On The Asymptotic Safety Scenario in Quantum Einstein Gravity

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Abstract

In this study the key features and consequences of the asymptotic safety scenario are highlighted as a possible means to renormalising quantum gravity in four dimensions. The Wilsonian renormalisation scheme is used to deal with the short distance fluctuations in the canonical field $\phi(q)$, where the pathological non-renormalisability of Einstein-Hilbert theory arises. The resulting renormalisation group flow equation analyses the implications of a non-trivial fixed point in the ultraviolet. It also links the macroscopic Einstein-Hilbert theory at large distances with a microscopic action at short distances. Though the renormalisation group equation is exact, it is solved numerically. Furthermore, a given approxiamtion scheme can be optimised by an appropriate choice of regulator, a term added to the coarse-grained effective action that acts as a momentum cut-off. Finally, the nature of the critical exponents are investigated at the Gaussian and the interacting fixed point with regards to finding an optimised trajectory connecting the infrared and ultraviolet physics.

Contents

1 Introduction

In reaching a unified theory of nature, one important challenge that remains in physics is to overcome the problem of merging gravity with quantum physics satisfactorily. Ultimately, the first step in the direction of a true Grand Unified Theory would be the union of quantum field theories, such as QED [1] and classical general relativity. Presently, the quantum field theoretic formalism in the form of the quantum gravity [2] is one such means of achieving such a union. However, the problem is that quantum gravity as a field theory suffers from pathological non-renormalisability [3]. In field theories such as QED, loop expansions in the Green's function give rise to divergent terms which may be regulated by introducing a physical cut-off. Adding a cut-off amounts to compensating for screening effects of self-energy propagation in the ultraviolet. This is could be due to having no a priori knowledge of the nature of interactions occuring at high energy. Another means is to compensate the Lagrangian directly by introducing extra terms countaining interactions that ought to cancel out the divergences at high energy. Theories that exhibit compliance to these rules are said to be **perturbatively renormalisable**, good examples being QED and non-abelian gauge theories like QCD [2]. Quantum gravity does not respond well to such means of renormalisation, however, it is possible there exists a scheme by which it may be renormalised non-perturbatively. This was suggested by S. Weinberg [2] early on as an **asymptotic safety** scenario, in a similar but distinct manner in which non-abelian gauge theories like QCD are asymptotically free.

The area of work covered in this article follows from studies done by [4], [5] [6], [7] and [8]. The studies focus on an aspect of asymptotic safety that involves stable fixed points under renormalisation scaling in four dimensions. Extensive discussions on asymptotic safety involve $2+\epsilon$ dimension expansion [9, 10], higher derivative expansion in terms of the Lagrangian [11],4+ higher dimensional analysis and dimensional reduction [8, 12]. By appropriate choice of action one can study the evolution of fixed points using exact renormalisation group equations [13, 14, 15, 16, 17, 11, 18, 19, 20], there are also intimate ties to thermal and statistical many-body physics [21, 22, 23], insinuated by analogies between the partition function and functional path integral [2.2]. There also exists a physical analogy between universality classes [21] and the onset of quantum fluctuations at a sub-Planckian level. To corroborate this fact, we treat that classical general relativistic physics as a low energy effective theory of some microscopic bare theory, so that treatment by an appropriate choice of renormalisation group retrieves the fundamental action which describes low energy physics, albeit a truncated Einstein-Hilbert action [15]. The microscopic theory used here shall be described by the coarse grained effecive action $\Gamma_k[\phi]$, while the effective action will be treated in the R truncation, that is, only terms first order in the Ricci scalar are considered in the action. By carefully restricting transformations under the renormalisation group, it is not unreasonable to assume scale dependence interpolates between these two limits, and that this makes quantum gravity quasi-renormalisable in the very least.

2 Renormalisation Group

2.1 Perturbative renormalisation

Dimensionally, it is clear that quantum gravity in 3+1 dimensions with Newton's constant G_N, will have canonical dimensions of $[G_N] = -2$ due to the units $G_N = 6.7 \times 10^{-39} GeV^{-2}$. The fundamental problem with quantum Einstein gravity (QEG) emanates from loop divergences. As it turns out, in its first order formalism QEG is not perturbatively renormalisable [24]. In this section renormalised perturbation theory is illustrated for the case of 'phi to the fourth theory $\frac{\lambda}{4!}\phi^4$' in QED†. Ideally one wants to calculate loop diagram integrals which are derived from the expansion of the connected n-point Green's function $\mathcal{G}_n(p_1, p_2, ..., p_n)$. However, for theories like $\lambda\frac{\phi^4}{4!}$, and indeed quantum gravity, it is here that divergences are manifested. This pathological behaviour is a result of integrating over undefined momenta p in the loop diagrams, where $\int \frac{d^d p}{p^n} \sim \int \frac{p^{d-1}}{p^n} dp$ yields a logarithmic divergence for $d = n$, and an order $\sim p^{d-n}$ divergence when $d > n$. This issue can be resolved by regulating the integral with a cut-off Λ, which imposes a finite value on $\mathcal{G}_n(p_1, p_2, ..., p_n)$. For the purposes of this argument it is easier to work with the two-point correlator $\mathcal{G}_n(p_1, p_2)$, which is written as a power series in $\Gamma_2(p^2)$‡, its one particle irreducible (1PI) correlator

$$\mathcal{G}_n(p_1, p_2) = \text{finite piece} \times \sum_n \left(\frac{1}{p^2 - m_0^2} . \Gamma_2(p^2) \right)^n \sim \frac{1}{p^2 - m_0^2 + \Gamma_2(p^2)} \tag{2.1}$$

The 1PI correlator $\Gamma_2(p^2)$ is regulated (i.e. it depends on Λ and is finite), which ultimately leads to a finite reduced correlator $\mathcal{G}_n(p_1, p_2)$. However, the pole of the propagator $\mathcal{G}_n$ is shifted due to the presence of $\Gamma_2(p^2)$ in the denominator. Any observable mass will correspond to the new pole, where $p^2 - m_0^2 + \Gamma_2(p^2) = 0$. As a result, the renormalised mass m_r is defined in terms of m_0 and $\Gamma_2(p^2)$, where $m_r = m_0 + \Gamma_2(p^2)$. On the other hand, by relating experimental scattering amplitudes to appropriate 1PI funcions, an expression for the renormalised coupling λ_r may be obtained. Finally, by series expanding $\mathcal{G}_n(p_1, p_2)$ in powers of $\Gamma_2(p^2)$ around $k^2 = m_r^2$,

† The term λ is the interaction coupling constant in bosonic quantum field theory. Later, λ will also be used to counter-balance matter attraction in the form of dimensional cosmological constant [2]

‡ The 1PI correlators represent Feynman diagrams that cannot be split into products of propagators. By ensuring that $\Gamma_2(k^2)$ is regulated, any Green's function that decomposes into products containing it can be taken for granted as being regulated.

the n-point correlator can re-expressed as,

$$\mathcal{G}_n(p_1, p_2) = \text{constant} \times \frac{Z_\phi}{p^2 - m_r^2} + \mathcal{O}(k^2) \tag{2.2}$$

Due to the renormalisation factor Z_ϕ in the numerator, the residue of the pole in $\mathcal{G}_n$ is not one. As a consequence, the self-scattering of the field must be ensured, in other words, $\langle\phi|\mathcal{G}_n(p_1, p_2)|\phi\rangle = 1$. Thus, the canonical field rescaling is introduced for both ϕ and $\mathcal{G}_n(p_1, p_2)$ whereby $\phi = \sqrt{Z_\phi}\,\phi_r$ preserves normalisation. In short, the terms m_r, λ_r and Z_ϕ are obtained from the shifted residue in $\mathcal{G}_n(p_1, p_2)$, experimental scattering amplitudes and field rescaling respectively. The original bare mass term m_0 in $\mathcal{G}_n$, also appears in the Lagrangian, along with the canonical field ϕ and bare the coupling λ_0,

$$\mathcal{L} = \frac{1}{2} Z_\phi \partial_\mu \phi_r \partial^\mu \phi_r + \frac{1}{2} Z_\phi m_0^2 \phi_r^2 + Z_\phi^2 \lambda_0 \frac{\phi^4}{4!}. \tag{2.3}$$

Unfortunately, both bare and renormalised terms now appear in the Lagrangian. Ideally, we would want the Lagrangian to describe a theroy in terms of only renormalised parameters, so that those parameters correspond to experimentally observable quantities. This problem is resolved by treating the terms Z_ϕ, $Z_\phi m_0^2$ and $Z_\phi^2 \lambda_0$ perturbatively such that, $Z_\phi = 1 + \delta Z_\phi$, $Z_\phi m_0^2 = m_r^2 + \delta m$ and $Z_\phi^2 \lambda_0 = \lambda_r + \delta\lambda$. Substituting into equation [2.3], splits the Lagrangian into two parts, its renormalised part and its **counterterms**,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_r \partial^\mu \phi_r + \frac{1}{2} m_r^2 \phi_r^2 + \lambda_r \frac{\phi^4}{4!} + \underbrace{\frac{1}{2} \delta Z_\phi \partial_\mu \phi_r \partial^\mu \phi_r + \frac{1}{2} \delta m \phi_r^2 + \delta\lambda \frac{\phi^4}{4!}}_{\mathcal{L}_{\text{counter}}} \tag{2.4}$$

The absorption of divergences coming from the bare theory into new counterterms in the Lagrangian is referred to as **renormalised perturbation theory**. These counterterms, which can be treated as interactions, give rise to new Feynman diagrams in the loop expansion. By virtue of this, gravity is renormalisable in only a very limited sense, that is, when the gravitational Lagrangian is linear in the Ricci curvature R, the theory is finite in the first order of the loop expansion [24]. However, at second order in the loop expansion the theory is already divergent. Furthermore, when coupled to matter fields the theory is entirely pathological, producing divergences immediately at first order in the loop expansion. Adding terms [3] quadratic in the Ricci scalar R^2 and the Weyl tensor $W(R_{\mu\nu}R^{\mu\nu}, R^2)$, absorbs the first order loop diveregence in the presence of arbitrary matter fields. The downside being the inclusion of terms quadratic in the metric $g_{\mu\nu}$ [25], which correspond to inpertinent degrees of freedom. Thus, it would seem reasonable to think that if quantum gravity is non-renormalisable perturbatively, the challenge remains to determine if it is non-perturbatively renormalisable.

