Understanding: D3-Brane Potentials from Fluxes in Ads/CFT

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Contents

1 Introduction

Inflation in the early universe is now an accepted cosmological fact supporting the homogeneity of the universe and the inhomogeneities of galaxies. Inflationary dynamics is controlled by Planck-suppressed contributions within an effective action. This motivates the building of models which can realize this inflation and handle the Planck-scale contributions. One natural starting point for this is to use string theory and quantum field theory since these theories are well placed to deal with Planck-scale, infra-red and ultra-violet scenarios. However, one major problem with such theories has been the sensitivity of inflation to Planck-scale physics; consequently, building a string theory based model to address all of these factors has proven highly challenging.

D. Baumann et al have been working on such a model, the D3 brane model within their paper - D3 brane potentials from fluxes in AdS/CFT - and their previous supporting papers. What has made their model particularly interesting and robust is that they have self-imposed the requirements that: the model should be consistent with four dimensional Conformal Field Theory in the spirit of the Maldacena AdS/CFT duality; should work successfully as a ten-dimensional supergravity and as a four-dimensional gauge theory, and finally, possess a high level of computability.

The problem I had in reading and trying to understand Baumann's paper was that there were many bridging areas which are beyond that covered by the Master's courses. Therefore, I am assuming that the reader has taken courses in String Theory, Supersymmetry, Differential Geometry, Advanced Field Theory, Differential Geometry and General Relativity. My task in this dissertation is to understand Baumann's paper and explain some of the more important bridging areas.

It makes sense to summarise the D3 brane model itself and provide a simple summary of the inter-relationships and results reported within Baumann's paper.

The D3 brane model:

The goal is to create an effective action for D3 branes in a flux compactification. It is assumed that all of the fluxes will be of an ISD and IASD (Imaginary Self Dual) type and that the presence of nonperturbative effects will cause the D3 brane to experience a potential. It is assumed that the model fits a Type IIB supergravity and that the fields satisfy the associated equations of motion. After compactification, a finite throat is glued into the compact space causing distortions of the supergravity fields in the ultraviolet regions of the throat. The finite warped region of the throat is then approximated by a non-compact deformed conifold whose solutions are explicitly known in terms of harmonic functions, Kahler forms and holomorphic forms. The fluxes, including the scalar mode, can be expressed in terms of these solutions. Stability of the solutions is important and leads to approximations to the associated perturbative expansions and places restrictions on the radial component and scalar modes. Only three types of fluxes, depending on the Hodge type, can be constructed and from these the D3 brane potential is known or can be derived.

Subsequently, there is one major enhancement, that of embedding a stack of D7 branes wrapping a four cycle, which when accompanied by gaugino condensation on the D7 branes induces an extra D3 brane potential.

Next, we consider the inter-relationships and results:

1) Conformal Field Theory: The above theory is a string theory model on $AdS \times T^{11}$. The CFT dual of this is an N=1 supersymmetric Yang Mills theory possessing gauge group and continuous global symmetries. The matching is between the low-dimension protected supermultiplets of operators in CFT and the supergravity three-form fluxes and scalar mode. This is mainly done using dimensional comparison rather than su-

persymmetry derivations. There is also a brief discursive comparison of the scalar potential between the two theories.

2) Superpotentials: That the D3 brane potential specified by a superpotential on the conifold can be geometrized. That is for a superpotential for a D3 brane on the conifold, there exists a supergravity solution in which the BI plus CS potential equals the F-term potential in 4d supergravity ie the F-term potential can be expressed in terms of a nonperturbative superpotential which is equal to the flux potential expressed in terms of holomorphic functions. After matching, this gives a flux representation in terms of the superpotential.

3) Gaugino condensation: Considering gauginos as a source of flux in Field Theory and the Bulk, results are established for a coupling in 4d between the gauginos of the D7 branes in a warped throat and CFT fields dual to fluxes in the throat and that this coupling relationship can be formulated from the BI plus CS couplings action involving gauginos and bulk fluxes

Next we comment on the structure of this dissertation. There is a section on the background concepts consisting of three topics which have been selected as being fundamental to the theory: Calbi-Yau manifolds, AdS/CFT duality and Fluxes. The depth discussed reflects their underlying importance. Following this, there are sections on: (1) the D3 brane model, which is covered in Baumann's chapters 2,3 and 4, (2) CFT, which is covered in Baumann's chapter 5, (3) Superpotentials from Fluxes, which is covered in Baumann's chapter's 6, and finally (4) Gauginos as a source of fluxes, which is covered in Baumann's chapter 7.

Finally, Baumann's paper has been written in the context of an overall physical cosmological structure, namely inflation cosmology which covers the inflation stage of the creation of the universe. The appendix briefly outlines the three main areas leading from inflation cosmology to Baumann's D3-brane model: the Dynamics of Inflation, Quantum Fluctuations of Inflation, and Inflation in String Theory.

2 Background Concepts

2.1 Calabi-Yau Manifolds

There is a good review on Calabi-Yau by B.R.Greene (1997) and this section has used results from that paper.

Compactified string theory requires that the compact portion of space-time meets various stringent constraints. Although there are various manifolds which satisfy these constraints, the most successful manifold is the Calabi-Yau Manifold. This is defined as an n-dimensional manifold which is compact, complex, *Kähler*, and has SU(d) holonomy. There are three kinds of manifolds: topological, differentiable and complex, which can be regarded as surfaces comprising sets of points which possess respectively continuity, smoothness and holomorphic structure. Compactness is a topological manifold property requiring any set of points in the manifold to be approximately continuous. Complex refers to a complex manifold which has structure allowing the existence of holomorphic functions; these are functions $h : C^{n/2} \to C^{n/2}$ such that $h : (z_i, \bar{z}_i) \to h(z_i)$ and similarly for anti-holomorphic functions $h : (z_i, \bar{z}_i) \to h(\bar{z}_i)$, ie the functions in the complex space are separable. These are adequate definitions for our purposes.

However, to define *Kähler* requires some knowledge of differential forms, homology, co-homology and the Hermitean metric. Denoting the complex tangent space by T_pX^C and the dual complex tangent space $T_p^*X^C$, q-forms can be constructed in terms of coefficients $w_{i_1..i_q}$

$$w = \sum w_{i_1..i_r\bar{j}_1...\bar{j}_{q-r}} dz^{i_1} \wedge dz^{i_2} \wedge .....dz^{i_r} \wedge d\bar{z}^{\bar{j}_1} \wedge ...d\bar{z}^{\bar{j}_{q-r}} \tag{1}$$

where the wedge products are the antisymmetric tensor products of the bases dz^i and $d\bar{z}^{\bar{j}}$. $\Omega^{r,s}(X)$ is used to denote the space of antisymmetric tensors with r holomorphic indices and s anti-holomorphic indices: $\bigwedge^r T^{*(1,0)}X \otimes \bigwedge^s T^{*(0,1)}X$

Next the concept of ordinary differentiation is generalised to give a real exterior differentiation map $d : \bigwedge^q T^*X \to \bigwedge^{q+1} T^*X$, defined by

$$d : w \to dw = \frac{\partial w_{i_1..i_q}}{\partial x^{i_{q+1}}} dx^{i_{q+1}} \wedge dx^{i_1} \wedge .....dx^{i_q} \tag{2}$$

For the complex case $\Omega^{r,s}(X) \to \Omega^{r+1,s}(X) \oplus \Omega^{r,s+1}(X)$, the $w^{r,s}$ form, the complex version, becomes

$$\begin{aligned} dw^{r,s} = & \frac{\partial w_{i_1...i_r\bar{j}_1..\bar{j}_s}}{\partial z^{i_{r+1}}} dz^{i_{r+1}} \wedge dz^{i_1} \wedge .....dz^{i_r} \wedge d\bar{z}^{\bar{j}_1} \wedge ...d\bar{z}^{\bar{j}_s} + \\ & \frac{\partial w_{i_1...i_r\bar{j}_1...\bar{j}_s}}{\partial \bar{z}^{i_{s+1}}} dz^{i_1} .....dz^{i_r} \wedge d\bar{z}^{\bar{j}_{s+1}} \wedge d\bar{z}^{\bar{j}^1} \wedge ...d\bar{z}^{\bar{j}_s} \end{aligned} \tag{3}$$

which can be written briefly as $dw^{r,s} = \partial w^{r,s} + \bar{\partial} w^{r,s}$. Now a q form is called closed if $dw_p = 0$ and a q-form is called exact if there exists a (q-1) form w_{q-1} such that $w_q = dw_{q-1}$. From differential geometry, there are some standard results. If a w form is closed, then w is exact or can be expressed as $d\beta$ where β is (q-1) form. Furthermore, there exists a (real) qth DeRham cohomology group $H_d^q(x)$, defined as the quotient space of closed q-forms to the space of exact q-forms ie

$$H_d^q(X, R) = \frac{(w|dw = 0)}{(\alpha|\alpha = d\beta)} \tag{4}$$

where w and α are q-forms and β is a (q-1) form. For a complex manifold, this can be generalised to the (r,s) Dolbeault cohomology group

$$H_{\bar{\partial}}^{r,s}(X,C) = \frac{(w^{r,s}|\bar{\partial}w^{r,s} = 0)}{(\alpha^{r,s}|\alpha^{r,s} = \bar{\partial}\beta^{r,s-1})} \tag{5}$$

where $w^{r,s}$ are as defined in $\Omega^{r,s}(X)$. These groups will provide parameters called Hodge numbers, explained more fully in the next section.

Next, we need the concept of the (complex) Hermitean metric, which is a map $g : T_pX^C \times T_pX^C \to C$ defined as

$$g = g_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}} + g_{\bar{i}j}d\bar{z}^{\bar{i}} \otimes dz^j \tag{6}$$

where g_{ij} are the metric concepts encountered in general relativity. In particular, in the Hermitean metric, there are no components $g_{ij} = g_{\bar{i}\bar{j}} = 0$. There is another way of expressing this result which says that if $\mathfrak{J}$ is a map acting on the tangent space and $\mathfrak{J}^2 = -1$ then $g(\mathfrak{J}v_1, \mathfrak{J}v_2) = g(v_1, v_2)$ for tangent vectors v_1 and v_2.

The tools are now in place to define the *Kähler* property for manifolds. Given a Hermitean metric, there is a form in $\Omega^{1,1}(X)$ defined as

$$J = ig_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}} - ig_{\bar{j}i}d\bar{z}^{\bar{j}} \otimes dz^i = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} \tag{7}$$

When J is closed ie dJ=0, J is called a *Kähler* form and X is called a *Kähler* manifold. Using second differentials, there is a *Kähler* potential K which can be constructed as follows. Since $dJ = 0$,

$$dJ = (\partial + \bar{\partial})ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} = 0 \;\; this \;\; implies \;\; \frac{\partial g_{i\bar{j}}}{\partial z^l} = \frac{\partial g_{l\bar{j}}}{\partial z^i} \tag{8}$$

(similarly with z and $\bar{z}$ interchanged). So there exists a potential, K, such that $g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}}$ or, in brief, $J = i\partial\bar{\partial}K$. One consequence of this is that the Christoffel symbols are either holomorphic or anti-holomorphic.

The last of the properties is Holonomy. This is best understood by description rather than a lengthy proof. Given a tangent vector $v \epsilon T_pX$, if it is parallel transported around a closed curve located at p, the orientation of v will change to say $v^{'}$. The group of transformations which transform v to $v^{'}$ is called the Holonomy Group. For an orientable, differentiable manifold, this group is SO(n), or a subgroup thereof. If X is a complex *Kähler* manifold, the holonomy group is U(n/2) and if n is even, the group becomes SU(n/2).

Calabi-Yau Manifolds - Hodge numbers and Moduli Space

The size and shape of the Calabi-Yau manifolds are usually identified by the Hodge numbers of the manifold's Cohomology group and the family of manifold parameters collectively referred to as the moduli space or Calabi-Yau space.

The Hodge number, denoted by $h_X^{r,s}$ is the dimension of the Dolbeault cohomology group $H_{\bar{\partial}}^{r,s}(X, C)$. However, these take on more significance for *Kähler* manifolds. Since Calabi-Yau manifolds look rather like a 3D fine-meshed fishing net drifting in the sea, the Hodge numbers are a measure of the size of the holes in the net and the net mesh size. Now the adjoint $d^{\dagger}$ is defined as

$$d^{\dagger} : w \longrightarrow d^{\dagger} w = -\frac{1}{(p-1)!} w^{\mu}_{\mu_1 \ldots \mu_{p-1};\mu} dx^{\mu_1} \wedge dx^{\mu_{p-1}} \tag{9}$$

where $w^{\mu}_{\mu_1 \ldots \mu_{p-1};\mu}$ is the covariant derivative of $w^{\mu}_{\mu_1 \ldots \mu_p}$. From differential geometry, there is a Hodge decomposition theorem, which states that any (r,s)-form can be written as

$$w^{r,s} = \bar{\partial}\alpha^{r,s-1} + \bar{\partial}^{\dagger}\beta^{r,s+1} + w^{'r,s} \tag{10}$$

where α is $(r, s-1)$ form, β is $(r, s+1)$ form, and $w^{'r,s}$ is a harmonic p-from. The latter is defined as a form satisfying $\Delta w^{'r,s} = 0$, where $\Delta = \bar{\partial}^{\dagger}\bar{\partial} + \bar{\partial}\bar{\partial}^{\dagger}$ is the Laplacian for (r, s) forms. For *Kähler* manifolds, the $h_X^{r,s}$ are the same as the dimension of the vector space of harmonic (r, s) forms on X. Other properties ensure that there are symmetry properties between r and s and also between n-r and n-s.

If the holonomy group is denoted by SU(d), where $d = n/2$, then the Hodge numbers are usually portrayed as a diamond, eg d=3, a commonly encountered case:

$$\begin{array}{ccccccc} & & & h^{3,3}=1 & & & \\ & & h^{3,2}=0 & & h^{2,3}=0 & & \\ & h^{3,1}=0 & & h^{2,2}=h^{1,1} & & h^{1,3}=0 & \\ h^{3,0}=1 & & h^{2,1} & & h^{1,2}=h^{2,1} & & h^{0,3}=1 \\ & h^{2,0}=0 & & h^{1,1} & & h^{0,2}=0 & \\ & & h^{1,0}=0 & & h^{0,1}=0 & & \\ & & & h^{0,0}=1 & & & \end{array} \tag{11}$$

As can be seen, for d=3 there are numerous possibilities and combinations of hodge numbers, whereas, in fact, for d=1 and d=2 the h numbers are unique.

