Slow-roll inflation and the primordial metric perturbation spectrum

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Author: Trevor Peter Hardcastle

Imperial College London

Abstract

Inflation is introduced as a solution to the horizon and flatness problems. The equation of state during inflation is shown to violate the strong energy dominance condition, and the de Sitter solution is used as a first approximation to estimate the duration of inflation at ~ 64 e-folds. A scalar field is introduced as the driving force for inflation and energy conservation is used to derive its equation of motion. The slow-roll approximation is then introduced and the slow-roll conditions for the field are derived. The scalar-vector-tensor decomposition of metric perturbations is introduced, and the behaviour of the perturbations, as well as the perturbations to the scalar field, under a generic infinitesimal gauge transformation, is derived. Gauge-invariant variables are then constructed and the longitudinal gauge is employed to derive an updated field equation of motion taking into account metric perturbations. The perturbed Einstein's equations are quoted in the longitudinal gauge and are used to supplement the perturbed equation of motion to provide the framework for a full dynamical description of the field/metric system. This system is solved for initial sub-horizon field perturbations in the adiabatic limit, and then solved in the super-horizon limit, using the horizon-crossing as a boundary condition. This is used to predict the power spectrum at the end of inflation as a function of perturbation scale. The spectrum is found to differ from the flat Harrison-Zeldovich spectrum with only a logarithmic scale-dependence.

Contents

Introduction

In 1981, Alan Guth based at SLAC published a paper outlining the first intuitive model of inflation theory which proposed to solve the horizon and flatness problems [5]. The idea was that the vacuum energy density of the universe drove an exponential expansion of spacetime. As it expanded, the vacuum energy density nevertheless remained constant, which can be understood from the point of view of energy conservation by remembering that the energy of the gravitational field is negative. The vacuum was then thought to decay, giving rise to a phase transition. However, he could not find a way to end inflation with a graceful exit. The problem was that in the phase which followed, vacuum bubbles formed which experienced wall-collisions giving rise to extreme inhomogeneity [6]. Shortly afterwards, Andrei Linde published a paper entitled ‘A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems’ [7]. It was a culmination of Guth’s original paper, but with the different postulate that the driving force for inflation was a scalar field rolling slowly down a potential with a sufficiently flat region. However, Linde’s model still relied on the assumption of highly homogeneous initial conditions (which may arbitrarily be identified with the Planck time $t_{Pl} \sim 10^{-43}$ s), prior to inflation. In principle, requiring uniform initial conditions appears to be philosophically sound and in accordance with the Copernican principle. However, it would have been more satisfactory to remove the need for this assumption. This was in fact achieved by Linde with his “chaotic inflation” model [8]. This model still used a slow-roll potential, but did not require any particular level of pre-inflation energy density uniformity. Subsequent research has led to inflation becoming generally accepted as the strongest current candidate which offers solutions to the cosmological puzzles.

A theory which explains the levels of homogeneity and isotropy is particularly interesting in the context of cosmological structure formation, and provides a foundation on which to study the subtle structure which does remain, namely the anisotropy of the CMB and the inhomogeneity of the matter distribution. The theory of inflation in the purely homogeneous case provides this foundation. Fluctuations can then be added to the theory to serve as the seeds for later formation of inhomogeneity and anisotropy. It should be noted that this is not an ad-hoc modification – the homogeneous case is not now believed to be realized in nature. Quantum uncertainty characterizes the very early universe so it would be unnatural for it to be completely void of fluctuations. A basic understanding of the consequences of inflation for structure formation can be achieved without need for rigorous applications of quantum theory however. The distribution of the physical fluctuations can be expressed in a general way as a Gaussian random field, characterized entirely by its variance, and the corresponding behaviour of the surrounding spacetime can be naturally described in the linear regime using the scalar-vector-tensor decomposition. The physical ingredient comes in the form of the slow-roll approximation, which is applied in this paper.

During the inflation epoch, these fluctuations are thought to have been amplified to super-horizon scales, generating a spectrum of large-scale primordial inhomogeneities by the end of inflation, which ultimately evolved into the large scale structure observed today. This paper uses a simple model of inflation to predict that spectrum. At recombination, the primordial distribution is thought to have decoupled into separate matter and radiation distributions which respectively evolved into the present day matter distribution and the CMB. The signals observed today were emitted at the recombination stage. The observed distributions of matter and radiation exhibit stark differences primarily because radiation pressure prevents radiation from forming localized inhomogeneities like those exhibited by matter, which is more subject to gravitational instability. However they are both thought to originate from the same primordial spectrum generated at the end of inflation, which is where the prediction of this spectrum, and therefore the aim of this paper, find their relevance.

1. Homogeneous inflation

1.1 Where does inflation fit into the big bang picture?

The conventional big bang theory relies on the assumption that all of the radiation in the universe was uniformly distributed at the Planck time, $t_{Pl} \sim 10^{-43}$ s, in order for the theory to stand up to observations today. At the very least, this assumption seems to be philosophically sound, although it seems unsatisfactory to have to assume such initial conditions without any physical justification. Modern inflation actually *predicts* the uniform generation of standard model particles during what is nowadays called the reheating phase which occurs at the end of inflation around 10^{-36} s, thereby removing the need for extreme uniformity in the pre-inflation stage. In this way, inflation is seen as the prequel to the conventional big bang. Modern particle physics predicts that at GUT energies ($\sim 10^{16}$GeV), physical states of matter exhibiting negative pressure may occur [10]. For the early universe, these energies are associated with the time $\sim 10^{-36}$ s. Despite the statistical improbability that may be associated with the occurrence of such a patch of material in the early universe, only a non-zero probability is required in order for it to occur *somewhere*, and the gravitational repulsion it generates then dominates the rest of the universe and becomes the driving force for the subsequent big bang [10].

1.2 The horizon problem and its solution in a nutshell

To a very good approximation, the entire observable universe is homogeneous and isotropic, so the region satisfying homogeneity and isotropy is at least as large as the current horizon scale, since identical conditions may, for all we know, exist beyond the horizon. In fact, a simple argument suggests that the observable universe (or any given causal patch) has always been smaller than the entire, global universe:

At a given initial time, t_i, a given length scale in spacetime was smaller than it is today, at t_0 , by the ratio of the corresponding scale factors, a_i and a_0. The initial conditions are identified as those which occur at the Planck time, so let $t_i = t_{Pl}$. So the length scale of the region of homogeneity is at least as large as $ct_0(a_i/a_0)$. The scale of the observable universe at the Planck time is $\sim ct_i$. So the ratio of the scale of the total homogeneous region compared to that of the causal region will indicate whether it is apparently impossible for a causal process to be responsible for homogeneity outside the horizon at the Planck time. The ratio is

$$\frac{l_i}{l_c} = \frac{ct_0}{ct_i}\frac{a_i}{a_0} \tag{1.1}$$

where $\frac{a_i}{a_0} = \frac{T_0}{T_{Pl}} = 10^{-32}$ [1], which gives $\frac{l_i}{l_c} = 10^{28}$.

This suggests a volume containing $\sim (10^{28})^3$ causally disconnected regions. So the causal region at the Planck time is much smaller than the homogeneous region, and the homogeneity which characterizes the global universe can apparently not be attributed to a causal process.

This raises the question of why the modes entering the horizon at recombination exhibit the same level of homogeneity as the rest of the observable universe at this time (the CMB survey effectively shows the universe at recombination). Obviously, observational astronomy has not been around for long enough to literally track the modes entering the horizon, but given that the entire observable universe today is composed of regions which entered the horizon in the past, the question of why the entire observable universe today is so homogeneous at all observable distance scales is an entirely equivalent one.

Inflation theory simply postulates that all regions of the observable universe *were* in fact in causal contact before the inflation mechanism occurred. The subsequent inflation that then occurred caused spacetime to expand at a rate far in excess of any causal signal propagating within it, resulting in a globally homogeneous yet causally disconnected state by the end of inflation.

1.3 The flatness problem and its solution in a nutshell

The flatness problem can be regarded as the problem of the requirement of extreme fine tuning of the initial velocity distribution. The initial velocity has to be so extremely finely tuned to avoid immediate collapse or expansion which is so fast as to make the universe empty too soon [2].

The problem can be expressed in a relativistic way in terms of the energy density parameter $\Omega(t)$. It seems unnatural that the universe should be so close to flatness today, given that $\Omega(t)$ evolves in time, causing the curvature of the universe to diverge away from flatness. $\Omega(t_i)$ must have been extremely close to unity since $\Omega(t_0)$ is still so close to unity today. The latest WMAP+BAO+SN measurement is $\Omega(t_0) = 1.0052 \pm 0.0064$ [11]. It can be shown using the 1st Friedmann equation that $\Omega(t_i)$ must differ from unity by a quantity less than 10^{-56} [A.1]. If $\Omega(t_i)$ was slightly larger than 1, it would have diverged to infinity, preventing galaxies from forming. If it was slightly smaller than 1, it would have converged to zero very quickly, also preventing galaxies from forming. Since galaxies have, in fact, formed in our universe and we are alive to talk about it, it is tempting to invoke the anthropic principle as an explanation. A physical explanation is always preferable to this however, and inflation provides just that.

The solution is that spacetime at the beginning of inflation, irrespective of its degree of curvature at this point, was rapidly expanded at a rate far exceeding the propagation of causal horizons so that the scale of curvature by the end of inflation was far in excess of the scale of any given causal patch. The universe has therefore been driven to flatness by the end of the inflation period

1.4 The variation of the Hubble radius

Another way of looking at the solution to the horizon problem is to allow the comoving Hubble radius to vary over the history of the universe in such a way as to allow the comoving particle horizon to become larger than the present Hubble radius. This gives rise to a scenario in which the entire observable universe can exhibit self-resemblance at all observable scales without the causality issues outlined in section 1.2. If it is assumed that the entire observable universe was formed from a primordial homogeneous region, existing amongst a larger region perhaps containing inhomogeneities, then this primordial region is causally separate from the outside since all universes undergoing accelerating expansion have an event horizon [A.2].

The comoving particle horizon can be written as

$$d_p(t) = \int_0^{a_0} \frac{da}{a(t)} \frac{1}{a(t)H} = \int_{-\infty}^{a_0} d(\ln a) \frac{1}{a(t)H} \; . \tag{1.2}$$

Figure 1 shows such a scenario in which $d_p(t_0) >> \frac{1}{a(t_0)H(t_0)}$ with a sub-horizon distance scale (mode) exiting the horizon during inflation, and reentering the horizon at a later stage.

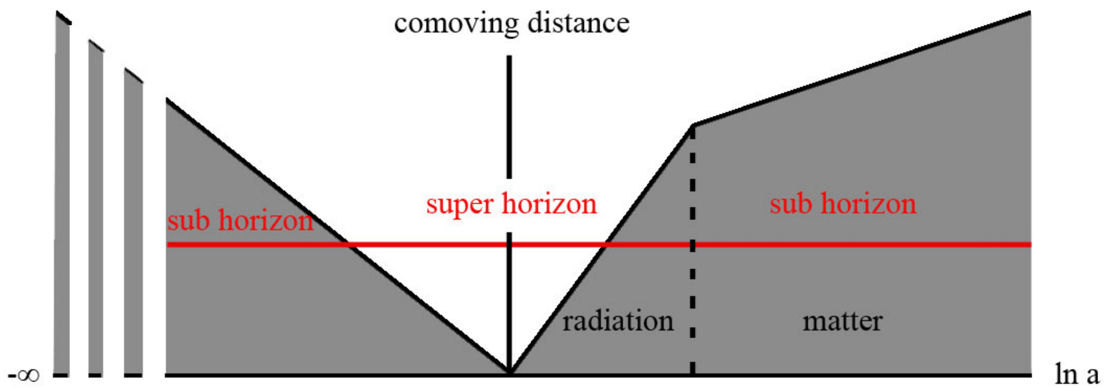

Fig. 1 – A general inflationary scenario in which a mode within the Hubble radius at an early time becomes causally disconnected by exiting the horizon, and reenters at a later stage due to the variation of the Hubble radius.

1.5 Looking towards a physical realization of inflation

It can be seen from the second Friedmann equation,

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a \tag{1.3}$$

, that ä > 0 is required during the inflationary period in order for the Hubble radius to decrease [A.3]. So the condition for gravity to be repulsive is that the strong energy dominance condition is violated. This can be written as [A.4]

$$\rho < -3p \tag{1.4}$$

Positive cosmological constant type matter satisfies condition (1.4):

$$\rho_\Lambda = -p_\Lambda \tag{1.5}$$

It can be shown that in order for the initial stages of inflation to be well described, the initial conditions must not deviate by more than 1% from the vacuum equation of state [3]. That is, the physical realization of the mechanism for inflation must strongly resemble positive cosmological type matter in the initial stages

The de Sitter solution to Einstein's equations can therefore be used as an approximate model in aid of investigating the initial conditions required for inflation.

1.6 Inflation induced by a scalar field

A scalar field, φ, can be identified as a physical mechanism to drive inflation.

The stress-energy tensor is of a scalar field φ is defined as

$$T^{\mu}{}_{\nu} \equiv \frac{\partial L}{\partial(\partial_{\mu}\varphi)}(\partial_{\nu}\varphi) - \delta^{\mu}{}_{\nu}V(\varphi) \tag{1.6}$$

, where L is the Lagrangian density of the scalar field:

$$L = \frac{1}{2}g^{\mu\nu}(\partial_{\mu}\varphi)(\partial_{\nu}\varphi) - V(\varphi) \tag{1.7}$$

Accordingly, the $T^{0}{}_{0}$ component (the Hamiltonian density) which can be identified as the energy density parameter of the scalar field ρ, and the $T^{0}{}_{i}$ components (the momentum density), which can be identified as the pressure of the scalar field, p, can both be calculated explicitly [A.5]:

$$\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi) \tag{1.8}$$

, and

$$p = \frac{\dot{\varphi}^2}{2} - V(\varphi)\,. \tag{1.9}$$

Combining (1.8) and (1.9) with condition (1.5) results in

$$\dot{\varphi}^2 << V(\varphi) \tag{1.10}$$

This is a necessary condition for inflation, but not a sufficient condition for inflation which ends in a graceful exit. The conditions required for inflation which ends in a graceful exit are provided by a slow-rolling scalar field, which is described in section 1.8.

1.7 Estimating the duration of the inflation epoch

Since the whole of the observed CMB is smooth, i.e. the homogeneous region is at least as large as the present day comoving Hubble radius, the distance scales which have been entering the horizon since the end of the inflationary period must have originally been sub-horizon (see section 1.5). Another way of saying this is that the ratio of the comoving Hubble radius today and at the end of inflation must equal the ratio of the comoving Hubble radius at the end of inflation and the Planck time. The Hubble radius today is related to the Hubble radius at the end of inflation by the ratio of the corresponding scale factors:

$$\frac{1}{a_0 H_0} = \frac{a_0}{a_f} \frac{1}{a_f H_f} \tag{1.11}$$

Progressing from here depends on choosing an equation of state of the universe for the period t_f - t_0. Assuming for the sake of example that the universe is radiation dominated for this period (for which a ~ 1/T, where T is the radiation temperature):

$$\frac{1}{a_0 H_0} = \frac{T_f}{T_0} \frac{1}{a_f H_f} \approx 10^{28} \frac{1}{a_f H_f} \tag{1.12}$$

, using $T_0 = 2.725$ K and $T_f \approx 1.2 \times 10^{28}$ K.