2.2 Wilsonian Renormalisation

Strictly speaking, there are two cut-offs employed in the Wilsonian renormalisation scheme (WRS). The physical cut-off Λ, which is used to isolate any divergences occuring in the ultraviolet $\phi(q \sim \Lambda)$, and the coarse graining k, which is used to supress fields with momenta $\phi(q < k)$. The coarse-graining is used to define a unique point at which relatively large oscillations in $\phi(q)$ first occur, and it is usually in the vicinity of the physical cut-off. In addition, there are unique physical systems whereby it is useful to think of the cut-off in terms of space-like separations and not energies. This can be seen through an anology between quantum field theory and statistical thermodynamics. In relativistic field theory, the reduced Green's function $\mathcal{G}_n$ can be constructed by taking successive derivatives of the generating functional $Z[J]$. The actual integral is in terms of the Fourier transformed $\phi(q)$, since Λ is wanted in terms of energy not distance, therefore,

$$Z[J] = \int_\Lambda \mathcal{D}\phi \; e^{i \int [\mathcal{L} + J\phi] \; d^d x} \, . \tag{2.5}$$

When defining a momentum range for $Z[J]$ in Minkowski space, the statement $k < \Lambda$ may lead to ambiguity for on-shell momenta. This ambiguity can be resolved by performing a Wick rotation on the path integral. Originally, $Z[J]$ is written in terms of a complex action in Minkowski space, but it can be translated to Euclidean space by employing a complex time variable $t \to -i\tau$. The Wick rotation relates a dynamic $n-1$ dimensional field theory in real time, to a static n dimensional statistical mechanical problem in complex time, by trading one time-like variable for a space-like one. In Euclidean space,

$$Z_\Lambda = \int_\Lambda \mathcal{D}\phi \; e^{-S[\phi]} \tag{2.6}$$

which is reminescent of a four dimensional partition function of the form $Z \sim \int e^{-\beta H}\, dE$. For example, in condensed matter systems the physical cut-off is the atomic spacing, so there can be no spin excitations between atoms. This would be the analogue for Λ in a field theoretical sense, though in such a case the cut-off would be in units of distance $1/\Lambda$ not energy. In a completely uncorrelated thermodynamic system, like a demagnetised ferromagnet, all fluctuations are of the order of a few atomic spacings. However, close to criticality T_c (the temperature at which the material becomes magnetised), the system may exhibit corrleations well beyond the atomic spacing $1/\Lambda$. For the modified Einstein-Hilbert action, k plays the role of such criticality. As another example, the scattering between phonons and electrons in a superconductor leads to an effective attraction between pairs of electrons. When the superconductor's temperature falls below T_c [21], they effective attraction results in the formation of bound Cooper pairs. The wavefunction order parameter $\psi(x)$ represents a measure of the superconducting phase. It exhibits the behaviour of non-locality as $T \to T_c$. Eventually, $\psi(x)$ extends uniformly throughout the material, except for a small region at the boundaries (called the penetration depth). This non-local behaviour of $\psi(x)$, over a narrow band of temperature close to T_c, would be analogous to the onset ultraviolet divergences in WRS.

Having defined the Wick rotated functional in Euclidean space, a suitable change of variable $\phi(q) \to \phi(q) + \phi_k(q)$ allows $Z_\Lambda[J]$ to be split into two parts giving,

$$Z_\Lambda = \int_{0<q<k} \mathcal{D}\phi \int_{k<q<\Lambda} \mathcal{D}\phi_k \; e^{-S[\phi+\phi_k]} \,. \tag{2.7}$$

The momentum dependence of the new variables $\phi(q)$ and $\phi_k(q)$ is as follows:

$$\phi(q) = \begin{cases} \phi(q), & \text{when } q < k \\ 0, & \text{when } q > k \end{cases}$$

$$\phi_k(q) = \begin{cases} 0, & \text{when } q < k \\ \phi_k(q), & \text{when } q > k \end{cases}$$

The pathological nature of ϕ_k can be isolated by perturbative means, allowing for the expansion of the exponential in Z_Λ in powers of ϕ_k. The scale dependence, otherwise mediated by the modes $\phi_k(q)$, and preserved in the evaluation of the integral over $\sim \int \mathcal{D}\phi_k$ is now contained

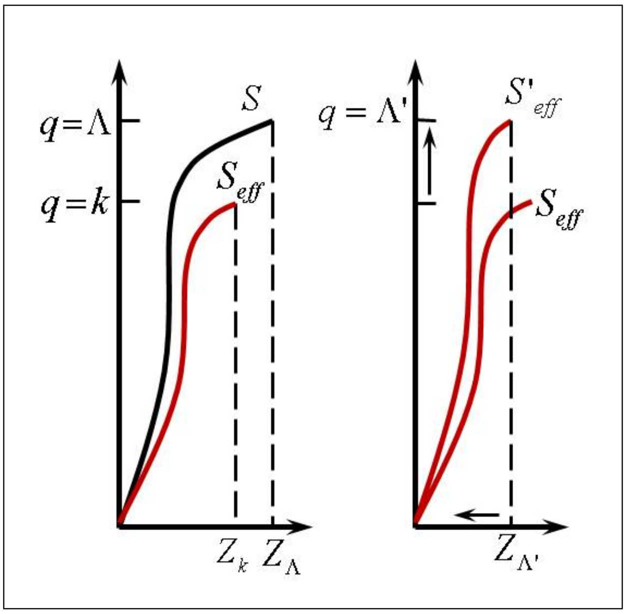


Figure 1: Illustration showing $Z_{0<q<\Lambda} \to Z_{\text{eff}}$ as a function of momentum scale. The initial action S is evaluated up to k with the domain $k < q < \Lambda$ added on as a perturbation (left). As a result, the new action describes a theory only up to k. The resulting transformations 'squeezes' (right) the length-scale and 'stretches' the momentum range to match a theory describing the entire domain S'_{eff}.

in ΔS_k. Therefore, the remainder of the functional is written in terms of an effecvtive action $S_{\text{eff}}[\phi] = S[\phi] + \Delta S_k$,

$$Z_k = \int_{0<q<k} \mathcal{D}\phi \; e^{-(S[\phi]+\Delta S_k)} = \int_k \mathcal{D}\phi \; e^{-S_{\text{eff}}[\phi]}. \tag{2.8}$$

The UV divergences have been absorbed into couplings defined in a new effective Lagrangian $S_{\text{eff}} = \int \mathcal{L}_{\text{eff}} \; d^d x$, where

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}[q, \phi(q), \phi_k(q), C] \tag{2.9}$$

and C represents a generic coupling the Lagrangian. In the expression for S_{eff}, the cut-off ΔS_k is used as a regulator in the truncated Einstein-Hilbert action later on in section [4.1]. The effective action is what determines the behaviour of fields at arbitrarily low energies in the EH formalism. However, by comparing equations [2.6] and [2.8], it is clear that all fields and couplings must be re-parametrised in order to offset the scale difference between the generating functional $Z_{q=\Lambda}$ and the coarse-grained functional $Z_{q=k}$. This can be done through a change of variable $q' \to \frac{q\Lambda}{k}$, a re-scaling $\phi' \to f(\frac{k}{\Lambda}, \delta Z)\phi$ and a re-coupling $C' = g(\frac{k}{\Lambda}, \delta Z)(C+\delta C)$. The term δZ_ϕ stands for the field renomalisation, δC comes from loop contributions in the expansion of $e^{-\Delta S_k}$ and $\frac{k}{\Lambda} = b$ is the dimensionless scale dependence. Strictly speaking, it is the function $f(b, \delta Z)$ that acts as the true field renormalization for ϕ, while $g(b, \delta C)$ is the transformation law for a generic coupling C. This transformation law is what is referred to as the **renormalisation group tranformation**. The strength of this approach lies in the ability to determine infinitesimal transformations of the coupling, which correspond to integrating over thin momentum shells. This gives rise to a differential equation of the form

$$\frac{dC}{d \ln b} = f(C) \tag{2.10}$$

The flow equation underlines the scale-dependent behaviours (or family of trajectories) of couplings in the Lagrangian, and leads to the Callan-Symanzik equation (section 3.4).