Turning to the moduli space, this is the family of parameters of the manifold. However, to understand its content, it is necessary to understand the concept of deformations of the complex structure of X. Given a Calabi-Yau manifold with a Hermitean metric g such that the Ricci metric vanishes ie $R_{i\bar{j}}(g) = 0$, by considering perturbations $g+\delta g$ to the internal space and the curvature tensor (and requiring Ricci flatness) leads to restrictions on g. For our purposes, the fact that matters is that $\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ is harmonic and, therefore, is related to $H^{1,1}_{\bar{\partial}}(X)$, while $\Omega_{ijk} g^{k\bar{k}} \delta_{\bar{k}\bar{l}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{l}}$ is related to $H^{2,1}_{\bar{\partial}}(X)$. Therefore, the metric perturbations are representatives of the cohomology classes ie there are two cohomology groups associated with deformations of Ricci flat metric space on X. In fact, because the resulting Hermitean metric is with respect to a different complex structure on X the cohomology group is actually $H^{2,1}_{\partial}(X)$. In summary, the moduli space comprises parameters from the deformations of the complex and the *Kähler* structures. These are *Kähler* manifolds in their own right. There are now two *Kähler* potentials respectively,

$$-\,ln(i\int_M \Omega\wedge\bar{\Omega}) \qquad and \qquad \int_M J\wedge J\wedge J \qquad (for\ a\ three\ fold) \tag{12}$$

where Ω is differential form $(d,0)$ and J is $(1,1)$ form. The parameter spaces are special *Kähler* manifolds. Consequently, there exists a pre-potential $F(z)$ and *Kähler* potential, K, satisfying

$$K = i(\bar{w}^j\frac{\partial F}{\partial w^j} - w^i\frac{\partial \bar{F}}{\partial \bar{w}^j}) \tag{13}$$

SU(3) structure manifolds

Under compactification, constructing the internal space-time manifold, while preserving N=1 supersymmetry, leads to the requirement that the supersymmetric infinitesimal variation of the gravitino field must vanish. Splitting this result between the internal and external spinors forces the internal piece to have SU(3) holonomy. In the case of Calabi-Yau 3-folds, the manifolds have $SU(3)$ structure and the invariant spinor is covariantly constant. The metric, actually the Levi-Civita connection, has $SU(3)$ holonomy. In this scenario, for this manifold with $SU(3)$ structure, there is a connection with $SU(3)$ holonomy, which may or may not have torsion,

Defining the torsion tensor as

$$T^p_{mn}\ \varepsilon\ \ \Lambda^1\otimes(\mathfrak{su}(3)\oplus\mathfrak{su}(3)^\perp) \tag{14}$$

where Λ is the space of 1-forms and mn span the space of 2-forms isomorphic to $\mathfrak{su}(6)$. The piece that matters for us is that $\mathfrak{su}(3)^\perp$ gives an intrinsic torsion tensor, defined:

$$T^{0p}_{mn}\ \varepsilon\ \ (\wedge^1\otimes\mathfrak{su}(3)^\perp) = (1\oplus 1)\ \oplus\ (8\oplus 8)\ \oplus\ (6\oplus\bar{6})\ \oplus\ 2(3\oplus\bar{3}) \tag{15}$$

$$W_1 \qquad\qquad W_2 \qquad\qquad W_3 \qquad\qquad W_4, W_5 \tag{16}$$

where the W_i are the five torsion classes, which appear in the covariant derivatives of the spinor. W_1 is complex scalar, W_2 is complex primitive (1,1) form, W_3 is real primitive $(2,1)+(1,2)$ form, W_4 and W_5 are real vectors. Although this is sketchy, the purpose is to provide a categorisation according to the W_i values, as follows. The

Manifold	Vanishing torsion class
Complex	$W_1 = W_2 = 0$
Symplectic	$W_1 = W_3 = W_4 = 0$
Half flat	$ImW_1 = ImW_2 = W_4 = W_5 = 0$
Special Hermitean	$W_1 = W_2 = W_4 = W_5 = 0$
Nearly Khaler	$W_2 = W_3 = W_4 = W_5 = 0$
Almost Kahler	$W_1 = W_3 = W_4 = W_5 = 0$
Kahler	$W_1 = W_2 = W_3 = W_4 = 0$
Calabi-Yau	$W_1 = W_2 = W_3 = W_4 = W_5 = 0$
Conformal Calabi-Yau	$W_1 = W_2 = W_3 = 3W_4 - 2W_5 = 0$

Table 1: Vanishing torsion classes in SU(3) structure manifolds

detailed construction is in *Graña* (2005).

Special Manifold Concepts needed for Baumann Paper

Within the Baumann paper, there are various manifold concepts of a more specialised nature. Two are dealt with here.

Conifold singularity: Calabi-Yau manifolds have numerous solutions arising from the many hodge numbers. In addition, Calabi-Yau compactifications will have singularities which may lead to the low energy effective action breaking down. A Conifold Singularity is an example of this. Suppose we have a holomorphic three-form, labelled Ω which is a function of the moduli space coordinates x_i and one/some of these coordinates vanish over some vanishing cycle expressed ie $x_i = \int_{van-cycle} \Omega$ vanish, there is a singularity. Typically, the reason for this is because the cycle is looped around a point, which collapses to singularity. By identifying the *Kähler* potential and metric near the singularity, the singularity can be fixed by process called deformation or resolution.

Kähler **cones:** The construction of these cones is best understood by example. Given a Ricci flat metric $R_{mn}(g) = 0$ and applying metric deformations leads to a Ricci of the form $R_{mn}(g + \delta g) = 0$. The latter can be expanded and give a condition $\nabla^2 \nabla_k \delta g_{mn} + 2(R^p_m)^q_n \delta_{pq} = 0$. The solution, δg (1,1) form, has to be harmonic and the $(g + \delta g)$ is a *Kähler* metric with *Kähler* form $J = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. This J satisfies positivity $\int_M J \wedge J ... J > 0$. The metric deformations which lead to this positivity form a *Kähler* cone. It is a cone because not only is J positive but so is rJ for positive r.

2.2 Ads/CFT Correspondence

There is a good review on AdS/CFT correspondence by E.D'Hoker (2002) and this section has used the results from that paper.

Maldecena (1998)conjectured that there is an equivalence between the ten dimensional type IIB superstring theory on the product space, Anti-de Sitter $Ads_5 \times S^5$ (AdS) and the four dimensional supersymmetric Yang-Mills theory with maximal N=4 Supersymmetry in its superconformal phase (SYM). The conjecture states that the two theories, including operator observables, correlation functions and dynamics are equivalent. The strong-form correspondence manifests itself through several parameters: L, the radii of anti-de Sitter space Ads_5 and S^5; the integer flux $N = \int_{s^5} F_5^+$, the type IIB 5-form flux through S^5; g_s, the string coupling in type IIB; α', the type

IIB plank length squared; and g_{YM}, the Yang-Mills coupling in superconformal phase (with gauge group $SU(N)$). The relationships linking these are $L^4 = 4\pi g_s N\alpha'^2$ and $g_{YM}^2 = g_s$. The equivalence refers to maps between the states and fields in superstring theory (on the curved manifold $AdS_5 \times S^5$) and the local gauge invariant operators in super Yang Mills theory. (There are also two other correspondences which relate the two theories in the $N \mapsto \infty$ expansion limits).

The AdS/CFT conjecture requires extending or summarising various concepts encountered on the courses: Super Yang Mills, its Lagrangian, conformal transformations, Chiral or BPS multiplets; N=4 Supersymmetry, in particular, its mass representations; D=10 Supergravity, its Lagrangian and a little D3 brane knowledge leading to the $AdS \times S$ background metric; and then finally the CFT/AdS global symmetries conjecture.

Firstly, $N = 4$ Super Yang Mills supersymmetries. The starting point is the Lagrangian, which is given by

$$\mathcal{L} = tr(-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \Sigma_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \Sigma_i D_\mu X^i D^\mu X^i + \Sigma_{abi} g\, C_i^{ab}\lambda_a[X^i, \lambda_b] + \Sigma_{abi} g\bar{C}_{iab}\bar{\lambda^a}[X^i, \bar{\lambda}^b] + \frac{g^2}{2}\Sigma_{ij}[X^i, X^j]^2) \tag{17}$$

The fields have the usual meanings except the only possibly unfamiliar symbols are θ_I, which is the instanton angle and the C's, which are the Clifford Dirac matrices for $SO(6)_R \sim SU(4)_R$.

From Supersymmetry, the Poincare symmetry comprises the Lorentz transformations of $SO(1,3)$ and translations of R^4 with generators $L_{\mu\nu}$ and P_μ respectively while the complexified Lorentz algebra is isomorphic to $SU(2) \times SU(2)$. In addition, the Superpoincare algebra which includes the spinor supercharges transform as Weyl spinors of $SO(1,3)$.

There is another symmetry, R-symmetry. The SUSY anti-commutation relationships, including the antisymmetric generators, Z^{ab}, called central charges (which commute with all generators) are,

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab} \tag{18}$$

Since these supercharges can be rotated into one another under the unitary group $SU(N)_R$, these automorphism symmetries are called R-symmetry. The gauge algebra of the $N = 4$ Gauge multiplet is $(A_\mu, \lambda^a_\alpha, X^i)$, where the A_μ is the gauge field, the λ's are the Weyl fermions and the X^i's are scalars; these transform under $SU(4)_R$ as the singlet, the **4**-rep and the **6**-rep respectively.

Next, the symmetries arising from conformal transformations must be considered. A general diffeomorphism on a manifold X is a differentiable map acting on local coordinates x^μ or infinitesimally by a vector field v^μ such that $\delta_v x^\mu = -v^\mu(x)$. Under a diffeomorphism, the metric on X transforms as $g_{\mu\nu} dx^\mu dx^\nu = g^{'}_{\mu\nu} dx^{'\mu} dx^{'\nu}$ and $\delta_\nu g_{\mu\nu} = \nabla_{(\mu} v_{\nu)}$. From this, a conformal transformation is a diffeomorphism which preserves the metric up to a scale factor and preserves angles ie $g^{'}_{\mu\nu} = \omega(x) g_{\mu\nu}$. For $X = R^D$, the conformal transformations have solutions referred to as dilations ($v^\mu = \lambda x^\mu$),(generators D), and special conformal ($v^\mu = 2c_\rho x^\rho x_\mu - x_\rho x^\rho c_\mu$), (generators ($K^\mu$)). For reference, the isometry solutions are translations (v^μ constant) and Lorentz ($v_\mu = \omega_{\mu\nu} x^\nu$). This leads to a new total symmetry, Conformal Symmetry, which comprises translations, Lorentz, dilations and special conformal transformations and is $SO(4) \sim SU(2,2)$

As mentioned, the Poincare supersymmetries are generated by the supercharges Q^a_α + c.c.. There are also Conformal Supersymmetries generated by the supercharges $S_{a\alpha}$ + c.c.; these arise because the Poincare supersymmetries and the special conformal transformations K^μ do not commute and these have their own anti-commutation relationships (which we do not need).

Combining the above gives a Superconformal Yang Mills symmetry group $SU(2,2 \mid 4)$, the superconformal group.

Diagrammatically, the SYM generators can be represented by a matrix showing:- the bosonic subalgebras $SO(2,4)$ in top left and bottom right, and spinor algebras **4** and **4*** in top right and bottom left respectively.

$$\begin{pmatrix} P_\mu \; K_\mu \; L_{\mu\nu} \; D & Q^a_\alpha \; \bar{S}^a_{\dot{\alpha}} \\ \bar{Q}_{\dot{\alpha}a} \; S_{\alpha a} & T^A \end{pmatrix}$$

Turning to the D=10 Supergravity Lagrangians and basic brane interactions. These are developed so as to fit in with the AdS/SYM correspondence. The bosonic low-energy action for type IIB supergravity in Einstein frame ($G_{E\mu\nu} = e^{-\phi/2}G_{\mu\nu}$) is given by (excluding fermions):

$$\begin{aligned} S_{IIB} = & \frac{1}{4\kappa_B^2}\int \sqrt{G_E}(2R_{G_E} - \frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{Im(\tau)^2} - \frac{1}{2} \mid F_1 \mid^2 - \frac{1}{2} \mid G_3 \mid^2 - - \frac{1}{2} \mid \tilde{F}_5 \mid^2) \\ & - \frac{1}{4i\kappa_B^2}\int A_4 \wedge \bar{G}_3 \wedge G_3 \end{aligned} \tag{19}$$

Type IIB supergravity also has several symmetries. The metric G_E and the anti-symmetric, self dual field A_4^+ are invariant under the non-compact symmetry group $SU(1,1) \sim SL(2,R)$. The dilaton-axion τ transforms under a Mobius transformation $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$ while the complex three-form G_3 transforms under a Mobius transformation $G_3 \to G_3' = \frac{c\bar{\tau}+d}{|c\tau+d|}G_3$. (The fermion field transformations are given in D'Hoker, but we do not need them).

Next, we turn to Dp branes. Any Dp brane has (p+1) dimensional flat hypersurface with Poincare invariance group $R^{p+1} \times SO(1,p)$ and transverse space of SO(D-p-1). Therefore, for type IIB, D=10 the symmetry group of a Dp brane is $R^{p+1} \times SO(1,p) \times SO(9-p)$. From the string theory course, a (p+1) form couples to geometrical objects Σ_{p+1} because the action $S_{p+1} = T_{p+1} \int_{\Sigma_{p+1}} A_{p+1}$ is invariant under abelian transformation ρ_p given by $A_{p+1} \rightarrow A_{p+1} + d\rho_p$. The associated field strength (p+2) form F_{p+2} has conserved flux. This is called a Dp brane because of the p space dimensions. Associated with the A_{p+1} gauge field is the magnetic dual A_{D-p-3} satisfying duality $dA^{mag}_{D-p-3} = *_D dA_{p+1}$ and the magnetic dual brane is D_{D-4-p}. One important point about D3 branes is that when D=10, p=3, the magnetic dual brane is also D3. Also, a key D3 brane property is that its metric has the form

$$ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y}^{1/2})(dy^2 + y^2 d\Omega_5^2) \tag{20}$$

where x^μ are $\|$ coords and y^u are $\perp$ coords and

$$H(\vec{y}) = 1 + \sum_I \frac{4\pi g_s N_I (\alpha')^2}{\mid \vec{y} - \vec{y}_I \mid^4} \tag{21}$$

where number of D3 branes is $N = \Sigma N_I$.

In addition, the D3 brane has vanishing G_3 and $g_s = e^\phi$.

By defining the radius L of the D3 brane solution by $L^4 = 4\pi g_s N l_p^4$, the metric can be written as

$$ds^2 = (1 + \frac{L^4}{y^4})^{-1/2} \eta_{ij} dx^i dx^j + (1 + \frac{L^4}{y^4})^{1/2} (dy^2 + y^2 d\Omega_5^2) \tag{22}$$

The metric becomes, in the limit: for $y \gg L$, a flat space-time R^{10}, for $y < L$, a throat, and for $y \ll L$, a compact singularity . However, letting $u = L^2/y$, the metric becomes

$$ds^2 = L^2 [\frac{1}{u^2} \eta_{ij} dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2] \tag{23}$$

The last term is the metric for S^5 and the first two terms are the metric for the hyperbolic Anti-deSitter AdS^5. These have symmetries $SO(6)$ and $SO(2,4)$ respectively. Adding the fermionic piece, results in the symmetry group of AdS also being $SU(2,2|4)$.

To summarise, the global unbroken symmetries of the two theories SYM and AdS must be the same. The superconformal global group of SYM (in the conformal phase) is $SU(2,2 \mid 4)$; this identifies the bosonic group as $SU(2,2) \times SU(4), \sim SO(2,4) \times SO(6)_R$; it arises from the conformal group $SO(2,4)$ times the automorphism group of the $N = 4$ Poincare SUSY algebra $SU(4)_R$. In contrast, the superconformal global group of AdS has a bosonic group given by the isometry group of the $AdS_5 \times S^5$ background; this has group symmetries $SO(2,4)$ and $SO(6)$ respectively. The completion to the full supergroup, $SU(2,2 \mid 4)$, is achieved because the N D3 branes preserve 16 of the Poincare Symmetries together with another 16 conformal supersymmetries.

To complete our AdS/SYM section, we look very briefly at operators and correlators.

The states in SYM are handled using operators. A superconformal primary operator $\mathcal{O} \neq 0$ is defined with properties: it commutes with the conformal supercharges S, and must not involve the gauginos or the gauge field strengths, the derivatives of scalars, or commutators of scalars - so it is only a function of the scalars X^i. In fact the $\mathcal{O}$ are symmetric trace of products of the scalars.

Now the unitary representations of the superconformal algebra maybe labelled by the quantum numbers of the bosonic group

$$SO(1,3) \times SO(1,1) \times SU(4)_R \tag{24}$$

$$(s_+, s_-) \qquad \Delta \qquad [r_1, r_2, r_3] \tag{25}$$

Various constraints on the Δ and the r_i lead to states referred to as BPS multiplets. As an example, 1/2 BPS operators can be defined as

$$\mathcal{O}_k(x) = \frac{1}{n_k} str(X^{\{i_1(x) \ldots X^{i_k}\}}) \tag{26}$$

Similarly, the 1/4 and 1/8 BPS operators can also be constructed.

For AdS, the contents of the irreducible representations of $SU(2,2 \mid 4)$ are needed. The type IIB massless supergravity and massive string degrees of freedom are described by fields φ living on $AdS_5 \times S^5$. Using coordinates z^μ and y^μ respectively, the fields can be matched to string degrees of freedom and expanded in terms of spherical harmonic functions on S^5.

$$\varphi(z, y) = \Sigma \varphi_\Delta Y_\Delta(y) \tag{27}$$

Δ are labels corresponding to the totally symmetric traceless representations of $SO(6)$, called scaling dimensions.

Using the SYM operators and AdS fields, D'Hoker shows the matching between the SYM operators and the Sugra fields. For example: This provides a mapping

SYM operator	Sugra	$SU(4)_R$
$\mathcal{O}_k^4 \sim tr(F_+ X^k)$	$A_{\mu\nu}$	$(0, k, 0)$

Table 2: Example of Super Yang Mills Operators and Supergravity Fields

between the AdS and SYM states.

Next AdS and SYM correlators. First we outline the mapping relationship between the two correlators. The AdS fields need changing by taking the 10-dimensional fields and *decomposing* them onto S^5, after which they are denoted by $\varphi_\Delta(z)$ on AdS_5, labelled by dimension Δ. Away from the interaction region, they behave as free fields $\varphi_\Delta^0(z)$; these are either normalizable z_0^Δ or non-normalizable $z_0^{4-\Delta}$. It is assumed that the normalizable fields determine the vacuum expectation values of the operators and the non-normalizable solutions represent the coupling of external sources to supergravity. Associated with the functions φ are *boundary fields*, $\bar{\varphi}_\Delta$ defined as

$$\bar{\varphi}_\Delta(\vec{z}) = \lim_{(z_0 \to 0)} \varphi(z_0, \vec{z}) z_0^{4-\Delta} \tag{28}$$

The action for the Type IIB supergravity on $AdS^5 \times S^5$ is denoted by $S[\varphi_\Delta]$.

