Conformal Field Theory

Foundations, Extensions and Boundaries

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To my parents, with gratitude

Contents

1 Introduction

1.1 Motivation

Conformal field theory (CFT) has been an important tool in theoretical physics in the last 30 years. This is linked to the fact that conformal field theories play an important role in many different areas of physics, for example in studying critical systems in statistical mechanics or as an example for exactly solvable models in algebraic/axiomatic quantum field theory. The mathematical structure of conformal field theories is studied in pure mathematics under the name vertex operator algebras. But it is string theory which especially in the recent past brought most interest to conformal field theory. We will take a moment to discuss the different appearances of CFT in more detail.

In statistical mechanics conformal field theories are used to describe the continuum limit of well-known models at their critical points. Perhaps the most prominent example in this regard is the Ising modell which corresponds to a certain minimal model in conformal field theory. At the critical point the correlation length which is the typical length scale over which the spins are correlated becomes infinite. Thus this scale becomes large in comparison to the lattice spacing, the other scale in the system and the whole system can be considered as scale invariant in this limit. This motivates the emergence of a conformal field theory in this particular example, but is by no means restricted to it. The idea of universality in statistical physics states that models at their critical points can be classified according to their behaviour at the critical point to fit in a discrete set of universality classes. As we will see conformal field theory is especially suited to two dimensions and may thus be helpful in the classification of universality classes of two-dimensional critical systems. A text which approaches CFT from the point of view of critical systems in statistical mechanics is the article [2] by Cardy. Additional material on this topic can be found in the book [18] by Di Francesco, Mathieu and Sénéchal.

Conformal field theories are also studied in the context of algebraic/axiomatic quantum field theory. One rigorous approach is given in the article [8] by Gaberdiel and Goddard. A bit less rigorous but with the same background is the review article [5] by Gaberdiel. In algebraic quantum field theory there is interest in CFT as an example for a two-dimensional quantum field theory because these theories are strongly constrained due to their high degree of symmetry and may thus serve as test ground for algebraic methods or axiomatic approaches. This may then help indirectly to gain a better understanding of higher dimensional theories. The articles [25, 12] may serve as starting points in this direction.

The impact of conformal field theory is not limited to the area of physics. The theory of vertex operator algebras in pure mathematics may be seen as mathematically rigorous formulation of the algebraic foundations which underlie conformal field theory. We cite [9] as an exemplary reference for the large number of mathematical texts on this subject.

The last motivation which was mentioned at the beginning of this section is string theory. String theory is inseparably linked to CFT via the worldsheet description of the string. The worldsheet is the two-dimensional surface which the string sweeps out while propagating through space-time and the formulation of string theory on the worldsheet is a conformal field theory. Another link is between so-called *boundary conformal field theories (BCFTs)* i.e. CFTs on manifolds with a boundary and Dirichlet branes in string theory. The last point to be mentioned is the AdS/CFT-Correspondence, a very active area of current research, where a (super) conformal field theory usually in 4 dimensions plays an important role on the gauge-theory side of the correspondence. As examples for references which contain sections about CFT in particular in regard to application in string theory we recommend the books [15, 21, 17, 13].

We conclude this section with a short word on the literature available on this topic in addition to the references cited above. A good starting point is perhaps the short review article [7] by Gaberdiel. Virtually any topic of the next chapter is treated in great detail in the book [18] by Di Francesco, Mathieu and Sénéchal and in the book [14] by Ketov. During the preparation of this thesis also the review articles [4, 27, 23, 10, 1] by Fuchs, Zuber, Schellekens, Ginsparg and Belavin, Polyakov and Zamolodchikov were very useful.

1.2 Outline

The following first part of the thesis will cover the foundations of conformal field theory in detail but we will restrict the discussion to the case where the chiral algebra coincides with the Virasoro algebra and we will cover mainly the sphere and the torus as underlying manifolds.

Our starting point will be the *conformal (symmetry) group* in d dimensions. This is the group of coordinate transformation under which the metric stays invariant up to a scale factor. We will find that the global conformal group contains in addition to the Poincaré group dilations and special conformal transformations. One significant feature of conformal field theories is that they severely restrict correlation functions between distinguished fields, the so-called *primary fields.* In fact 2- and 3-point functions are determined up to a constant just by giving the conformal dimensions of the fields involved. We will mostly focus on two-dimensional CFTs because conformal methods are especially powerful in this case as the conformal algebra is infinite dimensional.

The next step will be the transition from the classical field theory to the quantized theory. The well-known mechanism of time-ordering will carry over as radial ordering after mapping the two-dimensional cylinder to the complex plane. We will see that the transformation properties of a conformal field is linked to the singular terms occurring in its operator product expansion with the stress-energy tensor. The *operator product expansion (OPE)* is a central concept in conformal field theory. It indicates that the product of two conformal fields evaluated at different points can be expanded as another local field times a numerical coefficient depending on the difference between the points in question or written schematically

$$\phi_i(y)\phi_j(z) = \sum_k C_{ijk}(y-z)\phi_k(y). \tag{1.2.1}$$

In a two-dimensional CFT the OPE of the stress-energy tensor with itself will lead us to another central concept, the *Virasoro algebra*, giving the algebra of the conformal generators of the quantized theory i.e. of the modes L_n of the stress-energy tensor:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{0,m+n} \tag{1.2.2}$$

Afterwards we will discuss the construction of the Hilbert space of a conformal field theory, which means studying the representation theory of the Virasoro algebra. The representations will be highest-weight representations where highest weight states will be characterized by being annihilated by all L_n, $n > 0$ and correspond to primary fields via the state-operator correspondence. Descendant states can be obtained from highest weight states by acting with L_{-n} on them. A remarkable result in two-dimensional CFTs is that a general n-point function can be computed just from knowing the central charge c, the conformal weights of the corresponding primary fields and the coefficients appearing in the 3-point functions. The latter may be calculated in principle from the *conformal bootstrap* procedure.

The requirement of unitary representations will severely restrict the central charges and highest weights and will lead us for $c < 1$ to a discrete set of combinations of central charges c and highest weights $h_{p,q}$, the so-called *minimal models* which are examples for the so-called *rational conformal field theories.* The first minimal models can be identified with models known from statistical mechanics at their critical points e.g. with the Ising model.

We will also discuss conformal field theories on the torus, which can be represented by a complex modular parameter. Modular transformations of this parameter give rise to the same torus, but place constraints on the field content of a modular invariant CFT and link in particular the chiral and antichiral parts of the theory. We will discuss *fusion rules* and *Verlinde's formula* and conclude the chapter with a look at the simplest examples for conformal field theories namely free bosons and free fermions.

The next chapter is devoted to possible extensions of the Virasoro algebra which occur if there are more conserved currents in the theory in addition to the stress-energy tensor. In these cases the chiral algebra is extended but still contains the Virasoro algebra as a subalgebra. The representation theory is very similar to the representation theory of the Virasoro algebra discussed previously. After a discussion of general features we will treat the two most important extensions namely *Kac-Moody algebras* and *superconformal algebras* in more detail. As an explicit example for a conformal field theoretic model with extended symmetry algebra we will give a very brief introduction to WZW models.

Finally the third part of the thesis gives a flavour of another way of generalizing conformal field theory. We consider *boundary conformal field theories* which are defined on manifolds with a boundary. Our prototypical example will be a boundary conformal field theory defined on the upper half plane. We will discuss *conformal boundary conditions* and their implications for the boundary states. A consideration of partition functions will lead us to consistency conditions, the *Cardy conditions*. We conclude the thesis by a discussion of an explicit method for constructing the boundary states in a diagonal conformal field theory.

2 Foundations of CFT

This chapter is the main part of the thesis from the conceptual point of view. We will restrict ourselves to the case where the chiral algebra coincides with the Virasoro algebra and will restrict the manifolds which underlie the CFTs to be manifolds without boundaries. The generalizations to conformal field theories with extended symmetry algebras and boundary conformal field theory will be sketched in the two subsequent chapters.

2.1 The Conformal Symmetry Group

In this section we look at the *Conformal Group* in detail. We consider $\mathbb{R}^d$ with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ with signature (p, q) where $p+q = d$. We allow arbitrary dimensions d at this point, since it helps us to understand the special role of the $d = 2$ case later. Furthermore we can study the implications of conformal invariance in this more general setting first.

Under a general coordinate transformation $x^\mu \to x'^\mu$ the metric transforms according to

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{2.1.1}$$

A *conformal transformation* is defined as a mapping $x \to x'$ leaving the metric invariant up to a scale factor:

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu} \quad \text{where} \quad \Omega^2(x) = e^{\omega(x)} \tag{2.1.2}$$

Here we look at perturbations around a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$. We see immediately that the d dimensional Poincaré group forms a subgroup of the Conformal Group in d dimensions as these transformations leave $g_{\mu\nu}$ invariant.

It follows directly from the definition that conformal transformations preserve angles: Assuming we are dealing with $\mathbb{R}^d$ and $g_{\mu\nu} = \delta_{\mu\nu}$ then $\cos(\phi_{\measuredangle(a,b)}) = \frac{a \cdot b}{\sqrt{a^2 \cdot b^2}}$ (where $a \cdot b = g_{\mu\nu} a^\mu b^\nu$) remains invariant under a conformal transformation.

2.1.1 The (Global) Conformal Group in d dimensions

To identify the generators of the Conformal Group consider the variation of $g_{\mu\nu}$ under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$. To first order we have:

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon^\lambda(x) g_{\lambda\nu} - \partial_\nu \epsilon^\lambda(x) g_{\mu\lambda} - \epsilon^\lambda(x) \partial_\lambda g_{\mu\nu}(x) \tag{2.1.3}$$

In order to have a conformal transformation around a flat metric $\eta_{\mu\nu}$, we have to require that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega(x) \eta_{\mu\nu}. \tag{2.1.4}$$

The function ω can be fixed by taking the trace on both sides of the last equation which yields $\omega = \frac{d}{2}(\partial \cdot \epsilon)$. Thus we get

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{d}{2}(\partial \cdot \epsilon) \eta_{\mu\nu}. \tag{2.1.5}$$

If we apply the operator ∂_μ to the last equation, we get

$$\Box\epsilon = \big(\frac{d}{2} - 1\big)\partial_\nu(\partial \cdot \epsilon) = 0. \tag{2.1.6}$$

Acting with the operator $\Box$ on both sides of eq. (2.1.5) gives

$$\partial_\mu \Box \epsilon_\nu + \partial_\nu \Box \epsilon_\mu = \frac{d}{2}\eta_{\mu\nu}\Box(\partial \cdot \epsilon). \tag{2.1.7}$$

Combining the last two we obtain equations which constrains $(\partial \cdot \epsilon)$:

$$\big(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu\big)(\partial \cdot \epsilon) = 0 \tag{2.1.8}$$

$$(d-1)\cdot \Box(\partial \cdot \epsilon) = 0, \tag{2.1.9}$$

where (2.1.9) follows by taking the trace of eq. (2.1.8).

From eq. (2.1.9) we deduce that in $d = 1$ dimensions every smooth transformation is a conformal transformation. Furthermore these equations show that in $d > 2$ dimensions $\partial_\mu\partial_\nu(\partial \cdot \epsilon) = 0$ which means that $\partial \cdot \epsilon$ can be at most linear in x.

There are four different expressions for ϵ^μ which are consistent with eq. (2.1.5). The easiest way to find them is to start with a general ansatz for ϵ^μ which is at most quadratic in x and to derive conditions for the coefficients by plugging them back into eq. (2.1.5). The four cases are:

1. **Translations**: $\epsilon^\mu = a^\mu$ is generated by $P^\mu = -\mathrm{i}\partial^\mu$. Exponentiating gives finite translations $x^\mu \to x^\mu + A^\mu$

2. **Lorentz transformations**: $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ (where $\omega_{\mu\nu} = -\omega_{\nu\mu}$) is generated by $J^{\mu\nu} = \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$. Exponentiating gives Lorentz transformations $x^\mu \to$

$\Lambda^\mu{}_\nu x^\nu$ where $\Lambda^\mu{}_\nu \in SO(p,q)$ (for $q=0$ we have just rotations)

3. **Dilations**: $\epsilon^\mu = \lambda x^\mu$ is generated by $D = -\mathrm{i} x^\mu \partial_\mu$. Exponentiating gives scale transformations $x^\mu \to \Lambda x^\mu$.

4. **Special conformal transformations**: $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x$ is generated by $K^\mu = -\mathrm{i}(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu)$. The corresponding finite transformations are given by $x^\mu \to \frac{x^\mu - B^\mu x^2}{1 - 2B\cdot x + B^2 x^2}$. These transformations can be understood as a translation preceded and followed by an inversion $x^\mu \to \frac{x^\mu}{x^2}$.

The generators given above represent the action of these transformations on a space of functions. We say that $\hat{O}$ generates the transformation if $f(x) \to f(x) + i\epsilon \hat{O} f + \mathcal{O}(\epsilon^2)$.

We are rather familiar with finite and infinitesimal translations or Lorentz transformations and the corresponding generators. The result of the exponentiation of an infinitesimal dilation is easy to check. For the special conformal transformations it is easier to start with the finite transformation and to confirm that the corresponding infinitesimal transformation is given as claimed before. The generators obey the following commutation relations:

$$\begin{aligned} [D, P^\mu] &= \mathrm{i} P^\mu \quad [K^\mu, P^\nu] = 2\mathrm{i}(\eta^{\mu\nu} D - J^{\mu\nu}) \\ [D, K^\mu] &= \mathrm{i} K^\mu \quad [K^\rho, J^{\mu\nu}] = \mathrm{i}(\eta^{\rho\mu} K^\nu - \eta^{\rho\nu} K^\mu) \\ [P^\rho, J^{\mu\nu}] &= \mathrm{i}(\eta^{\rho\mu} P^\nu - \eta^{\rho\nu} P^\mu) \\ [J^{\mu\nu}, J^{\rho\sigma}] &= \mathrm{i}(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho}) \end{aligned} \tag{2.1.10}$$

The algebra of generators is isomorphic to $so(p+1, q+1)$ which can be made explicit by reorganizing the generators and checking that these generators obey the $so(p+1, q+1)$ commutation relations [18]. Here we will simply convince ourselves that the dimensions agree in both cases:

$$\underbrace{p+q}_{\text{translations}} + \underbrace{\frac{1}{2}(p+q)(p+q-1)}_{\text{Lorentz transformations}} + \underbrace{1}_{\text{dilations}} + \underbrace{p+q}_{\text{SCTs}} = \frac{1}{2}(p+q+2)(p+q+1) \quad (2.1.11)$$

We call this algebra the *conformal algebra* and the corresponding group the *(global) conformal group*. The designation global distinguishes these transformations from the larger transformation group in the case of $d = 2$ dimensions which we will discuss in a moment.

2.1.2 Correlation functions

As one example for the strong constraints which conformal symmetry places on the underlying (classical) field theory we discuss the form of correlation functions consistent with conformal symmetry. In d dimensions we look at the n-point correlation function of (spinless) *quasiprimary fields* i.e. of fields that transform under a global conformal transformation $x \to x'$ as

$$\phi_i(x) \to \phi_i'(x') = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_i/d} \phi_i(x'), \quad (2.1.12)$$

where Δ_i is the so-called *scaling dimension* of ϕ_i. In a conformally invariant theory the correlation functions involving these fields then satisfy

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \cdots \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d} \langle \phi_1(x_1') \dots \phi_n(x_n') \rangle. \quad (2.1.13)$$

In particular we look at two-point functions $\langle \phi_1(x_1)\phi_2(x_2) \rangle$ which are restricted by inserting expressions for translations and rotations in eq. (2.1.13) to be functions of

$|x_1 - x_2|$. The Jacobians simply give a factor 1 in this case. Next we insert dilations $x \to \lambda x$ into the equation and deduce that the 2- point functions have to be of the form

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{2.1.14}$$

where C_{12} is a numerical coefficient which depends on the normalization of the fields ϕ_1 and ϕ_2. The last condition on the form of the 2-point function is imposed by inserting a special conformal transformation $x^\mu \to \frac{x^\mu - B^\mu x^2}{1-2B\cdot x - B^2x^2}$ into eq. (2.1.13). For these transformation we find the Jacobian $\left|\frac{\partial x'}{\partial x}\right| = (1 - 2B\cdot x + B^2x^2)^{-d}$ and get the following condition from eq. (2.1.13):

$${\alpha_1}^{\Delta_1}{\alpha_2}^{\Delta_2} = (\alpha_1\alpha_2)^{\frac{1}{2}(\Delta_1+\Delta_2)}, \tag{2.1.15}$$

where $\alpha_i = 1 - 2B\cdot x_i + B^2x_i^2$. This condition requires $\Delta_1 = \Delta_2$ so we obtain the final result:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta}} & \text{if} \quad \Delta_1 = \Delta_2 = \Delta \\ 0 & \text{else} \end{cases} \tag{2.1.16}$$

This result is a demonstration of the strong constraints conformal invariance imposes on a conformal field theory because it follows from symmetry properties alone and does for example not depend on the way how the theory is formulated. An argument similar to the one presented above gives us the possible form of 3-point functions in a conformally invariant field theory [18]:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{2.1.17}$$

where $x_{ij} = |x_i - x_j|$. One might suspect that all n-point will have the structure of eq. (2.1.17) namely being determined by the conformal weights of the scaling dimensions up to a constant. The analysis of the 4-point functions shows that this is not the case. The reason is that we can form another kind of conformally invariant quantities with four points, the so-called *anharmonic ratios* or *cross-ratios*. For n distinct points there are $n(n-3)/2$ independent of these ratios which take the form $\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}$ for a collection of pairwise distinct indices i, j, k, l.

