Black Holes, Dualities and the non-linear sigma model

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Contents

Chapter 1

Introduction

The dream of all theoretical physicists around the world is to develop a theory that provides a unified description of the four fundamental forces. Over the last century, many developments have been made in this direction, mainly based on the hypothesis that our spacetime has extra dimensions curled up in a very small compact manifold. Assuming this, spacetime symmetries in the compact dimensions can be interpreted as internal symmetries from the lower-dimensional point of view. Using the reversed argument, one may unify the internal and spacetime symmetries of a lower-dimensional theory into spacetime symmetries of a higher-dimensional system. As a result of this, the eleven-dimensional M-theory and the ten-dimensional string theory came up to be the best candidates for unification. But if these theories are indeed fundamental and they describe our four-dimensional universe, there should be a way to extract lower-dimensional theories from them. The first to introduce such a method were Kaluza and Klein (with many contributions by Pauli) and although there have been many developments and advances since their days, the general procedure bears their names.

The Kaluza-Klein dimensional reduction will be studied in detail in chapter 2. We first present the general idea of this method, which is nothing but a compactification on a compact Manifold M together with a consistent truncation to the massless sector (in other words, we expand the higher-dimensional fields into fourier modes, of which only the massless ones are included in the effective, lower-dimensional theory). Special emphasis is given to the scalar lagrangian emerging from the lower-dimensional theory as it carries all the information about the residual symmetries. We express it in terms of the so-called coset representative $\mathcal{V}$, which is a matrix representing points on the scalar manifold (the scalar manifold is the manifold parameterized by the scalar fields of the theory), and we note that it is manifestly invariant under the action of a symmetry group G

$$\mathcal{V} \to \mathcal{V}\Lambda, \qquad \mathcal{L} \to \mathcal{L}, \tag{1.1}$$

where $\Lambda \epsilon G$. But in order for $\mathcal{V}\Lambda$ still to represent points on the scalar manifold, we have to do a compensating local transformation O

$$\mathcal{V} \to O\mathcal{V}\Lambda, \qquad \mathcal{L} \to \mathcal{L}, \tag{1.2}$$

where $O \epsilon K$ and K is the maximal compact subgroup of G. We also note that G has a transitive action on the scalar manifold and thus we deduce that the latter can in fact be identified with the coset manifold G/K with a global G symmetry. In the last section of chapter 2, we investigate the evolution of the supergravity cosets(summarized in table 2.1) through the dimensions using Dynkin diagrams.

In the next chapter, we explore stationary solutions of supergravity. The reason why we restrict our discussion to this kind of solution and do not study supergravity solutions in general is just a matter of taste. We introduce dimensional reduction along the time direction, which differs from the usual reduction described in the previous chapter only on the fact that the original theory is now reduced on a manifold with a Minkowskian signature instead of Euclidean (this gives rise to some extra minus signs). The coset manifold in this case is given by G/K^*, where K^* is the non-compact form of K, and the metric $G_{AB}(\phi)$ on it is now indefinite. This recasts the problem in terms of a particular type of a non-linear sigma model with gravitational constraints. Then, we specialize our discussion for four-dimensional theories and we explore in detail the four-dimensional, Einstein-Maxwell example by solving explicitly the equations of motion that arise(up to simplifying assumptions). Surprisingly, we end up with the well-known Reissner-Nordström, charged black hole solution.

In chapter 4, we explore the properties of the solutions of N-extended, four-dimensional supergravity theories that are already G_4-symmetric in 4 dimensions. We define the Komar mass m, NUT charge n and the electric q_I and magnetic p_I charges related to the four-dimensional field strengths. Upon reduction, the symmetry group is enhanced to G and because of the G-invariance of the three-dimensional theory, we can introduce a "conserved" charge matrix C that satisfies the so-called characteristic equation. As we will discuss later, this equation selects out of all G-orbits the acceptable ones and restricts the scalar charges to be functions of (m, n, q_I, p_I) only. The charge matrix C is associated to a charge state $|C>$, which transforms as a $Spin^*(2N)$ chiral spinor (for N-extended supergravity, the group K^* is the product of $Spin^*(2N)$ with a symmetry group determined by the matter content of the theory). For asymptotically flat solutions, the BPS condition is equivalent to an algebraic "Dirac" equation

$$(\epsilon^i_a a_i + \Omega_{a\beta}\epsilon^\beta_i a^i)|C>= 0, \tag{1.3}$$

where a_i and a^i are the lowering and raising operators and ϵ^i_a and ϵ^a_i are the asymptotic supersymmetric parameters.

In the final chapter, we will *briefly* discuss the attractor formalism. A central question in black hole thermodynamics concerns the statistical interpretation of the black hole entropy. String theory has provided new insights here, which enable the identification of the black hole entropy as the logarithm of the degeneracy of states dQ of charge Q belonging to a certain system of microstates; in string theory these microstates are provided by the states of wrapped brane configurations of given momentum and winding. But, there is the dangerous possibility that the entropy of the black hole may depend on parameters that are continuous, namely the value of the scalar fields at infinity(ie the so-called moduli). This would be a problem since the number of microstates with given charges is an integer that should not depend on parameters that can be varied continuously -it should only depend on quantities that take discrete values, such as the electric and magnetic charges and the angular momenta. As it turns out, the entropy of a black hole is determined by the behavior of the solution at the horizon of the black hole (not at infinity) and the values of the scalars there $\phi|_{horizon}$ are completely determined by discrete quantities, such as the charges. In other words, ϕ is determined by a differential equation whose solution flows to a definite value at the horizon, regardless of its boundary value at infinity. This solution is called an attractor and its existence is necessary for a microscopic description of the black hole entropy to be possible.

It should be emphasized that the work presented here is by no mean original.

Chapter 2

Kaluza-Klein Dimensional Reduction on S^1 and T^n

The Kaluza-Klein(KK) compactification of the standard extra dimensions was extensively studied in [1, 2, 3, 4]. In the following sections, we will discuss only some basic aspects of this theory. We will restrict ourselves to studying reduction only on the circle S^1 and on the n-dimensional torus T^n. At the same time, we will discuss in detail the duality symmetries of the dimensionally reduced theories and the scalar coset manifold in various dimensions. We will *not* discuss dimensional reduction on other, more complicated manifolds (eg the Calabi-Yau threefold CY_3), brane-world Kaluza-Klein reduction and we will leave aside compactification in the presence of fermions(fermions will be consistently ignored through out this discussion). We will also *not* talk about solution oxidation.

2.1 Kaluza-Klein Dimensional Reduction on S^1

Dimensional Reduction of the Einstein-Hilbert Lagrangian on S^1:

For simplicity, we will first study the reduction on a circle S^1. As all the theories to be considered are theories of gravity plus additional terms, a good starting point would be to demonstrate how the dimensional reduction of gravity proceeds. In D+1 dimensions, the Einstein gravity is described by the so-called Einstein-Hilbert

Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R}, \tag{2.1}$$

where $\hat{R}$ is the Ricci scalar and $\hat{g}$ represents the determinant of the metric in D+1 dimension (throughout this dissertation, we will denote higher-dimensional fields with a hat). Now, split the (D+1)-dimensional coordinates $\hat{x}^{\hat{M}}$ into (x^M, z), where z is the dimension to be compactified on a circle of radius L and x^M parameterizes the D-dimensional spacetime transverse to z. Then, expand all the components of the metric as fourier series

$$\hat{g}_{\hat{M}\hat{N}}(x,z) = \sum_n \hat{g}^{(n)}_{\hat{M}\hat{N}}(x) \exp^{inz/L}, \tag{2.2}$$

with the constraint $\hat{g}_{\hat{M}\hat{N}}(x,0) = \hat{g}_{\hat{M}\hat{N}}(x, 2\pi nL)$ due to the circle. By doing this, we just replace the higher-dimensional metric with an infinite number of fourier fields labeled by n. One may argue that the modes corresponding to n=0 are massless, while the fields with $n \neq 0$ have enormous masses $m = \frac{n}{L}$(if L is really small) and thus can be neglected in the lower-dimensional theory. The argument goes as follows:

let $\hat{\phi}$ be a massless scalar field in D+1 dimensions satisfying the Klein-Gordon(K-G) field equation

$$\partial_{\hat{M}}\partial^{\hat{M}}\hat{\phi} = 0.$$

If we compactify the z coordinate as before and then we fourier expand $\hat{\phi}$ in terms of lower-dimensional fields ϕ_n, we obtain:

$$\hat{\phi}(x,z) = \sum_n \phi_n(x) e^{inz/L}.$$

Substituting this fourier expansion in the K-G equation, we find that $\phi_n(x)$ satisfies the equation

$$\partial_M \partial^M \phi_n - \frac{n^2}{L^2}\phi_n = 0.$$

This equation can be recognized as the K-G equation of a massive scalar field $\phi_n(x)$ in D spacetime dimensions. Thus, we conclude that when a (D+1)-dimensional scalar is compactified on a circle, we acquire a tower of D-dimensional scalar fields that are z-independent and have mass equal to n/L. The usual Kaluza-Klein approach is to assume that the radius, L, of the compactification circle S^1 is very small so that we do not observe the extra dimension. This means that the masses, m, of the modes that correspond to $n \neq 0$ are extremely large and accordingly will not play any role in the effective D-dimensional theory.

The argument stated above can not be applied naively for the metric field, as $\hat{g}_{\hat{M}\hat{N}}$ can not be interpreted as a metric anymore from the D-dimensional point of view. Due to the splitting of the D + 1 coordinates, the components of the metric can be decomposed as shown below

$$\hat{g}_{\hat{M}\hat{N}} = \begin{pmatrix} \hat{g}_{zz} & \hat{g}_{Mz} \\ \hat{g}_{zM} & \hat{g}_{MN} \end{pmatrix} = \begin{pmatrix} \phi & \mathcal{A}_M \\ \mathcal{A}_M & \hat{g}_{MN} \end{pmatrix}, \tag{2.3}$$

and thus, from the lower-dimensional perspective, the metric $\hat{g}_{\hat{M}\hat{N}}$ is seen as a D-dimensional metric $\hat{g}_{MN}$, a KK gauge field $\mathcal{A}_M = \hat{g}_{Mz}$ and a dilaton scalar field $\phi = \hat{g}_{zz}$ (from a more mathematical point of view, this split can also be seen as a result of basic group theory [5] -see Appendix). Taking this fact into account, we decompose the metric $\hat{g}_{\hat{M}\hat{N}}$ into fourier modes as shown in table 2.1 [2]. We note that the infinite tower of massive modes, corresponding to $n \neq 0$, is constituted by massive spin-2 particles, that acquire there mass via a higher-order Higgs mechanism. Just like before, their masses are enormous and thus they can be neglected in the effective action.

In other words, the Kaluza-Klein reduction is a compactification together with a consistent truncation to the massless sector. By the term consistent truncation, we understand a restriction on the variables such that the solutions to the equations of

n	D+1	DOF	D Field	DOF	Physical spectrum
			g_{MN}	$\frac{(D-2)(D-1)}{2}-1$	massless graviton
0	$\hat{g}^{(0)}_{\hat{M}\hat{N}}$	$\frac{(D-1)(D)}{2}-1$	$\mathcal{A}_M$	D-2	massless KK vector field
			ϕ	1	massless scalar
			$g^{(n)}_{MN}$	$\frac{(D-2)(D-1)}{2}-1$	
0	$\hat{g}^{(n)}_{\hat{M}\hat{N}}$	$\frac{(D-1)(D)}{2}-1$	$\mathcal{A}^{(n)}_M$	D-2	massive, spin-2 graviton
			$\phi^{(n)}$	1	

Table 2.1: Decomposition of the (D+1)-dimensional graviton into D-dimensional fields and the physical spectrum

motion for the restricted variables are also solutions to the equations for the unrestricted variables. This ensures that the solutions of the lower-dimensional theory lie in a particular class of solutions of the higher-dimensional theory. Our ansatz is simply to take the components of the metric $\hat{g}^{(0)}_{\hat{M}\hat{N}}$ to be independent of the compactified coordinates, just like $\phi_0(x)$ was z-independent.

In order to proceed, we need to redefine the fields emerging from the reduction of the metric; although the field definitions given by (2.3) are the most logical ones, they are not the most convenient. A more appropriate choice, that respects the symmetries of the system, is given in the literature [1, 2, 3, 4]: we choose to parameterize the (D+1)-dimensional metric in terms of the lower dimensional fields, *all of which are required to be z-independent*, as

$$d\hat{s}^2 = e^{2a\phi}ds^2 + e^{2\beta\phi}(dz + \mathcal{A}_\mu dx^\mu)^2. \tag{2.4}$$

a and β are constants, free to choose in a way that suits us. Using this ansatz and setting $\beta = -(D-2)a$ in order to get a "clean" D-dimensional gravity term, we find that(up to a total derivative)

$$\mathcal{L} = \sqrt{-\hat{g}}R(\hat{g}) = \sqrt{-g}\bigg(R(g) - (D-1)(D-2)a^2\nabla_M\phi\nabla^M\phi - \frac{1}{4}e^{-2(D-1)a\phi}\mathcal{F}_{\mathcal{MN}}\mathcal{F}^{\mathcal{MN}}\bigg), \tag{2.5}$$

where $\mathcal{F} = d\mathcal{A}$. In order to get the usual normalization for the dilaton ϕ, we have to make an appropriate choice of a

$$a^2 = \frac{1}{2(D-1)(D-2)}. \tag{2.6}$$

$$\mathcal{L} = \sqrt{-\hat{g}}R(\hat{g}) = \sqrt{-g}\Big(R(g) - \frac{1}{2}\nabla_M\phi\nabla^M\phi - \frac{1}{4}e^{-2(D-1)a\phi}\mathcal{F}_{\mathcal{MN}}\mathcal{F}^{\mathcal{MN}}\Big) \tag{2.7}$$

In the language of differential forms, equations (2.7) can be rewritten in the following form.

$$\boxed{\mathcal{L} = R\star 1 - \frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{-2(D-1)a\phi}\star\mathcal{F}_{[2]}\wedge\mathcal{F}_{[2]}} \tag{2.8}$$

Thus, the reduction of the (D+1)-dimensional Einstein-Hilbert gravity gives rise to a D-dimensional Einstein-Maxwell-Scalar system, whose dynamics are governed by the above lagrangian.

At this point, we need to pause and draw attention on two things. Firstly, in order to simplify the calculation of the spin connection, curvature and Ricci scalar $\hat{R}$, we need to use a veilbein basis. According to [1], an appropriate choice is the following.

$$\hat{e}^\alpha = e^{a\phi}e^\alpha, \qquad \hat{e}^z = e^{\beta\phi}(dz + \mathcal{A}) \tag{2.9}$$

$$\hat{R} = e^{-2a\phi}\Big(R - \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi) + (D-3)a\partial_\mu\partial^\mu\phi\Big) - \frac{1}{4}e^{-2Da\phi}\mathcal{F}^2 \tag{2.10}$$

The second point has to do with the consistency of the truncation. If we compute the equation of motion of the reduced theory with respect to the scalar fields ϕ, we will get the following result.

$$\partial_\mu\partial^\mu\phi = -\frac{1}{2}(D-1)e^{-2(D-1)a\phi}\mathcal{F}^2$$

Therefore, it would have been inconsistent to assume an ansatz (2.4) such that $\phi = 0$, as the source term on the right-hand side of the above equation forces ϕ to be non-trivial. Having this in mind, one may now ask whether is consistent to neglect the massive sector as this is effectively equivalent to setting all the massive fields to zero. To answer this question we need to understand that each fourier mode corresponds to a certain U(1) irreducible representation characterized by the integer number n (remember that the symmetry group of the circle is U(1)). If $n = 0$, the field corresponds to a U(1) singlet and if $n \neq 0$, the mode corresponds to a non-singlet. Consequently, in this case, the truncation to the massless sector is guaranteed to be consistent as we have ignored *all* the non-singlet fields. Note that in other cases, the consistency is not so trivial to prove.

Symmetries of the Dimensionally Reduced Einstein-Hilbert Lagrangian:

First of all, we note that the Einstein-Hilbert lagrangian, given by (2.1), has a general coordinate covariance. The relevant transformations, in their infinitesimal form, are the following

$$\delta\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^P\partial_P\hat{g}_{\hat{\mu}\hat{\nu}} + \hat{g}_{\hat{P}\hat{\nu}}\partial_\mu\hat{\xi}^P + \hat{g}_{\hat{\mu}\hat{P}}\partial_\nu\hat{\xi}^P, \qquad \delta\hat{x}^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}}, \tag{2.11}$$

where $\hat{\xi}^{\hat{\mu}}$ are arbitrary functions of D+1 coordinates. In order to determine the symmetries of the D-dimensional theory, we observe that the most general transformations that preserves the metric ansatz (2.4) are

$$\hat{\xi}^\mu = \xi^\mu(x), \qquad \hat{\xi}^z = cz + \lambda(x), \tag{2.12}$$

where ξ^μ and λ are functions of the D-dimensional coordinates and c is a constant. Thus, using the equations (2.11) and (2.12), we calculate $\delta\hat{g}_{\mu\nu}$, $\delta\hat{g}_{\mu z}$ and $\delta\hat{g}_{zz}$. Keeping in mind the definitions of the lower-dimensional fields $g_{\mu\nu}$, $\mathcal{A}_\mu$ and ϕ, we can determine their infinitesimal transformations. This procedure was followed in [1, 3]; the authors concluded that the lower-dimensional theory (2.7) has a general coordinate covariance involving D coordinates (described by $\xi^\mu(x)$), a local U(1) gauge invariance of the KK vector field (described by $\lambda(x)$) and yet another symmetry related to constant shifts of the dilaton accompanied with appropriate constant scaling of the KK vector field (described by c).

$$\phi \to \phi + c, \qquad \mathcal{A}_\mu \to e^{c(D-1)a}\mathcal{A}_\mu \tag{2.13}$$

Accordingly, we conclude that the symmetries of the reduced theory are just a *residue* of the symmetries of the initial theory; the original general coordinate covariance involves coordinate reparametrization by arbitrary functions of D+1 coordinates, while the symmetries of the D-dimensional theory involve only D coordinates.

Another symmetry that needs to be discussed is the rescaling symmetry of the higher-dimensional equation of motion, ie of the Einstein equation $\hat{R}_{MN}-\frac{1}{2}\hat{R}\hat{g}_{MN}=0$. Under the action of this global symmetry, the metric and the relevant curvature quantities are rescaled as follows

$$\begin{aligned} &\hat{g}_{\hat{M}\hat{N}} \to k^2 \hat{g}_{\hat{M}\hat{N}} && \hat{R}^{\hat{M}}{}_{\hat{N}\hat{P}\hat{Q}} \to \hat{R}^{\hat{M}}{}_{\hat{N}\hat{P}\hat{Q}} \\ &\sqrt{-\hat{g}} \to k^{D+1}\sqrt{-\hat{g}} && \hat{R}_{\hat{M}\hat{N}} \to \hat{R}_{\hat{M}\hat{N}} \\ & && \hat{R} \to k^{-2}\hat{R}. \end{aligned} \tag{2.14}$$

In infinitesimal form, the above transformations are translates into $\delta\hat{g}_{\hat{M}\hat{N}}=2\alpha\hat{g}_{\hat{M}\hat{N}}$, where α is a constant. Of course, this is not a valid symmetry of the relevant Lagrangian; the Lagrangian is uniformly rescaled.