Now for SYM. Define the generating functional, (defined in AFT), by $\Gamma[\bar{\varphi}]$ in terms of the operators by

$$exp\{-\Gamma[\bar{\varphi}]\} =< exp\{\int_{\partial_H} \bar{\varphi}\mathcal{O}\} > \tag{29}$$

Then the mappings between SYM and AdS correlators are

$$\Gamma[\bar{\varphi}] = ExtrS[\varphi_\Delta] \tag{30}$$

where the extremum is over all fields φ_Δ which have associated boundary fields.

There are also AdS rules for Type IIB supergravity, equivalent to the Feynmann rules, called Witten rules and these correspond to Witten circles, such as the 2-pt function $\ominus$ and 4-pt function $\otimes$; The interior of the circle is the interior of AdS and the boundary of the circle is boundary of AdS. These are given in D'Hoker.

2.3 Fluxes

There is a good review on Flux Compactification by M. Gra*ñ*a (2005) and this section has used the results from that paper.

Fluxes make an important contribution to any theory seeking physical reality because they can, amongst others things: make a variable contribution to the cosmological constant, stabilise moduli, generate warped metrics, give a positive contribution to the energy-momentum tensor, be turned on by D-brane sources, generate potentials and superpotentials, partially break supersymmetry giving vacuum expectation values to massless fields. So for instance, given flux-less Calabi-Yau compactifications, if the supersymmetry is broken by non-perturbative effects, this results in a negative cosmological constant; this can then be countered by turning on certain fluxes which make an offsetting positive contribution to the cosmological constant. This section is mainly discursive as almost all proofs are extensive.

First we explain the meaning of fluxes. In the absence of sources, the integral of the field strengths will vanish. However, when sources are present, the integral of the field strength over either a compact cycle A_K and or a non-compact cycle B^K maybe non-zero, lead to electric and magnetic fluxes. (This is similar to standard results on fluxes in electromagnetism). Accordingly, for Type IIB theory, the electric and magnetic fluxes are defined, for each field strength, as

Magnetic Fluxes	Electric Fluxes
$\int_A H_3 \sim m^K$	$\int_B H_3 \sim e_K$
$\int_A \hat{F}_3 \sim m^K_{RR}$	$\int_B \hat{F}_3 \sim e_{RRK}$
$\int_A \hat{F}_2 \sim m^a_{RR}$	$\int_B \hat{F}_4 \sim e_{RRa}$

Table 3: Magnetic and Electric Fluxes for each Field Strength

(K and a values are known). Although F_1 and F_5 can be similarly defined, for Calabi-Yau 3-folds, they are not defined due to the absence of appropriate cycles. The NS field strengths are defined by $H = dB$ while the RR field strengths are defined by $F^{10} = dC + me^B - H \wedge C$ and $\hat{F} = dC + me^B$, (where 10 indicates the dimension, m is called *mass* parameter and B and C are the usual massless bosonic fields).These results are integral relationships; however, using some standard results on integrals over cycles, the field strengths can be expanded as combinations of the magnetic and electric fluxes without integrals.

For **supersymmetric** backgrounds, it is natural to assume that the ten dimensional warped metric consists of a separate four-dimensional space-time, which is Minkowski, dS_4 or AdS_4, together with an internal six-dimensional metric. If we require our theory to satisfy reality and maximal supersymmetry, then the vacuum expectation value of the fermionic fields will have to vanish ie the background must be purely bosonic. This is done in the usual way by finding the supersymmetric transformations of the gravitino and dilatino spinors and setting the variation equal to zero. The former leads to the vanishing of a supersymmetric parameter $\nabla_m \epsilon = 0$; splitting this ten-dimensional result into external and internal components, the latter forces the internal manifold to have a covariantly constant spinor. Calabi-Yau manifolds admit a covariantly constant spinor and so satisfy this condition for the internal space.

When the fluxes are turned on, there are numerous resulting supersymmetric backgrounds depending on the combination of fluxes which have been activated (and also depending on the associated underlying torsion classes). Each resulting background scenario is categorised by its appropriate *vacua*'s properties. The Type IIB solutions have been analysed by Maldacena-Nunez, Klenbanov-Strassker (KS) Pochinski-Strassler. The proofs and analysis are lengthy and are not needed.

Essentially, as before, the supersymmetric transformations for the gravitinos and dilatinos are derived but these now including a contribution from the fluxes; in particular, the gravitino equations for KS, type B, are

$$\delta\psi_M = \nabla_m \epsilon + \frac{1}{4} H_M P_\epsilon + \frac{1}{16} e^{\phi} \Sigma_n \not{F}_{2n} \Gamma_M P_n \epsilon \tag{31}$$

where P,H denotes function of Γs. For the NS and RR fluxes, the Supersymmetric transformations for the gravitino ψ_M and dilatino λ are re-expressed in terms of spinors on the internal manifold and complicated coefficients Q,T and A for the NS fluxes; these, when analysed, lead to multiple conditions.

Given the complexity, we briefly comment on KS and show its parameter constraints in terms of the field strengths and torsion classes. Per Gra*ñ*a, this class is a non-compact Type IIB in which the underlying Calabi-Yau is the conifold. The solution is compactified by adding orientifold 3-planes - ie it can be used as local IR throat geometry of the compact Calabi-Yau. The conditions for the Type IIB Minkowski fall into four types labelled A, B, C and ABC according to the values of certain complex parameters. KS is a type B, which is analysed by SU(3) representation in table 4.

IIB	$TypeB$
1	$W_1 = F_3^{(1)} = H_3^{(1)} = 0$
8	$W_2 = 0$
6	$W_3 = 0, e^{\phi} F_3^{(6)} = \mp * H_3^{(6)}$
3	$e^{\phi} F_5^{(\bar{3})} = \frac{2}{3}\, i\bar{W}_5 = iW_4 = -2i\, \bar{\partial} A, \bar{\partial}\phi = 0$

Table 4: IIB N=1 vacua for type B

There are numerous other backgrounds but this is sufficient for our purposes.

Now we consider the effect of fluxes on **Einsteins** equations. The four-dimensional components of the Einsteins fluxless equations are

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \tilde{g}_{\mu\nu}(\nabla^2 A + 2(\nabla A)^2) = T_{\mu\nu} - \frac{1}{8} e^{2A} \tilde{g}_{\mu\nu} T_L^L \tag{32}$$

where a tilda means the internal metric and T is the energy-momentum tensor. Applying a $\tilde{g}^{\mu\nu}$ contraction to this gives

$$\tilde{R} + e^{2A}(-T_\mu^\mu + (1/2)T_L^L) = 4(\nabla^2 A + 2(\nabla A)^2) = 2e^{-2A}\nabla^2 e^{2A} \tag{33}$$

For Minkowski and de Sitter compactifications, $\tilde{R} \geq 0$. Defining $\hat{T} = (-T_\mu^\mu + (1/2)T_L^L)$ and the energy momentum tensor in terms of an n-flux F, the internal and external fluxes are separable within this equation:

$$\hat{T}_{int} = \frac{n-1}{2n} F^2 \geq 0 \qquad \hat{T}_{ext} = -\frac{9-n}{2n} F^2 \geq 0 \tag{34}$$

So all of the internal and external fluxes give strictly positive to the trace of energy momentum tensor (ignoring a few exceptions).

The expression for $\tilde{R}$ above, when integrated over the internal manifold, has a right side which vanishes and a left side which is non-negative. So without local flux sources, the deSitter space which is positive on the left side, cannot fit this equation. This is equivalent to saying that the internal spaces has a positive curvature on left side.

The solution to this dilemma is to accept that the external space has negative curvature such as in AdS (*eg* $AdS_5 \times S^5$), or, possibly Minkowski which has zero contribution. Including local sources has the effect of adding a local energy momentum tensor which may be positive such as in theories with Dp branes, negative in theories such as orentifold planes, or zero in theories such as D7.
The effect of the local source on the $\tilde{R}$ equation is:

$$\tilde{R} + \frac{1}{2} e^{2A} (\tilde{T}^{flux} + \tilde{T}^{local\ flux}) = 2e^{-2A} \nabla^2 e^{2A} \tag{35}$$

So local sources of flux provides flexibility to the theory.

The above is not very user-friendly. It can be re-formulated into restrictions on the local sources. For instance, in the case of supersymmetry and specialising to Type IIB class B, Einstein's equation and the Bianchi identities and the equation of motion for the fluxes can be used to derive direct conditions on the localised sources as: (i) the warp factor and four form potential satisfy $e^{4A} = f$ (per table 4 type B). (ii) the complex three form flux is imaginary self dual (iii) the inequality is saturated. So there exists flux and torsion class conditions for the various background solutions and also restrictions to be satisfied by the local sources of flux.

Now for the **flux induced superpotential**. Firstly, Orientifolds are included here because they include a negative contribution to the $\tilde{R}$ equation and help stabilise some moduli. We will need to consider type IIB moduli, the potential, and then the superpotential.

Building on our earlier general discussion on moduli, the Type IIB moduli arranged in N=2 multiplets (built from the Calabi-Yau hodge numbers) is given in table 5. The moduli labels are by convention. So, for instance, the scalars in the vector

Multiplet	hodge number	*moduli*
gravity multiplet	1	$(g_{\mu\nu}, V_1^0)$
vector multiplets	$h^{(2,1)}$	(V_1^k, z^k)
hypermultiplets	$h^{(1,1)}$	(v^a, b^a, c^a, ρ_a)
tensor multiplet	1	(B_2, C_2, ϕ, C_0)

Table 5: Type IIB moduli for N=2 multiplets

multiplet moduli space are the complex structure deformations z^k. Both scalars in this multiplet span the special K$\ddot{a}$hler manifold of complex dimension $h^{(2,1)}$. The v^a are the K$\ddot{a}$hler deformations of the metric. The scalars in the hypermultiplet span a quaternionic manifold with the dimension of its coordinates as $h^{(1,1)}$.

Including Orientifolds, projects out certain moduli. At the sametime, the original N=2 multiplets become N=1 gravity, vector and chiral multiplets in table 6.

Multiplet	hodge number	*moduli*
gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h^{(2,1)}$	V_1^k
chiral multiplet	$h^{(2,1)}$	z^k
"	$h^{(1,1)}$	(v^a, b^a, c^a, ρ_a)
"	1	(ϕ, C_0)

Table 6: Type IIB moduli with O3-orientifolds for N=1 multiplets

Next we need Kähler coordinates which, in terms of these moduli, are z^k and $\tau = C_0 + ie^{-\phi}$ and $G^a = c^a - tb^a$ and

$$T_\alpha = \frac{1}{2}K_\alpha + i\rho_\alpha - \frac{i}{2(\tau - \bar{\tau})}K_{abc}G^b(G - \bar{G})^c \tag{36}$$

where K_{abc} and K_α are intersection numbers (of wedge products). From this the Kähler potential can be constructed as

$$K_{O3} = -ln[-i\int \Omega(z) \wedge \bar{\Omega}(\bar{z})] - ln[-i(\tau - \bar{\tau})] - 2ln\frac{1}{6}K(\tau, G, T) \tag{37}$$

The inclusion of fluxes is achieved by modifying the forms C_2 in RR flux and B_2 in NS flux to include magnetic and electric fluxes. This is done by introducing the transformations for $dC_2 \to dC_2 + m^K_{RR} - e_{RRK}$ and $dB_2 \to dB_2 + m^K\alpha_K - e_{RRK}\beta^K$ into the Type IIB Lagrangian. The effect of the RR flux is to induce mass terms for the tensor B_2 and the effect of the NS fluxes is for electric fluxes to generate a potential for the scalars in the vector muliplet and for the axion-dilaton and for the magnetic fluxes to induce mass terms for the tensor C_2. The NS flux induced potential is

$$V_{NS} = -\frac{e^{4\phi}}{2K}\;(C_0^2 + \frac{e^{-2\phi}}{2K})\; e_K(ImM)^{KL}e_L \tag{38}$$

Finally, there is a Type IIB superpotential for compactifications of Calabi-Yau 3-folds including O3 orientifolds generated by fluxes as

$$W_{O3} = \int G_3 \wedge \Omega = (e_{KRR} - i\tau e_K)Z^K - (m^K_{RR} - i\tau m^K)F_K \tag{39}$$

M is a known complex matrix. This superpotential depends on complex structure moduli and dilaton-axion. However this superpotential does not depend on the Kähler moduli v^α and ρ_α and the B_2 and C_2 moduli b^a and c_a.

Lastly, we discuss **moduli stabilisation** with **non-perturbative corrections to the superpotential**. One of the main reasons compactification with fluxes is included in many theories is because the presence of flux induced potentials causes some of the moduli of the Calabi-Yau compactifications to stabilise. So if the flux generated potential for the moduli has a local minima, the moduli will stabilise at the local minima. As indicated in the previous section, the Type IIB superpotential for compactifications of Calab-Yau 3-folds and Calabi-Yau O3 generated by the fluxes is $W_{O3} = \int G_3 \wedge \Omega$

The key fact is that from the supersymmetric Minkowski vacuum conditions $W = 0$ and $DW = 0$, there are $2h^{(2,1)} + 2$ equations but none of them involve the *Kähler* moduli (v^α, ρ_α) and (b^a, c^a) which therefore remain unfixed. However, turning on appropriate fluxes can fix some of the $2h^{(2,1)} + 2$ real moduli of complex structure moduli z^k and the dilaton-axion, τ. For instance, given a flux configuration profile $(e_K, m^K, e_{RR,K}, m^K_{RR})$ they should fix H_3 and $\hat{F}_3$ and hence G_3 since $G_3 = \hat{F}_3 - \tau H_3$, but in fact the flux profile only fixes some of the components of G_3 moduli - a partial success. Orientifolds can produce further stabilisations if necessary.

Moving onto **non-perturbative corrections to the superpotential**, we have seen that fluxes are not usually enough to stabilise all moduli and in the above example, fluxes left the *Kähler* moduli unfixed. Non-perturbative corrections to the *Kähler* potential and superpotential can be used to stabilise the remaining moduli eg using D3 branes. A simple example explains how this works. For D3 branes wrapping a four cycle, the superpotential adds a term $W_{np} = B_n e^{-2\pi n^\alpha T_\alpha}$, where T_α are the *Kähler* moduli defined earlier. Combing the fluxes and D3 brane instantons leads to total superpotential contribution $W = W_0 + Be^{-2\pi T}$ where W_0 is flux contribution, B_n is one loop determinants. Applying $DW = 0$ a value for W_0 which leads to minimum potential $V_{min} = -\frac{2\pi^2 B^2 e^{-4\pi\sigma_{crit}}}{3\sigma_{crit}}$ and $\sigma = ReT$. More on this later.

3 D3 Brane Model

3.1 Discussion on the D3 Brane Model in 10d Supergravity

This section deals with Chapters 2,3,4 of Baumann's paper. The two main underlying references for the D3 Brane model are: Hierarchies from fluxes in string compactifications by S Giddings (2002) and On D3 brane potentials in Compactifications with Fluxes and Wrapped D-branes by D Baumann (2006).

First we construct the geometrical construction of Baumann's model outlined in the Introduction and explain the meaning and properties of the various fields and space-time geometries included within his method.

The model is built as follows. In the no-scale structure, there is no potential between any ISD fluxes and sources and the D3 brane. To create a potential on the D3 brane, the no scale structure must be broken by (say) stabilising some of the K*ä*hler moduli by nonperturbative effects; in this case the D3 brane experiences a potential from the fluxes and scalar mode sources. The overall goal is to derive such a potential in ten-dimensional supergravity by a perturbative expansion approach around the zero flux and scalar mode solution. The expansion must be done in a controlled and predictive manner to avoid instabilities, flux violations or uncontollable distortions.