The 4-point function is determined up to an arbitrary function of the two independent anharmonic ratios [18]:

$$\langle \phi_1(x_1) \cdots \phi_4(x_4) \rangle = f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{23}x_{14}} \right) \prod_{1 \leq i < j \leq 4} {x_{ij}}^{\Delta/3 - \Delta_i - \Delta_j}, \tag{2.1.18}$$

where $f(x, y)$ is an arbitrary function and $\Delta = \Delta_1 + \ldots + \Delta_4$. We find similar expression for a general n- point function, which is determined by conformal invariance up to an arbitrary function of the anharmonic ratios.

2.1.3 The Witt algebra

For the rest of the thesis we will now concentrate on two-dimensional conformal field theories. Already by looking at eq. (2.1.9) we can infer that the conformal field theory in two dimensions plays a special role among the conformal field theories. In two dimensions and for an euclidean metric $\eta_{\mu\nu} = \delta_{\mu\nu}$ the condition (2.1.5) reads:

$$\partial_1 \epsilon^1 = \partial_2 \epsilon^2 \quad \partial_2 \epsilon^1 = -\partial_1 \epsilon^2 \tag{2.1.19}$$

These conditions have the form of the Cauchy-Riemann differential equations. Hence we choose complex variables, which will allow us to make use of methods from complex analysis:

$$\begin{aligned} z &\equiv x^z = x^1 + \mathrm{i}x^1 \quad \bar{z} \equiv x^{\bar{z}} = x^0 - \mathrm{i}x^1 \\ \partial &\equiv \partial_z = \frac{\partial}{\partial x^z} = \frac{\partial x^\mu}{\partial x^z}\partial_\mu = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2) \\ \bar{\partial} &\equiv \partial_{\bar{z}} = \frac{\partial}{\partial x^{\bar{z}}} = \frac{\partial x^\mu}{\partial x^{\bar{z}}}\partial_\mu = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2) \\ \epsilon &\equiv \epsilon^z = \frac{\partial x^z}{\partial x^\mu}\epsilon^\mu = \epsilon^1 + \mathrm{i}\epsilon^2 \\ \bar{\epsilon} &\equiv \epsilon^{\bar{z}} = \frac{\partial x^{\bar{z}}}{\partial x^\mu}\epsilon^\mu = \epsilon^1 - \mathrm{i}\epsilon^2 \end{aligned} \tag{2.1.20}$$

Note that the components of the metric in these coordinates are given by

$$g_{\bar{z}\bar{z}} = g_{zz} = \frac{\partial x^\mu}{\partial x^z}\frac{\partial x^\nu}{\partial x^z}g_{\mu\nu} = 0 \quad \text{and} \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{\partial x^\mu}{\partial x^z}\frac{\partial x^\nu}{\partial x^{\bar{z}}}g_{\mu\nu} = \frac{1}{2}, \tag{2.1.21}$$

which may sometimes be a source of confusion. In these variables the conditions from (2.1.19) take the form

$$\partial_{\bar{z}}\epsilon(z,\bar{z}) = 0 \quad \partial_z\bar{\epsilon}(z,\bar{z}) = 0. \tag{2.1.22}$$

This means that $\epsilon(z,\bar{z})$ is a function of z alone, which is called a *holomorphic function* in the physics literature whereas in mathematical texts often the term meromorphic is used. $\bar{\epsilon}(z,\bar{z})$ only depends on $\bar{z}$, which is called *antiholomorphic*.

Note that z and $\bar{z}$ are treated as independent variables here. This is a perfectly acceptable approach if we image looking at a field theory defined on $\mathbb{C}^2$ instead of $\mathbb{C} \simeq \mathbb{R}^2$. This allows us to treat the holomorphic and the antiholomorphic parts independently. At the end we can still impose the reality condition $\bar{z} = z^*$ to return to the physically relevant case. We will often write down only the holomorphic contributions. The corresponding antiholomorphic contributions can then be obtained by adding bars at appropriate places.

An infinitesimal conformal transformation is given by $z \to z + \epsilon(z)$ where

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \quad \text{and} \quad \bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1}. \tag{2.1.23}$$

It is generated by $l_n = -z^{n+1}\partial$ and $\bar{l}_n = -\bar{z}^{n+1}\bar{\partial}$ which follows from

$$\begin{aligned}\phi(z', \bar{z}') &= \phi(z, \bar{z}) \\ &= \phi(z, \bar{z}) - \epsilon(z)\partial\phi(z, \bar{z}) - \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}) + \dots \\ &= \phi(z, \bar{z}) + \sum_{n\in\mathbb{Z}} (\epsilon_n l_n \partial + \bar{\epsilon}_n \bar{l}_n \bar{\partial})\phi(z, \bar{z}) + \dots .\end{aligned} \tag{2.1.24}$$

The generators form the so-called *Witt algebra* which is infinite dimensional unlike the global conformal algebra we encountered previously:

$$\begin{aligned}[l_n, l_m] &= (n - m)\, l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n - m)\, \bar{l}_{n+m} \\ [l_n, \bar{l}_m] &= 0 \quad \text{where} \quad n, m \in \mathbb{Z}\end{aligned} \tag{2.1.25}$$

In the quantum theory the analogue of the Witt algebra is the Virasoro algebra which is a central extension of the Witt algebra.

Note that the transformations generated by l_n respectively $\bar{l}_n$ are only globally defined on the complex plane if $n = -1, 0, 1$. The transformations generated by $l_n + \bar{l}_n$ and $\mathrm{i}(l_n - \bar{l}_n)$ preserve the real line (c.f. our previous discussion on treating holomophic and antiholomorphic parts independently), for example $l_0 + \bar{l}_0$ generates dilations and

$\mathrm{i}(l_0 - \bar{l}_0)$ rotations in the complex plane. This subgroup generated by $l_n + \bar{l}_n$ and $\mathrm{i}(l_n - \bar{l}_n)$ for $n = -1, 0, 1$ is in fact the global conformal group we discussed earlier.

Formulated in complex coordinates the transformations in the subgroup generated by l_{-1}, l_0, l_1 can be written as the group of Möbius transformations:

$$z \to \frac{az+b}{cz+d} \quad \text{where} \quad ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \tag{2.1.26}$$

The reason why we can impose the condition $ad - bc = 1$ is, that a global conformal transformations is parametrized by 3 complex parameters whereas the map (2.1.26) without the condition imposed admits four parameters. The group of Möbius transformations is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$ where the factor $\mathbb{Z}_2$ originates from the fact that a Möbius transformation and the transformation with the signs of all parameters inverted are identical. As locally $SL(2,\mathbb{C}) \simeq SO(3,1)$ our result here is consistent with our result $so(p+1, q+1)$ for $p = 2$ and $q = 0$ from the preceding section.

2.1.4 Correlation functions in two dimensions

In this section we will have another look at correlation functions, but not in the general case of d dimensions like in section 2.1.2 but with focus on the two-dimensional theory. The main difference to the previous considerations will be that we do not restrict ourselves to fields with vanishing conformal spin. We will only cite the results in this section for comparison with the results from section 2.1.2 and for later reference. As these results are of central importance their derivation is sketched in appendix A.

The 2-point functions are restricted to the form

$$\boxed{\langle \phi_1(z_1)\phi_2(z_2)\rangle = C_{12}(z_1 - z_2)^{-2h}(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}}} \tag{2.1.27}$$

for $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$. In a theory with only a finite number of primary fields we can chose their normalization such that $C_{ij} = \delta_{ij}$. The 3-point functions are of the form

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = C_{ijk}\, z_{12}^{h_3-h_1-h_2}\, z_{23}^{h_1-h_2-h_3}\, z_{31}^{h_2-h_3-h_1} \cdot \bar{z}_{12}^{\bar{h}_3-\bar{h}_1-\bar{h}_2}\, \bar{z}_{23}^{\bar{h}_1-\bar{h}_2-\bar{h}_3}\, \bar{z}_{31}^{\bar{h}_2-\bar{h}_3-\bar{h}_1}, \quad (2.1.28)$$

where $z_{ij} = z_i - z_j$ and $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. The coefficients C_{ijk} depend on the normalization of the 2-point function which is chosen as above.

Our results can be understood as follows: The global conformal group is generated by translations, rotations/dilations and special conformal transformations. A global conformal transformation is fixed by the images of three points $\zeta_1 = f(z_1), \zeta_2 = f(z_2)$ and $\zeta_3 = f(z_3)$ in the complex plane. The three equations $\zeta_i = \frac{az_i+b}{cz_i+d}$ together with the condition $ad - bc = 1$ fix the transformation. The 3-point function is determined if we know its values at just 3 points.

This argument already indicates that the 4-point function will not be fixed in a similar way. The 4-point functions in a conformally invariant two-dimensional field theory are constrained to the form (c.f. eq. (2.1.18))

$$\langle \phi_1(z_1, \bar{z}_1) \cdots \phi_4(z_4, \bar{z}_4)\rangle = f(x, \bar{x}) \prod_{1\le i<j\le 4} {z_{ij}}^{h/3-h_i-h_j}\, \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}, \quad (2.1.29)$$

where $h = \sum_i h_i$ and $\bar{h} = \sum_i \bar{h}_i$ and the anharmonic ratios are given by $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $\bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}$. Note that for $h - \bar{h} = 0$, which we will call vanishing *conformal spin* in section 2.2.2, all relations reduce to the expressions obtained in section 2.1.2.

2.2 Operator formalism

The aim of this section is to look at the quantization of two dimensional conformal field theories in detail. We will see that the procedure of time-ordering in usual quantum field theory corresponds to radial ordering of operators defined on the complex plane. The process is called *radial quantization*. The main result of this section is the *Virasoro algebra* which is formed by the conformal generators of the quantized theory.

2.2.1 The Stress-Energy tensor

Definition and properties

The stress-energy tensor is of special importance in conformal field theories because operator product expansions with it determine the transformation behaviour of other fields under conformal transformations. In a d-dimensional field theory we may define the stress-energy tensor via Noether's theorem or as variation of the action with respect to the metric:

$$\delta S = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \tag{2.2.1}$$

In general the two definitions of the stress-energy tensor do not coincide, especially in the definition via Noether's theorem the stress-energy tensor does not have to be symmetric, but can be made so by adding the derivative of an appropriate tensor which does not spoil energy-momentum conservation, whereas in our definition $T^{\mu\nu}$ is manifestly symmetric.

From scaling invariance in a CFT we can deduce that $T^{\mu\nu}$ has to be traceless: $T^{\mu}{}_{\mu} = 0$. Energy-momentum conservation implies $\partial_\mu T^{\mu\nu} = 0$.

The Stress-Energy tensor in two dimensions

Now we specialise to the case $d = 2$ and rewrite the stress-energy tensor in terms of complex coordinates:

$$\begin{aligned} T_{zz} &= \frac{\partial x^\mu}{\partial x^z}\frac{\partial x^\nu}{\partial x^z}T_{\mu\nu} = \frac{1}{4}(T_{11} - T_{22} + 2\mathrm{i}T_{12}) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{11} - T_{22} - 2\mathrm{i}T_{12}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}T^\mu{}_\mu = 0 \end{aligned} \tag{2.2.2}$$

Here we used the fact that $T_{\mu\nu}$ is traceless and symmetric. Energy momentum conservation together with the traceless condition yields (c.f. eq. (2.1.20))

$$\partial_{\bar{z}}T_{zz} = 0 \quad \partial_z T_{\bar{z}\bar{z}} = 0, \tag{2.2.3}$$

which implies that $T(z) \equiv T_{zz}(z)$ is holomorphic and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$ is antiholomorphic. Thus we can expand them in modes which can be expressed as contour integrals using Cauchy's integral formula:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n \quad \text{where} \quad \hat{L}_n = \oint \mathrm{d}z\, T(z) z^{n+1} \\ T(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\hat{\bar{L}}_n \quad \text{where} \quad \hat{\bar{L}}_n = \oint \mathrm{d}\bar{z}\, \bar{T}(\bar{z})\bar{z}^{n+1} \end{aligned} \tag{2.2.4}$$

Conserved currents

Noether's theorem implies that we can associate conserved currents to continuous symmetries. In the case of conformal symmetry these are given by

$$J_\mu(\epsilon) = T_{\mu\nu}\epsilon^\nu. \tag{2.2.5}$$

The current is conserved because

$$\partial^\mu J_\mu = (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}\partial^\mu \epsilon^\nu = 0. \tag{2.2.6}$$

The first term vanishes because $T_{\mu\nu}$ is conserved and the second term because $T_{\mu\nu}$ is symmetric and traceless using eq. (2.1.5). In $d = 2$ dimensions the current (2.2.5) can be written as

$$J_z = T_{zz}\epsilon(z) \quad J_{\bar{z}} = \bar{T}_{\bar{z}\bar{z}}\bar{\epsilon}(\bar{z}). \tag{2.2.7}$$

Here current conservation $\partial_{\bar{z}} J_z = \partial_z J_{\bar{z}} = 0$ is manifest because $J_{\bar{z}}$ and J_z are (anti-) holomorphic. The conserved charges are then given by

$$Q_\epsilon = \int \frac{\mathrm{d}z}{2\pi \mathrm{i}}\, \epsilon(z) T(z) \quad \text{and} \quad Q_{\bar{\epsilon}} = \int \frac{\mathrm{d}\bar{z}}{2\pi \mathrm{i}}\, \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}). \tag{2.2.8}$$

These are the generators of infinitesimal conformal transformations in the sense of

$$\delta_{\epsilon,\bar{\epsilon}}\, \phi(w, \bar{w}) = [Q_\epsilon, \phi(w, \bar{w})] + [Q_{\bar{\epsilon}}, \phi(w, \bar{w})]. \tag{2.2.9}$$

2.2.2 Radial quantization

Schwinger's time splitting technique

We consider a two dimensional quantum field theory with time coordinate τ and space coordinate σ. We assume the signature to be euclidean which corresponds to the situation after performing a Wick rotation $\tau \to -\mathrm{i}\tau$ and furthermore we will compactify the space direction by identifying $\sigma \sim \sigma + L$. The cylinder created in this

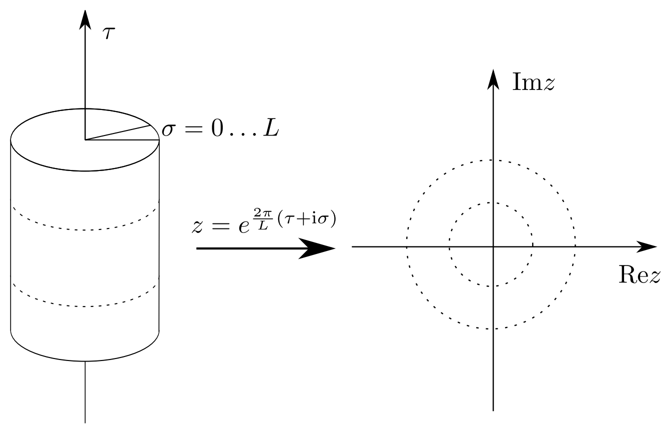

Figure 2.1: Mapping the cylinder to the complex plane

way can be mapped to the extended complex plane using the conformal map

$$(\tau, \sigma) \to z = e^{\frac{2\pi}{L}(\tau + \mathrm{i}\sigma)} \quad \text{and} \quad (\tau, \sigma) \to \bar{z} = e^{\frac{2\pi}{L}(\tau - \mathrm{i}\sigma)}. \tag{2.2.10}$$

This is shown in figure 2.1. The surface in the infinite past corresponds to $z = 0$ and the infinite future to $z = \infty$ on the extended complex plane. Slices of equal time on the cylinder correspond to circles centred at $z = 0$.

Time-ordering is a well-known concept in usual QFTs and carries over as radial ordering in the complex domain. Therefore we define a *radial ordering operator* $\mathcal{R}$:

$$\mathcal{R}\psi(z)\phi(w) = \begin{cases} \psi(z)\phi(w) & \text{if } |w| < |z| \\ \pm\phi(w)\psi(z) & \text{if } |z| < |w|, \end{cases} \tag{2.2.11}$$

where the minus sign is used if both ψ and ϕ are fermionic fields.

Now we want to evaluate the variation of $\phi(z, \bar{z})$ under an infinitesimal conformal transformation generated by Q_ϵ given in eq. (2.2.8). This can be written as a contour

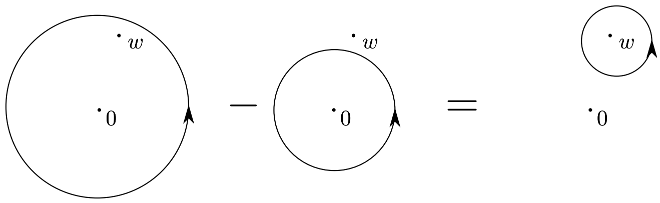

Figure 2.2: Deforming the integration contour

integral of the stress-energy tensor with the field in question:

$$\begin{aligned}\delta_\epsilon \phi(w,\bar{w}) &= [Q_\epsilon, \phi(w,\bar{w})] \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z\, \epsilon(z) \left(T(z)\phi(w,\bar{w}) - \phi(w,\bar{w})T(z)\right) \\ &= \frac{1}{2\pi \mathrm{i}} \left(\oint_{\odot_0 : |z|>|w|} \mathrm{d}z - \oint_{\odot_0 : |z|<|w|} \mathrm{d}z \right) \epsilon(z) \mathcal{R}(T(z), \phi(w,\bar{w})) \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\odot_w} \mathrm{d}z\, \epsilon(z) \mathcal{R}(T(z), \phi(w,\bar{w}))\end{aligned} \tag{2.2.12}$$

In the last step we used Cauchy's theorem to deform the integration path in a path enclosing w like it is shown in figure 2.2. We clearly get a non-vanishing contribution if the operator product expansion is singular as $w \to z$. Thus we should be interested in analysing operator product expansions of the fields with the stress-energy tensor T or $\bar{T}$ respectively in order to understand their transformation properties under conformal transformations.