If we take a linear combination of the dilaton shifting symmetry(seen as a residual symmetry of the higher-dimensional metric) and the higher-dimensional metric resealing

$$\delta\hat{g}_{\hat{M}\hat{N}} = c\delta^z_{\hat{M}}\hat{g}_{\hat{N}z} + c\delta^z_{\hat{N}}\hat{g}_{\hat{M}z} + 2\alpha\hat{g}_{\hat{M}\hat{N}} \tag{2.15}$$

for appropriate value of $\alpha=-c/(D-1)$, we get a purely internal symmetry of the reduced Einstein-Hilbert lagrangian that leaves the lower-dimensional metric invariant $\delta g_{\mu\nu}=0$ and acts only on the other fields; this symmetry is the so-called R-symmetry. The "orthogonal" combination, given by $\alpha=-c$, is a scaling symmetry that acts on all the lower-dimensional fields and resales them according to the number of indices that they carry; for instance

$$g_{MN} \to k^2 g_{MN}, \qquad \mathcal{A}_m \to k\mathcal{A}_m. \tag{2.16}$$

This is the so-called "trombone" symmetry.

Dimensional Reduction of Form Fields on S^1:

Now, one needs to establish the reduction ansatz for the (D+1)-dimensional antisymmetric gauge field $\hat{A}_{[n-1]}$, where the index n-1 in square brackets denotes that

$\hat{A}$ is an (n-1)-index form field. As all the indices are antisymmetrized, only one of them can be equal to z and thus, in D dimensions, we will have an (n-1)-form and an (n-2)-form gauge field (this can also be resulted using group theory arguments [5]). Once again, we will adopt the notation of [1].

$$\hat{A}_{[n-1]} = A_{[n-1]} + A_{[n-2]} \wedge dz, \quad \Rightarrow \quad \hat{F}_{[n]} = d\hat{A}_{[n-1]} = dA_{[n-1]} + dA_{[n-2]} \wedge dz \tag{2.17}$$

This parametrization is not the most convenient one as, upon dimensional reduction, it will lead to the appearance of terms with the gauge field $A_{[n-1]}$ undifferentiated (these are the so-called Chern-Simons terms). A more convenient choice of ansatz is obtained if we add and subtract a term from equation (2.17), so that we get:

$$\hat{F}_{[n]} = dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]} + dA_{[n-2]} \wedge (\mathcal{A}_{[1]} + dz) = F_{[n]} + F_{[n-1]} \wedge (\mathcal{A}_{[1]} + dz), \tag{2.18}$$

where $\mathcal{A}$ is the KK vector emerging from the reduction of the metric (2.4), $F_{[n]} = dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]}$ and $F_{[n-1]} = dA_{[n-2]}$. Now, keeping this in mind, we can dimensionally reduce the kinetic term of an n-form field strength in (D+1) dimensions.

$$\mathcal{L} = -\frac{\sqrt{-\hat{g}}}{2n!} \hat{F}^2_{[n]} = -\frac{\sqrt{-g}}{2n!} e^{-2(n-1)a\phi} F^2_{[n]} - \frac{\sqrt{-g}}{2n!} e^{-2(D-n)a\phi} F^2_{[n-1]} \tag{2.19}$$

In the language of differential forms, equation (2.19) can be rewritten in the following form.

$$\boxed{\mathcal{L} = -\frac{1}{2} e^{-2(D-1)a\phi} \star F_{[n]} \wedge F_{[n]} - \frac{1}{2} e^{-2(D-n)a\phi} \star F_{[n-1]} \wedge F_{[n-1]}} \tag{2.20}$$

Dimensional Reduction of the eleven-dimensional Supergravity on S^1:

The eleven-dimensional supergravity is the field theory whose bosonic sector is comprised of the metric tensor (ie a graviton) and a 4-form field strength and whose fermionic sector contains only one fermionic field with spin 3/2(ie a gravitino). If we constraint ourselves just to the bosonic sector, the dynamics of the system are controlled by the following Lagrangian density

$$L = R \star 1 - \frac{1}{2} \star F_{[4]} \wedge F_{[4]} + \mathcal{L}^{(11)}_{FFA}, \tag{2.21}$$

where, as usual, R is the Ricci scalar, $F_{[4]}$ is the 4-form field strength and $\mathcal{L}^{(11)}_{FFA}$ is the Chern-Simons term in 11 dimensions. By compactifying the above theory on S^1, we will obtain the bosonic sector of the ten-dimensional type IIA supergravity theory, given by the following lagrangian,

$$\mathcal{L} = R\star 1 - \frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{\frac{3}{2}\phi}\star\mathcal{F}_{[2]}\wedge\mathcal{F}_{[2]} - \frac{1}{2}e^{\frac{1}{2}\phi}\star F_{[4]}\wedge F_{[4]} - \frac{1}{2}e^{-\phi}\star F_{[3]}\wedge F_{[3]} + \mathcal{L}^{(10)}_{FFA}, \tag{2.22}$$

where $F_{[4]} = dA_{[3]} - dA_{[2]}\wedge\mathcal{A}_{[1]}$ is the 4-form field strength, $F_{[3]} = dA_{[2]}$, $\mathcal{A}_{[1]}$ is the KK vector field emerging from the reduction of the metric and $\mathcal{L}^{(10)}_{FFA}$ is the Chern-Simons term in 10 dimensions.

Observe that the reduced theory is invariant under the action of the "trombone" symmetry

$$g_{MN} \to k^2 g_{MN}, \qquad A_{m_1\ldots m_n} \to k^n A_{m_1\ldots m_n}.$$

It can also be shown that the equations of motion of the eleven-dimensional supergravity symmetry respect the rescaling symmetry (2.14). Thus, we may conclude that the ten-dimensional supergravity has a global internal symmetry(generated by c) such that

$$\begin{aligned} \phi \to \phi + c, \quad A_{[3]} \to e^{-\frac{c}{4}}A_{[3]}, \\ \mathcal{A}_{[1]} \to e^{-\frac{3c}{4}}\mathcal{A}_{[1]}, \quad A_{[2]} \to e^{\frac{c}{2}}A_{[2]}. \end{aligned} \tag{2.23}$$

2.2 Kaluza-Klein Dimensional Reduction on T^n

There are two equivalent ways to dimensionally reduce a (D+n)-dimensional theory on an n-dimensional Euclidean torus $T^n = S^1X...XS^1$:

- According to [2], the torus reduction can be done in just one step using the following metric ansatz

$$ds^2_{D+n} = e^{2a\phi}ds^2_D + e^{2b\phi}M_{mn}(dx^m + \mathcal{A}^m)(dx^n + \mathcal{A}^n), \tag{2.24}$$

  where M_{mn} is a symmetric matrix of scalars with unit determinant $detM = 1$, m,n= (D+1) ,..,(D+n),

$$a = \frac{n}{2(D+n-2)(D-2)}, \qquad b = -\frac{(D-2)a}{n}.$$

- We can successively repeat Kaluza-Klein dimensional reduction on a circle S^1 for arbitrary n times [1].

In what follows, we will concentrate on the second approach.

At each reduction step, for example the i'th step, one obtains a KK vector field $\mathcal{A}^i_{[1]}$ and a dilaton ϕ from the reduction of the metric. We will also have 0-forms $\mathcal{A}^i_{[0]j}$, called axions, coming from the reduction of the KK vector field $\mathcal{A}^i_{[1]}$, where $i < j$ (the KK vector field is created at the i'th step and then is reduced to give an axion at the j'th step). In total, from the metric reduction, we obtain n dilatons, n vector fields and $n(n-1)/2$ axions (axions always appear differentiated). In addition, we have to consider how p-forms in D+n dimensions reduce. We know from our previous discussion that upon dimensional reduction on S^1, a p-form gives a p-form and a (p-1)-form. If $i < p$, at the i'th step of reduction on a torus, we will have a p-form, (p-1)-forms,(p-2)-forms,...,(p-i)-forms. If $i > p$, we will have a p-form, (p-1)-forms,(p-2)-forms,..., 0-forms with varying multiplicities.

Reduction of the eleven-dimensional Supergravity on T^n:

The bosonic sector of the eleven-dimensional supergravity is given by (2.21). If we reduced the 3-form contained in the supergravity multiplet, we will get the potentials $\mathcal{A}_{[3]}$, $\mathcal{A}_{i[2]}$, $\mathcal{A}_{ij[1]}$ and $\mathcal{A}_{ijk[0]}$, where the indices i,j,k are antisymmetrized and can be thought of as directions on the reduction torus. According to [3], the reduced supergravity theory in 11-n dimensions will be given by the following lagrangian.

$$\begin{aligned}
\mathcal{L} &= R\star 1 - \tfrac{1}{2}\star d\overrightarrow{\phi}\wedge d\overrightarrow{\phi} - \tfrac{1}{2}\sum_i e^{\overrightarrow{b}_i\overrightarrow{\phi}}\star\mathcal{F}^i_{[2]}\wedge\mathcal{F}^i_{[2]} - \tfrac{1}{2}\sum_{i<j} e^{\overrightarrow{b}_{ij}\overrightarrow{\phi}}\star\mathcal{F}^i_{j[1]}\wedge\mathcal{F}^i_{j[1]} \\
&\quad -\tfrac{1}{2}e^{\overrightarrow{a}\overrightarrow{\phi}}\star F_{[4]}\wedge F_{[4]} - \tfrac{1}{2}\sum_i e^{\overrightarrow{a_i}\overrightarrow{\phi}}\star F_{i[3]}\wedge F_{i[3]} - \tfrac{1}{2}\sum_{i<j} e^{\overrightarrow{a_{ij}}\overrightarrow{\phi}}\star F_{ij[2]}\wedge F_{ij[2]} \\
&\quad -\tfrac{1}{2}\sum_{i<j<k} e^{\overrightarrow{a_{ijk}}\overrightarrow{\phi}}\star F_{ijk[1]}\wedge\ F_{ijk[1]} + \mathcal{L}^{(11-n)}_{FFA},
\end{aligned} \tag{2.25}$$

where $\mathcal{L}^{(11-n)}_{FFA}$ is the Chern-Simons term in (11-D) dimensions, given in [3], and $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$, $\vec{a}_{ijk}$, $\vec{b}_i$ and $\vec{b}_{ij}$ are the dilaton vectors. These vectors are constants and they characterize the strength of the gauge fields-dilatons couplings. Out of them, only $\vec{a}$ and $\vec{a}_i$ are independent.

$$\begin{aligned} \vec{b}_i &= -\vec{a}_i + \vec{a}, & \vec{b}_{ij} &= -\vec{a}_i + \vec{a}_j, \\ \vec{a}_{ij} &= \vec{a}_i + \vec{a}_j - \vec{a}, & \vec{a}_{ijk} &= \vec{a}_i + \vec{a}_j + \vec{a}_k - 2\vec{a}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \vec{a}_i \cdot \vec{a}_j &= 2\delta_{ij} + 2\tfrac{2(6-D)}{(D-2)}, \\ \vec{a} \cdot \vec{a} &= \tfrac{2(11-D)}{(D-2)}, \\ \vec{a} \cdot \vec{a}_i &= \tfrac{2(8-D)}{(D-2)}. \end{aligned} \tag{2.27}$$

Note that the form fields in equation (2.25) are pure exterior derivatives of the relevant gauge fields plus KK corrections; their explicit forms can be found in [3, 4]. For instance,

$$\hat{F}_{[4]} = \quad F_{[4]} + F^i_{[3]} \wedge h^i + \tfrac{1}{2} F^{ij}_{[2]} \wedge h^i \wedge h^j + \tfrac{1}{6} F^{ijk}_{[1]} \wedge h^i \wedge h^j \wedge h^k, \tag{2.28}$$

where $h^i = dz^i + \mathcal{A}^i_{[1]} + \mathcal{A}^i_{[0]j} dz^j$.

The metric in D dimensions is related to the metric in 11 dimensions in the following way

$$ds_1^2 1 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds_D^2 + \sum_i e^{2\vec{\gamma_i}\cdot\vec{\phi}} (h^i)^2, \tag{2.29}$$

where

$$\begin{aligned} \vec{g} &= 3(s_1, s_2, \ldots, s_{11-D}),\ s_i = \sqrt{\tfrac{2}{(10-i)(9-i)}} \\ \vec{\gamma}_i &= \tfrac{1}{6}\vec{g} - \tfrac{1}{2}\vec{f}_i, \\ \vec{f}_i &= (0, .., 0, (10-i)s_i, s_{i+1}, ..s_{11-D}) \end{aligned}$$

Symmetries of the Reduced eleven-dimensional Supergravity:

Firstly, we will discuss the symmetries of the terms coming from the reduction of the eleven-dimensional gravity term, ie the first four terms of the lagrangian (2.25):

- The initial Einstein-Hilbert term has a general coordinate covariance of 11 coordinates, while the gravity term of the reduced theory has a general coordinate covariance involving arbitrary functions of only 11-n coordinates. Generally, the story goes just like our discussion for the S^1 case with the only difference being that now we have n coordinates z^i and thus $\hat{x}^{\hat{\mu}}$ is split as (x^μ, z^i). We first identify the higher-dimensional transformations that respect the metric ansatz

$$\hat{x}^\mu(x,z) = \xi^\mu(x), \qquad \hat{x}^i(x,z) = \Lambda^i{}_j z^j + \lambda^i(x), \tag{2.30}$$

($\Lambda^i{}_j$ are constants) and then, using the equations (2.11) and (2.30), we calculate $\delta\hat{g}_{\mu\nu}$, $\delta\hat{g}_{\mu i}$ and $\delta\hat{g}_{ij}$. Having in mind the definitions of the lower-dimensional fields that emerge from $\hat{g}_{\hat{\mu}\hat{\nu}}$, we find their infinitesimal transformations. Following this procedure, we conclude that the terms coming from the reduction of the initial Einstein-Hilbert lagrangian exhibit local general coordinate covariance involving 11-n coordinates (described by $\xi^\mu(x)$) and local $U(1)$ gauge invariance of the n KK vector fields (described by $\lambda^i(x)$). The only parameters left to be discussed are the $\Lambda^i{}_j$, which are a generalization of the parameter c we talked about in the previous section.

- In the case of the S^1 reduction, we have seen that if we took a linear combination of the symmetry generated by c and the rescalling symmetry of the higher-dimensional Einstein equation, we could extract a purely internal symmetry of the lower-dimensional theory that leaves the metric invariant, while shifting the dilaton and rescalling the gauge fields by a constant. Something similar to this happens when reducing on T^n. If we combine the generalized version of the dilaton shifting symmetry, given by the group $\Lambda^i{}_j \epsilon GL(n,R)$, and the scaling symmetry of the 11-dimensional Einstein equation, we will extract an internal symmetry of the terms coming from the reduction of the initial Einstein-Hilbert lagrangian that leaves the metric invariant and describes a constant shift of the dilatons along with appropriate rescaling of the form fields.

But, are these symmetries valid symmetries of the total lagrangian (2.25)? As we have seen in the case of S^1 reduction, the reduced eleven-dimensional theory was invariant under the dilaton field shifting symmetry. As this symmetry, in the context of the T^n reduction, has been generalized into a global $GL(n,R) = R \times SL(n,R)$, we expect the reduced eleven-dimensional symmetry to be invariant under the action of this group. This is indeed a general feature of reduced theories of gravity coupled to other matter fields. Note that one usually can only be sure of the $SL(n,R)$ part of the symmetry, as the existence of the R factor depends on having the extra scaling symmetry of the higher-dimensional equations of motion. We should also note that, in low dimensions, we typically have bigger symmetries as there exists a symmetry enhancement due to the dualization of the form fields to scalars. In those cases, the full details of the maximal supergravity reduction, including the Chern-Simons term given in [1, 3], are necessary in order to specify the symmetry of the reduced theory.

Application: Reduction of the eleven-dimensional Supergravity on T^2

A special case of what has been discussed above is the reduction on T^2. Now, the reduction of the (D+2)-dimensional metric gives a D-dimensional metric, two KK gauge field denoted by $\mathcal{A}^i_\mu$ (i=1,2), a two-component dilaton vector $\overrightarrow{\phi}$ and one axion χ -see Appendix. Thus, the reduced Einstein-Hilbert lagrangian becomes

$$\mathcal{L} = R\star 1 - \frac{1}{2}\star d\overrightarrow{\phi}\wedge d\overrightarrow{\phi} - \frac{1}{2}\sum_i e^{\overrightarrow{c}_i\overrightarrow{\phi}}\star \mathcal{F}_{i[2]}\wedge\mathcal{F}_{i[2]} - \frac{1}{2}e^{\overrightarrow{c}\overrightarrow{\phi}}\star d\chi\wedge d\chi, \qquad (2.31)$$

where $\mathcal{F}^1_{[2]} = d\mathcal{A}^1_{[1]} - d\chi\wedge\mathcal{A}^2_{[1]}$, $\mathcal{F}^2_{[2]} = d\mathcal{A}^2_{[1]}$ and $\vec{c}, \vec{c}_i$ are constant functions of D

$$\begin{aligned}\vec{c}_1 &= \big(-\sqrt{\tfrac{2D}{D-1}}, -\sqrt{\tfrac{2}{(D-1)(D-2)}}\big),\\ \vec{c} &= \big(-\sqrt{\tfrac{2D}{D-1}}, \sqrt{\tfrac{2(D-2)}{(D-1)}}\big),\\ \vec{c}_2 &= \big(0, -\sqrt{\tfrac{2(D-1)}{(D-2)}}\big).\end{aligned} \qquad (2.32)$$

If we perform an appropriate rotation of ϕ_1 and ϕ_2

$$\begin{aligned}\phi &= -\tfrac{1}{2}\sqrt{\tfrac{2D}{(D-1)}}\phi_1 + \tfrac{1}{2}\sqrt{\tfrac{2(D-2)}{(D-1)}}\phi_2,\\ \varphi &= -\tfrac{1}{2}\sqrt{\tfrac{2(D-2)}{(D-1)}}\phi_1 - \tfrac{1}{2}\sqrt{\tfrac{2D}{(D-1)}}\phi_2,\end{aligned} \qquad (2.33)$$

the above lagrangian simplifies a lot

$$\begin{aligned}\mathcal{L} &= R\star 1 - \tfrac{1}{2}\star d\phi\wedge d\phi - \tfrac{1}{2}\star d\varphi\wedge d\varphi - \tfrac{1}{2}e^{\phi+\sqrt{D/(D-2)}\varphi}\star\mathcal{F}^1_{[2]}\wedge\mathcal{F}^1_{[2]}\\ &-\tfrac{1}{2}e^{-\phi+sqrD/(D-2)\varphi}\star\mathcal{F}^2_{[2]}\wedge\mathcal{F}^2_{[2]} - \tfrac{1}{2}e^{2\phi}\star d\chi\wedge d\chi.\end{aligned} \qquad (2.34)$$

Note that the (D+2)-dimensional metric is related to the D-dimensional metric via

$$ds^2_{D+2} = e^{-\frac{2\varphi}{\sqrt{D(D-2)}}} ds^2_D + e^{\sqrt{\frac{(D-2)}{D}}\varphi}\left(e^{\phi}(dz_1+\mathcal{A}^1_{(1)}+\chi dz_2)^2+e^{-\phi}(dz_2+\mathcal{A}^2_{(1)})^2\right). \quad (2.35)$$

The field φ parameterizes the volume of the torus, as it appears in the metric ansatz as an overall multiplicative factor of the internal compactified metric, and thus, is called a "breathing" mode. One the other hand, ϕ and χ characterize the shape of the torus: ϕ determines the radii of the two circles of the torus and χ determines the angle between them. All three of them, φ, ϕ, χ, determine completely the torus.