So the factor by which the Hubble radius must have decreased during the inflationary period is to the order of 10^{28}.

The assumption that the equation of state during the inflationary period is (1.5), which corresponds to the de Sitter solution, allows for the change in the comoving Hubble radius during inflation to be calculated.

The general solution for the scale factor in the de Sitter universe is

$$a(t) = \frac{\exp(H_\Lambda t)}{2H_\Lambda} - \frac{\exp(-H_\Lambda t)}{2H_\Lambda}. \tag{1.13}$$

The second term is zero for the K = 0 case and H_Λ is the constant Hubble parameter.

Since the Hubble parameter is constant during this stage, the scale factor must increase by a factor of at the order of 10^{28}or more.

$$\frac{a(t_f)}{a(t_i)} = \frac{\exp(H_\Lambda t_f)}{2H_\Lambda} \Bigg/ \frac{\exp(H_\Lambda t_i)}{2H_\Lambda} = \exp(H_\Lambda (t_f - t_i)) \approx 10^{28} \tag{1.14}$$

This gives $(t_f - t_i) = 64/H$, or 64 e-folds. Inflation is only realized if condition (1.10) is satisfied for at least approximately 64 Hubble times.

$$t_f - t_i = \frac{64}{H} \tag{1.15}$$

Late into the inflation period, the scale factor must eventually reach a stage where ä < 0, decelerating in such a way as to tend towards the observed Hubble expansion today. This is not the case in the de Sitter solution, which serves as a first approximation.

1.8 The slow-roll approximation and an expression for the scale factor

Obviously, the inflation stage is radically different to the universe today. If inflation is to be successfully integrated into conventional big bang cosmology, it must preserve the original aspects of conventional big bang cosmology. The expansion of the universe in inflation models must converge towards the Hubble expansion observed today. The stage where inflation dies out is the graceful exit stage, and this is a key requirement of the theory. In Guth's original paper, the physical realization of the required state of negative pressure came in the form of a scalar field trapped in a metastable vacuum. The ending of the inflation stage can be accounted for by allowing the scalar field to decay away to the true vacuum.

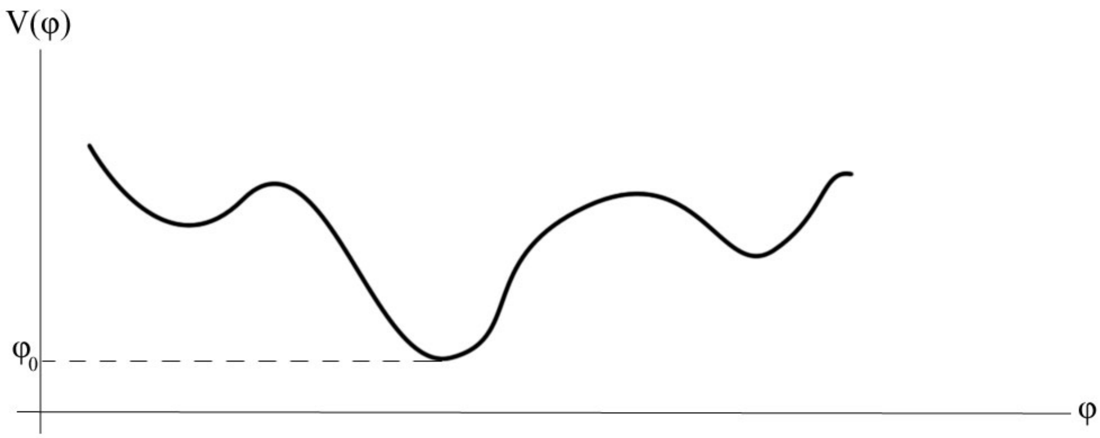

Fig. 2 – An arbitrary potential with a true vacuum and false vacua, satisfying $\dot{\varphi}^2 < V(\varphi)$

However, this model could not be made to end in a graceful exit and the newer model developed by Andrei Linde was adopted. It used the slow-roll approximation, a basic summary of which follows.

By substituting the first Friedmann equation into the second, the following expression of energy conservation can be found [A.6]:

$$\dot{\rho} = -3H(\rho + p) \tag{1.16}$$

Substituting (1.8) and (1.9) into this energy conservation equation results in [A.7]

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 \tag{1.17}$$

This is the general equation of motion describing the scalar field during inflation. At this point, the set of possible inflationary scenarios remains large since any combination of field φ and potential V(φ) which satisfies (1.17), subject to condition (1.10) will result in inflation. It will be convenient later in the report to express (1.17) in conformal time [A.8]:

$$\varphi'' + 2H\,\varphi' + a^2 \frac{\partial V}{\partial \varphi} = 0 \qquad (1.18)$$

The field equation of motion, (1.17), is identical to that of a damped (classical) harmonic oscillator. For example, if $V(\varphi) = \frac{m^2}{2}\varphi^2$, (1.17) reads

$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0 \qquad (1.19)$$

, where m^2 is equivalent to the constant undamped angular frequency and the constant damping ratio is identified as $\frac{3H}{2m}$.

A basic sketch of φ versus t for different values of the damping ratio show how the field can undergo slow-roll decay.

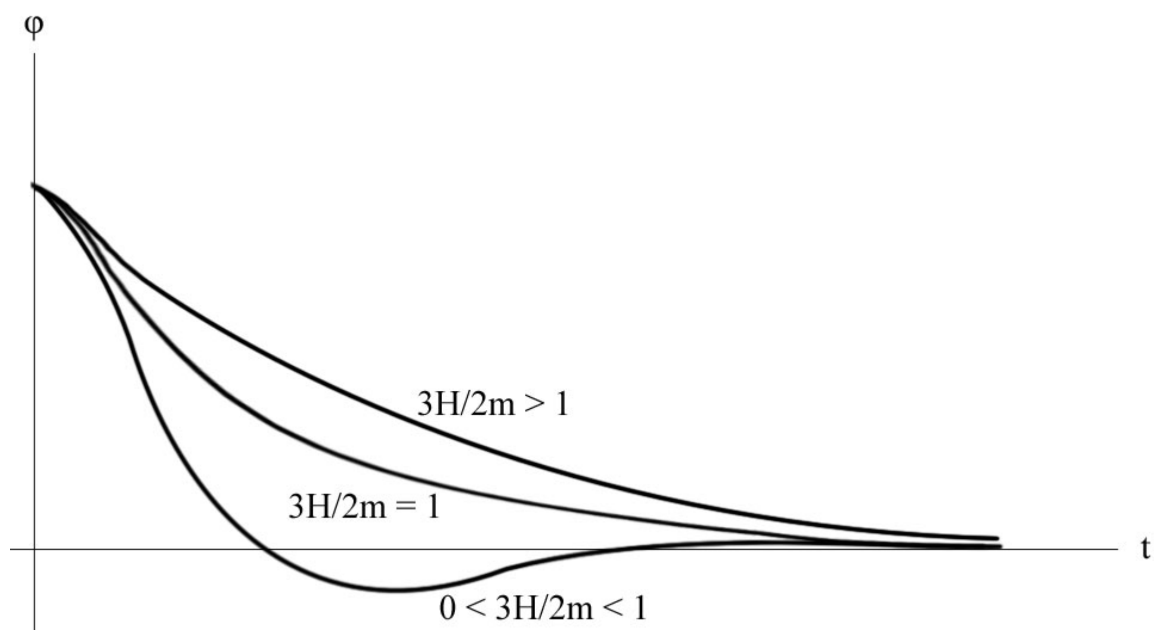

Fig. 3 – Sketch of solutions to (1.19) with different values of the damping ratio, resulting in slow roll decay

A slow-roll approximated solution for the scale factor purely as a function of φ can be obtained by allowing the damping term to be particularly large compared to the acceleration term. That is,

$$3H\dot{\varphi} >> |\ddot{\varphi}|. \qquad (1.20)$$

Equation (1.20) defines the slow-roll case. For this case, the equation of motion (1.17) reads

$$3H\dot{\varphi}+V,_{\varphi}\approx 0\,. \tag{1.21}$$

The 1st Friedmann equation can be simplified by using the inflation condition (1.10) to omit the $\dot{\varphi}^2$ term:

$$H^2=\frac{8\pi G}{3}\rho_{\varphi}=\frac{8\pi G}{3}\left(\frac{\dot{\varphi}^2}{2}+V(\varphi)\right)\approx\frac{8\pi G}{3}V(\varphi) \tag{1.22}$$

By combining (1.21) and (1.22) with the Hubble parameter definition, $H=\frac{\dot{a}}{a}$, the following equation is obtained [A.9]:

$$-8\pi GV(\varphi)=\frac{d(\ln a)}{d\varphi}\frac{\partial V}{\partial\varphi} \tag{1.23}$$

This can be integrated to yield the scale factor as a function of φ:

$$a(\varphi)=a_i\exp\left(\int_{\varphi}^{\varphi_i}d\varphi 8\pi GV(\varphi)\left(\frac{\partial V(\varphi)}{\partial\varphi}\right)^{-1}\right) \tag{1.24}$$

1.9 The scale factor for a power law potential

For the more specific case of a potential which depends on some arbitrary power of φ, as such:

$$V(\varphi)=\frac{\Lambda\varphi^n}{n} \tag{1.25}$$

, where Λ a constant. The scale factor at time t then takes the form [A.10]

$$a(\varphi)=a_i\exp\left(\frac{4\pi G}{n}\left(\varphi^2(t_i)-\left(\varphi^2(t)\right)\right)\right). \tag{1.26}$$

2. Inhomogeneous Inflation

2.1 Introducing perturbations

Despite the striking levels of homogeneity in the CMB, there is nevertheless some remaining structure. This can be integrated into the homogeneous inflation theory rather elegantly with the idea that quantum fluctuations during the inflationary period are amplified into large-scale inhomogeneities and can be thought of as the “seeds” of the present day structure.

To work towards studying a simplified instance of this, it is first necessary to study how perturbations to spacetime, and to the physical entities that occupy it, can be described in a general relativistic context.

Gauge freedom in general relativity means that perturbations apparently depend on the choice of coordinates, which would be physically absurd. To clarify this situation, gauge-invariant variables must be formulated which fully describe the perturbations in the spacetime metric, $g_{\mu\nu}$ and the scalar field φ.

2.2 General metric perturbations in flat spacetime

The metric for an unperturbed universe is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.1}$$

The perturbations to the metric can be considered as an addition to the above background metric so can be added separately:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \delta g_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.2}$$

In the linear regime, the components of the perturbation term are much smaller than the background metric:

$$\left|\delta g_{\mu\nu}\right| << \left|g_{\mu\nu}\right| \tag{2.3}$$

The background metric obviously satisfies homogeneity and isotropy. This means that the perturbations can be naturally categorized using the scalar-vector-tensor decomposition. Explicitly:

$$\delta g_{00} = 2a^2\phi \tag{2.4}$$

$$\delta g_{0i} = a^2\left(B,_i + S_i\right) \tag{2.5}$$

$$\delta g_{ij} = a^2\left(2\psi\delta_{ij} + 2E,_{ij} + F_i,_j + h_{ij}\right) \tag{2.6}$$

, which satisfy the 6 conditions

$$\begin{aligned} S^i,_i &= 0 \\ F^i,_i &= 0 \\ h_i^i = h^i{}_j,_i &= 0 \end{aligned} \tag{2.7}$$

, where ψ , ϕ , B and E are scalars, S_i and F_i are 3-vectors and h_{ij} is a tensor.

The matrix representing the metric perturbation therefore has $4^2 - 6 = 10$ independent components.

The scalar perturbations, ψ, ϕ, B and E, are of primary interest as they have the biggest effect on structure formation in the universe, whereas the vector perturbations have been shown to have no effect on structure formation at all [12]. Tensor perturbations are produced in the form of gravitational waves, although due to the current experimental inaccessibility of gravitational waves, they are not considered in this paper.

For flat spacetime, the unperturbed metric takes the standard Minkowski form (using conformal time):

$$ds^2 = a^2\left(d\eta^2 - \delta_{ij}dx^i dx^j\right). \tag{2.8}$$

The scalar perturbations can be selected from the full set of perturbations (2.4) – (2.6) by setting the vectors S_i and F_i, and the tensor h_{ij}, all equal to zero. What remains can then be integrated into the background metric (2.8) to give the full metric with scalar perturbations [A.11]:

$$ds^2 = a^2\left[(1+2\phi)d\eta^2 + 2B,_i\, dx^i d\eta + \left((2\psi - 1)\delta_{ij} + 2E,_{ij}\right)dx^i dx^j\right] \tag{2.9}$$

2.3 The behaviour of metric perturbations under gauge transformations

Under a general gauge transformation,

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha \tag{2.10}$$

, where the ξ^α components are infinitesimal, the unperturbed metric transforms as

$$\tilde{g}_{\mu\nu}(\tilde{x}^\rho) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x^\rho). \tag{2.11}$$

A metric with a small perturbation, $\delta g_{\mu\nu}$,

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu} \tag{2.12}$$

, where the notation ${}^{(0)}g_{\mu\nu}$ denotes the unperturbed flat background metric,

$${}^{(0)}g_{00} = a^2(\eta) \qquad {}^{(0)}g_{ij} = -a^2(\eta)\delta_{ij} \qquad {}^{(0)}g_{i0} = {}^{(0)}g_{0i} = 0, \tag{2.13}$$

, transforms under the gauge transformation (2.10) as [A.12]:

$$\widetilde{g}_{\mu\nu}(\widetilde{x}^{\rho}) \approx {}^{(0)}g_{\mu\nu}(x^{\rho}) - \xi^{\beta}{}_{,\nu}\,{}^{(0)}g_{\mu\beta} - \xi^{\alpha}{}_{,\mu}\,{}^{(0)}g_{\alpha\nu} + \delta g_{\mu\nu} \tag{2.14}$$

The LHS of (2.14) can written in its own right as a background metric in the new coordinates with the perturbation metric in the new coordinates, as such:

$$\widetilde{g}_{\mu\nu}(\widetilde{x}^{\rho}) = {}^{(0)}g_{\mu\nu}(\widetilde{x}^{\rho}) + \delta\widetilde{g}_{\mu\nu} \tag{2.15}$$

(2.14) can be compared with (2.15) to yield the infinitesimal gauge transformation law for the perturbation to the metric [A.13]:

$$\delta\widetilde{g}_{\mu\nu} = \delta g_{\mu\nu} - {}^{(0)}g_{\mu\nu,\gamma}\xi^{\gamma} - \xi^{\beta}{}_{,\nu}\,{}^{(0)}g_{\mu\beta} - \xi^{\alpha}{}_{,\mu}\,{}^{(0)}g_{\alpha\nu}\,. \tag{2.16}$$

2.4 The behaviour of perturbations to scalars under gauge transformations

The scalar field driving inflation, φ, is of primary physical interest, so its behaviour under a gauge transformation will be needed. To find the behaviour of a perturbation to a scalar quantity under the gauge transformation (2.10), a scalar quantity is split into background and perturbation parts.

$$q(x^{\rho}) = {}^{(0)}q(x^{\rho}) + \delta q \tag{2.17}$$

The gauge transformation law can then be found by similar reasoning to that used for the metric. The result is [A.14]:

$$\delta\widetilde{q} = \delta q - {}^{(0)}q_{,\gamma}\xi^{\gamma} \tag{2.18}$$

The behaviour of a covector can also be calculated in a similar way [A.15], and this will come in useful later in the paper.

$$\delta\widetilde{q}_{\mu} = \delta q_{\mu} - {}^{(0)}q_{\mu,\gamma}\,\xi^{\gamma} - {}^{(0)}q_{\gamma}\xi^{\gamma}{}_{,\mu} \tag{2.19}$$

2.5 Explicit expressions for metric perturbations under gauge transformations

In order to derive the explicit transformation laws for the metric perturbations, it is necessary to consider the separate components of the infinitesimal function, ξ^{α}, in the following way [4]:

$$\xi^i = \xi^i_\perp + \zeta^{,i} \tag{2.20}$$

, where $\xi^i_\perp$ has zero divergence and ζ is a scalar function [A.16].