Primary fields

A field $\Phi(z,\bar{z})$ which transforms as

$$\boxed{\Phi(z,\bar{z}) \to \left(\frac{\partial f(z)}{\partial z}\right)^h \left(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\right)^{\bar{h}} \Phi\left(f(z), \bar{f}(\bar{z})\right)} \tag{2.2.13}$$

under a conformal mapping $z \to f(z)$ is called *primary field* and $(h, \bar{h})$ where $h, \bar{h} \in \mathbb{R}$ are called *conformal dimensions* of Φ. If this relation holds only for $f \in SL(2, \mathbb{C})$ the field is called *quasi-primary*.

Dilations are generated by $L_0 + \bar{L}_0$ and map $z \to \lambda z$ ($\lambda \in \mathbb{R}$). Under these transformations a (quasi-)primary field transforms as $\Phi \to \lambda^{h+\bar{h}}\Phi$ and the combination $h + \bar{h}$ is thus called *scaling dimension*. Rotations $z \to e^{\mathrm{i}\theta} z$ are generated by $\mathrm{i}(L_0 - \bar{L}_0)$ and (quasi-)primary fields transform according to $\Phi \to e^{\mathrm{i}\theta(h-\bar{h})}\Phi$. This justifies the designation *conformal spin* for the combination $h - \bar{h}$.

Under an infinitesimal conformal transformation of the form $z \to z + \epsilon(z)$, $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ the variation of a primary field Φ is

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w, \bar{w}) = \Big((h\partial\epsilon + \epsilon\partial) + (\bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial})\Big)\Phi(w, \bar{w}). \tag{2.2.14}$$

If we compare this result to the last line in eq. (2.2.12) we can infer the operator product expansion of a primary field with the stress-energy tensor using the residue theorem:

$$\boxed{\mathcal{R}(T(z)\Phi(w, \bar{w})) \sim \frac{h}{(z-w)^2}\Phi(w, \bar{w}) + \frac{1}{z-w}\partial_w\Phi(w, \bar{w})}, \tag{2.2.15}$$

where we considered for notational simplicity only contributions from the holomorphic parts. The OPE in eq. (2.2.15) and the corresponding OPE with $\bar{T}$ can be used as an alternative definition of a primary field. The symbol $\sim$ in eq. (2.2.15) denotes equality up to terms which are regular as $w \to z$. We will use this notation in the following when we consider operator product expansions.

2.2.3 Operator product expansions and Ward identities

The Operator product expansion (OPE)

In the last section we saw one example for an operator product expansion of a primary field with the stress-energy tensor. This concept is valid under more general circumstances and is a central concept in conformal field theory. In the following we will omit the radial ordering operator for notational convenience and simply assume that operator products are always radially ordered.

We study the short-distance limit of a (radially ordered) product of two operators $\Phi(z)$ and $\Psi(w)$. In general this has the form

$$\Phi(z)\Psi(w) = \sum_{\lambda} C_\lambda(z-w) O_\lambda(w), \tag{2.2.16}$$

where $\{O_\lambda\}$ is a complete set of local operators and C_λ are (potentially singular) $\mathbb{C}$-valued coefficient functions of $(z-w)$.

Expressions like eq. (2.2.16) always have to be understood as being inserted into arbitrary correlation functions i.e. in the sense of

$$\lim_{z\to w} \langle \big(\Phi(z)\Psi(w) - \sum_{\lambda} C_\lambda(z-w) O_\lambda(w)\big)\phi_1(x_1)\cdots\phi_N(x_N)\rangle = 0. \tag{2.2.17}$$

Of particular interest is the OPE of the stress-energy tensor with itself as this determines the transformation properties of $T(z)$ under a conformal transformation. Therefore we consider the variation of a primary field ϕ under two successive infinitesimal conformal transformations. Using the relation $[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{(\partial\xi_1\xi_2 - \partial\xi_2\xi_1)}$ applied to ϕ and eq. (2.2.12) and comparing both sides of the equation, we can derive an expression for the OPE of the stress-energy tensor with itself. We find that T has to be of weight $h = 2$, but that the argument remains valid if the OPE contains a quartic term with a yet unfixed coefficient c in addition to the terms from eq. (2.2.15):

$$\boxed{T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{\partial T(w)}{z-w}} \tag{2.2.18}$$

This is in fact the most general OPE for a conformal field of weight $h = 2$ which is consistent with scaling invariance and analyticity.

As pointed out above, apart from the first term involving c we recognize the OPE of a primary field of weight $(2,0)$. For non-vanishing c we conclude that T is not a primary field. The *central charge* or *conformal anomaly* c does not occur in the classical theory and is thus a purely quantum effect. As we will see in section 2.7.2 the central charge is proportional to the Casimir energy i.e. the change in the vacuum energy caused by imposing periodicity conditions on the cylinder. In section 2.9 we will calculate the central charge in free theories with help of Wick's theorem.

Note that in string theory the content of the theory is chosen such that the overall central charge vanishes i.e. the theory possesses full conformal invariance. The simplest example is the quantization of the bosonic string where the extra contributions to the central charge from the ghost fields which arise during the gauge fixing procedure exactly cancel the central charge $c = 26$ from the matter degrees of freedom in the critical dimension.

The Ward identities

In this subsection we look at insertions of the stress-energy tensor into general n-point functions, which will give the so-called *Ward identities*. Thus we look at correlation functions of the form

$$F^n(z, z_i, \bar{z}_i) = \langle T(z)\phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n)\rangle, \tag{2.2.19}$$

where ϕ_i is a primary field of weight $(h_i, \bar{h}_i)$ for $i = 1, \ldots, N$. Therefore we consider for a closed path γ encircling $z_1, \ldots, z_N$ and calculate

$$\begin{aligned}
&\langle \oint_\gamma \mathrm{d}z\, T(z)\epsilon(z)\phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n)\rangle \\
&\quad = \sum_{i=1}^n \langle \phi_1(z_1, \bar{z}_1) \cdots \left(\oint_{\odot z_i} \mathrm{d}z\, T(z)\epsilon(z)\phi_i(z_i, \bar{z}_i) \right) \cdots \phi_n(z_n, \bar{z}_n)\rangle \\
&\quad = \sum_{i=1}^n \langle \phi_1(z_1, \bar{z}_1) \cdots \delta_\epsilon \phi_i(z_i, \bar{z}_i) \cdots \phi_n(z_n, \bar{z}_n)\rangle \\
&\quad = \sum_{i=1}^n \langle \phi_1(z_1, \bar{z}_1) \cdots \left(\oint_{\odot z_i} \mathrm{d}z\, \epsilon(z) \left(\frac{h_i}{(z - z_i)^2} + \frac{\partial_{z_i}}{z - z_i} \right) \phi_i(z_i, \bar{z}_i) \right) \cdots \phi_N(z_n, \bar{z}_n)\rangle.
\end{aligned} \tag{2.2.20}$$

In the last step we used eq. (2.2.14) and rewrote the result as a contour integral. As the relation holds for arbitrary $\epsilon(z)$ we obtain the *conformal Ward identities*:

$$\boxed{\langle T(z)\phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n)\rangle = \sum_{i=1}^n \left(\frac{h_i}{(z - z_i)^2} + \frac{\partial_{z_i}}{z - z_i} \right) \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n)\rangle} \tag{2.2.21}$$

This result states that $F^n(z, z_i, \bar{z}_i)$ seen as a function of z while fixing all other variables is a meromorphic function with poles at the insertion points z_i of the fields ϕ_i. The residues can be calculated from the n-point correlation functions using eq. (2.2.21).

2.2.4 The Virasoro algebra

Now we have the necessary tools to investigate the algebra of the conformal generators in the quantized theory i.e. the commutators of the modes L_n of the stress-energy tensor defined in eq. (2.2.4):

$$\begin{aligned}[\hat{L}_n, \hat{L}_m] &= \left[\frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z\, T(z) z^{n+1}, \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}w\, T(w) w^{m+1}\right] \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}w\, w^{m+1} \frac{1}{2\pi \mathrm{i}} \oint_{\odot_w} \mathrm{d}z\, z^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{\partial_w T(w)}{z-w}\right) \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}w\, w^{m+1} \left(\frac{c}{12} w^{n-2}(n^3 - n) + 2T(w)(n+1)w^n + \partial_w T(w) w^{n-1}\right)\end{aligned} \tag{2.2.22}$$

In the second line we used the argument from eq. (2.2.12) again and inserted the OPE from eq. (2.2.18). The only contributions come from the singular parts of the OPE here. The first contour integral can be evaluated using the residue theorem and e.g. $\mathrm{Res}_{z=w} \frac{z^{n+1}}{(z-w)^m} = \frac{1}{(m-1)!} \lim_{z \to w} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} z^{n+1}$. We integrate the first term in the last line of eq. (2.2.22) again by applying the residue theorem. The remaining two terms can be combined after integrating by parts the third addend. The final result is the *Virasoro algebra*:

$$\boxed{[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}(m^3 - m)\delta_{0,m+n}} \tag{2.2.23}$$

To justify the use of the term algebra, we should consider c as additional generator which commutes with all other elements of the algebra i.e. $[\hat{L}_n, c] = 0$. For the antiholomorphic generators $\hat{\bar{L}}_n$ we obtain the same algebra with central charge[1] $\bar{c}$ and

[1] We know from (2.2.2) that $T + \bar{T}$ is real and hence we deduce from the reality of the OPE $(T + \bar{T})(T + \bar{T})$ that $c = \bar{c}$.

we have $[\hat{L}_n, \hat{\bar{L}}_m] = 0$ as there are no singular terms in the OPE of $T(z)$ with $\bar{T}(\bar{z})$. So in total we find two commuting copies of the Virasoro algebra to which we will refer in the following as holomorphic/antiholomorphic or equivalently as chiral/antichiral algebras.

We note that the commutation relations of the finite dimensional $sl(2,\mathbb{C})$ subalgebra generated by $\hat{L}_{-1}, \hat{L}_0, \hat{L}_1$ are not affected by the central charge term.

2.3 Representation theory

In the last section we finally derived the Virasoro algebra which determines the commutation relations of the conformal generators. Now we will proceed by showing how the Hilbert space of the theory is constructed.

We represent some observable algebra $\mathcal{W}$ (which will be identical with the Virasoro algebra for the remainder of this chapter) on a representation space $\mathcal{H}$. We demand that the representation is *unitary*[2] and assume in the following that $\mathcal{H}$ is a Hilbert space. Finally we impose the *spectrum condition* on the space of physical states which implies that the energy (i.e. the L_0 eigenvalue) is bounded from below.

We look at a special class of representations namely the *highest weight representations*[3]. All states in a *highest weight* or *Verma module* V_h are obtained from a so-called *highest weight state* $|h\rangle$ which we label by its L_0 eigenvalue (choosing L_0 as Cartan

[2]In the context of quantum field theories this is an obvious requirement which is linked to the probability interpretation. There are some interesting non-unitary models in statistical physics which may motivate the study of non-unitary representations. In this thesis however we will restrict our attention to unitary representations.

[3]Highest weight representations exist for any observable algebra which possesses a triangular decomposition [4] like the Virasoro algebra.

subalgebra). The highest weight state is characterized by

$$L_0|h\rangle = h|h\rangle, \quad L_n|h\rangle = 0 \quad \text{for} \quad n > 0. \tag{2.3.1}$$

We obtain the other elements of the Verma module V_h by acting with all possible combinations of L_{-n} $(n > 0)$ on the highest weight state $|h\rangle$. There is a natural inner product defined on V_h if we set $\langle h|h\rangle = 1$ and represent the Virasoro generators obeying $L_n^\dagger = L_{-n}$. In general these representations will not be irreducible which is linked to the fact that the inner product defined in this way is degenerate (c.f. section (2.3.4). We may then construct irreducible Verma modules by considering the quotient space of the original Hilbert space by the space spanned by all null vectors.

The Hilbert space can be decomposed into a (possibly infinite) direct sum of irreducible highest weight modules V_h of the Virasoro algebra:

$$\mathcal{H} = \bigoplus_h V_h \tag{2.3.2}$$

The highest weight vector $|h\rangle$ is both a highest weight vector with respect to the Virasoro algebra and with respect to the $sl(2,\mathbb{C})$ subalgebra generated by $\{L_{-1}, L_0, L_1\}$.

So far we considered only the chiral half of the actual conformal field theory. As the chiral and the antichiral algebra commute we simply have to duplicate the whole structure and demand e.g. that $|h,\bar{h}\rangle$ is a highest weight state both with respect to the chiral and the antichiral algebra (with L_0 and $\bar{L}_0$ eigenvalues h and $\bar{h}$ respectively). The full Hilbert space then decomposes into a direct sum of tensor products

$$\mathcal{H} = \bigoplus_{h,\bar{h}} M_{h\bar{h}} V_h \otimes \overline{V}_{\bar{h}}, \tag{2.3.3}$$

where $M_{h\bar{h}}$ denotes the multiplicity with which a certain tensor product occurs.

2.3.1 The state-operator correspondence

In this section we make the connection between the highest weight states and the primary fields in a conformal field theory. The key is the *state-operator correspondence* which links conformal fields with states in the Hilbert space and in particular primary fields with highest weight states. Let Φ be a conformal field with weight $(h, \bar{h})$. Then we define the state $|h, \bar{h}\rangle$ as the asymptotic state

$$|h, \bar{h}\rangle = \lim_{z,\bar{z}\to 0} \Phi(z, \bar{z})|0\rangle. \tag{2.3.4}$$

The state $|0\rangle$ corresponding to the identity operator is called *vacuum state* and is assumed to be unique in the following. The conditions imposed on the vacuum state can be derived from the condition that $T(z)|0\rangle$ should be non-singular at $z = 0$ (c.f. eq.(2.2.4)):

$$L_n|0\rangle = 0 \quad \text{for} \quad n > -1 \tag{2.3.5}$$

In particular we notice that the vacuum state is annihilated by L_{-1}, L_0, L_1 which means that the vacuum is $SL(2, \mathbb{C})$ invariant. Another point worth noting is that we cannot demand $L_n|0\rangle = 0$ for all $n \in \mathbb{Z}$ which might have been an intuitive guess, because this would be inconsistent with the central charge term in the Virasoro algebra in eq. (2.2.23). Here we use the notation $|0\rangle$ as a shorthand for the tensor product $|0\rangle \otimes \overline{|0\rangle}$ consisting of the respective vacuum states for the chiral/antichiral Virasoro algebra.

In order to define the inner product we first define the adjoint operator $A^\dagger$ of an operator $A(z,\bar{z})$ with conformal dimension $(h,\bar{h})$ as

$$A^{\dagger}(z,\bar{z}) = z^{-2\bar{h}}\bar{z}^{-2h}A\Big(\frac{1}{\bar{z}},\frac{1}{z}\Big). \tag{2.3.6}$$

This definition may look unnatural but is in fact closely related to our choice of an euclidean signature. The euclidean version of the time evolution operator e^{-Ht} differs from the time evolution operator in Minkowski space-time $e^{-\mathrm{i}Ht}$ by a factor of i which has to be compensated in the definition of the adjoint by an euclidean time inversion $\tau \to -\tau$ which corresponds $z \to \frac{1}{\bar{z}}$ on the complex plane. Note that unlike in usual QFT the asymptotic states are defined as eigenstates of the full Hamiltonian and not of some non-interacting free Hamiltonian.

If we apply the definition given above to the (holomorphic) stress-energy tensor $T(z)$ which has conformal weight (2,0), we get

$$T(z)^{\dagger} = \sum_{m\in\mathbb{Z}} \frac{L_m^{\dagger}}{\bar{z}^{m+2}} = \sum_{m\in\mathbb{Z}} \frac{L_m}{\bar{z}^{m+2}}\bar{z}^{-4}. \tag{2.3.7}$$

From the hermiticity of T it follows $L_n^{\dagger} = L_{-n}$ i.e. the Virasoro modes are represented just as demanded in the previous section.

Also for quasi-primary fields Φ of weight $(h,\bar{h})$ we can define asymptotic *in- and out- states* via

$$|\Phi\rangle = \lim_{z,\bar{z}\to 0}\Phi(z,\bar{z})|0\rangle \quad \langle\Phi| = \lim_{z,\bar{z}\to 0}\langle 0|\Phi^{\dagger}(z,\bar{z}) \tag{2.3.8}$$

and we have $|\Phi\rangle^{\dagger} = \langle\Phi|$. But if Φ is only quasi-primary and not primary the state is not a highest weight state with respect to the Virasoro algebra but with respect to the $sl(2,\mathbb{C})$ subalgebra.

2.3.2 Primary fields

In this section we convince ourselves that the state corresponding to a primary field Φ of weight $(h_\Phi, \bar{h}_\Phi)$ is actually a highest weight state with the expected L_0 and $\bar{L}_0$ eigenvalues h_Φ and $\bar{h}_\Phi$ respectively. Starting from eq. (2.2.12) with $\epsilon(z) = z^{n+1}$ and inserting the OPE for the primary field Φ with the stress-energy tensor eq. (2.2.15) one obtains

$$[\hat{L}_n, \Phi(z)] = z^n[z\partial + (n+1)h_\Phi]\Phi(z). \tag{2.3.9}$$

It is another way of defining a primary field to require that the relation (2.3.9) holds for all modes n. It can then be used to verify that

$$L_0|\Phi\rangle = h_\Phi|\Phi\rangle \quad \text{and} \quad L_n|\Phi\rangle = 0 \quad \text{for} \quad n > 0, \tag{2.3.10}$$

which shows that the primary field Φ indeed gives rise to an highest weight state with L_0 eigenvalue h_Φ. In a similar way one shows that $|\Phi\rangle$ is a highest weight state with respect to the antichiral Virasoro algebra.