If we now restrict our attention to the scalar lagrangian of the reduced Einstein-Hilbert lagrangian, we note that φ decouples from the other 2 fields.

$$\mathcal{L} = \frac{1}{2}\star d\phi \wedge d\phi - \frac{1}{2}\star d\varphi \wedge d\varphi - \frac{1}{2}e^{2\phi}\star d\chi \wedge d\chi \quad (2.36)$$

Combine χ and ϕ into a complex scalar field $\tau = \chi + ie^{-\phi}$. The above lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\star d\varphi \wedge d\varphi - \frac{1}{2\tau_2^2}\star d\overline{\tau} \wedge d\tau, \quad (2.37)$$

where $\tau_2 = e^{-\phi}$. Now, it is easy to determine the symmetries of the scalar lagrangian:

- A global shift symmetry of φ: $\varphi \to \varphi + c$, accompanied by appropriate constant scalings of the other potentials.
- Also, the scalar lagrangian is invariant under $\tau \to \frac{a\tau+b}{c\tau+d}$, where $ad - bc = 1$. This symmetry can be recognized as the $SL(2,R)$.

Therefore, the overall global symmetry of the scalar lagrangian is $R \times SL(2,R) = GL(2,R)$. But, is this symmetry a valid symmetry of the full lagrangian (2.31)? If we make the following field redefinition

$$\mathcal{A}^1_{[1]} \to \mathcal{A}^1_{[1]} + \chi\mathcal{A}^2_{[1]} \quad \Rightarrow \quad \mathcal{F}^1_{[2]} = d\mathcal{A}^1_{[1]} + \chi\mathcal{A}^2_{[1]},$$

we note that (2.31) is left invariant under the action of $SL(2,R)$, if the KK gauge fields transformation as follows

$$\begin{pmatrix} \mathcal{A}^2_{[1]} \\ \mathcal{A}^1_{[1]} \end{pmatrix} \longrightarrow (\Lambda^T)^{-1} \begin{pmatrix} \mathcal{A}^2_{[1]} \\ \mathcal{A}^1_{[1]} \end{pmatrix}, \quad (2.38)$$

where $\Lambda \epsilon SL(2,R)$. Thus, the symmetry of the reduced Einstein-Hilbert lagrangian is the same as the symmetry of the scalar lagrangian. This is a general result: *the symmetry of the full lagrangian is determined just by looking at the scalar lagrangian.*

One may now ask what is the symmetry of the reduced eleven-dimensional supergravity on T^2. The answer here is really simple if one observes that the reduced eleven-dimensional theory and the reduced Einstein-Hilbert lagrangian share the same scalar Lagrangian (upon reduction on T^2, the eleven-dimensional 3-form does not generate any scalars). Therefore, the symmetry is also $GL(2,R)$.

Another general result is that the scalars transform non-linearly, while the higher-rank potentials lie in linear representations of the symmetry group. For instance, when compactifying the eleven-dimensional supergravity on the torus, we find that the one 1-form $A_{[1]}$ and the one 3-form $A_{[3]}$ transform as singlets, while the two 2-forms $A^i_{[2]}$ and the two 1-forms $\mathcal{A}^i_{[1]}$ transform as doublets.

2.3 Scalar Coset Manifolds: a scan through various dimensions

In order to acquire a better understanding of the structure of the global symmetry, Cremmer, Julia, Lü and Pope studied the scalar lagrangians emerging from supergravity theories in various dimensions [1, 3]. In this section, we will present some of their work.

Let us start our discussion by looking at the SL(2,R) example of the previous section. The generators of the SL(2,R) group satisfy the following commutation relations

$$[h, E_\pm] = \pm 2E_\pm, \qquad [E_+, E_-] = h, \tag{2.39}$$

where h is the generator forming the Cartan subalgebra and (E_+, E_-) are the raising and lowering operators (SL(2,R) is a rank 1 group). A convenient way of representing these generators is the following

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.40}$$

Now, we can calculate the coset representative $\mathcal{V}$ and the Maurer-Cartan form $d\mathcal{V}\mathcal{V}^{-1}$.

$$\mathcal{V} = e^{\frac{1}{2}\phi h} e^{\chi E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}, \tag{2.41}$$

$$d\mathcal{V}\mathcal{V}^{-1} = \begin{pmatrix} \frac{1}{2}d\phi & e^{\phi}d\chi \\ 0 & -\frac{1}{2}d\phi \end{pmatrix} = \frac{1}{2}d\phi h + e^{\phi}d\chi E_+, \tag{2.42}$$

where $\mathcal{V}$ is a matrix representing points on the scalar manifold. By the term scalar manifold, we mean the manifold parameterized by the scalar fields of the theory -in this case, by ϕ and χ. Using the above definitions, we can calculate the followings

$$\mathcal{M} = \mathcal{V}^T\mathcal{V} = \begin{pmatrix} e^{\phi} & e^{\phi}\chi \\ e^{\phi}\chi & e^{-\phi} + e^{\phi}\chi^2 \end{pmatrix}, \qquad \mathcal{M}^{-1} = \begin{pmatrix} e^{-\phi} + e^{\phi}\chi^2 & -e^{\phi}\chi \\ -e^{\phi}\chi & e^{\phi} \end{pmatrix}. \tag{2.43}$$

The scalar lagrangian, given by (2.37), can be translated into this language as

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi + \frac{1}{4}tr(\partial\mathcal{M}^{-1}\partial\mathcal{M}). \tag{2.44}$$

Now, the global SL(2,R) symmetry is manifested as

$$\begin{aligned} &\mathcal{V} \to \mathcal{V}\Lambda, \\ &\mathcal{M} \to \Lambda^T\mathcal{M}\Lambda, \\ &\mathcal{L} \to \mathcal{L}, \end{aligned} \tag{2.45}$$

where $\Lambda\epsilon SL(2,R)$. The only problem with the above derivation is that although $\mathcal{V}$ is upper-triangular, $\mathcal{V}\Lambda$ is not and thus it does *not* correspond to points on the scalar manifold anymore. In order to fix this, we have to do a compensating local transformation $\mathcal{O}$ that acts on $\mathcal{V}$ from the left. The transformation law is now the following

$$\mathcal{V} \to \mathcal{V}' = \mathcal{O}\mathcal{V}\Lambda, \tag{2.46}$$

where $\mathcal{O}$ is by definition a matrix such that $\mathcal{V}'$ is upper-triangular. This is a unique, orthogonal matrix $\mathcal{O}\epsilon O(2)$ that depends on ϕ and χ. It is easy to show that the transformation (2.46), which is equivalent to the transformations of the scalar fields given in the previous section, leaves the scalar lagrangian invariant.

Note that for any pair of fixed ϕ and χ, we can get any other pair using the SL(2,R) transformations. In other words, SL(2,R) has a transitive action on the scalar manifold and thus, we may specify points on the scalar manifold by the coset $\frac{SL(2,R)}{O(2)}$. Consequently, we deduce that we can identify the dilaton-axion scalar manifold with the coset manifold accompanied by a global SL(2,R) symmetry.

Now, one may consider what happens upon dimensional reductions on a higher-dimensional torus. There, the story is much more complicated as there are additional axionic fields coming from the reduction of the KK gauge field and additional dilatons. For instance, if we descend from D=11 down to D=8 on T^3, we have 3 dilatons $\overrightarrow{\phi}$ emerging from the reduction of the metric, 3 axions $\mathcal{A}^i_{j[0]}$ coming from the reduction of the KK gauge field and another axion $A_{[0]}$ coming from the reduction of the 3-form. The scalar lagrangian controlling the interactions of the fields mentioned above is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\star d\phi_1\wedge d\phi_1 - \frac{1}{2}\sum_{i=2,3}\star d\phi_i\wedge d\phi_i - \frac{1}{2}\sum_{i<j} e^{\overrightarrow{b}_{ij}\cdot\overrightarrow{\phi}}\star\mathcal{F}^i_{j[1]}\wedge\mathcal{F}^i_{j[1]} \\ &- \frac{1}{2}e^{2\phi_1}\star F_{[1]}\wedge F_{[1]}, \end{aligned} \tag{2.47}$$

where $\mathcal{F}^1_{[1]2} = d\mathcal{A}^1_{[0]2}$, $\mathcal{F}^2_{[1]3} = d\mathcal{A}^2_{[0]3}$ and $\mathcal{F}^1_{[1]3} = d\mathcal{A}^1_{[0]3} - \mathcal{A}^2_{[0]3}d\mathcal{A}^1_{[0]2}$. If we do an orthogonal transformation, we can make the dilaton vectors become $\overrightarrow{b}_{12} = (0,1,\sqrt{3})$, $\overrightarrow{b}_{23} = (0,1,-\sqrt{3})$ and $b_{13} = (0,2,0)$. Thus, we see that the axion $A_{[0]}$ and the dilaton ϕ_1 decouple from the rest of the scalars. If we compare the part of the lagrangian that controls their dynamics with the equation (2.37), we conclude that these fields parameterize an $\frac{SL(2,R)}{O(2)}$ coset manifold with a global SL(2,R) symmetry. This leaves the question of understanding the part of the coset parameterized by the remaining 5 scalars. In order to answer this, we have to define quantities similar to those defined for the T^2 example.

By observing the second and third terms in equation (2.47), we expect to find an SL(3,R) symmetry involving only (ϕ_2, ϕ_3) (we recognize $\overrightarrow{b}_{ij}$ to be the positive roots of SL(3,R) -the simple ones are b_{12} and b_{23}). We may introduce positive-root generators E_i^j, with $i < j$, associated with the root vectors and a Cartan generator vector $\overrightarrow{h}$ such that

$$[\vec{h}, E_i^j] = \overrightarrow{b}_{ij} E_i^j, \qquad [E_i^j, E_l^k] = \delta_k^j E_i^l - \delta_i^l E_k^j. \tag{2.48}$$

A convenient choice for these generators is the following. The three raising/lowering operators are given by

$$E_1^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, E_1^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.49}$$

while the two Cartan operators are

$$h_1 = diag(1, 0, -1), \qquad h_2 = \frac{1}{\sqrt{3}} diag(1, -2, 1). \tag{2.50}$$

To find the coset representative $\mathcal{V}$, we just exponentiate the Cartan and the positive-root generators.

$$\begin{aligned} &\mathcal{V} = e^{\frac{1}{2}\overrightarrow{\phi}\cdot\overrightarrow{h}} e^{\mathcal{A}^2_{3[0]} E_2^3} e^{\mathcal{A}^1_{3[0]} E_1^3} e^{\mathcal{A}^1_{2[0]} E_1^2} \\ &d\mathcal{V}\mathcal{V}^{-1} = \tfrac{1}{2} d\overrightarrow{\phi} \cdot \overrightarrow{h} + \sum_{i<j} e^{\overrightarrow{b}_{ij}\cdot\overrightarrow{\phi}} \mathcal{F}^i_{j[1]} E_i^j \\ &\mathcal{M} = \mathcal{V}^T \mathcal{V} \end{aligned} \tag{2.51}$$

Note that the exponentials in equation (2.51) do not commute among each other because of the Baker-Campbell-Hausdorf formula. Choosing a certain way to organize the exponentials corresponds to a certain parametrization of the coset space. Now, it is straight forward to show that the scalar lagrangian (2.47) can be written as

$$\mathcal{L} = \frac{1}{4} tr(\partial\mathcal{M}^{-1}\partial\mathcal{M}). \tag{2.52}$$

We consider a general transformation $\Lambda \epsilon SL(3, R)$ acting on the coset representative $\mathcal{V}$ from the right and a compensating transformation $\mathcal{O} \epsilon O(3)$ acting on $\mathcal{V}$ from the left

$$\mathcal{V} \to \mathcal{V}' = \mathcal{O}\mathcal{V}\Lambda, \tag{2.53}$$

such that $\mathcal{V}'$ is upper-triangular just like $\mathcal{V}$ itself; this means that $\mathcal{V}'$ can be interpreted as a different point on the coset manifold. The global SL(3,R) symmetry is now manifested

$$\mathcal{M} \to \Lambda^T \mathcal{M} \Lambda \qquad \mathcal{L} \to \mathcal{L}. \tag{2.54}$$

To sum up, in the case of compactification of the eleven-dimensional supergravity on T^3, the seven-dimensional coset manifold is $\frac{SL(3,R)}{O(3)} \times \frac{SL(2,R)}{O(2)}$ and the total symmetry of the lagrangian is $SL(2,R) \times SL(3,R)$.

But, is it always that easy to determine the needed compensating transformation? No, but thankfully group theory comes to our rescue:

Iwasawa Theorem: *Every element g in a group G, that can be obtained by exponentiating the lie algebra $\mathcal{G}$, can be expressed as*

$$g = g_K g_H g_N,$$

where g_K is in the maximal compact subgroup K of G, g_H is in the maximal torus of G and g_N is in the exponentiation of the positive-root part of the algebra g of G.

Our coset representative is constructed by exponentiating the Cartan subalgebra and the full set of positive-root generators, ie $\mathcal{V} = g_H g_N$. If we act on the right of $\mathcal{V}$ by Λ, we obtain a new element of the group $\mathcal{V}\Lambda$. Invoking the above theorem, $\mathcal{V}\Lambda = g_K {g'}_H {g'}_N = g_K \mathcal{V}'$. This assures us that we will always be able to find an element $g_K \epsilon K$ such that $g_K \mathcal{V}' = \mathcal{V}\Lambda$. Thus, the coset manifold is of the form G/K, where K is the maximal compact subgroup of G.

The next question that arises is how to identify the group G in each dimension. The answer is easy: we just identify the dilaton vectors that correspond to simple roots of the lie algebra and then we look up to find to which algebra they correspond.

	G	K	dim(G/K)
D=10	O(1,1)	-	-
D=9	GL(2,R)	O(2)	3
D=8	SL(3,R)× SL(2,R)	SO(3)×SO(2)	7
D=7	SL(5,R)	SO(5)	14
D=6	O(5,5)	O(5)×O(5)	25
D=5	$E_{6(+6)}$	USp(8)	42
D=4	$E_{7(+7)}$	SU(8)	70
D=3	$E_{8(+8)}$	SO(16)	128

Table 2.2: Cosets for maximal supergravity in Minkowskian signature

The scalar coset manifolds coming from toroidal dimensional reduction of the eleven-dimensional supergravity are summarized in the table 2.2. Just by observation of that table, we note that the D=6 case begins to indicate a new phenomenon: it appears to have a mismatch of fields -see Appendix.

- When compactifying on T^5, we obtain 10 1-forms from the reduction of the 3-from and 5 KK gauge fields from the reduction of the metric. So, in total, we have 15 1-forms that should be transforming in a linear representation of the group $O(5,5)$. The problem is that O(5,5) does *not* have a 15-dimensional representation. But, in 6 dimension, the 3-form is dual to a vector field. This adds one more 1-form to give in total 16. The O(5,5) group has a 16-dimensional representation: the spinor representation.

- Also, upon compactification on T^n with n=(2,3,4,5,6,7,8), we obtain (1,4,10,20, 35,56,84) axions in total. On the other hand, the number of positive roots of the groups G are (1,4,10,20,36,63,120) respectively. Thus, we conclude that the numbers of axions do not much for $D < 6$ or in other words, the dimensions of the coset manifolds do not match the total number of scalars obtained by reducing. The solution to this problem is again dualization. For the case D=5, we have 6 dilatons, 35 axions and one 3-form that dualises to an axion. In total, 42 scalars(36 axions) as we wanted. For the case D=4, we have 7 dilatons, 56 axions and 7 2-form that dualises to axions: 70 scalars(63axions) in total, as we wanted. For the case D=3, we have 8 dilatons, 84 axions and 36 1-form that dualises to axions: 128 scalars(120 axions) in total, as we wanted.

Note that in order to obtain the precise form of the group G is needed to take into consideration the Chern-Simons term as well as it provides additional couplings between scalars and form fields.

In order to be more explicit, we shall explore the D=5 case in detail. We will only consider the scalar sector, since it governs the global symmetry of the entire theory, and the 3-form potential term. First of all, we need to show how dualisation of $A_{[3]}$ works. The terms in the lagrangian that involve the latter are the following

$$\mathcal{L}(F_{[4]}) = \frac{1}{2}e^{\overrightarrow{a}\cdot\overrightarrow{\phi}} \star F_{[4]} \wedge F_{[4]} - \frac{1}{72}A_{ijk[0]}dA_{lmn[0]} \wedge F_{[4]}\epsilon^{ijklmn}, \qquad (2.55)$$

where $F_{[4]} = dA_{[3]}$. In the dualisation process, the Bianchi identity $dF_{[4]}$ and the field strength equation of motion interchange roles. To achieve this, we treat $F_{[4]}$ as the fundamental field and we impose the Bianchi identity by adding a term $-\chi dF_{[4]}$ containing a Lagrange multiplier χ. Thus, equation (2.55) becomes

$$\mathcal{L}'(F_{[4]}) = -\frac{1}{2}e^{\overrightarrow{a}\cdot\overrightarrow{\phi}} \star F_{[4]} \wedge F_{[4]} - \frac{1}{72}A_{ijk[0]}dA_{lmn[0]} \wedge F_{[4]}\epsilon^{ijklmn} - \chi dF_{[4]}. \qquad (2.56)$$

Obviously, the equation of motion with respect to the fields χ is the Bianchi identity, while the equation of motion with respect to $F_{[4]}$ is a purely algebraic equation

$$e^{\overrightarrow{a}\cdot\overrightarrow{\phi}} \star F_{[4]} = d\chi - \frac{1}{72}A_{ijk[0]}dA_{lmn[0]}\epsilon^{ijklmn}. \qquad (2.57)$$

We now introduce a new quantity $G_{[1]} \equiv e^{\overrightarrow{a}\cdot\overrightarrow{\phi}} \star F_{[4]}$. If we solve the latter for $F_{[4]}$, we get $F_{[4]} = e^{-\overrightarrow{a}\cdot\overrightarrow{\phi}} \star G_{[1]}$. In terms of $G_{[1]}$, the lagrangian (2.56) becomes

$$\mathcal{L}'(F_{[4]}) = -\frac{1}{2}e^{-\overrightarrow{a}\cdot\overrightarrow{\phi}} \star G_{[1]} \wedge G_{[1]}. \qquad (2.58)$$

What we have just done is to replace the 4-form field strength with a 1-form $G_{[1]}$, or in other words, we have replaced the 3-form gauge field $A_{[3]}$ with an axion χ. Note the change of the sign of the dilaton vector $\overrightarrow{a}$.