With the help of (2.4) – (2.6) and (2.20) the metric perturbation gauge transformation law (2.16) can then be used to yield explicit expressions for the metric perturbations. The results are [A.17]:

$$\delta\tilde{g}_{00} = 2a\big([a\phi] - (a\xi)'\big) \tag{2.21}$$

$$\delta\tilde{g}_{0i} = a^2\left([B,_i + S_i] + \left(\xi^i_\perp\right)' + \left(\zeta' - \xi^0\right),_i\right) \tag{2.22}$$

$$\delta\tilde{g}_{ij} = a^2\left(\left[2\psi\delta_{ij} + 2E,_{ij} + F_i,_j + F_j,_i + h_{ij}\right] + \frac{2a'}{a}\xi^0\delta_{ij} + 2\zeta,_{ij} + \xi_{\perp i,j} + \xi_{\perp j,i}\right) \tag{2.23}$$

, where the untransformed parts, $\delta g_{\mu\nu}$, are left in square brackets.

2.6 Explicit expressions for scalar perturbations under gauge transformations

The primary scalar quantity of interest in the context of inflation is the scalar field φ. The gauge transformation law for perturbations to a physical scalar quantity, equation (2.17), can be used to find the explicit behaviour of φ under such a transformation. By giving φ a perturbation, namely $\delta\varphi$, and writing the field as the sum of a homogeneous component and a perturbed component as such [A.18]:

$$\varphi = \varphi_0(\eta) + \delta\varphi(\underline{x}, \eta) \tag{2.24}$$

, the explicit transformation law results immediately [A.19]:

$$\delta\tilde{\varphi} = \delta\varphi - \varphi_0'\xi^0 \tag{2.25}$$

2.7 Forming gauge-invariant variables

The issue of distinguishing fictitious, gauge-dependent perturbations from physical ones is solved by picking gauge-invariant variables with which to express the metric perturbations and field perturbations. Generally, in practice, this means defining new gauge-invariant parameters as linear combinations of the original system parameters, where the number of new parameters is equal to the number of degrees of freedom in the system. To find the number of degrees of freedom existent in the scalar metric perturbations, equations (2.4) – (2.6) are combined with (2.21) – (2.23) to find the number of unique parameters on which the gauge-transformed scalars, ψ, ϕ, B and E

depend. The gauge transformation properties of ψ, ϕ, B and E are shown below [A.20]:

$$\tilde{\psi} = \psi + a'\xi^0 \tag{2.26}$$

$$\tilde{E} = E + \zeta \tag{2.27}$$

$$\tilde{B} = B + \zeta' - \xi^0 \tag{2.28}$$

$$\tilde{\phi} \rightarrow \phi - \frac{1}{a}(a\xi^0)' \tag{2.29}$$

There are 2 unique parameters, ξ^0 and ζ, so the whole perturbed metric can therefore be described fully using 2 gauge-invariant variables. By definition, these are arbitrary, so the way they are defined comes down to convenience.

It is easy to see that choosing

$$\zeta = -E \qquad\qquad \xi^0 = B - E' \tag{2.30}$$

causes $\tilde{B}$ and $\tilde{E}$ to vanish. $\tilde{\phi}$ and $\tilde{\psi}$ remain non-zero, and these will be the 2 gauge-invariant variables. By relabelling them as $\tilde{\phi} \rightarrow \Phi$ and $\tilde{\psi} \rightarrow \Psi$ (to signify their importance as the gauge-invariant variables), they are

$$\Phi = \phi - \frac{1}{a}\left(a\left(B - E'\right)\right)' \tag{2.31}$$

$$\Psi = \psi + \frac{a'}{a}\left(B - E'\right). \tag{2.32}$$

It can be shown that (2.31) and (2.32) are the simplest linear combination which exhibit gauge-invariance [4].

Using the choice (2.30) the gauge-invariant field perturbation (2.25) reads:

$$\delta\tilde{\varphi} = \delta\varphi - \varphi_0'\left(B - E'\right) \tag{2.33}$$

2.8 The Longitudinal Gauge

Now that gauge-invariant variables have been established, a particular gauge can be picked to work with. For what lies ahead, the longitudinal gauge will work best. The longitudinal gauge (or the conformal Newtonian gauge) is the most physically intuitive one to work with since there exist natural correspondences between classical Newtonian parameters and the perturbation parameters.

It is defined by:

$$B = E = 0 \tag{2.34}$$

, which, by quick inspection of (2.31) and (2.32) implies [A.21]:

$$\Psi = \psi \tag{2.35}$$

$$\Phi = \phi . \tag{2.36}$$

The scalar perturbation metric (2.9) is therefore greatly simplified in the longitudinal gauge, and reduces to [A.22]:

$$ds^2 = a^2\left[\left((1+2\phi)d\eta^2\right) + (2\psi - 1)\delta_{ij}dx^i dx^j\right] \tag{2.37}$$

In this gauge, the field perturbation (2.33) takes on the trivial form:

$$\delta\tilde{\varphi} = \delta\varphi \tag{2.38}$$

2.9 Deriving an equation of motion for the perturbations to φ

The aim of this section will be to derive an updated version of the equation of motion (1.18) for the scalar field φ, which describes the scalar perturbations to φ, and the scalar perturbations to the metric. It will therefore be an equation in $\delta\varphi$ and the metric perturbation parameters B, E, ψ and ϕ.

It can be derived directly from the Klein-Gordon equation, and this will be done here. However since the metric now has perturbations, it is necessary to use the Klein Gordon equation in curved spacetime:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi\right) + \frac{\partial V}{\partial\varphi} = 0 \tag{2.39}$$

The general scalar-perturbed metric, (2.9), is quite complicated due to its non-diagonal components, and direct substitution into (2.39) gives rise to a very complicated equation, which is not practical to work with. However, the longitudinal gauge, in which B = E = 0, comes to the rescue here by simplifying the metric enough for direct substitution to be practical.

Since scalar perturbations are only fully described by metric perturbations supplemented with perturbations to physical scalar quantities, it is also necessary to substitute the perturbed field (2.24) to yield the perturbed equation of motion. (It should be noted that this only results in a complete description of the kinematics for the field perturbations. Einstein's equations are needed to couple the field to spacetime, giving a fully dynamical description of the whole system – see later section).

The scalar-perturbed metric in the longitudinal gauge is

$$g_{00} = a^2(1+2\phi) \qquad g_{ij} = a^2(2\psi - 1)\delta_{ij} \qquad g_{0i} = 0 \tag{2.40}$$

, with

$$\sqrt{-g} = ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}\,. \tag{2.41}$$

Substituting this metric along with the perturbed field (2.24) into the Klein Gordon equation (2.39) gives an expression which, through quite a long process of simplification, can be reduced down to [A.23]:

$$\begin{aligned}&\varphi_0'' + 2H\,\varphi_0' + a^2V,_{\varphi} + \delta\varphi'' + 2H\,\delta\varphi' - \nabla^2\delta\varphi + a^2V,_{\varphi\varphi}\,\delta\varphi - \varphi_0'(3\psi' + \phi')\\&+ 2a^2V,_{\varphi}\,\phi = 0\end{aligned} \tag{2.42}$$

It is important to note here that the first 3 homogeneous background terms are independent of the perturbed terms. That is, the homogeneous terms can be extracted to recover the original homogeneous equation of motion (1.18):

$$\varphi_0'' + 2H\,\varphi_0' + a^2V,_{\varphi} = 0 \tag{2.43}$$

This leaves the equation of motion for the perturbations in their own right:

$$\delta\varphi'' + 2H\,\delta\varphi' - \nabla^2\delta\varphi + a^2V,_{\varphi\varphi}\,\delta\varphi - \varphi_0'(3\psi' + \phi') + 2a^2V,_{\varphi}\,\phi = 0 \tag{2.44}$$

2.10 Perturbing Einstein's equations

Einstein's equations are

$$G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu} \tag{2.45}$$

These can be expressed with perturbations by adding linear perturbation terms:

$$G^{\mu}_{\nu} + \delta G^{\mu}_{\nu} = 8\pi\left(T^{\mu}_{\nu} + \delta T^{\mu}_{\nu}\right) \tag{2.46}$$

With the unperturbed background metric (2.13), the components of G^{μ}_{ν} are as follows [4]:

$$G_0^0 = \frac{3\mathcal{H}^2}{a^2} \qquad G_i^0 = 0 \qquad G_j^i = \frac{1}{a^2}\left(2\mathcal{H}' + \mathcal{H}^2\right)\delta_{ij} \tag{2.47}$$

It follows from (2.45) that the components of the stress-energy tensor T_ν^μ must satisfy

$$T_i^0 = 0 \qquad T_j^i \propto \delta_j^i \tag{2.48}$$

In the same way that the perturbations to the metric and the scalar field φ were considered separately above, so too are the perturbations to the stress-energy tensor, and Einstein's tensor G_ν^μ:

$$\delta G_\nu^\mu = 8\pi \delta T_\nu^\mu \tag{2.49}$$

The behaviour of the perturbations to the stress-energy tensor, δT_ν^μ, under the gauge transformation (2.10), is most easily calculated by considering the individual components. The δT_0^0 in its own right is just a scalar, the δT_i^0 components can be considered as the components of a vector etc. Then the results (2.18) and (2.19) can be used to find their transformation behaviour in gauge-invariant form. The results are not derived here but they are [4]:

$$\overline{\delta T}{}_0^0 = \delta T_0^0 - \left({}^{(0)}T_0^0\right)'\left(B - E'\right) \tag{2.50}$$

$$\overline{\delta T}{}_i^0 = \delta T_i^0 - \left({}^{(0)}T_0^0 - \frac{1}{3}{}^{(0)}T_j^j\right)\left(B - E'\right)' \tag{2.51}$$

$$\overline{\delta T}{}_j^i = \delta T_j^i - \left({}^{(0)}T_0^0\right)'\left(B - E'\right) \tag{2.52}$$

The same procedure is done for δG_ν^μ:

$$\overline{\delta G}{}_0^0 = \delta G_0^0 - \left({}^{(0)}G_0^0\right)'\left(B - E'\right) \tag{2.53}$$

$$\overline{\delta G}{}_i^0 = \delta G_i^0 - \left({}^{(0)}G_0^0 - \frac{1}{3}{}^{(0)}G_j^j\right)\left(B - E'\right)' \tag{2.54}$$

$$\overline{\delta G}{}_j^i = \delta G_j^i - \left({}^{(0)}\overline{G}{}_0^0\right)'\left(B - E'\right) \tag{2.55}$$

This allows for the Einstein equations for the perturbations to be written in gauge-invariant form:

$$\overline{\delta G}_{\nu}^{\mu} = 8\pi \overline{\delta T}_{\nu}^{\mu} \tag{2.56}$$

The scalar perturbation metric in the longitudinal gauge, equation (2.34), allows for the components of $\overline{\delta G}_{\nu}^{\mu}$ to be calculated in terms of the parameters Ψ and Φ. These come out as [4]:

$$\nabla^2\Psi - 3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) = 4\pi a^2 \overline{\delta T}_0^0 \tag{2.57}$$

$$\nabla_i\left(\Psi' + \mathcal{H}\Phi\right) = 4\pi a^2 \overline{\delta T}_i^0 \tag{2.58}$$

$$\begin{aligned}&\left(\Psi'' + \mathcal{H}\left(2\Psi + \Phi\right)' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{2}\nabla^2\left(\Phi - \Psi\right)\right)\delta_{ij} \\ &-\frac{1}{2}\nabla_i\nabla_j\left(\Phi - \Psi\right) = 4\pi a^2 \overline{\delta T}_j^i\end{aligned} \tag{2.59}$$

The spatial components of the perturbation to the stress-energy tensor are all zero for $i \neq j$, which corresponds to zero shear stress. This is significant as it results in the following simplification:

$$\Psi = \Phi \tag{2.60}$$

With this simplification, the equation of motion (2.44) now reads

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2V,_{\varphi\varphi}\delta\varphi - 4\varphi_0'\Phi' + 2a^2V,_{\varphi}\Phi = 0\,. \tag{2.61}$$

By using (1.6) to find explicit expression for perturbations to the stress-energy tensor, and also using (2.60), the RHS of equation (2.58) (which is the most useful for the following section) then takes the following form [4]:

$$\Phi' + \mathcal{H}\Phi = 4\pi\varphi_0'\delta\varphi \tag{2.62}$$

A complete description of cosmological perturbations induced by a scalar field requires coupling of the physical field with the spacetime perturbations. This means combining Einstein's equations expressed in terms of $\delta\varphi$ with the equation of motion for $\delta\varphi$. In the case of scalar perturbations, the equations of interest are the equation of motion, (2.61), and the stress-energy-tensor perturbations (2.57) – (2.59).

3. Structure Formation

3.1 The latest measurements of anisotropy in the CMB

The present day matter power spectrum is measured using galaxy redshift surveys. However, redshift surveys are more time consuming than angular surveys and suffer from the problem that accurate measurements of galaxy recession velocities do not translate into unambiguous measurements of radial distance, due to the peculiar velocities associated with the galaxies being observed [13].

An angular power spectrum of the CMB captures the structure of anisotropy in a much more revealing way. The CMB exhibits a blackbody spectrum, which is determined solely by its temperature. Its temperature has been measured by the FIRAS instrument on the COBE satellite as 2.725 +/- 0.002 K in the wavelength range 0.5 - 5 mm [14]. The CMB anisotropy becomes clear on a contrast ratio of $\sim 10^{-5}$. Figure 4 shows the CMB angular temperature distribution going through the stages of subtracting superfluous signals, using the data collected by WMAP. The range of the temperature deviations shown on the bottom right panel is approximately 0.0005 K [15].