2.3.3 Descendant fields

Descendant or *secondary fields* originate from non-singular terms of the OPE of the field in question with the stress-energy tensor. For example we can extract such fields from the OPE of a primary field Φ with the stress-energy tensor (c.f. eq. (2.2.15)):

$$T(w)\Phi(z,\bar{z}) = \sum_{n\geq 0}(w-z)^{n-2}\hat{L}_{-n}\Phi(z,\bar{z}), \tag{2.3.11}$$

where

$$\Phi^{(-n)}(z,\bar{z}) \equiv \hat{L}_{-n}\Phi(z,\bar{z}) = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}w \frac{1}{(w-z)^{n-1}} T(w)\Phi(z,\bar{z}) \tag{2.3.12}$$

By comparison with eq. (2.2.15) we find explicit expressions for the first two descendants:

$$\Phi^{(0)}(z,\bar{z}) = h\,\Phi(z,\bar{z}) \quad \text{and} \quad \Phi^{(-1)}(z,\bar{z}) = \partial_z \Phi(z,\bar{z}) \tag{2.3.13}$$

On the level of the Hilbert space these fields then correspond to states obtained by acting with L_{-n} for $n > 0$ on a highest weight state:

$$\Phi^{(-n)}(0,0)|0\rangle = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}w \frac{1}{(w-z)^{n-1}} T(w)\Phi(0,0)|0\rangle = L_{-n}\Phi(0,0)|0\rangle \tag{2.3.14}$$

We can obtain further descendants from the fields occurring in the OPE of a given descendant field with the stress-energy tensor. These fields are then denoted by $\Phi^{(-n_1,\ldots,-n_N)}$ and correspond to states $L_{-n_1}\ldots L_{-n_N}\Phi(0,0)|0\rangle$ on the Hilbert space. $\sum_i n_i$ is called *level* of the descendant of the primary field Φ. We may assume that $n_1 \geq \ldots \geq n_N \geq 1$ because an arbitrary state can be written as linear combination of ordered states using the Virasoro algebra.

A simple example of a descendant field which is present in any conformal field theory is given by the stress-energy tensor itself being the level 2 descendant of the identity operator:

$$1^{(-2)}(z,\bar{z}) = \hat{L}_{-2}\,1(z,\bar{z}) = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}w \frac{1}{w-z} T(w) = T(z) \tag{2.3.15}$$

One can easily deduce the conformal weight $(h+n,\bar{h})$ of a level n descendant of a primary field of weight $(h, \bar{h})$ using the relation $[L_0, L_n] = nL_n$ from the Viraso algebra eq. (2.2.23). This is another way of showing that the stress-energy tensor T has conformal weight $(2, 0)$.

Correlation functions involving descendant fields

A special property of a CFT in two dimensions is that we can calculate correlators involving descendant fields from correlators just involving primary fields. To justify this claim we look at n-point correlation functions involving one descendant field. With $X(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) = \phi_1(z_1, \bar{z}_1) \cdots \phi_{n-1}(z_{n-1}, \bar{z}_{n-1})$ we have using eq. (2.3.12)

$$\begin{aligned} \langle X\phi^{(-m)}(z,\bar{z})\rangle &= \oint_{\odot_z} \frac{\mathrm{d}w}{2\pi \mathrm{i}} (w-z)^{-m+1} \langle XT(z)\phi(w)\rangle \\ &= -\oint_{\odot_{z_1 \cdots z_{n-1}}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} (w-z)^{-m+1} \sum_{i=1}^{n-1} \left(\frac{1}{w-z_i} \partial_{z_i} + \frac{h_i}{(w-z_i)^2} \right) \langle X\phi(z,\bar{z})\rangle \\ &= \mathcal{L}_{-m} \langle X\phi(z,\bar{z})\rangle. \end{aligned} \tag{2.3.16}$$

The contour in the first line encircles z only. We can see it as a contour of opposite direction encircling $z_1, \ldots z_{n-1}$ and use the residue theorem together with the OPE for primary fields in eq. (2.2.15) to arrive at the second resp. third line.

The differential operators[4] $\mathcal{L}_{-m}$ from the last line are given by

$$\mathcal{L}_{-m} = \sum_{i=1}^{n-1} \left(\frac{(m-1)h_i}{(z_i - z)^m} - \frac{1}{(z_i - z)^{m-1}} \partial_{z_i} \right). \tag{2.3.17}$$

[4]A short note on the notation used here and in the following: We distinguish the notation $\hat{L}_n$ for the operator modes acting on conformal fields, L_n for the Virasoro generators acting on states of the respective Hilbert spaces and $\mathcal{L}_n$ for the realization as differential operator acting on correlation functions.

This means the evaluation of a correlation function involving a descendant field $\phi^{(-m)}$ reduces to applying the differential operator $\mathcal{L}_{-m}$ to the correlation function $\langle X\phi\rangle$ which involves only primary fields. Note that the operator $\mathcal{L}_{-1}$ is equivalent to ∂_z when applied to correlation functions, because the operator $\partial_z + \partial_{z_1} + \ldots \partial_{z_{n-1}}$ annihilates any correlation function by translation invariance (c.f. eq. (A.1.2)). The procedure discussed here generalizes to descendant fields of the form $\phi^{(-n_1,\ldots,-n_N)}$ and also to correlation functions involving multiple descendant fields [18].

2.3.4 Null state decoupling

We start this section by defining a *null state*. A null state is a level N descendant state $|\chi\rangle$ of a highest weight state $|h\rangle$ satisfying

$$L_0|\chi\rangle = (h+N)|\chi\rangle, \quad L_n|\chi\rangle = 0 \quad \text{for} \quad n > 0. \tag{2.3.18}$$

This implies in particular that $\langle\psi|\chi\rangle = 0$ for any state $|\psi\rangle$ of the Verma module. The definition above means that $|\chi\rangle$ is a descendant state of $|h\rangle$ but at the same time a highest weight state with its own corresponding Verma module whose states are orthogonal to the states of the original module. This is why we speak of *null state decoupling*. Setting $|\chi\rangle = 0$ corresponds to taking the quotient of the original module by the Verma module associated to the null state. Eq. (2.3.18) then ensures the conformal invariance of the condition $|\chi\rangle = 0$.

For further discussions about existence of null states and their implications it is enlightening to study a simple example. Therefore let us look at a general level 2 descendant $|\chi\rangle = L_{-2}|h\rangle + \alpha {L_{-1}}^2|h\rangle$ and find conditions under which this is a null state. To ensure that (2.3.18) is satisfied we only have to check that $L_1|\chi\rangle = 0$

and $L_2|\chi\rangle = 0$ as L_1 and L_2 generate the Virasoro algebra in an algebraic sense. The first equation leads to the condition $\alpha = -3/(2\cdot(2h+1))$ and the second to $c = 2(-6\alpha h - 4h) = 2h(5-8h)/(2h+1)$ where in both cases the Virasoro algebra was used. For this central charge c the state $L_{-2}|h\rangle - \frac{3}{2(2h+1)}L_{-1}^2|h\rangle$ is a *null descendant at level 2* or also called *degenerate at level 2*.

We can rewrite this as an expression for the primary field $\phi(z,\bar{z})$ corresponding to $|h\rangle$:

$$\hat{L}_{-2}\phi(z,\bar{z}) = \frac{3}{2(2h+1)}\hat{L}_{-1}{}^2\phi(z,\bar{z}) = \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\phi(z,\bar{z}), \tag{2.3.19}$$

where we used (2.3.13). In this way we obtain differential equations for correlation functions involving ϕ using the definition of $\mathcal{L}_{-2}$ in eq. (2.3.17):

$$\begin{aligned}
&\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi_{n-1}(z_{n-1},\bar{z}_{n-1})\rangle \\
&\quad= \langle(\hat{L}_{-2}\phi(z,\bar{z}))\phi_1(z_1,\bar{z}_1)\dots\phi_{n-1}(z_{n-1},\bar{z}_{n-1})\rangle \\
&\quad= \mathcal{L}_{-2}\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi_{n-1}(z_{n-1},\bar{z}_{n-1})\rangle \\
&\quad= \sum_{1\le j<n}\left(\frac{h_j}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_j\right)\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi_{n-1}(z_{n-1},\bar{z}_{n-1})\rangle
\end{aligned} \tag{2.3.20}$$

Thus all n-point functions involving ϕ have to obey the second order partial differential equation above. In a similar way every primary field ϕ with a degenerate state at a level m gives rise to a m^{th} order differential equation for the n-point functions involving the corresponding primary field.

Solving these differential equations is actually a very powerful approach for calculating analytical expressions for n-point functions involving primary fields with null descendants [10, 1, 18]. In [10] this is shown explicitly for the Ising model. Even if for example the 3-point functions are not fixed completely by these equations, we obtain selection rules indicating which coefficients C_{ijk} may be non-zero.

2.4 Unitarity, Kac Determinant and Minimal models

2.4.1 Unitarity and the Kac Determinant

In section 2.3 we required the representation to be unitary but did not have a closer look at the implications of this requirement yet. As Virasoro representations are specified by the conformal weight h of the highest weight state (once again restricting to the chiral sector) and the central charge c the requirement of unitarity will simply lead to constraints on the possible combinations of the parameters c and h.

The first requirement comes from the simple observation that

$$0 \leq \langle h|L_n L_{-n}|h\rangle = \langle h|[L_n, L_{-n}]|h\rangle = \left(2nh + \frac{c}{12}(n^3 - n)\right)\langle h|h\rangle. \tag{2.4.1}$$

From $n = 1$ we learn that the highest weight h has to be non-negative. Furthermore we can infer from the limit of large n that also the central charge c has to be non-negative.

More systematically we analyse the *Gram matrix* G whose elements are formed by taking inner products between all basis states i.e. $G_{ij} = \langle i|j\rangle$ if we denote the basis states of the Verma module by $|i\rangle$. G is obviously a hermitian matrix and thus can be diagonalized. Writing a general vector in terms of the basis in which G is diagonal we conclude that there are negative norm states if and only if G has negative eigenvalues.

The structure of the Virasoro algebra requires G to be block diagonal with non-zero entries occurring only between descendants of the same level. We denote the corresponding submatrices at level n by $G_{(n)}$. For example the first two of these matrices are given by

$$\begin{aligned} G_{(1)} &= \langle h|L_1 L_{-1}|h\rangle = 2h, \\ G_{(2)} &= \begin{pmatrix} \langle h|L_1^2 L_{-1}^2|h\rangle & \langle h|L_1^2 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1}^2|h\rangle & \langle h|L_2 L_{-2}|h\rangle \end{pmatrix} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h+c/2 \end{pmatrix}. \end{aligned} \tag{2.4.2}$$

A necessary condition for $G_{(2)}$ to be positive definite is that the determinant

$$\begin{aligned} \det G_{(2)} &= 2 \cdot (16h^3 - 10h^2 + 4h^2 c + 2hc) \\ &= 32(h - h_{1,1})(h - h_{2,1})(h - h_{1,2}) \\ &\text{where} \quad h_{1,1} = 0 \quad h_{1,2}, h_{2,1} = \frac{1}{16}(5-c) \mp \frac{1}{16}\sqrt{(1-c)(25-c)} \end{aligned} \tag{2.4.3}$$

is positive.

The procedure can be generalized to the submatrix $G_{(n)}$ involving level n descendants. For the determinant of this matrix, the so-called *Kac-determinant*, there is a formula due to Kac [10]:

$$\boxed{\begin{aligned} &\det G_{(n)} = \alpha_n \prod_{\substack{p,q \geq 1 \\ pq \leq n}} (h - h_{p,q}(c))^{P(n-pq)}, \text{ where} \\ &h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad \text{and} \quad c = 1 - \frac{6}{m(m+1)} \end{aligned}} \tag{2.4.4}$$

Here α_n is a positive constant and $P(k)$ denotes the number of partitions of the integer k. The conformal weights $h_{p,q}$ and the central charge c are given above as functions of a generally complex variable m.

In the (c, h)-plane, the Kac-determinant vanishes along the curves $h = h_{p,q}(c)$ which are therefore called *vanishing curves*. The Verma modules corresponding to these values of c and h are reducible. We may obtain irreducible modules by taking quotients by the null subspaces contained in the Verma module (c.f. section 2.3.4).

It is important to note that a non-negative determinant is only a necessary condition for having a non-negative definite matrix and thus a unitary representation as the matrix may have an even number of negative eigenvalues.

We can use the explicit expressions for the Kac determinant in eq. (2.4.4) to learn more about unitary representations for different central charges c. One can show that the vanishing curves do not intersect the region $c \geq 1, h \geq 0$ and that the determinant $\det G_{(n)}$ is strictly positive in this region. In the literature [18] it is shown that for at least one point in that region the matrices $G_{(n)}$ are actually positive definite. Thus we conclude that all representations with $c \geq 1, h \geq 0$ are unitary.

The case $c < 1$ is much more difficult to analyse. A graphical argument given in [18] shows that all points in the (c, h) plane with $0 \leq c < 1$ and $h > 0$ which do not lie on a vanishing curve correspond to non-unitary representations. The same is true for points on the vanishing curves themselves apart from the so-called *first intersections of vanishing curves*. A first intersection of a given vanishing curve is the intersection point with another vanishing curve of the same level which lies closest to the axis $c = 1$. Hence we expect to find unitary representations only for a discrete set of parameters. These are given by eq. (2.4.4) with the restrictions $m, p, q \in \mathbb{N}$, $m \geq 2$, $1 \leq p \leq m$, $1 \leq q < p$. The argument presented in this section is useful to rule out non-unitary representations but does not allow us to prove unitarity. In appendix B we will see that there is an explicit realization of these models as coset theories which form unitary representations.

Theories which satisfy the conditions on central charge and highest weight from the last paragraph are known as *minimal models*. The minimal models are a subset of the *rational conformal field theories* which possess by definition only a finite number of primary fields. Correspondingly the direct sum in the decomposition (2.3.3) is finite. The attribute rational stems from the fact that they occur at rational values for c and h.

Surprisingly the first minimal models can be associated with the continuum limit of well-known models from statistical physics at their critical points. For example $m = 2, 3, 4$ and thus $c = \frac{1}{2}, \frac{7}{10}, \frac{4}{5}$ can be associated with the Ising model, the tricritical Ising model and the 3-state Potts model at their respective critical points [1, 10, 18].

2.4.2 The Ising model

We will sketch briefly the explicit identification of operators in the Ising model with the left-right symmetric conformal fields $\Phi_{p,q}(z,\bar{z}) = \phi_{p,q}(z)\bar{\phi}_{p,q}(\bar{z})$ where $\Phi_{1,1}$ has conformal weight $(0,0)$, $\Phi_{2,1}$ $(\frac{1}{2},\frac{1}{2})$ and $\Phi_{1,2}$ $(\frac{1}{16},\frac{1}{16})$.

In the Ising model we have the spin $\sigma = \pm 1$ as order parameter with $\langle\sigma\rangle = 0$ in the high temperature and $\langle\sigma\rangle \neq 0$ in the low temperature regime. The typical behaviour away from the critical point is $\langle\sigma_n\sigma_0\rangle \sim \exp(-|n|/\xi)$ where ξ is the correlation length. When approaching the critical point the correlations length tends to infinity and the correlation is described by a power law with critical exponent η i.e. $\langle\sigma_n\sigma_0\rangle \sim \frac{1}{|n|^{d-2+\eta}}$. We also have $\langle\epsilon_n\epsilon_0\rangle \sim \langle\sigma_n\sigma_{n+1}\sigma_0\sigma_1\rangle \sim \frac{1}{|n|^{2(d-1/\nu)}}$.

For the two-dimensional Ising model the critical exponents are known from statistical mechanics [18] to be $\eta = \frac{1}{4}$ and $\nu = 1$ hence $\langle\sigma_n\sigma_0\rangle \sim \frac{1}{|n|^{\frac{1}{4}}}$ and $\langle\epsilon_n\epsilon_0\rangle \sim \frac{1}{|n|^2}$. By comparing with the exponents occurring in the two point functions of the primary fields with itself (2.1.27) we identify the spin operator σ with $\Phi_{1,2}$ and the energy operator ϵ with $\Phi_{2,1}$. The identity operator is naturally identified with $\Phi_{1,1}$.

Both $\Phi_{1,2}$ and $\Phi_{2,1}$ possess null descendants at level 2. According to our discussion in section 2.3.4 these give rise to differential equations for the n-point functions involving these fields. In [10] this is used to derive analytical expressions for 4-point functions like $G^{(4)} = \langle \sigma\sigma\sigma\sigma \rangle$.

2.5 Correlation functions revisited

The actual task in investigating a quantum field theory is to calculate correlation functions as these correspond to physically measurable quantities. During our discussion about conformal invariance at the very beginning of this chapter we already saw that the form of n-point function is strongly constrained in a conformally invariant field theory. Nevertheless the correlators are not completely fixed by conformal invariance. Hence our aim is to specify the information which fixes the *operator algebra* i.e. the operator product expansion (including finite parts of the expansions) of all primary fields with each other.