Next, the bulk region of space is assumed to be ten-dimensional Type IIB Supergravity with its well defined fields and properties. A finite throat is glued into the compact space which causes distortions of the supergravity fields. These distortions cause the D3 brane located in the throat to experience a potential. The compact space is initally a Calabi-Yau manifold but loses its structure as a result of the distortions. In fact, while the associated perturbed fluxes and scalar modes couple to the D3 brane,

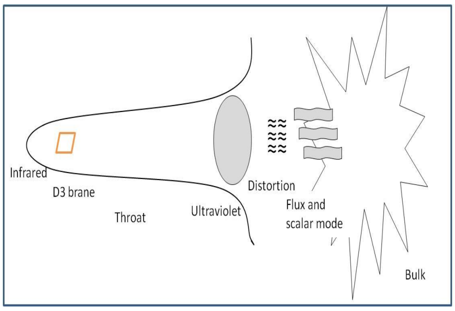

Figure 1: Insertion of throat causing distortions of the fluxes and scalar modes leading to D3 brane potential

the dilaton and unwarped metric do not couple to the D3 brane. As a result, new terms appear in the D3 brane Lagrangian and in the known D3 brane potential expressions.

The UV end of the throat is where the geometry continuously merges with the supergravity bulk while the IR end of the throat limits to a singularity. It is assumed that the D3 brane is sited deep towards the tip of the throat.

Next there are some assumptions about the UV region and the throat which are summarised as follows: that the UV perturbations occur in the supergravity bulk perturbing the fluxes and scalar modes; that the finite warped region of the throat may be approximated by a noncompact warped deformed conifold solution; and that UV solutions may be approximated in the infrared by a solution parametrized by a few dominant modes due to the effects of radial scaling and smallest scaling dimensions; the presence of the brane is said to backreact on the geometry leading to 4d/6d warped line element.

So simplistically (figure 1): Starting with Type IIB Supergravity fields with homogenous geometry, a throat with a D3 brane situated towards the tip and UV region towards the base is inserted into the supergravity. As a result there are supergravity flux and scalar field perturbations which creates a force field which is experienced by the D3 brane in the form of a potential. The throat is then replaced by a warped deformed conifold ie by an object with known geometry and mathematics. The D3 brane potential is calculated from three additive parts arising from fluxes, scalar modes, and curvature.

Now we discuss some of the above concepts in more detail. First the throat. To build on the AdS/CFT section, a simple throat is an alternative nomenclature for the near horizon solution at which a radial coordinate $\rightarrow 0$. To consider a very simple metric $ds^2 = -\frac{y^2}{Q^2}dt^2 + \frac{Q^2}{y^2}dy^2$ plus a symmetrical angular piece, as $y \rightarrow 0$, the time piece disappears and the spatial piece becomes an infinitely long radial tube called a throat. More complicated throats can be constructed which leave both terms intact by taking a different form in the small and large limits. In the model, a conifold throat structure is inserted into the supergravity bulk. A conifold (more on this later) is a cone over a five-dimensional manifold with a metric $dy^2 + y^2 d\Omega^2$. But from the AdS/CFT section, the presence of D3 brane ensures that its structure is split into coordinates parallel to and perpendicular to the D3 brane. As given in that section, the D3 brane metric is

$$ds^2 = (1 + \frac{L^4}{y^4})^{-1/2}\eta_{ij}dx^i dx^j + (1 + \frac{L^4}{y^4})^{1/2}(dy^2 + y^2 d\Omega_5^2) \tag{40}$$

where the perpendicular coordinates are the same as the conifold. As noted, $y \gg L$, $y < L$ and $y \ll L$ yield different metrics with different geometries.

Next we consider the meaning of no scale models and structure. Typically a no scale model refers to nonsupersymmetric solutions with vanishing cosmological constant and radial modulus. No scale structure similarly means that the D3 brane (probe) in an ISD flux compactification experiences no force or potential and can be placed at any point in a compact space with no resulting energy change ie there is no potential between the D3 brane and ISD fluxes and sources. This means that presence of the D3 brane provides a zero potential basis.

The no-scale structure can be broken in several ways so that the D3 brane experiences a potential from the fluxes and sources via, for instance, Kahler moduli stabilisation eg by nonperturbative effects (or for instance by the inclusion of antibrane).

Next we consider the stability of the throat solutions in the presence of UV perturbations. Ideally the D3 brane potential would be fully specified in terms of fluxes, brane positions and scalar modes. But the approach of the model is to use UV deformations of various fields and fluxes and arrive at a perturbative expansion of the form $V(\phi) = \Sigma c_i \frac{\phi^{\Delta_i}}{M_{UV}^{\Delta_i - 4}}$ where ϕ is a normalized field as a function of the D3 brane position and $M_{UV}^{\Delta_i - 4}$ is UV mass scale (related to r_{UV}, the UV location of the throat merging with the compact bulk). r is used instead of y to emphasise that it is the radial component. Determining the scaling dimensions Δ_i will provide a well defined power series for calculation or model development. The radial r is constrained within the throat by the condition $r_{IR} \ll r \ll r_{UV}$. The leading terms in the series will be remote from the UV due to "filtering by RG flows" and the D3 brane must be reasonably far from the IR because of "conifold deformities". Baumann comments that his paper focusses on non-normalizable perturbations, corresponding to deformations of the gauge theory Lagrangian sourced by effects in the compact bulk, since these encode the effects of Planck-scale physics in the form of the above potential expansion. And as mentioned in the Introduction, the aim is to control such terms in the effective action.

There is also an assumption that the finite throat configuration is in a stabilized compactification, the supersymmetry breaks controllably in the bulk, and that there is a moduli potential to prevent decompactification. This amounts to requiring the four dimensional potential energy must be bounded. D Baumann (2009) found that under suitable conditions this can be expressed in terms of the scalar modes as $\Phi_-(r) < \Phi_+^{(0)}(r_{IR}) \leq \Phi_+^{(0)}(r_{UV})$; this provides a general constraint that the expansion has exponentially small coefficients on flux Λ and scalar mode Φ_- in the UV region.

Finally from the DBI and CS terms in the action for the D3 brane, the potential felt by the D3 brane is $T_3\Phi_-$ (to be explained later).

3.2 Constructing Flux solutions Λ in the Conifold

The aim here is not to simply repeat the mathematics in Baumanns chapters 2, 3 and 4 but to explain the key steps and highlight the key assumptions used in the derivations of the Flux solutions in the Conifold. The starting point is the bosonic low energy action for Type IIB supergravity, which in the Einstein frame is given by

$$\begin{aligned} S_{IIB} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{|g|}[R_{10} - \frac{\partial_M \tau \partial^M \bar{\tau}}{2Im(\tau)^2} - \frac{G_3 \cdot \bar{G}_3}{12Im(\tau)} - \frac{\tilde{F}_5^2}{4 \cdot 5!}] \\ + \frac{1}{8i\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{Im(\tau)} + S_{local} \end{aligned} \tag{41}$$

We need to define the three key physical fields, including fluxes, which will be required in the flux solution derivation:

- the three-form ISD and IASD fluxes $G_\pm$, where $G_\pm = (\star_6 \pm i)G_3$ and G_+ is the ISD component and G_- is the IASD component. Flux G_3 is defined as $G_3 = F_3 - \tau H_3$ where F_3 and H_3 are the RR and NS three form fluxes dC_2 and dB_2 respectively from IIB supergravity;

- the scalar mode $\Phi_\pm$, where $\Phi_\pm = e^{4A} \pm \alpha$. Φ_- will be the most important as the integral of its Laplacian vanishes. e^{4A} is the warp factor and α is the four-form potential;

- the mixed flux Λ, where $\Lambda = \Phi_+ G_- + \Phi_- G_+$, which couples and mixes the warp factor, the four potential and the ISD and IASD fluxes.

In addition, there are three physical concepts: the axion-dilation field $\tau = C_0 + ie^{-\phi}$, the Ricci tensor R and the self dual five-form flux F_5, $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$ which we do not need. Next, these physical concepts are expressed as perturbative expansions so that at zero order they have no impact on the geometry or the D3 brane potential. The G fluxes can be regarded as similar to the field strength forms from electromagnetism and at zero order are switched off. The negative scalar mode is a function of the warp factor and four potential, but conditions on the local electromagnetic tensors require $e^{4A} = \alpha$ at zero order. So the Baumann model is controlled by first and higher order terms in the perturbative expansions; these affect the geometry, the potentials, and other fields etc in a controlled manner.

The equations of motion provide fundamental information for any field. S Giddings (2002) derived them for the scalar mode Φ_- and mixed flux Λ from the 5-form flux Bianchi Identity $d\tilde{F}_5 = H_3 \wedge F_3 + local$ and the 3-form flux Bianchi Identity $dF_3 = dH_3 = 0$ (and Einsteins equations). He obtained:

$$\nabla^2 \Phi_- = \frac{e^{8A+\phi}}{24}|G_-|^2 + e^{-4A}|\nabla\Phi_-|^2 + R_4 + S_{local} \qquad and \tag{42}$$

$$d\Lambda = \frac{id\tau}{2Im(\tau)} \wedge (\Lambda + \bar{\Lambda}) = 0 \tag{43}$$

Next consider the perturbations. The fluxes are switched on and the scalar mode (background) is disturbed; this is equivalent to considering the first and second order terms in the perturbative expansions and assessing their relative magnitudes (∇ is also assumed to have an expansion). The assumptions affecting the Φ_- equation of motion are that: the curvature and local contributions are small; at first order, $\nabla^2_{(0)}\Phi^{(1)}_- = 0$; and $\Phi^{(1)}_- = 0$ (this is an optional assumption). Consequently, the Φ_- equation of motion becomes

$$\nabla^2_{(0)}\Phi^{(2)}_- = \frac{g_s}{96}(\Phi^{(0)}_+)^2|G^{(1)}_-|^2 \tag{44}$$

which implies that perturbations of G_- arise at linear order acting as sources for second order scalar mode.

Applying the same constraints to the Λ equation of motion leads to treating the second term as small in equation (43), ie

$$d\Lambda_{(1)} = 0 \quad with \quad \Lambda_{(1)} = \Phi^{(0)}_+ G^{(1)}_-. \tag{45}$$

Substituting this into the above equation (44) for $\nabla^2_{(0)}\Phi^{(2)}_-$ gives a result $\nabla^2_{(0)}\Phi^{(2)}_- = \frac{g_s}{96}|\Lambda_{(1)}|^2$ which has a physical interpretation, namely, that the $\Lambda_{(1)}$ can acts as a source for $\Phi^{(2)}_-$ field ie $\Lambda_{(1)}$ acts as source perturbation. However, the $\Lambda_{(1)}$ is IASD of first order with respect to background metric, while Φ_+ is at zero order. This also implies that the IASD flux solution is not affected by the explicit form of perturbed metric. As a consequence, the distortions disturb the manifold from being Calabi-Yau but this does not affect the perturbation expansions.

Solving the $\nabla^2_{(0)}\Phi^{(2)}_-$ equation (44) was undertaken in Baumann's paper (chapter 3) using Greens functions for G_3 and G_-, but this led to a solution in terms of unknown eigenfunctions and unknown spectrum of eigenvalues. Therefore, Baumann decided to use the equation $d\Lambda_{(1)} = 0$ (45) for which explicit solutions for Λ can be derived.

Now, we need to understand the conifold, which, from linear algebra, has known solutions. This is a singular non-compact Calabi-Yau threefold in C^4 such that $\Sigma z_a^2 = 0$. This can be re-configured as a cone over a five dimensional manifold with five angular coordinates $\Psi(\theta_i, \phi_i, \psi)$, with i=1,2 of $T^{1,1}$. The line element over $T^{1,1}$ is $dr^2 + r^2 d\Omega^2_{T^{1,1}}$

where $r^3 = (\frac{3}{2})^{3/2}\Sigma|z_a|^2$. The cone is noncompact but is assumed to smoothly join the compact space smoothly at r_{UV}. A stack of D3 branes placed at the singularity $z_a = 0$ backreacts on the geometry producing the ten-dimensional warped line element

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A_{(0)}}(dr^2 + r^2 d\Omega^2_{T^{1,1}}) \tag{46}$$

which is $AdS_5 \times T^{1,1}$, such that $e^{-4A_{(0)}}(r) = \frac{L^4}{r^4}$ and $L^4 = \frac{27\pi}{4}g_s N(\alpha^{'})^2$.

The harmonic function solutions f on the conifold satisfy $\nabla^2 f = 0$ and are well documented and will form an important part of the flux Λ solutions. They take the form:

$$f(r,\Psi) = \Sigma f_{LM}(\frac{r}{r_{uv}})^{\Delta_f(L)}\Upsilon_{LM}(\Psi) \tag{47}$$

where $\Delta_f(L)$ are radial scaling dimensions satisfying $\Delta_f(L) = -2 + \sqrt{H+4}$ and $H = 6[j_1(j_1+1) + j_2(j_2+1) - R_f^2/8]$. The $L = (j_1, j_2, R_f)$ are positive quantum numbers satisfying group theoretic selection rules for $SU(2) \times SU(2) \times U(1)$. An example of allowed harmonic functions spectrum is $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2})$((table 7)

Δ_f	j_1	j_2	R_f
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 7: Example of harmonic functions spectrum on conifold

Finally, we still need to solve $d\Lambda = 0$. Baumann draws on the results and solutions from linear algebra which state that the explicit solutions for flux perturbations on a Calabi-Yau can be constructed from the Kähler from $J_{\alpha\bar{\beta}} = ig_{\alpha\bar{\beta}}$, the holomorphic (3,0) form $\Omega_{\alpha\beta\gamma} = q\epsilon_{\alpha\beta\gamma}$ with $q\bar{q} = detg$ and the harmonic functions f. For the closed flux equation $d\Lambda = 0$, $(\star_6\Lambda = -i\Lambda)$, there are only three possible solution types, as follows:

- Series I: Flux $\Lambda_I = \nabla\nabla f_1 \cdot \bar{\Omega}$ of Hodge type $(1,2)$;

- Series II: Flux $\Lambda_{II} = (\partial + \bar{\partial})(f_2 + \frac{1}{2}k^\alpha \partial_\alpha f_2) \wedge J + \partial(\bar{\partial} f_2 \wedge \bar{\partial} k)$ of Hodge type (2,1)+(1,2);

- Series III: Flux $\Lambda_{III} = (2h + k^\alpha \partial_\alpha h)\Omega + (\bar{\partial} h \cdot w) \wedge J + \bar{\partial}(\bar{\partial} f_3 \cdot w) \wedge \bar{\partial} k$ of Hodge type (3,0) + (2,1)+ (1,2).

where w, k and h are known functions of Ω, metrics and harmonic functions. Proving that these satisfy the equation is done by verification. Completeness is more difficult but is not needed here. These Series have solutions with a known spectrum $\delta_I = 1 + \Delta_f$, $\delta_{II} = 2 + \Delta_f$ and $\delta_{III} = 3 + \Delta_f$ where $\Delta_f = \sqrt{H+4} - 2$ is the scaling dimension of harmonic function. For example (table 8):

Series	Δ	j_1	j_2	R_f	Chirality
Series I	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	chiral
Series II	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	chiral
Series III	$\frac{9}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	3	chiral

Table 8: Example of Series spectrum

The chiral modes will be particularly useful later because their flux perturbations take a simple form as the harmonic functions are holomorphic functions and each series is of a distinct Hodge type.

$$\Lambda_I^{(1,2)} = \nabla^2 \hat{f}_1 \cdot \Omega \qquad \Lambda_{II}^{(2,1)} = \partial \hat{f}_2 \wedge J \qquad \Lambda_{III}^{(3,0)} = \hat{f}_3 \Omega \tag{48}$$

where the above $\hat{f}_i$ have been redefined $\hat{f}_1 = f_1$, $\hat{f}_2 = f_2 + \frac{1}{2}k^\alpha \partial_\alpha f_2$ and $\hat{f}_3 = 6f_3 + 5k^\alpha \partial_\alpha f_3 + k_\alpha \partial_\alpha (k_\beta \partial_\beta f_3)$.

The flux solutions will be used extensively in the following sections, and in particular, in the AdS/CFT correspondence and the D3 brane potential.

3.3 D3 Brane potential

Deriving the D3 brane potential is by direct computation. First we note that the D3 brane potential is made up, additively, of 3 parts: mixed flux and harmonics, and curvature ie $V(\phi) = V_\Lambda + V_R + V_H$.

First, $V(\phi) = V_\Lambda$. The approach is to express the potential as a perturbative expansion of a product of: the radial functions to a dimensional scaling power times the products of angular functions.

The Green's function solution for the scalar mode $\nabla^2 \Phi_- = \frac{g_s}{96} | \Lambda |^2$ equation of motion is

$$\Phi_- = \frac{g_s}{96} \int d^6 y^{'} G(y; y^{'}) | \Lambda |^2 (y^{'}) + \Phi_H(y) \tag{49}$$

where the Green's function satisfies $\nabla^2_y \ G(y; y^{'}) = \delta(y - y^{'})$ and $\Phi_{Homog'ous}$ satisfies $\nabla^2_y \ \Phi_H(y) = 0$. D Baumann (2009) derived the Greens function for the singular conifold as

$$G(y; y^{'}) = \Sigma \Upsilon_{LM}(\Psi) \Upsilon^*_{LM}(\Psi^{'}) g_L(r, r^{'}) \tag{50}$$

where $g_L(r, r^{'})$ are known functions of radial scaling dimension $\Delta(L)$.