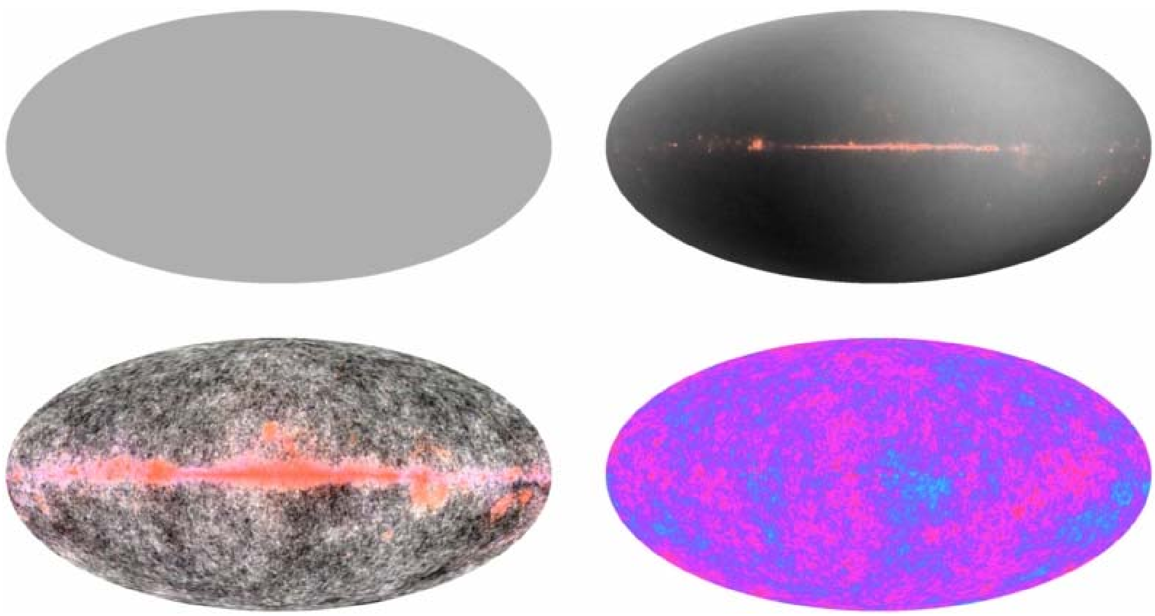

Fig. 4 - The top left panel is the raw CMB map, showing the 'white noise' signal. The top right panel shows the average-subtracted signal, which shows the Milky Way, and has a clear dipole pattern caused by the Doppler shift associated with the Earth's own peculiar velocity. The bottom left panel shows the dipole-subtracted map with a higher contrast ratio. The Milky Way signal is subtracted to give the bottom right panel, which displays the temperature deviations from the measured average. Image taken from http://www.astro.ucla.edu/~wright/CMB-DT.html

The angular power spectrum is conventionally expressed as an expansion in multipoles, or spherical harmonics, whose coefficients, labelled l, correspond to the intensity of the corresponding l-pole exhibited in the map. A peak around $l = k$ would indicate a higher incidence of structures corresponding to an angular size (180/k)°. Figure 5 shows the latest measurement of the CMB angular power spectrum.

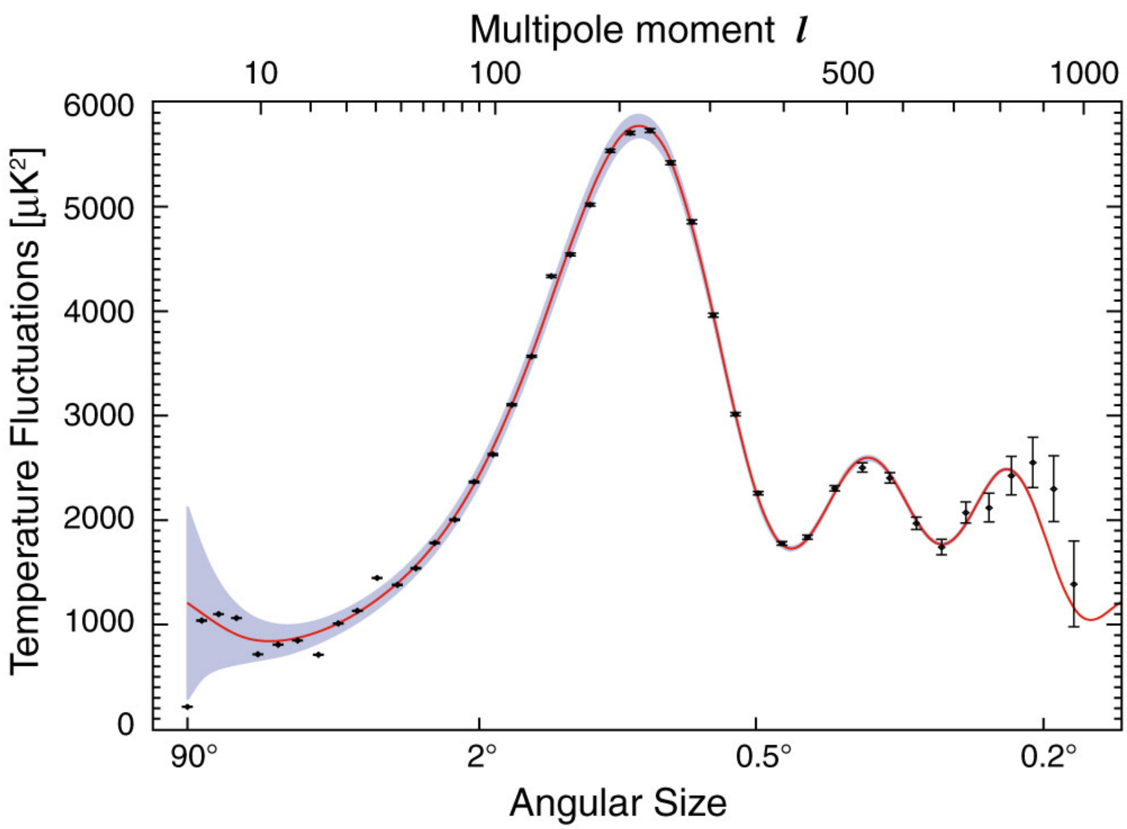

Fig. 5 - The angular power spectrum of the CMB measured by using the latest 5-year data from WMAP. The red curve shows the prediction of a ΛCDM inflation model prediction. Image taken from http://map.gsfc.nasa.gov.

There are localized peaks, but given that the temperature variations are to the order of 10^{-5}, the basic message to be taken from the angular power spectrum of the CMB, at least for the purposes of this paper, is that the post-inflation primordial spectrum may be expected to not show any significant scale dependence.

It was mentioned in the introduction that the field perturbations are treated as Gaussian. Since all non-linear perturbations have been neglected, the Fourier modes behave independently so it is easy to deal with the remaining linear perturbations in Fourier space. Any spatial function $f(x)$ can be expressed as an integral over its Fourier modes. The spatial two-point correlation function of the function f is a Gaussian random field. Such a field is completely characterized by its variance, which for the function $f(x)$ is the expectation value of two of its Fourier coefficients:

$$\left\langle f_{\underline{p}} f_{\underline{q}} \right\rangle = \sigma^2{}_{\underline{p}} \delta\left(\underline{p} + \underline{q}\right) \tag{3.0}$$

, where $\sigma^2{}_{\underline{p}}$ is the variance.

With the identification $\sigma^2{}_{\underline{k}} = \left|\Phi_k\right|^2$, the quantity of interest for calculating the power spectrum for the metric perturbations will be the dimensionless variance:

$$\delta_\Phi^2(k) \equiv \frac{\left|\Phi_k\right|^2 k^3}{2\pi^2} \tag{3.1}$$

3.2 Perturbations on the quantum scale

In section 1.1.5, the notion that sub-horizon distance scales are inflated out of the horizon to become super-horizon was touched upon. That notion can now be applied in a specific way in the context of vacuum fluctuations in the inflaton field, with a wavelength, λ, which characterizes the scale of the fluctuation, playing the role of the sub-horizon distance scale. These vacuum fluctuations are rapidly redshifted in proportion to the scale factor during inflation, so inhomogeneities induced by these fluctuations therefore grow (in the comoving sense) at superluminal speeds. It is in this sense that the fluctuations in φ can be considered as the primordial seeds for the present day structure of the universe.

3.3 Finding a solution for $\delta\varphi$ in the sub-horizon limit

The primary task here is to simplify the equation of motion (2.61) and the Einstein equation (2.62) using the sub-horizon limit, the slow-roll approximation and the assumption of zero shear stress, in aid of finding a solution for $\delta\varphi$.

λ is (initially) much smaller than the Hubble radius and increases proportionally with the scale factor. This is the sub-horizon limit.

$$\frac{1}{H} >> \lambda = \frac{2\pi a}{k} \sim \frac{a}{k} \tag{3.2}$$

, where k is the wavevector corresponding to λ.

Also, in the slow-roll approximation [A.24],

$$\frac{1}{H} \approx a\left|\eta\right|. \tag{3.3}$$

Since the equation of motion (2.61) contains only linear perturbation terms, the perturbation modes behave independently in Fourier space. It is therefore useful, with the help of condition (3.3), to express the small λ condition (3.1) in terms of k:

$$k|\eta| >> 1 \tag{3.4}$$

The potential is still arbitrary, so long as the slow-roll conditions are satisfied, so it should be included in the final answer for the power spectrum. To this end, the 1st Friedmann equation can be combined with the slow-roll conditions (1.10) and (1.20) to eliminate the φ terms and express the slow-roll approximation purely in terms of the potential, and its derivatives [A.25]:

$$\left(\frac{V,_{\varphi}}{V}\right)^2 << 1 \tag{3.5}$$

$$\left|V,_{\varphi\varphi}\right| << |V| \sim H^2 \tag{3.6}$$

Equation (1.10) converted into conformal time also comes in useful:

$$(\varphi_0')^2 << a^2 V \tag{3.7}$$

The other approximation to be used is the approximate oscillatory behaviour of the gravitational field in the slow-roll case. It is a matter of straightforward inspection of (2.62) to see that Φ behaves as [A.25b]:

$$\Phi' \sim k\Phi \Rightarrow \Phi \sim \frac{\varphi_0'\delta\varphi}{k} \tag{3.8}$$

3.4 Calculating the primordial power spectrum

The approach in this section will be to solve for $\delta\varphi$ in the sub-horizon limit, and then to solve for $\delta\varphi$ and Φ in the super-horizon limit and use Einstein's equations to find the spectrum of metric perturbations Φ in terms of $\delta\varphi$.

Since the equation of motion (2.61) contains only linear perturbation terms, the perturbation modes behave independently in Fourier space. Each mode can be labelled with its corresponding wavevector and considered separately. It is therefore useful to perform a Fourier transform on (2.61) and seek a solution as a function of k. It should also be noted that under a Fourier transform, F, $\nabla^2 \xrightarrow{F} -k^2$.

Before this is done however, the equation can be largely simplified using the approximations (3.2), (3.5), (3.6), (3.7) and (3.8). Doing this shows the last 3 terms of

(2.61) to be negligible compared to the first 3. The most straightforward way to show this is to show that each of the last 3 terms is negligible compared to the spatial derivative term, $\nabla^2 \delta\varphi$ (this is easiest since ∇^2 corresponds directly to k^2, which is very large), and then to show that the terms $\delta\varphi''$ and $2H\,\delta\varphi'$ are of the same order of magnitude as $\nabla^2 \delta\varphi$. The details of this process are shown in [A.26]. Thus only the first 3 terms remain and the equation reduces to:

$$\delta\varphi_k'' + 2H\,\delta\varphi_k' - k^2 \delta\varphi_k \approx 0 \tag{3.9}$$

As the subscript labels now indicate, the solution to this equation will correspond to a particular value of k, so will necessarily be a function of k. It is in fact the equation of a simple harmonic oscillator with a variable mass, which can be made obvious by rescaling

$$a\delta\varphi_k \equiv u_k \tag{3.10}$$

to get [A.27]

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k = 0. \tag{3.11}$$

The frequency is clearly $\sqrt{k^2 - \frac{a''}{a}}$ and carries time dependence so let

$$\omega(k,\eta) = \sqrt{k^2 - \frac{a''}{a}}. \tag{3.12}$$

It can be shown that in the adiabatic limit [A.28],

$$k^2 >> \frac{a''}{a} \tag{3.13}$$

so that in fact, $\omega(k,\eta) \to \pm k$, causing equation (3.11) to reduce to

$$u_k'' + k^2 u_k = 0 \tag{3.14}$$

, which has the straightforward solution

$$u_k = a\delta\varphi_k = \left(A_k \exp(ik\eta) + B_k \exp(-ik\eta)\right)\Big|_{k|\eta|>>1} \approx A_k \exp(ik\eta) \tag{3.15}$$

The considerations so far have been purely classical, but it is essential to recognize $\delta\varphi$ as a quantum field for the sake of deciding on a suitable boundary condition to evaluate A_k.

In light of this, a short digression into how the solution u_k should be regarded from the point of view of QFT follows. This will also shed some light on the physical meaning of the adiabatic assumption employed in finding the solution (3.15).

<u>*Quantum digression*</u>

Under quantization, u_k will become a sum of creation and annihilation operators:

$$u_k \rightarrow \hat{u}_k = a\delta\hat{\varphi}_k = U(k,\eta)\hat{a}_k + U^*(k,\eta)\hat{a}_k{}^{\dagger} \qquad (3.16)$$

, with the canonically quantized scalar field commutation relations:

$$\left[\hat{a}_k, \hat{a}_{k'}{}^{\dagger}\right] = (2\pi)^3 \delta^{(3)}(k-k') \qquad \left[\hat{a}_k, \hat{a}_{k'}\right] = \left[\hat{a}_k{}^{\dagger}, \hat{a}_{k'}{}^{\dagger}\right] = 0 \qquad (3.17)$$

, where $U(\omega,\eta)$ is an amplitude to be found.

The power spectrum is defined using in terms of correlation functions. The bridge joining the quantum fluctuation $\delta\varphi$ and the primordial power spectrum will therefore come in the form of a correlation function for the (rescaled) field perturbation (c.f. (3.1)):

$$\delta_{u_k}^2(k) = \langle\Omega|\hat{u}_k\hat{u}^{\dagger}{}_{k'}|\Omega\rangle\frac{k^3}{2\pi^2} \qquad (3.18)$$

, where $|\Omega\rangle$ denotes the vacuum taking into account interactions.

Here the notation k' does not denote a conformal time derivative, but is merely used to distinguish the two momentum labels in the commutation relations.

In the sub-horizon limit, $k|\eta| >> 1$, the metric perturbations may be neglected on curvature scales associated with the initial scale of the field perturbation. That is to say that the standard Minkowski metric (2.8) is a good approximation of the metric on such scales. This is the adiabatic limit, and it implies that all of the states of $\delta\varphi$ are non-interacting and in the ground state, so the vacuum $|\Omega\rangle$ can in fact be treated as the non-interacting vacuum $|0\rangle$. This makes it easy to extract the amplitude $U(k,\eta)$ from the vacuum correlation function using the canonical commutation relations [A.29]:

$$\langle 0|\hat{u}_k\hat{u}^{\dagger}{}_{k'}|0\rangle = (2\pi)^3|U(k,\eta)|^2 \qquad (3.19)$$

It can be seen by comparing (3.1) and (3.18) that under the canonical quantization procedure

$$|\delta\varphi_k|^2 = a^2|u_k|^2 \rightarrow a^2(2\pi)^3|U(k,\eta)|^2 \tag{3.20}$$

, so the operator $\hat{u}_k$ is seen to produce a superposition of plane waves:

$$\hat{u}_k = A_k \exp(ik\eta)\hat{a}_k + B^*{}_k \exp(-ik\eta)\hat{a}^\dagger{}_k \tag{3.21}$$

, with

$$\hat{u}_k\Big|_{k|\eta|>>1} \approx A_k \exp(ik\eta)\hat{a}_k \qquad \hat{u}^\dagger{}_k\Big|_{k|\eta|>>1} \approx A^*{}_k \exp(-ik\eta)\hat{a}_k{}^\dagger \tag{3.22}$$

This should be compared to (3.15), which concludes the digression.

