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Cosmological inflation and Q-balls

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To those who support me.

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1 Introduction

Cosmology is the study of spacetime evolution. The ultimate goal would be to have a complete history of the universe, with a deep understanding of how it happened to exist and models for the origin of structure.
The problem of the origin requires a fundamental theory of spacetime, exact at the smallest scales, a theory of quantum gravity. That issue will not be considered in the following.
The origin of structure, however, can be studied with models based on verified theories such as General Relativity (GR) and Quantum Field Theory (QFT), or even Classical Field Theory (CFT). That is because structures are bound states, which are only possible at energies relatively lower than the Planck scale (the scale where GR and QFT breakdown).
Standard cosmology describes the evolution of the universe starting from a gaseous state, at energies of the order of $10^{6\pm3}GeV$. At those energies, the universe is $10^{-18\pm6}s$ old and is radiation dominated. From then on the evolution of the universe is understood in terms of GR and QFT. The universe expands, and is very close to flat. A brief review of standard cosmology is given in Chapter 2.
However, that is not the full story. Standard cosmology requires extremely precise initial conditions to evolve into the universe we observe today. Furthermore, the homogeneity on scales that contain causally disconnected regions cannot be explained within standard cosmology. Inflation is the most successful idea in solving these problems. The principles of inflation are explained in Chapter 3. Since it was first presented by Guth in the early 1980's, enormous amounts of data confirmed its preditions.
Inflation takes place in the energy range $10^{18} - 10^{13\pm3}GeV$, corresponding to the time range $10^{-42} - 10^{-32\pm6}$. At those energies, the appropriate physics is not known as they have not yet been explored experimentally. However, it is possible to extrapolate from the physics we understand, assuming that the same general techniques and principles still apply.

Inflationary model building is a very rich field, many things can be done and many results can be obtained. Hence, it is important to narrow the landscape of possible models. This can be done computing observed quantities within the model. For example, building models on the origin of large bound states in the universe, such as galaxies and clusters, links inflationary models to observations. Hence, observing the structures around us allows to reduce the number of viable models by imposing constraints these have to satisfy.

In the last section of Chapter 4, we solve numerically the equations of motion for the hybrid inflation potential. The solutions found are Q-balls, spherically symmetric bound states that oscillate with a constant phase in time. They have already been proven to exist in supersymmetric F-term hybrid inflation [1], hence our work may be considered as a corollary of that result. Note, however, that the differential equations solved in [1] are different from the ones we solve here.

These solutions arise in theories with non-topological charge. They are bound states that minimize the energy for a fixed charge Q. Their stability and properties are related to the total charge and energy carried by the solution. For example, if we consider a theory with one field and a $U(1)$ global symmetry, the Q-ball will be stable if the ratio of the energy to the charge of the Q-ball is smaller than the mass of the field. In the language of QFT, they are stable if the energy of the Q-ball is lower than the energy that the particles would have if they were randomly distributed over space. The properties of Q-balls and their relevance to inflation are discussed in detail in Chapter 4. In particular, we discuss how Q-balls could be responsible for an important fraction of the dark matter of the universe. This issue is related to their stability. Assuming that most, if not all, of the charge that survived at the end of inflation, went into Q-balls, and if those are stable, then it is natural to think of them as dark matter: they are massive and, because of their stability, interact with the surrounding structures exclusively via gravity.

In this dissertation we won't treat supersymmetric models of inflation, nor will we consider supergravity theories. Some may argue that a review of present cosmology cannot be complete without a treatment of these. We agree. However, their treatment is not relevant to the main result presented in this work, that is the existence and stability of Q-balls in hybrid inflation.

In the following we use natural units where the speed of light and the Boltzmann constant are set to 1, $c = k_B = 1$.

2 Standard Cosmology

In order to get some work done, it is important to find out what the main properties of the universe are. Observations, in particular the identical temperature of cosmic microwave background radiation coming from causally disconnected regions, suggest that the universe is homogeneous and isotropic on large scales, that means it is translation and rotation invarariant.

2.1 Friedmann-Robertson-Walker Spacetime

A formal description of an isotropic and homogeneous spacetime was first published in the 1920's by Friedmann. He found a solution to the theory of General Relativity with metric:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right] \tag{2.1}$$

where $a(t)$ is the scale factor: it determines the relative size of spacelike hypersurfaces Σ at different times. The curvature parameter k is $+1, 0$ or -1 for closed, flat or open hypersurfaces Σ. Equation (2.1) is expressed in terms of comoving coordinates: as $a(t)$ increases the universe expands, but r, θ, ϕ remain fixed for galaxies and observers as long as there aren't any forces acting on them. The physical distance is obtained by multiplying with the scale factor, $r_{phys} = a(t)r$. The metric (2.1) may be written as

$$ds^2 = dt^2 - a^2(t)[d\chi^2 + \Phi_k(\chi^2)d\Omega] \tag{2.2}$$

where

$$r^2 = \Phi_k(\chi^2) = \begin{cases} \sinh^2\chi & \text{if} \quad k = -1, \\ \chi^2 & \text{if} \quad k = 0, \\ \sin^2\chi & \text{if} \quad k = +1. \end{cases}$$

All information on the evolution of an homogeneous and isotropic universe is enclosed in the function $a(t)$. Its form depends on the matter content of the universe.

2.2 Conformal time and horizons

Causality in the FRW spacetime (2.1) is determined by the propagation of light. Photons travel along null geodesics, $ds^2 = 0$. In order to simplify the following analysis, let's define conformal time:

$$\tau = \int \frac{dt}{a(t)}. \tag{2.3}$$

In terms of τ, the metric (2.2) becomes

$$ds^2 = a^2(\tau)[d\tau^2 - (d\chi^2 + \Phi_k(\chi^2)d\Omega)]. \tag{2.4}$$

For radial propagation we have

$$ds^2 = a^2(\tau)(d\tau^2 - d\chi^2), \tag{2.5}$$

which is the static Minkowski metric multiplied by the time dependent scale factor. Hence the radial null geodesics in the FRW spacetime are given by

$$\chi(\tau) = \mp\tau + \text{constant}. \tag{2.6}$$

2.2.1 Particle Horizon

A photon emitted at an initial time t_i and reaching us now, at time t_0, will have travelled a maximum comoving distance given by

$$x_p = \int_{t_i}^{t_f} \frac{dt}{a(t)} = \tau_f - \tau_i. \tag{2.7}$$

Assuming that the universe started with the initial condition $a(0) = 0$, we can define the physical particle horizon to be

$$r_p = a_0 \int_0^{t_0} \frac{dt}{a(t)}. \tag{2.8}$$

The particle horizon determines the size of the observable universe and will be fundamental in the discussion of inflation.

2.2.2 Event Horizon

The event horizon is the locus of points from which a signal sent at a given time τ will never reach an observer in the future. In comoving coordinates this means that

$$\chi > \chi_e = \int_{\tau}^{\tau_{max}} d\tau = \tau_{max} - \tau, \tag{2.9}$$

where τ_{max} indicates the end of time. The physical size of the event horizon is obtained by multiplying with the scale factor.

2.2.3 Redshift

The redshift z of light coming from a cosmological source is defined to be

$$1 + z = \frac{\lambda_{obs}}{\lambda_{emit}}, \tag{2.10}$$

where λ_{obs} is the wavelength measured by the observer and λ_{emit} is the wavelength at the point of emission.
It is practical to express the redshift in terms of the expansion parameter. It is straightforward, since $\lambda \propto a$, that the redshift of light emitted at time t_1 is given by

$$1 + z = \frac{a(t_0}{a(t_1)}. \tag{2.11}$$

2.3 Evolution: Einstein Equations

The dynamics of a FRW spacetime are given by the Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.12}$$

where G is the gravitational constant, $T_{\mu\nu}$ is the stress-energy tensor and $G_{\mu\nu}$ is the Einstein tensor.