3 Quantum Einstein Gravity

3.1 Asymptotic safety

Singularities arising from sub-Planckian quantum fluctuations of the canonical field $\phi(q)$, are systematically absorbed via iterative integrations of the path integral functional $Z[J]$. Each time the integral over the momentum shell $k < q < \Lambda$ is carried out, the couplings in the original action or Lagrangian can be redefined in terms of a new effective action $S_{\text{eff}}[\phi]$ or new effective Lagrangian $\mathcal{L}_{\text{eff}}$. The momentum cut-off Λ is mandatory, without it divergences will occur due to contributions from field modes with momenta approaching the ultraviolet limit can be taken to infinity. Moreover, it defines an energy-scale above which the physics described by the original action is no longer reliable, albeit due to self-energy or screening effects [1] occuring in the ultraviolet. However, some theories like QCD possess aymptotic freedom, which means that there need not be a cut-off imposed on the upper limit of integration, in other words $\Lambda \to \infty$ (though QCD is asymptotically free in the infrared). The reason for this is that the flow or trajectory for such theories approaches a stable point (QED is believed to possess one such stability point in the ultraviolet). By the same token, the aim is to remove the cut-off in the truncated EH formalism altogether by taking $\Lambda \to \infty$. This can be made possible only if there exists a fixed point in the ultraviolet limit. A fixed point represents the invariance of a theory under renormalisation group transformations. The universal Gaussian fixed point is a simple fixed point corresponding to an uncoupled kinetic theory with $\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi$. Given, $\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi + \delta m^2 \phi^2 + \cdots$ where $\delta m \ll 1$, an arbitrary coupling constant will either converge or diverge away from the fixed point depending on whether the coupling is renormalisable or supereonrmalisable. The mass coupled Lagrangian will diverge away from the trivial Gaussian point, for this reason it is essential that there be a non-trivial fixed point to impede the flow into the ultraviolet. Evidence for such a fixed point is the main purpose of this article, following on from studies made in [4],[5],[6] and [8].

The generalised criteria, first introduced by Weinberg [2], for dealing with the non-perturbative renormalisability of theories such as QEG, is referred to as **asymptotic safety**. These requirements can be summarised as follows:

(i) As mentioned previously, there must be a non-trivial fixed point in the ultraviolet. This means that in the space of all possible theories spanning close to $g_* \neq 0$, there exists at least one trajectory that approaches the fixed point asymptotically.

(ii) There must also be an interpolarity between the familiar long-distance action which takes effect at low energy, and the microscopic effective action in the ultraviolet limit. Thus, the two fixed points can be connected via an explicit renormalisation group flow.

(iii) Finally, a finite number of unstable trajectories towards low energy is required in the vicinity of the fixed point, in order that only a finite number of parameters need to be fine-tuned experimentally. If not, an arbitrary point away from the trivial or non-trivial fixed point $g_* \neq 0$ will define an unstable flow away towards the ultraviolet (see section [5.1]).

3.2 Quantum effective action

In classical thermodynamics, a pair of conjugate variables such as pressure and temperature, and their respective thermodynamic potential, can be exchanged for a different pair of conjugate varibales, like the entropy and volume, via a Legendre transformation[1]. Likewise, one can relate energy density functional $W_k[J]$ in the terms of its current density, to the effective action $\Gamma[\phi]$ in terms its canonical field. The energy functional is related to the Wick rotated Schwinger functional [4] via $Z_k[J] = e^{W_k[J]}$. As a result, the effective action has the same boundary conditions as a canocical action in the absence of sources, $\left.\frac{\delta\Gamma[\phi(x)]}{\delta\phi(x)}\right|_{J(x)=0} = 0$. The source modified Schwinger functional is defined as,

$$\exp W_k[J] = \int \mathcal{D}\phi \, \exp\left(-S[\phi] - \Delta S_k + \mathrm{Tr}\left\{\phi^\dagger J\right\}\right) . \tag{3.1}$$

The trace on the source term represents an integral over momenta. The added ΔS_k contains the **regulator** which acts as a Wilsonian cutoff, $\Delta S_k \sim \int \phi^\dagger R_k \phi$. It has the following properties: (i) R_k vanishes when $k \to 0$; (ii) it is finite when $q \approx 0$; (iii) R_k diverges as $k \to \Lambda$. The Legendre transformation leads to a coarse grained effective action defined as,

$$\Gamma[\phi] = -W_k[J] - \Delta S_k[\phi] + \mathrm{Tr}\left\{\phi^\dagger J\right\} , \tag{3.2}$$

The structure of equation [3.1] is similar to that of equation [2.8], with the exception of the source term and the nature of the added cut-off. Whereas the cut-off used in WRS is a sharp cut-off in the form a Heaviside function, the regulator $R(q^2)$ is smoothly varying function with boundary conditions that mimic the behaviour of the sharp cut-off. Moreover, the fundamental difference in the phenomenological effects of the effect action used in [3.1] comes from the utilisation of the limits $\Lambda \to \infty$ and $k \to 0$, the latter by of virtue of the asymptotic safety and the former based [4], [5] and [6]. The idea is to ensure that at low energies the truncated Einstein-Hilbert action is retrieved [26, 27], while at high energies the theory is controlled by the UV fixed point, rendering the cut-off Λ redundant. An alternative view would be to consider the Einstein-Hilbert action as an effective theory, which is retrieved only by performing Wilsonian integrals on some **ansatz** fundamental theory. If the WRS were to be employed, the quantum corrections from such a theory, resulting from integrating out $\phi_k(q)$ modes between $k < q < \Lambda$, would not suffice if the Einstein-Hilbert action were to be reproduced as an effective action. This is merely due to the fact that EH theory is a low energy effective theory. Only an integral over all momenta, or low momenta in the very least, would yield the EH action as an effective theory. This is the reason for the boundary conditions $k \to 0$ and $\Lambda \to \infty$ employed in [4]. If such conditions were enforced then, $\Gamma[\phi]$ would represent a full quantum effective action with interpolariting between ultraviolet and infrared, in contrast to the Wilsonian scheme whereby the effective action only encodes quantum information in a thin momentum shell.

3.3 Einstein-Hilbert action

In order to determine $Z_k[J]$ in terms of the relevant degrees of freedom, which would correspond to the metric field, the path integral must be written in terms of the graviton field. Since the metric encodes these symmetries, it is useful to define the functional $Z[\eta_{\mu\nu}]$, where $\eta_{\mu\nu}$ is spacetime dependent in its most general realisation. Thus, when dealing with gravity the generating function is determined using the background field formalism [13, 14], where ϕ is replaced with $\eta_{\mu\nu}(x^\alpha)$ and $\eta_{\mu\nu}(x^\alpha) = \bar{g}_{\mu\nu}(x^\alpha) + h_{\mu\nu}(x^\alpha)$, so that

$$Z[\eta_{\mu\nu}] = \int \mathcal{D}\eta_{\mu\nu} \; \exp\left(-S[\eta_{\mu\nu}]\right) . \tag{3.3}$$

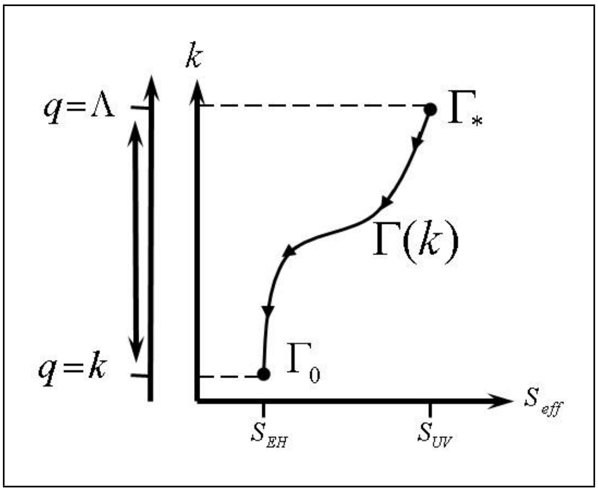


Figure 2: Diagrams $\Gamma_{EH} \to \Gamma_{UV}$

Here $\overline{g}_{\mu\nu}(x^\alpha) = \langle g_{\mu\nu} \rangle$ is the background metric, and $h_{\mu\nu}$ is the non-trivial fluctuation in the field. The combined invariance of $h_{\mu\nu}$ and $\overline{g}_{\mu\nu}(x^\alpha)$ under conformal transformation, allows the functional to be gauge fixed by adding the gauge condition $F_\mu(\overline{g}, h) = 0$ and the Fadeev-Popov ghost action $S_{\text{gh}}[h, \overline{g}, C, \overline{C}]$. Here $F_\mu(\overline{g}, h)$ is defined as some construct in terms of covariant derivatives, and C and $\overline{C}$ are the Fadeev-Popov fields. The modified Einstein-Hilbert action [14, 15, 16, 17, 6, 8] is given as

$$\Gamma[\phi] = \int d^d x \sqrt{g} \left(\frac{1}{16\pi G_N} (-R + 2\Lambda) + S_{\text{gf}} + S_{\text{gh}} + S_{\text{source}} \right), \tag{3.4}$$

where Λ and G_N are the unrenormalised cosmological and gravitational constant, S_{gf} is the gauge fixing action

$$S_{\text{gf}} = \frac{1}{2\alpha} \int \mathcal{L}_{\text{gf}}\, d^d x; \qquad \mathcal{L}_{\text{gf}} = \sqrt{\overline{g}} \overline{g}_{\mu\nu} F^\mu F^\nu, \tag{3.5}$$

S_{gh} is the ghost action

$$S_{\text{gh}} = -\frac{1}{\kappa} \int d^d x \overline{g}_{\mu\nu} \overline{C}^\mu \frac{\partial F_{mu}}{\partial h_{\mu\nu}} \mathcal{L}_{\text{C}}(\overline{g}_{\mu\nu} + h_{\mu\nu}) \tag{3.6}$$

and S_{source} is the source term. It is convenient to have $\Gamma[\phi]$ in terms of the metric and the metric's average expectation, so $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = \overline{g}\mu\nu + \overline{h}_{\mu\nu} \,. \tag{3.7}$$

where $\overline{h}_{\mu\nu} = \langle h_{\mu\nu}\rangle$ is the expectation in the fluctuation, and $g_{\mu\nu}$ is the expectation in the full metric $\langle \eta_{\mu\nu}\rangle$. The Legendre transform Γ_k of the energy functional $W_k[J]$ implies that Γ_k depends on $\overline{h}$ and $\overline{g}$, as a result of which is the redefinition of $\Gamma[g;\overline{g}]$ by the change of variable $\overline{h} = g - \overline{g}$. Subsequently, the effective action in terms of g only is obtained in the limit, $\Gamma[g] = \lim_{k\to\infty} \Gamma[0,\overline{g}]$.