The potential is derived from the self and cross products of fluxes of the Series I, II, III. The form of the spectrum for the scalar mode Φ_- can be deduced by noting that it must involve the product of radial functions from each flux Λ_i andΛ_j resulting in a radial function to a power involving the sum of separate scaling dimensions. Similarly the angular piece must be an overlap of the angular functions. So the spectrum for Φ_- is of the form

$$\Phi_- = \Sigma_{\delta_i, \delta_j} r^{\Delta(\delta_i, \delta_j)} h_{(\delta_i, \delta_j)}(\Psi) \tag{51}$$

where the h's are angular products of Green's functions, the deltas are scaling dimensions of Λ's and $\Delta = \delta_i + \delta_j - 4$. Using the deltas from earlier flux Series gives

$\Delta_\Lambda = 1, 2, \frac{5}{2}, \sqrt{28} - \frac{5}{2}, ...$ These are the permitted powers for the potential ie.

$$V_\Lambda(\phi) = b_1 \phi^1 j_1(\Psi) + b_2 \phi^2 j_2(\Psi) + b_{\frac{5}{2}} \phi^{\frac{5}{2}} j_{\frac{5}{2}}(\Psi) + ... \tag{52}$$

where $\phi = T_3 r^2$, T_3 is a function of the coupling, and j_k denote the overlap angular functions.

As mentioned earlier, the chiral flux solutions for each flux series is of a separate Hodge type. So there are no mixed terms between the different fluxes and for the chiral case, the flux squared is the sum of Λ's squared ie

$$|\Lambda|^2 = |\Lambda_I|^2 + |\Lambda_{II}|^2 + |\Lambda_{III}|^2 \tag{53}$$

$$|\Lambda|^2 = 6 g^{\alpha\bar{\alpha}} g^{\beta\bar{\beta}} \nabla^2_{\alpha\beta} f_1 \overline{\nabla^2_{\alpha\beta} f_1} + 12 g^{\alpha\bar{\alpha}} \nabla_\alpha f_2 \overline{\nabla_\alpha f_2} + 6|f_3|^2 \tag{54}$$

From which the flux induced potential is

$$\Phi_- = \frac{g_s}{96}[3 g^{\alpha\bar{\alpha}} \nabla_\alpha f_1 \overline{\nabla_\alpha f_1} + 12|Re f_2|^2 + 6\nabla^{-2}|f_3|^2 \tag{55}$$

Next we consider the Ricci curvature additional contribution to the potential ignoring the presence of the IASD Λ flux . The Φ_- equation of motion becomes

$$\nabla^2 \Phi_- = R_4 \quad and \quad R_4 = \frac{4}{M_{pl}^2}(V_0 + T_3 \Phi_-) \tag{56}$$

where the latter uses Friedmann's equation for 4d deSitter and the D3 brane potential is $T_3 \Phi_-$.

Solving this is by direct mathematical computation with all steps included in the Baumann's paper (chapter 4). Essentially the steps are: express Φ_- as product of radial and angular parts where the radial part Q satisfies

$$\frac{d^2 Q}{dr^2} + \frac{5}{r}\frac{dQ}{dr} - \frac{H}{r^2} Q = \lambda Q \tag{57}$$

and H is the same H as defined in the Series spectra. This is the modified Bessel equation satisfied by Bessel functions I_n (ignoring the divergent piece). Using the potential for the curvature is $V_R(\phi) = V_0 + T_3\Phi_-$ and substituting the I_n gives

$$V_R(\phi) = 2V_0\,\frac{M_{pl}^2}{\phi^2}\,\Sigma_L I_n(\frac{2\phi}{M_{pl}})\,h_L(\Psi) \tag{58}$$

which gives the terms in the potential as

$$V_R = V_0(1 + \frac{\phi^2}{3M_{pl}^2}..) \sim c_2\phi^2 \tag{59}$$

to first order.

Lastly $V(\phi) = V_H$. The harmonic potential piece was derived by Baumann (2009) as

$$V_H = V_0 + a_{\frac{3}{2}}\phi^{\frac{3}{2}}h_{\frac{3}{2}}(\Psi) + a_2\phi^2 h_2(\Psi) \tag{60}$$

The D3 brane potential is the sum of these three potential series.

4 Conformal Field Theory

This section deals with Chapter 5 of Baumann's paper. The two main underlying references for the D3 Brane model are: Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity by I Klebanov (1998) and Spectrum of Type IIB Supergravity on $\mathrm{AdS}_5 \times T^{11}$ Predictions on N=1 SCFT by A Ceresole (1999).

As indicated in the Introduction and the section on the AdS/CFT correspondence, there are several levels of matching between the Superconformal Gauge Theory/Supergravity. Our matching will be primarily using CFT operators.

To summarise Baumann's chapter 5. Baumann briefly sets up the Conformal Field Theory operators using Supersymmetric Nonabelian Gauge Theory consisting of two doublets of chiral superfields and the chiral gauge field strength superfields for the two gauge symmetries. The theory focusses on the protected operators of the chiral, conserved and semi-conserved types, since these operators have well defined dimensions as opposed to unprotected operators whose dimensions become anomalous in limit of 't Hooft coupling. The treatment in the paper is to match the scalar mode and flux Series I,II,III with the protected CFT operators by comparing the quantum numbers of the continuous global and gauge group symmetries together with the associated scaling dimensions. Baumann briefly sketches how to analytically take a supersymmetry operator involving just chiral superfields and derive the associated flux Series I, but He does not track from supersymmetry operators involving chiral and vector superfields to flux Series I, II and III since this appears to digress from the main theme and appears to be a specialist area.

As an additional comparison, Baumann compares the results with findings in Ceresole's paper on Supergravity on $\mathrm{AdS}_5 \times T^{11}$ and undertakes a discussion matching of the flux induced potential from chiral perturbations for the three series arising from superpotential perturbations.

Chapter 5 has a large underlying knowledge infrastructure. To understand it further, my discussion focusses on specific selected areas: a summary of the supersymmetric non-abelian gauge theory and the motivation for symmetries and superfields; Ceresole's AdS/CFT correspondence for IIB Supergravity on AdS $\times$ T^{11} multiplets; and finally an understanding of Baumann's matching tables and related matters.

4.1 Non-Abelian Supersymmetry

From the Supersymmetry course, to construct the N=1 SUSY Abelian gauge theory we need a chiral superfield, Φ and U(1) vector (gauge) superfield denoted by V. These fields have transformation rules of $\Phi \to e^{iq\Lambda}\Phi$ and $V \to V - i(\Lambda - \Lambda^\dagger)$ where Λ is chiral superfield.

The Lagrangian of a supersymmetric Abelian gauge theory has a *Kähler* potential term $\Phi^\dagger e^{qV}\Phi$, a kinetic term $(W^\alpha W_\alpha)$ where the W_α is the chiral field strength superfield defined as $W_\alpha = (-\frac{1}{4})\bar{D}^2 D_\alpha V$ and a superpotential (holomorphic) $(W(\Phi))$, where, to restrict the gauge freedom, V is assumed to be in the Wess-Zumino gauge. Finally, to get the Lagrangian terms, we extract the D term from the *Kähler* potential and the F term from the superpotential and kinetic term (and ignore the Fayet-Lliopoulos term which drops out in non-abelian case). Integrating over superspace, leads to an Action:

$$S = \int d^4x d^4\theta [(\Phi^\dagger e^{qV}\Phi) + (W^\alpha W_\alpha) + (W(\Phi) + \ hc)] \tag{61}$$

To change the above to the non-Abelian case, the usual method is adopted: express the gauge fields as matrices, use spinor indices and take the trace, ie

- replace the charge by generators of the gauge group $-\frac{1}{2}q \rightarrow T^a_{ij}$ and change the covariant derivative accordingly
- the field strength W_α becomes $W_\alpha = \bar{D}^2(e^V D_\alpha e^{-V})$ ie $W^\alpha W_\alpha \rightarrow \frac{1}{g^2}Tr(W^\alpha W_\alpha)$

From which the N=1 global SUSY and gauge covariant action becomes (for i chiral superfields):

$$S = \int d^4x d^4\theta \ [K(\Phi_i^\dagger, e^{2qV}\Phi_i) + f_{ab}(\Phi_i)(W^{a\alpha}W^b_\alpha) + (W(\Phi_i) + \ hc)] \tag{62}$$

Therefore, to set up such a Field theory requires determining the global and gauge symmetries and its chiral and vector superfields. Klebanov and Witten (1998) (KW) have done this for three branes at a Calabi-Yau singularity, which has a structure similar to that required in Chapters 2,3 and 4 and so it is useful to outline the group structure and in particular, how it is then formulated into superfields and vector fields and the format of the CFT operators.

KW explain that the Type IIB theory backgrounds $AdS_5 \times S^5$ has N=4 SU(N) gauge theory and preserves the maximal number of supersymmetries while $AdS_5 \times X_5$, where X_5 is an Einstein manifold, often preseves no supersymmetry and that Field Theories are best constructed from manifolds which have some supersymmetry left unbroken. Therefore a manifold between these two extremes is desirable and T^{11} is a 'good candidate'. T^{11} has a group structure given by the coset space $SU(2) \times SU(2)/U(1)$ corresponding to the Type IIB on $AdS_5 \times S^5/Z_2$. As was indicated earlier, placing the N D3 branes near the conical singularity leads to a conelike metric of dimension 6 - which in the near horizon becomes $AdS^5 \times T^{11}$. KW comment that the holonomy of the cone can be SU(3) (Calabi-Yau threefold) or SU(2) which, because of the number of unbroken symmetries, respectively results in an N=1 and N=2 superconformal field theory. Since T^{11} has $SU(2) \times SU(2) \times U(1) = SO(4) \times U(1)$ symmetry this is a suitable choice for the structure of the singularity.

Ceresole (1999) and Baumann (2010) have each chosen two chiral superfields; the logic is simple but key. The singularity coordinates $\Sigma_{i=1}^{4}\ z_i^2 = 0$ can be transformed into an alternative format $z_1 z_2 - z_3 z_4 = 0$ which can then be solved as $z_1 = A_1 B_1, z_2 = A_2 B_2, z_3 = A_1 B_2, z_4 = A_2 B_1$ ie 'pairing the four coordinates' This parametrization of the conifold suggests using two chiral superfields A_k and B_l, with $k, l = 1, 2$.

The geometry of T^{11} has $SU(2) \times SU(2) \times U(1)_R$ continuous global symmetry. So the chiral fields A_k and B_l, which are assigned an R-symmetry value of $\frac{1}{2}$ must transform under (2,0,$\frac{1}{2}$) and (0,2,$\frac{1}{2}$).

In addition from the string theory course, parallel three branes have symmetry U(1) $\times$ U(1), the second one viewed as coming from the unbroken U(1) on the three-brane volume and for N D3 parallel branes the gauge group is U(N) $\times$ U(N) which in the infrared becomes SU(N) $\times$ SU(N). The chiral fields A_k, k=1,2 transform in the $(N, \bar{N})$ representation and B_l transforms in the $(\bar{N}, N)$ representation.

KW also explains the thinking behind the format of CFT chiral operators. The underlying idea, motivated by the traceless symmetric polynomials of scalar fields in the N=4 SYM theory, is that because each chiral superfield is assigned an R-charge of 1/2, forming the operator TrA_kB_l will give an R-charge 1, dimension $\frac{3}{2}$ and occur in the (2,2) of $SU(2) \times SU(2)$. Generalising this to multiple products of pairs AB will give towers of operators of the form $TrA_{k_1}B_{l_1}...A_{k_n}B_{l_n}$ with an R-charge of n, dimension $\frac{3n}{2}$ and occur in the (n+1,n+1) of $SU(2) \times SU(2)$.

Also, for each of the gauge group SU(N) symmetries, there are two vector superfields V_1 and V_1, from which can be constructed the chiral gauge field strength superfields $W_\alpha^{(1)}$ and $W_\alpha^{(2)}$ where $W_\alpha^i = \bar{D}^2(e^{V_i} D_\alpha e^{-V_i})$ for i=1,2.

So in summary, there is a continuous global symmetry SU(2) $\times$ SU(2) $\times$ $U(1)_R$ which comes from the geometry of T^{11} and gauge group symmetry SU(N) $\times$ SU(N) which comes from the presence of N D3 branes. This is the group structure quoted by Baumann in Chapter 5.

4.2 Constructing Multiplets in IIB Supergravity on AdS_5 $\times$ T^{11}

In presenting his scalar mode and flux matching table to CFT tower operators, Baumann cross-references to Ceresole's multiplets derived for $AdS_5 \times T^{11}$. Ceresole also uses new operators formed from the products of chiral and vector fields. For our purposes we need to broadly understand Ceresole's construction of the nine families of multiplets for the harmonics of the coset space T^{11} used in the Baumann matching. And then understand his construction of the sequences of (chiral) superfields corresponding to the hypermultiplets and tensor multiplets in the AdS bulk. (This is the tower of operators used by Baumann in Chapter 5).

To avoid deviating too far in our multiplet construction, there is a key result by Ceresole that: in a KK compactification (ie five-dimensional), the equations of motion for the ten-dimensional fields ϕ fluctuations lead to equations

$$(\Box_x^{[\Lambda]} + \Box_y^{[\lambda_1,\lambda_2]})\phi^{[\Lambda]}_{[\lambda_1,\lambda_2]}(x,y) = 0 \tag{63}$$

where coordinates x are for AdS_5 and y are for T^{11} and $[\Lambda]$ has three labels, energy E_0 and spin quantum numbers s1 and s2 of SU(2,2) and $[\lambda_1, \lambda_2]$ are spin labels of SO(5). The boxes are the kinetic operators of 5d internal and 5d external spaces and the fields $\phi^{[\lambda_1,\lambda_2]}(x,y)$ can be expanded in terms of the harmonics of T^{11}. The harmonics of T^{11} are either $Y^{j,l}_{[\lambda_1,\lambda_2]}$ where the lower labels must be both integer or half integer and add to 0,1,2 or fragmented $Y^{j,l,r}_{(q)}(y)$ where j,l are spin quantum numbers of SU(2), r is quantum number of $U(1)_R$, and q is $U(1)_H$ charge. There are six such harmonics, $Y_{[0,0]}, Y_{[1,0]}, Y_{[1,1]}, Y_{[2,0]}, Y_{[\frac{1}{2},\frac{1}{2}]}, Y_{[\frac{3}{2},\frac{1}{2}]}$ each with its own Laplace-Beltrami differential equation and mass matrices.

Focussing on the scalar harmonic $Y_{[0,0]}$, the Laplace-Beltrami differential equation gives $\Box Y_0^{(j,l,r)} = H_0(j,l,r)Y_0^{(j,l,r)}$ with eigenvalue $H_0 = 6(j(j+1)+l(l+1)-\frac{r^2}{8})$ as well as masses for the five-dimensional fields. (This H_0 is used by Baumann in Chapter 5). There are similar results for the spinor and vector harmonics giving eigenvalues and masses. Ceresole presents her multiplets categorised by (E_0, s_1, s_2) and E_0 is expressed in terms of H_0. Since we will focus on the Vector Multiplet I later, this corresponds to $(s_1 = \frac{1}{2}, s_2 = \frac{1}{2})$. We extract the b(ottom) and ϕ_μ values from the Ceresole's Vector Multiplet I of table 7, and note that the (j,l,r) (where r corresponds to the highest spin dependence is buried within H_0). This Vector Multiplet I is used most frequently

(s_1, s_2)	E_0^s	R-sym	field	Mass
$(\frac{1}{2}, \frac{1}{2})$	$E_0 + 1$	r	ϕ_μ	$H_0 + 12 - 6\sqrt{H_0 + 4}$
$(0, 0)$	E_0	r	ϕ	$H_0 + 16 - 8\sqrt{H_0 + 4}$

Table 9: Extract from Vector multiplet I with $E_0 = \sqrt{H_0 + 4} - 2$

by Baumann in cross-referencing.