2.3.1 Evolution in FRW

The Einstein equations for this metric give the evolution of the scale factor (Friedmann equations):

$$\ddot{a} = -\frac{(\rho + 3p)a}{2M_P^2} + \frac{a\Lambda}{3}, \tag{2.13}$$

$$H^2 + \frac{k}{a^2} \equiv (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{\rho}{M_P^2} + \frac{\Lambda}{3}, \tag{2.14}$$

where ρ is the energy density of matter in the universe, p its pressure and $M_P = \sqrt{\frac{3}{8\pi G}}$ is the reduced Planck mass. $H = \frac{\dot{a}}{a}$ is the Hubble parameter, which generally depends on time. Λ is the cosmological constant. That term is often ignored, but increasing observational evidence suggests that it might not be zero. The Einstein equations give the following energy conservation law:

$$\dot{\rho}a + 3(\rho + p)\dot{a} = 0. \tag{2.15}$$

It is equivalent to the energy conservation law for adiabatic expansion, $dE = -pdV$, where $E = V\rho$ is the energy in a comoving volume $V \propto a^3$. The expansion of the universe must be adiabatic because heat cannot flow. To solve this equation one needs to know the equation of state of the matter in the universe. Let's assume that the equation of state is of the form $p = w\rho$. We may then solve for the energy density to find

$$\rho \approx a^{-3(1+w)}. \tag{2.16}$$

For instance, nonrelativistic cold matter with the equation of state $p = 0$ gives

$$\rho \approx a^{-3}, \tag{2.17}$$

and a gas of photons with $p = \frac{\rho}{3}$ gives

$$\rho \approx a^{-4}. \tag{2.18}$$

We may then use (2.14) to solve for small a:

$$a \approx t^{\frac{2}{3(1+w)}}. \tag{2.19}$$

Thus, for nonrelativistic cold matter

$$a \approx t^{\frac{2}{3}}, \tag{2.20}$$

and for a gas of photons

$$a \approx t^{\frac{1}{2}}. \tag{2.21}$$

In both cases, there exists a time $t = 0$ such that the scale factor vanishes and the energy density becomes infinite. That time corresponds, in the Friedmann model, to the so-called cosmological singularity and that is where modern theories fail to describe the structure of spacetime.

2.3.2 Critical Density and Density Parameter

From the Friedmann equation we can see that, for a given value of H, there exist a particular density, called the critical density ρ_c, such that the hypersurface Σ is flat when the cosmological constant is zero. It is given by

$$\rho_c = M_P^2 H^2. \tag{2.22}$$

From a practical point of view, it is often convenient to work with the density parameter $\Omega = \frac{\rho}{\rho_c}$. Every component of the universe contributing to its energy content has its own density parameter. The cosmological constant leads to a contribution $\Omega_\Lambda = \frac{\Lambda}{3H^2}$, and $\Omega_{total} = \Omega + \Omega_\Lambda$. In terms of the density parameter, the Friedmann equation becomes

$$\Omega_{total} - 1 = \frac{k}{3H^2}. \tag{2.23}$$

If $\Omega_{total} = 1$ the hypersurface Σ is flat and it remains flat forever; otherwise, Ω_{total} is time dependent.

2.4 Eras

A realistic model of the dynamics of the universe must take into account that the energetic content evolves with time, so that evolution can be split into different epochs.

2.4.1 Radiation Domination Era

The standard cosmology model starts with an era of radiation domination at energies of the order of $100GeV$. The universe is filled with an ultra-relativistic gas of photons, electrons, quarks, their antiparticles and other particles. They are in thermal equilibrium, with zero chemical potential. Hence their momentum distribution has the blackbody form. The energy density of a collection of ultrarelativistic particles at temperture T is given by

$$\rho = \frac{\pi^2 g_*}{30} T^4 \tag{2.24}$$

and their number density is

$$n = \frac{\zeta(3) g_*}{\pi^2} T^3. \tag{2.25}$$

where g_* is the number of spin states of the particle, the Boltzmann constant has been set equal to 1 and $\zeta(3) = 1.202$. For a gas composed by different species, g_* is given by a sum over the species, with a weight of $\frac{7}{8}$ for fermions.

The number of degrees of freedom $g_*(T)$ depends on the relevant physics at a given temperature. For instance, in the Standard Model of particle physics, at high temperature $g_* = 106.75$ [2]. Possible extensions such as supersymmetry or Grand Unified Theories might increase g_* up to several hundreds. As the temperature drops, the degrees of freedom diminish, the particles becoming nonrelativistic.

In a radiation dominated universe in thermal equilibrium, the entropy density s can be derived from the second law of thermodynamics, $dE = TdS - PdV$, where V is a comoving volume as before, $E = \rho V$ and $S = sV$. Rewriting this law in terms of the relevant quantities, remembering that ρ depends only on T and that $\rho = 3p$ for an ultrarelativistic gas, we find:

$$s = \frac{\rho + p}{T} = \frac{2\pi^2}{45} g_* T^3. \tag{2.26}$$

Hence, ignoring small variation in the proportianality factor, $T \propto \frac{1}{a}$.
As the temperature drops, at $T \approx 1MeV$, electrons and positrons become nonrelativistic and annihilate, except for those electrons coupled to a proton. A process of primordial nucleosynthesis starts and the universe goes through a smooth transition from radiation domination to matter domination.

2.4.2 Radiation-Matter Transition

During the transition, the scale factor evolution will depend both on radiation and matter energy density. In equations (2.17) and (2.18) we found how $a(t)$ evolves with time. In terms of conformal time, these equations become

$$\text{matter:} \qquad a \propto t^{2/3} \propto \tau^2 \tag{2.27}$$

$$\text{radiation:} \qquad a \propto t^{1/2} \propto \tau \tag{2.28}$$

Hence, including both matter and radiation, the Friedmann equation (2.14) with $\Lambda = 0$ becomes

$$H^2 = \frac{\rho_{eq}}{M_P^2} \left[(\frac{a_{eq}}{a})^3 + (\frac{a_{eq}}{a})^4 \right], \tag{2.29}$$

where a_{eq} is the scale factor at matter-radiation equality and ρ_{eq} is the energy density at that time. Using conformal time, we can find an exact solution [2]:

$$\frac{a(\tau)}{a_{eq}} = (2\sqrt{2} - 2)\frac{\tau}{\tau_{eq}} + (1 - 2\sqrt{2} + 2)(\frac{\tau}{\tau_{eq}})^2, \tag{2.30}$$

$$\tau_{eq} = \frac{2\sqrt{2} - 2}{a_{eq}} \frac{M_P}{\sqrt{\rho_{eq}}}. \tag{2.31}$$

2.4.3 Matter Domination Era

We already know the solution for the critical density case: $a \propto t^{2/3}$. Nevertheless, many observations suggest that the density of matter in the universe is less than the critical density. Hence, we are faced with two possibilities: an open universe or a flat universe with non-zero cosmological constant.

Open Universe

In that case, $k = -1$ and the curvature term in the Friedmann equation goes as a^{-2}, whereas the matter energy density goes as a^{-3}. Hence the curvature term dominates and leads to a late-time solution $a \propto t$. Nevertheless, knowing that the present universe is not far from the critical density, let's consider both terms. The solution, using conformal time, is [2]

$$a(\tau) = \frac{\Omega_0 H_0^2}{2}(\cosh \tau - 1) \tag{2.32}$$

where the subscript $_0$ means we are taking the present time values of these quantities.

The evolution of Ω in terms of the redshift, assuming the universe only contains norelativistic matter, is

$$\Omega(z) = \Omega_0 \frac{1+z}{1+\Omega_0 z}. \tag{2.33}$$

Flat Universe with Cosmological Constant

Here, $k = 0 \neq \Lambda$. The Friedmann equation reads

$$H^2 = \frac{\rho}{M_P^2} + \frac{\Lambda}{3}. \tag{2.34}$$

If the universe is dominated by the cosmological constant, the solution is an exponential expansion rate $a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}} t\right)$. Such a solution is called a De Sitter spacetime.

Even if the cosmological contant is nonzero in our universe, it cannot be dominant: we must take into account matter contributions. However, analytic solutions are not available. As above, it is possible to compute the dependance of Ω on the redshift [2]:

$$\Omega(z) = \Omega_0 \frac{(1+z)^3}{1 - \Omega_0 + (1+z)^3 \Omega_0}. \tag{2.35}$$

The density tends to critical when $(1 + z^3) \gg 1/\Omega_0 - 1$.

3 Inflation

3.1 Motivation

Historically, inflation has been proposed as a solution to a series of problems of standard cosmology. These problems concern the precision of the initial conditions required to give rise to the properties of the universe. In standard cosmology only a very narrow set of initial conditions can evolve into a universe with the properties observed nowadays. It is a rather philosophical question to ask whether or not our universe is just a lucky accident. However, a theory that explains these initial conditions as a result of a dynamical process, *i.e.* inflation, is particularly attractive.