3.4 The Callan-Symanzik Equation

The canonical dimension of an interaction term in any theory is evaluated in units of $[m]$, where $[m^d] = d$. The action has zero mass dimensions $[S] = \left[\int d^dx\, \mathcal{L}\right] = 0$, therefore the Lagrangian must have dimensions $[\mathcal{L}] = -d$. Using equation [3.4], the gravitational coupling's canonical dimension is determined to be $[G_N] = 2-d$, which is negative for $d > 2$. Under the renormalisation group, couplings with positive mass dimensions are referred to as essential (super-renormalisable) couplings. They correspond to unstable trajectories under the RG transformations, that is, they grow with every iteration of the re-scaling. Whereas, an inessential (non-renormalisable) coupling with negative mass dimension has a stable or diminishing trajectory, and is also responsible for the non-renormalisabilty of quantum gravity in $d > 2$ dimensions. In two dimensions $[G_N] = d_{\text{cr}}$ the coupling is said to be marginal (renormalisable), that is, invariant under RG transformations. All couplings can be written in terms of a general vector $\vec{g}$, so that only the attribute of $\vec{g}\,'(\Lambda', G', ...) \to \vec{g}(\Lambda, G, ...)$ under RG transformations is relevant. The uncoupled kinetic theory always represents a trivial fixed point under the renormalisation group where $\vec{g}_0 = 0$, and the non-trivial point is denoted by g_* . A mathematically succint way of describing trajectories of the vector $\vec{g}$ is the Callan-Symanzik beta function

$$\beta_{\vec{g}} \equiv k\frac{\partial \vec{g}(k)}{\partial k} = \frac{\partial \vec{g}(k)}{\partial \ln k}. \tag{3.8}$$

At a given vector point in the vector field defined by $\vec{g}$, $\beta_{\vec{g}}$ defines a vector in the direction of $\ln k$, while its zeros indicate where the fixed points of a theory lie. The nature of both fixed points is investigated following arguments in [4],[5], [6] and [8]. First, the scale dependent renormalised coupling is introduced $G(k) = Z_{\text{gr}}^{-1}(k)G$. The graviton is the carrier of the metric degrees in QEG, so $Z_{\text{gr}}(k)$ is the graviton wave function renormalisation factor and k is the momentum scale dependence. Subsequently, the dimensionless gravitational coupling is given as $g(k) = k^{(d-2)}G(k)$. For small values of $k = k_0 \ll \Lambda$ (corresponding to low energy physics), the gravitational coupling $G(k_0)$ behaves like the dimensionful Newton's constant G_N. Conventionally, in a field renormalisation where $Z_{\text{gr}}(k)$ is introduced, the anomalous dimension η must also be introduced in order to take in to account the mixing of kinetic and potential terms in the curvature scalar, so η is given as

$$\eta = -k\frac{\partial}{\partial k}\ln Z_{\text{gr}}(k). \tag{3.9}$$

In terms of the dimensionless coupling g, the Callan-Symanzik equation reads as

$$\beta_g = k\frac{\partial g(k)}{\partial k} = (d-2+\eta)g(k) \tag{3.10}$$

The anamalous dimension plays an important role in compensating for the shift in the critical exponent of the field at the fixed point. Since there is no rescaling of the field at the Gaussian fixed point, then $\eta = 0$ when $g = g_0$. However, when $g = g_*$ the anomalous dimension must be equal to $2-d$ for the beta function to vanish. Given that $g(k_0) \to g_0$ when $k = k_0$, $\eta \approx 0$ and so the beta function simplifies to $\beta_g \approx (d-2)g(k_0)$. This shows a unique fixed point at $g_0 = 0$, implying that the Gaussian fixed corresponds to the low-energy Einstein-Hilbert theory. Alternatively, as $k \to \Lambda$ the dimensionful coupling $G(k) = g(k)/k^{d-2}$ diminishes indicating weak coupling in the space of theories close to g_* . It is also noteworthy that as $k \to 0$, $G(k)$ diverges, isinuating possible modifications to extremely long range effects of gravity.

4 Group Flow Equations

4.1 Properties of the flow equation

The standard action used in this study is the much simpler truncated Einstein-Hilbert action [14, 15, 16],

$$\Gamma_k[\phi] = \Gamma[\phi] = \frac{1}{16\pi G_N} \int d^d x \sqrt{g}\,(-R + 2\Lambda) \tag{4.1}$$

governed by the flow equation [4, 5, 22, 19]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left\{ \frac{1}{\Gamma_k^{(2)} + R_k} \right\} \partial_t R_k \tag{4.2}$$

where $-R$ is the Ricci scalar corresponding to metric degrees of freedom in equaton [4.1]; $t = \ln k$, $\Gamma_k^{(2)} = \delta^2 \Gamma_k / \delta\phi(p)\delta\phi(q)$ in equation [4.2] and the trace denotes an integral over all fields and momenta. Though equation [4.2] is an **exact** differential equation, it is typically solved numerically. Analyses on the exact renormalisation group equation (ERGE) are given in [11] and [17]. In this article only the simplified coupled system in the form of two beta functions is solved numerically.

(i) Since the flow equation is exact and well behaved, it can be viewed as a smooth path running between the ultraviolet physics and low energy classical general relativity [4].

(ii) The right hand side of the equation of motion given in [4.2] vanishes in two regions of momenta; (i) for small momenta when $q^2 \ll k^2$ and $R \sim 0$; (ii) for large momenta when $k^2 \ll q^2$. On the other hand, it is peaked for momentum values in the range $q^2 \approx k^2$. This is due to a unique normalisation condition enforced on the regulator at $R_k(q^2 \approx k^2)$.

(iii) In section 5.1, rather than work with the flow equation dirtectly, the interacting fixed points are investigated at the low-energies and high energies. Since $\Gamma_k[\phi]$ encodes information about both, it is possible to obtain reasonable approximations for the trajectories from the coupled equations for $\vec{g} = \vec{g}(g, \lambda, ...)$.

(iv) The reliability of the approximations depend on the choice regulator. Appropriate regulators are chosen based on optimisation techniques given in [7] and [26], where $R_k(q^2) = (k^2 - q^2)$ is used as the optimal cut-off for $\Gamma_k[\phi]$.

(v) Finally, the global symmetry of the theory must satisfy diffeomorphic invariance. The use of modified Ward-Takahashi identities ensures that the truncation of the EH action is still a valid approximation, in order that it may satisfy diffeomorphic invariance.

4.2 Regulator optimisation

Opimisation of the regulator term $\int d^dq\, \phi^\dagger(q) R_k(q)\phi(q)$ leads to the choice of regulator $R_k(q^2) = (k^2 - q^2)\,\theta(k^2 - q^2)$ [4, 26]. The regulator is a smoothly varying function [22], in contrast to the sharp cut-off used conventional the renormalisation scheme. This is important because the flow equation is solved by approximative means. It also affects how quickly one gets from the microscopic theory to the classical theory, due to the use of the truncated Einstein-Hilbert action. So given a particular functional form of $R_k(q^2)$, one can 'tweak' it in order to maximise convergence towards the physics in question [7, 26]. The following set of requirements are necessary to ensure interpolarity between the infrared and the ultraviolet regimes. Given the effective propagator of the form

$$P_{\text{eff}}(q^2) = \frac{1}{q^2 + m^2 + R_k(q^2)}, \tag{4.3}$$

the regulator is required to be finite $R_k(q^2) > 0$ for vanishing field $\phi(q = 0)$, or else P_{eff} will diverge for stationary massless modes. The next requirement is that when the cut-off is taken to zero the regulator should vanish, so that $\lim_{k\to 0} \Gamma_k[\phi] \to \Gamma_{\text{EH}}[\phi]$ desciribes classical general relativity. Finally, in order that $\lim_{k\to\Lambda} \Gamma_{\text{eff}}[\phi] \to \Gamma_{\text{UV}}[\phi]$, the regulator is required to diverge $R_k(q^2) \to \infty$, where Γ_{UV} is defined as the microscopic effective action.