Next Ceresole draws on some Lie Algebra standard results for the given global and gauge symmetries, and chiral and vector fields, and states their gauge transformations. From these, Gauge covariant combinations can be derived:

$$W_\alpha(AB)^k = W_\alpha^1(AB)^k \qquad W_\alpha(BA)^k = W_\alpha^2(BA)^k \tag{64}$$

$$Ae^V\bar{A}e^{-V} = Ae^{V_2}\bar{A}e^{-V_1} \qquad Be^V\bar{B}e^{-V} = Be^{V_2}\bar{B}e^{-V_1} \tag{65}$$

But the real strength is that many more Gauge covariant combinations or towers can be built by intermultiplying. For instance $Ae^{V_2}\bar{A}\bar{B}e^{-V_2}B$ is a Gauge covariant combination. Guided by this, many towers were constructed and categorised into four (of which Baumann uses three) types of protected operators.

- Chiral superfields, S defined by $\bar{D}_{\dot{\alpha}}S_{(\alpha_1 \dots \alpha_{2s_1})}(x, \vartheta, \bar{\vartheta}) = 0, \quad s_2 = 0$
- Conserved superfields, J defined by $D^{\alpha_1}J_{(\alpha_1 \dots \alpha_{2s_1}, \dot{\alpha}_1 \dots \dot{\alpha}_{2s_2})}(x, \vartheta, \bar{\vartheta}) = 0,$ and $\bar{D}^{\dot{\alpha}}J_{(\alpha_1 \dots \alpha_{2s_1}, \dot{\alpha}_1 \dots \dot{\alpha}_{2s_2})}(x, \vartheta, \bar{\vartheta}) = 0$
- Semi-conserved superfields, L defined by $\bar{D}^{\dot{\alpha}}L_{(\alpha_1 \dots \alpha_{2s_1}, \dot{\alpha}_1 \dots \dot{\alpha}_{2s_2})}(x, \vartheta, \bar{\vartheta}) = 0$

Respectively these have r-values: $r = \frac{2}{3}\Delta$, $r = \frac{2}{3}(s_1 - s_2)$ and $\Delta = (2 + s_1 + s_2)$, $r = \frac{2}{3}(\Delta - 2 - 2s_2)$ and $\Delta = (2 + s_1 + s_2)$. where $J = \{J_a, J_b\}$ and $J_a = Ae^V\bar{A}e^{-V}$, $J_b = Be^V\bar{B}e^{-V}$,$L_{\dot{\alpha}} = e^V\bar{W}_{\dot{\alpha}}e^{-V}$

Ceresole constructed towers for these Chiral, Conserved and Semi-conserved superfields (for k a positive integer).

Chiral	
$S^k = Tr(AB)^k$	$\Delta_k = \frac{3}{2}k$
$T^k_\alpha = Tr[W_\alpha (AB)^k]$	$\Delta_k = \frac{3}{2} + \frac{3}{2}k$
$\Phi^k = Tr[W^\alpha W_\alpha (AB)^k]$	$\Delta_k = 3 + \frac{3}{2}k$

Table 10: Protected Towers of Chiral Superfields

Conserved		Semi-conserved	
$J^k = Tr[J(AB)^k]$	$\Delta_k = 2 + \frac{3}{2}k$	$L^{1,k}_{\dot\alpha} = Tr[L_{\dot\alpha}(AB)^k]$	$\Delta_k = \frac{3}{2} + \frac{3}{2}k$
$J^k_{\alpha\dot\alpha} = Tr[J_{\alpha\dot\alpha}(AB)^k]$	$\Delta_k = 3 + \frac{3}{2}k$	$L^{2,k}_{\alpha} = Tr[W_\alpha J(AB)^k]$	$\Delta_k = \frac{7}{2} + \frac{3}{2}k$
$I^k = Tr[JW^2(AB)^k]$	$\Delta_k = 5 + \frac{3}{2}k$	$L^{2,k}_{\dot\alpha} = Tr[L_{\dot\alpha}W^2(AB)^k]$	$\Delta_k = \frac{9}{2} + \frac{3}{2}k$

Table 11: Protected Towers of Conserved and Semi-Conserved Superfields

Cresole found constraints for the associated labels, constructed towers for non-chiral operators and produced extensive multiplet tables.

4.3 Matching Supergravity Scalar Φ_- and Flux G_- Modes and CFT Operators

In Baumann Chapter 5, he presents matching between

- Φ_- and G_- Series and CFT operators - and some matching to Ceresoles multiplets.
- Chiral Flux induced potential and superpotential perturbations of operators

albeit, that the level of matching varies significantly from detailed to discursive. Also the tables are presented as separate complete entities and hence some values cannot be cross matched. This suggests that some matching really means demonstrating consistency rather than total matching.

For each of the above, the aim is to drill down and explain the level of matching by taking a specific (easy) example.

In Baumann chapter 3, as well as expanding the harmonic functions f on the conifold in terms of angular harmonics, there are associated radial scaling dimensions $\Delta_f(L)$, defined as: $\Delta_f(L) = \sqrt{H(j_1, j_2.R_f) + 4} - 2$ where $H(j_1, j_2.R_f) = 6[j_1(j_1+1) + j_2(j_2+1) - \frac{R_f^2}{8}]$. from which the spectrum of harmonic functions on conifold for chiral $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2})$ and $(j_1 = 1, j_2 = 1)$ is (table 12)

Δ_f	j_1	j_2	R_f
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
3	1	1	2

Table 12: Extract from Baumann table 2, spectrum of harmonics

In defining Series I flux Hodge type (1,2) as $\Lambda_I = \nabla\nabla f_1 \cdot \bar{\Omega}$, the dimension of the dual field theory is $\delta_I = 1 + \Delta_f$ and $R = R_f - 2$, from which the spectrum of Series I for chiral modes is (table 13)

δ_f	j_1	j_2	R
$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1
4	1	1	0

Table 13: Extract from Baumann table 3, Series I, $\Lambda_I = \nabla\nabla f_1 \cdot \bar{\Omega}$

In setting up the CFT Chiral operators in Chapter 5, the simplest case is found by multiplying the two chiral superfields A and B and extracting the bottom and ϑ^2 component and taking the trace ie $[Tr(AB)]_{b,\vartheta^2}$ and assigning $f_1 = (AB)$ in the above table. So Baumann's Φ_- and G_- tables (between his tables 5 and 6) with k=1 gives:

Scalar/Flux	Defined as	Operator	Δ	j_1	j_2	R
$\Phi_-(k=1)$	(AB)	$Tr[AB]_b$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
G_- (k=2)	$\nabla\nabla(AB)^2 \cdot \bar{\Omega}$	$Tr[AB]^2_{\vartheta^2}$	4	1	1	0

Table 14: Extract from Baumann Φ_- and G_- tables and CFT operators f=(AB)

which can be seen to equal the first line of table 12 and the second line of table 13.

Then in tables 6 and 7, Baumann states that both belong to Ceresoles Vector Multiplet I. This is immediately apparent since E_0 and Δ_f are equal. All other levels of matching requires skill at products of chiral and vector superfields and gauge field strength superfields.

There is a physical interpretation for any form of matching which in the above case amounts to: although the flux form is valid for any Calabi-Yau manifold since it was created from the three building blocks for Calabi-Yau cone ie Kähler form, three form Ω and harmonic functions, it is necessary to focus on the conifold in order to obtain a radial scaling of the flux solution and the relevant symmetries and quantum number relationships. Also the f_1 has to be holomorphic of the form $(AB)^k$ in order for the matching to occur. In comparison on the CFT side, for instance, the superpotential perturbations $\int d^2\theta Tr[AB]^k$ project out the θ^2 corresponding to supersymmetry of unperturbed CFT.

Baumann makes other self-contained comments about the Ceresole's various multiplets which add no value in repeating here.

Turning to scalar potential matching. Baumann's treatment is primarily a discussion. I shall just highlight a couple of points.

As mentioned earlier, when the specific forms of the fluxes were given for Series I,II,III ie $\Lambda_I, \Lambda_{II}, \Lambda_{III}$, there was also a subset corresponding to chiral perturbations in which the harmonic functions f_1, f_2, f_3 are holomorphic. In this case, the flux induced potential was

$$\Phi_- = \frac{g_s}{96}[3g^{\alpha\bar{\alpha}}\nabla_\alpha f_1 \overline{\nabla_\alpha f_1} + 12|Ref_2|^2 + 6\nabla^{-2}|f_3|^2 \tag{66}$$

For the matching to CFT operators, (briefly).

- Assign $f_1 \sim (AB)^k$ to the first term which is equivalent to evaluating the superpotential perturbation $\int d^2\theta (AB)^k$ (F term)
- Assign $f_2 \sim (AB)^k$ to the second term which is equivalent to evaluating the superpotential perturbation $\int d^2\theta Tr[W_+^\alpha (AB)^k Y_\alpha]$ where Y_α are spurion fields (ie $Y_\alpha = [Y_\alpha]_b + [Y_\alpha]_\theta \theta + [Y_\alpha]_{\theta^2}\theta^2)$, followed by some supersummetry manipulations
- This matching is lengthy but is similar to the second term method.

Baumann's matching between $AdS_5 \times T^{11}$ with N=1 SYM focuses on the matching between supergravity Φ_- modes and G_- flux modes and the CFT operators. The technique used is primarily quantum number and dimension matching with only limited use of CFT operator to supergravity modes using supersymmetric techniques. Noting that the object is to understand the chapter 5 methods rather than independently verify the results, this completes our summary.

5 D3 Brane Superpotentials from Fluxes

This section deals with Chapter 6 of Baumann's paper. The main underlying references for the D3 Brane Superpotentials are: AdS strings with Torsion: Non-complex Heterotic Compactifications by A.R.Frey (2005), On D3 brane Potentials in Compact-

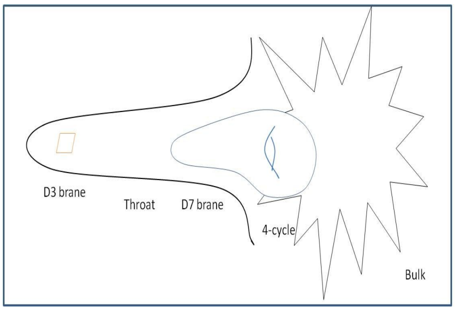

Figure 2: Insertion of throat causing distortions of the fluxes and scalar modes leading to D3 brane potential

ifications with Fluxes and Wrapped D-branes by D Baumann (2006) and Towards an Explicit Model of D-brane Inflation by D Baumann (2007).

Baumann's Chapter 6 is a short chapter which in summary (i) states that for a superpotential interaction in a conifold gauge theory, there is a $G_{(1,2)}$ flux which *geometrizes* the superpotential. (ii) for a stack of D7 Branes wrapping a four cycle, gaugino condensation on the D7 Branes induces a non-trivial potential on the D3 Brane. (iii) for a specified superpotential of a D3 Brane in a conifold, there exists a Born-Infeld plus Chern Simons potential of the D3 Brane which gives an F-term potential in 4d supergravity - and again the superpotential is *geometrized.*

An immediate clarification is that geometrizes roughly means that the superpotential is embedded into some dimensional geometry; simplistically, this means that the D3 Brane potential can be expressed in terms of the superpotential. The main features of D7 branes is covered in part (ii), although a few D7 brane facts will be needed in part (i).

Figure 2 provides simple sketch of the D7 brane geometry (gauginos presence implied).

5.1 D3 Brane Potential in terms of Superpotential

So far, D3 Brane potential solutions have been found in various forms for various scenarios from the equation of motion for Φ_- containing flux sourced terms taking the basic form

$$\nabla^2\Phi_- = \frac{g_s}{96}|\Lambda|^2 \tag{67}$$

Resulting solutions include the following types:

1) Green's function solution

$$\Phi_- = \frac{g_s}{96}\int d^6y^{'}G(y;y^{'})|\ \Lambda\ |^2(y^{'}) + \Phi_H(y) \tag{68}$$

2) chiral perturbations for Series I

$$\Phi_- = \frac{g_s}{96}[3g^{\alpha\bar{\alpha}}\nabla_\alpha f_1\overline{\nabla_\alpha f_1}] \tag{69}$$

3) non-vanishing 4d Ricci scalar,

$$V_R(\phi) = 2V_0\ \frac{M_{pl}^2}{\phi^2}\ \Sigma_L I_n(\frac{2\phi}{M_{pl}})h_L(\Psi) \tag{70}$$

Having established the CFT relationships in the previous section, the natural extension is to see if the D3 Brane potential can expressed in terms of the CFT Superpotential, W, assumed to be a function of the holomorphic chiral fields. Baumann's Chapter 6 puts D7 Branes at the centre of his discussion. The key result is that the D3 Brane potential can be expressed in terms of the superpotential, given by

$$V = g^{\alpha\bar{\beta}}\nabla_\alpha W\overline{\nabla_\beta W} \tag{71}$$

Baumann considers the D7 brane scenario: for global-supersymmetry on a D3 brane probing a non-compact cone with D7 branes, then for a superpotential W, there is a solution of 10d equations of motion such that the Series I fluxes give rise to the superpotential. The superpotential W is assumed to proportional to the holomorphic function within the Series I flux.

We need three facts about D7 branes for this section. Early on it was stated that perturbations of the dilaton and metric do not affect the flux induced potential on the D3 brane. However, D7 branes can source dilaton terms leading to a holomorphic axion-dilaton within a compactification. As a result, the Λ flux equation is modified and becomes

$$d\Lambda + \partial\phi \wedge (\Lambda + \bar{\Lambda}) = 0 \tag{72}$$

The second fact is that the Ricci for the internal space is given by $R_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \phi$ and thirdly the Φ_- equation of motion becomes

$$\nabla^2 \Phi_- = \frac{e^\phi}{96} |\Lambda|^2 \tag{73}$$

Now we demonstrate that

- suitable fluxes $\Lambda_{\alpha\bar{\beta}\bar{\gamma}} = g_s e^{-\phi} \nabla_\alpha \nabla_\sigma f g^{\sigma\bar{\rho}} \bar{\Omega}_{\bar{\rho}\bar{\beta}\bar{\gamma}}$ and $\Lambda_{\alpha\beta\gamma} = g_s \overline{\nabla_\sigma f g^{\sigma\bar{\rho}} \nabla_{\bar{\rho}} e^{-\phi}} \Omega_{\alpha\beta\gamma}$, which now include a dilaton factor due to the D7 brane, satisfy the Λ flux equation
- and the $V = g^{\alpha\bar{\beta}} \nabla_\alpha W \overline{\nabla_\beta W}$, where W is assumed to be a holomorphic function f, satisfies the Φ_- equation.

Using the properties of the IASD forms, covariant derivatives and substituting for Ω, the above (1,2) form can be expressed in terms of simple derivatives to give $\Lambda_{\alpha\bar{\beta}\bar{\gamma}} = \partial_\alpha (\partial_\sigma f g^{\sigma\bar{\rho}} \bar{q} \epsilon_{\bar{\rho}\bar{\beta}\bar{\gamma}}) e^{-\phi}$. Then assuming that the (3,0) takes the form $\Lambda_{\alpha\beta\gamma} = \psi q \epsilon_{\alpha\beta\gamma}$, substituting into the Λ equation gives a pair of equations

$$\partial_{\bar{\alpha}} \psi q + \partial_\beta \phi \partial_{\bar{\alpha}} [\overline{\partial_\sigma f g^{\sigma\bar{\beta}} \bar{q}}] e^{-\phi} = 0 \tag{74}$$

$$- \partial_\beta [\partial_\alpha (\partial_\sigma f g^{\sigma\bar{\beta}}) e^{-\phi}] + \partial_\alpha \phi \bar{\psi} \bar{q} = 0 \tag{75}$$

which are satisfied provided $\psi = \overline{\partial_\sigma f g^{\sigma\bar{\beta}} \partial_{\bar{\beta}} e^{-\phi}}$. This briefly outlines that the fluxes with the above form satisfy the equations of motion when modified to reflect the presence of the D7 brane.

Next, to show that V expressed in terms of the superpotential satisfies the Φ_- equation, and remembering that the superpotential W $\sim$ to holomorphic potential f (and $W^2 = \frac{T_3}{8} f^2$). The left hand side of Φ_- equation is

$$\nabla^2 V = \nabla^2 [g^{\sigma\bar{\beta}} \nabla_\alpha f \overline{\nabla_\beta f}] = g^{\rho\bar{\sigma}} g^{\alpha\bar{\beta}} (2\nabla_\rho \nabla_\alpha f \overline{\nabla_\sigma \nabla_\beta f} + 2\nabla_{\bar{\sigma}} \nabla_\alpha f \nabla_\rho \nabla_{\bar{\beta}})$$
$$+ g^{\alpha\bar{\beta}} \nabla^2 \nabla_\alpha f \nabla_{\bar{\beta}} + g^{\alpha\bar{\beta}} \nabla_\alpha f \nabla^2 \nabla_{\bar{\beta}} \bar{f} \qquad (76)$$

from which second term vanishes because f is holomorphic. The remaining terms after some algebra give $\frac{1}{3!} e^\phi |\Lambda^{(1,2)}|^2 + \frac{2}{3!} e^\phi |\Lambda^{(3,0)}|^2$, which is the right hand side.