Now that we have found one more axion, we need one more positive-root generator, denoted by J, that will correspond to the $-\overrightarrow{a}$ dilaton vector (in addition to the Cartan generators $\overrightarrow{h}$ and positive-root generators $E_i{}^j$ and E^{ijk}) and will satisfy the following commutation relations

$$[\vec{h}, J] = -\vec{a}J, \qquad [E_i{}^j, J] = 0, \qquad [E^{ijk}, J] = 0, \qquad [E^{ijk}, E^{lmn}] = -\epsilon^{ijklmn}J. \tag{2.59}$$

The coset representative will be

$$\mathcal{V} = e^{\frac{1}{2}\overrightarrow{a}\cdot\overrightarrow{H}} \prod_{i<j} e^{\mathcal{A}^i{}_{j[0]}E_i{}^j} \prod_{i<j<k} e^{A_{ijk[0]}E^{ijk}} e^{\chi J}. \tag{2.60}$$

As the maximal compact subgroup of $E_{6(+6)}$ is USp(8), which is no longer orthogonal, the definition of $\mathcal{M}$ will be slightly different than before

$$\mathcal{M} = \tau(\mathcal{V}^{-1})\mathcal{V}, \tag{2.61}$$

where τ is the Cartan involution operator. The action of this operator is to reverse the sign of every non-compact generator, while leaving the sign of the compact generators unchanged. At this point, it would be wise to mention that in the process of toroidal compactification of the eleven-dimensional supergravity we are always dealing with maximally non-compact groups. This means that all the Cartan generators are non-compact, while the rest are equally split into compact and non-compact. This is the reason for the existence of the extra subscript (6) on E_6: it denotes the existence of 6 more non-compact Cartan generators.

Now, we can express the scalar lagrangian in terms of $\mathcal{M}$

$$\mathcal{L} = \frac{1}{4}tr(\partial\mathcal{M}^{-1}\partial\mathcal{M}), \tag{2.62}$$

which is manifestly invariant under the action of global E_6. The coset manifold in this case is $\frac{E_6}{USp(8)}$.

Compactification on T^7 and T^8 proceeds in a very similar manner.

2.4 Duality group evolution: tri-graded structure

In this section, we will attempt to find a link between the duality groups summarized in table (2.2). This problem was also addressed by K.S.Stelle in [7]. In order to investigate this possibility, let us consider the compactification of the eleven-dimensional supergravity on T^4 and then compactify again on S^1.

Upon reduction from D=11 down to D=7, we obtain 14 scalars that parameterize the scalar coset $\frac{SL(5,R)}{SO(5)}$: 4 dilaton, 6 $\mathcal{A}^{ij}_{[0]}$ and 4 $A^{ijk}_{[0]}$; 10 1-forms that lie in the representation $\underline{10}$ of SL(5,R): 4 $\mathcal{A}^{i}_{[1]}$ and 6 $A^{i}_{[1]}$; and 5 2-forms that lie in the $\underline{5}$ representation of SL(5,R): 4 $A^{i}_{[2]}$ and another 2-form obtained from dualisation of the 3-form -see Appendix. If we now reduce again on S^1, we obtain 1 more dilaton coming from the reduction of the metric and 10 more axions coming from the reduction of the 1-forms. As stated at the end of the previous section, the duality group evolves from $SL(5,R)$ in 7 dimensions to $SO(5,5)$ in 6 dimensions.

The D=6 algebra can be presented in a tri-graded structure, with the grading generator gl_1 corresponding to the new dilaton. This gl_1 operator along with the sl_5 generators form the zero weight of the D=6 algebra. We also have 10 non-zero weight generators corresponding to the 10 new axions. Thus, the D=6 algebra S0(5,5) may be presented as

$$\overline{10}^{(-n)} \oplus (gl_1 \oplus sl_5)^{(0)} \oplus 10^{(n)}, \tag{2.63}$$

where $R^{(n)}$ denotes the R representation of sl_5 with (n) gl_1 grading weights. The reason why the $gl_1 \oplus sl_5$ subalgebra has a zero weight is because the metric, from which the gl_1 dilaton emerges, is invariant under the transformations generated by the sl_5 algebra and thus gl_1 and sl_5 commute among themselves (recall the definition of R-symmetry). To determine n, we have to consider how the SO(5,5) algebra acts on 2-forms and 1-forms. In 6 dimensions, we have:

- 5 self-dual 2-forms emerging directly from the 5 2-forms in 7 dimension. Thus, they will lie in 5^q and $\overline{5}^{(-q)}$ representations, with only one of the two being independent. Let this be $\overline{5}^{(-q)}$(q is to be determined).

- 1 KK vector field arising from the reduction of the metric, 10 1-forms emerging directly from the 10 1-forms in 7 dimension and another 5 coming from the 5 2-forms in 7 dimensions. Thus, they will lie in the $1^{(-p)} \oplus (10)^{(n-p)} \oplus \overline{5}^{(-q+p)}$ representation, with p to be determined. In order to make the assignments, we take into account the relation between the $10^{(n)}$ axions and the $10^{(n-p)}$ 1-forms: the axions have one extra internal index pointing in the 7th direction and this index corresponds to adding gl_1 weight p. The same argument holds for the $\overline{5}^{(-q+p)}$ 1-forms and the $\overline{5}^{(-q)}$ 2-forms. Finally, the $10^{(n)}$ axions must rotate $1^{(-p)}$ into $10^{(n-p)}$.

To determine the exact values of the grading assignments, we just require the 2-forms and 1-forms to fall in representations of the D=6 algebra.

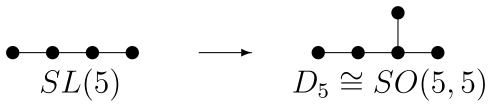
- From basic group theory [5], we know that the product of the 1-form representations $(10)^{(n-p)} \times 10^{(n)}$ must give a $\overline{5}^{(-q+p)}$. Thus, 2(n-p)=-q.
- Also, the product of the 2-form representation $(10)^{(n)} \times \overline{5}^{(-q)}$ must give a 5^{q}. Thus, n=2q and p=5q/2. If we choose q to be q=2, then n=4 and p=5.

Consequently, the D=6 algebra S0(5,5) may be presented as

$$\overline{10}^{(-4)} \oplus (gl_1 \oplus sl_5)^{(0)} \oplus 10^{(4)}. \tag{2.64}$$

The 1-forms form the representation 16_c of S0(5,5): $1^{(-5)} \oplus (10)^{(-1)} \oplus \overline{5}^{(3)}$ and the 2-forms form the 10 of S0(5,5): $\overline{5}^{(-2)} \oplus 5^{(2)}$. The 25 scalars parameterize the coset manifold $\frac{SO(5,5)}{SO(5)\times SO(5)}$ and can be arranged in terms of the 16_c using the parabolic gauge. One S0(5) may be used to gauge away 10 out of the 24 sl_5 generators. We have 14 generators left, corresponding to the 14 scalars in D=7 (remember that in D=7 the coset manifold is $\frac{SL(5,R)}{So(5)}$). Another S0(5) may be used to gauge away the negatively graded $\overline{10}^{(-4)}$ generators. We still have the positively graded generators $10^{(4)}$, corresponding to the 10 new axions that emerge, and the gl_1 generator, corresponding to the new dilaton that appears.

In terms of Dynkin diagrams, the duality group evolution just described can be presented as

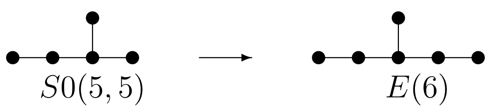


Now, compactify one more dimension: we obtain 1 more dilaton and 16 more axions. From the table in section (2.3), we expect the duality group to evolve from SO(5,5) to $E_{6,(6)}$. The D=5 algebra, as before, can be presented in a tri-graded structure; the grading generator corresponds to the new dilaton just like before. This gl_1 operator along with the $so(5,5)$ generators form the weight zero sector of the D=5 algebra. We also have 16 non-zero weight generators corresponding to the 16 new axions. Thus, the D=5 algebra $e_{6,(6)}$ may be presented as

$$16_a^{(-3)} \oplus (gl_1 \oplus s0(5,5)^{(0)} \oplus 16_c^{(3)} \tag{2.65}$$

The 1-forms lie in the representation 27 of $e_{6,(6)}$: $1^{(-4)} \oplus (16)^{(-1)} \oplus 10^{(2)}$, as we have 1 KK vector field arising from the reduction of the metric, 16 1-forms emerging directly from the 16 1-forms in 6 dimension and another 10 coming from the reduction and dualisation of the 2-forms. Note that the exact grading assignments are determined by the requirement that the 1-forms form a representation of the D=5 algebra. The 42 scalars parameterizing the coset manifold $\frac{E_{6(+6)}}{USp(8)}$ can be arranged in terms of the 27 using the parabolic gauge. The generators of USp(8) divided into those lying in the $SO(5) \times SO(5)$, which are used to gauge away 20 generators of the $so(5,5)$, and those lying outside of it, which are used to gauge away the negatively graded $16^{(-3)}$ generators. Thus, we are left with the 25 scalars parameterizing the coset manifold $\frac{SO(5,5)}{SO(5)\times SO(5)}$, with the dilaton corresponding to gl_1 and the 16 axions corresponding to $16_c^{(3)}$.

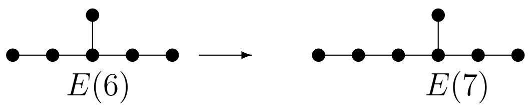
The relevant Dynkin diagrams are the following.



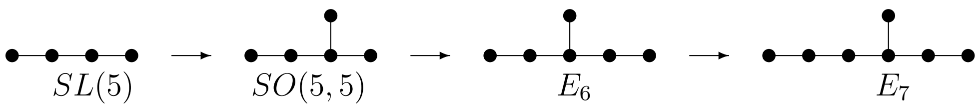
If you do another compactification on S^1 (in order to descend down to 4 dimensions) and use similar arguments to those stated above, we find that the duality algebra structure is the following

$$\overline{27}^{(-2)} \oplus (gl_1 \oplus e_{6(+6)})^{(0)} \oplus 27^{(2)} \tag{2.66}$$

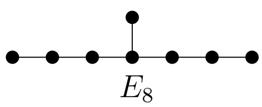
The self-dual 1-forms lie in the representation 56 of the algebra of $E_{6(+6)}$: $1^{(-3)} \oplus (27)^{(-1)} \oplus \overline{27}^{(1)} \oplus 1^{(-3)}$. This evolution can be represented in terms of Dynkin diagrams as follows.



If we put together all the Dynkin diagrams that correspond to the duality groups of D=7,6,5,4, we get a really enlightening picture.



Finally, note that the duality group in 3 dimensions is $E_{8(8)}$ and the relevant Dynkin diagram is the following.



Chapter 3

Stationary Solutions

In this chapter, we will discuss stationary solutions of supergravity theories [4, 7, 8, 9]. In order to do that, we will need to introduce dimensional reduction along the time direction. For simplicity, we will restrict ourselves on studying spherically symmetric solutions and we will mainly focus on four-dimensional theories.

3.1 Timelike Reduction and the relevant duality groups

In this section, we will explore compactification along the time direction. This kind of reduction is really similar to the one described in the previous chapter and thus we will be brief; we will basically follow [7]. Firstly, we need to set the ansatz for the metric and the 1-form gauge field.

- Upon timelike reduction of a (D+1)-dimensional, Minkowskian metric, we obtain a D-dimensional Euclidean metric γ_{ij}, a KK scalar ϕ and a KK vector field $\mathcal{A}_i$. Remember that all the lower-dimensional fields are required to be independent of the compactified coordinate; in our case, the fields need to be time independent. The ansatz is the following

$$d\hat{s}^2 = -H^{2a}(dt + \mathcal{B}_i dx^i)(dt + \mathcal{A}_j dx^j) + H^{2\beta}\gamma_{ij}dx^i dx^j, \tag{3.1}$$

where $a^2 = [2(D-1)(D-2)]^{-1}$ and $b = -(D-2)a$, just like before, and H is a general harmonic function, i=1..D.

- Upon reduction of a (D+1)-dimensional gauge field $\hat{A}_\mu$, a D-dimensional gauge field A_i and an axion U emerge. The reduction ansatz is the following

$$\hat{A}_{\hat{\mu}} dx^{\hat{\mu}} = U(dt + \mathcal{A}_i dx^i) + A_i dx^i, \tag{3.2}$$

where $\mathcal{A}_i$ is the KK gauge field coming from the reduction of the metric and $\hat{\mu}$=1..(D+1). Both, A_i and U, are considered to be time independent.

Having set the above, we can now compactify the Einstein-Hilbert lagrangian, given by (2.1). The reduced lagrangian is the following

$$\mathcal{L} = \sqrt{\gamma}(R(\gamma) - \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{4}e^{b\phi}\mathcal{F}_{ij}\mathcal{F}^{ij}), \tag{3.3}$$

where $\mathcal{F} = d\mathcal{A}$. We can also compactify the standard, kinetic term of the gauge field $\mathcal{L} = -\frac{1}{4}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}}$, where $\hat{F}_{\hat{\mu}\hat{\nu}} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$ as usual. If we assume flat space, then

$$\mathcal{L} = -\frac{1}{4}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\mu\nu} = -\frac{1}{2}\big(\hat{F}_{0i}\hat{F}^{0i} + \hat{F}_{ij}\hat{F}^{ij}\big) = \frac{1}{2}\partial_i U\partial_i U - \frac{1}{4}F_{ij}F_{ij}. \tag{3.4}$$

Note that the last term of (3.3) and the first term of (3.4) come with a wrong sign; the extra minus sign arises from the single η^{00} used to contract the indices. Systematic analysis of the terms coming with the wrong sign, yielded the following general rule: if the rank of a tensor field is reduced upon timelike dimensional reduction by an odd number, then the kinetic term of that field comes with the wrong sign. For example, the gauge field emerging from the reduction of the metric ($2 \to 1$) and the scalar coming from the reduction of the gauge field ($1 \to 0$) will have kinetic terms with a plus sign instead of the usual minus.

As we have seen, essentially things proceed just like before, for spacelike dimensional reduction, with the difference being some extra minus signs. In table 3.1, we can see the duality groups of the theories emerging after spacelike and timelike reduction; we note that the group G is left unchanged, while the divisor group K is exchanged for a non-compact from K^* of K. Because of this fact, the metric on the scalar manifold G/K^* is now indefinite.

	G	K	K^*
D=10	O(1,1)	-	-
D=9	GL(2,R)	O(2)	O(1,1)
D=8	SL(3,R)$\times$ SL(2,R)	SO(3)$\times$SO(2)	S0(2,1)$\times$ SO(1,1)
D=7	SL(5,R)	SO(5)	SO(3,2)
D=6	O(5,5)	O(5)$\times$O(5)	SO(5,C)
D=5	$E_{6(+6)}$	USp(8)	USp(4,4)
D=4	$E_{7(+7)}$	SU(8)	SU*(8)

Table 3.1: Supergravity sigma-model symmetries for spacelike and timelike dimensional reduction

3.2 Single-center and Multi-center, spherically symmetric solutions

In what follows we will be particularly interested in the sector of the D-dimensional reduced lagrangian (on the time direction) that contains the gravity term and the non-linear sigma model term, ie

$$\mathcal{L}_\sigma = \sqrt{\gamma}\Big(R(\gamma) - \frac{1}{2}G_{AB}(\phi)\partial_i\phi^A\partial_j\phi^B\gamma^{ij}\Big), \tag{3.5}$$

where the scalar fields ϕ^A parameterize the scalar manifold G/K^*, G_{AB} is the indefinite metric on the scalar manifold and γ^{ij} is the D-dimensional Euclidean metric. The field equations arising from the above lagrangian are the following

$$\begin{aligned} \frac{1}{\sqrt{\gamma}}\nabla_i\Big(\sqrt{\gamma}\gamma^{ij}G_{AB}(\phi)\partial_j\phi^B\Big) &= 0, \\ R_{ij}(\gamma) &= \frac{1}{2}G_{AB}(\phi)\partial_i\phi^A\partial_j\phi^B, \end{aligned} \tag{3.6}$$

where ∇_i is a doubly covariant derivative; its action on coset space vectors gives $\nabla_i V_A = \partial_i V_A - \Gamma^D_{AE}(G)\partial_i\phi^E$, where $\Gamma^D_{AE}(G)$ is the usual Cristoffel connection for the metric of the scalar manifold G_{AB}. The above equations were rewritten by Clement using a matrix representation $M(x^M)$ of points on the coset manifold G/K^*[8]

$$\nabla_i\Big(M^{-1}\nabla^i M\Big) = 0,$$
$$R_{ij} = -\frac{1}{4}Tr(\nabla_i M\nabla_j M^{-1}), \tag{3.7}$$

where M is such that the line element on the target space can be written as $ds^2 = G_{AB}d\phi^A d\phi^B = -\frac{1}{4}Tr(dMdM^{-1})$.

Now, we make two simplifying assumptions:

- ϕ^A depends on x^i only through an intermediate scalar function $\sigma(x)$, ie $\phi^A(x) = \phi^A(\sigma(x))$.
- $\sigma(x)$ is a function of $r = \sqrt{x^i x^i}$, ie we are looking for a spherically symmetric solution.

The equations of motion, given by (3.6), become

$$\nabla^2\sigma\frac{d\phi^A}{d\sigma} + \gamma^{ij}\partial_i\sigma\partial_j\sigma\big[\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(G)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma}\big] = 0, \tag{3.8}$$
$$R_{ij} = \frac{1}{2}G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma}\partial_i\sigma\partial_j\sigma. \tag{3.9}$$

Using the Bianchi identity, $\nabla^i(R_{ij} - \frac{1}{2}\gamma_{ij}R) = 0$, the latter equation can be rewritten as

$$\frac{1}{4}\frac{d}{d\sigma}\big(G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma}\big)(\nabla_i\sigma\partial_i\sigma\partial_j\sigma) = 0. \tag{3.10}$$

If we require the properties of $\sigma(x^i)$ and $\frac{d}{d\sigma}$ to be separated, the above equations yield the following ones

$$\nabla^2\sigma = 0 \quad\Rightarrow\quad \sigma = h + \frac{q}{r^{D-2}}$$
$$\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(G)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma} = 0, \tag{3.11}$$

along with the gravitational constraint

$$\frac{d}{d\sigma}\Big(G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma}\Big) = 0 \quad\Rightarrow\quad G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma} = 2v^2 = const, \tag{3.12}$$

where h, q and v are constants and D denotes the space dimensions. Equation (3.11) tells us that $\sigma(x)$ is a harmonic function that acts as a map from the D-dimensional Euclidean space to the target space and $\phi^A(\sigma)$ is a geodesic on the coset manifold with respect to the affine parameter $\sigma(x)$. Note that G_{AB} is indefinite and thus, three types of geodesics may exist: spacelike if $v^2 > 0$, timelike if $v^2 < 0$ and null if $v^2 = 0$.