Now returning to the issue of deciding on a boundary condition, without any rigorous quantum theory, the evaluation of A_k can only be an estimate. It will not be derived here but simply quoted [4]:

$$A_k = \frac{1}{\sqrt{k}} \Rightarrow u_k = a\delta\varphi_k = \frac{1}{\sqrt{k}}\exp(ik\eta) \tag{3.23}$$

Using (3.1), the spectrum evaluated for the initial quantum scale fluctuation then reads:

$$\delta^2_{\delta\varphi}(k) = \frac{\left|\frac{u_k}{a}\right|^2 k^3}{2\pi^2} = \frac{k^2}{2\pi^2 a^2} \tag{3.24}$$

In order to predict the scale dependence of the post-inflation spectrum, it is now necessary to consider the behaviour of the fluctuation as it is stretched to super-horizon scales.

When the fluctuation crosses the horizon, $k = 2\pi aH \sim aH$, which can be applied to (3.24), resulting in a boundary condition to be used in the super-horizon case:

$$\delta^2_{\delta\varphi}(k)\Big|_{k\sim aH} = \frac{1}{2\pi^2}H^2\Big|_{k\sim aH} \sim H^2{}_{k\sim aH} \tag{3.25}$$

Now we want to study the evolution of the perturbation as it grows to a super-horizon scale. For this purpose, it is easiest to express the equation of motion (2.61) and the Einstein equation (2.62) in physical time [A.30]:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \nabla^2\delta\varphi + V,_{\varphi\varphi}\delta\varphi - 4\dot{\varphi}_0\dot{\Phi} + 2V,_{\varphi}\Phi = 0 \tag{3.26}$$

$$\dot{\Phi} + H\Phi = 4\pi\dot{\varphi}_0\delta\varphi \tag{3.27}$$

On the super-horizon scale, the $\nabla^2\delta\varphi$ term can be neglected on intuitive grounds. As the initial plane wave fluctuation (3.23) is stretched to the super-horizon scale, it is clear that spatial derivative terms will become negligibly small. The slow-roll approximation (1.20) can then be used to neglect the $\delta\ddot{\varphi}$ term and the $\dot{\Phi}$ term, giving

$$3H\delta\dot{\varphi} + V,_{\varphi\varphi}\delta\varphi + 2V,_{\varphi}\Phi \approx 0 \tag{3.28}$$

$$H\Phi \approx 4\pi\dot{\varphi}_0\delta\varphi . \tag{3.29}$$

These can be further simplified [A.31]:

$$3H\frac{\partial}{\partial t}\left(\frac{\delta\varphi}{V,_{\varphi}}\right) + 2\Phi \approx 0 \tag{3.30}$$

$$H\Phi \approx 4\pi\dot{V}\left(\frac{\delta\varphi}{V,_{\varphi}}\right) \tag{3.31}$$

Now by using the simplified Friedmann equation $H \approx \sqrt{\frac{8\pi G}{3}V(\varphi)}$, (3.30) and (3.31) can be combined into a single equation [A.32] which is easily solved to give the behaviour of the perturbation $\delta\varphi$ and the metric perturbation Φ,

$$\frac{d}{dt}\left(\frac{V\delta\varphi}{V,_{\varphi}}\right) = 0 \Rightarrow \frac{V\delta\varphi}{V,_{\varphi}} = C \tag{3.32}$$

, where C is the integration constant

.In Fourier space this reads:

$$\delta\varphi_k = C_k\frac{V,_{\varphi}}{V} \tag{3.33}$$

It is assumed here that the amplitude of the fluctuation here has the minimum amplitude $|\delta\varphi_k| = \frac{1}{a\sqrt{k}}$ (equation 3.23) at horizon crossing so that the horizon crossing boundary condition (3.25) can be used to evaluate C_k. Equating this with $\delta\varphi_k = C_k \frac{V_{,\varphi}}{V}$ at horizon crossing gives $\left.\frac{1}{a\sqrt{k}}\frac{V}{V_{,\varphi}}\right|_{k\sim aH} = C_k$, so the fluctuation amplitude in the super-horizon limit reads:

$$\delta\varphi_k = \left.\left(\frac{aV}{V_{,\varphi}}\right)\right|_{k\sim aH} \frac{1}{\sqrt{k}}\left(\frac{V_{,\varphi}}{V}\right) = \left.\left(\frac{V}{V_{,\varphi}}\right)\right|_{k\sim aH} \frac{H}{k^{3/2}}\left(\frac{V_{,\varphi}}{V}\right) \tag{3.34}$$

Finally, this can be substituted back into the Einstein equation (3.31) (in k space) to obtain Φ_k in terms of V [A.33]:

$$\Phi_k \approx -\frac{1}{2}\left(\frac{V_{,\varphi}}{V}\right)\delta\varphi_k = -\left.\left(\frac{V}{V_{,\varphi}}\right)\right|_{k\sim aH} \frac{H}{2k^{3/2}}\left(\frac{V_{,\varphi}}{V}\right)^2 \tag{3.35}$$

To determine the power spectrum at the *end* of inflation, it is necessary to decide what exactly is meant by the end. Throughout this paper, the inflation stage has been defined by the slow-roll approximation, so, looking at the form of equation (3.35), it is obvious that allowing the slow-roll condition (3.5) to be violated is an easy way of defining the end of inflation. This can be expressed as such:

$$\left(\frac{V_{,\varphi}}{V}\right)^2 \sim \left(\frac{V_{,\varphi}}{V}\right) \sim 1 \tag{3.36}$$

The post-inflation power spectrum is therefore defined as the spectrum at the time when (3.36) first becomes satisfied. Remembering that the perturbation crosses the horizon before the end of inflation, so that the $\left.\left(\frac{V}{V_{,\varphi}}\right)\right|_{k\sim aH}$ term still satisfies $\frac{V}{V_{,\varphi}} >> 1$, this gives:

$$\Phi_k \sim \left.\left(\frac{V}{V_{,\varphi}}\right)\right|_{k\sim aH} \frac{H}{k^{3/2}} \tag{3.37}$$

, with the corresponding power spectrum:

$$\left.\delta_{\Phi}^{2}(k)\right|_{V_{,\varphi}\sim V}=\frac{|\Phi_k|^2 k^3}{2\pi^2}\sim\left(\left.\left(\frac{V}{V_{,\varphi}}\right)\right|_{k\sim aH}\right)^2 H^2\sim\left(\left.\left(\frac{V^{3/2}}{V_{,\varphi}}\right)\right|_{k\sim aH}\right)^2 \tag{3.38}$$

, where $H\approx\sqrt{\frac{8\pi G}{3}V(\varphi)}$ was used.

A potential needs to be picked now to see the scale-dependence of this spectrum. For the power-law potential (1.25) with $\Lambda=m^2$ and n = 2, the potential is that of a massive scalar field. In this case the power spectrum is:

$$\delta_{\Phi}^{2}(k)\sim\left(\left.\left(\frac{m^3\varphi^3}{2^{3/2}m^2\varphi}\right)\right|_{k\sim aH}\right)^2\sim m^2\left(\left.\varphi^2\right|_{k\sim aH}\right)^2 \tag{3.39}$$

Now equation (1.26), which expresses the scale factor as a function of φ, can finally be put to use. By expressing the field in (3.39) in terms of the scale factor and using equation (3.2) to relate the scale factor to the wavelength of the perturbation in question, the power spectrum (3.39) can be expressed as a function of the perturbation wavelength [A.34]:

$$\delta_{\Phi}^{2}(k)\sim m^2\left(\ln(\lambda H)\right)^2 \tag{3.40}$$

The Hubble parameter is evaluated at horizon crossing, although is essentially constant at all times during inflation.

This is the key result of the paper. The effect of the logarithm is to suppress perturbations for all values of λ. This brings large scale perturbations into the regime of smaller ones, and the resulting spectrum is nearly flat, i.e. approximately scale-invariant.

4. Concluding remarks

The prediction for the primordial spectrum differs from the perfectly flat Harrison-Zeldovich spectrum by a logarithmic scale-dependence. However, the obvious first remark to be made is that the predicted primordial spectrum (3.40) is many calculations short of being an analytic prediction of the observed CMB spectrum. The spectrum occurs immediately before reheating, when the age of the universe is ~10^{-35} s, whereas the measured spectrum of the CMB shows the universe at recombination (assuming zero scattering along the way), i.e. at an age of approximately 377000 years [16]. Of course, a lot could happen in 377000 years so the natural thing to ask is how the spectrum might be preserved, or otherwise, during this time period, in such a way as to evolve into the observed CMB spectrum.

When the inflation paradigm first emerged around 1982, the field φ was thought to decay during the subsequent reheating phase into standard model particles via field oscillations, typically ~ $O(10^3)$ oscillations [17], which could be described in the conventional perturbative way [18]. In 1997, it was found that coherent field effects such as parametric resonance may have been responsible for causing φ to decay within fewer oscillations than that predicted by perturbative methods [19], [20]. The term 'preheating' was coined for this process. The preheating stage, should it have been physically realized, is thought to have implications for baryogenesis [17], which occurs after (p)reheating. Faster decay mechanisms have been considered [21], which involve tachyonic decay. The decay mechanism used depends on the inflation model being studied.

In fact, it can be shown that the primordial spectrum is largely unaffected by the reheating phase and furthermore, that all subsequent phases leading up to recombination are not particularly significant to the produced spectrum [22]. This fact allows for a fairly accurate spectrum prediction to be made with the simplification of assumed instantaneous recombination. With or without the assumption of instantaneous recombination, spectrum predictions depend crucially on the Hubble radius at the time of recombination, which distinguishes frozen-out perturbations which were super-horizon by the time of recombination from perturbations which were sub-horizon by this time.

Sub-horizon modes at recombination require a comprehensive description of the transfer and growth functions in the full Einstein-Boltzmann system. This has in fact been done analytically by Mukhanov [22], although the full calculation still involves some degree of fitting the prediction to observed data. The parameters on which the calculation relies are density parameters associated with dark energy and baryonic matter, as well as the temperature measurements of the CMB measured by WMAP, and the calculation is very sensitive to these parameters [22]. The extra complexity of the sub-horizon spectrum can be appreciated by referring to figure 5. The spectrum for angles below ~0.87°, which corresponds to the Hubble scale at recombination [22], shows a more intricate set of angular peaks.

Modes which were super-horizon at recombination are not subject to the evolution effects of sub-horizon modes so in principle are easier to predict. However, there is no free lunch with super-horizon modes either, due to the cost of cosmic variance on the accuracy of observations corresponding to smaller multipoles.

There is one parameter which is a completely free parameter in all models of inflation, which is the objective amplitude of fluctuations, and therefore the amplitude of measured temperature anisotropies. The predicted curve can be normalized to provide a best-fit curve, and it turns out that the normalized spectrum fits very well with the observed data, indicating a high level of predictive power for relative amplitudes [22].

One factor which was ignored in this paper was tensor contributions to the spectrum. No unambiguous measurements of gravitational waves yet exist, however studies of the relative significance of scalar and tensor perturbations to the spectrum are ongoing

[9][23][24]. Other generally overlooked factors include multiple-field inflation models, approaches based on string theory and the multiverse.

Extensions to this paper could involve further investigation of any of the issues highlighted in the above paragraphs, and would serve to test the robustness of the prediction of the primordial spectrum made in this paper in the context of different models.

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Appendix

A.1

The 1st Friedmann equation is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

, which can be written as

$$\Omega(t) - 1 = \frac{K}{(Ha)^2}$$

, so

$$\Omega(t) - 1 = \frac{\left(H(t_0)a(t_0)\right)^2}{\left(H(t_i)a(t_i)\right)^2}\left(\Omega(t_0) - 1\right) \leq 10^{-56}.$$

A.2

The event horizon of a universe is

$$d_e(t) = a(t)\int_t^{t_{\max}} \frac{dt}{a(t)}$$

If the universe in question is undergoing accelerated expansion, this integral is convergent and there exists a finite event horizon. For two events separated by an interval $d_e(t)$, the expansion of the intermediate spacetime separating the two events prevents any future interaction.

The particle horizon changes with time and is equal to

$$a(t)\int_t^{t_0} \frac{dt}{a(t)} \sim \frac{a(t)}{a_i} d_e(t)$$

A.3

The Hubble radius at time t is

$$\frac{1}{a(t)H(t)}$$

Hence an accelerating scale factor is required for the Hubble radius to decrease.

A.4

2nd Friedmann equation:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right)a$$

Strong energy dominance condition:

$$\rho + 3p > 0$$

A.5

$$T^{\mu}{}_{\nu} \equiv \frac{\partial L}{\partial\left(\partial_{\mu}\varphi\right)}\left(\partial_{\nu}\varphi\right) - \delta^{\mu}{}_{\nu}L$$

$$=\left(\frac{\partial}{\partial(\partial_\mu\varphi)}\left[\frac{1}{2}g^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi)-V(\varphi)\right]\right)(\partial_\nu\varphi)-\delta^\mu{}_\nu\left(\frac{1}{2}g^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi)-V(\varphi)\right)$$

$$=\frac{1}{2}\left(g^{\mu\beta}(\partial_\beta\varphi)+g^{\alpha\mu}(\partial_\alpha\varphi)\right)(\partial_\nu\varphi)-\frac{1}{2}\left(\partial^2\varphi\right)\delta^\mu{}_\nu+\delta^\mu{}_\nu V(\varphi)$$

$$=\left(\partial^\mu\varphi\right)(\partial_\nu\varphi)-\frac{1}{2}\left(\dot{\varphi}^2\right)\delta^\mu{}_\nu+\delta^\mu{}_\nu V(\varphi)$$

In the last line, $\varphi=\varphi(t)\Rightarrow\nabla\varphi=0$ was used.