3.1.1 Flatness Problem

We know that nowadays the universe is very close to flat. The flatness problem arises from the fact that a nearly flat universe evolves away from flatness as time flows. Recall the Friedmann equation in terms of the density parameter (2.23)

$$\Omega_a - 1 = \frac{\rho_c - \rho(a)}{\rho_c} = \frac{k}{3H^2}. \tag{3.1}$$

A flat universe corresponds to $\Omega = 1$ and is a stable solution. However, if we consider a non-flat universe, the quantity $|\Omega - 1|$ diverges with time, since the Hubble radius $(aH)^{-1}$ grows with time. Observations tell us that $|\Omega_0 - 1| \leq 0.1$, which implies that at earlier times it must have been even closer to 1. For example, at the time of nucleosinthesis, when $t \approx 1s$ and $T \approx 0.1MeV$, it implies that [3]

$$|\Omega_{nuc} - 1| \leq 10^{-16} \tag{3.2}$$

and at the Planck scale [3]

$$|\Omega_{Planck} - 1| \leq 10^{-61}. \tag{3.3}$$

Such finely tuned initial conditions seem extremely unlikely.

3.1.2 Causality Problem

The particle horizon determines the region of spacetime which is causally connected at a given time. Assuming the universe begun at $t = 0$, the maximum comoving causal distance in terms of the Hubble radius is given by

$$\tau \equiv \int_0^{t_0} \frac{dt}{a(t)} = \int_0^{a_0} \frac{da}{Ha^2} = \int_{-\infty}^{\ln a_0} \frac{d\ln a}{aH}. \tag{3.4}$$

Hence, the causal distance grows as the Hubble radius grows. Putting this toghether with the fact that in standard cosmology the Hubble radius grows monotonically with time, this means that comoving scales entering the horizon today have never been in causal contact before. Put in these terms, the homogeneity of the CMB radiation looks like an incredible coincidence.

3.1.3 Small-scale Inhomogeneities

Stars, galaxies and clusters of galaxies are the striking evidence of the inhomogeneity of small-scale structures in the universe. Within standard cosmology, the only way to account for such structures, is, as for the other problems, to absorb the informations in the initial conditions. Again, this implies a fine tuning, as inhomogeneities tend to grow over time under the effect of gravity.

3.2 Principles of Inflation

As anticipated in the previous section, inflation is a dynamic solution to the problem of intitial conditions in standard cosmology. It is evident that both the horizon and flatness problems arise because the Hubble radius $(aH)^{-1}$ is strictly increasing. This suggests that the standard cosmology problems can be solved by inverting the behaviour of the comoving Hubble radius in the very early universe ($t \approx 10^{-42}$). Hence, during inflation, we require

$$\frac{d}{dt}(aH)^{-1} < 0. \tag{3.5}$$

Under such condition, both the flatness and the causality problems are easily solved. Let's first take a look at the flatness problem; the effective result of a decreasing Hubble radius is that Ω is driven towards 1, as can be directly seen from equation (3.1). Thus, inflation drives the universe towards flatness rather than away from it. In addition to that, the dramatic reduction of the comoving Hubble radius implies that regions which were in causal contact before inflation, were driven apart during the last-named. This allows the present observable universe to originate from a region that was inside the Hubble radius at the beginning of inflation, explaining the homogeneity of the CMB.

As can be seen directly by deriving equation (3.5), a shrinking Hubble radius is equivalent to an accelerating expansion, that is $\ddot{a} > 0$. Then, equation (2.13), with Λ absorbed into ρ and p, implies

$$\rho + 3p < 0. \tag{3.6}$$

ρ being always positive, inflation must be generated by a field that has a negative pressure. There is a particular type of field allowing this property: the scalar field.

3.2.1 Scalar Fields and Inflation

Scalar fields are the most simple type of fields. They correspond to spin-0 particles and, even though they are believed to have a fundamental role in the process of symmetry breaking in the standard Model of particle physics, they have not yet been observed. Nevertheless, their simple nature and their properties made them very popular among particle physicsts and cosmologists.

In order to study the properties of a real scalar field ϕ, we need the relevant lagrangian density:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi). \tag{3.7}$$

The stress-energy tensor for this lagrangian density, as obtained via general relativity, is given by

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)\right]. \tag{3.8}$$

For a perfect fluid, in a comoving frame, we have

$$\rho_\phi = \frac{1}{2}\dot{\phi} + V(\phi), \tag{3.9}$$

$$p_\phi = \frac{1}{2}\dot{\phi} - V(\phi). \tag{3.10}$$

Hence the equation of state is

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi} - V(\phi)}{\frac{1}{2}\dot{\phi} + V(\phi)}. \tag{3.11}$$

Substituting the expressions for the energy density and pressure in the Friedmann equations, we can derive the equations of motion. Assuming a spatially flat universe, we have

$$H^2 = \frac{1}{M_P^2}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \tag{3.12}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}. \tag{3.13}$$

Inflation can happen only if equation (3.6) is satisfied. That implies $\dot{\phi}^2 < V(\phi)$. The scalar field that generates inflation is called the inflaton.

3.3 Slow-roll Inflation

A very useful approximation for analyzing inflation is the slow-roll approximation. It consists of ignoring the last term of equation (3.12) and the first term of equation (3.13), leaving

$$H^2 \approx \frac{V(\phi)}{M_P^2}, \tag{3.14}$$

$$3H\dot{\phi} \approx -V'(\phi), \tag{3.15}$$

where the prime denotes derivation with respect to ϕ. This approximation is valid when the following two conditions hold:

$$\epsilon(\phi) \equiv \frac{M_P^2}{6}\left(\frac{V'}{V}\right)^2 \ll 1 \qquad \text{and} \qquad |\eta(\phi)| \equiv \left|\frac{M_P^2}{3}\frac{V''}{V}\right| \ll 1. \tag{3.16}$$

These conditions are necessary for the slow-roll approximation to be valid. However, they are not sufficient as they only restrict the form of the potential. The equations of motion being of second order, there is freedom in the choice of $\dot{\phi}$, and, a priori, it can be chosen so as to violate the approximation. Therefore it is necessary to assume that the solution to the equations of motion (3.12) and (3.13) satisfies (3.16). This assumption can be proven to be true by considering the attractor behaviour of these solutions, *i.e.* the fact that solutions with different initial conditions rapidly converge. This property is of vital importance to the predictive power of inflationary models, since the initial conditions are unknowable.
It is easy to proof that these conditions are necessary for inflation. For this purpose, let's rewrite the condition for inflation as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0. \tag{3.17}$$

If $\dot{H}$ is positive, this is obviously satisfied. However it would imply $p < -\rho$ in a general relativity theory. That is incompatible with a scalar field, hence we do not explore this possibility. The other possibility is

$$-\frac{\dot{H}}{H^2} \approx \epsilon. \tag{3.18}$$

Thus, if the slow-roll approximation (3.16) is valid, inflation is guaranteed. The slow-roll conditions allow to easily check whether a given potential is suitable for inflation and at which value of the inflaton it might occur. For example, a simple mass term, $V(\phi) = m^2\phi^2$, satisfies the slow-roll conditions if $\phi^2 > \frac{1}{4\pi G}$. For such a potential inflation will continue as the scalar field rolls down the potential and ends as it approaches its minimum.
In most inflationary models, inflation ends when the slow-roll conditions are violated

$$\epsilon(\phi_{end}) \approx 1. \tag{3.19}$$

However, as mentioned before, this condition is necessary but not sufficient. In theory, inflation can continue after the slow-roll conditions are violated. In practice, the amount of inflation that occurs in these circumstances is very small compared to the amount occurred when the conditions were satisfied. A natural way to quantify the amount of inflation is the ratio of the scale factor at the final time, a_{end}, to its value before inflation, a_i. Since it is a large number, the logarithm is taken to give the number of e-folds N:

$$N(t) \equiv \ln \frac{a_{end}}{a_i} = \int_{t_i}^{t_{end}} H dt = \int_{\phi_i}^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi \approx \int_{\phi_{end}}^{\phi_i} \frac{V}{V'} d\phi, \tag{3.20}$$

where the last relation is true in the slow-roll approximation and ϕ_{end} is defined by $\epsilon(\phi_{end}) = 1$. In order to solve the flatness and causality problems, the total number of e-folds must exceed 60.