This outlines the justification for $V = g^{\alpha\bar{\beta}} \nabla_\alpha W \overline{\nabla_\beta W}$, at least for the D7 brane. Generalising to general W appears to follow the same method.

5.2 Superpotentials from D7 branes

Turning to part (ii), our objective is to explain the main D7 features and justify that the brane superpotential takes the form $W = W_0 + W_{np}(z_\alpha, \rho)$, where the various terms are to be explained.

First lets clarify the difference between perturbative and nonperturbative effects. There are action terms, such as the Chern-Simons terms and localized p-brane terms, which give rise to perturbative expansions in powers of the couplings, g_s and α' (defined earlier), which are determined by the Planck scales. Often, they appear in the Kähler potential but not in the superpotentials. In contrast non-perturbative effects arise in well-defined constructed situations such as gaugino condensation, instantons and composite goldstinos. We are interested in the first - gaugino condensation.

The key features, for our purposes, of gaugino condensate are found in Frey (2005). These are that in pure N=1 non-abelian gauge theory, the gauge field becomes strongly coupled at energy scale Λ and the gaugino condenses leading to

$$< \chi\chi >\sim \Lambda^3 \sim M_{UV}^3 e^{-\frac{1}{bg^2}} \tag{77}$$

where M_{UV} is the UV cut off scale, b is "one loop determinant", g is 4d (YM) gauge coupling. The reason it condenses is that the underlying running coupling has an inverse dependence which becomes strong at scale leading to Λ having exponential dependence. At low scales, when coupled with the auxiliary field it takes on the form of superpotential, which is often stated that the condensate induces a condensation superpotential

$$W \sim e^{-\frac{1}{bg^2}} \tag{78}$$

We know that in 4d N=1 theory, fluxes induce a flux superpotential W of the form

$$W = \int_M G \wedge \Omega \tag{79}$$

where G (comprising RR and NS fluxes) is coupled to the geometry via the 3-form Ω. This fixes the dilaton and complex structure moduli but leaves the volume modulus unfixed ie in no-scale compactification. Baumann creates his model so that each of these terms is well defined. This model is the D3 brane model described earlier but he now inserts a holomorphically-embedded stack of D7 branes wrapping a four cycle, where the latter resides partially in throat and in the bulk. Finally the presence of the gaugino condensation induces a D3 brane potential. In fact, combining the superpotentials fixes all of the Kähler moduli, in particular the volume modulus, and the superpotential becomes

$$W = W_0 + \mathbb{A}(X)e^{-a\rho} \tag{80}$$

where the first term comes from the flux superpotential and is a constant W_0 after stabilisation of the complex structure moduli. The second term is the condensation superpotential W_{np} in which the exponential ρ is the Kähler modulus associated with the volume over the four cycle and the $\mathbb{A}(X)$ takes on the structure of the model. In Baumann's case, the four-cycle volume depends on the D3 brane position which forces the condensation superpotential coefficient to depend on the D3 brane position and is

$$\mathbb{A}(z_\alpha) = \mathbb{A}_0 h(z_\alpha)^{\frac{1}{N_c}} \tag{81}$$

for N_c D7 branes and $\mathbb{A}_0 \propto N_c^2$. The term, holomorphic embedding, provides a condition on the geometry $h(z_\alpha) = 0$.

We note that Frey deals with gaugino condensation with an H flux in heterotic supergravity leading to $AdS_4 \times X^6$. From the effective supergravity action for bosonic field and gaugino, he derives the string frame SUSY dilatino, gaugino and gravitino variations leading to the torsion dependent manifolds and expressions for gaugino condensate, condensate scales, volume modulus, and Kähler potential leading to AdS_4, but this takes us too far from our discussion.

Whilst the role of the gaugino condensate is clear, the role of the D7 branes needs explaining; these are:

The action for gauge fields on D7 branes wrapping a four-cycle Σ_4 is given by

$$S = \frac{1}{g_7^2} \int_{\Sigma_4} d^4\xi \sqrt{g^{ind}} h(Y) \cdot \int d^4x \sqrt{g} g^{\mu\alpha} g^{\nu\beta} Tr F_\mu F_{\alpha\beta} \tag{82}$$

where ξ_i are coordinates on Σ_4 and g_{ind} is the induced metric on from g_{ij}, Y are internal space coordinates and h(Y) determines the geometry. The warped volume of Σ_4 is given by

$$V^{\omega}_{\Sigma_4} = \int_{\Sigma_4} d^4\xi \sqrt{g^{ind}} h(Y) \tag{83}$$

and the gauge coupling of 4d theory is, from the above action

$$\frac{1}{g^2} = \frac{V^{\omega}_{\Sigma_4}}{g_7^2} = \frac{T_3 V^{\omega}_{\Sigma_4}}{8\pi^2} \tag{84}$$

Using this expression for the warped volume, the modulus of the gaugino superpotential in $SU(N_{D7})$ super YM with M_{UV} cut off takes the precise form

$$|W_{np}| = 16\pi^2 M_{UV}^3 \; exp(-\frac{8\pi^2}{N_{D7} g^2}) \propto exp(-\frac{T_3 V^{\omega}_{\Sigma_4}}{N_{D7}}) \tag{85}$$

The D3 brane adds $SU(N_{D7})$ flavour to the $SU(N_{D7})$ gauge theory whose mass is a holomorphic function of the D3 brane coordinates. A displacement of a D3 brane in the compactification creates a slight distortion δh of the warped background and so perturbs the warped volumes of the four-cycles ie

$$V^{\omega}_{\Sigma_4} = \int_{\Sigma_4} d^4Y \sqrt{g^{ind}(X;Y)} \delta h(X;Y) \tag{86}$$

This change in volume enables the dependence of the superpotential on D3 location X to be determined. Rephrasing this, the D3 brane location at point X in 6d internal space, coords Y, acts as a point source for the perturbation δh and so δh is Green's function of the Laplacian with a background charge ρ

$$-\nabla^2_Y \delta h(X;Y) = c[\frac{\delta^6(X-Y)}{\sqrt{g(Y)}} - \rho(Y)] \tag{87}$$

Solving for δh and integrating over the four-cycle gives A(X).

The solution of δh which is effectively the same as considering the corrections to the warped volume are an inverse power function of the radial coordinate and therefore justify that the effects of interest are deep within the infrared region of the throat.

5.3 Flux Potential in terms of Superpotentials

Now we consider part (iii). Baumann states that for any specified superpotential for a D3 brane in the conifold, there exists a noncompact supergravity solution in which the Born-Infeld plus Chern-Simons potential of a D3 brane probe precisely equals the F-term potential $V_F[W(z_\alpha), K(z_\alpha, \bar{z}_\alpha)]$ computed in 4d supergravity with the superpotential W and Kähler as input.

In other words, The D3 brane potential derived for Series I and II fluxes $\Lambda_I = \nabla\nabla f_1 \cdot \bar{\Omega}$ and $\Lambda_{II} = \partial f_2 \wedge J$,where f_1 amd f_2 are holomorphic functions and given by

$$\Phi_- = \frac{g_s}{32}[g^{\alpha\bar{\beta}}\nabla_\alpha f_1 \overline{\nabla_\beta f_1} + 2|f_2|^2] \tag{88}$$

can be matched to the 4d supergravity F-term potential

$$V_F = \frac{\kappa^2}{12\sigma^2}\frac{e^{-2a\sigma}}{\gamma}[g^{\alpha\bar{\beta}}\mathbb{A}_\alpha\bar{\mathbb{A}}_{\bar{\beta}} + 2a\gamma(a\sigma+3)\mathbb{A}\bar{\mathbb{A}} - a\gamma(\bar{\mathbb{A}}g^{\alpha\bar{\beta}}k_{\bar{\beta}}\mathbb{A}_\alpha + cc)] + harmonic \tag{89}$$

The actual coefficient matching is clearly explained by Baumann and it adds no value to repeat it here. But the answer is an expression for Φ_- in terms of $\mathbb{A}$ The task here is to explain the F-term potential. Following Baumann (2007), the standard place to start is by observing that from the Superpotential and Kähler potential, the N=1 supergravity potential in the action can be expressed in terms of the superpotential W and Kähler K as follows

$$V_F = e^{\kappa^2 K}[D_\Sigma L^{\Sigma\bar{\Omega}}\overline{D_\Omega W} - 3\kappa^2 W\bar{W}] \tag{90}$$

where the Z^Σ is function of the volume and coordinates modulus and $D_\Sigma W$ is covariant derivative and the Kähler K is given by

$$K = -\frac{3}{\kappa^2}log[\rho + \bar{\rho} - \gamma k(z_\alpha, \bar{z_\alpha})] \tag{91}$$

where the Kähler potential is of Calabi-Yau manifold, γ is a constant. The related Kähler metric is known in terms of k_γ which leads to a potential

$$V_F = \frac{\kappa^2}{3U^2}[.... + (k^{\alpha\bar{\delta}}k_{\bar{\delta}}\overline{W_{,\rho}}W_{,\alpha} + cc) + \frac{1}{\gamma}k^{\alpha\bar{\beta}}W_{,\alpha}\overline{W_{,\beta}}] \tag{92}$$

where the dotted terms are the KKLT F-term potential and the displayed terms are the nonperturbative potential depending on the brane position. To match the two expressions for V_F is lengthy but involves redefining U, σ, extracting the exponential from the superpotential ie $W_{np}(z_\alpha) = \mathbb{A}(z_\alpha)e^{-a\rho}$ and carrying out algebra manipulation.

6 Gauginos as a Source of Flux: Field Theory v Bulk Perspectives

This section deals with Chapter 7 of Baumann's paper. Baumann's chapter 7 has two brief sections of a general discursive nature on non-commutative superpotentials and the role of ten-dimensional equations of motion for fluxes incorporating fermion expectation values that are nonvanishing but my assessment is that these are ponderings for the future rather than tangible results. There is a third brief section on gaugino condensation as a source of fluxes from a field theory perspective and a bulk perspective which leads to tangible results and this is the focus below.

Field Theory Perspective

Intriligator (1995) covers extensively the dynamics of 4d supersymmetric gauge theories in his lectures. In particular, he derives an effective action (his section §4) generated by instantons and associated with gaugino condensation and gives the 4d SUSY low energy WZ term in terms of the gaugino mass term. What is fascinating is that from group QCD symmetries, he derives the exact dynamic superpotential (inclusion here adds no value) leading to the result adapted by Baumann which includes a gauge coupling homolomorphic function f of chiral superfields of the D7 brane theory.

$$L \supset \int d^2\theta f(\Phi) W^\alpha W_\alpha \ + cc \ \sim \ \lambda\lambda \int d^2\theta f(\Phi) = F_\Phi \frac{\partial f}{\partial \Phi}\lambda\lambda \tag{93}$$

Regarding this equation as a source for F_Φ ie $\int d^2\theta W(\Phi)$ gives

$$\frac{\partial W}{\partial \Phi} = -\frac{\partial f}{\partial \Phi}\lambda\lambda \tag{94}$$

consistent with the standard gaugino condensate expression $W = N_c\lambda\lambda$. From the Flux potentials section, for series I and using local coords instead of superfield

$$\frac{\partial W}{\partial z_\alpha} = T_3\sqrt{\frac{g_s}{32}}\frac{f_1}{\partial z_\alpha} \quad \Rightarrow \quad T_3\sqrt{\frac{g_s}{32}}\frac{f_1}{\partial z_\alpha} = -\frac{\partial f}{\partial z_\alpha}\lambda\lambda \tag{95}$$

D.Baumann (2006) found a specific form for the coupling function for D7 brane embedded along a divisor in terms of an homlomorphic equation $h(z_\alpha) = 0$ as $f = 2\pi\rho - log(h(z_\alpha))$ with the result

$$T_3\sqrt{\frac{g_s}{32}}\frac{f_1}{\partial z_\alpha} = \lambda\lambda\frac{\partial(logh)}{\partial z_\alpha} \tag{96}$$

The physics description for this is: on the left are the $G_{(1,2)}$ fluxes for Series I in the warped throat arising from 10d Supergravity while on the right, there is a coupling in 4d between the gauginos condensate of the D7 branes arising from the 4d SUSY and the gauge coupling constant arising from the D7 brane embedded in the divisor. Baumann suggests that there should be a coupling term of the form $\int d^8\xi G_3\lambda\lambda$ representing the interaction between the 10d three form flux fields and the gaugino condensate in the presence of D7 branes (possibly embedded in the divisor).

Bulk Perspective

Cámara (2004) expands the DBI and CS actions for D7 and D3 branes and computes field forms for the resulting interactions. From this paper Baumann extracts the tree level coupling action involving G_3 flux and gaugino condensate $\lambda\lambda$ as

$$L = 16\zeta \int_{\Sigma} \sqrt{g} G_3 \cdot \Omega \bar{\lambda}\bar{\lambda} \sim \int_M G_3 \wedge \Omega(\bar{\lambda}\bar{\lambda}\delta^{(0)}) \tag{97}$$

where $\zeta = T_3 \sqrt{\frac{g_3}{32}}$ Taking the variation with respect to C_2 and B_2 gives

$$16 \int_M \delta G_3 \wedge \Omega(\bar{\lambda}\bar{\lambda}\delta^{(0)}) \tag{98}$$

The variation for the bulk is

$$-\frac{g_s}{4\kappa_{10}^2} \int_M \delta G_3 \wedge \bar{\Lambda} \tag{99}$$

In the case of Series I, some manipulation leads to the relationship

$$\nabla^2 f_1 = \frac{4\pi}{\zeta} \lambda\lambda\delta^{(0)} \tag{100}$$

which has a solution

$$f_1 = \frac{2}{\zeta} \lambda\lambda Re(log(h(z_\alpha))) \tag{101}$$

The physics description for this is: on the left are the $G_{(1,2)}$ fluxes for Series I while on the right are again gaugino condensates but this time sourced from the tree level coupling arising in the expansion from the D7 brane fermions and bulk fields.

These two expressions from Field Theory and Bulk Perspective are equivalent. This can be re-phrased to imply that the potential from the 4d SUSY is the same as that from 10d Sugra in flux background sourced by gaugino condensate.

7 Conclusion and The Future

Conclusion

The Baumann 2010 paper under review in this dissertation, D3 brane Potentials from Fluxes in AdS/CFT, is the latest in a series of papers over the last six years which have been constructing a string theory based model capable of controlling the inflationary dynamics in cosmology by creating Planck-suppressed contributions to the effective action. The resulting D3 brane model has an overriding objective of formulating D3 brane potentials from physical scenarios that possess a high degree of computability which are capable of feeding into cosmological theories. Whilst there are related papers dealing with the cosmological aspects of the models, this paper - and my interest - focus on the model construction.

The strength of the model's evolutionary path has been underpinned by ensuring consistency of the complementary descriptions, namely, 10d supergravity, 4d conformal field theory and 4d supergravity. The achievements of the paper can be summarised

- From Type IIB supergravity, following compactification from the D3 branes and throat geometry, the resulting non-normalizable perturbations of the background leads to closed three-form IASD flux solutions and a spectrum of contributions to the D3 brane potential.
- Systematic matching of the flux solutions to sources for dual operators in conformal field theory by the construction of complete matching tables between the fluxes and scalar modes and associated protected chiral, conserved, semi-conserved CFT operators.
- For any superpotential for a D3 brane in the conifold, the F-term potential in

4d supergravity can be geometrized from a 10d background of IASD fluxes. And that gaugino condensation on D7 branes wrapping a four-cycle sources the IASD fluxes in ten dimensions. Finally, that the scalar potential takes the same form in 4d SUSY or 10d brane in flux background sourced by gaugino condensation.

The Future

There have been no further publications on the D3 brane model since the current paper. However, the authors have indicated that a paper would be forthcoming on: Flux Duals of Non-Perturbative Effects on D7 branes. This appears to be looking for a more comprehensive geometric treatment of the D7 branes. There is also the possibility of including the multitude of other sugergravity tools such as antibranes, orientifolds and so on, into the model.

It is also noted that, for the six authors of the the current publication: D. Baumann has published a paper on: Desensitizing Inflation from the Planck Scale, which looks at the role of the Inflaton interacting with the conformal sector; I Klebanov has published a paper on: M-Branes and Metastable States, which looks at supersymmetry deformation of M-theory solutions in higher dimensional conifolds; the others have no related or no papers published since this paper.