2.5.1 Orthogonality of Verma modules

First we choose a normalization of the primary fields in the way discussed in section 2.1.4 namely such that the normalization factors C_{ij} are given by $C_{ij} = \delta_{ij}$. This is orthonormality for primary fields in the sense of 2-point functions. This immediately leads to orthonormality of the corresponding highest weight states:

$$\langle h_i, \bar{h}_i | h_j, \bar{h}_j \rangle = \lim_{w,\bar{w} \to \infty} w^{2h_i} \bar{w}^{2\bar{h}_i} \langle \phi_i(w, \bar{w}) \phi_j(0, 0) \rangle = \delta_{ij} \qquad (2.5.1)$$

A simple consequence is orthogonality of the associated Verma modules in the sense that the scalar product of any state from one Verma module with an arbitrary state from a second distinct Verma module vanishes. This follows by using the Virasoro algebra, the definition of a highest weight state and their orthogonality property.

2.5.2 OPE and 3-point function coefficients

Using the orthogonality property of Verma modules established above we will prove in this section that the most singular term in the OPE of two fields coincides with the coefficient of the corresponding 3-point function.

The starting point is the most general expression for the OPE of two primary fields ϕ_1 and ϕ_2 which is compatible with scaling invariance:

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p \sum_{\{k,\bar{k}\}} C_{12}^{p\{-k,-\bar{k}\}} z^{h_p-h_1-h_2+K} \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+\bar{K}} \phi_p^{(\{-k,-\bar{k}\})}(0,0) \quad (2.5.2)$$

Here a short word on notation is necessary. The first sum is over the primary fields in the theory. The notation $\{k,\bar{k}\}$ from the second sum is adopted from [1, 18] and stands for an arbitrary collection of non-negative integers $k_1,\dots k_N;\bar{k}_1,\dots \bar{k}_M$ and $\phi_p^{(\{-k,-\bar{k}\})}$ is a descendant field with the notation from section 2.3.3 with levels $K=\sum_i k_i$ and $\bar{K}=\sum_i \bar{k}_i$ respectively. To make contact with the coefficients occurring in the 3-point functions, we look at the 3-point function involving another primary field ϕ_r:

$$\begin{aligned} \langle \phi_r \phi_1(z,\bar{z})\phi_2\rangle &= \lim_{w,\bar{w}\to\infty} w^{2h_r}\bar{w}^{2\bar{h}_r}\langle \phi_r(w,\bar{w})\phi_1(z,\bar{z})\phi_2(0,0)\rangle \\ &= C_{r12}\, z^{h_r-h_1-h_2}\, \bar{z}^{\bar{h}_r-\bar{h}_1-\bar{h}_2}, \end{aligned} \quad (2.5.3)$$

where we used the general expression for the 3-point function from eq. (2.1.28). There is another way to calculate the same correlator by using the expansion in eq. (2.5.2). We find

$$\begin{aligned}\langle\phi_r\phi_1(z,\bar z)\phi_2\rangle &= \sum_p\sum_{\{k,\bar k\}} C_{12}^{p\{-k,-\bar k\}} z^{h_p-h_1-h_2+K}\bar z^{\bar h_p-\bar h_1-\bar h_2+\bar K}\langle\phi_r|\phi_p^{(\{-k,-\bar k\})}\rangle \\ &= C_{12}^{r\{0,\bar 0\}} z^{h_r-h_1-h_2}\,\bar z^{\bar h_r-\bar h_1-\bar h_2},\end{aligned} \tag{2.5.4}$$

where the second line follows from the orthogonality of the Verma modules and the orthonormality of the highest weight states discussed before. Comparing the last two equations yields the desired result

$$C_{r12} = C_{12}^{r\{0,0\}}, \tag{2.5.5}$$

which means that the coefficient of the most singular term in the OPE is given by the coefficient in the corresponding 3-point function.

2.5.3 Determining the operator algebra

Descendant fields can only be correlated if the corresponding primary fields are. Thus the coefficients $C_{ij}^{r\{k,\bar k\}}$ which have to be determined in order to obtain the operator algebra may be written as

$$C_{ij}^{r\{-k,-\bar k\}} = C_{rij}\beta^{r\{-k\}}\bar\beta^{r\{-\bar k\}}, \tag{2.5.6}$$

which is known as BPZ-theorem. Holomorphic and antiholomorphic parts may be treated independently and hence factorize. In [14, 1] it is sketched how one can derive recursive relations between the coefficients $\beta^{r\{-k\}}$ which may then be used together with the initial value $\beta_{ij}^{r\{0\}} = 1$ to determine all coefficients $\beta^{r\{-k\}}$ (as functions of the central charge and the conformal weights of the primary fields only c.f. [1]) and hence to derive the operator algebra.

Knowing the complete operator algebra we can in principle calculate any n-point function by successive use of the expansion eq. (2.5.2) and our knowledge about calculating correlation functions involving descendant fields from correlation functions involving only primary fields (c.f. section 2.3.3).

The correlation functions in a conformal field theory and thus the theory itself are completely determined by the central charge c, the conformal weights of the primary fields $(h_i, \bar{h}_i)$ and the coefficients C_{ijk} of the 3-point functions. One way of determining the remaining coefficients C_{ijk} is known as *conformal bootstrap* and will be discussed in the following section.

2.6 Conformal blocks and Crossing symmetry

In this section we will look at crossing symmetry which originates from the associativity of the operator product expansion. In particular we will look at two different ways to evaluate 4-point functions. This procedure will give us an infinite number of equations involving the coefficients occurring in the 4-point functions. The starting point of our discussion is the general 4-point function $\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\phi_4(z_4, \bar{z}_4)\rangle$. By $SL(2, \mathbb{C})$ invariance we may fix 3 points. The 4-point function then depends on two single coordinates z and $\bar{z}$. The conventional choice is $z_1 \to \infty$, $z_2 = 1$, $z_3 = z$ and $z_4 = 0$. The 4-point function $\mathcal{G}^{21}_{34}(z, \bar{z})$ can be written in terms of so-called *conformal* *blocks* $\mathcal{F}^{21}_{34}(p;z)$ and $\bar{\mathcal{F}}^{21}_{34}(p;\bar{z})$:

$$\begin{aligned}
\mathcal{G}^{21}_{34}(z,\bar{z}) &= \sum_p C_{p34}C_{p12}\mathcal{F}^{21}_{34}(p;z)\bar{\mathcal{F}}^{21}_{34}(p;\bar{z}) \\
\mathcal{F}^{21}_{34}(p;z) &= z^{h_p-h_3-h_4}\sum_{\{k\}}\beta^{p\{-k\}}_{34}\, z^K \frac{\langle h_1|\phi_2(1,1)L_{-k_1}\ldots L_{-k_N}|h_p\rangle}{\langle h_1|\phi_2(1,1)|h_p\rangle} \\
\bar{\mathcal{F}}^{21}_{34}(p;\bar{z}) &= \bar{z}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\sum_{\{\bar{k}\}}\bar{\beta}^{p\{-\bar{k}\}}_{34}\, \bar{z}^{\bar{K}} \frac{\langle \bar{h}_1|\phi_2(1,1)\bar{L}_{-\bar{k}_1}\ldots \bar{L}_{-\bar{k}_M}|\bar{h}_p\rangle}{\langle \bar{h}_1|\phi_2(1,1)|\bar{h}_p\rangle}
\end{aligned} \tag{2.6.1}$$

We can check this relation by direct calculation using eq. (2.5.2) and eq. (2.5.6):

$$\begin{aligned}
&\sum_p C_{p34}C_{p12}\mathcal{F}^{21}_{34}(p;z)\bar{\mathcal{F}}^{21}_{34}(p;\bar{z}) \\
&= \sum_p \frac{C_{p12}}{\langle h_1,\bar{h}_1|\phi_2(1,1)|h_p,\bar{h}_p\rangle} z^{h_p-h_3-h_4}\,\bar{z}^{\bar{h}_p-\bar{h}_3-\bar{h}_4} \sum_{\{k,\bar{k}\}}(C_{p34}\beta^{p\{-k\}}_{34}\,\bar{\beta}^{p\{-\bar{k}\}}_{34}) \\
&\qquad\qquad \cdot z^K\,\bar{z}^{\bar{K}}\langle h_1,\bar{h}_1|\phi_2(1)\phi_p^{(\{-k,-\bar{k}\})}(0,0)|0\rangle \\
&= \langle h_1,\bar{h}_1|\phi_2(1,1)\sum_p\sum_{\{k,\bar{k}\}} C^{p\{-k,-\bar{k}\}}_{34}\, z^{h_p-h_3-h_4+K}\,\bar{z}^{\bar{h}_p-\bar{h}_3-\bar{h}_4+\bar{K}}\,\phi_p^{(\{-k,-\bar{k}\})}(0,0)|0\rangle \\
&= \langle h_1,\bar{h}_1|\phi_2(1,1)\phi_3(z,\bar{z})\phi_4(0,0)|0\rangle \\
&= \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\,\bar{z}_1^{2\bar{h}_1}\langle\phi_1(z_1,\bar{z}_1)\phi_2(1,1)\phi_3(z,\bar{z}),\phi_4(0,0)\rangle \\
&= \mathcal{G}^{21}_{34}(z,\bar{z})
\end{aligned} \tag{2.6.2}$$

The conformal blocks $\mathcal{F}^{21}_{34}(p;z)$ can in principle be calculated from the definition (2.6.1) by commuting the Virasoro generators through the field ϕ_2 or by expanding in powers of z and solving for the coefficients starting from the definition [18, 14].

The particular order in which the fields occur in the 4-point function should have no effect on the final result (up to signs when dealing with fermions). The underlying

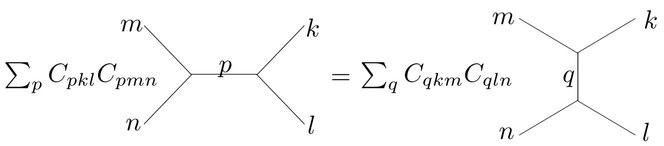

Figure 2.3: Diagrammatic representation of the Crossing symmetry

reason is the associativity of the operator product expansion. Under the $SL(2,\mathbb{C})$ transformation $z \to 1-z$ we have

$$\mathcal{G}^{21}_{34}(z,\bar{z}) = \mathcal{G}^{41}_{32}(1-z, 1-\bar{z}) \tag{2.6.3}$$

or equivalently written in terms of conformal blocks (after replacing $1,2,3,4$ by generic labels n,m,k,l)

$$\sum_p C_{pkl}C_{pmn}\mathcal{F}^{mn}_{kl}(p;z)\bar{\mathcal{F}}^{mn}_{kl}(p;\bar{z}) = \sum_q C_{qkm}C_{qln}\mathcal{F}^{ln}_{km}(q;1-z)\bar{\mathcal{F}}^{ln}_{km}(q;1-\bar{z}). \tag{2.6.4}$$

These relations are called *crossing symmetries* in analogy to the same term in quantum field theory. The analogy becomes even more obvious if we represent equation (2.6.4) diagrammatically as shown in figure 2.3 corresponding to Feynman diagrams in QFT.

The above relations constrain the coefficients C_{ijk} and the conformal dimensions of the primary fields which appear in the conformal blocks. Decoupling equations (c.f. section 2.3.4) for primary fields with null descendants place further constraints on the conformal blocks. It is believed that these equations are sufficient to calculate the values of the coefficients C_{ijk} in a general conformal field theory although this is very complicated in practice and was only accomplished for a few special cases. This method is known as *conformal bootstrap*.

2.7 CFT on the torus and Modular invariance

So far we only looked at conformal field theory on the complex plane which is topologically equivalent to the Riemann sphere and thus a genus zero Riemann surface. In this section we will consider conformal field theories defined on Riemann surfaces of genus $g > 1$. In string theory this is a quite natural generalization as higher genus surfaces correspond to loop diagrams in a perturbative expansion. In the theory of critical phenomena at least the case $g = 1$ is of physical relevance as the torus is equivalent to a plane with periodic boundary conditions in two directions.

It should however be pointed out that modular invariance is not a property of a generic conformal field theory as such but has to be imposed as an additional requirement for CFTs with respect to a particular application e.g. in string theory.

We will limit ourselves to the case $g = 1$ namely the CFT on the torus, on the one hand because this is the simplest higher genus surface but on the other hand because it is believed that there are no more constraints coming from higher genus ($g > 2$) modular invariance [5]. The generalization to higher genus Riemann surfaces which is particularly relevant for string theory is discussed for example in [23, 17].

2.7.1 Representing the torus via a complex lattice

The most convenient way of representing a torus for use in conformal field theory is in terms of a complex plane modulo a lattice. Hence we essentially see the torus as a parallelogram with vertices $(0, 1, \tau, \tau + 1)$ where opposite sides are identified. On the complex plane this amounts to seeing the torus $\mathfrak{T}$ as complex plane with the identification $\omega \sim \omega + n + m\tau$ where $\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{C}$ with $\tau_1, \tau_2 \in \mathbb{R}$, $n, m \in \mathbb{Z}$ and hence $\mathfrak{T} = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. The lattice is shown in figure 2.4. This construction

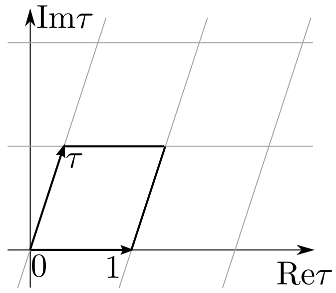

Figure 2.4: Representing the torus as complex plane modulo a lattice

immediately gives us a complex structure on the torus which is necessary to define a conformal field theory on it. The parameter τ which parametrizes the torus is called *modular parameter*.

2.7.2 The Partition function

We are interested in deriving an expression for the partition function i.e. for the vacuum functional as a function of the modular parameter $L \cdot \tau$ which is given by

$$Z(\tau) = \int \mathcal{D}\phi \, e^{-S(\phi)} = \mathrm{Tr}\, e^{-\tau_2 LH} e^{-\tau_1 LP} \tag{2.7.1}$$

in generalization of the well-known relation $Z = \mathrm{Tr}\, e^{-\beta H}$ from ordinary quantum mechanics. Here H generates a translation along the imaginary and P generates a translation along the real axis in the complex plane.

The next step is to find expressions for the Hamiltonian and momentum operator on the cylinder in terms of Virasoro modes on the plane. The Hamiltonian and the momentum operator are generators of time (τ) and space (σ) translation respectively.

They can be calculated from the stress-energy tensor [2] via

$$H = \frac{1}{2\pi}\int_0^L \mathrm{d}\sigma\, T^{\mathrm{Cyl}}_{\tau\tau}(\sigma) \quad \text{and} \quad P = \frac{1}{2\pi}\int_0^L \mathrm{d}\sigma\, T^{\mathrm{Cyl}}_{\tau\sigma}(\sigma). \tag{2.7.2}$$

We will need an expression for the relation between the stress-energy tensor T on the plane and the corresponding quantity T^{Cyl} on the cylinder. Thus we have to study the transformation of the stress-energy under finite conformal transformations first.

The transformation properties of the stress-energy tensor T under an infinitesimal conformal transformation can be deduced using the OPE of the stress-energy tensor with itself eq. (2.2.18) and formula (2.2.12):

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)T(z) + \frac{c}{12}\frac{\mathrm{d}^3\epsilon(z)}{\mathrm{d}z^3} \tag{2.7.3}$$

The corresponding finite transformation under $z \to f(z)$ is more complicated because of the central term in the OPE and is given by

$$T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}S(f(z), z), \tag{2.7.4}$$

where $S(f(z), z) = \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}$ is the so-called *Schwartzian derivative* which has the property that it vanishes identically if and only if f is a global conformal transformation i.e. of the form $f(z) = \frac{az+b}{cz+d}$.

Now we can use eq. (2.7.4) to derive an expression for the stress-energy tensor T^{Cyl} on the cylinder where $w(z) = \frac{L}{2\pi}\ln z$:

$$T^{\mathrm{Cyl}}(w) = \left(\frac{2\pi}{L}\right)^2\left(z^2 T(z) - \frac{c}{24}\right) \tag{2.7.5}$$

Returning to our original question we may represent the cylinder as an infinite strip $-\infty < \tau < \infty$, $0 < \sigma < L$ in the real (τ, σ)-plane with the identification $(\tau, 0) \sim (\tau, L)$. We may consider the complex plane of the complex variable $w = \tau + \mathrm{i}\sigma$ instead. We demonstrate the calculation for the Hamiltonian H:

$$\begin{aligned} H &= \frac{1}{2\pi} \int_0^L \mathrm{d}\sigma \, T_{\tau\tau}^{\mathrm{Cyl}}(\sigma) \\ &= \frac{1}{2\pi} \int_0^L \mathrm{d}\sigma \, (T^{\mathrm{Cyl}}(\sigma) + \bar{T}^{\mathrm{Cyl}}(\sigma)) \\ &= \frac{1}{2\pi \mathrm{i}} \int_0^{\mathrm{i}L} \mathrm{d}w \, T^{\mathrm{Cyl}}(w) - \frac{1}{2\pi \mathrm{i}} \int_0^{-\mathrm{i}L} \mathrm{d}\bar{w} \, \bar{T}^{\mathrm{Cyl}}(\bar{w}) \\ &= \frac{L}{2\pi} \Big(\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{T^{\mathrm{Cyl}}(z)}{z} + \oint \frac{\mathrm{d}\bar{z}}{2\pi \mathrm{i}} \frac{\bar{T}^{\mathrm{Cyl}}(\bar{z})}{\bar{z}} \Big). \end{aligned} \tag{2.7.6}$$

Using (2.7.5) and (2.2.4) we obtain finally

$$H = \frac{2\pi}{L} \Big((L_0 - \frac{c}{24}) + (\bar{L}_0 - \frac{\bar{c}}{24}) \Big). \tag{2.7.7}$$

A similar calculation for P reveals

$$P = -\frac{2\pi \mathrm{i}}{L} \Big((L_0 - \frac{c}{24}) - (\bar{L}_0 - \frac{\bar{c}}{24}) \Big). \tag{2.7.8}$$

We obtain the following expression for the partition function in eq. (2.7.1):

$$\begin{aligned} Z(\tau) &= \mathrm{Tr}\, e^{2\pi \mathrm{i}\tau (L_0 - \frac{c}{24})} e^{-2\pi \mathrm{i}\bar{\tau}(\bar{L}_0 - \frac{\bar{c}}{24})} \\ &= \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \\ &= q^{-\frac{c}{24}} \bar{q}^{-\frac{\bar{c}}{24}} \mathrm{Tr}\, q^{L_0} \bar{q}^{\bar{L}_0}, \end{aligned} \tag{2.7.9}$$

where $q = e^{2\pi \mathrm{i}\tau}$ and $\bar{q} = e^{-2\pi \mathrm{i}\bar{\tau}}$. Note that the trace is over the physical states in the

Hilbert space and thus does not include null-states.