3.3.1 Hamilton-Jacobi Formulation

A different way to formulate inflation, which turns out to allow easier derivations of many results, is the Hamilton-Jacobi formulation (Salopek and Bond 1990). We present it here because it allows an exact derivation of the slow-roll conditions.
The idea is to consider ϕ as a time parameter, using $\frac{\partial}{\partial t} = \dot{\phi}\frac{\partial}{\partial \phi}$. Differentiating (3.12) with respect to time and substituting in (3.13), we obtain

$$2\dot{H} = -\frac{\dot{\phi}^2}{M_P^2}. \tag{3.21}$$

Furthermore, we have

$$2H'(\phi) = -\frac{\dot{\phi}}{M_P^2}, \tag{3.22}$$

which may be rewritten as

$$\dot{\phi} = -2M_P^2 H'(\phi). \tag{3.23}$$

Using this result, we can rewrite the Friedmann equation as follows

$$[H'(\phi)]^2 - \frac{3}{2M_P^2}H^2(\phi) = -\frac{1}{2M_P^4}V(\phi). \tag{3.24}$$

This is the Hamilton-Jacobi equation. Now $H(\phi)$, rather than $V(\phi)$, is the fundamental quantity. H being a geometrical quantity, inflation is more naturally understood in these terms. Through equation (3.24), we can see that to a given $H(\phi)$ there corresponds only one potential $V(\phi)$. In addition to that, equation (3.23) can be integrated to find $\phi(t)$, which allows to derive $H(t)$, which can be used to find $a(t)$. Hence, the Hamilton-Jacobi formalism provides a direct method to compute inflationary solutions.
We can now wirte down a different version of the slow-roll parameters:

$$\epsilon_H = 2M_P^2 \left(\frac{H'(\phi)}{H(\phi)}\right)^2, \tag{3.25}$$

$$\eta_H = 2M_P^2 \frac{H''(\phi}{H(\phi)}. \tag{3.26}$$

In the slow-roll approximation, we have $\epsilon_H \to \epsilon$ and $\eta_H \to \eta - \epsilon$. As opposed to the derivation of ϵ and η, we did not require the slow-roll approximation to be valid. Hence, results that were approximate in terms of $V(\phi)$ can be exactly derived in terms of $H(\phi)$. First of all, the definition of inflation is now given by

$$\ddot{a} > 0 \Rightarrow \epsilon_H < 1, \tag{3.27}$$

and the number of e-folds

$$N \equiv \ln \frac{a(t_{end})}{a(t)} = \int_t^{t_{end}} H dt = -\frac{1}{2M_P^2}\int_\phi^{\phi_{end}} \frac{H}{H'} d\phi. \tag{3.28}$$

3.3.2 Evolution of Scales

During inflation the Hubble radius decreases. Hence, a scale that is smaller than the Hubble radius before inflation may evolve to be bigger by the end of it. Knowing how a given scale evolves is extremely important to the discussion of density perturbations and more generally the origin of structures in the universe. To do that, we need a model for the evolution of the universe up to the present.

We define a scale by its comoving wavenumber, k, arising from a Fourier decomposition of the density perturbation the scale corresponds to. The scale equals the Hubble radius when $k = aH$.

If we choose the simplest cosmological model, the evolution can be divided as follows:

- From the time the scale k^{-1} equals the Hubble radius, t_k, to the end of inflation at t_{end}.
- From the end of inflation until the Hot Big Bang begins at t_{reh}. Here, for simplicity, we assume the universe is matter dominated at that time.
- From the end of reheating to the time of matter-radiation equality, t_{eq}.
- From t_{eq} to the present, t_0.

Assuming instantaneous transitions between the phases and measuring all quantities relative to the present comoving Hubble scale $(a_0 H_0)^{-1}$, we have

$$\frac{k}{a_0 H_0} = \frac{a_k H_k}{a_0 H_0} = \frac{a_k}{a_{end}} \frac{a_{end}}{a_{reh}} \frac{a_{reh}}{a_{eq}} \frac{a_{eq}}{a_0} \frac{H_k}{H_0}. \tag{3.29}$$

Note that the first fraction on the right hand side gives the number of e-foldings $N(k)$ that occur after the scale k equals the Hubble radius. Inserting the characteristic values, we obtain

$$N(k) = 62 - \ln \frac{k}{a_0 H_0} - \ln \frac{10^{16} GeV}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{end}^{1/4}} - \frac{1}{3} \ln \frac{V_{end}^{1/4}}{\rho_{eq}^{1/4}}. \tag{3.30}$$

The values of the energy scales connected with inflation are not known, hence the last three terms in equation (3.30) don't have numerical values.

However, in most inflationary models, they are not expected to be too large. Knowing the exact number of e-foldings at which the present Hubble scale $k = a_0 H_0$ equalled the Hubble scale during inflation is not necessary. In the framework of standard cosmology, this number is usually taken to be 50. Roughly, it is always contained in the range of $40 - 60$ e-foldings, the precise value depending on the details of reheating and the post-inflationary thermal history of the universe.

3.3.3 Initial Conditions

An important feature of inflation is that it does not depend on the initial conditions. Hence, our region of the universe retains no memory of the pre-inflation era. However, we should not ignore this era completely; a model of inflation consists not only of a potential and a way of ending inflation, but also of specifying how the inflaton finds itself slow-rolling down the potential when our region of the universe leaves the horizon.

Generally, an era of inflation is supposed to begin at the Planck scale, corresponding to $V^{1/4} \approx M_P$. This is appealing first of all because,in the case $\Omega > 1$, it prevents the universe from collapsing within a Planck time or so, unless we require a value finely tuned to be close to 1. However, this might not be a problem if the universe has a chaotic geometry, with open and closed regions.

Furthermore, inflation prevents inhomogeneities to enter homogeneous regions. Assuming inhomogeneities propagate at a speed of order $c = 1$, a region that is homogeneous at time t will remain so until time t_2 if its initial size is bigger than

$$r(t) = a(t) \int_t^{t_2} \frac{dt}{a(t)} = a(t) \int_a^{a_2} \frac{da}{a^2 H}. \tag{3.31}$$

If an era of inflation begins at the Planck scale, and we take t_2 as the end of inflation, the integral is dominated by the lower limit, giving $r(t) \approx H^{-1}(t)$. That means the inhomogeneity travels about a Hubble distance in the first Hubble time, but then it stops. Hence the homogeneous region need not be much bigger than the Hubble radius.

On the other hand, if inflation starts at a later time, the integral is dominated by the upper limit and $r(t) \gg H^{-1}(t)$. Hence the inhomogeneity propagates indefinetely and the homogeneous region has to be many orders bigger that the Hubble radius to survive.

It is important to understand that this first era of inflation is not necessary to solve the FRW cosmology problems. Nevertheless, it allows to create the suitable initial conditions to conventional inflation: large homogeneneous patches surrounded by inhomogeneities.

A very widely accepted proposal is that conditions at the Planck scale are chaotic, in the sense that the inflaton field is spread over a wide range of values. Assuming that, it is easy to imagine that in some regions the field will have the right values to trigger inflation.

Since perturbations tend to dissappear during this early era of inflation, they don't have to be particularly small during the pre-inflationary era. Even if they were of order of unity when the energy density of the inflaton is well below the Planck scale, they would not lead to consequences which are not compatible with today's observations. Accepting perturbations of high magnitude as a working hypothesis leads to the eternal inflation scenario; quantum effects are dominant on large scales and the inflaton might be rolling down or climbing up the potential. In regions where the energy density is higher, space expands more rapidly; hence the physical volume of the universe would be dominated by regions where the inflaton moves up the potential. Parts of the universe would inflate forever, constantly emitting regions where the field classically rolls down the potential and triggers conventional inflation. Our observable universe could be in any of these regions.

Eternal inflation allows to intuitively understand that our observable universe is not a just lucky accident. Nevertheless, even if the eternal scenario is not right, it is not necessary for inflation to be common near the Planck scale.

Many things might happen between the Planck scale and the time our region of the universe leaves the horizon. Generally the simplest possibility is assumed: the energy density continues to be dominated by the inflaton, with recurrent inflation eras. Whatever happens, our region of the universe is assumend to undergo an era of slow-roll inflation starting when it is inside the Hubble radius and ending some tens of e-folds after it left the horizon.

3.4 Origin of Structures

Historically, inflation was formulated to solve the problem of the origin of an homogeneous universe. However, the true merit of inflation is that it provides a fruitful framework for building theories on the origin of structures. Structures are due to inhomogeneities created by the quantum fluctuations of the inflaton about its vacuum state.
These fluctuations generate a primeval density perturbation, that will eventually evolve into the structures we observe today. The evolution of the perturbation depends on the amount and real nature of dark matter and on the value of the cosmological constant.
Building models on the origin of structures is extremely important. It allows to constrain inflationary models. Furthermore, structures are all around us and ignoring them is just not a viable possibility.
A lot of work is being done on that topic. For further information, we refer the reader to [2].

3.5 Examples: Exact Solutions

Most of the times, the slow-roll approximation works so well that there is no need for more information. The misestimation of the e-folds due to the non-necessity of the slow-roll conditions to inflation is small compared to the inherent uncertainty of cosmological quantities. However, it is always useful to study simple cases for which analytic solutions exist. Many such examples are known.

