms of different field quanta, necessary when phase transitions involving particle condensation take place. Bogoliubov transformations serve as the tool to change between those descriptions.

Those first chapters are expected to be a motivation for what is meant to be the central topic of the dissertation, the unitary inequivalence of representations in quantum field theories. That is a necessary aspect for their success in accounting for the rich phenomena that matter exhibits in its diverse phases, but simultaneously

entails a drawback. Our current lack of understanding in how to cope with the inequivalence poses an obstacle for a fully axiomatic formulation of QFT, which remains as an open problem.

As a final remark, it's interesting to note the philosophical questions which these discussions relate to. On the one hand, quantum field theory took us closer to understand elementary particles, and seems to give confidence in regarding them as fundamental. On the other hand, quantum field theory in the condensed matter systems we've encountered does a bit the opposite, feels like it takes away some of the fundamental nature of the constituents of matter. QFT definitely is and has meant a great progress in science, but in a sense the overall situation might not resemble big development in the inconclusive question of how fundamental elementary particles are. Another fascinating issue is that of the possible need for a paradigm shift in order to deal with certain aspects of nature [2]. The phrase *a qualitative change from a quantitative change* is something we've seen realized in terms of phonons and other emergent phenomena. The emergence of a given behaviour in a complex or many-body system, when being something unexpected from knowledge of the physics of its constituents, and even being indescribable in terms of them, is indeed an amazing facet of nature. Reductionism, the perspective with the motto *a system is the sum of its parts*, is the view opposed to emergence or holism. Though the reductionist attitude has been incredibly fruitful along the history of science, the need of new approaches is usually a symptom of good sense.

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