If now the first term is expanded as follows:

$$\left(\partial^\mu\varphi\right)(\partial_\nu\varphi)=\left(\frac{1}{2}\left(\partial^\alpha\varphi\right)(\partial_\alpha\varphi)+V(\varphi)+\frac{1}{2}\left(\partial^\alpha\varphi\right)(\partial_\alpha\varphi)-V(\varphi)\right)\left(\frac{\partial^\mu\varphi}{\sqrt{\left(\partial^\alpha\varphi\right)(\partial_\alpha\varphi)}}\right)\left(\frac{\partial_\nu\varphi}{\sqrt{\left(\partial^\alpha\varphi\right)(\partial_\alpha\varphi)}}\right)$$

$$=\left(\frac{1}{2}\dot{\varphi}^2+V(\varphi)+\frac{1}{2}\dot{\varphi}^2-V(\varphi)\right)\left(\frac{\partial^\mu\varphi}{\sqrt{\dot{\varphi}^2}}\right)\left(\frac{\partial_\nu\varphi}{\sqrt{\dot{\varphi}^2}}\right)$$

The stress-energy tensor now reads:

$$T^\mu{}_\nu=\left(\frac{1}{2}\dot{\varphi}^2+V(\varphi)+\frac{1}{2}\dot{\varphi}^2-V(\varphi)\right)\left(\frac{\partial^\mu\varphi}{\sqrt{\dot{\varphi}^2}}\right)\left(\frac{\partial_\nu\varphi}{\sqrt{\dot{\varphi}^2}}\right)-\left(\frac{1}{2}\left(\dot{\varphi}^2\right)+V(\varphi)\right)\delta^\mu{}_\nu$$

This can be identified as the stress-energy tensor for a perfect fluid with 4-velocity V^μ:

$$T^\mu{}_\nu=(\rho+p)V^\mu V_\nu-p\delta^\mu{}_\nu$$

, leading to the identifications

$$\rho=\frac{1}{2}\dot{\varphi}^2+V(\varphi)\,,\qquad\qquad p=\frac{1}{2}\dot{\varphi}^2-V(\varphi)\,.$$

A.6

First taking the square root of the first Friedmann equation, and taking the time derivative:

$$\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right)=\frac{\partial H}{\partial t}=\frac{\partial}{\partial t}\sqrt{\frac{8\pi G}{3}\rho}=\sqrt{\frac{2\pi G}{3}}\frac{\dot{\rho}}{\sqrt{\rho}}$$

On the other hand,

$$\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right)=\frac{\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2$$

Combining these two:

$$\frac{\ddot{a}}{a}=\sqrt{\frac{2\pi G}{3}}\frac{\dot{\rho}}{\sqrt{\rho}}+\left(\frac{\dot{a}}{a}\right)^2$$

So equating the LHS with the second Friedmann equation gives:

$$\sqrt{\frac{2\pi G}{3}}\frac{\dot{\rho}}{\sqrt{\rho}}+\left(\frac{\dot{a}}{a}\right)^2=-\frac{4\pi G}{3}(\rho+3p)$$

Now using $H^2=\frac{8\pi G}{3}\rho$ and $H^2=\left(\frac{\dot{a}}{a}\right)^2$ causes this equation to reduce to the energy conservation equation:

$$\dot{\rho}=-3H(\rho+p)$$

A.7

Substituting equations (1.8) and (1.9), the expressions for the energy density and pressure in terms of φ, into the energy conservation equation:

$$\dot{\rho}=-3H(\rho+p)$$

, gives:

$$\dot{\rho}=-3H\left(\frac{\dot{\varphi}^2}{2}+V(\varphi)+\frac{\dot{\varphi}^2}{2}-V(\varphi)\right)=-3H\dot{\varphi}^2$$

which can be compared with the time derivative of (1.8):

$$\frac{\partial}{\partial t}\left(\frac{\dot{\varphi}^2}{2}+V(\varphi)\right)=\ddot{\varphi}+\frac{\partial V(\varphi)}{\partial\varphi}\dot{\varphi}$$

, to give

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 .$$

A.8

To convert (1.17) into conformal time, use $d\eta a(t) = dt$ to yield, term by term:

$$\ddot{\varphi} = \frac{1}{a^2}\varphi'' - \frac{1}{a^2}\dot{a}\varphi'$$

$$3H\dot{\varphi} = 3\mathcal{H}\,\varphi'$$

, and substitute into (1.17) to obtain

$$\varphi'' + 2\mathcal{H}\,\varphi' + a^2\frac{\partial V}{\partial \varphi} = 0$$

, where $\mathcal{H} \equiv \dfrac{a'}{a}$ and $H \equiv \dfrac{\dot{a}}{a}$

A.9

By using some basic algebra to expand the definition of the Hubble parameter as such:

$$H = \frac{\dot{a}}{a} = \frac{d(\ln a)}{dt} = \frac{d(\ln a)}{d\varphi}\frac{d\varphi}{dt}$$

, then using the simplified equation of motion (1.21) to substitute for $\dfrac{d\varphi}{dt}$:

$$= \frac{d(\ln a)}{d\varphi}\left(\frac{-1}{3H}\frac{\partial V}{\partial \varphi}\right)$$

and using the simplified Friedmann equation (1.22) to substitute for H:

$$= \frac{d(\ln a)}{d\varphi}\left(\frac{-1}{3}\sqrt{\frac{3}{8\pi G}}\frac{\partial V}{\partial \varphi}\right)$$

which can be equated with the square root of (1.22) to give

$$\frac{d(\ln a)}{d\varphi}\left(\frac{-1}{3}\sqrt{\frac{3}{8\pi G}}\frac{\partial V}{\partial \varphi}\right) = \sqrt{\frac{8\pi G}{3}V(\varphi)}$$

, which simplifies to

$$-8\pi G V(\varphi) = \frac{d(\ln a)}{d\varphi}\frac{\partial V}{\partial \varphi}.$$

A.10

Starting from

$$H = \frac{d(\ln a)}{d\varphi}\left(\frac{-1}{3}\sqrt{\frac{3}{8\pi G}}V,_{\varphi}\right)$$

as obtained in [A.9] and then substituting in $V(\varphi) = \frac{\Lambda\varphi^n}{n}$ gives

$$H = -\frac{d(\ln a)}{d\varphi}\left(\sqrt{\frac{n\Lambda}{24\pi G}}\varphi^{\frac{n-2}{2}}\right)$$

Then substituting $V(\varphi) = \frac{\Lambda\varphi^n}{n}$ into the square root of (1.22) gives

$$H = \sqrt{\frac{8\pi G V(\varphi)}{3}} = \sqrt{\frac{8\pi G\Lambda n}{3}}\varphi^{\frac{n}{2}}$$

These can be equated to yield

$$-\frac{d(\ln a)}{d\varphi}\frac{n}{8\pi G} = \varphi$$

which integrates to give

$$a(\varphi) = a_i \exp\left(\frac{4\pi G}{n}\left(\varphi^2(t_i) - \left(\varphi^2(t)\right)\right)\right)$$

A.11

The background metric:

$$^{(0)}g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The perturbation metric with vector and tensor perturbations neglected:

$$\delta g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 2\phi & B,_x & B,_y & B,_z \\ B,_x & 2\psi + 2E,_{xx} & E,_{xy} & E,_{xz} \\ B,_y & E,_{yx} & 2\psi + 2E,_{yy} & E,_{yz} \\ B,_z & E,_{zx} & E,_{zy} & 2\psi + 2E,_{zz} \end{pmatrix}$$

The complete metric:

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 1+2\phi & B,_x & B,_y & B,_z \\ B,_x & (2\psi - 1) + 2E,_{xx} & E,_{xy} & E,_{xz} \\ B,_y & E,_{yx} & (2\psi - 1) + 2E,_{yy} & E,_{yz} \\ B,_z & E,_{zx} & E,_{zy} & (2\psi - 1) + 2E,_{zz} \end{pmatrix}$$

A.12

Using the general gauge transformation (2.10) to see that

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\beta} = \delta^\alpha_\beta - \xi^\alpha,_\beta$$

, and substituting this into the transformation law for the metric (2.11) yields the following:

$$\tilde{g}_{\mu\nu} = {}^{(0)}\tilde{g}_{\mu\nu} + \delta\tilde{g}_{\mu\nu} = \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\right)\left({}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}\right)$$
$$= \left(\delta^\alpha_\mu - \xi^\alpha,_\mu\right)\left(\delta^\beta_\nu - \xi^\beta,_\nu\right)\left({}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}\right)$$
$$= \delta^\alpha_\mu \delta^\beta_\nu {}^{(0)}g_{\alpha\beta} - \delta^\alpha_\mu \xi^\beta,_\nu {}^{(0)}g_{\alpha\beta} - \xi^\alpha,_\mu \delta^\beta_\nu {}^{(0)}g_{\alpha\beta} + \xi^\alpha,_\mu \xi^\beta,_\nu {}^{(0)}g_{\alpha\beta}$$
$$+ \delta^\alpha_\mu \delta^\beta_\nu \delta g_{\alpha\beta} - \delta^\alpha_\mu \xi^\beta,_\nu \delta g_{\alpha\beta} - \xi^\alpha,_\mu \delta^\beta_\nu \delta g_{\alpha\beta} + \xi^\alpha,_\mu \xi^\beta,_\nu \delta g_{\alpha\beta}$$
$$= {}^{(0)}g_{\mu\nu} - \xi^\beta,_\nu {}^{(0)}g_{\mu\beta} - \xi^\alpha,_\mu {}^{(0)}g_{\alpha\nu} + \delta g_{\mu\nu} + O(\xi^2 g) + O(\xi\delta g) + O(\xi^2 \delta g)$$

Neglecting the non-linear terms in ξ and g, this approximates to

$$\tilde{g}_{\mu\nu}(\tilde{x}^\rho) \approx {}^{(0)}g_{\mu\nu}(x^\rho) - \xi^\beta,_\nu {}^{(0)}g_{\mu\beta} - \xi^\alpha,_\mu {}^{(0)}g_{\alpha\nu} + \delta g_{\mu\nu}$$

, which is equation (2.14).

A.13

Equating the RHS of (1.31) and (1.32) gives

$$^{(0)}g_{\mu\nu}(\tilde{x}^{\rho})+\delta\tilde{g}_{\mu\nu}\approx{}^{(0)}g_{\mu\nu}(x^{\rho})-\xi^{\beta}{}_{,\nu}{}^{(0)}g_{\mu\beta}-\xi^{\alpha}{}_{,\mu}{}^{(0)}g_{\alpha\nu}+\delta g_{\mu\nu}$$

which can be simplified by substituting

$$^{(0)}g_{\mu\nu}(x^{\rho})\approx{}^{(0)}g_{\mu\nu}(\tilde{x}^{\rho})-{}^{(0)}g_{\mu\beta,\gamma}\xi^{\gamma}$$

into the RHS and cancelling out the resultant $^{(0)}g_{\mu\nu}(\tilde{x}^{\rho})$ terms to give equation (1.33).

A.14

The reasoning used here is similar to that in A.13. The Jacobian for a scalar quantity is just 1, so the general transformation law is

$$\tilde{q}(\tilde{x}^{\rho})=q(x^{\rho})={}^{(0)}q(x^{\rho})+\delta q\,.$$

So by expressing the scalar perturbation in the new coordinates,

$$\tilde{q}(\tilde{x}^{\rho})={}^{(0)}q(\tilde{x}^{\rho})+\delta\tilde{q}$$

, equating for $\tilde{q}(\tilde{x}^{\rho})$ and using the fact that

$$^{(0)}q(x^{\rho})\approx{}^{(0)}q(\tilde{x}^{\rho})-{}^{(0)}q_{,\gamma}\xi^{\gamma}$$

to cancel the $^{(0)}q(\tilde{x}^{\rho})$ terms, gives equation (1.33), which is the gauge transformation law for the scalar perturbations:

$$\delta\tilde{q}=\delta q-{}^{(0)}q_{,\gamma}\xi^{\gamma}\,.$$

A.15

Starting with $\tilde{q}_{\mu}(\tilde{x}^{\rho})=\frac{\partial x^{\nu}}{\partial\tilde{x}^{\mu}}q_{\nu}(x^{\rho})$ and splitting the vector $q_{\nu}(x^{\rho})$ into background and perturbation parts gives:

$$\tilde{q}_{\mu}(\tilde{x}^{\rho})=\frac{\partial x^{\nu}}{\partial\tilde{x}^{\mu}}\left({}^{(0)}q_{\nu}(x^{\rho})+\delta q_{\nu}\right)$$

Also splitting it into background and perturbation parts in the new coordinates:

$$\tilde{q}_\mu(\tilde{x}^\rho)={}^{(0)}q_\mu(\tilde{x}^\rho)+\delta\tilde{q}_\mu$$

, and equating for the LHS gives

$${}^{(0)}q_\mu(\tilde{x}^\rho)+\delta\tilde{q}_\mu=\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\left({}^{(0)}q_\nu(x^\rho)+\delta q_\nu\right):$$

Now using (2.10) to express the Jacobian in terms of the infinitesimal function ξ:

$$\frac{\partial x^\nu}{\partial\tilde{x}^\mu}=\delta^\nu_\mu-\xi^\nu{}_{,\mu}$$

and the fact that

$${}^{(0)}q_\nu(x^\rho)\approx{}^{(0)}q_\nu(\tilde{x}^\rho)-{}^{(0)}q_{\nu,\gamma}\xi^\gamma$$

and substituting these both gives:

$${}^{(0)}q_\mu(\tilde{x}^\rho)+\delta\tilde{q}_\mu=\left(\delta^\nu_\mu-\xi^\nu{}_{,\mu}\right)\left({}^{(0)}q_\nu(\tilde{x}^\rho)-{}^{(0)}q_{\nu,\gamma}\xi^\gamma+\delta q_\nu\right)$$

Neglecting the quadratic term in ξ gives the covector transformation law:

$$\delta\tilde{q}_\mu=\delta q_\mu-{}^{(0)}q_{\mu,\gamma}\xi^\gamma-{}^{(0)}q_\gamma\xi^\gamma{}_{,\mu}$$

A.16

This is the natural way to split a vector into its irreducible components, a.k.a. the fundamental theorem of vector calculus, which states that any vector field can be split into the sum of a divergenceless vector field and the gradient of a scalar function.

A.17

Using (1.33) and (1.26):

$$\begin{aligned}
\delta\widetilde{g}_{00} &= \delta g_{00} - {}^{(0)}g_{00,\gamma}\xi^{\gamma} - {}^{(0)}g_{0\beta}\xi^{\beta}{}_{,0} - {}^{(0)}g_{\alpha 0}\xi^{\alpha}{}_{,0} \\
&= 2a^2\phi - {}^{(0)}g_{00,0}\xi^0 - {}^{(0)}g_{00,i}\xi^i - {}^{(0)}g_{00}\xi^0{}_{,0} - {}^{(0)}g_{0i}\xi^i{}_{,0} - {}^{(0)}g_{00}\xi^0{}_{,0} - {}^{(0)}g_{i0}\xi^i{}_{,0} \\
&= 2a^2\phi - 2aa'\xi^0 - 0 - a^2\xi^{0'} - 0 - a^2\xi^{0'} \\
&= 2a(a\phi - \left(a\xi^0\right)'
\end{aligned}$$

Similarly,

$$\begin{aligned}
\delta\widetilde{g}_{0i} &= \delta g_{0i} - {}^{(0)}g_{0i,\gamma}\xi^{\gamma} - {}^{(0)}g_{\alpha i}\xi^{\alpha}{}_{,0} - {}^{(0)}g_{0\beta}\xi^{\beta}{}_{,i} \\
&= a^2(B_{,i} + S_i) - {}^{(0)}g_{0i,0}\xi^0 - {}^{(0)}g_{0i,j}\xi^j - {}^{(0)}g_{0i}\xi^0{}_{,0} - {}^{(0)}g_{ji}\xi^j{}_{,0} - {}^{(0)}g_{00}\xi^0{}_{,i} - {}^{(0)}g_{0j}\xi^j{}_{,i} \\
&= a^2(B_{,i} + S_i) - 0 - 0 - 0 - (-a^2\delta_{ji}\left(\xi_{\perp}^j + \zeta^{,j}\right)' - a^2\xi^0{}_{,i} \\
&= a^2\left(B_{,i} + S_i + \xi_{i\perp}{}_{,0} + (\zeta_{,i} - \xi^0)_{,i}\right)
\end{aligned}$$

$$\begin{aligned}
\delta\widetilde{g}_{ij} &= \delta g_{ij} - {}^{(0)}g_{ij,\gamma}\xi^{\gamma} - {}^{(0)}g_{\gamma j}\xi^{\gamma}{}_{,i} - {}^{(0)}g_{i\delta}\xi^{\delta}{}_{,j} \\
&= a^2\left(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + h_{ij}\right) - \left(-a^2\delta_{ij}\right)'\xi^0 - \left(\partial_k\left(-a^2\delta_{ij}\right)\right)\left(\xi_{\perp}^k + \zeta^{,k}\right) - 0 \\
&\quad - \left(\delta_{kj}\left(-a^2\right)\right)\left(\partial_i\left(\xi_{\perp}^k + \zeta^{,k}\right)\right) - \left(a^2\delta_{ik}\right)\left(\partial_j\left(\xi_{\perp}^k + \zeta^{,k}\right)\right) \\
&= a^2\left(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij} + \frac{2a'}{a}\xi^0\delta_{ij} + \xi_{\perp i,j} + \xi_{\perp j,i} + 2\zeta_{,ij}\right)
\end{aligned}$$

A.18

The perturbed component $\delta\varphi$ is written as $\delta\varphi(\underline{x},\eta)$ above to clarify that the perturbations have both spatial and temporal dependence. φ_0, being the homogeneous component of the field, has no spatial dependence by definition.