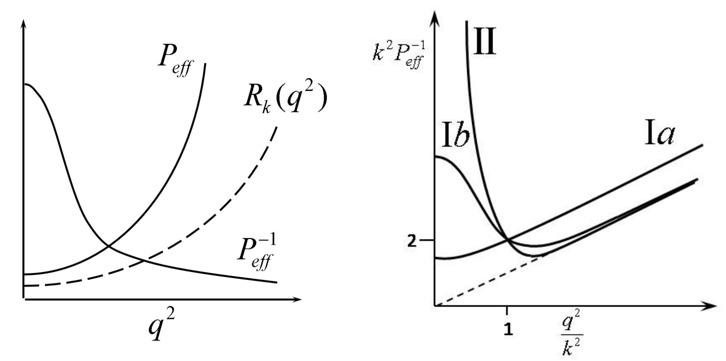
Further optimisation requires the introduction of a dimensionless inverse effective propagator and a dimensionless regulator[26], given as

$$\mathcal{P}_{\text{eff}}^{-1} \equiv \frac{1}{k^2 P_{\text{eff}}(q^2)} = \frac{q^2}{k^2} + \frac{m^2}{k^2} + \frac{1}{k^2} R_k(q^2) \tag{4.4}$$

and

$$R_k(q^2) = q^2 r\left(\frac{q^2}{k^2}\right) \tag{4.5}$$

respectively. Based on the three conditions on the regulator, a heuristic scale dependence of $P_{\text{eff}}^{-1} = R_k(q^2) + q^2 + m^2$ and $P_{\text{eff}} = [R_k(q^2) + q^2 + m^2]^{-1}$ is illustrated in figure 3. By looking at



(a) P_{eff} and $R_k(q^2)$

(b) Class Ia, Ib and II, showing the relative optimisations of massive and the massless modes. Note that all function coincide at $q^2 = k^2$

Figure 3

the behaviour of $\mathcal{P}_{\text{eff}}^{-1}$ as a function of $y = q^2/k^2$, different classes can be determined for different types of regulators[26]. The regulators at this stage are still undefined, $\mathcal{P}_{\text{eff}}^{-1}$ simply denotes the generalized form of the function at important values of q^2/k^2. Figure [3(b)] shows the three different functions of $\mathcal{P}_{\text{eff}}^{-1}$ for three unique classes of arbitrary regulators. At large momenta the inverse propagator's functional dependence linearises $\mathcal{P}_{\text{eff}}^{-1} \to q^2/k^2$ for all classes of $R_k(q^2)$.

as $q^2 \to 0$ $\quad q^2/k^2 \to 0$ $\quad \mathcal{P}_{\text{eff}}^{-1} \approx [R_k(q^2) + m^2]/k^2$

as $k^2 \to \Lambda$ $\quad R_k(q^2) \to \infty$ $\quad \mathcal{P}_{\text{eff}}^{-1} \approx R_k(q^2)/k^2$

as $q^2/k^2 \to \infty$ $\quad \mathcal{P}_{\text{eff}}^{-1} \approx q^2/k^2$

Such criteria causes a discrepancy in the relation between the regulators' and the effective propagators' scale dependencies. For $k \sim \Lambda$ and $q^2/\Lambda^2 \to 0$, the regulator $R_k(q^2)$ approaches it boundary value at a different rate than when $q^2/k^2 \to 0$ for $k \ll \Lambda$. This is resolved by imposing a normalisation condition on $R_k(q^2)$, such that $R_k(q^2 \approx k^2) = k^2$. Thus, different classes of $R_k(q^2)$ can be defined as functions of $\mathcal{P}_{\text{eff}}^{-1}$ against q^2/k^2 in a consistent way (illustrated figure 3(b)). With those conditions set in place, the optimised regulator is determined by choosing one whose minimum (shown by the curves defined on $\mathcal{P}_{\text{eff}}^{-1}$) is as large as possible. This suggests that of the three classes, class Ib is optimised with respect to class Ia for massive modes and class II is optimised for massless. The optimisation parameter C_{opt} is given for a several functions $r(y)$ in [22, 7], while the optimised regulator used for the quantum effective action $\Gamma_k[\phi]$ corresponds to $r(y) = \left(y^{-1} - 1\right) \theta(k^2 - q^2)$.

5 Analytical Fixed Points

In this section the behaviour of the two coupling constants appearing in equation [4.1] are investigated under the RG transformations. In order to do this, it is necessary to possess their Callan-Symanzik flow equations [4, 5, 6], which are given by the coupled beta function system given in section [5.1]. For an appropriate choice of the gauge fixing parameter $\alpha \to \infty$ [6], the gauge fixing term in [3.4] vanishes and the beta functions are given in the equations [5.4] and [5.3]. It turns out that in order for the coarse-grained action to obey modified Ward-Takahashi [11] identities, the interactions coming from the Fadeev-Popov sector must also be supressed. As final points, only the scalar part of the optimised regulator is used as $R_{\text{opt}} = (k^2 - q^2)\,\theta(k^2 - q^2)$ [26, 27], and the Einstein-Hilbert action is defined in the absence of matter fields $S_{\text{source}} = 0$. Certain numerical factors [6] namely, $\frac{1}{\alpha}$ have been supressed in the definition of the dimensionless couplings and the constant $c_d = (\sqrt{(4\pi)^{d-2}}\Gamma(\frac{d}{2}+2)$ in the definition of g in the beta functions, where $g \to g/c_d$.

5.1 Phase diagram

In d-dimensional Euclidean space, the gravitational and cosmological dimensionless couplings are given as

$$g_k = k^{d-2}G(k) = k^{d-2}Z_G^{-1}G_R(k), \tag{5.1}$$

$$\lambda = k^{-2}\Lambda_k, \tag{5.2}$$

where k is the scale dependence and d is the canonical dimension. The beta functions are

$$\beta_g = \frac{\partial g}{\partial t} = (d-2)g + \frac{2(d-2)(d+2)g^2}{2(d-2)g - (1-2\lambda)^2}, \tag{5.3}$$

$$\beta_\lambda = \frac{\partial \lambda}{\partial t} = -2\lambda + \frac{g}{2}d(d+2)(d-5) - d(d+2)g\frac{(d-1)g + \frac{1}{d-2}(1 - 4\frac{d-1}{d}\lambda)}{2g - \frac{1}{d-2}(1-2\lambda)^2}\,. \tag{5.4}$$

From β_g the anomalous dimension η can be determined,

$$\eta = \frac{\beta_g}{g} + 2 - d = \frac{(d-2)(d+2)g}{(d-2)g - (1-2\lambda)^2} \,. \tag{5.5}$$

The anomalous dimension is divergent when $g = g_{\rm cr} = \frac{1}{2}(1-2\lambda)^2/(d-2)$, this provides an explicit boundary $g < g_{\rm cr}$ within which the trajectory of dimensionless coupling is valid. The anomalous dimension vanishes for $d = 2$ and $g = g_0 = 0$, reinforcing the statement made in section [3.4] that gravity is power-counting renormalisable at the Gaussian fixed point and when $d = d_{\rm cr}$. Given the pair of equations in [5.4], there is ample information to determine the value of the interacting fixed point in the specified dimensions. The dimensionally invariant Gaussian fixed point is trivially $(0,0)$. The non-trivial fixed point can be obtained by equating β_g and β_λ to zero and solving the pair of equations simultaneously to eliminate g, leaving λ in terms of a quadratic equation in d. When $\beta_g = 0$, $g_*(\lambda) = (1/4d)(1-2\lambda)^2$, substituting this into β_λ yields,

$$\beta_\lambda = \frac{1}{4}(d-4)(d+1)(1-4\lambda-4\lambda^2) - 2d\lambda + \frac{d}{2}. \tag{5.6}$$

The roots of the quadratic equation in λ are substituted into $g_*(\lambda)$, to obtain

$$g_* = \frac{(\sqrt{d^2-d-4} - \sqrt{2d})^2}{2(d-4)^2(d+1)^2}, \qquad \lambda_* = \frac{d^2-d-4-\sqrt{2d(d^2-d-4)}}{2(d-4)(d+1)} \tag{5.7}$$

It is clear that the only real solutions occur when $d > 4$, however, in four dimensions the quadratic piece in equation [5.6] vanishes simplifying its solution to $(g_*, \lambda_*) = \left(\frac{1}{64}, \frac{1}{4}\right)$. Furthermore, for simplicity in forthcoming linearisation analyses, the couplings and the coupled beta functions are redefined as the following binary vectors,

$$\vec{g} = \begin{pmatrix} g \\ \lambda \end{pmatrix}, \qquad \vec{\beta} = \begin{pmatrix} \beta_g \\ \beta_\lambda \end{pmatrix} \tag{5.8}$$

Finally, the coupled beta functions are also evaluated in four dimensions,

$$\beta_g = 2g + \frac{24g^2}{4g - (1-2\lambda)^2} \tag{5.9}$$

$$\beta_\lambda = -2\lambda - 12g - \frac{24g[6g + (1-3\lambda)]}{4g - (1-2\lambda)^2} \tag{5.10}$$

With the above definitions it is now convenient to analyse the flow of the couplings under RG