3.5.1 Power-law Inflation

The most famous exact solution power-law inflation(Lucchin and Matarrese 1985). This model consists of the potential

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_P}\right), \tag{3.32}$$

where V_0 and p are constants. The spatially flat equation of motion are

$$\left[\partial_\mu \partial^\mu + V_0 \left(-\sqrt{\frac{2}{p}}\frac{1}{M_P}\right) \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_P}\right)\right] \phi = 0. \tag{3.33}$$

This equation is exactly solvable and it has the particular solution

$$a = a_0 t^p, \tag{3.34}$$

$$\frac{\phi}{M_P} = \sqrt{2p} ln \left(\sqrt{\frac{V_0}{p(3p-1)}} \frac{t}{M_P} \right). \tag{3.35}$$

The general solution can be also found in parametric form. However, any solution fastly converges to the particular solution (3.35), confirming the attractor behaviour of inflationary solutions.
The slow-roll parameters con be easily verified to be $\epsilon = \eta/2 = 1/p$. Hence, provided that $p > 1$, this solution satisfies the conditions for inflation. However, note that the slow-roll parameters do not depend on ϕ. That means that the conditions for inflation are always satisfied and inflation never comes to an end in this model, unless extra-physics intervenes to change this situation.

3.5.2 Intermediate Inflation

Another example is intermediate inflation (Barrow 1990; Muslimov 1990). It arises from the potential

$$V(\phi) \propto \left(\frac{\phi}{M_P} \right)^{-\beta} \left(1 - \frac{\beta^2 M_P^2}{6\phi^2} \right), \tag{3.36}$$

where $\beta = 4(f^{-1} - 1)$ and $0 < f < 1$. It implies an expansion

$$a(t) \propto \exp(At^f), \qquad A > 0. \tag{3.37}$$

The slow-roll parameters ϵ and η are both proportional to $1/\phi^2$, and, provided that ϕ is big enough, they are less that unity.

Other examples of exact solutions, which we don't mention here, exist in the literature.

3.6 Hybrid Inflation

Many models of inflation can be built. Broadly, these can be distinguished in two classes. Single field models have a potential $V(\phi)$ that is dominated by the slow-rolling inflaton field. The minimum corresponds to the vev of ϕ. Inflation ends as the field approaches its vev because the slow-roll conditions eventually fail to be satisfied.

In hybrid models the potential depends on at least another field, χ, in addition to the inflaton ϕ. We will consider here the case in which the potential depends just on these two fields [4]. The defect field χ is initially held practically constant by its interaction with ϕ; thus, its contribution to $V(\phi, \chi)$ is constant. As ϕ slowly rolls down the potential and reaches a critical value ϕ_c, χ is destabilized and relaxes to its true minimum.

If we take ϕ to be a complex field and χ to be a real scalar, the general potential of a two-field hybrid inflation model is

$$V(|\phi|, \chi) = m_\phi^2 |\phi|^2 + g^2 |\phi|^2 \chi^2 + \frac{\lambda}{4}(\chi^2 - v^2)^2. \tag{3.38}$$

The defect field χ has an effective mass $m_\chi^2 = -\lambda v^2 + g^2|\phi|^2$. When inflation begins, $|\phi|^2 \gg \lambda v^2 = |\phi_c|^2$. The χ field is fixed at the origin. Hence, the inflaton evolves in the effective potential

$$V(|\phi|) = V_0 + m^2 |\phi|^2. \tag{3.39}$$

With such a potential, slow-roll inflation can take place. It goes on until the effective mass of the defect field becomes negative, *i.e* when $\phi < \phi_c$. Then, χ evolves to its true minimum, $\chi = \pm v$ and so does the inflaton, reaching $\phi = 0$.

This model has three independent parameters, for example m, m_χ and λ. These can be chosen to fit particle theory requirements and, at the same time, bounds imposed by observations (COBE normalization). A reasonable choice could be $m \approx m_\chi = 100 GeV$ and $v \approx M_P$.

For inflation to end promptly when the inflaton reaches the critical value ϕ_c, v should be significantly smaller than M_P. Hence, the choice $v \approx M_P$ implies that inflation will go on for a few e-foldings after ϕ rolls below its critical value.

The potential $V(|\phi|, \chi)$ has the only unusual feature that there is no term proportional to ϕ^4. Such a term would spoil the model, unless we require an extremely small coupling associated with that term. However, in the context of supersymmetry, the problem does not arise as there are many directions in the field space (flat directions) where this term does not appear.

4 Q-balls

Q-balls are soliton solutions to a whide class of field theories in four space-time dimensions with unbroken continuous global symmetries [5]. For a fixed charge, the Q-ball is the ground state solution, so that its existence and stability are related to the conservation of the charge.
In realistic physical theories, Q-balls may arise in supersymmetric generalizations of the Standard Model of particle physics with flat directions in their potentials [6]. Q-balls are allowed, as we show in the following, in hybrid inflation potentials as well, where there is a global $U(1)$ charge associated with the inflaton field.
The importance of Q-balls in cosmology is related to their stability. If stable Q-balls are formed in the early universe, they may represent an important contribution to the dark matter content of the universe. Stable Q-balls can also trigger explosions of neutron stars, if created inside the stars, by decreasing their mass by absorbing baryons [7]. Even if Q-balls are not stable, they can arise in interesting phenomena; if Q-balls decay after the electroweak symmetry breaking phase transition, they can prevent the baryon number from being erased by transitions violating the baryon number conservation (sphaleron transitions) [8]. Decaying Q-balls may also explain the baryon to dark matter content of the universe [8], if their decay results in the production of dark matter in the form of the lightest supersymmetric particles.
In this chapter, we will study the classical properties of these solutions.

4.1 Properties of Q-balls

The simplest theory with Q-balls is a $U(1)$ invariant field theory, with lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - V(|\phi|), \tag{4.1}$$

where ϕ is a complex field. The $U(1)$ symmetry implies that $\mathcal{L}$ is invariant under

$$\phi \to e^{i\alpha}\phi, \tag{4.2}$$

where α is a constant. The energy is given by

$$E = \int d^3x[|\dot{\phi}|^2 + |\nabla\phi|^2 + V(|\phi|)]. \tag{4.3}$$

The $U(1)$ symmetry is associated to the current

$$j_\mu = -i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*). \tag{4.4}$$

The corresponding conserved charge is

$$Q = -i\int d^3x(\phi^*\dot{\phi} - \phi\dot{\phi}^*). \tag{4.5}$$

A Q-ball is a solution that, for a given charge Q, minimizes the energy. Such a solution can be found using Lagrange multipliers. We want to minimize

$$E_\omega[\phi, \dot{\phi}, \omega] = E - \omega\left[-i\int d^3x(\phi^*\dot{\phi} - \phi\dot{\phi}^*) - Q\right] \tag{4.6}$$

with respect to all arguments. Minizing with respect to the time derivative and assuming that the mimimal energy configuration is spherically symmetric, implies that

$$\phi = e^{i\omega t}\frac{\rho(r)}{\sqrt{2}}. \tag{4.7}$$

Substituting this into (4.6), we obtain

$$E_\omega = 4\pi\int drr^2\left[\frac{1}{2}\left(\frac{\partial\rho}{\partial r}\right)^2 - \omega^2\frac{1}{2}\rho^2 + V(\rho)\right] + \omega Q, \tag{4.8}$$

where Q takes the form

$$Q = 4\pi\omega\int drr^2\rho^2. \tag{4.9}$$

Minimizing with respect to ρ gives the Q-ball equation

$$\frac{\partial^2\rho}{\partial r^2} + \frac{2}{r}\frac{\partial\rho}{\partial r} = \frac{\partial V}{\partial\rho} - \omega^2\rho. \tag{4.10}$$

We are looking for solutions which satisfy the boundary conditions $\rho(r) = 0$ as $r \to \infty$ and $\partial\rho/\partial r = 0$ as $r \to 0$. For a given global charge Q there exists a unique Q-ball solution. The Q-ball is stable if $E/Q < m_\phi$. The most stable solution is obtained by minimizing E_ω with respect to ω. If we interpret ρ as the position of a particle and r as a time, equation (4.10) is analogous to the Newtonian equation of motion for a particle of unit mass subject to a viscous force, with viscosity inversely proportional to time, evolving in the effective potential $\frac{1}{2}\omega^2\rho^2 - V$.
The existence of Q-balls implies some constraints on the potential $V(\rho)$ and ω:

1. The effective mass of ρ must be negative. Assuming that $V(0) = V'(0) = 0$ and $V''(0) = \omega_+^2$, we can deduce that $\omega < \omega_+$.