A.19

Using the result (2.18):

$$\delta\widetilde{\varphi} = \delta\varphi - \left(\varphi_0\right)_{,0}\xi^0 - \left(\varphi_0\right)_{,i}\xi^i = \delta\varphi - \varphi_0'\xi^0$$

A.20

- For $\phi \to \widetilde{\phi}$

(1.26a) written in some new, gauge-transformed coordinates, reads:

$$\delta g_{00} = 2a^2\tilde{\phi}$$

(1.37) reads

$$\delta g_{00} = 2a\left([a\phi] - (a\xi^0)'\right)$$

So equating δg_{00} for these two:

$$\tilde{\phi} = \frac{\delta g_{00}}{2a^2} = \frac{1}{2a^2}2a\left([a\phi] - (a\xi^0)'\right) = \phi - \frac{1}{a}(a\xi^0)'$$

- For $B \to \tilde{B}$

As above, equating δg_{0i} from (1.26b) and (1.38)

$$\delta g_{0i} = a^2\left(\tilde{B},_i + \tilde{S}_i\right)$$

$$\tilde{B},_i + S_i' = \frac{1}{a^2}\left(a^2\left(\left[B,_i + S_i\right] + \xi_\perp^{i\,\prime} + \left(\zeta' - \xi^0\right),_i\right)\right) = B,_i + S_i + \xi_\perp^{i\,\prime} + \zeta',_i - \xi^0,_i$$

This is a vector equation in S_i and a scalar equation in $B,_i$, so picking out the scalar equation:

$$\left(\tilde{B}\right),_i = \left(B + \zeta' - \xi^0\right),_i \Rightarrow \tilde{B} = B + \zeta' - \xi^0.$$

- For $E \to \tilde{E}$, $\psi \to \tilde{\psi}$

Equating δg_{ij} from (1.26c) and (1.39):

$$a^2\left(2\tilde{\psi}\delta_{ij} + 2\tilde{E},_{ij} + \tilde{F}_i,_j + \tilde{h}_{ij}\right) = \left(a^2\left(\left[2\psi\delta_{ij} + 2E,_{ij} + F_i,_j + F_j,_i + h_{ij}\right] + \frac{2a'}{a}\xi^0\delta_{ij} + 2\zeta,_{ij} + \xi_{\perp i,j} + \xi_{\perp j,i}\right)\right)$$

Picking out the two relevant equations:

$$\tilde{\psi}\delta_{ij} = \left(\psi + \frac{a'}{a}\xi^0\right)\delta_{ij} \Rightarrow \tilde{\psi} = \psi + \frac{a'}{a}\xi^0$$

$$\widetilde{E},_{ij} = (E+\zeta),_{ij} \Rightarrow \widetilde{E} = E+\zeta$$

A.21

$$B = E = 0 \Rightarrow B,_0 = E,_0 = 0$$

This causes (1.46) and (1.47) to reduce to $\Phi = \phi$ and $\Psi = \psi$ respectively.

A.22

$$B = E = 0 \Rightarrow B,_i = E,_i = B,_{ij} = E,_{ij} = 0$$

This causes the perturbed metric (1.27b) to reduce to

$$ds^2 = a^2\left[\left((1+2\phi)d\eta^2\right) + (2\psi-1)\delta_{ij}dx^i dx^j\right].$$

A.23

The Klein Gordon equation is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right) + \frac{\partial V}{\partial\varphi} = 0.$$

It can be expanded out as such:

$$= \frac{1}{\sqrt{-g}}\left(\partial_\mu\left(\sqrt{-g}\right)\right)\left(g^{\mu\nu}\partial_\nu\varphi\right) + \left(\partial_\mu\left(g^{\mu\nu}\right)\right)\partial_\nu\varphi + \partial^2\varphi + \frac{\partial V}{\partial\varphi} = 0.$$

Since in the longitudinal gauge, $g_{\mu\nu} = 0$ for $\mu \neq \nu$, writing out the above equation in components is not too tedious:

$$\frac{1}{\sqrt{-g}}\left[\left(\partial_0\left(\sqrt{-g}\right)\right)\left(g^{00}\partial_0\varphi\right) + \left(\partial_i\left(\sqrt{-g}\right)\right)\left(g^{ij}\partial_k\varphi\right)\delta^k{}_j\right] + \left(\partial_0\left(g^{00}\right)\right)\partial_0\varphi + \left(\partial_i\left(g^{ij}\right)\right)\left(\partial_k\varphi\right)\delta^k{}_j$$

$$+\partial^2\varphi + \frac{\partial V}{\partial\varphi} = 0$$

, where ∂_0 denotes a derivative with respect to conformal time.

Now substituting the metric in, which in the longitudinal gauge, comes in the following form:

$$g_{00} = a^2(1+2\phi) \qquad g_{ij} = a^2(2\psi - 1)\delta_{ij} \qquad g_{0i} = 0$$

$$\sqrt{-g} = ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}$$

, the equation looks like

$$\frac{1}{ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}}\left(\partial_0\left(ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}\right)\right)\left(\frac{1}{\left(a^2(1+2\phi)\right)}\partial_0\varphi\right) \tag{a}$$

$$+\frac{1}{ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}}\left(\nabla\left(ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}\right)\right)\left(\frac{1}{\left(a^2(2\psi-1)\right)}\nabla\varphi\right) \tag{b}$$

$$+\left(\partial_0\left(\frac{1}{\left(a^2(1+2\phi)\right)}\right)\right)\partial_0\varphi+\left[\left(\nabla\left(\frac{1}{\left(a^2(1+2\phi)\right)}\right)\right)\cdot(\nabla\varphi)\right]+\frac{\partial^2\varphi}{\partial\eta^2}+\nabla^2\varphi+\frac{\partial V}{\partial\varphi} \tag{c}$$

$$=0$$

, where the lines of the equation are labelled (a) – (c).

Taking it line by line:

Line (a):

$$\left(\partial_0\left(ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}\right)\right)\left(\frac{1}{\left(a^2(1+2\phi)\right)}\partial_0\varphi\right)$$

$$=ia^4\left(4\frac{a'}{a}(1+2\phi)^{1/2}(2\psi-1)^{3/2}+\frac{(2\psi-1)^{3/2}}{(1+2\phi)^{1/2}}\phi'+3(1+2\phi)^{1/2}(2\psi-1)^{1/2}\psi'\right)\left(\frac{1}{\left(a^2(1+2\phi)\right)}\varphi'\right)$$

So line (a) reads:

$$=\frac{ia^4\varphi'\left(4H\,(1+2\phi)^{1/2}(2\psi-1)^{3/2}+\frac{(2\psi-1)^{3/2}}{(1+2\phi)^{1/2}}\phi'+3(1+2\phi)^{1/2}(2\psi-1)^{1/2}\psi'\right)}{ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}\left(a^2(1+2\phi)\right)}$$

This reduces down to

$$= \frac{\varphi'}{a^2(1+2\phi)}\left(4H \;+ \frac{\phi'}{(1+2\phi)} + \frac{3\psi'}{(2\psi-1)}\right).$$

Line (b):

$$\left(\nabla\left(ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}\right)\right)\left(\frac{1}{\left(a^2(2\psi-1)\right)}\nabla\varphi\right)$$

$$= ia^4\left(\frac{(2\psi-1)^{3/2}}{(1+2\phi)^{1/2}}\nabla\phi + 3(1+2\phi)^{1/2}(2\psi-1)^{1/2}\nabla\psi\right)$$

So line (b) reads:

$$= \frac{ia^4\nabla\varphi\left(\frac{(2\psi-1)^{3/2}}{(1+2\phi)^{1/2}}\nabla\phi + 3(1+2\phi)^{1/2}(2\psi-1)^{1/2}\nabla\psi\right)}{ia^4(1+2\phi)^{1/2}(2\psi-1)^{3/2}}$$

and reduces to

$$= \frac{\nabla\varphi}{a^2(1+2\phi)}\left(\frac{\nabla\phi}{(1+2\phi)} + \frac{3\nabla\psi}{(2\psi-1)}\right).$$

Line (c):

$$\left(\partial_0\left(\frac{1}{\left(a^2(1+2\phi)\right)}\right)\right)\partial_0\varphi + \left[\left(\nabla\left(\frac{1}{\left(a^2(1+2\phi)\right)}\right)\right)\cdot(\nabla\varphi)\right] + \frac{\partial^2\varphi}{\partial\eta^2} + \nabla^2\varphi + \frac{\partial V}{\partial\varphi}$$

reduces to

$$= \frac{2}{a^2}\left(\phi' - H\left(\frac{1}{1+2\phi}\right)\right)\varphi' - \frac{2}{a^2(1+2\phi)^2}(\nabla\phi)\cdot(\nabla\varphi) + \frac{\partial^2\varphi}{\partial\eta^2} + \nabla^2\varphi + \frac{\partial V}{\partial\varphi}$$

Putting all this together:

The Klein Gordon equation now reads:

$$\frac{\varphi'}{a^2(1+2\phi)}\left(4\mathcal{H}+\frac{\phi'}{(1+2\phi)}+\frac{3\psi'}{(2\psi-1)}\right)+\frac{\nabla\varphi}{a^2(1+2\phi)}\left(\frac{\nabla\phi}{(1+2\phi)}+\frac{3\nabla\psi}{(2\psi-1)}\right)$$
$$+\frac{2}{a^2}\left(\phi'-\mathcal{H}\left(\frac{1}{1+2\phi}\right)\right)\varphi'-\frac{2}{a^2(1+2\phi)^2}(\nabla\phi)\cdot(\nabla\varphi)+\frac{\partial^2\varphi}{\partial\eta^2}+\nabla^2\varphi+\frac{\partial V}{\partial\varphi}$$
$$=0$$

Factorizing for φ' and $\nabla\varphi$, this becomes:

$$\varphi'\left[\left(\frac{1}{a^2(1+2\phi)}\right)\left(4\mathcal{H}+\frac{\phi'}{(1+2\phi)}+\frac{3\psi'}{(2\psi-1)}\right)+\frac{2}{a^2}\left(\phi'-\mathcal{H}\left(\frac{1}{1+2\phi}\right)\right)\right] \quad \text{(a)}$$

$$+(\nabla\varphi)\cdot\left[\frac{1}{a^2(1+2\phi)}\left(\frac{\nabla\phi}{(1+2\phi)}+\frac{3\nabla\psi}{(2\psi-1)}\right)-\frac{2}{a^2(1+2\phi)^2}(\nabla\phi)\right] \quad \text{(b)}$$

$$+\frac{\partial^2\varphi}{\partial\eta^2}+\nabla^2\varphi+\frac{\partial V}{\partial\varphi}=0 \quad \text{(c)}$$

From (1.25a), it is obvious that ψ and ϕ satisfy $\psi << 1$ and $\phi << 1$, so the remaining denominators can be expressed as power series:

Line by line once again:

Line (a):

$$\varphi'\left[\left(\frac{1}{a^2(1+2\phi)}\right)\left(4\mathcal{H}+\frac{\phi'}{(1+2\phi)}+\frac{3\psi'}{(2\psi-1)}\right)+\frac{2}{a^2}\left(\phi'-\mathcal{H}\left(\frac{1}{1+2\phi}\right)\right)\right]$$

$$=\frac{\varphi'}{a^2}\begin{bmatrix}\left(1-2\phi+O(\phi^2)\right)\left(4\mathcal{H}+\phi'\left(1-2\phi+O(\phi^2)\right)-3\psi'\left(1+2\psi+O(\psi^2)\right)\right)\\+2\left(\phi'-\mathcal{H}\left(1-2\phi+O(\phi^2)\right)\right)\end{bmatrix}$$
$$\approx\frac{\varphi'}{a^2}\left[2\mathcal{H}(1+2\phi)+3\phi'-3\psi'\right]$$

Line b:

By neglecting all non-linear terms:

$$(\nabla\varphi)\cdot\left[\frac{1}{a^2(1+2\phi)}\left(\frac{\nabla\phi}{(1+2\phi)}+\frac{3\nabla\psi}{(2\psi-1)}\right)-\frac{2}{a^2(1+2\phi)^2}(\nabla\phi)\right]$$
$$\approx 0$$

KG now reads:

$$= \frac{\varphi'}{a^2}\left[2H\ (1+2\phi)+3\phi'-3\psi'\right]+\varphi''+\nabla^2\varphi+\frac{\partial V}{\partial\varphi}=0$$

Now considering the field broken into its background and perturbed components:

$$\varphi = \varphi_0(\eta)+\delta\varphi(\underline{x},\eta)$$

The following simplifications/modifications can be made:

1. $\varphi' = \frac{\partial}{\partial\eta}\left(\varphi_0(\eta)+\delta\varphi(\underline{x},\eta)\right)=\varphi_0'+\delta\varphi'$

2. $\nabla\varphi = \nabla\left(\varphi_0(\eta)+\delta\varphi(\underline{x},\eta)\right)=\nabla(\delta\varphi)$

3. $\varphi'' = \frac{\partial^2}{\partial\eta^2}\left(\varphi_0(\eta)+\delta\varphi(\underline{x},\eta)\right)=\varphi_0''+\delta\varphi''$

4. $\nabla^2\varphi = \nabla^2\left(\varphi_0(\eta)+\delta\varphi(\underline{x},\eta)\right)=\nabla^2(\delta\varphi)$

5. The potential term also changes under substitution of the perturbed field:

$$V(\varphi(\eta))\rightarrow V(\varphi_0(\eta),\delta\varphi(\eta,\underline{x}))=V_0(\varphi_0(\eta))+\delta V(\delta\varphi(\eta,\underline{x}))$$

Using this, and the fact that δV can be written as

$$\delta V = \frac{dV}{d\varphi}\delta\varphi$$

, the potential term changes as follows:

$$\frac{\partial V}{\partial\varphi}\rightarrow\frac{\partial}{\partial\varphi}\left(V_0+\frac{dV}{d\varphi}\delta\varphi\right)=\frac{\partial V_0}{\partial\varphi}+\frac{\partial}{\partial\varphi}\left(\frac{dV}{d\varphi}\delta\varphi\right)=\frac{\partial V_0}{\partial\varphi}+\frac{\partial^2 V}{\partial\varphi^2}\delta\varphi+\frac{\partial V}{\partial\varphi}\frac{\partial(\delta\varphi)}{\partial\varphi}$$

Substituting all of this once again into the KG equation yields equation (1.50):

$$\varphi_0''+2H\,\varphi_0'+a^2V,_{\varphi}+\delta\varphi''+2H\,\delta\varphi'-\nabla^2\delta\varphi+a^2V,_{\varphi\varphi}\delta\varphi-\varphi_0'(3\psi'+\phi')+2a^2V,_{\varphi}\phi=0$$

A.24

Substituting $\frac{dt}{a} = \frac{da}{\dot{a}a} = \frac{da}{Ha^2}$ into $\eta \equiv \int \frac{dt}{a}$ gives $\eta \approx \frac{1}{H}\int \frac{da}{a^2} = \frac{-1}{aH}$, where the fact that H is virtually constant during inflation has been used.