Writing the geodesic equation (3.11) and the gravitational constraint (3.12) in matrix terms, we get

$$\frac{d}{d\sigma}\Big(M^{-1}\frac{dM}{d\sigma}\Big) = 0, \quad \Rightarrow \quad M = Ae^{B\sigma},$$
$$R_{ij} = \frac{1}{4}Tr(B^2)\nabla_i\sigma\nabla_i\sigma, \tag{3.13}$$

where $A\epsilon G/H^*$ and $B\epsilon g$ are constant matrices such that $M\epsilon G/H^*$; A and B are related to the asymptotic values of the scalar fields. We observe that in the particular case that $Tr(B^2) = 0$, the Euclidean space is flat and the geodesics of the target space are null $ds^2 = G_{AB}d\phi^A d\phi^B = -\frac{1}{4}Tr(dMdM^{-1}) = 0$. Non-trivial, null geodesics exist only for non-compact manifolds and are really important as they lead to a class of solutions that contains the supersymmetric BPS black holes. The flatness of the three-dimensional space supports the anticipation that the attractive/repulsive forces associated with different charges are mutually balanced.

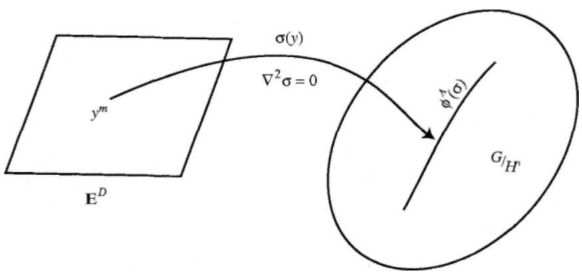


Figure 3.1: Harmonic map from E^D to a null Geodesic to $G/K*$

The above discussion was generalized by Clement for solutions involving multiple harmonic maps $\sigma_a(x)$ (multi-charge solutions)[8]. If the field equations can be reduced to a set of decoupled harmonic equations without assuming spherical symmetry

$$\nabla^2\sigma_a = 0 \quad \Rightarrow \quad \sigma_a = h_a + \frac{q_a}{|\vec{x} - \vec{x}_a|^{D-2}}, \tag{3.14}$$

then, the matrix

$$M = Aexp\Big(\sum_a B_a \sigma_a\Big) \tag{3.15}$$

solves the field equations $\nabla^i(M^{-1}\nabla_i M) = 0$ provided that the commutator $[B_a, B_b]$ commutes with B_c, ie $[[B_a, B_b], B_c] = 0$. In other words, the geodesics $\phi^A(\sigma)$ of the single-center case are replaced by a totally geodesic submanifold of G/K^* (a submanifold is called totally geodesic if every geodesic in it is also a geodesic of the whole manifold).

The condition $[[B_a, B_b], B_c] = 0$ allows one to write

$$M = Aexp\Big(-\frac{1}{2}\sum_{c>b}\sum_b [B_b, B_c]\sigma_b\sigma_c\Big)\prod_a e^{B_a\sigma_a}. \tag{3.16}$$

The matrix current is

$$M^{-1}\nabla_i M = \sum_a B_a \nabla_i \sigma_a - \frac{1}{2}[B_b, B_c](\sigma_b \nabla_i \sigma_c - \sigma_c \nabla_i \sigma_b) \tag{3.17}$$

and is conserved provided that σ_a are harmonic functions. The gravitational constraint can also be translated into this language

$$R_{ij} = \frac{1}{4}\sum_{a,b} tr(B_a B_b)\nabla_i \sigma_a \nabla_j \sigma_b. \tag{3.18}$$

If the D-dimensional space is Ricci flat, then $tr(B_a B_b) = 0$ and the geodesics generated by any affine parameter $\sigma_a(x)$ are all null.

3.3 Stationary life in four dimensions: reduction to E^3 and duality enhancement

Now, we are going to specialize the above discussion for four-dimensional Minkowsky spacetime reduced to three-dimensional Euclidean space. This case was examined by Breitenlohner, Maison and Gibbons in [9].

The reduced Einstein-Hilbert lagrangian is given by equation (3.3)

$$\mathcal{L} = \sqrt{-g}\hat{R}(\hat{g}) = \sqrt{\gamma}(R(\gamma) - \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{4}e^{-2\phi}\mathcal{F}_{ij}\mathcal{F}^{ij}),$$

where we have set b=-2. Note that in three dimensions, the 1-forms are dual to scalars and thus the above system can be interpreted as a 2-scalar fields system. In order to dualise the 1-form, we have to use a Lagrange multiplier χ and treat $\mathcal{F}_{ij}$ as the fundamental quantity. Therefore, we add to the above lagrangian the term $\sqrt{\gamma}\chi\epsilon^{ijk}\partial_i\mathcal{F}_{jk}$, which if intergraded by parts gives $-\sqrt{\gamma}\partial_i\chi\epsilon^{ijk}\mathcal{F}_{jk}$. The equation of motion with respect to the field $\mathcal{F}_{jk}$ is now $\mathcal{F}_{jk} = e^{2\phi}\partial_i\chi\epsilon^{ijk}$. Eliminating $\mathcal{F}_{jk}$ from the equation (3.3), we obtain

$$\mathcal{L} = \sqrt{\gamma}\Big(R(\gamma) - \frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{2}e^{2\phi}\partial_i\chi\partial^i\chi\Big). \tag{3.19}$$

Note the sign-change of the last term of (3.19): the bad + sign flipped back to the normal -. This is due to the extra minus sign that always emerges upon dualisation. For instance, if we were examining the kinetic term of a 1-form field in D=3 dimensions, upon dualisation of the field into a scalar, the sign of the kinetic term would flip from the usual minus to the "bad" plus. Observe also the changed of the dilaton coupling $e^{-2\phi} \to e^{2\phi}$.

The theory given by (3.19) was studied in section (2.1) in detail. The conclusion that we reached was that the two scalars parameterize an $\frac{SL(2,R)}{SO(2)}$ coset manifold with an SL(2,R) global symmetry. This is the so-called Ehlers group and it can be written as $SL(2,R) = 1^{(-2)} \oplus 1^{(0)} \oplus 1^{(2)}$, where the zero-graded generator corresponds to the Cartan generator. In other words, the stationary solutions of pure gravity theories in 4 dimensions yield a formulation of the theory as an SL(2,R) non-linear sigma model coupled to three-dimensional gravity.

This property generalizes for the case of theories of gravity coupled to matter which are already G_4 symmetric in 4 dimensions [7, 9]. Consider a four-dimensional theory that already has a G_4 symmetry and contains 1-forms in the l_4 representation of G_4. Now, reduce this theory to the three-dimensional Euclidean space E_3. The scalars emerging from the reduction of the four-dimensional 1-forms admit a (constant) shifting symmetry, as a result of the global gauge transformations in 4 dimensions. The scalars obtained by dualising the three-dimensional 1-form fields also admit a shifting symmetry as dualisation leaves undetermined constants. All together, these shifting symmetries transform in the l_4 representation of G_4. The commutators of the Ehlers group generators with the generator of the shifting symmetries generate other generators also in l_4. Thus, the G_4 symmetry is enhanced to a symmetry G with the following penta-graded structure:

$$g \cong sl(2,R) \oplus g_4 \oplus (2l_4) \cong 1^{(-2)} \oplus l_4^{(-1)} \oplus (1+g_4)^{(0)} \oplus l_4^{(1)} \oplus 1^{(2)},$$

where the grading generator corresponds to the Cartan subalgebra generator of the Ehlers group, as before. The resulting three-dimensional theory is described by the coset representative $\mathcal{V}\epsilon G/K^*$ and a three-dimensional metric. Note that according to our previous discussion on dualisation, the induced metric is positive definite for the Ehlers and $G_4 \ominus K_4$ sectors, but negative definite for the l_4 Einstein-Maxwell sector (as the scalars emerging from the four-dimensional Maxwell fields come with the wrong sign either because of the nature of the timelike reduction, or because of the dualization process). Some examples of the symmetry enhancement just described are summarized in table 3.2.

The Einstein-Maxwell example:

The Einstein-Maxwell lagrangian in 4 dimensions is the following.

$$\mathcal{L} = \sqrt{\hat{g}}(\hat{R} - \frac{1}{2}\hat{F}^2), \tag{3.20}$$

where $\hat{F}$ is the 2-form field strength of the gauge field $\hat{A}$. Using the ansatzs described previously, we dimensionally reduce this theory along the time direction.

$$\mathcal{L} = \sqrt{\gamma}\Big(R(\gamma) - \frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{2}e^{-2\phi}\partial_i\chi\partial^i\chi + \frac{1}{2}e^{-\frac{1}{2}\phi}\gamma^{ij}\partial_iU\partial_jU - \frac{1}{4}e^{-\frac{1}{2}\phi}F_{ij}F^{ij}\Big) \tag{3.21}$$

Let the Euclidean, three-dimensional metric γ_{ij} be spherically symmetric(not flat):

$$ds^2 = \gamma_{ij}dx^idx^j = dr^2 + f(r)^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{3.22}$$

Minkowsky theory	$\frac{\mathbf{G_4}}{\mathbf{K_4}}$	$\frac{\mathbf{G_3}}{\mathbf{K_3}}$ **spatial reduction**	$\frac{\mathbf{G_3}}{\mathbf{K_3}}$ **timelike reduction**
n+4 dimensional Einstein gravity reduced to D=4	$\frac{GL(n)}{SO(n)}$	$\frac{SL(n+2)}{SO(n+2)}$	$\frac{SL(n+2)}{SO(n,2)}$
Einstein-Maxwell N=2 supergravity	no scalars in D=4	$\frac{SU(2,1)}{S\left(U(2)\times U(1)\right)}$	$\frac{SU(2,1)}{S\left(U(1,1)\times U(1)\right)}$
N=4 supergravity	$\frac{SO(6)\times SO(2,1)}{SO(6)\times SO(2)}$	$\frac{SO(8,2)}{SO(8)\times SO(2)}$	$\frac{SO(8,2)}{SO(6,2)\times SO(2)}$
N=4 supergravity and 6 N=4 super Maxwell	$\frac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$	$\frac{SO(8,8)}{SO(8)\times SO(8)}$	$\frac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$
N=8 supergravity	$\frac{E_{7(+7)}}{SU(8)}$	$\frac{E_{8(+8)}}{SO(16)}$	$\frac{E_{8(+8)}}{SO^*(16)}$

Table 3.2: Symmetry enhancement of 4-dimensional theories with original $\frac{G_4}{K_4}$ symmetry.

The equations of motion of the reduced theory are as follows

$$f^{-2}\frac{d}{dr}(f^2\frac{d\phi^A}{dr}) + \Gamma^A_{BC}(\phi)\frac{d\phi^B}{dr}\frac{d\phi^C}{dr} = 0, \tag{3.23}$$

$$R_{rr} = -2f^{-1}\frac{d^2 f}{dr^2} = G_{AB}(\phi)\frac{d\phi^A}{dr}\frac{d\phi^B}{dr},$$

$$R_{\theta\theta} = R_{\varphi\varphi} = f^{-2}(\frac{df}{dr}\cdot\frac{df}{dr} - 1) = 0. \tag{3.24}$$

The latter equation yields $f^2(r) = (r - r_0)^2 + c^2$. Thus, if we introduce the new, harmonic function $\sigma = -\int\limits_r^\infty f^{-2}(s)ds$ and assume that $\phi(r) = \phi(\sigma(r))$, equation (3.23) becomes

$$\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(\phi)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma} = 0. \tag{3.25}$$

This equation tells us that $\phi^A(\sigma(r))$ is a geodesic and $\sigma(r)$ is an affine parameter. As discussed in the previous section, the scalar coset in this case is given by $\frac{G}{K^*} = \frac{SU(2,1)}{S\left(U(1,1)\times U(1)\right)}$ and corresponds to signature diag(++- -). If we restrict ourselves to static solutions, then the coset manifold becomes $\frac{G}{K^*}\Big|_{static} = \frac{SO(2,1)}{SO(1,1)}$ and corresponds to signature diag(+ -); we can do this because a magnetic field can be eliminated by a duality transformation. The metric G_{AB} in this restricted target space, which in fact is the two-dimensional De Sitter space, is the following $(A, B = 1, 2, \phi^1 = \varphi, \phi^2 = \chi)$

$$ds^2 = \frac{d\phi^2}{2\phi^2} - \frac{2d\chi^2}{\phi}. \tag{3.26}$$

The geodesic equations on this space are

$$\ddot{\varphi} - \varphi^{-1}\dot{\varphi}^2 - 2\dot{\chi}^2 = 0, \qquad \ddot{\chi} - \varphi^{-1}\dot{\varphi}\dot{\chi} = 0, \tag{3.27}$$

where the dot denotes differentiation with respect to σ. Solving the above equations with boundary conditions $(\varphi, \chi) = (1, 0)$ at $\sigma = 0 \leftrightarrow r \to \infty$, we obtain

$$\varphi(\sigma) = \frac{sinh^2\beta}{sinh^2(\beta - v\sigma)},$$
$$\chi(\sigma) = \frac{sinh(v\sigma)}{sinh(\beta - v\sigma)}, \tag{3.28}$$

where $v^2 = \frac{1}{2}G_{AB}\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma}$.

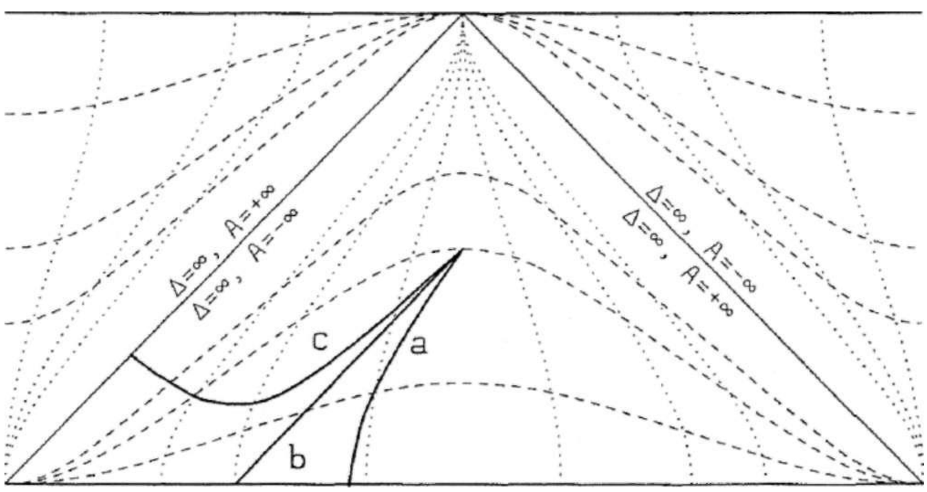


Figure 3.2: Carter-Penrose diagram for the two-dimensional de Sitter space. The curves a,b,c are examples of timelike, null and spacelike geodesics corresponding to the solution found; $\Delta \equiv \varphi, A \equiv \chi$

The above solution can be characterized according to the value of v to which it corresponds.

a. if $v^2 > 0$, the solution is the non-extremal Reissner-Nordström black hole and goes from $(\varphi, \chi) = (1, 0)$ to the horizon where $\varphi = 0$.

b. if $v^2 = 0$, the solution is the extremal Reissner-Nordström black hole and goes from $(\varphi, \chi) = (1, 0)$ to a degenerate horizon where $\varphi = 0$.

c. if $v^2 < 0$, the solution is the hyper-extremal Reissner-Nordström black hole. For a finite value of the affine parameter σ, it goes from $(\varphi, \chi) = (1, 0)$ to a naked singularity where $\varphi = \infty$. Note that these geodesics are obtained by analytic continuation of the $v^2 > 0$ case.

The above solution has mass $m = v \coth \beta$ and charge $q = \frac{v}{\sinh \beta}$. Obviously, the mass m and the charge q satisfy the equation $m^2 - q^2 = v^2$.

Chapter 4

Masses, Charges and Supersymmetry

In the following chapter, we will explore the basic properties of asymptotically flat, stationary solutions of four-dimensional theories. We will restrict our discussion to theories that already have a G_4 symmetry in 4 dimensions and contain 1-forms in the l_4 representation of G_4. G_4 will be assumed to be a semi-simple Lie group. Further details on the subject can be found in [7, 10, 11].

Note that although in the previous chapters we have discussed in detail spacelike and timelike dimensional reduction using the concept of an ansatz, in what follows it will be easier to use the concept of killing vectors: exact solutions of theories of gravity coupled to matter are generally known only when the corresponding metric admits a certain number of killing vectors, and furthermore these isometries leave invariant the various matter fields of the theory. But the existence of k such killing vectors is equivalent to the solution being independent of the k corresponding coordinates in such a way that the solution can be interpreted in D-k dimensions. If all the killing vectors are spacelike, then the fields of the dimensionally reduced theory are defined on a (D-k)-dimensional spacetime and the Hamiltonian of the theory is positively definite. In contrast, if one of the killing vectors is timelike, the fields of the reduced theory are defined in an Euclidean-signature space and the action of the dimensionally reduced theory is indefinite (of course, this is not a problem as we are only interested in the classical equations of motion in an Euclidean-signature reduced theory that has no problem of instabilities -we have assumed a stationary spacetime).

For instance, when compactifying on S^1, the imposition of the Kaluza-Klein metric ansatz is the same as demanding the metric field to admit one killing vector $\hat{k}_{\hat{\mu}}$

$$\mathcal{L}_{\hat{k}}\hat{g}^{\hat{\mu}\hat{\nu}} = 0, \tag{4.1}$$

and that all other fields have vanishing Lie derivatives with respects to that killing vector. If we take $\hat{k}_{\hat{\mu}} = \delta^{\mu}_{z}$, the above requirement is exactly equivalent to demanding z-independence, as discussed in chapter 2.

4.1 Masses and Charges

In the special case of timelike reduction from four-dimensional Minkowsky spacetime to three-dimensional Euclidean space, the Komar 2-form is defined as $K = \partial_\mu k_\nu dx^\mu \wedge dx^\nu$ for a timelike killing vector k and is invariant under the action of the timelike isometry. Assuming that the scalar dual to the KK vector field falls off at infinity as O(1/r), the asymptotic fall off is slow enough to generate non-vanishing Komar mass and NUT charge as a result of the Gauss flux law

$$m = \frac{1}{8\pi}\int_{\partial\Sigma_3} S^* \star K, \qquad n = \frac{1}{8\pi}\int_{\partial\Sigma_3} S^* K, \tag{4.2}$$

where S^* is the pullback to a section of the spatial boundary of Σ_3 and $\star K$ is the hodge dual of K in 4 dimensions.

The field strength equations of motion can be brought to the form $d \star \tilde{F} = 0$, while the Bianchi identity is $d\tilde{F} = 0$, where $\tilde{F} = \frac{\partial \mathcal{L}}{\partial \tilde{F}}$ is in fact a linear combination of the four-dimensional field strengths. These equations permit one to define electric and magnetic charges as follows

$$q = \frac{1}{2\pi}\int_{\partial\Sigma_3} S^* \star \tilde{F}, \qquad p = \frac{1}{2\pi}\int_{\partial\Sigma_3} S^* \tilde{F}. \tag{4.3}$$

These quantities transform together in the representation l_4 of G_4.