2.7.3 Modular invariance on the torus

Obviously there is no one-to-one correspondence between the modular parameter τ and the complex structure of the torus. The group of transformations of τ which give rise to the same complex structure are called *modular transformations*. The lattice generated by 1 and τ is given by $\{m \cdot 1 + n \cdot \tau | m, n \in \mathbb{Z}\}$. Clearly we get the same lattice if we replace τ by $\tau + 1$ which corresponds to the modular transformation $\mathcal{T} : \tau \to \tau + 1$.

We might as well consider the equivalent parallelogram formed by $(0, \tau, \tau+1, 2\tau+1)$. To bring this into the conventional form with one vertex at 1 we have to multiply by $\frac{1}{\tau+1}$ which clearly leaves the complex structure of the torus invariant. This gives rise to a torus with modular parameter $\frac{\tau}{\tau+1}$. Instead of the modular transformation $\mathcal{X} : \tau \to \frac{\tau}{\tau+1}$ which corresponds to the aforementioned transformation one conventionally considers the combination $\mathcal{S} = \mathcal{T}^{-1}\mathcal{X}\mathcal{T}^{-1} : \tau \to -\frac{1}{\tau}$ instead. The two modular transformations

$$\begin{gathered} \mathcal{T} : \tau \to \tau + 1 \quad \text{and} \quad \mathcal{S} : \tau \to -\frac{1}{\tau} \\ \text{with} \quad (\mathcal{S}\mathcal{T})^3 = 1, \quad \mathcal{S}^2 = 1 \end{gathered} \tag{2.7.10}$$

generate the *modular group* of the torus. A general element is given by

$$\tau \to \frac{a\tau + b}{c\tau + c} \quad \text{where} \quad a, b, c, d \in \mathbb{Z}, ad - bc = 1. \tag{2.7.11}$$

The group is isomorphic to $SL(2, \mathbb{Z})/\mathbb{Z}_2$.

2.7.4 Characters and their transformation properties

As it is known from general representation theory the *character* characterizes a particular representation. In the context of (chiral) Virasoro representation we define the character of a highest weight representation (h) via

$$\chi_{(h)}(\tau) = \mathrm{Tr}_{(h)} e^{2\pi \mathrm{i}\tau(L_0 - \frac{c}{24})} = q^{h-\frac{c}{24}} \sum_{n\geq 0} d(n) q^n \tag{2.7.12}$$

where the trace is over all states contained in the Verma module associated to h. The second equation follows for a highest weight representation with highest weight h if we denote the degeneracy of the Verma module at level n by $d(n)$. The reason for including the factor of $\frac{c}{24}$ into the definition above is linked to the particular transformation properties of these characters under modular transformations.

Following from the decomposition of the Hilbert space into a direct sum of tensor products of chiral and antichiral Verma modules in eq. (2.3.3) the partition function from eq. (2.7.9) can be written as sum over products of characters of the chiral and antichiral representations:

$$Z(\tau, \bar{\tau}) = \sum_{i,j} M_{ij} \chi_i(\tau) \bar{\chi}_j(\bar{\tau}) \tag{2.7.13}$$

The entries M_{ij} are non-negative integers and give the multiplicity with which a particular representation (i.e. a tensor product of a representation of the holomorphic and of the antiholomorphic Virasoro algebra) occurs. The existence and uniqueness of the vacuum requires $M_{00} = 1$. The further entries are however strongly constrained by the requirement of modular invariance.

First we a have to know how characters transform under the generating transformations $\mathcal{S}$ and $\mathcal{T}$ of the modular group. For $\mathcal{T}$ it is fairly easy to see from eq. (2.7.12) that $\chi_i(\tau+1) = e^{2\pi \mathrm{i}(h_i - \frac{c}{24})}\chi_i(\tau)$ and thus

$$\chi_i(\tau+1) = \sum_j T_{ij}\chi_j(\tau) \quad \text{where} \quad T_{ij} = \delta_{ij} e^{2\pi \mathrm{i}(h_i - \frac{c}{24})}. \tag{2.7.14}$$

The characters transform also linearly under the modular transformation $\mathcal{S}$ although this is highly non-trivial to see[5]:

$$\chi_i\Big(-\frac{1}{\tau}\Big) = \sum_j S_{ij}\chi_j(\tau), \tag{2.7.15}$$

where S is a symmetric and unitary matrix. Using the unitarity of S and T the modular invariance of (2.7.13) is found to be equivalent to the conditions

$$[M, T] = 0 \quad \text{and} \quad [M, S] = 0. \tag{2.7.16}$$

These conditions are trivially satisfied for $M_{ij} = \delta_{ij}$. Such a theory is called *diagonal CFT*. One example is the Ising model from section 2.4 which contains only primary fields of weights $(0,0)$, $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{16},\frac{1}{16})$.

There may of course be other non-diagonal solutions M of eq. (2.7.16) for given matrices S and T. For rational CFTs there typically is only a finite number of solutions. These have been classified for the minimal models [18] and a few affine theories (ADE classification [27, 5]).

[5] In [18] this is proven in the context of minimal models by explicitly calculating the matrix S. For the general case c.f. [5] and references therein.

2.8 Fusion rules

Often we are not interested in the full complexity of the operator product expansions of all fields in the theory. It may be sufficient to know which conformal fields may appear in certain operator product expansions.

First we define the concept of a *conformal family.* To the conformal family $[\phi]$ of a primary field ϕ belong the field itself and its descendant fields.

Then we denote the *Fusion product* of two conformal families $[\phi_1]$ and $[\phi_2]$ by

$$[\phi_i] \times [\phi_j] = N_{ij}{}^{k}[\phi_k]. \tag{2.8.1}$$

The Fusion rule coefficients $N_{ij}{}^{k}$ are non-negative integers and have to be understood as multiplicities counting distinct ways two fields from the left hand side may couple to give a field contained in the conformal family $[\phi_k]$ from the right hand side. $N_{ij}{}^{k} > 0$ means in particular that a field from the conformal family $[\phi_k]$ occurs in the operator product expansion of some fields from the conformal family $[\phi_i]$ with some field from $[\phi_j]$. Following from the arguments in section 2.5.2 $N_{ij}{}^{k} = 0$ means that the corresponding coefficients in the 3-point functions C_{ijk} have to vanish.

2.8.1 Verlinde's formula

Remarkably there is a formula due to Verlinde[6] which links the fusion rule coefficients $N_{ij}{}^{k}$ with an expression involving the modular transformation matrix S defined in eq. (2.7.15):

$$\boxed{N_{ij}{}^{k} = \sum_n \frac{S_{in}S_{jn}(S^\dagger)_{nk}}{S_{0n}} = \sum_n \frac{S_{in}S_{jn}S_{kn}}{S_{0n}}}, \tag{2.8.2}$$

[6]c.f. [4] for a list of proofs of Verlinde's formula and further references or [18]

where the second equality follows from the fact that pure Virasoro representations are self-conjugate ($S^2 = 1$), S is symmetric and unitary and thus real. We can interpret ${N_{ij}}^k$ as components $(N_i)_{jk}$ of a collection of matrices labelled by the index i. This allows us to rewrite eq. (2.8.2) as

$$(S^\dagger N_i S)_{mn} = (S_{im}/S_{0m})\delta_{mn}, \tag{2.8.3}$$

which means that S simultaneously diagonalizes all matrices N_i.[7]

2.9 Free field representations

We conclude this chapter by discussing some simple examples namely free bosons and free fermions. A discussion of the ghost system which is important for the application in string theory but which is not covered here can be found in [17, 21, 15, 18]. Both examples are not only interesting theories for the application of the theory developed so far but also serve as starting point for the construction of other theories (e.g. for theories with background charge or orbifolds).

2.9.1 The free boson

The action for a free boson in two dimensions is given by

$$S = \alpha \int \mathrm{d}^2x\, \partial_\mu X \partial^\mu X = \frac{1}{4\pi} \int \mathrm{d}^2z\, \partial X \bar{\partial} X, \tag{2.9.1}$$

[7] It is not the fact that such a matrix exists, which follows from the commutativity of the fusion ring and the consequence that its representations (and thus in particular the adjoint representation) are isomorphic to direct sums of irreducible one-dimensional representations [4, 5], but the fact that it is precisely the matrix S defined in the context of modular transformations that makes Verlinde's formula so remarkable.

where $\mathrm{d}^2z = \mathrm{i}\mathrm{d}z \wedge \mathrm{d}\bar{z} = 2\mathrm{d}x^1\mathrm{d}x^2$ and we set $\alpha = \frac{1}{4\pi l_s^2} = \frac{1}{8\pi}$ which corresponds to setting $l_s^2 = 2$ in the convention used in string theory. The operators ∂ and $\bar{\partial}$ are defined in (2.1.20). The propagator is found to be given up to a constant by

$$\langle X(z,\bar{z})X(z',\bar{z}')\rangle = -\ln|z-z'|^2. \tag{2.9.2}$$

This can be checked by using the identity [18] $\partial\frac{1}{\bar{z}} = 2\pi\delta^{(2)}(z,\bar{z})$ where $\delta^{(2)}(z,\bar{z}) = \frac{1}{2}\delta^{(2)}(x^1,x^2)$ to calculate $\partial\bar{\partial}\ln|z|^2 = \partial\bar{\partial}\ln\bar{z} = \partial\frac{1}{\bar{z}} = 2\pi\delta^{(2)}(z,\bar{z})$.

We can split X into an holomorphic and antiholomorphic part via $X(z,\bar{z}) = x(z) + \bar{x}(\bar{z})$ and find the propagators

$$\langle x(z)x(w)\rangle = -\ln(z-w) \quad \langle\bar{x}(\bar{z})\bar{x}(\bar{w})\rangle = -\ln(\bar{z}-\bar{w}) \quad \langle x(z)\bar{x}(\bar{w})\rangle = 0. \tag{2.9.3}$$

Because of the logarithmic singularities $x(z)$ is not a conformal field but $\partial x(z)$ and the vertex operators $:e^{\mathrm{i}\alpha x(z)}:$ are. By differentiating (2.9.3) we find

$$\langle\partial x(z)\partial x(w)\rangle = -\frac{1}{(z-w)^2} \quad \langle\bar{\partial}\bar{x}(\bar{z})\bar{\partial}\bar{x}(\bar{w})\rangle = -\frac{1}{(\bar{z}-\bar{w})^2}. \tag{2.9.4}$$

From the action we derive the (holomorphic) stress-energy tensor

$$T(z) = -\frac{1}{2}:\partial x\partial x := \lim_{w\to z} -\frac{1}{2}\partial x(z)\partial x(w) + \frac{1}{2(z-w)^2}, \tag{2.9.5}$$

where *normal ordering* is defined as above by subtracting the singular parts of the OPE of the field with itself. For the transition to the quantized theory we had to normal order to ensure a vanishing vacuum expectation value. We work out the OPE with the stress-energy tensor T using Wick's theorem:

$$
\begin{aligned}
T(z)\partial x(w,\bar{w}) &= -\tfrac{1}{2} : \partial x(z)\partial x(z) : \partial x(w) \\
&= - : \partial x(z)\partial x(z) : \partial x(w) + \dots \\
&= \partial x(z)\frac{1}{(z-w)^2} + \dots \\
&\sim \frac{\partial x(w)}{(z-w)^2} + \frac{\partial(\partial x(w))}{z-w}
\end{aligned}
\tag{2.9.6}
$$

The last line follows from a Taylor expansion of $\partial x(z)$ around $z = w$. This shows that $\partial x(z)$ is a primary field with conformal weight $h = 1$. We can also work out the OPE of the stress-energy tensor with itself:

$$
\begin{aligned}
T(z)T(w) &= \tfrac{1}{4} : \partial x(z)\partial x(z) :: \partial x(w)\partial x(w) : \\
&= \tfrac{2}{4} : \partial x(z)\partial x(z) :: \partial x(w)\partial x(w) : + \tfrac{4}{4} : \partial x(z)\partial x(z) :: \partial x(w)\partial x(w) : + \dots \\
&= \frac{1}{2}\frac{1}{(z-w)^4} - \frac{1}{(z-w)^2} : \partial x(z)\partial x(w) : + \dots \\
&= \frac{1}{2}\frac{1}{(z-w)^4} - \frac{1}{(z-w)^2} : \partial x(w)\partial x(w) : - \frac{1}{(z-w)} : \partial\partial x(w)\partial x(w) : + \dots \\
&\sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}.
\end{aligned}
\tag{2.9.7}
$$

The OPE has exactly the form anticipated in eq. (2.2.18) and we can immediately identify the central charge $c = 1$. For the vertex operators $: e^{\mathrm{i}\alpha x(z)} :$ we find a conformal weight $h = \alpha^2/2$ using similar methods [18].

We can expand the primary field $\mathrm{i}\partial x$ into modes via

$$\mathrm{i}\partial x = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} \quad \text{where} \quad \alpha_n = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \mathrm{i}\partial x(z)\, z^n. \tag{2.9.8}$$

The OPE $\partial x(z)\partial x(w) \sim -\frac{1}{(z-w)^2}$ derived from eq. (2.9.4) then leads to the commutation relations $[\alpha_n, \alpha_m] = n\delta_{n+m,o}$ for the modes.

We might also consider a free boson with twisted boundary conditions but we will postpone the discussion of twisted boundary conditions until the next section.

2.9.2 The free fermion

We start with the action for a free Majorana-Weyl fermion in two dimensions:

$$S = \alpha \int \mathrm{d}^2 x\, \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi = \frac{1}{4\pi} \int \mathrm{d}^2 z\, (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) \tag{2.9.9}$$

The gamma matrices γ^0 and γ^1 obey the Dirac algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ (euclidean metric) where we chose the particular representation $\gamma^0 = \sigma^1$ and $\gamma^1 = \sigma^2$ in terms of Pauli matrices. The second equality in eq. (2.9.9) then results from writing the two-component spinor Ψ as $\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$ and choosing $\alpha = \frac{1}{4\pi}$. We find the propagators

$$\langle \psi(z)\psi(w)\rangle = \frac{1}{z-w} \quad \langle \bar{\psi}(\bar{z})\bar{\psi}(\bar{w})\rangle = \frac{1}{\bar{z}-\bar{w}} \quad \langle \psi(z)\bar{\psi}(\bar{w})\rangle = 0 \tag{2.9.10}$$

and the (anti)holomorphic stress-energy tensor is given by

$$T(z) = -\frac{1}{2} :\psi(z)\partial\psi(z): \quad \text{and} \quad \bar{T}(\bar{z}) = -\frac{1}{2} :\bar{\psi}(\bar{z})\bar{\partial}\bar{\psi}(\bar{z}):. \tag{2.9.11}$$

Using Wick's theorem and the propagators from above we can work out the OPEs just like in the last section for example:

$$\psi(z)\psi(w) \sim \frac{1}{z-w} \tag{2.9.12}$$
$$T(z)\psi(w) \sim \frac{1}{2}\frac{1}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial\psi(w) \tag{2.9.13}$$
$$T(z)T(w) \sim \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.9.14}$$

We find that $\psi(z)$ and $\bar{\psi}(\bar{z})$ are primary fields of weight $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively. The central charge $c = \frac{1}{2}$ is found from the most singular part in the OPE of the stress-energy tensor with itself.

The $\mathbb{Z}_2$ symmetry of the action allows us to define periodic or antiperiodic boundary conditions for $\psi(z)$ on the cylinder. As $h = \frac{1}{2}$ these correspond to antiperiodic or periodic boundary conditions on the plane (c.f. (2.2.13) under the map (2.2.10) with $h = \frac{1}{2}$). These are called *Neveu-Schwarz (NS)* and *Ramond (R) sectors*:

$$\mathrm{NS}: \psi(e^{2\pi \mathrm{i}}z) = \psi(z) \tag{2.9.15}$$
$$\mathrm{R}: \psi(e^{2\pi \mathrm{i}}z) = -\psi(z) \tag{2.9.16}$$

On the level of modes this corresponds to

$$\mathrm{NS}: \psi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi_n z^{-n-\frac{1}{2}} \qquad \text{where} \quad \psi_n = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}}\psi(z) z^{n-\frac{1}{2}}, \tag{2.9.17}$$
$$\mathrm{R}: \psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-\frac{1}{2}} \qquad \text{where} \quad \psi_n = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}}\psi(z) z^{n-\frac{1}{2}}. \tag{2.9.18}$$

Note that $\psi(z)$ is single-valued in the NS-sector but double-valued with a branch cut singularity in the R-sector. From the OPE we find the anti-commutation relations $\{\psi_n, \psi_m\} = \delta_{n+m,0}$.