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9 Appendix - Physical Significance

The main reference for this Physical Significance section are the TASI Lectures on Inflation by D.Baumann. ArXiv: hep-th/0907.5424 (2009).

The Dynamics of Inflation

The history of the Universe after the first 10^{-10} secs is based on generally well understood and experimentally tested laws of physics. After the first 3 minutes, at an appropriate energy level of 0.1 MeV, the strong interaction is significant and protons and neutrons combine into the light elements in accordance with Big Bang Nucleosynthesis; after 10^4 years and at 1eV, equal radiation and matter densities result in charged matter particles and photons coupling in the plasma to cause density fluctuations; after 10^5 years, and at 0.1 eV, electrons and protons combine into neutral hydrogen atoms from which photons decouple and form the free-streaming cosmic microwave background. Some of the important stages are summarised in Baumann's TASI evolution table, a selection of which are included in table 15:

However, prior to 10^{-10} secs is still speculative. What is believed to happen is that there is a Planck Epoch ($< 10^{-43}$ secs, 10^{18} GeV) during which string theory applies, followed by a Grand Unification period ($< 10^{-36}$ secs, 10^{15} GeV) followed by an Inflation period ($< 10^{-34}$ secs, 10^{15} GeV). Inflation is described as a period

Event	Time	Energy
SUSY breaking (?)	$< 10^{-10}$ s	> 1 TeV
Electroweak Unification	10^{-10} s	1 TeV
Quark-Hadron transition	10^{-4} s	10^2 MeV
Nucleon Freeze Out	0.01 s	10 MeV
Neutrino Decoupling	1 s	1 MeV
BBN	200 s	0.1 MeV
Matter-Radiation Equality	10^4 years	1 eV
Recombination	10^5 years	0.1 eV
Galaxy Formation	10^8 years	-

Table 15: Selected Physical Significance Events after 10^{-10} secs

of exponential expansion. During the Inflation period, it is believed that microscopic quantum fluctuations are transformed by the inflationary expansion into macroscopic fluctuations manifested in the form of density fluctuations and gravitational waves. Although subsequent 10^{-10} secs events affect these fluctuations and waves, these can be stripped from current observations to obtain their original form.

We need to develop some language and physics to describe Inflation. From the general relativity course, the FRW metric arises naturally from the homogeneity and isotropy of the large scale universe such that the evolution of the universe is encompassed within the scale factor $a(t)$ and the expansion rate, the Hubble parameter H, given by $H = \frac{\dot{a}}{a}$; the form of $a(t)$ is determined by the matter content of the universe via Einsteins field equations.

The comoving (particle) horizon τ is the causal horizon or maximum distance a light ray can travel over an interval. In terms of a and H, this is $\tau = \int d(ln\ a)\frac{1}{aH}$ where the $(aH)^{-1}$ is called the comoving Hubble radius which for Big Bang grows monotonically resulting in $\tau \propto a$ when radiation dominates and $\tau \propto a^{1/2}$ when matter dominates. So the comoving horizon grows monotonically. Now consider- if particles are separated by distances greater than τ, they never could have communicated with one another, if they are separated by distances greater than $(aH)^{-1}$, they cannot communicate now, but particles causal contact could have occurred earlier. One possible scenario for this is: while a grows exponentially during inflation, if H is constant, the comoving Hubble radius decreases during inflation. which can be expressed as $\frac{d}{dt}(\frac{1}{aH}) < 0$; using Friedmann's Equations, this implies $\frac{d^2a}{dt^2} > 0$.

From this, there are three equivalent conditions for Inflation, known respectively as Decreasing comoving horizon ($\frac{d}{dt}(\frac{1}{aH}) < 0$), Accelerated expansion (-$\frac{\ddot{a}}{(aH)^2} < 0$ or $\epsilon = -\frac{\dot{H}}{H^2} < 1$) and Negative pressure ($p < -\frac{1}{3}\rho$).

A few more facts are needed for the later sections. There is a curvature parameter Ω_k which is a function of the ratio of the current energy density and the critical energy density (the latter equals $3H_0^2$), which can be re-expressed as the difference between the average potential energy and the average kinetic energy of a region of space. Using a modified definition for Ω, defined as $1 - \Omega(a) = \frac{-k}{(aH)^2}$ and assuming it is time dependent, leads to an instability around the currently observed unit value called near-flatness. The presence of Inflation manages this apparent instability since by using one of the conditions - that the comoving Hubble radius is decreasing - drives the universe towards flatness (rather than away from it). One final expression is the number of e-folds $N = \int_t H dt$.

Now the simplest field dynamics model of Inflation uses a scalar parameter ϕ and a simple gravitational Einstein-Hilbert action plus a scalar field kinetic term plus potential term. Making assumptions that the metric is FRW and the energy-momentum tensor is a perfect fluid, leads to many equations and conditions. But our focus is on accelerated expansion within Inflation which is determined by the slow roll parameters $\epsilon \ll 1$ and $\eta \ll 1$,

$$\epsilon_V = \frac{M_{pl}^2}{2}(\frac{V_{,\phi}}{V})^2 \qquad \eta_V = M_{pl}^2(\frac{V_{,\phi\phi}}{V}) \tag{102}$$

which tells us that Inflation ends when these approach 1.

Clearly the form of the inflationary potential function $V(\phi)$ is important as it needs to satisfy the slow roll conditions. Commonly used is the chaotic Large-Field Inflation potential $V(\phi) = \lambda_p \phi^p$, (where λ_p is the Inflation self coupling) which has the characteristics: the slow roll parameters are small for super Planckian field values ($\phi \gg M_{pl}$), $\lambda_p \ll 1$, the potential energy density is sub-Planckian ie $V \ll M_{pl}^4$ and quantum gravity effects are assumed not important. This treatment is classical.

Quantum Fluctuations during Inflation

This section sketches how quantum fluctuations during Inflation can lead to a primordial fluctuation spectra. Again, our main interest will be in setting up the language for later use.

First, there are two gauge invariant combinations of metric and matter perturbations: the curvature perturbation on uniform density hypersurfaces defined by $-\zeta = \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho$ which measures the spatial curvature of constant density hypersurfaces (parametrised by Ψ) and is constant for certain matter perturbations; and the comoving curvature perturbation $R = \Psi - \frac{H}{(\bar{\rho}+\bar{p})}\delta q$ which measures the spatial curvature of co-moving hypersurfaces. These can be shown to be equal during slow roll inflation and both can be perturbatively expanded using Einsteins equations. The critical statistical measure of the primordial scalar fluctuations is the power spectrum of R, or ζ,

$$< R_k R_{k'} >= (2\pi)^3 \delta(k + k') P_R(k) \qquad \Delta_R^2 = \frac{k^3}{2\pi^2} P_R(k) \tag{103}$$

where $< R_k R_{k'} >$ is the ensemble average of the fluctuations. The perturbations separate into scalar and tensor power spectra.

Briefly, the theory involves: taking the unperturbed slow roll model of the inflation action, expanding to second order, promoting the scalar to quantum operators, imposing initial conditions, and solving the mode equations, and then evaluating the power scalar spectrum. Its the final result we need:

$$\Delta_R^2(k) = \frac{H^2}{(2\pi)^2} \frac{H^2}{\dot{\phi}^2} \tag{104}$$

evaluated at the horizon crossing. There is a similar expression for the power tensor spectrum. From which, after some algebra, we are left with expressions for the power spectra of scalar and tensor fluctuations.

$$\Delta_s^2(k) = \frac{1}{8\pi^2} \frac{H^2}{M_{pl}^2} \frac{1}{\epsilon} |_{k=aH} \quad \Delta_t^2(k) = \frac{2}{\pi^2} \frac{H^2}{M_{pl}^2} |_{k=aH} \tag{105}$$

where $\epsilon = -\frac{dlnH}{dN}$ The final step for us is that in the slow roll case, in which H can be expressed in terms of V

$$\Delta_s^2(k) = \frac{1}{24\pi^2} \frac{V}{M_{pl}^4} \frac{1}{\epsilon} |_{k=aH} \quad \Delta_t^2(k) = \frac{2}{3\pi^2} \frac{V}{M_{pl}^4} |_{k=aH} \tag{106}$$

This finally gives us the main result that: the power spectra for the scalar and tensors are explicitly dependent on the potential's shape ie that H is a measure of the scale of the potential. ϵ and η are a measure of the first and second derivatives of the potential. And so measurements of the amplitude and scale dependence of the cosmological perturbations provide information about the potential driving the inflationary expansion.

From the ratio r of the tensor to scalar power spectra, assuming Δ_s^2 is fixed, and $\Delta_t^2 \propto H^2 \propto V^2$, gives a direct measure of the energy scale of Inflation

$$V^{1/4} = (\frac{r}{0.01})^4 10^{16} GeV \tag{107}$$

which is interpreted that for large values of $r \geq 0.01$ correspond to inflation at GUT scale energies.

Rewriting r as $r = \frac{8}{M_{pl}^2}(\frac{d\phi}{dN})^2$ provides a (Lyth) Bound for the time field evolution between CMB fluctuations excited between the horizon at N_{cmb} and the end of inflation N_{end} given by

$$\frac{\Delta\phi}{M_{pl}} \sim (\frac{r}{0.01})^{1/2} \tag{108}$$

Again $\Delta\phi > M_{pl}$ for large scale inflation.

The above historical scalar and tensor power spectra, together with equivalent power bispectra (which arise from non-Gaussian effects), must be adjusted for time transfer functions and dark matter transfer functions (etc) to allow for known physical events over the last 10^9 years (such as CMB polarization sourced from Thomson scattering) and then matched to current CMB anisotropies and fluctuations and the galaxy power spectrum. The results are close and support the Inflation model. For instance, Ω, an indicator for the spatial geometry of the universe, has an experimental value of 1 ± 0.02 and an Inflation predicted value of 1 ± 10^{-5}. As the TASI lectures comment: current observations are in good agreement with the Inflation predictions. The universe is essentially flat with a spectrum of nearly scale-invariant, Gaussian and adiabatic density fluctuations. But future tests are still necessary including: understanding the B-modes of CMB polarization, which are a unique signature of the inflationary gravitational waves as they are a direct measure of the energy of inflation; and the overall effects of non-Gaussianicity as such lesser effects act as a constraining factor for the inflation field.

Inflation in String Theory

From the previous sections, the structure of the inflationary potential function feeds directly into the power spectra. The underlying field content and interactions will be existing at energies approaching the Planck scale and these are arguably best modeled by string theory.

Now we first remind ourselves that Inflation requires a potential that is nearly flat in Planck units ie

$$\epsilon_V = \frac{M_{pl}^2}{2}(\frac{V_{,\phi}}{V})^2 \ll 1 \qquad \eta_V = M_{pl}^2(\frac{V_{,\phi\phi}}{V}) \ll 1 \tag{109}$$

Introducing the high scale cut off Λ as the mass of the lightest particle that is not in the spectrum of the low energy theory and coefficient operators existing within the low energy which can be used to describe the high- scale physics above the cut off. Particles above the cut off are said to lie in UV-completion. Integrating out particles of mass $M \geq \Lambda$ gives rise to operators of the form $\frac{O_\delta}{M^{\delta-4}}$, where δ is the mass dimension of the operator. However, for inflation, the flatness of the potential requires $\delta \leq 6$, resulting in Planck-suppressed operators, of the form $\frac{O_6}{M^2}$. Consequently, these will contribute to the Lagrangian. In the numerator, two dimensions can be replaced with ϕ^2, leaving four dimension operator with a vacuum expectation value comparable to the inflation energy density $\sim V$, this impacts the eta parameter by order one. This is the eta problem.

In fact, a more detailed treatment shows that $\Lambda \geq H$ and $m_\phi \ll H$ leads to $\Delta\eta = \frac{\Delta m_\phi^2}{3H^2} \geq 1$ contradicting Inflation conditions. The solution is to introduce some symmetry or fine tune the action. For N=1 Supergravity, the F-term potential in the scalar potential is

$$V_F = e^{\frac{K}{M_{pl}^2}}[K^{\varphi\bar{\varphi}} D_\varphi W \overline{D_\varphi W} - \frac{3}{M_{pl}^2}|W|^2] \tag{110}$$

where φ is complex scalar field. Expanding K leads to terms in the inflationary Lagrangian of the form

$$L = -\partial\phi\partial\bar{\phi} - V_0(1 + \frac{\phi\bar{\phi}}{M^2{}_{pl}}) \tag{111}$$

from which $O_6 = V_0\phi\bar{\phi}$ and is a large contribution to the eta parameter. To fix this, we need the F-term potential to be negligible or the inflaton must not appear in the F-term potential.

Next, to derive the Lyth Bound, we used

$$\frac{\Delta\phi}{M_{pl}} \sim (\frac{r}{0.01})^{1/2} \tag{112}$$

For $r > 0.01$, epsilon and eta must be very much less than 1 over the super-Planck range $\Delta\phi > M_{pl}$. To achieve this, the flatness of the inflation potential must be dynamically and sensitively controlled over the specified range because interacting terms between the inflaton and other fields leads to self coupling and mass changes which in turn changes the inflaton potential. In summary, the effective Lagrangian receives substantial corrections from a series of higher dimension operators and symmetries at the Planck scale.

The strength of string theory compactification from 10d to 4d is that the kinetic terms and scalar potentials can be determined in terms of the moduli (scalars). In turn, if the values of the moduli are known, this fixes the parameters of the four-dimensional theory. There is an energy cost for deforming the compactification in the presence of D branes and quantized fluxes which forces some of the moduli fields to become massive. Usually the Hubble scale is used as the cut off between light and heavy fields. Such moduli stabilization also contributes to the eta problem.

Now that the necessary language has been developed, we consider the two aspects of Baumann's D3 brane model - its ability to satisfy the above Inflation bound condition and its potential content.

Inflation is tested as follows

- the inflaton kinetic term determined by the DBI action for D3 brane leads to the product form $\phi^2 = T_3 r^2$ where r is the radial coordinate. Using the length of the throat as r_{UV} gives $\Delta\phi^2 < T_3 r_{UV}^2$

- the volume of the internal space V_6 is bounded below by the volume of the throat ie $V_6 > V_{6\,throat} = 2\pi^4 g_s N(\alpha^{'})^2 r_{UV}^2$, where N measures the background flux and $N \gg 1$

- Combining these gives $\frac{\Delta\phi}{M_{pl}} < \frac{2}{\sqrt{N}}$. That is the inflaton variation will always be sub-Planckian $\Delta\phi \ll M_{pl}$. These arguments are geometrical and ensure that D3 brane model fits the Inflation condition.

The potential is broken down as follows. Inflation proceeds as a D3 brane moves radially inward in the throat region towards the throat tip (where an anti-D3 brane is situated). The inflation potential is assumed to be weak Coulomb; but moduli stabilization introduces new terms into the inflaton potential, the most important of which is the inflaton mass term arising from the supergravity F-term potential of the form $H_0^2\phi^2 = \frac{1}{3}V_0\frac{\phi^2}{M_{pl}^2}$, and finally other additional contributions to the potential arising from all other sources. This can be written as

$$V(\phi) = V_0(\phi) + H_0^2\phi^2 + \Delta V(\phi) \tag{113}$$

which has an equivalent eta parameter format

$$\eta(\phi) = \eta_0 + \frac{2}{3} + \Delta\eta(\phi) \tag{114}$$

Since $\eta_0 \ll 1$, all correction terms ΔV must be considered to ensure that eta is well behaved. Baumann's main ΔVpotentials (section 3.3)are:

- The potential of the D3 branes, impervious to other fields, $V_{D3}(\phi) = T_3\Phi_-$
- Induced $\Phi-$ perturbations, $\nabla^2\Phi_- = \frac{1}{24}|G_-|^2 + R$ where Ricci R is the square of the Hubble parameter H (equation 42). The solutions are harmonic expansions with a spectrum of eigenvalues expessible in terms of scaling dimensions Δ

  (i) Homogeneous solution (right side=0): this gives expansion terms corresponding to $\Delta = \frac{3}{2}, 2$ (equation 60).

  (ii) Inhomogeneous solution (right side $\sim$ R)-Curvature induced correction: this gives an inflaton mass correction (equation 59)

  (iii) Inhomogeneous solution (right side $\sim$ G_-^2)-Fluxed induced corrections: this gives expansion terms corresponding to $\Delta = 1, \frac{5}{2}$ (equation 52).

So the discrete spectrum of correction terms to the Inflaton potential are $\Delta = 1, \frac{3}{2}$,2 and $\frac{5}{2}$. Further terms are encountered in the Baumann's paper but these are of a similar nature and format. The eta problem and its expression must be evaluated for each of the contributing correction terms to ensure Inflation conditions are compliant.