Now, we want to study the above charges from the three-dimensional point of view. According to our previous discussion, the reduced theory can be described in terms of the coset representative $\mathcal{V} \epsilon G/K^*$. The Maurer-Cartan form $\mathcal{V}^{-1}d\mathcal{V}$, which is valued in the Lie algebra g of the duality group G, can be decomposed as $\mathcal{V}^{-1}d\mathcal{V} = Q + P$, where $Q = Q_i dx^i$ in k^* and $P = P_i dx^i$ in $g \ominus k^*$. In terms of these quantities, the Bianchi identity can be written as

$$dQ + Q^2 = -P^2 \tag{4.4}$$

and the three-dimensional scalar field equations can be written as

$$d \star P + \{Q, \star P\} = 0, \tag{4.5}$$

where $\star$ is now the Hodge star in 3 dimensions. Note that equation (4.5) can also be written as follows

$$d \star \mathcal{V} P \mathcal{V}^{-1} = 0. \tag{4.6}$$

Thus, one can consider $\star J = \star \mathcal{V} P \mathcal{V}^{-1}$ as the 2-form dual of the conserved current J associated with the G-invariance of the three-dimensional theory (J is valued in the lie algebra g). But since the three-dimensional theory is Euclidean (ie there is no time coordinate), we can *not* speak of a conserved charge. Nevertheless, $\star \mathcal{V} P \mathcal{V}^{-1}$ is d-closed and thus the integral of this 2-form on a given cycle is independent of the choice of cycle representative, ie one can talk of independence of the section s chosen as $\partial\Sigma_3$ in (4.2). Thus, subject to appropriate asymptotic conditions, m, n, q^I and p^I are conserved in this sense. The d-closure of $\star J$ allows one to define a conserved charge matrix

$$C = \frac{1}{4\pi} \int_{\partial\Sigma_2} \star \mathcal{V} P \mathcal{V}^{-1}, \tag{4.7}$$

which transforms in the adjoint representation of G. For asymptotically flat solutions,

$\mathcal{V}$ can be normalized in such a way to tend asymptotically to the identity matrix. Then, the charge matrix C can be read off the asymptotic value of P

$$P = C\frac{dr}{r^2} + O(r^{-3}) \tag{4.8}$$

and is valued in $g \ominus k^*$. Recalling that $g = sl(2, R) \oplus g_4 \oplus (2l_4)$ and that k^* is used to gauge away the k_4 components of g_4, the negative-graded l_4 and the so(2) components of the Ehlers sl(2,R), one has

$$g \ominus k^* = (sl(2, R) \ominus so(2)) \oplus l_4 \oplus (g_4 \ominus k_4). \tag{4.9}$$

Thus, the computation of C permits one to define its $sl(2, R) \ominus o(2)$ component as the Komar mass and the Komar NUT charge, and its l_4 components as the electromagnetic charges. The remaining $g_4 \ominus k_4$ scalar charges come from the Noether current related to the original, G_4-invariance of the four-dimensional theory. The associated 3-form J_3 transforms in the adjoint representation of G_4 and satisfies $dJ_3 = 0$ on shell. However, J_3 can not be written as a local function of fields and their derivatives in four dimensions. Nevertheless, for stationary solutions, $i_k J_3$ can be written in terms of the pull-backs of the fields of the three-dimensional theory. The integral of the pull-back on any 2-cycle

$$\frac{1}{4\pi}\int_\sigma s \star i_k J_3 \tag{4.10}$$

is independent of the choice of the representative of that cycle. It is important to note that the scalar charges are *not* independent. In supergravity theories, the imposition of characteristic equations on C will lead to scalar charges being functions of the electromagnetic charges and gravitational charges.

We can summarize what has been said above by the following equation

$$\star\mathcal{V}P\mathcal{V} = 4s^* \star Kh - 4s^*K(e + f) + s^* \star F + s^*F + s^*i_k J_3 + O(r^{-2}), \tag{4.11}$$

where $s^* \star F + s^*F$ is the electromagnetic current, which transforms in the l_4 representation of G_4 and is understood to be valued in the corresponding generators of G with the appropriate normalization; h,e and f are the usual generators of the $SL(2, R)$ group

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

The charge matrix C is associated with single-point solutions of the 3-dimensional theory, called instantons, that correspond to particle-like solutions in 4 dimensions. The independent parameters needed to describe these solutions are the conserved charges in four dimensions: the mass, the NUT charge and the electromagnetic charges (the $g_4 \ominus k_4$ scalar charges are not independent due to the characteristic equation constraint). The electromagnetic charges and the non-compact generators of K^* transform in the l_4 representation of G_4. The action of these generators on the electromagnetic charges is linear in the scalar and gravitational charges in such a way that, if $m^2 + n^2 \neq 0$, one can always find a generator that acts on the electromagnetic charges just by shifting them. This generator forms the abelian S0(1,1) subgroup of K^* and, if $TrC^2 > 0$, it permits one to cancel the electromagnetic charges. It was also proven in [9] that static solutions without electromagnetic charges are regular outside the horizon only if the scalar fields are constant throughout spacetime. Thus, all static solutions with $TrC^2 > 0$ that have no singularities outside the horizon lie on the K^*-orbit of the Schwarzschild solution (the electromagnetic charges are equal to zero for this solution).

4.2 Characteristic Equations

In Weyl canonical coordinates, the four-dimensional metric is given by

$$ds^2 = f^{-1}\big(e^{2k}(dx^2 + d\rho^2) + \rho^2 d\varphi\big) + f(dt + Ad\varphi)^2. \tag{4.12}$$

For axisymmetric stationary solutions f, k and A are functions of x and ρ: $f(x, \rho)$, $k(x, \rho)$ and $A(x, \rho)$. Using these coordinates, according to [7, 10], the coset representative $\mathcal{V}$ for the Schwarzschild solution of mass m can be written as

$$\mathcal{V}_{Schwarschild} = exp(\frac{1}{2}ln\Big(\frac{\rho - m}{\rho + m}h\Big), \tag{4.13}$$

where h is the non-compact generator of the Ehlers sl(2,R). From this, one obtains directly that the charge matrix is $C = mh$. In the penta-graded decomposition of g, the non-compact generator h in the adjoint representation is given by h=diag(2,1,0,-1,-2), where 1 is the identity on l_4 and 0 acts on $g_4 \oplus \{h\}$. It is trivial to show that h satisfies the quintic equation $h^5 = 5h^3 - 4h$. For the duality group E_8, the fundamental and the adjoint representations coincide and thus the quintic equation is also valid for the fundamental representation of h. Consequently, the charge matrix for the Schwarzschild solution and for other solutions on a G-orbit passing through Schwarzschild satisfy the so-called characteristic equation

$$C^5 = 5c^2C^3 - 4c^4C, \tag{4.14}$$

where the square of the so-called BPS parameter c is $c^2 = \frac{1}{trh^2}trC^2$ (the normalization chosen here is such that $c^2 = m^2$ for Schwarzschild). The characteristic equation selects out of all G-orbits the acceptable ones and constrains the $g_4 \ominus k_4$ charges to be functions of (m, n, q^I, p_I). Unacceptable orbits contains exclusively solutions with naked singularities. Note that the same equation is obtained starting from the Kerr solution.

The quintic characteristic equation holds in all cases, but for three-dimensional systems with less than E_8 symmetry. If we are dealing with a theory other than N=8 supergravity with real forms in $E_{8(8)}$ or N=2 supergravity with real forms in $E_{8(-24)}$, a stronger cubic equation is obtained

$$C^3 = c^2C. \tag{4.15}$$

This is true because in these cases, the fundamental representation of G admits a tri-graded decomposition such that h=diag(1,0,-1) and thus $h^3 = h$.

Note that owing to the indefinite metric on the coset manifold $g\ominus k^*$, the trace trC^2 (and thus c^2) can assume either sign. However, the negative values of c^2 correspond to hyper-extremal solutions that will not be considered here. Hence, we will take the BPS parameter to be non-negative.

4.3 Extremal Solutions

In the case of non-rotating black holes, the BPS parameter c^2 is the same as the target space velocity v^2 discussed in the previous chapter: $c^2 = v^2$ and is related to the extremality of the solutions. Look at the Maxwell-Einstein theory as the simplest example with indefinite sigma model metric, $\frac{G}{K^*} = \frac{SU(2,1)}{S(U(1,1)\times U(1))}$. In this case, the charge matrix is

$$C_{Einstein-Maxwell} = \begin{pmatrix} m & n & -\frac{z}{\sqrt{2}} \\ n & -m & i\frac{z}{\sqrt{2}} \\ \frac{\bar{z}}{\sqrt{2}} & i\frac{\bar{z}}{\sqrt{2}} & 0 \end{pmatrix}, \tag{4.16}$$

where z=q+ip is the complex electromagnetic charge. For this charge matrix, $c^2 = \frac{1}{trh^2} trC^2 = m^2 + n^2 - z\bar{z}$. Thus, the solutions fall into three categories depending on the value of c^2: if $c^2 > 0$, the solution is non-extremal; if $c^2 = 0$, the solution is extremal and finally, if $c^2 < 0$, the solution is hyper-extremal. Note that the hyper-extremal solutions have naked singularities, while the extremal and non-extremal solutions have their singularities cloaked by horizons.

In general, we define the K^*-invariant, extremality parameter $\kappa = \sqrt{c^2 - a^2}$, where a is the angular momentum in mass units. For asymptotically Taub-NUT black holes $\kappa = \frac{kA}{4\pi}$, where A is the horizon area and k is the surface gravity.

For extremal, non-rotating black holes $c^2 = 0$. Thus, equation (4.14) yields $C^5 = 0$, while (4.15) gives us $C^3 = 0$. Consequently, the characteristic equations express the degree of nilpotency of the charge matrix which is in turn related to the BPS degree of the solution. Note that for rotating, extremal black holes $c \neq 0$.

4.4 Supersymmetric Solutions

Upon timelike dimensional reduction, the R-symmetry group U(N) of N-extended four-dimensional supergravity theories is enhanced to the larger $G = SO^*(2N)$ group (non-compact for $N > 1$). The non-compact form of the maximal subgroup is $K^* = Spin^*(2N) \times K_0$, where K_0 is a symmetry group determined by the matter content of the four-dimensional theory. The group of automorphisms of the 2N-extended superalgebra in 3 dimensions is the product of the three-dimensional rotation group SU(2) and the R-symmetry group $Spin^*(2N)$. The bosons and the fermions in 3 dimensions are related to spinor representations of $Spin^*(2N)$ and transform into each other by the action of the 2N-extended supersymmetry, with the supersymmetry parameters belonging to the vector representation of the $SO^*(2N)$ group. Note that $Spin*(4) = SU(1,1) \times SU(2)$, $Spin*(6) = SU(1,1)$ and $Spin*(8) = Spin(6,2)$.

Now, let us construct the representations of the group $Spin*(2N)$ starting from its Clifford algebra

$$\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}, \tag{4.17}$$

where $I, J = 1, .., 2N$. From these, we construct the raising and lowering operators

$$a_i = \frac{1}{2}(\Gamma_{2i-1} + i\Gamma_{2i}), \qquad a^i = (a_i)^\dagger = \frac{1}{2}(\Gamma_{2i-1} - i\Gamma_{2i}), \tag{4.18}$$

that satisfy the following anticommutation relations

$$\{a_i, a_j\} = \{a^i, a^j\} = 0, \qquad \{a_i, a^j\} = \delta_i^j. \tag{4.19}$$

Then, we require that there exists a vacuum state $|0>$, such that $a_i|0>$=0, and we build all the other representations by acting on the vacuum state with creation operators.

For $N \geq 5$, there is no independent supermatter and thus, one can represent P_μ(coming from the decomposition of the Maurer-Cartan form) in terms of a state created by the a^i:

$$|P_\mu >= P_\mu^{(0)} + P_{\mu ij}^{(2)} a^i a^j + P_{\mu ijkl}^{(4)} a^i a^j a^k a^l + ..)|0> . \tag{4.20}$$

For $N < 5$, one can have independent supermatter and thus the state carries an extra label for the K_0 representation $|P_\mu, A>$. The charge matrix C can also be represented as a state $|C>$ transforming as a $Spin^*(2N)$ chiral spinor

$$|C >= (\omega + Z_{ij} a^i a^j + \Sigma_{ijkl} a^i a^j a^k a^l + ..)|0>, \tag{4.21}$$

where $\omega = m + in$ is the complex gravitational charge (mass and NUT charge), $Z_{ij} = Q_{ij} + iP_{ij}$ are the electromagnetic charges and Σ_{ijkl} are the scalar charges. If we require the dilatino fields $|\chi>_a$ to be left invariant under the residual, unbroken supersymmetry $\delta|\chi>_a=0$, we find that

$$e^{\mu}_{\alpha}\sigma^{a\ \beta}_{\alpha}(\epsilon^{i}_{\beta}a_i+\Omega_{\beta\gamma}\epsilon^{\gamma}_{i}a^i)|P_\mu>=0. \tag{4.22}$$

The charge matrix sector must satisfy the following equation, which is the so-called Supersymmetric Dirac equation,

$$\boxed{(\epsilon^{i}_{a}a_i+\Omega_{a\beta}\epsilon^{\beta}_{i}a^i)|C>=0} \tag{4.23}$$

where ϵ^{i}_{a} and ϵ^{a}_{i} are the asymptotic values of the killing spinor for the unbroken supersymmetry as $r\to\infty$ and $\Omega_{a\beta}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is an SU(2) invariant. Note that for $N<5$, the state $|C>$ requires an extra label and thus, the relevant condition is

$$\boxed{(\epsilon^{i}_{a}a_i+\Omega_{a\beta}\epsilon^{\beta}_{i}a^i)|C,A>=0}. \tag{4.24}$$

The importance of the last two equations is major as they contain all the information about the supersymmetric solutions of N-extended supergravity theories needed for a complete analysis.

In this language, the extremality condition is translated into a null condition:

$$c^2=0\Leftrightarrow<C|C>=0. \tag{4.25}$$

If this equation is satisfied, then the first order equation that follows from the vanishing of the supersymmetry variation of the gravitinos(and involves the killing spinor)can be integrated. On the other hand, the supersymmetric Dirac equation is equivalent to the requirement that $|C>$ is a pure spinor of $Spin^*(2N)$ and it has the consequence that the characteristic equations can be solved in terms of rational functions. Extremality, i.e. (4.25), coincides with the supersymmetry Dirac equation for $N\leq 5$ pure supergravities, but for the extremal, non-supersymmetric solutions of N=6 and N=8 or of theories with independent vector supermultiplets, the Dirac equation turns out to be a stronger condition.

In order to be more precise, we calculate the extremality parameter for pure supergravity theory with $N \leq 5$ and we find that it factorized as expected

$$c^2 = \frac{|\omega|^2 - |z_1|^2)(|\omega|^2 - |z_2|^2)}{|\omega|^2}. \tag{4.26}$$

The extremality condition $c^2 = 0$ gives either $|\omega|^2 = |z_1|^2$ or $|\omega|^2 = |z_2|^2$. Thus, some degree of supersymmetry follows since a BPS bound is saturated. This factorization does not occur with vector multiplet or N=6,8.

4.5 BPS Geology

As was explained by G.Bossard, H.Nicolai and K.S.Stelle in [10] and [11], the moduli space M of spherically symmetric solutions may be decomposed into strata of various BPS degrees.

$$M = \bigcup_{n \epsilon I} M_n \tag{4.27}$$

These strata, denoted by M_n, are such that their intersections are empty and the intersection of the closure of a stratum $\bar{M}_n$ with another stratum M_m is either empty or M_m itself.

Let M_0 be the non-BPS stratum and M_n be n/N BPS. The main stratum M_0 corresponds to a solution with $c^2 \neq 0$ and has the following structure

$$M_0 = R_+^* \times \frac{K^*}{K_4}, \tag{4.28}$$

where K^*/K_4 corresponds to the conserved four-dimensional charges of a given charge matrix C and R_+^* corresponds to the non-zero value of the BPS parameter. The stratum M_n of n/N BPS degrees can be given as the coset space

$$M_n = K^*/J_n, \tag{4.29}$$

where J_n is the isotropy subgroup that leaves invariant a given charge matrix C.

The moduli space dimensions of some strata of pure supergravity theories, summarized in [10], are given in table 4.1. In order to explain how these dimensions are calculated, let us examine the N=8, non-extremal M_0 case. The maximal subgroup

	N=2	N=3	N=4	N=5	N=6	N=8
dimM_0	4	8	14	22	34	58
dimM_1	3	7	13	21	33	57
dimM_1^0					32	56
dimM_2			8	16	26	49
dimM_4					17	29

Table 4.1: Dimensions of strata in pure supergravities

	N=2	N=3	N=4
M_1	$C^2 = 0$	$C^2 = 0$	$C^3 = 0$
M_2			$C^2 = 0$

Table 4.2: The nilpotency degree of C for N=2,3,4.

J_0 of the non-compact Lie group $E_{8(8)}$ has 190 generators and thus the coset manifold $M_0 = E_{8(8)}/J_0$ is 248-190=58 dimensional (note that non-compact Lie groups may have larger proper subgroups than compact Lie groups). One of those 58 parameters that parameterize M_0 is the BPS parameter c^2. If we restrict ourselves to the submanifold where $c^2 = 0$, we get the 57-dimensional manifold M_1.

As it turns out, the order of the stratum n is related to the nilpotency degree of the charge matrix C in a G-invariant way. In the table 4.2, we can see the nilpotency degree of C for N=2,3,4. For N=2,3, the condition $C^2 = 0$ implies that the last non-trivial stratum is M_1. For N=4, $C^2 = 0$ on M_2 and thus the latter is the last non-trivial stratum. For $N \geq 5$, the nilpotency degree in the fundamental representation is not enough to characterize the degree of the stratum and thus we also need to consider the nilpotency degree of C in the adjoint representation .

Chapter 5

A second approach to the subject: the attractor formalism

So far we have only discussed the pure field-theoretic approach of black holes emerging from supergravity theories. According to this description, the extra dimensions are compactified on some internal manifold X, as discussed in chapter 2, leading to effective field theories in lower dimensions; in this case, our interest is specialized to four dimensions. Locally, the original spacetime is a product of $M^4 \times X$, where M^4 denotes the four-dimensional spacetime. This means that at every point x^μ of M^4 there exists a corresponding, internal space X , whose size is such that it can not be directly observable. However, the space X may be different at each point in M^4: moving through M^4 one may encounter spaces X that are not necessarily equivalent (but they belong to a well-defined class of fixed topology parameterized by certain scalar fields). Hence, when dealing with a solution of a four-dimensional theory that is not constant in M^4, each patch in M^4 corresponding to a certain internal manifold X has a non-trivial image in the scalar manifold of the theory.