A.25

1. Using $3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} \approx 0$, $\dot{\varphi}^2 << V(\varphi)$ reads $\left(\frac{(-V,_{\varphi})^2}{3H^2}\right) << V(\varphi)$. Then $H \approx \sqrt{\frac{8\pi V(\varphi)}{3}}$ can be substituted which results in equation (3.5): $\left(\frac{V,_{\varphi}}{V}\right)^2 << 1$.

2. Taking the time derivative of $3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} \approx 0$ gives $\frac{\ddot{\varphi}}{\dot{\varphi}} \approx \frac{-V,_{\varphi\varphi}}{3H}$ Substituting this into $3H\dot{\varphi} >> |\ddot{\varphi}|$ gives $3H >> \left|\frac{-V,_{\varphi\varphi}}{3H}\right|$. Then using $H \approx \sqrt{\frac{8\pi V(\varphi)}{3}}$ to eliminate H this results in $\left|\frac{V,_{\varphi\varphi}}{V}\right| << 1$

It is clear also from the 1st Friedmann equation that $V \sim H^2$:

$$H^2 = \frac{8\pi G}{3}\rho_{\varphi} = \frac{8\pi G}{3}\left(\dot{\varphi}^2 + V\right)$$

$$\dot{\varphi}^2 << V \Rightarrow V \sim H^2$$

So $|V,_{\varphi\varphi}| << |V|$ can be extended to $|V,_{\varphi\varphi}| << |V| \sim H^2$.

A.25b

The square root of the slow-roll condition (3.7) is

$|\varphi_0'| << |a\sqrt{V}| \sim aH \sim \mathcal{H}$, where $H = \sqrt{\frac{8\pi GV}{3}} \sim \sqrt{V}$ was used.

So the φ_0' term in the Einstein equation (2.62) can be neglected, giving:

$$\Phi' = 4\pi\varphi_0'\delta\varphi - \mathcal{H}\Phi \approx -\mathcal{H}\Phi$$

A.26

$$\delta\varphi'' + 2\mathcal{H}\,\delta\varphi' - \nabla^2\delta\varphi + a^2V_{,\varphi\varphi}\,\delta\varphi - 4\varphi_0'\Phi' + 2a^2V_{,\varphi}\,\Phi = 0$$

It should also be noted that under a Fourier transform, F, $\nabla^2 \xrightarrow{F} k^2$.

- To show that $|a^2V_{,\varphi\varphi}\,\delta\varphi| << |\nabla^2\delta\varphi|$:

$$|a^2V_{,\varphi\varphi}\,\delta\varphi| << |a^2V\delta\varphi| \sim |a^2H^2\delta\varphi| << |k^2\delta\varphi|$$

$$\therefore |a^2V_{,\varphi\varphi}\,\delta\varphi| << |\nabla^2\delta\varphi|$$

, where (in order) $V_{,\varphi\varphi} << V$, $V \sim H^2$, and $k >> aH$ were used.

- To show that $2a^2V_{,\varphi}\,\Phi << \nabla^2\delta\varphi$:

$$2\Phi a^2V_{,\varphi} << 2\Phi a^2V \sim 2\Phi a^2H^2 << 2\Phi k^2$$

$$\therefore 2\Phi a^2V_{,\varphi} << \nabla^2\delta\varphi$$

, where (in order) $V_{,\varphi} << V$, $V \sim H^2$ and $k >> aH$ were used.

- To show that $-4\varphi_0'\Phi' << \nabla^2\delta\varphi$:

$$|-4\varphi_0'\Phi'| \sim |-4\varphi_0'k\Phi| \sim |-4(\varphi_0')^2\,\delta\varphi| << |-4a^2V\delta\varphi| \sim |-4a^2H^2\delta\varphi| << |-4k^2\delta\varphi|$$

$$\therefore |-4\varphi_0'\Phi'| << |\nabla^2\delta\varphi|$$

, where (in order) $\Phi' \sim k\Phi$, $\Phi \sim \dfrac{\varphi_0'\delta\varphi}{k}$, $(\varphi_0')^2 << a^2V$, $V \sim H^2$ and $k >> aH$ were used.

All that remains to do now is to show that $\delta\varphi''$ and $2\mathcal{H}\,\delta\varphi'$ are of the same order of magnitude as $\nabla^2\delta\varphi$:

$$2H\,\delta\varphi' = 2H\left(\frac{\Phi k}{\varphi_0'}\right)' = \Phi'\left(\frac{k}{\varphi_0'}\right) + \Phi\left(\frac{k'}{\varphi_0'} - k(\varphi_0')^2\varphi_0''\right)$$
$$= \Phi'\left(\frac{k}{\varphi_0'}\right) + \frac{\Phi k^2}{\varphi_0'} - k\Phi(\varphi_0')^2\varphi_0''$$

, where $\Phi' \sim k\Phi$ was used. There is no need to continue with any algebra here since it is readily seen that the terms in k are not negligibly small, so the $2H\,\delta\varphi'$ can not be neglected.

Doing the same procedure for $\delta\varphi''$ produces a similar result with the same conclusion.

A.27

$$\delta\varphi_k = \frac{u_k}{a}$$
$$\delta\varphi_k' = \frac{u_k'}{a} - u_k\frac{a'}{a^2}$$

$$\delta\varphi_k'' = \frac{u_k''}{a} + u_k'\left(\frac{-2a'}{a^2}\right) + u_k\left(\frac{2a'}{a^3} - \frac{a''}{a}\right)$$

Substituting into (1.76):

$$\delta\varphi_k'' + 2H\,\delta\varphi_k' + k^2\delta\varphi_k$$
$$= \frac{u_k''}{a} + u_k'\left(\frac{-2a'}{a^2}\right) + u_k\left(\frac{2a'}{a^3} - \frac{a''}{a}\right) + 2\frac{a'}{a}\left(\frac{u_k'}{a} - u_k\frac{a'}{a^2}\right) + k^2\frac{u_k}{a}$$
$$= u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k$$
$$= 0$$

A.28

Slow-roll approximation gives $\eta \approx \frac{-1}{\dot{a}} = \frac{a}{a'} \Rightarrow \frac{a''}{a} \sim \frac{1}{\eta^2}$

So the $k|\eta| >> 1$ limit reads $k^2 >> \frac{a''}{a}$

A.29

Using $\left[\hat{a}_k, \hat{a}_{k'}{}^\dagger\right] = (2\pi)^3\delta^{(3)}(k-k')$ and $\left[\hat{a}_k, \hat{a}_{k'}\right] = \left[\hat{a}_k{}^\dagger, \hat{a}_{k'}{}^\dagger\right] = 0$:

$$\langle 0|\hat{u}_k \hat{u}^\dagger{}_{k'}|0\rangle = \langle 0|\left(U(\omega,\eta)\hat{a}_k + U^*(\omega,\eta)\hat{a}_k{}^\dagger\right)\left(U^*(\omega',\eta)\hat{a}_{k'}{}^\dagger + U(\omega',\eta)\hat{a}_{k'}\right)|0\rangle$$

$$= \langle 0|\begin{pmatrix} U(\omega,\eta)U^*(\omega',\eta)\hat{a}_k\hat{a}_{k'}{}^\dagger + U(\omega,\eta)U(\omega',\eta)\hat{a}_k\hat{a}_{k'} \\ +U^*(\omega,\eta)U^*(\omega',\eta)\hat{a}_k{}^\dagger\hat{a}_{k'}{}^\dagger + U^*(\omega,\eta)U(\omega',\eta)\hat{a}_k{}^\dagger\hat{a}_{k'} \end{pmatrix}|0\rangle$$

$$= U(\omega,\eta)U^*(\omega',\eta)\langle 0|\hat{a}_k\hat{a}_{k'}{}^\dagger|0\rangle$$

Inserting the term $-\hat{a}_{k'}{}^\dagger\hat{a}_k$ here makes no difference due to its annihilating effect on the vacuum, so a commutator can be formed:

$$= U(\omega,\eta)U^*(\omega',\eta)\langle 0|\left[\hat{a}_k, \hat{a}_{k'}{}^\dagger\right]|0\rangle$$

$$= U(\omega,\eta)U^*(\omega',\eta)\langle 0|(2\pi)^3\delta(k-k')|0\rangle$$

$$= (2\pi)^3|U(\omega,\eta)|^2$$

A.30

Using

$$\delta\varphi'' = a^2\delta\ddot{\varphi} + \dot{a}\delta\varphi'$$

$$H\,\delta\varphi' = Ha^2\delta\dot{\varphi}$$

etc., (2.61) and (2.62) then respectively are:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \nabla^2\delta\varphi + V,_{\varphi\varphi}\,\delta\varphi - 4\dot{\varphi}_0\dot{\Phi} + 2V,_\varphi\,\Phi = 0$$

$$\dot{\Phi} + H\Phi = 4\pi\dot{\varphi}_0\delta\varphi$$

A.31

Having neglected the terms $\nabla^2\delta\varphi$, $\delta\ddot{\varphi}$ and $\dot{\Phi}$, (3.26) and (3.27) simplify to

1. $3H\delta\dot{\varphi} + V,_{\varphi\varphi}\,\delta\varphi + 2V,_\varphi\,\Phi \approx 0$
2. $H\Phi \approx 4\pi\dot{\varphi}_0\delta\varphi$

1. The first equation can be simplified with the following manipulations:

Divide by $V,_{\varphi}$ (obviously $V,_{\varphi} \neq 0$ during inflation, otherwise there would be no slow-roll at all and no inflation) to get:

$$3H\frac{\delta\dot{\varphi}_0}{V,_{\varphi}} + \frac{V,_{\varphi\varphi}\,\delta\varphi}{V,_{\varphi}} + 2\Phi \approx 0$$

Modify the $V,_{\varphi\varphi}$ term by substituting $V,_{\varphi\varphi} = \frac{\dot{V},_{\varphi}}{\dot{\varphi}_0}$ (chain rule) and $V,_{\varphi} \approx -3H\dot{\varphi}_0$ (the slow-roll equation of motion, (1.21)) in succession to obtain

$$3H\frac{\delta\dot{\varphi}}{V,_{\varphi}} - \frac{3H\delta\varphi}{V,_{\varphi}}\dot{V},_{\varphi} + 2\Phi \approx 0$$

The first two terms can be identified as the derivative of a product, to give equation (3.30):

$$3H\frac{\partial}{\partial t}\left(\frac{\delta\varphi}{V,_{\varphi}}\right) + 2\Phi \approx 0$$

2. The second equation is modified by using $\dot{\varphi}_0 = \frac{\dot{V}}{V,_{\varphi}}$ (chain rule) to give (3.31):

$$H\Phi \approx 4\pi\dot{\varphi}_0\delta\varphi = 4\pi\dot{V}\left(\frac{\delta\varphi}{V,_{\varphi}}\right)$$

A.32

Substituting Φ from (3.31) into (3.30):

$$3H\frac{\partial}{\partial t}\left(\frac{\delta\varphi}{V,_{\varphi}}\right) + 2\left(\frac{4}{H}\pi\dot{V}\left(\frac{\delta\varphi}{V,_{\varphi}}\right)\right) \approx 0$$

Then substituting the Friedmann equation $H \approx \sqrt{\frac{8\pi G}{3}V(\varphi)}$:

$$V\frac{\partial}{\partial t}\left(\frac{\delta\varphi}{V,_{\varphi}}\right) + \dot{V}\left(\frac{\delta\varphi}{V,_{\varphi}}\right) = \frac{d}{dt}\left(\frac{V\delta\varphi}{V,_{\varphi}}\right) \approx 0$$

, which is equation (3.32)

A.33

$\dot{\varphi}_0$ can be eliminated from equation (3.29) by using $3H\dot{\varphi} + V_{,\varphi} \approx 0$:

$$\Phi_k \approx \frac{4\pi}{H}\dot{\varphi}_0 \delta\varphi_k = \frac{4\pi}{H}\left(\frac{-V_{,\varphi}}{3H}\right)\delta\varphi_k$$

Then by using the Friedmann equation $H \approx \sqrt{\frac{8\pi}{3}V(\varphi)}$, H can be eliminated:

$$\Phi_k \approx \frac{4\pi}{H}\dot{\varphi}_0 \delta\varphi_k = \frac{4\pi}{3H^2}\left(-V_{,\varphi}\right)\delta\varphi_k = -\frac{1}{2}\left(\frac{V_{,\varphi}}{V}\right)\delta\varphi_k$$

A.34

Equation (1.26) is:

$$a_f(\varphi) = a_i \exp\left(\frac{4\pi G}{n}\left(\varphi^2(t_i) - \left(\varphi^2(t_f)\right)\right)\right)$$

First, the times t_f and t_i need to be specified. Set t_f as the time at the end of inflation, and set t_i as the time at horizon-crossing as such:

$$t_f = t_{V_{,\varphi} \sim V} \qquad\qquad t_i = t_{k \sim aH}$$

This gives $\varphi^2(t_i) = \varphi^2{}_{k\sim aH}$ and $\varphi^2(t_f) = \varphi^2(t_{V_{,\varphi}\sim V})$,

By taking the logarithm of the scale factor expression,

$$\ln a_f - \ln a_i = 2\pi\left(\varphi^2{}_{k\sim aH} - \varphi^2(t_{V_{,\varphi}\sim V})\right) \sim \varphi^2{}_{k\sim aH} - \varphi^2(t_{V_{,\varphi}\sim V})$$

, and substituting in $a \sim \lambda k$, the LHS reads:

$$\begin{aligned} \ln a_f - \ln a_i &= \ln\left(\lambda k|_{k\sim aH}\right) - \ln a_i \\ &= \ln\left(\lambda H a_i\right) - \ln a_i \\ &= \ln(\lambda H) \end{aligned}$$

If $\varphi^2(t_{V_{,\varphi}\sim V})$ is neglected, then the following is obtained:

$$\varphi^2{}_{k\sim aH} \approx \ln(\lambda H)$$

The Hubble parameter here is its value at the end of inflation. Its value is virtually constant during the inflation stage though, so it can be relabelled as its value at horizon crossing: $H \sim \frac{k}{a}$.