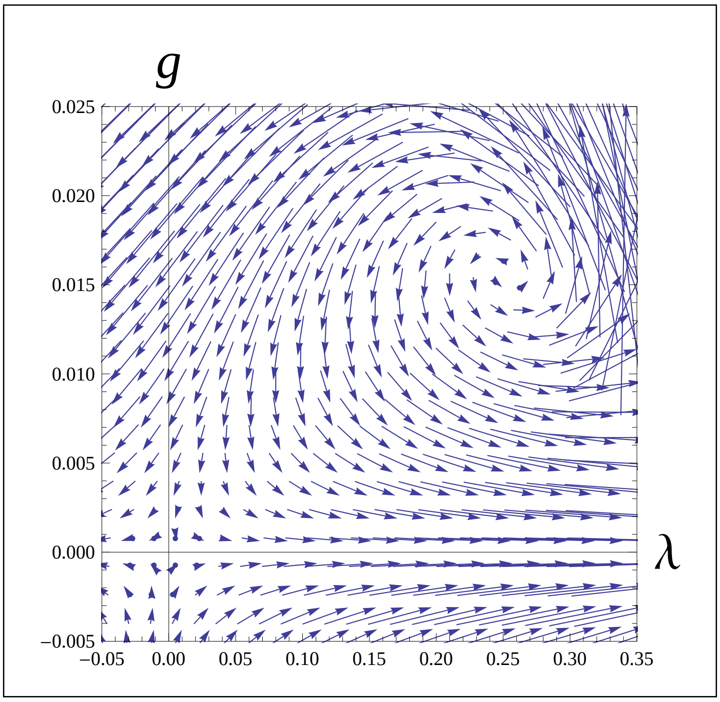


Figure 4: Illustration of the vector field $\vec{\beta}$. Values of the $\vec{\beta}$ were evaluated over a carefully chosen range to ensure that $g < g_{\rm cr}(\lambda) = \frac{1}{4}(1-2\lambda)^2$. The data was obatined by taking a set grid points over a range $0 < g < 0.025$ and $0 < \lambda < 0.34$ using step-sizes of 0.0005 and 0.005 respectively.

transformations. The quantity $\vec{\beta}$ indicates the direction of the flow at given point on the vector field defined by $\vec{g}$. The convention used here represents the transition from the high energy microscpic limit $k \to \infty$ to the infrared $k \to 0$, as a result $\vec{\beta}$ has been defined on the negatives of the beta functions $(-\beta_g, -\beta_\lambda)$. The vector field clearly manifests interesting structure in the vicinity of $\vec{g} \approx (0,0)$ and the interval $\vec{g} \approx ([0.014, 0.017], [0.24, 0.28])$, corresponding to $g = g_0$ and $g = g_*$ respectively. Its behaviour close to the fixed points is shown more detail in figures [5(a)] and [5(b)]. Moreover, it can be seen in the relativie sizes of the vectors that the flow diverges as one approaches the forbidden region $g_{\rm cr}$. Strictly speaking, since the flow vanishes at the fixed points they can only reached by the trajectories asymptotically. If the directions of the flows were positivley outgoing in the usual sense (that is emanating away from the trivial fixed point and towards the non-trivial), then it would be reasonable to assume that a well chosen tracjectory $\vec{g}$ would safely approach the ultraviolet asymptotically. This well chosen path is in fact called the separatrix line[4, 5]. It is also worth pointing out that in the region of $g \ll 1$, the vector field lies solely in the direction of λ. It would seem that in this limit λ's behaviour is consistent with that of an essential coupling. Furthermore, the regions of the vector field for which $g < 0$ and $g > 0$ are completely isolated. In other words, no trajectory lying above the origin can end up below it, a fact also highlighted by the scale dependence of $g = G(k)/k^{d-2} > 0$. This is not the case for the line $\lambda = 0$, where a number of trajectories beginning in the quadrant $\vec{g} = (-g, -\lambda)$ end up in the adjacent quadrant. $\vec{g}(-g, \lambda)$. This does not pose any phenomenological issues as we are only interested in $\lambda > 0$ and $g > 0$. The flow in the vicinity of the non-trivial fixed point shows that not all of the trajectories end up at the Gaussian. Indeed, only one particular tracjetory will approach g_0 infinitesimally, this path is defined as the separatrix line [4]. By introducing a third

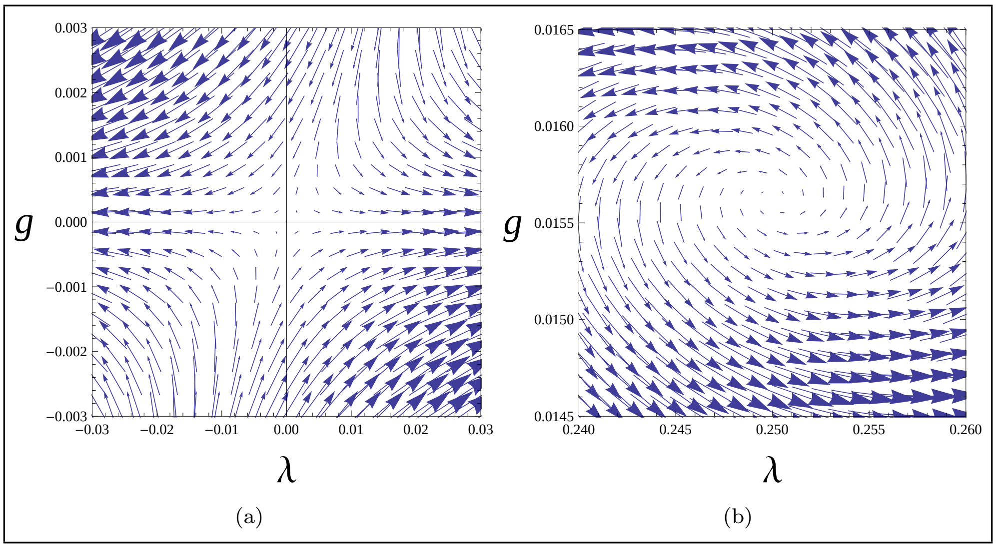


Figure 5: (a) Gaussian fixed point $g = g_0$ (left): The flow pattern indicates some line that separates unstbale trajectories in the left an right directions. This line is in fact the separatrix line[4, 6, 26], and it connects the high energy and the low energy physics. (b) Non-trivial fixed point $g = g_*$ (right): The pattern shows unique spiral trajectories as a result of the nature of scaling exponents; and diminishing magnitude of the vector field approaching the fixed point

parameter z, a three dimensional vector field can be defined using the coupled beta function β_g and β_λ. Then, it is possible to obtain $\hat{\beta}$ as a function of $\beta_g(g,\lambda)$, $\beta_\lambda(g,\lambda)$ and β_z, in the directions $\hat{g}$, $\hat{\lambda}$ and $\hat{z}$ respectively. The inclusion of z allows for an extra component function, albeit an empty one, $\beta_z(g,\lambda)$† to be defined along the $z-axis$, where $\hat{\beta} = \beta_g(g,\lambda)\hat{g} + \beta_\lambda(g,\lambda)\hat{\lambda} + \beta_z(g,\lambda)\hat{z}$ and therefore,

$$\hat{\beta}(g,\lambda) = \begin{cases} \beta_g(g,\lambda)\hat{g} = \left(2g + \dfrac{24g^2}{4g-(1-2\lambda)^2}\right)\hat{g} \\ \beta_\lambda(g,\lambda)\hat{\lambda} = \left(-2\lambda - 12g - \dfrac{24g(6g+(1-3\lambda))}{4g-(1-2\lambda)^2}\right)\hat{\lambda} \\ \beta_z(g,\lambda)\hat{z} = 0 \end{cases} \tag{5.11}$$

Having defined $\hat{\beta}$, the perpendicular vorticity of the vector field can be obtain through,

$$\begin{aligned} \vec{\nabla}\times\hat{\beta} = \Bigg[-12 - \frac{72g}{-\frac{1}{2}(1-2\lambda)^2+2g} + \frac{48(\frac{1}{2}(1-3\lambda)+3g)}{(-\frac{1}{2}(1-2\lambda)^2+2g)^2} - \\ \frac{24(\frac{1}{2}(1-3\lambda)+3g)}{-\frac{1}{2}(1-2\lambda)^2+2g} + \frac{96g^2(1-2\lambda)}{(-(1-2\lambda)^2+4g)^2}\Bigg]\hat{z}\,. \end{aligned} \tag{5.12}$$

Curl of $\hat{\beta}$ was evaluated over a range of values spanning up to and including the interacting fixed point g_*. The trend indicated that $\vec{\nabla}\times\hat{\beta}$ remains fairly stable close to the Gaussian fixed point, whereas there is a dramatic increase in the size of curl as on approaches the interacting fixed point. This increase continues until $\vec{\nabla}\times\hat{\beta}$ becomes singular in the forbidden region marked out by the parabola $g_{\rm cr}$. As a result, this approach proves to be an inconclusive way of observing any structure around the interacting fixed point g_*, since the curl function has no local maxima or minima in the domain of validity.