We may understand the anti-periodic case by introducing a *twist field* σ with OPE

$$\psi(z)\sigma(w) \sim (z-w)^{-\frac{1}{2}}\mu(w), \tag{2.9.19}$$

where μ is another (conformal) twist field with the same conformal weight as σ. It can be shown by considering OPEs with the stress-energy tensor that the conformal weight h_σ has to be $\frac{1}{16}$. Because of its square root singularity in the OPE the twist may be understood as boundary changing field and we can consider $\sigma(0)|0\rangle$ as new vacuum state to which only R-fields may be applied (which then results in single-valued correlation functions). The 2-point correlator in the R-sector can then be written as

$$\langle\psi(z)\psi(w)\rangle_R = \langle 0|\sigma(\infty)\psi(z)\psi(w)\sigma(0)|0\rangle = \frac{\frac{1}{2}(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}})}{z-w}, \tag{2.9.20}$$

where we used the mode expansion (2.9.18) together with $\psi_0^2 = \frac{1}{2}$ to derive the second equality. In the same way we can work out the 2-point correlator in the NS-sector again and find that it is indeed given by (2.9.10). As expected the short-distance behaviour agrees in both cases.

Let us have a look at the ground state in the Ramond sector. First we introduce the fermion parity operator $(-1)^F$ by requiring $\{(-1)^F, \psi(z)\}$=0 and $\left((-1)^F\right)^2 = 1$. On the level of modes this corresponds to $\{(-1)^F, \psi_n\} = 0$. The action of ψ_0 does not change the L_0 eigenvalue and thus we need a ground state which represents the two-dimensional Clifford algebra given by ψ_0 and $(-1)^F$. The smallest non-trivial representation is two-dimensional. The states $|+\rangle$ and $|-\rangle$ obey

$$(-1)^F|\pm\rangle = \pm|\pm\rangle \quad \psi_0|\pm\rangle = \frac{1}{\sqrt{2}}|\mp\rangle. \tag{2.9.21}$$

We may identify $|+\rangle$ with $\sigma(0)|0\rangle$ and $|-\rangle$ with $\mu(0)|0\rangle$ from above.

It is instructive to study also the free fermion on the torus [23, 10], where we can find 4 different spin structures depending on whether we choose periodic or anti-periodic boundary conditions in the space and time direction. We can define specialized characters for each of these types by inserting or not inserting the fermion parity operator $(-1)^F$ into the trace which is taken in the periodic or anti-periodic sector. These characters can be written in terms of the Jacobi theta functions and the Dedekind eta function which possess nice transformation properties under modular transformations.

3 Extensions of the Virasoro algebra

Conformal field theories may contain additional conserved currents apart from the stress-energy tensor. The corresponding algebras which extend the Virasoro algebra can be classified according to their conformal spin. First we will look at general features of conformal field theories with extended algebras and in particular at the implications for their representation theory. Then we will have a closer look at Kac-Moody algebras and superconformal algebras, which are very important for applications e.g. in string theory.

3.1 General procedure

This first section is based on the discussion in [23]. We assume that there is an additional conserved (holomorphic) current $J(z)$. The Virasoro generators L_n are the modes of the stress-energy tensor T which is of conformal spin 2. When discussing extensions we may have bosonic ($h-\bar{h} \in \mathbb{N}$), fermionic ($h-\bar{h} \in \mathbb{N}+\frac{1}{2}$) or parafermionic ($h - \bar{h} \in \mathbb{Q}$) currents. In the following we will only consider bosonic and fermionic currents. Restricting ourselves to the holomorphic part again we have the following classification:

$h = \frac{1}{2}$	free fermions
$h = 1$	affine Lie Algebras/ Kac Moody algebras
$h = \frac{3}{2}$	superconformal algebras
$h = 2$	Virasoro tensor products
$h > 2$	W-Algebras (c.f. [26, 14])

Just like in the case of the stress-energy tensor before (c.f. eq. (2.2.4)), we can decompose the current into modes:

$$J(z) = \sum_r z^{-r-h} \hat{J}_r \quad \text{where} \quad \hat{J}_r = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} J(z) z^{r+h-1} \tag{3.1.1}$$

For integer h the sum is restricted to integer values $r \in \mathbb{Z}$. For half-integer values of h the sum may run over integer (R- sector) or half- integer values (NS- sector) as we already saw it in our discussion of the free fermion in section 2.9.2.

As the currents are conformal fields of weight h we can use the usual technique to work out the commutator between the modes of the current and the Virasoro generators. Therefore we make use of eq. (2.2.12) and the OPE in eq. (2.2.15) and get after an integration by parts:

$$[\hat{L}_n, \hat{J}_r] = (n(h-1) - r)\hat{J}_{r+n} \tag{3.1.2}$$

In particular if we look at the case $n = 0$ we infer that applying $\hat{J}_{-r}$ to some conformal field increases its conformal weight by r. In a later section we will also work out commutation relations between the current modes themselves, which can be calculated in the same way as before once the OPE of the currents with themselves is specified. These modes form an algebra whose representations we will discuss in the next section.

3.1.1 Representations of the extended algebra

The aim of this section is not to give a complete overview, but rather to stress similarities of the representation theory of the extended algebra to the representation theory of the Virasoro algebra discussed in section 2.3. And even in this regard this section is by no means complete. For example [14] discusses many more concepts which generalize to extended algebras with only small modifications such as Ward identities, the form of 2-point functions, null state decoupling or crossing symmetry.

The construction of representations of the extended algebra follows the discussion in section 2.3. On highest weight states $|\Phi\rangle$ of the extended algebra we now impose $J_r|\Phi\rangle = 0$ for all $r > 0$ in addition to $L_n|\Phi\rangle = 0$ for all $n > 0$. A primary field which corresponds to a highest weight state via the state-operator correspondence will be characterized by a particular OPE with the additional current and its OPE with the stress-energy tensor given in eq. (2.2.15). Generalizing (2.3.11) we require for integer-moded currents

$$J(z)\Phi(w,\bar{w}) = \frac{1}{(z-w)^h}\tilde{\Phi}(w,\bar{w}) + \cdots , \qquad (3.1.3)$$

where $\tilde{\Phi} = \hat{J}_0\,\Phi$ has the same conformal weight as Φ and the dots denote terms which are are less singular than the first term. Note that the first term only indicates the most singular term which might appear in the OPE. For example if J_0 annihilates the state $\Phi(0,0)|0\rangle$ the leading power in the OPE could be at most $(-h+1)$.

For half-integer currents we may have integer-moded (Ramond) operators or half-integer-moded (Neveu-Schwarz) operators. The first case is discussed above. In the latter case the most singular part in the OPE is given by

$$J(z)\Phi(w,\bar{w}) = \frac{1}{(z-w)^{h-\frac{1}{2}}}\tilde{\Phi}(w,\bar{w}) + \cdots , \tag{3.1.4}$$

where $\tilde{\Phi} = \hat{J}_{-\frac{1}{2}}\Phi$ is a conformal field with conformal weight increased by $\frac{1}{2}$ compared to Φ. Just like in the discussion above we note that the first term is not present if $J_{-\frac{1}{2}}$ annihilates $\Phi(0,0)|0\rangle$.

Another change in comparison to the representation of the pure Virasoro algebra concerns the descendant states. Descendant states are now obtained by applying combinations of J_{-r} with $r \geq 0$ and L_{-n} with $n > 0$ on a highest weight state corresponding to some primary field.

We conclude this section with a short remark on rational conformal field theories. These were defined in section 2.4 as conformal field theories with a finite number of primary fields. Primary fields should now be understood as primary fields with respect to a possibly extended algebra as defined in this section. This means that there are CFTs which contain an infinite number of (Virasoro-) primary fields but which are however rational CFTs with respect to some extended symmetry algebra.

3.1.2 Modular invariance

We briefly discuss implications of modular invariance for theories with extended algebras [18] to illustrate the similarities to our earlier discussion in section 2.7.

In the pure Virasoro case we found the result that the 2-point function vanishes unless both conformal fields have equal conformal weights. In the presence of extended symmetries the constant appearing in the 2-point function which is not fixed by conformal invariance vanishes unless the two fields are charge conjugates which we define by the normalization of the non-zero 2-point functions i.e. $\langle \phi_i \phi_j \rangle \sim \delta_{ji^*}$. Then we define a symmetric matrix $C_{ji} = \delta_{ji^*}$ which obeys $C^2 = 1$ and maps every field to its charge conjugate.

When we look at transformations of characters under modular transformations the definitions of the matrices T from eq. (2.7.14) and S from eq. (2.7.15) remain valid with the only difference that the matrix S may no longer be fixed completely. This is the case because there may be multiple representations of the extended algebra with the same character. A way of circumventing this is to define non-specialized characters [27] which include generators of the Cartan subalgebra of the extended algebra and to require that S is a symmetric and unitary matrix.

The relations among the generators given in (2.7.10) are now modified in the following way:

$$(ST)^3 = S^2, \quad S^2 = C \tag{3.1.5}$$

Under these more general conditions *Verlinde's formula* from eq. (2.8.2) still holds in the form

$$N_{ij}{}^{k} = \sum_n \frac{S_{in} S_{jn} S^\dagger_{nk}}{S_{0n}}. \tag{3.1.6}$$

The indices of $N_{ij}{}^{k}$ can be lowered using the matrix C which yields

$$N_{ijl} = C_{lk} N_{ij}{}^{k} = \sum_n \frac{S_{in} S_{jn} S_{nl}}{S_{0n}}, \tag{3.1.7}$$

where we used $S = CS^\dagger$ which follows from (3.1.5) by multiplying with $S^\dagger$.

3.2 Kac-Moody algebras

As first application of the general concepts discussed in the last section we consider a collection J^a of conformal fields of weight $(1,0)$. The operator product expansion between two of these fields is given by

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{\mathrm{i}f^{abc}}{z-w}J^c, \tag{3.2.1}$$

because by dimensional reasons the first term involving $(z-w)^{-2}$ has to multiply an operator of spin 0 and thus has to be proportional to the identity operator. The second term has to be multiplied with an operator of spin 1 which has to be one of the currents itself.

As the J^a are bosonic fields we conclude that κ has to be symmetric and f^{abc} has to be antisymmetric in the first two indices. Using the associativity of the operator product expansion one can show that f^{abc} satisfies the Jacobi identity and can thus be seen as structure constants of some Lie algebra $\mathfrak{g}$ with corresponding Lie group G which is assumed to be compact which corresponds to a positive definite Killing metric. The symmetric matrix κ is identified with the Cartan-Killing-metric.

It is possible to choose a basis in which κ is proportional to the identity operator in each simple component and thus given by $\kappa^{ab} = \delta^{ab}K$ if we restrict ourselves to simple Lie algebras like it will be done in the following. We may now insert the mode expansion (3.1.1) with $h = 1$ and work out their commutators in the usual way using eq. (2.2.12) and the OPE eq. (3.2.1):

$$\boxed{[J^a_m, J^b_n] = \mathrm{i}f^{abc}J^c_{m+n} + mK\delta^{ab}\delta_{m+n,0}} \tag{3.2.2}$$

These are the commutation relations of a *centrally extended loop algebra*[1]. Together with the Derivation operator $D = -L_0$ which is present in any conformal field theory and which satisfies $[D, J_n^a] = nJ_n^a$ and $[K, D] = 0$ we obtain the structure of an *affine Lie algebra*. An affine Lie algebra is a special *Kac-Moody algebra*, but especially in physics literature both terms are often used synonymously.

Note that the zero modes J_0^a form an ordinary Lie algebra, the so-called *horizontal Lie subalgebra*, with the structure constants f^{abc} from the Lie algebra $\mathfrak{g}$ mentioned above.

Turning to representation theory we infer from the regularity of $J(z)|0\rangle$ at $z = 0$ that the vacuum state has to satisfy

$$J_n^a|0\rangle = 0 \quad \text{for} \quad n \geq 0. \tag{3.2.3}$$

On primary fields we now impose in addition the condition that it has to have the following OPE consistent with eq. (3.1.3):

$$J^a(z)\Phi_{(r)}(w, \bar{w}) \sim \frac{t^a_{(r)}}{z - w}\Phi_{(r)}(w, \bar{w}) \tag{3.2.4}$$

This should be seen as $\Phi_{(r)}$ transforming under some representation (r) of $\mathfrak{g}$ where $t^a_{(r)}$ are understood as representation matrices. This becomes even more transparent on the level of the corresponding highest weight states. Just like for the pure Virasoro case we define the (multiplet of) highest weight states corresponding to the primary field $\Phi_{(r)}$ as

$$|\Phi_{(r)}\rangle = \Phi_{(r)}(0)|0\rangle. \tag{3.2.5}$$

[1]For details on this construction c.f. [11, 4, 18]

Using eq. (3.2.3) and eq. (3.2.4) we have

$$J_0^a|\Phi_{(r)}\rangle = t^a_{(r)}|\Phi_{(r)}\rangle, \quad J_n^a|\Phi_{(r)}\rangle = 0 \quad \text{for} \quad n > 0 \tag{3.2.6}$$

i.e. the highest weight states transform into themselves under a representation (r) of the horizontal Lie subalgebra.

3.2.1 The Sugawara construction

Until now the stress-energy tensor and the additional currents seemed to be completely independent objects. However there is a method, the *Sugawara construction*, which allows us to construct the stress-energy tensor from a combination of Kac-Moody generators. As the stress-energy tensor has conformal weight $(2,0)$ and the currents are of weight $(1,0)$ the most natural ansatz is to write T as the normal ordered sum of terms bilinear in the currents:

$$T_{\text{SW}}(z) = \frac{1}{\beta}\sum_{a=1}^{\dim G} :J^a(z)J^a(z):, \tag{3.2.7}$$

where normal ordering should be understood in the sense of operator modes here:

$$:J_n^aJ_m^b: := \begin{cases} J_n^aJ_m^b & \text{if} \quad n<0 \\ J_m^bJ_n^a & \text{if} \quad n\geq 0 \end{cases} \tag{3.2.8}$$

Following [10] we can fix the constant β by requiring that the currents J^a have conformal weight (1,0) with respect to T_{SW}.

From (3.2.7) we get in particular the following expression for L_{-1}:

$$L_{-1} = \frac{1}{\beta}\sum_{a=1}^{\dim G}\sum_{n=-\infty}^{\infty} :J^a_{n-1}J^a_{-n}: \tag{3.2.9}$$

Using eq. (3.2.6) we can work out the action of this operator on a highest weight state $|\Phi_{(r)}\rangle$:

$$L_{-1}|\Phi_{(r)}\rangle = \frac{2}{\beta} J^a_{-1} t^a_{(r)} |\Phi_{(r)}\rangle \tag{3.2.10}$$

Now we apply the operator J^b_1 to both sides of this equation. The left hand side yields

$$J^b_1 L_{-1}|\Phi_{(r)}\rangle = 1 \cdot J^b_0 |\Phi_{(r)}\rangle = t^b_{(r)}|\Phi_{(r)}\rangle, \tag{3.2.11}$$

where we made use of eq. (3.1.2) for $h = 1$. Here our requirement of having spin 1 currents was actually invoked. The right hand side can be evaluated using the commutation relations (3.2.2):

$$\begin{aligned} \frac{2}{\beta} J^b_1 J^a_{-1} t^a_{(r)}|\Phi_{(r)}\rangle &= \frac{2}{\beta}(\mathrm{i} f^{bac} J^c_0 + K\delta^{ab}) t^a_{(r)}|\Phi_{(r)}\rangle \\ &= \frac{2}{\beta}(\mathrm{i} f^{bac} t^{[c}_{(r)} t^{a]}_{(r)} + K t^b_{(r)})|\Phi_{(r)}\rangle \\ &= \frac{2}{\beta}\left(\frac{1}{2} f^{acb} f^{acd} t^d_{(r)} + K t^b_{(r)}\right)|\Phi_{(r)}\rangle \\ &= \frac{2}{\beta}(\frac{1}{2} C_A + K) t^b_{(r)}|\Phi_{(r)}\rangle \end{aligned} \tag{3.2.12}$$

Hence consistency of (3.2.11) and (3.2.12) requires

$$\beta = C_A + 2K = 2(g^\vee + K), \tag{3.2.13}$$

where C_A is the quadratic Casimir of the adjoint representation[2] defined by $f^{abc} f^{abd} = C_A \delta^{dc}$. In a calculation similar to the free boson case in eq. (2.9.7) we can calculate

[2]This is equal to twice the dual Coxeter number $g^\vee$ of $\mathfrak{g}$ if the longer roots of $\mathfrak{g}$ are normalized to square length 2.

the central charge [18]:

$$c_{\mathrm{SW}} = \frac{K \dim G}{K + C_A/2} = \frac{K \dim G}{K + g^\vee} \tag{3.2.14}$$

So far the level K was assumed to be arbitrary. Unitarity restricts K to be a positive integer[3].