Viewed from the higher-dimensional perspective, the fields and in particular the four-dimensional spacetime metric will vary nontrivially over M^4: when moving towards the center of the black hole, the gravitational fields will become stronger and therefore the local product structure $M^4 \times X$ could break down. Kaluza-Klein theories do *not* have much to say about what happens in that case, beyond the fact that the four-dimensional solutions can be lifted to the higher-dimensional ones (oxidation of solutions).

However, the effective field-theoretic approach just described does *not* take into account a special feature of string theory: extended objects, like the branes, may carry global degrees of freedom, as well as local degrees of freedom. This is basically due to the fact that they can wrap themselves around non-trivial cycles of the internal space X. This wrapping tends to take place at a particular point in M^4 and therefore, from the higher-dimensional perspective, this will be seen as a pointlike object: a black hole. Thus, we are dealing with two complementary pictures of the black hole solutions: the first one is based on general relativity, where black holes emerge as global solutions of the Einstein equations and we shall refer to it as the *macroscopic description* (field-theoretic); the other one is based on the entanglement of an extended object on a cycle of the internal space and does not immediately involve gravitons. This description will be referred to as *microscopic* (statistical).

Although it is difficult to understand how these two approaches are related, a connection must definitely exist as gravitons are closed strings that interact with the wrapped branes. As these interactions are governed by the string coupling constant g, we expect the latter to be the link between the two descriptions. In principle, it is very difficult to determine how this link works and therefore, a realistic comparison between microscopic and macroscopic results is usually impossible. However, this is not the case for extremal BPS black holes: it has been shown that the results obtained by these two alternative descriptions coincide and thus, new insights about black holes wes obtained.

To understand how the wrapped branes represent themselves in the field-theoretic description, it is important to realize that the massless four-dimensional fields are associated with harmonic forms on X (remember that an important step of the Kaluza-Klein procedure is the expansion of the higher-dimensional fields in terms of fourier modes). Thus, the higher-dimensional fields $\hat{\phi}(x, y)$ decompose into the massless fields $\phi^A(x)$ according to (schematically)

$$\hat{\phi}(x,y) = \sum_A \phi^A(x)\omega_A(y), \tag{5.1}$$

where $\omega_A(y)$ denotes independent harmonic forms on X. Harmonic forms are in one-to-one correspondence with the so-called cohomology groups consisting of equivalence classes of forms that are closed but not exact. The number of independent harmonic forms of a given degree is equal to the Betti numbers, which are fixed by the topology of the spaces X. The above expression, when substituted into the action of the higher-dimensional theory, leads to interactions of the fields ϕ^A proportional to the following coupling constants,

$$C_{ABC\ldots} \propto \int_X \omega_A \wedge \omega_B \wedge \omega_C \ldots. \tag{5.2}$$

These constants are the so-called intersection numbers.

Due to de Rhams' theorem, there exists a dual relationship between the harmonic p-forms ω and the $(d_X - p)$-cycles on X, where d_X is the dimension of X. We can therefore choose a homology basis for the $(d_X - p)$-cycles dual to the basis adopted for the p-forms. Denoting this basis by Ω_A, the wrapping can now be characterized by writing the corresponding cycle P in terms of the homology basis

$$P = p^A \Omega_A, \tag{5.3}$$

where p^A are integers counting how many times the extended object is wrapped around each cycle. Form the field-theoretic point of view, p^A are the magnetic charges related to the effective action. The electric charges are already an integer part of the effective action, because they are associated with the residual symmetries of the lower-dimensional theory.

5.1 Microscopic description: Black holes in M-theory

In this section, we will examine microscopically the black holes of M-theory; we will try to compute the entropy of the black hole S by counting the microscopic degrees of freedom. Doing so, the need for the existence of attractors will be obvious. The work presented here can be found in [12].

The massless content of M-theory is given by the eleven-dimensional supergravity, which is invariant under 32 supersymmetries. Now, consider the compactification of this theory on an internal space which is the product of the Calabi-Yau threefold (a Ricci flat, three-dimensional complex manifold denoted by CY_3) and a circle S_1. The reduced, four-dimensional theory is now invariant under 8 supersymmetries that are encoded into two independent Lorentz spinors. Hence, the effective four-dimensional field theory will be some version of N = 2 supergravity.

The 5-brane M5 solution of M-theory is the microscopic object that leads to the appearance of black holes: it is wrapped on a 4-cycle P of the CY_3 manifold. Equivalently, having in mind the link between M-theory and type IIA string theory (the latter one is the strong coupling limit of the first one), one may study this class of black holes in type-IIA string theory, where a 4-brane D4 is wrapped on the 4-cycle P.

As was already explained in the introduction, the massless modes of the effective field theory correspond to harmonic forms on the CY_3 space; they do not depend on the S_1 coordinate. For instance, the KK gauge field $\mathcal{A}^0_\mu$ related to the compactification on S_1 corresponds to a 0-form on CY_3. This field couples to the electric charge q_0, which is associated with the momentum modes on S_1. In addition, there also exist 2-form fields on CY_3 that correspond to vector gauge fields A^A_μ originated from the 3-form in 11-dimensions. These 2-forms are dual to 4-cycles and thus the wrapping numbers p^A, related to the wrapping of the 5-brane on these 4-cycles, appear in the effective field theory as magnetic charges coupling to the vector gauge fields A^A_μ. Because of the fact that the product of three 2-forms defines a 6-form and 6-forms can be integrated over the CY_3 space, there exist non-trivial intersection numbers C_{ABC} that will serve as three-point couplings of the effective field theory.

Now, let us try to count the microscopic degrees of freedom. These degrees are associated with the massless excitations of the wrapped five-brane on the 4-cycle(M5 is a (5+1)-dimensional object) and are described by a (1+1)-dimensional superconformal field theory (note that the cycle P must correspond to a holomorphically embedded complex submanifold in order to preserve 4 supersymmetries). Thus, we are dealing with a closed string with left- and right-moving states. The 4 supersymmetries of this conformal field theory are reflected in one of these two sectors, say the right-handed one. As conformal theories in 1+1 dimensions are characterized by a central charge, both the right- and the left-handed sectors are characterized by central charges (c_R and c_L respectively) that can be expressed in terms of the wrapping numbers p^A, the intersection numbers C_{ABC} and the second Chern class c_{2A}.

$$\begin{aligned} c_L &= C_{ABC}p^Ap^Bp^C + c_{2A}p^A \\ c_R &= C_{ABC}p^Ap^Bp^C + \tfrac{1}{2}c_{2A}p^A \end{aligned} \tag{5.4}$$

The above results are really hard to prove and hold only if the wrapping integers p^A are *large*. Supposing that this is true, every generic deformation of P will be smooth and thus it will be possible to relate the topological properties of the 4-cycle with the topological information of CY_3.

From general arguments, it follows that we can always find a supersymmetric state in the right-moving sector with a given momentum q_0. The states in the left-handed sector are not supersymmetric and have a certain degeneracy depending on the value of q_0. Thus, we can speak of towers of degenerate states invariant under 4 supersymmetries, that are built on supersymmetric states in the right-moving sector. Using Cardy's formula, the degeneracy of states for fixed, but large momentum (large as compared to c_L), equals $e^{2\pi\sqrt{|q_0|c_L/6}}$. This leads to the following expression for the entropy

$$S_{micro}(p,q) = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|(C_{ABC}p^Ap^Bp^C + c_{2A}p^A)}, \tag{5.5}$$

where $\hat{q}_0 = q_0 + \frac{1}{2}C^{AB}q_Aq_B$ and C^{AB} is the inverse of $C_{AB} = C_{ABC}p^C$. The shifting of q_0 is due to the fact that the electric charges associated with the vector gauge fields A^A_μ will interact with the two-brane M2 solution of the eleven-dimensional supergravity. At this point, we should stress that the term in (5.5) which is proportional to the triple intersection number C_{ABC} is obviously the leading contribution whereas the others are subleading.

As shown by equation (5.5), the microscopic expression for the black hole entropy depends only on the charges. On the other hand, following a field-theoretic calculation, we may end up with a formula giving the entropy S as a function of other quantities as well (such as the values of the moduli fields at the horizon). *To establish any agreement between these two descriptions, the scalar fields must take fixed values at the horizon which may only depend on the charges.* As it turns out, this is indeed the case for extremal black hole solutions: the values taken by the fields at the horizon are independent of their asymptotic values at spatial infinity and are given by the ratio of the charges. In the next section, we will demonstrate this in the context of specific examples.

5.2 BPS attractors

According to the no-hair theorem, there is a limited number of parameters that completely specify the black hole solutions. These parameters include the mass, the electric and magnetic charges, the angular momentum and the asymptotic values of the scalar fields. It appears that for BPS extremal black holes one can prove a new, stronger version of the no-hair theorem: Black hole solutions near the horizon are characterized only by those discrete quantities which correspond to conserved charges. This is basically due to the fact that by the time the scalar field reaches the horizon, it loses all the information about its initial condition (scalar hair), even though the dynamics are strictly deterministic. In other words, ϕ gets a definite value at the horizon, regardless of its boundary value at infinity that may change continuously. This solution is called an attractor and its existence is the essence of the attractor mechanism.

Example: Dilatonic Black Hole attractor

Here we review work by Ferrara and Kallosh. In section 4 of [17], the authors investigate a simple example of the attractor mechanism given by the N=4 dilatonic black holes of the heterotic string theory in a supersymmetric spacetime. Our aim is to verify that the dilaton field ϕ obtains a definite value on the black hole horizon, depending only on the electric q and magnetic p charges of the black hole and not on its value at the spatial infinity.

The lagrangian to be considered is the part of the SO(4) version of the N=4 four-dimensional supergravity action without an axion:

$$\mathcal{L} = \frac{1}{16}\sqrt{-g}\Big(R - 2\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\big(e^{-2\phi}F^{\mu\nu}F_{\mu\nu} + e^{2\phi}\bar{G}^{\mu\nu}\bar{G}_{\mu\nu}\big)\Big) \tag{5.6}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor and $\bar{G}^{\mu\nu}$ is an electromagnetic dual field tensor given by

$$\bar{G}^{\mu\nu} = \frac{i}{2\sqrt{-g}}e^{-2\phi}\epsilon^{\mu\nu\lambda\delta}\bar{F}_{\lambda\delta}. \tag{5.7}$$

Note that $F^{\mu\nu}$ and $\bar{G}^{\mu\nu}$ are independent fields. The equations of motion of the theory described above are the following

$$\nabla_\rho\partial^\rho\phi = \frac{1}{4}(e^{-2\phi}F^{\mu\nu}F_{\mu\nu} - e^{2\phi}\bar{G}^{\mu\nu}\bar{G}_{\mu\nu}) \tag{5.8}$$

$$\partial_\rho(\sqrt{-g}e^{-2\phi}F^\rho_\nu) = 0 \tag{5.9}$$

$$\partial_\rho(\sqrt{-g}e^{-2\phi}\bar{G}^\rho_\nu) = 0 \tag{5.10}$$

$$\begin{aligned} G_{\mu\nu} = 8\pi T_{\mu\nu} = \;& \tfrac{1}{2}\big(g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \tfrac{1}{4}g_{\mu\nu}(e^{-2\phi}F^{\rho\sigma}F_{\rho\sigma} - e^{2\phi}\bar{G}^{\rho\sigma}\bar{G}_{\rho\sigma}) \\ & +(e^{-2\phi}F^\rho_\mu F_{\mu\rho} + e^{2\phi}\bar{G}^\rho_\mu\bar{G}_{\mu\rho} - 2\partial_\mu\phi\partial_\nu\phi) \end{aligned} \tag{5.11}$$

For extreme supersymmetric dilatonic black holes, the fields are built out of two harmonic functions H_1 and H_2:

$$\begin{aligned} &ds^2 = e^{2U}dt^2 - e^{-2U}dx^2 \\ &e^{-2U} = H_1H_2, \quad e^{2\phi} = \tfrac{H_2}{H_1}, \\ &F = d\psi\wedge dt, \qquad \bar{G} = d\chi\wedge dt, \\ &\psi = \pm H_1^{-1}, \qquad \chi = \pm H_2^{-1}. \end{aligned} \tag{5.12}$$

Using the gauge field equations (5.9) and (5.10), we determine the harmonic functions in terms of isotropic coordinates

$$H_1 = e^{-\phi_0} + \frac{|q|}{r}, \qquad H_2 = e^{\phi_0} + \frac{|p|}{r}, \tag{5.13}$$

where ϕ_0 is the matter field at infinity. Then, the scalar field equation (5.8) becomes

$$\phi'' + \frac{2\phi'}{r} = \frac{1}{2}\frac{(H_1')^2 H_2^2 - (H_2')^2 H_1^2}{H_1^2 H_2^2}. \tag{5.14}$$

The above equation along with the outputs of the Einstein equation (which are too large to display) gives the following result

$$\phi = \frac{1}{2} lnH_2(r) - \frac{1}{2} lnH_1(r), \tag{5.15}$$

where the explicit forms of H_1 and H_2 are given by (5.13). Calculating now the quantity $e^{-2\phi}$, we find that

$$e^{-2\phi} = \frac{H_1}{H_2} = \frac{e^{-\phi_0} + \frac{|q|}{r}}{e^{\phi_0} + \frac{|p|}{r}}. \tag{5.16}$$

If we plot $e^{-2\phi}$ with respect to r for various ϕ_0, we note that all curves converge to the value $\frac{|q|}{|p|}$ on the horizon(r=0) regardless of the value of the field at infinity.

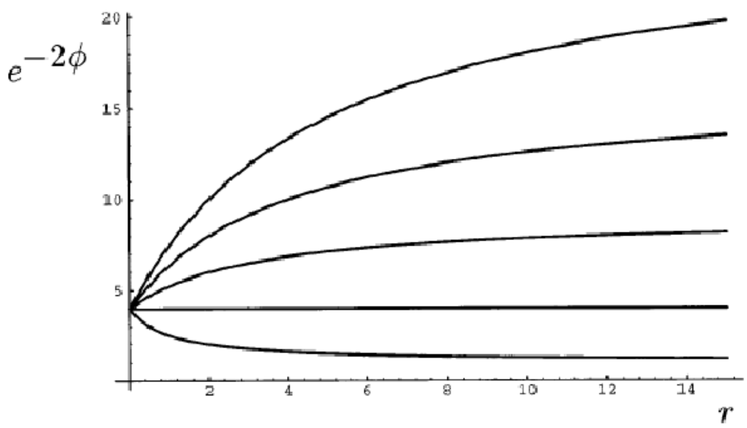


Figure 5.1: Evolution of the dilaton from various initial conditions at infinity to a common fixed point at r=0

The mass M and the dilaton charge Σ of the black hole are related to the U(1) electric $|q|$ and magnetic $|p|$ charges

$$M = \frac{1}{2}(e^{-\phi_0}|p| + e^{\phi_0}|q|), \qquad \Sigma = \frac{1}{2}(e^{-\phi_0}|p| - e^{\phi_0}|q|). \tag{5.17}$$

Thus, we note that the dilatonic black hole is completely characterized by three independent parameters: the two charges q, p and the value of the dilaton at infinity ϕ_0. But, for the black hole solution near the horizon, the value of the dilaton at infinity becomes irrelevant, as it is totally determined by the electric and magnetic charges $e^{-2\phi}\Big|_{fixed} = \frac{|q|}{|p|}$.

Example: N=2 supergravity attractor

Ferrara, Kallosh and Strominger introduced the N=2 supergravity attractor in their original paper [14]. Later on, this attractor was further studied in [14, 15, 16]. In what follows, we will study the supersymmetric black holes arising from four-dimensional N=2 supergravity theories coupled to n N=2 vector multiplets. For simplicity, we will only consider static, spherically symmetric, asymptotically flat black hole solutions.

The supergravity theory is defined in term of a projective covariantly holomorphic section $(X^\Lambda(\phi^i), -\frac{i}{2}F_\Lambda(\phi^i))$ of an Sp(2n+2) vector bundle over the scalar manifold parameterized by ϕ^i, $\Lambda, i = 1..n$. Given this section, one can construct the entire scalar and vector parts of the action. In some cases the theory can be described in terms of a covariantly holomorphic function F(X) of degree two

$$F_\Lambda(\phi^i) = F_\Lambda(X(\phi^i)) = \frac{\partial F(X)}{\partial X^\Lambda}. \tag{5.18}$$

We also introduce the inhomogeneous coordinates

$$Z^\Lambda = \frac{X^\Lambda(\phi^i)}{X^0(\phi^i)}, \qquad Z^0 = 1, \tag{5.19}$$

which are considered to be invertible (special geometry: [13]). For such theories, it was shown that a static, spherically symmetric metric admitting supersymmetries can be written as

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\Big(c^4\frac{dr^2}{sinh^4(cr)} + \frac{c^2}{sinh^2(cr)}d\Omega^2_{S^2}\Big), \tag{5.20}$$

where U is the wrapping parameter and c is the extremality parameter. For spherically symmetric solutions, U is a function only of a radial coordinate r.