2. The minimum of $V(\rho)/\rho^2$ must be attained at some positive value of ρ, name it ρ_0. If $V(\rho_0)/\rho_0^2 = \omega_-^2$, then the existence of the solution implies that $\omega_- < \omega < \omega_+$.

In the limit $\omega \to \omega_-$, the profile function $\rho(r)$ is constant within a certain radius R, say $\rho = \rho_0$, and is zero outside the radius (thin-wall approximation). The transition zone connecting these two regions is of thickness of the order of ω_+^{-1}. In this limit the charge takes the form

$$Q = \frac{4}{3}\pi R^3 \omega_0 \rho_0^2 \tag{4.11}$$

and the energy is given by

$$E = Q\sqrt{\frac{2V(\rho_0)}{\rho_0^2}}. \tag{4.12}$$

On the other hand, when $\omega \to \omega_+$, the profile function goes to zero very rapidly (thick-wall approximation).
In general, the main properties of Q-balls remain in the quantum theory. Stable solutions may become metastable, with lifetimes depending on the values of the coupling constants.

4.2 Existence and Stability

In this section we will provide two theorems [5] that ensure the existence and stability of Q-balls for sufficiently large Q.

The first theorem is about the initial conditions. A set of initial conditions is of the Q-ball type if $\phi(t=0,\mathbf{x}) = \phi(r)$ and $\dot{\phi}(t=0,\mathbf{x}) = i\omega\phi(r)$, where ω is a contant and $\phi(r)$ is a positive, monotonically decreasing function.

Theorem 1: For any theory of the type (4.1), with $V \geq 0$, given a set of initial conditions, with some Q and E, there exists another set of initial conditions with same Q and smaller or equal E.

Proof: The energy and charge are given respectively by 4.3 and 4.5. In addition to that let's define

$$I \equiv 2\int d^3x\phi\phi^*, \tag{4.13}$$

and

$$\omega \equiv \frac{Q}{I}. \tag{4.14}$$

The Scwharz inequality gives

$$Q^2 \leq 2I\int d^3x\dot{\phi}\dot{\phi}^*. \tag{4.15}$$

This provides us with a bound on $\dot{\phi}$ for fixed ϕ and Q. The inequality is satisfied if and only if

$$\dot{\phi} = i\omega\phi. \tag{4.16}$$

In the following, we will assume that $\dot{\phi}$ has been chosen to satisfy (4.15). The energy functional becomes

$$E_Q = \int d^3x[|\nabla\phi|^2 + V(|\phi|)] + \frac{Q^2}{2I}. \tag{4.17}$$

We now choose the following parametrization for ϕ:

$$\phi = e^{i\theta}\tilde{\phi}, \quad \tilde{\phi} \geq 0. \tag{4.18}$$

The only term depending on θ in the energy is the spatial derivative term

$$E_Q = \int d^3x\tilde{\phi}^2(\nabla\theta)^2 ... \tag{4.19}$$

It follows that the energy can always be minimized by choosing θ to be constant and keeping $\tilde{\phi}$ fixed. Without loss of generality, we set $\theta = 0$. The initial conditions become

$$\phi = \tilde{\phi}, \tag{4.20}$$

and

$$\dot{\phi} = i\omega\tilde{\phi}. \tag{4.21}$$

For any positive function of position, vanishing at infinity, as $\tilde{\phi}(\mathbf{x})$ is, the spherical rearrengement $\rho(r)$ is defined as the spherically symmetric monotinacally decreasing function satisfying

$$\mu_L\{\mathbf{x}|\rho(\mathbf{x}) \geq \epsilon\} = \mu_L\{\mathbf{x}|\tilde{\phi}(\mathbf{x}) \geq \epsilon\}, \quad \text{for any } \epsilon > 0, \tag{4.22}$$

where μ_L is the Lebesgue measure. It follows from the definition of ρ that I and the integral of V remain unchanged if we replace $\tilde{\phi}$ by its spherical rearrengement. We can now use the theorem by Glaser, Grosse, Martin and Thirring on spherical rearrengements, which states that

$$\int d^3x(\nabla\tilde{\phi})^2 \geq \int d^3x(\nabla\rho)^2, \tag{4.23}$$

to conclude that we can always minimize the energy by choosing $\tilde{\phi}$ to be shperically symmetric and monotonically decreasing. This result, together with (4.20) and (4.21) ends the proof. □

However, this theorem is not sufficient to prove the existence of Q-balls. We need stronger constraints on the potential.

Definition: A potential V can display Q-balls if:

1. $V(0) = 0$ and V is positive everywhere else. It must be C^2. $V'(0) = 0$ and $V''(0) = \mu^2 > 0$.
2. The minimum of $V/|\phi|^2$ is attained at some value ϕ_0 greater than 0.
3. There exist three positive numbers a, b and c such that $\mu^2|\phi|^2 \leq \min(a, b|\phi|^c)$.

The first two conditions are the same as those quoted in the previous section. Condition 2 tells us that the potential should fall below $\mu^2|\phi|^2$ for some value of the field. Condition 3 makes sure it does not fall too far.

Theorem 2: If V can display Q-balls, there exists $Q_{min} \geq 0$ such that for any $Q \geq Q_{min}$ there are initial values of the Q-ball type that minimize E for that given Q. Furthermore, these are the initial conditions for a Q-ball solution to the equations of motion.

The proof of Theorem 2 is laborious. The idea is to minimize some ingeniously defined functionals of the field, whose infimums bound the minimum of the energy, using a parametrization of the Q-ball type (Theorem 1 guarantees that this parametrization exists). For the complete proof, we refer the interested reader to the original article [5]. Theorem 2 guarantees that the Q-ball minimizes the energy for a given charge. In the previous section we showed that Q-balls are stationary points of the energy functional, but we had no guarantee that they actually are minima. Without Theorem 2, there is no guarantee that minima even exist. For example, in the free theory $V = \mu^2|\phi|^2$, the infimum of E for fixed Q is $\mu|Q|$. We can come arbitratily close to this lower bound. However, there is no set of initial conditions that exactly attains it because to attain it ϕ has to have a vanishing gradient, which is not possible if $Q \neq 0$.

4.3 Semi-analytic Example

We reproduce here work done in [9]. Consider the potential

$$V(\rho) = \rho^2[1 + (1 - \rho^2)^2]. \tag{4.24}$$

Here, $\omega_- = \sqrt{2}$ and $\omega_+ = 2$, hence stable Q-balls exist in the range $\sqrt{2} < \omega < 2$. It is possible to obtain an ansatz for the profile function by using a semi-Bogomolny argument [10] in the energy functional. Using such argument, the energy can be written as

$$\begin{aligned} E_B &= \left(\frac{\omega}{2} + \frac{1}{\omega}\right) + 4\pi \int drr^2 \left(\frac{(\rho')^2}{2} + \rho^2(1-\rho^2)^2\right) \\ &= \left(\frac{\omega}{2} + \frac{1}{\omega}\right) + 4\pi \int drr^2 \left(\frac{\rho'}{\sqrt{2}} + \rho(1-\rho^2)\right)^2 - 4\pi\sqrt{2} \int drr^2\rho'\rho(1-\rho^2) \\ &\geq \left(\frac{\omega}{2} + \frac{1}{\omega}\right) - 4\pi\sqrt{2} \int drr^2\rho'\rho(1-\rho^2). \end{aligned} \tag{4.25}$$

The equality holds when the total square term is zero. This requirement gives the semi-Bogomolny equation

$$\rho' = -\sqrt{2}\rho(1-\rho^2). \tag{4.26}$$

This equation has the solution

$$\rho(r) = \frac{1}{\sqrt{1 + C_B \exp(2\sqrt{2}r)}}, \qquad C_B > 0. \tag{4.27}$$