5.2 Stability matrix and critical exponents

In this section a more consistent way of working out the precise structure of the ultraviolet and infrared fixed points is discussed. It involves obtaining eigenvalues of the stability matrix S, derived from the second derivatives of the beta functions. The stability matrix is obtained by

†Not to be confused with the beta coupling used to investigate stability of the non-trivial fixed under the variation of R^i truncation for $i = 1, 2, \cdots$

taylor expanding the beta functions around either of the fixed points up to leading order,

$$\beta_g(g,\lambda) = \beta_g(g_i,\lambda_i) + \partial_g\beta_g\delta g + \partial_\lambda\beta_g\delta\lambda + \mathcal{O}(\delta g^2,\delta\lambda^2)\cdots \tag{5.13}$$

$$\beta_\lambda(g,\lambda) = \beta_\lambda(g_i,\lambda_i) + \partial_g\beta_\lambda\delta g + \partial_\lambda\beta_\lambda\delta\lambda + \mathcal{O}(\delta g^2,\delta\lambda^2)\cdots \tag{5.14}$$

where $\beta_g(g_i,\lambda_i) = \beta_\lambda(g_i,\lambda_i) = 0$ represents the fixed point values for $i = 1,2$ at the Gaussian and interacting fixed point respectively. The Taylor expansion can be written more succintly,

$$\vec{\beta} \approx \vec{\beta}_i + \underbrace{\begin{pmatrix} \partial_g\beta_g & \partial_\lambda\beta_g \\ \partial_g\beta_\lambda & \partial_\lambda\beta_\lambda \end{pmatrix}}_{S} \begin{pmatrix} \delta g \\ \delta\lambda \end{pmatrix} \tag{5.15}$$

where $\vec{\beta}_i = (\beta_g(g_i,\lambda_i), \beta_\lambda(g_i,\lambda_i))$. The second derivatives of the beta functions describe the flow at a given point as stable or unstable. If the value $\partial_m\beta_n$ for $m = n$ at a given point is positive, then the flow is unstable (away from the point), whilst the flow is stable (towards the point) when $\partial_m\beta_n < 0$ for $m = n$. However, it turns out to be useful to have an expression for the stability of a trajectory in the direction of the infinitesimal stability vector $\delta\vec{g}$. Though the stability matrix equation simplifies due to $\vec{\beta}_i = 0$, the cross terms $\partial_m\beta_n$ where are $m \neq n$ and $m \equiv g$; $n \equiv \lambda$, are non-vanishing for $\delta\vec{g}$ in the vicinity of $\vec{\beta}_i$. By diagonalising S, the eigenvalues of in the linearised basis can be used to work out the stability vector at any given point.

The eigenvalue decomposition of S in the vicinity of the fixed points, linearises the flow equation in order that the critical exponents may be evaluated [19]. In its diagonal basis,the stability matrix looks like $M^{-1}SM = \tilde{S}$. The matrix M is used to rotate the basis vectors, $(g,\lambda) \mapsto (g',\lambda')$, so that equation [5.15] becomes $\vec{\beta} \approx \vec{\beta}_i + M^{-1}SM\delta\vec{g}$. Then multiplying through by M gives

$$\partial_t \begin{pmatrix} g' \\ \lambda' \end{pmatrix} = M\vec{\beta}_i + \begin{pmatrix} \partial\beta'_g/\partial g' & 0 \\ 0 & \partial\beta'_\lambda/\partial\lambda' \end{pmatrix} \begin{pmatrix} \delta g' \\ \delta\lambda' \end{pmatrix}, \tag{5.16}$$

where the right hand side is the beta function $\vec{\beta}\,'$ and $\delta\vec{g}\,'$ is the stability vector in the diagonal basis. The eigenvalues of $\tilde{S}$ are $\partial\beta'_g/\partial g'$ and $\partial\beta'_\lambda/\partial\lambda'$, corresponding to simplified flow equations $\partial_t g' = (\partial\beta'_g/\partial g')\delta g'$ and $\partial_t\lambda' = (\partial\beta'_\lambda/\partial\lambda')\delta\lambda'$. The solutions are exponential functions of the form

$$g'(t) = g'(t_0)\, e^{(\partial\beta'_g/\partial g')t} \tag{5.17}$$

$$\lambda'(t) = \lambda'(t_0)\, e^{(\partial\beta'_\lambda/\partial\lambda')t} \tag{5.18}$$

The critical exponents are then given by the eigenvalues; $\partial\beta'_g/\partial g' = -\theta_g$ and $\partial\beta'_\lambda/\partial\lambda' = -\theta_\lambda$.

5.3 Diagonalised coordinates

The following analysis considers what the effects of linearisation of the basis vectors are at the Gaussian and non-trivial fixed point, with the aim of producing a suitable pair of eigenvalues, in order to define the coupled parametric equations. Once these equations are obtained, the pair can be reverted to their original basis using the rotation matrix M. Evaluating the stability matrix at the Gaussian yields the stability matrix S_0, which diagonalises to $\tilde{S}_0$ with eigenvalues ± 2, where $\partial\beta'_g/\partial g' = -\theta_g = -2$ and $\partial\beta'_\lambda/\partial\lambda' = -\theta_\lambda = 2$. Equations [5.18] become $g'(t) = g'(t_0)\exp[-2t]$ and $\lambda'(t) = \lambda'(t_0)\exp[2t]$, eliminating t yields the reciprocal function $g' = g'(t_0)\lambda'(t_0)(\lambda')^{-1}$. A family of solutions can be obtained depending on the choice of initial conditions of the stability vector components $g'(t_0)$ and $\lambda'(t_0)$. Thus, it is possible to obtain the phase diagram around the Gaussian fixed point g_0 in terms of explicit trajectories using critical exponents, while the amplitude and direction of the flow is given by $\tilde{S}\delta\vec{g}'$ directly. The eigenvalues of S at the Gaussian fixed are simple to calculate, due the fact that S is upper triangular. However, at the intercating fixed point the stability matrix is neither triangular or symmetric, which means its eigenvalues are not necessarily real. Table [5.3] shows the two stability matrices and their corresponding eigenvalue decomposition.

S_0	$\tilde{S}_0$	S_*	$\tilde{S}_*$
$\begin{pmatrix} 2 & 0 \\ 228 & -2 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} -\frac{8}{3} & -\frac{1}{3} \\ \frac{5084}{3} & -\frac{2}{3} \end{pmatrix}$	$\begin{pmatrix} \frac{5}{3}(-1+i\sqrt{203}) & 0 \\ 0 & \frac{5}{3}(-1-i\sqrt{203}) \end{pmatrix}$

The critical exponents at the interacting fixed point are given by $-\theta_g = 5(-1+i\sqrt{203})/3$ and $-\theta_\lambda = 5(-1-i\sqrt{203})/3$. In terms of the complex argument $\theta = \theta' \pm i\theta''$, the exponentials in [5.16], become $\exp[\theta' \pm i\theta''] = \exp[\theta'](\cos\theta'' + i\sin\theta'')$, where $\theta' = -\frac{5}{3}$ and $\theta'' = \pm\frac{5}{3}\sqrt{203}$.

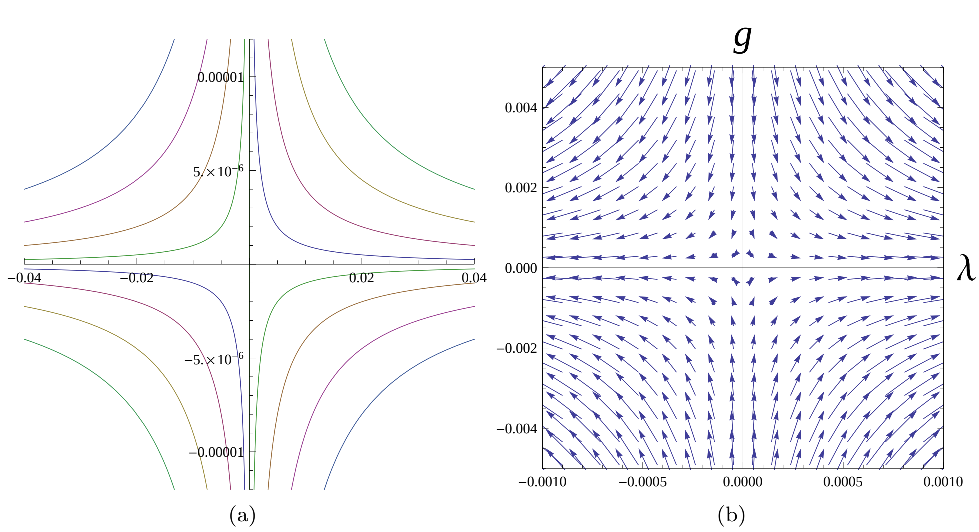


Figure 6: (a) Family of reciprocal functions from the linearisation of the Gaussian fixed; (b) The nature of the flow here is identical to that of [5(a)] up to coordinate rotation

The linearised beta functions are transformed to,

$$g'(t) = \quad g'(t_0)e^{-\theta_g t} \quad = g'(t_0)e^{-5t/3}\left(\cos\frac{5\sqrt{203}}{3}t + i\sin\frac{5\sqrt{203}}{3}t\right) \tag{5.19}$$

$$\lambda'(t) = \quad \lambda'(t_0)e^{-\theta_\lambda t} \quad = \lambda'(t_0)e^{-5t/3}\left(\cos\frac{5\sqrt{203}}{3} - i\sin\frac{5\sqrt{203}}{3}\right) \tag{5.20}$$

The implications of [5.20] are as follows:

(i) Real roots in θ imply that $g'(t)$ and $\lambda'(t)$ are real valued functions, and S is a real symmetric matrix. The scale dependence t can be eliminated from the linearised equations, to obtain $g \sim \lambda^{a/b}$.

(ii) If the roots are purely imaginary, it means that both $g'(t)$ and $\lambda'(t)$ are complex phasors. Therefore, the trajectories are circles modulated by the $g'(t_0)$ and $\lambda'(t_0)$ respectively.

(iii) Finally, for complex roots both functions exhibit spiral trajectories emanating from $g'(t_0)$ and $\lambda'(t_0)$ respectively. This is the origin of the spiral like behaviour seen in [5(b)].