The dynamics of the bosonic sector of this theory are controlled by the following four-dimensional, pure geodesic lagrangian density [15]

$$\mathcal{L} = -\frac{R}{2} + g_{i\bar{j}}\partial_\mu\phi^i\partial_\nu\bar{\phi}^{\bar{j}} + I_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} + R_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\mu\nu}, \tag{5.21}$$

where $I_{\Lambda\Sigma}(\phi)$ and $R_{\Lambda\Sigma}(\phi)$ are functions of ϕ and $g_{i\bar{j}}$ is the metric on the scalar manifold. If we now integrate the lagrangian (5.21) over $R_t \times S^2$ and we discard infinite integration constants, we get the following effective one-dimensional lagrangian

$$\mathcal{L} = (U'(r))^2 + g_{i\bar{j}}\phi'^i\bar{\phi}'^{\bar{j}} + e^{2U}V_{BH}(\phi, q, p) - c^2. \tag{5.22}$$

The first and last terms of the above lagrangian come from the Einstein-Hilbert term, while the second term comes from the kinetic term of the scalar fields. The black hole effective potential V_{BH} comes from the vector fields terms and it is positive semi-definite. Note that the theory described by this effective lagrangian gives the same equations of motion as the initial theory only if the following Hamiltonian constraint is satisfied

$$(U'(r))^2 + g_{i\bar{j}}(\phi^i(r))'(\bar{\phi}^{\bar{j}}(r))' + e^{2U}V_{BH}(\phi, q, p) = c^2. \tag{5.23}$$

V_{BH} is the effective black hole potential and it can be identified with the symplectic invariant I_1 of the special geometry

$$V_{BH} = I_1 = |\mathcal{Z}|^2 + 4g_{i\bar{j}}\partial_i|\mathcal{Z}|\partial_{\bar{j}}|\mathcal{Z}|. \tag{5.24}$$

$\mathcal{Z}$ is the central charge and is given by

$$\mathcal{Z} = e^{\frac{K(\phi\bar{\phi}}{2}}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda), \tag{5.25}$$

where K is the Kähler potential. L^Λ and M_Λ form a symplectic section (L^Λ, M_Λ) $\Lambda = 1, .., n$ and satisfy the symplectic constraint $i(\bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda) = 1$. Using some properties of the special geometry and of the central charges and their covariant derivatives, the lagrangian (5.22) can be brought to the following form for an extremal solution (c=0)

$$\mathcal{L} = \Big((U' \pm e^U |\mathcal{Z}|)^2 + \big|\phi^{i'} \pm 2e^U g_{i\bar{j}} \partial_{\bar{j}} |\mathcal{Z}|\big|^2 \mp 2\frac{d}{dr}(e^U |\mathcal{Z}|) \Big). \tag{5.26}$$

The equations of motion of this theory

$$U'' = e^{2U} V_{BH}, \qquad \phi'' + \Gamma^i_{jk} \phi^{j'} \phi^{k'} = e^{2U} g^{ij} \partial_j V_{BH} \tag{5.27}$$

coincide with the equations of motion of the initial theory upon satisfaction of the following first-order flow equations

$$U' = e^U |\mathcal{Z}|, \tag{5.28}$$

$$\phi^{i'} = 2e^U g_{i\bar{j}} \partial_{\bar{j}} |\mathcal{Z}|. \tag{5.29}$$

The value of the ADM mass turns out to be $M = e^U|\mathcal{Z}|$. If we now differentiate the last equation with respect to r and substitute U' given by equation (5.28), we obtain a second-order differential equation that involves only ϕ. This equation can be thought of as a generalized geodesic equation describing the evolution of the scalar field as it approaches the center of the black hole. Setting initial conditions for ϕ^i at infinity (r=∞), we fix $\phi^{i'}$ in terms of the charges by (5.29), while ϕ^i evolves until it runs on a fixed point at the horizon

$$\phi^i|_{fixed} = \frac{q^i}{q^0}. \tag{5.30}$$

Thus, we've reached the same conclusion as for dilatonic black holes: the values of the scalar fields at the fixed point depend on the charges, but not on the asymptotic values of the scalars. Note that according to the behavior of the central charge at the horizon

$$D^i \mathcal{Z}|_{fixed} = 0, \tag{5.31}$$

where D^i is the covariant derivative with respect to the Kähler connection. Using properties of the special geometry, we find that

$$D^i \mathcal{Z}|_{fixed} = 0 \qquad \Rightarrow \qquad \frac{\partial V_{BH}}{\partial \phi^i}\Big|_{fixed} = 0. \tag{5.32}$$

Therefore, we are also allowed to conclude that the effective potential is extremized (only) at the fixed points of the scalar manifold. Thus the argument can be reversed:

the fixed values of the moduli are defined by extremization of the potential. To find whether the extremum is a maximum or a minimum, we have to check the sign of second derivative of the potential

$$\bar{\partial}_{\bar{i}}\partial_j V_{BH}\Big|_{fixed} = 2G_{\bar{i}j}V_{BH}\Big|_{fixed}. \tag{5.33}$$

Thus, the sign of the second derivative is defined by the sign of the metric on the scalar manifold at the critical point: if the metric is positive, the potential reaches its *unique* minimum at the fixed point; if the scalar metric is singular or it changes the sign, the potential is extremized outside the range of applicability of the special geometry.

5.3 Extended Supergravity attractors

In this section, we will discuss four-dimensional $N > 2$ supergravities theories such that their scalar fields parameterize a coset manifold G/K. Such theories were studied by Ferrara, Kallosh an Gibbons in [15].

Note that for $N \leq 4$, the divisor group decomposes into $K = K_{aut} \times K_{matter}$, where $K_{aut} = SU(N) \times U(1)$ is the automorphism subgroup of K and K_{matter} is determined by the matter content of the four-dimensional theory(if any).

The graviphoton central charges $\mathcal{Z}_{AB}$ and the matter charges $\mathcal{Z}_I$ are given by the following equations

$$\begin{aligned} \mathcal{Z}_{AB} &= f^{\Lambda}_{AB}q_{\Lambda} - h_{\Lambda AB}p^{\Lambda}, \\ \mathcal{Z}_I &= f^{\Lambda}_I q_{\Lambda} - h_{\Lambda I}p^{\Lambda}, \end{aligned} \tag{5.34}$$

where q_{Λ} and p^{Λ} are the electric and magnetic charges respectively and $(f^{\Lambda}_{AB}, h_{\Lambda AB})$ and $(f^{\Lambda}_I, h_{\Lambda I})$ form symplectic sections (A,B are indices in the antisymmetric representation of K_{aut} and I is in the fundamental representation of H_{matter}). The covariant derivatives of the central charges with respect to the geometric formulation of extended supergravities are as follows.

$$\nabla \mathcal{Z}_{AB} = \frac{1}{2}\bar{\mathcal{Z}}^{CD} P_{ABCD} + \bar{\mathcal{Z}}_I P^I_{AB},$$
$$\nabla \mathcal{Z}_J = \frac{1}{2}\bar{\mathcal{Z}}^{CD} P^I_{CD} + \bar{\mathcal{Z}}_J P^{JI}, \tag{5.35}$$

where P_{ABCD}, P^I_{AB} and P^{JI} are the components of the coset veilbein P

$$P = \begin{pmatrix} P_{ABCD} & P_{ABJ} \\ P_{JAB} & P^{IJ} \end{pmatrix} \tag{5.36}$$

Because of the manifestly symplectic form of the supergravity, there exists a simple and completely general expression for the black hole potential

$$V_{BH} = \frac{1}{2}\mathcal{Z}_{AB}\bar{\mathcal{Z}}^{AB} + \mathcal{Z}^I \bar{\mathcal{Z}}_I. \tag{5.37}$$

In order to proceed, we will need to consider two cases:

- **$\mathbf{N \geq 2}$ supergravities with $\mathcal{Z} = \mathbf{0}$.** The effective black hole potential is now given by $V_{BH} = \frac{1}{2}\mathcal{Z}_{AB}\bar{\mathcal{Z}}^{AB}$ and the covariant derivative of the central charges is $\nabla \mathcal{Z}_{AB} = \frac{1}{2}\bar{\mathcal{Z}}^{CD} P_{ABCD}$. If we now calculate the derivative of the potential and we set it to zero in order to get the extremum

$$\partial_i V_{BH} = \frac{1}{4}\bar{\mathcal{Z}}^{CD}\bar{\mathcal{Z}}^{AB} P_{ABCD,i} + \frac{1}{4}\mathcal{Z}_{AB}\mathcal{Z}_{CD} P^{ABCD}_{,i} = 0, \tag{5.38}$$

  we find that the above equation has solutions only if the central charge matrix has only one non-zero eigenvalue, while all the others vanish (this is due to the fact that P_{ABCD} is completely antisymmetric).

- **Matter coupled $\mathbf{N \geq 2}$ supergravities with $\mathcal{Z} \neq \mathbf{0}$**. In this case, if we set the derivative of the black hole potential (5.37) to zero, we obtain the following

$$\begin{aligned}\partial_i V_{BH} \;&= \tfrac{1}{4}\bar{\mathcal{Z}}^{CD}\bar{\mathcal{Z}}^{AB} P_{ABCD,i} + \tfrac{1}{4}\mathcal{Z}_{AB}\mathcal{Z}_{CD} P^{ABCD}_{,i} \\ &\quad + \tfrac{1}{2}\bar{\mathcal{Z}}^I P^I_{AB,i}\bar{\mathcal{Z}}^{AB} + \tfrac{1}{2}\mathcal{Z}^I P^{AB}_{I,i}\mathcal{Z}^{AB} = 0.\end{aligned} \tag{5.39}$$

  The above equation is solved, if the central charge matrix has only one non-zero eigenvalue, while all the others and the matter charges vanish.

Thus, the construction set for the BPS attractor applies also for the case of N-extended supergravity theories.

5.4 Non-BPS attractors

The attractor mechanism seems to be related to the extremality rather than to the supersymmetry property of a given solution, as an attractor behavior can also be found for non-BPS extremal solutions (only if the effective black hole potential is extremized on the horizon and the Hessian matrix $\partial_i \partial_j V_{BH}|_{horizon}$ is positive definite). Although the BPS and non-BPS cases share some common features, the non-BPS extremal black holes are not expected to satisfy a first order differential equation, as this is a feature that arise as a consequence of the supersymmetry transformations of the fermions. Instead, one expects them to satisfy a second order differential equation. However, as was shown by Ceresol, D'Auria and Ferrara in [16], there is also a first-order formalism, that identically solves the equations of motion and can be related to the attractor behavior. This is where "fake superpotentials" W and "fake" supergravities come into the discussion; fake supergravities are theories that although are not supersymmetric in general, they present first-order equations for the metric and the scalar fields that "look like" BPS equations. These equations originate from the vanishing action of certain operators on spinor parameters.

The lagrangian (5.22)

$$\mathcal{L} = (U'(r))^2 + g_{i\bar{j}} \phi'^i \bar{\phi}'^{\bar{j}} + e^{2U} V_{BH}(\phi, q, p) - c^2,$$

is quite general, as it can describe any four-dimensional theory of gravity provided that the effective black hole potential is tuned with the theory under consideration. For instance, for appropriate V_{BH}, the above lagrangian can describe non-supersymmetric theories and supersymmetric theories for which the hamiltonian constraint admits multiple solutions. The only difference now is that the effective black hole potential is *not* equal to $V_{BH} = |\mathcal{Z}|^2 + 4 g_{i\bar{j}} \partial_i |\mathcal{Z}| \partial_{\bar{j}} |\mathcal{Z}|$ as before. For vanishing extremality parameter c, the Hamiltonian constraint

$$(U'(r))^2 + g_{i\bar{j}} \phi'^i \bar{\phi}'^{\bar{j}} + e^{2U} V_{BH}(\phi, q, p) = 0$$

can be solved by a real superpotential $W(\phi, \bar{\phi})$ such that

$$U' = \pm e^U W, \qquad \phi^{i'} = \pm 2 e^U g^{i\bar{j}} \partial_{\bar{j}} W, \qquad V_{BH} = W^2 + 4 g^{i\bar{j}} \partial_i W \partial_j W. \tag{5.40}$$

The equations of motion (5.27) are identically satisfied, if the above equations are true. So, we have reached the conclusion that there exists a class of non-BPS extremal black holes that satisfy first-order equation. These first-order equations do not imply anymore preserved supersymmetries as they do *not* coincide with the killing spinor equations and the supersymmetry rules. That is the reason why we refer to them as "fake" supergravities. The critical points are now specified by $D_i W|_{fixed} = 0$.

At this point, we should make two final comments. Firstly, we should note that the BPS attractor, discussed in the previous section, can be seen as a special case of the non-BPS case with $W = \mathcal{Z}$. Also, we have to emphasize the reason why we restrict our discussion to extremal solutions. This is because it is really difficult to find a first-order equation that solves the equations of motion for the scalar fields if $c \neq 0$ (for the wrap-factor, U, this can be done easily, even if $c \neq 0$).

5.5 Attractors and Entropy

In order to understand the black hole entropy in terms of the attractor mechanism, Ferrara and Kallosh introduced in [17] a principle of a minimal central charge $\mathcal{Z}$ and they found a formula giving the Bekenstain-Hawking entropy S. Later on, their formulation was generalized in terms of the effective black hole potential in order to include the non-BPS case. In four dimensions, the entropy S (in Planck units) is given by

$$S = \frac{A}{4} = \pi V_{BH}^{fixed}, \tag{5.41}$$

For extremal BPS black hole $V_{BH} = |\mathcal{Z}|^2 + |D_i \mathcal{Z}|^2 \;\; \Rightarrow \;\; V_{BH}^{fixed} = |\mathcal{Z}^{fixed}|^2$ as at the fixed point $D^i \mathcal{Z}|_{fixed} = 0$. For the non-BAPS case, $V_{BH} = W^2 + 4g^{i\bar{j}} D_i W D_j W \;\; \Rightarrow$ $V_{BH}^{fixed} = W^2$ as $D^i W|_{fixed} = 0$.

Example: Dilatonic Black Hole entropy

In the context of four-dimensional dilatonic black holes, the minimal central charge principle goes as follows. The modulus of he central charge $|\mathcal{Z}|$ is equal to the mass M given by equation (5.17)

$$M = |\mathcal{Z}| = \tfrac{1}{2}(e^{-\phi_0}|p| + e^{\phi_0}|q|).$$

If we extremize the modulus of the central charge with respect to the string coupling constant $g = e^{\phi_0}$,

$$\frac{\partial|\mathcal{Z}|}{\partial g} = \frac{1}{2}\frac{\partial}{\partial g}\Big(\frac{1}{g}|p| + g|q|\Big) = -\frac{1}{g^2}|p| + |q| = 0, \tag{5.42}$$

we find that the extremum of $|\mathcal{Z}|$ occurs at

$$g^2_{fixed} = \Big|\frac{p}{q}\Big|. \tag{5.43}$$

Inserting this value into the central charge formula given by (5.5), we find that $|\mathcal{Z}_{fixed}| = \sqrt{pq}$ and therefore the black hole entropy, given by equation (5.41), is

$$S = \frac{A}{4} = \pi|\mathcal{Z}_{fixed}|^2 = \pi|pq|. \tag{5.44}$$

This result coincides with the results obtained by completely different methods.

Example: Supergravity N=2 Black hole entropy.

In general, for N=2 supergravity theories coupled to n N=2 vector multiplet, the modulus of the central charge is again equal to the mass M of the black hole, but in this case the mass is a function of the charges $|p|$ and $|q|$ and of the scalar field ϕ^i through the holomorphic symplectic section $(X^\Lambda(\phi^i), -\frac{i}{2}F_\Lambda(\phi^i))$

$$M = \Big|\mathcal{Z}(\phi^i, \bar{\phi}^i, q, p)\Big| = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda), \tag{5.45}$$

where L^Λ and M_Λ form a symplectic section (L^Λ, M_Λ) that satisfies the symplectic constraint $i(\bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda) = 1$ $(\Lambda = 1, .., n)$.

Now, following the same procedure as before, we differentiate the modulus of the central charge with respect to the moduli and we set it equal to zero in order to extremize.

$$\frac{\partial}{\partial\phi^i}|\mathcal{Z}| = 0 \tag{5.46}$$

It turns out that $\phi^i|_{fixed}$ is a function of the charges and therefore $\mathcal{Z}_{fixed} = \mathcal{Z}(p, q)$. As a consequence, the Bekenstein-Hawking entropy $S = \pi|\mathcal{Z}_{fixed}|^2$ and the area are just functions of p and q.

Appendix A

Bosonic content of supergravity theories in various dimensions

In what follows, we will need the concept of the little group. Assuming a (D+1)-dimensional spacetime with Minkowskian signature, the little group is the maximal subgroup of SO(D,1) that leaves the momentum of a particle invariant. Thus, if the particle is massless, the little group is S0(D-1) and if the particle is massive, it is S0(D). As was shown by Wigner in 1939 [6], just by knowing the representations of the little group, we can construct the representations of the full SO(D,1).

The bosonic content of the 11-dimensional supergravity is a graviton with 44 degrees of freedom lying in the 2nd rank, symmetric and traceless representation of the little group SO(9) and a 3-form gauge field with 84 degrees of freedom lying in the 3rd rank, antisymmetric representation of SO(9). In the following table, we reduce the graviton and the 3-form in order to find the bosonic content of supergravity theories in less than 11 dimensions. To do this, we follow two simple rules related to the reduction of SO(D-1) representation into representations of SO(D-2):

1. when we reduce a (D+1)-dimensional graviton, we get a D-dimensional graviton, a KK gauge field and a dilaton.

2. when we reduce a (D+1)-dimensional n-form, we get a D-dimensional n-form and a D-dimensional (n-1)-form.

	D=11	D=10	D=9	D=8	D=7	D=6	D=5	D=4	D=3
graviton g_{MN}	1	1	1	1	1	1	1	1	1
KK field $\mathcal{A}_{[1]}$		1	2	3	4	5	6	7	
dilaton ϕ		1	2	3	4	5	6	7	8
axions $\mathcal{A}_{[0]}$			1	3	6	10	15	21	28+8
3-form $A_{[3]}$	1	1	1	1				-	-
2-form $A_{[2]}$		1	2	3	4+1	5			-
1-form $A_{[1]}$			1	3	6	11	16+5	21	
0-form $A_{[0]}$				1	4	10	21	42	63+21

We observe that in 7 dimensions a new phenomenon occurs and the reason is dualization. In 7 dimensions the little group for massless fields is S0(5) and accordingly, the 3-form is dual to a 2-form. The same thing happens both in 5 and 3 dimensions: in 5 dimensions, the 2-form is dual to a 1-form and in 3 dimensions, the 1-form is dual to a 0-form. Note also the existence of (-). This is due to the fact that S0(n) does not have representations with more that n indices antisymmetrized.

Finally, note that the reduced theory is much more complicated than the initial one. This is true in compactification on any manifold and is due to the fact that the number of fields proliferates as the dimensions reduce. This is the main reason why physicists spend so much time studying higher-dimensional theories instead of concentrating just on four dimensions.

Bibliography

[1] C.N.Pope, *IHP lectures on Kaluza-Klein Theory,* http://faculty.physics.tamu.edu/pope/ihlec.pdf

[2] Tomas Ortin, *Gravity and Strings*, Cambridge Univesity Press, 2004

[3] E. Cremmer, B. Julia, H. Lü, C.N. Pope, *Dualisation of Dualities I. and II.,* hep-th/9710119; hep-th/9806106

[4] K.S.Stelle, *Brane Solution in Supergravity,* hep-th/9803116

[5] R.Slansky, *Group Theory for Unified Model Building,* North Holland Publishing Company-Amsterdam 1981

[6] E. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group,* The Annals of Mathematics, Second Series, Vol. 40, No. 1 (Jan., 1939), pp. 149-204

[7] K.S.Stelle, *Dubna Lecture notes 2009: Dualities, Branes and Black Holes*

[8] G. Clement and D. Gal'tsov, *Stationary BPS solutions to dilaton-axion gravity,* Phys. Rev. D 54, 6136 (1996), hep-th/9607043

[9] P.Breitenlohner, D.Maison, G. Gibbons *4-Dimensional Black Holes from Kaluza-Klein Theories,* Commun. Math. Phys. 120, 295-333 (1988)

[10] G.Bossard, H.Nicolai, K.S.Stelle, *Universal BPS structure of stationary supergravity solutions,* hep-th/09024438

[11] G.Bossard, H.Nicolai, *Multi-black holes from nilpotent Lie algebra orbits,* hep-th/0906.1987

[12] B. de Wit, *BPS Black Holes,* hep-th/0704.1452

[13] A.Ceresole, R.D'Auria, S.Ferrara, *The Symplectic Structure of N=2 Supergravity and its central extension,* hep-th/9509160

[14] S.Ferrara, R.Kallosh, A.Strominger, *N=2 extremal Black Holes,* hep-th/9508072

[15] S. Ferrara, G.Gibbons, R.Kallosh, *Black Holes and Critical Points in Moduli Space,* hep-th/9702103

[16] A.Ceresole, G.Dall'Agata, *Flow Equations for Non-BPS Extremal Black Holes,* hep-th/0702088

[17] S. Ferrara, R.Kallosh, *Supersymmetry and attractors,* hep-th/9602136