It satisfies the boundary condition $\rho(0) = 1/\sqrt{1+C_B}$ and $\rho(\infty) = 0$, whereas $\rho'(0) = -\sqrt{2}C_B/(1+C_B)^{3/2}$. Inside the Q-ball, we have $\rho' \approx 0$ and $\rho \approx 1$, while outside the Q-ball we have $\rho' = 0$ and $\rho = 0$. Hence (4.27) describes correctly the Q-ball outside and inside the radius. However, in the transition zone, this description is not accurate.
For the profile function (4.27), the charge and energy take the form

$$Q_B = -\frac{\pi\omega}{2\sqrt{2}}\left(\frac{\pi^2}{6}\ln(C_B) + \frac{1}{6}[\ln(C_B]^6 + Li_3[-C_B]\right),$$

$$E_B = \left(\frac{\omega}{2} + \frac{1}{\omega}\right)Q + \frac{\sqrt{2}\pi}{4}\left(\frac{\pi^2}{6} + \frac{1}{2}[\ln(C_B]^2 + \ln\left(1 + \frac{1}{C_B}\right) + Li_2[-C_B]\right), \tag{4.28}$$

where $Li_n(z) = \int_0^z dy \frac{Li_{n-1}(y)}{y}$ and $Li_1 = \ln(1-y)$ is the n-logarithmic function.
The next step is to minimize the energy E_B with respect to C_B keeping the charge Q_B constant. The semi-Bogomolny argument should be only valid in the limit $f'(r) \to 0$ ar $r \to 0$, *i.e.* when $C_B \approx 0$. In this limit, the logarithms dominate the n-logarithms, since these functions tend to zero as polynomials. Hence, substituting $z = -\ln(C_B)$, for $z > 0$ we have

$$Q_B = \frac{\pi\omega z}{12\sqrt{2}}\left(\pi^2 + z^2\right),$$

$$E_B = \left(\sqrt{\omega}2 + \frac{1}{\omega}\right)Q_B + \frac{\sqrt{2}\pi}{4}\left(\sqrt{\pi^2 + 3z^2}6 + z\right). \tag{4.29}$$

Solving ω in terms of Q_B and substituting into the energy gives

$$E_B = \frac{6\sqrt{2}Q_B^2}{\pi z(\pi^2+z^2)} + \frac{\pi z(\pi^2+z^2)}{12\sqrt{2}} + \frac{\sqrt{2}\pi}{4}\left(\frac{\pi^2+3z^2}{6} + z\right). \tag{4.30}$$

Minimizing the energy with respect to z, we find the frequency ω and the charge Q_B:

$$\omega^2 = 2 + \frac{12(1+z)}{\pi^2 + 3z^2}, \tag{4.31}$$

$$Q_B^2 = \frac{\pi^2 z^2 (\pi^2 + z^2)^2}{144(\pi^2 + 3z^2)}(6 + 6z + \pi^2 + 3z^2). \tag{4.32}$$

Solving equation (4.30) to find z, gives

$$z = \frac{2}{\omega^2 - 2}\left(1 + \sqrt{1 - \frac{\pi^2}{12}(\omega^2-2)^2 + (\omega^2 - 2)}\right). \tag{4.33}$$

In the limit $\omega \to \sqrt{2}$, z goes to infinity, consistently with the result (4.29). Hence, the semi-Bogomolny argument is valid in the thin-wall approximation.

Now we can eliminate z from the equations and find E_B and Q_B in terms of ω only. Then, susbtituting Q_B in E_B, we find that the energy is

$$E_B = \sqrt{2}Q_B + \frac{3^{2/3}\pi^{1/3}}{2^{7/6}}Q_B^{2/3} + \frac{5\pi^{2/3}}{2^{11/6}3^{2/3}}Q_B^{1/3} - \frac{\pi(4+3\pi^2)}{36\sqrt{2}} + O(Q_B^{-1/3}). \tag{4.34}$$

The agreement of the approximated expression (4.33) and the numerical simulations performed by the authors of the article [9] is very good and goes beyond the expected range. For further comments and details we refer the reader to the original article.

4.4 Q-balls and Hybrid Inflation

The analysis in section 4.1 applies to the hybrid inflation potential (3.38):

$$V(|\phi|, \chi) = m_\phi^2|\phi|^2 + g^2|\phi|^2\chi^2 + \frac{\lambda}{4}(\chi^2 - v^2)^2, \tag{4.35}$$

with the difference that instead of a single equation of motion we have a system of two coupled equations. If we assume that $\phi = e^{i\omega t}\frac{\rho(r)}{\sqrt{2}}$, they are

$$\frac{\partial^2\rho}{\partial r^2} + \frac{2}{r}\frac{\partial\rho}{\partial r} = -\omega^2\rho + m_\phi^2\rho + g^2\chi^2\rho, \tag{4.36}$$

$$\frac{\partial^2\chi}{\partial r^2} + \frac{2}{r}\frac{\partial\chi}{\partial r} = g^2\rho^2\chi + \lambda(\chi^2 - v^2)\chi, \tag{4.37}$$

The energy and charge take the form:

$$E = 4\pi \int drr^2 \left[-\omega^2 \frac{\rho}{2} + \frac{1}{2}\left(\frac{\partial \rho}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial \chi}{\partial r}\right)^2 + V(\rho,\chi) \right], \tag{4.38}$$

$$Q = 4\pi\omega \int drr^2 \rho^2. \tag{4.39}$$

4.4.1 Numerical Solutions

We base our approach on work done in [1], where Q-ball solutions have been found numerically for the supersymmetric F-term hybrid inflation field equations.

In order to solve numerically the equations (4.35) and (4.36), we need to choose the free parameters of the model in a convenient way. First of all, the mass of the inflaton m_ϕ is set to zero. Note, however, that the inflaton field still has a mass arising from the term $g^2|\phi|^2\chi^2$ in the potential. Furthermore, we choose $g^2 = 2\lambda$. The mass scale v can be eliminated by choosing units such that $M = \frac{v}{\sqrt{2}} = 1$. The coupling constant g can be absorbed in the variable by defining $\tilde{r} = gr$. Then, our system of equations becomes

$$\frac{\partial^2 \rho}{\partial \tilde{r}^2} + \frac{2}{\tilde{r}}\frac{\partial \rho}{\partial \tilde{r}} = -\tilde{\omega}^2\rho + \chi^2\rho, \tag{4.40}$$

$$\frac{\partial^2 \chi}{\partial \tilde{r}^2} + \frac{2}{\tilde{r}}\frac{\partial \chi}{\partial \tilde{r}} = \rho^2\chi + (\chi^2 - 1)\chi. \tag{4.41}$$

These equations correspond to the Q-ball equations in the case $g = 1$ in $M = 1$ units. The energy and charge become

$$E = \frac{4\pi}{g} \int d\tilde{r}\tilde{r}^2 \left[-\tilde{\omega}^2 \frac{\rho}{2} + \frac{1}{2}\left(\frac{\partial \rho}{\partial \tilde{r}}\right)^2 + \frac{1}{2}\left(\frac{\partial \chi}{\partial \tilde{r}}\right)^2 + \tilde{V}(\rho,\chi) \right] = \frac{\tilde{E}}{g}, \tag{4.42}$$

where

$$\tilde{V}(\rho,\chi) = \left(\frac{\chi^2}{2} - 1\right)^2 + \frac{1}{2}\rho^2\chi^2, \tag{4.43}$$

and

$$Q = \frac{4\pi\tilde{\omega}}{g^2} \int d\tilde{r}\tilde{r}^2\rho^2 = \frac{\tilde{Q}}{g^2}. \tag{4.44}$$

$\tilde{E}$ and $\tilde{Q}$ are the energy and charge of the $g = 1$ Q-ball. The inflaton mass in given by $m = gv$. Hence, for $g = 1$, $m = \sqrt{2}$.

ω	ρ_0	χ_0	E	Q	$E/(Qm)$	r_{end}
1.3	1.458824	0.69888	118.969	77.996	1.079	6.3
1.25	1.68464	0.552551	139.301	91.997	1.071	8
1.17	2.075345	0.334867	166.998	111.11	1.063	8
1.1	2.37263	0.20285	203.627	138.759	1.038	8
1.0	2.79282	0.0816	276.707	198.247	0.987	10
0.9	3.2339	0.023758	387.716	298.445	0.919	8
0.8	3.73708	0.0041436	564.719	475.804	0.839	8
0.7	4.34405	3.16×10^{-4}	860.183	809.284	0.752	8
0.6	5.1397	5.4×10^{-6}	1393.32	1502.51	0.656	8
0.58	5.51005	1.03×10^{-6}	1614.7	1831.04	0.624	6.5

Table 4.1: Values of Q-ball solutions for equations (4.40) and (4.41)

The procedure to solve equations (4.40) and (4.41) numerically consists of choosing ω, the initial values ρ_0 and χ_0 at $r = r_{min}$ and then imposing the initial conditions that $d\rho/dr(r_{min}) = 0$ and $d\chi/dr(r_{min}) = 0$. We chose $r_{min} = 0.1$. The initial values of the fields need to be chosen with an extreme precision as the solution varies a lot for small changes in those values. Once we have obtained the solution we calculate E and Q and check if $E/Q < m$, *i.e.* if the solution is stable. The same process is repeated for different values of ω.