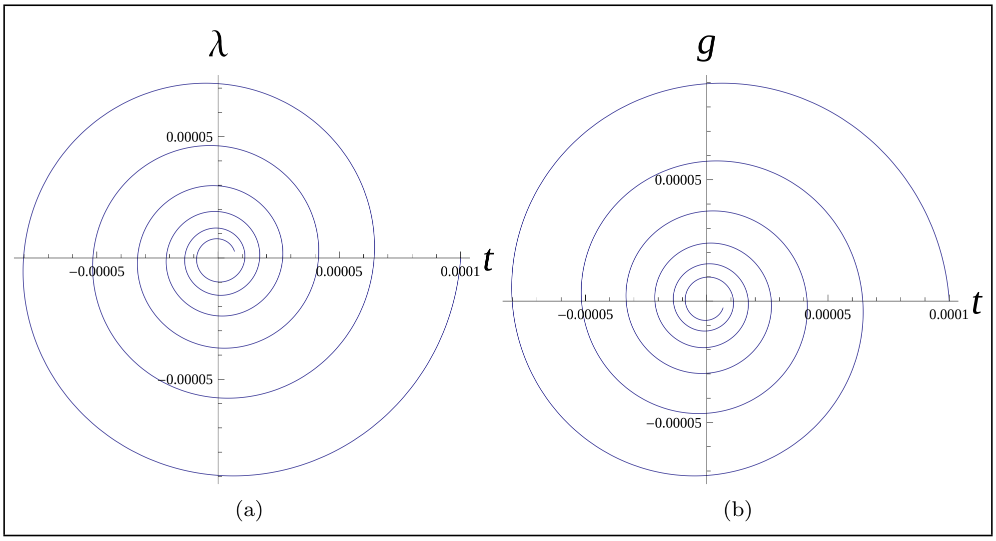


Figure 7: Individual parmetrisation of the λ and g evaluated at a single point, they full would contain many such parametrisations. It interesting that g and λ rotate in opposite directions as result of relative phases. This may have implications on the nature of the spiral trajectories in [4]

5.4 Standard coordinates

It is possible to obtain the behaviour for both trajectories in their original coordiante system by applying the simple counter-rotation,

$$M^{-1}\partial_t \begin{pmatrix} g' \\ \lambda' \end{pmatrix} = M^{-1}M\vec{\beta}_i + M^{-1} \begin{pmatrix} \partial\beta'_g/\partial g' & 0 \\ 0 & \partial\beta'_\lambda/\partial\lambda' \end{pmatrix} \begin{pmatrix} \delta g' \\ \delta\lambda' \end{pmatrix} . \tag{5.21}$$

Once again the non-trivial fixed point poses a problem owing to the fact both $g'(t)$ and $\lambda'(t)$ are complex functions, and consequently do not possess real valued parametrisations of the logarithmic momentum t. It follows that it is much easier to evaluate directly the trajectories in their original coordinate system by numerical means. To this end, the coupled beta function numerical solutions have been determined in original coordinates $\vec{g} = (g, \lambda)$. Figure [8] shows the demarcation of flows to left and right of the separatrix line. The reliability of the approximation method is compromised close to $g = g_{\mathrm{cr}}$, as a result of function interpolation between increasingly divergent estimation terms. This can be seen in the form of "unsmoothness" of the individual trajectories in figure [7] in the vicinity of the parabolic function. The results clearly show that

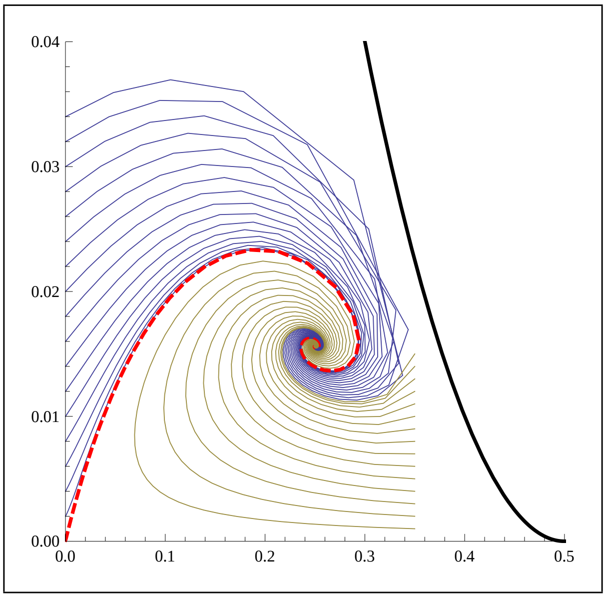


Figure 8: The full flow trajectories showing the separatrix line (red); the regions roughly the right of that (yellow); and the regions to its left (blue). Also the forbidden region is marked by the parabolic (black)

nature of the optimised flow is indeed confined to the separatrix. In other words, for a given choice of fine-tuned parameters in a given region, the resulting theories will divergent if those parameters lie initially on the right of the separatrix (in which case the flow diverges via yellow) or the left (in which case it follows the blue lines). The flow in this case is meant in an outward sense originating from the Gaussian, since separating out right-unstable and left-unstable flows at $k \to \infty$ is made difficult due to the fact that their trajectories spiral infinitely towards the interacting fixed point.

6 Conclusions

The properties of the possible fixed points in QEG have been studied under the renomralisation group and have found to possess, in the very least basic properties that satisfy preliminary requirements of asymptotic safety in $d = 4$ dimensions, albeit in the presence of a truncated EH action. A number of explicit trajectories were evaluated, and it was determined that at least one trajectory ought to connect the ultraviolet and the infrared physics. In the presence of an appropriate regulator scheme the low energy dimensional scaling behaves like classical general relativity in the very least. The results from figure [4] would suggest there indeed exists a flow connecting the micrscopic and macroscopic physics. By observing figure [5(a)] it can be seen that all trajectories are also isolated from negative values of the cosmological constant, while a narrow band of trajectories are unstable in the left and right directions of $g \to \pm\infty$. On the other hand, the structure in the region of the non-trivial fixed point indicates intricate scaling behaviour in the microscopic action. In addtion, in both the original and diagonal basis the nature of the flow bares the similarity of being separated into isolated regions of flows above and below the line $g = g' = 0$, however the former shows a slightly odd symmetric behaviour. Though the curl $\vec{\nabla} \times \hat{\beta}$ does not indicate any local maxima or minimum which might highlight the scaling behaviour at the non-trivial fixed point, it does show that the vorticity is markebly smaller in the regions of the Gaussian fixed point.

The values of critical exponent have been determined at both fixed points and have been used to deduce the explicit nature of the trajectories in both vicinites. In the low energy limit the functional dependence $g = g(\lambda; t)$ is simpler than the functional dependence in the high energy limit, where the trajectories exhibit a spiral nature. Table [6] shows a summary of the critical exponent results,

S-element	$-\theta$	ordinate
$\left(\dfrac{\partial \beta'_g}{\partial g'}\right)_{\vec{g}_0}$	-2	$g = g' = 0$
$\left(\dfrac{\partial \beta'_\lambda}{\partial \lambda'}\right)_{\vec{g}_0}$	2	$\lambda = \lambda' = 0$
$\left(\dfrac{\partial \beta'_g}{\partial g'}\right)_{\vec{g}_*}$	$\frac{5}{3}(-1 + i\sqrt{203})$	$g = g' = \frac{1}{64}$
$\left(\dfrac{\partial \beta'_\lambda}{\partial \lambda'}\right)_{\vec{g}_*}$	$\frac{5}{3}(-1 - i\sqrt{203})$	$\lambda = \lambda' = \frac{1}{4}$

Naturally, an extension to the above study would be the inclusion of higher order terms in R into the Lagrangian, specifically, the quadratic term R^2. If this were the case, then one could investigate whether the nature fixed point changes much under such variation. To do this a third invariant would have to be introduced into the Einstein-Hilbert action such that

$$\Gamma_k[\phi] = \Gamma[\phi] = \frac{1}{16\pi G_N} \int d^d x \sqrt{g}\,(-R + 2\Lambda) + XR^2. \tag{6.1}$$

Here, X is the new coupling constant and its dimensionless form is given by $X = k^{4-d}\chi$. Now we have a third beta function β_χ, and a third component to the vector $\vec{g} = \vec{g}(g, \lambda, \chi)$, which would result in a flow in a three dimensional vector space. Studies carried out in [13, 16, 11] investigate the nature of the RG flow under R^2 truncation. Furthermore, the inclusion of terms quadratic in curvature would also include $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\delta}R^{\mu\nu\rho\delta}$. The analysis involving such terms is technically demanding due to the following:

(i) The projection technique [16] which maps the RG flow onto a space of maximally symmetric sub-spheres is degenerate when dealing with (curvature)^2 terms because they all have canonical scaling k^{4-d}. Whereas, $\sqrt{g}$ and $\sqrt{g}R$ have unique scalings 0 and -2 respectively.

(ii) Projecting each of the three terms unto their own space leads to the insertion asymmetric spaces

(iii) The Gaussian fixed point can include all three (curvature)^2 terms due to its scaling invariance, but the interacting fixed points differently with the inclusion of different curvature)^2 terms

However, analysis on RG flow in [4, 5, 11] indicate that the non-trivial fixed point is still stable in the presence of R^2 inclusion and that the most defining instability arises from the gauge fixing sector. It is clear that such factors still pose pertinent issues, however, for the time being it would seem the under appropriate approximation there is good evidence supporting asymptotic safety in Quantum Einstein Gravity.

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