In table 4.1 the energy and charge of a set of Q-ball solutions are given. We span the region $0.58 < \omega < 1.3$. Stable Q-balls ($E/(Qm) < 1$) exist for $\omega \lesssim 1.08$. Fitting a power-law to the energy versus charge graph, we find $E = 3.4606Q^{0.822}$. That means that Q-balls carrying a big charge are stable.

The energies and charge of Q-balls with different values of g can be calculated from equations (4.42) and (4.43). The values of ω for which stable Q-balls exist are $omega \lesssim 1.08g$.

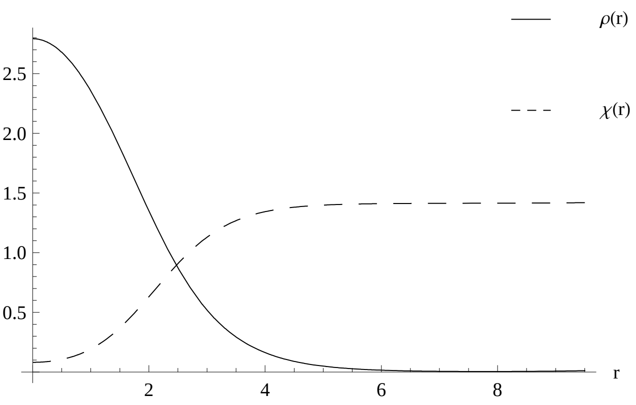

Figure 4.1: Q-ball solution for $\omega = 1.0$, $g = 1$ and $v^2 = 2$.

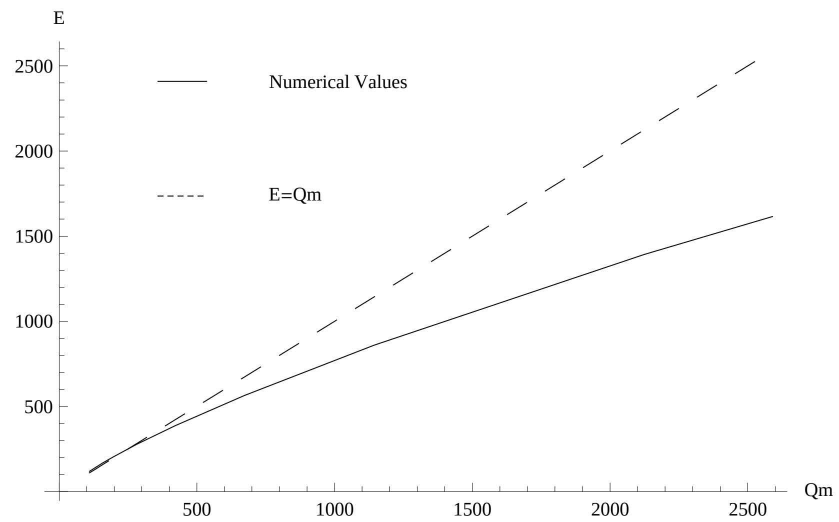

Figure 4.2: Energy vs charge times mass. The numerical values have been fitted with the power law $E = 3.4606Q^{0.822}$.

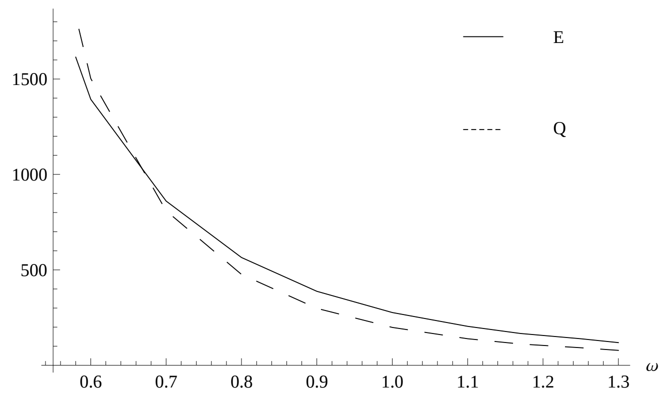

Figure 4.3: Energy and charge times mass vs ω. Note that both graphs diverge as ω approaches 0.5.

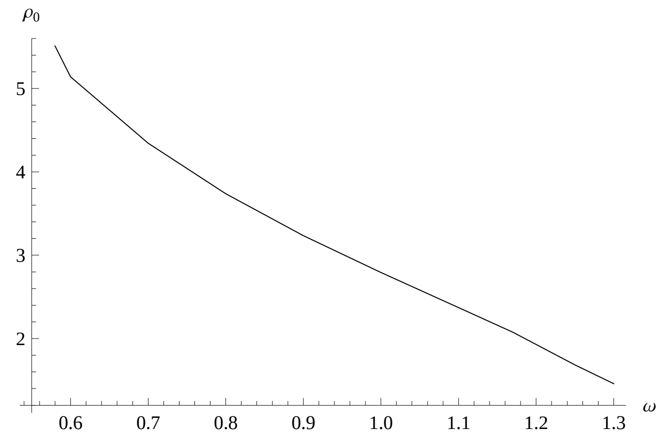

Figure 4.4: ρ_0 vs ω.

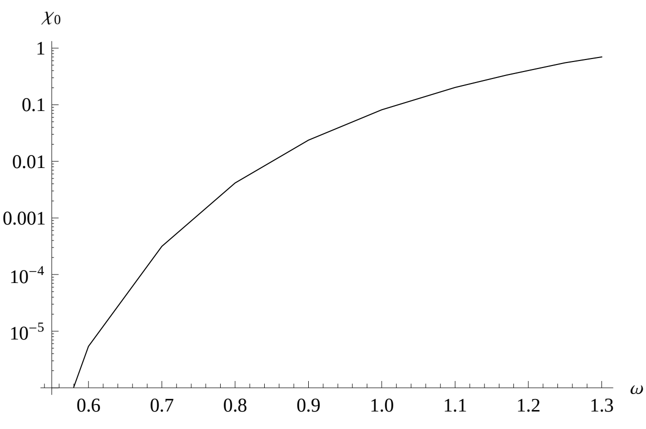

Figure 4.5: χ_0 vs ω.

5 Conclusions

In the previous chapters, we gave a review of standard cosmology, inflation and presented some results on Q-balls applied to cosmology.
As pointed out before, standard cosmology allows to understand the evolution of spacetime starting from an initial state which has to be specified. For a description of our universe, the initial state is that of a radiation dominated universe. The question of how this initial state happened to exist, is not addressed within standard cosmology. Instead, a very precise fine-tuning of the free parameters is required.
That is where inflation comes into the game. It provides a dynamical solution to the problem of initial conditions by requiring that, for a certain period of time, the Hubble sphere shrinks instead of expanding as it does in standard cosmology. During the process many scales leave the horizon, and, by the end of inflation the region contained in the Hubble sphere is highly homogeneous. Subsequently, when inflation ends and the Hubble sphere starts to expand again, inhomogeneities re-enter the horizon, allowing the formation of structures. Hence, inflation not only provides a solution to the problems of FRW cosmology; it also provides a framework for building models on the origin of structures. That is the ground where most developments and results are expected in the future, especially in the form of constraints imposed by observation.
In the last chapter, we showed that Q-balls may evolve from the hybrid inflation potential. These Q-balls carry a $U(1)$ charge, associated to the inflaton field. Furthermore, we found that the ratio of energy to charge scales as $E/(Qm) \propto Q^{-0.188}$, which means that large Q-balls are stable.
Large Q-balls are bound states with large mass, and, if stable, they could be a candidate for dark matter. Hence, our results lead to interesting ideas. If a region of the universe, with a certain charge density, undergoes a period of hybrid inflation, its charge density will be dramatically reduced during the process. Nevertheless, it is possible to build models with very high initial

charge density, so that, by the end of inflation, the amount of charge contained within a Hubble radius allows the creation of stable Q-balls. These Q-balls, formed at the end of inflation, may account for an important fraction of dark matter in the universe.

Dark matter accounts for 22% of the energetic content of the universe. This imposes a constraint on the initial charge density of a hypothetical model. The risk is to require a fine-tuning of the initial charge density in order to obtain the desired quantity of dark matter, in the form of Q-balls, at the end of inflation. Since inflation was born to solve a fine-tuning problem, it would be difficult to accept a fine tuning of the initial charge density without perplexities. However, that problem could be addressed from a statistical point of view: if many regions underwent a period of inflation, we could as well be living in the region that happened to have the initial charge density that leads to the right amount of Q-balls to explain dark matter.

The relevance of Q-balls to dark matter remains an open question. Interesting developments are expected in the future.

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