3.2.2 Application: The WZW model

In this section we introduce an explicit example for a conformal field theoretic model with additional conserved currents. The *Wess-Zumino-Witten (WZW) model* can be formulated in terms of an action as a nonlinear sigma model supplemented by a topological term know as *Wess-Zumino topological term.* The action is given by [16]

$$\begin{aligned} S_{\mathrm{WZW}} =& \frac{k}{16\pi} \int_{S^2} \mathrm{d}^2x \, \mathrm{Tr}(\partial_a g \partial^a g^{-1}) \\ &+ \frac{k}{24\pi} \int_{B^3} \mathrm{d}^3y \, \epsilon^{abc} \, \mathrm{Tr}(g^{-1}(\partial_a g) g^{-1}(\partial_b g) g^{-1}(\partial_c g)). \end{aligned} \tag{3.2.15}$$

The field g defined on the sphere S^2 takes values in some semisimple Lie group G and obeys the boundary conditions $g(\tau, 0) = g(\tau, 2\pi)$. Note that the integrand of the second term is a total derivative. Thus Stokes theorem allows us to rewrite this term as an integral over the boundary S^2 of the ball $B^3 = \{y \in \mathbb{R}^3 : |y|^2 \leq 1\}$. The coefficients

[3] We present this argument following [17, 10] in a footnote as it requires more knowledge about representation theory than assumed as prerequisite for the rest of the thesis. Consider for a root of $\mathfrak{g}$ the $su(2)$ subalgebra generated by $E_1^{-\alpha}$ E_{-1}^{α} and $(k - \alpha \cdot H_0)$. The latter is identified with $2J_z$ and we require from our knowledge about $su(2)$ representations that the highest weight state $|m\rangle$ has an integer eigenvalue under $(k - \alpha \cdot H_0)$ thus $k - \alpha \cdot m \in \mathbb{Z}$ or $k \in \mathbb{Z}$ as $\alpha \cdot m \in \mathbb{Z}$. Furthermore from $||E_{-1}^{\alpha}|m\rangle||^2 = \langle m|E_1^{-\alpha} E_{-1}^{\alpha}|m\rangle = (k - \alpha \cdot m)|||m\rangle||^2 \geq 0$ follows $k \geq \alpha \cdot m$. We maximize this inequality by taking α to be the highest root and m to be the highest weight in the given representation. For fixed value of k only a finite number of highest weights can satisfy this inequality. Thus the theory is a rational CFT.

of both terms were chosen already such that the model becomes conformally invariant also in the quantized theory [16]. Topological quantization requires k to be an integer [14]. Apart from conformal invariance the remarkable property of this model is the invariance under transformations of the form

$$g(z = x^1 + \mathrm{i}x^2, \bar{z} = x^1 - \mathrm{i}x^2) = \Omega(z)g(z, \bar{z})\bar{\Omega}^{-1}(\bar{z}), \tag{3.2.16}$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are arbitrary G-valued functions of z and $\bar{z}$ respectively. The currents associated to this symmetry are given by [16]

$$J(z) = -\frac{k}{2}\partial g g^{-1} \quad \text{and} \quad \bar{J}(\bar{z}) = -\frac{k}{2}g^{-1}\bar{\partial}g \tag{3.2.17}$$

and are conserved ($\bar{\partial}J = \partial\bar{J} = 0$) which follows from the equations of motion [18]. We can expand the currents in components J^a in terms of generators $t^a \in \mathfrak{g}$ of G via $J = \sum_a J^a t^a$. The components can then be shown to obey an OPE of the form (3.2.1) with level $K = k$. The same is true $\bar{J}^a$ and hence the two conserved currents generate two independent Kac-Moody algebras.

More detailed introductions to WZW models in the context of conformal field theory can be found for example in [11, 18, 4, 14].

3.3 Superconformal algebras

In this section we will look at superconformal field theories i.e. theories which have at least one additional conserved current of weight $h = \frac{3}{2}$. Due to space limitations we will just discuss the $\mathcal{N} = 1$ *superconformal algebra*. For theories with extended supersymmetry we refer the reader to the literature [14, 15, 17]. For a superspace formulation compare [14]. We just consider one chiral half of the conformal field theory as this corresponds for example to the situation found in heterotic string theories.

As an illustration we start with a very simple example. Consider the action of a free boson and a free Majorana-Weyl fermion:

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z(\partial X\bar{\partial}X + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}). \tag{3.3.1}$$

The action is invariant under the infinitesimal (left-moving) supersymmetry transformation parametrized by the Grassmann valued field $\epsilon(z)$:

$$\delta_\epsilon X = \epsilon(z)\psi \quad \delta_\epsilon\psi = -\epsilon(z)\partial X \quad \delta_\epsilon\bar{\psi} = 0 \tag{3.3.2}$$

The associated conserved current is given by $G = \mathrm{i}\psi\partial X$. We work out the following OPEs involving the stress-energy tensor $T = -\frac{1}{2} : \partial X\partial X : -\frac{1}{2} : \psi\partial\psi :$ using Wick's theorem:

$$T(z)T(w) \sim \frac{\frac{3}{4}\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{3.3.3}$$

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \tag{3.3.4}$$

$$G(z)G(w) \sim \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w} \tag{3.3.5}$$

with $\hat{c} = 1$. The general $\mathcal{N} = 1$ superconformal algebra exists for arbitrary $\hat{c}$ and obeys the OPE [14] given above. Note in particular that (3.3.4) means that $G(z)$ is a (Virasoro-) primary field with $h = \frac{3}{2}$. The value $\hat{c}$, sometimes called *supercharge*, is related to the central charge c via $\frac{3}{2}\,\hat{c} = c$. In our example we find a central charge of $\frac{3}{2} = 1 + \frac{1}{2}$ as expected for an action which is just the sum of a free boson and a free fermion action.

As before we can decompose G into modes via

$$G(z) = \sum_n G_n z^{-n-\frac{3}{2}} \quad \text{where} \quad G_n = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, G(z) z^{n+\frac{1}{2}}. \tag{3.3.6}$$

Just like in the case of the free fermion treated earlier we can impose anti-periodic or periodic boundary conditions for $G(z)$ on the cylinder corresponding to the Ramond- or Neveu-Schwarz sector. In the first case the sum in eq. (3.3.6) runs over integer values and in the second case over half-integer values. From the OPE given above we can work out the algebra of modes in the usual way:

$$\begin{aligned} [L_n, L_m] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_{m+n,0} \\ [L_n, G_r] &= \left(\frac{n}{2} - r\right)G_{n+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \end{aligned} \tag{3.3.7}$$

We get anti-commutation relations between the modes of G because of the additional sign in the definition of the radial ordering operator (2.2.11). The result is the $\mathcal{N} = 1$ superconformal algebra. It hold both for the Neveu-Schwarz sector ($r, s \in \mathbb{Z} + \frac{1}{2}$) and the Ramond sector ($r, s \in \mathbb{Z}$).

The representation theory follows the general principles sketched in section 3.1.1. For $\hat{c} < 1$ i.e. $c < \frac{3}{2}$ we find just like in the pure Virasoro theory (c.f. eq. (2.4.4)) only a discrete set of charges $\hat{c}$ for which unitary representations can occur [14]:

$$\hat{c} = 1 - \frac{8}{(l+2)(l+4)}, \tag{3.3.8}$$

where l is a non-negative integer.

4 Boundary Conformal Field Theory

Boundary conformal field theory (BCFT) is the study of conformal field theories on manifolds with a boundary. Apart from a straightforward generalization in a mathematical sense of CFT to manifolds with boundaries, there are physical reasons for considering these theories. In string theory there is a characterization of D-Branes by the property that open strings may end on them. From the worldsheet perspective they can be described by a boundary conformal field theory [6]. Another application of BCFT is the study of quantum impurity problems in condensed matter physics [22]. The main references for this chapter are the review articles [3, 27, 20, 22] and the corresponding chapter in [18]. In the whole section we will restrict ourselves to the situation discussed in the first part of the thesis and not consider BCFTs with extended symmetry algebras.

The first question we have to address is how to impose boundary conditions consistently. The only possibility in a general CFT is to impose conditions on components of the stress-energy tensor normal/parallel to the boundary. We impose the *conformal boundary condition* $T_{\perp\|} = T_{\|\perp} = 0$ which has the physical interpretation of no momentum flowing across the boundary if the time axis is chosen parallel to the boundary.

4.1 BCFT on the upper halfplane

It is useful to discuss the situation with a prototypical example in mind. A convenient choice here is the upper half plane for which according to eq. (2.2.2) the conformal boundary conditions become

$$T(z) = \bar{T}(z) \quad \text{for} \quad z \in \mathbb{R}. \tag{4.1.1}$$

This means that we can obtain $\bar{T}(\bar{z})$ by analytic continuation of $T(z)$ in the lower half plane and thus define $T(z)$ in the lower half plane via $T(z) = \bar{T}(\bar{z})$. In analogy to the theory in the bulk we define Virasoro modes L_n via

$$\begin{aligned} L_n &= \int_{C^+} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, T(z) z^{n+1} - \int_{C^+} \frac{\mathrm{d}\bar{z}}{2\pi\mathrm{i}}\, \bar{T}(\bar{z}) \bar{z}^{n+1} \\ &= \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, T(z) z^{n+1}, \end{aligned} \tag{4.1.2}$$

where C^+ is a half circle in the upper half plane centred at the origin and the contour of the last integral is a circle around the origin. The modes L_n form a Virasoro algebra just like in ordinary CFT. The crucial difference here is that we obtain only one Virasoro algebra and not two commuting copies as before which is due to the additional condition on the real line which constrains the conformal transformations.

In the following we will consider the upper half-plane punctured at the origin which introduces a distinction between the negative and the positive real axis and allows us to define different boundary conditions on the positive/negative real axis. The Virasoro algebra formed by the modes from eq. (4.1.2) is not spoiled by this construction.

4.2 BCFT on the annulus and the Cardy conditions

In section 2.7 we saw how the requirement of modular invariance on the torus lead to restrictions on the field content of the theory. This is the motivation for considering a BCFT defined on the annulus which is obtained by identifying left and right edges in a rectangle of height L and width T. We will label the two boundary conditions on the boundaries of the strip as a, b. The outline of the argument is as follows: We derive two different expressions for the partition function of the theory and obtain consistency conditions from them, the so-called *Cardy conditions*. In string theoretical terms we would refer to the first construction as tree-level propagation of a closed string and would call the second construction a one-loop evolution of an open string.

We start with the theory on the upper half plane (corresponding to the complex variable z) with boundary conditions labelled b, a for the negative and positive real axis. The half plane can be conformally mapped to an infinite strip via $w = \frac{L}{\pi} \log z$. We obtain the Hamiltonian H_{ba} by calculations similar to those leading to eq. (2.7.7) using the transformation properties of the stress-energy tensor

$$T^w(w) = \left(\frac{\pi}{L}\right)^2 \left(z^2 T^z(z) - \frac{c}{24}\right) \tag{4.2.1}$$

and eq. (4.1.2):

$$H_{ba} = \frac{\pi}{L}\left(L_0^z - \frac{c}{24}\right) \tag{4.2.2}$$

Similar to the torus we considered before we make the transition to the theory on the annulus by restricting the the w-plane to $0 \leq \operatorname{Re} w \leq T$; $0 \leq \operatorname{Im} w \leq L$ and identifying $0 + \mathrm{i}l \sim T + \mathrm{i}l$ for $0 \leq l \leq L$. Via $\zeta = e^{-2\pi \mathrm{i} w/T}$ this domain is mapped to an area of the shape of a circular ring in the complex ζ-plane. The respective stress-energy tensors are related via

$$T^w(w) = \Big(\frac{2\pi}{T}\Big)\Big(-\zeta^2 T^\zeta(\zeta) + \frac{c}{24}\Big). \tag{4.2.3}$$

The conformal boundary conditions (4.1.1) translate to the conditions

$$T^\zeta(\zeta)\zeta^2 = \bar{T}^\zeta(\bar{\zeta})\bar{\zeta}^2 \quad \text{for} \quad |\zeta| = 1, e^{2\pi L/T}. \tag{4.2.4}$$

Using the mode expansion of the stress-energy tensor (2.2.4) we find that the *boundary states* $|a\rangle$ and $|b\rangle$ which represent the boundary conditions a, b after radial quantization on the Hilbert space of the bulk theory have to satisfy

$$L^\zeta_n|a\rangle = \bar{L}^\zeta_{-n}|a\rangle \quad \text{and} \quad L^\zeta_n|b\rangle = \bar{L}^\zeta_{-n}|b\rangle \quad \text{where}\, n \in \mathbb{Z}. \tag{4.2.5}$$

4.2.1 First construction

The first way of calculating the partition function is from the perspective of the infinitely long strip. The generator of a translations along the strip is H_{ab} from (4.2.2) and we have to propagate by T. Hence the first expression for the partition function is

$$Z^{(1)}_{ab}(q) = \mathrm{Tr}\, e^{-TH_{ba}} = \sum_i n^i_{ab}\chi_i(q) = \sum_{i,j} n^i_{ab} S_{ij}\chi_j(\tilde{q}), \tag{4.2.6}$$

where $q = \exp(2\pi\mathrm{i}\tau)$, $\tau = \mathrm{i}T/2L$, $\tilde{q} = \exp(-2\pi\mathrm{i}/\tau)$ and $\chi_i(q) = q^{-\frac{c}{24}}\mathrm{Tr}_i q^{L_0}$. The last equality follows by using the transformation properties (2.7.15) of the characters under the modular transformation $\mathcal{S}$. The positive integers n^i_{ab} are the analogues of the

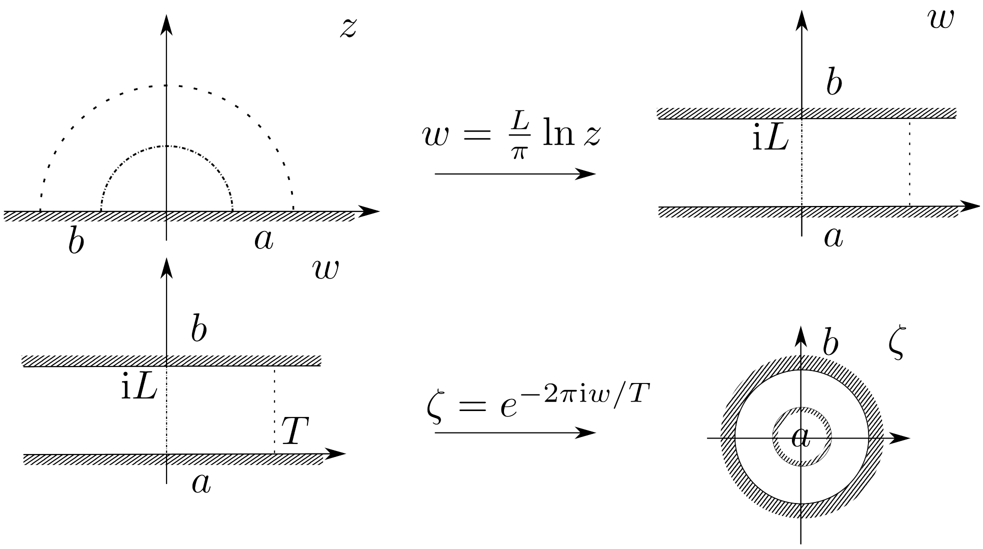

Figure 4.1: The annulus in different coordinates

coefficients M_{ij} in (2.7.13) and indicate how many copies of a particular representation are contained in the spectrum. Note that here the partition function is only linear in the characters as the Hilbert space decomposes into a direct sum of irreducible representations of just one Virasoro algebra.

4.2.2 Second construction

For the second construction assume that the time axis coincides with the symmetry axis of the cylinder. The theory on the annulus may then be interpreted as CFT of the circle propagated by L. The boundaries a, b, then correspond to boundary states $|a\rangle, |b\rangle$ in the full Hilbert space. These have to satisfy the conformal boundary condition (4.2.5). The constraint is satisfied by the so-called *Ishibashi (coherent) states* [27]

$$|h\rangle\rangle = \sum_k |h;k\rangle \otimes \overline{|h;k\rangle}, \tag{4.2.7}$$

where $|h;k\rangle$ denotes the k^{th} vector in an orthonormal basis of the Verma module corresponding to the highest weight state $|h\rangle$.

The Hamiltonian $H = \frac{2\pi}{T}(L_0^\zeta + \bar{L}_0^\zeta - \frac{c}{12})$ of the bulk theory (c.f. eq. (2.7.7)) generates translations along the symmetry axis of the cylinder which corresponds to translations parallel to the imaginary axis in the w-plane. The partition function may then be written as matrix element of the time-evolution operator $\exp(-HL)$:

$$Z_{ab}^{(2)}(q) = \langle a|\exp(-HL)|b\rangle = \langle a|(\tilde{q})^{\frac{1}{2}(L_0^\zeta + \bar{L}_0^\zeta - c/12)}|b\rangle \tag{4.2.8}$$

Now we expand[1] the boundary states in terms of the Ishibashi states e.g. $|a\rangle = \sum_h |h\rangle\rangle\langle\langle h|a\rangle$. With the property $\langle\langle h|(\tilde{q})^{\frac{1}{2}(L_0^\zeta + \bar{L}_0^\zeta - c/12)}|h'\rangle\rangle = \delta_{hh'}\chi_h(\tilde{q})$ which holds in diagonal CFTs we obtain the following expression for the partition function:

$$Z_{ab}^{(2)}(q) = \sum_h \langle a|h\rangle\rangle\langle\langle h|b\rangle\chi_h(\tilde{q}) \tag{4.2.9}$$

4.2.3 The Cardy conditions

The consistency of the expressions (4.2.6) and (4.2.9) for a diagonal CFT together with the linear independence of the characters finally leads us to the *Cardy conditions*:

$$\boxed{n_{ab}^i = \sum_j S_{ij}\langle a|j\rangle\rangle\langle\langle j|b\rangle} \tag{4.2.10}$$

[1]It is not obvious that the Ishibashi states form a basis of the space of boundary states [27]. The completeness is simply taken as an assumption here.

4.3 Explicit construction of boundary states

4.3.1 General construction

In this last section we present an explicit construction based on the modular transformation matrix S to obtain the boundary states within a diagonal CFT. First we set $\langle\langle h|0\rangle = (S_{0h})^{\frac{1}{2}}$ and thus

$$|0\rangle = \sum_h (S_{0h})^{\frac{1}{2}} |h\rangle\rangle. \tag{4.3.1}$$

From the Cardy conditions (4.2.10) together with the properties of S we obtain $n^h_{00} = \delta^h_0$. In the next step we set $\langle\langle h|h'\rangle = S_{h'h}/(S_{0h})^{\frac{1}{2}}$ for $h' \neq 0$ and therefore we get

$$|h'\rangle = \sum_h S_{h'h}/(S_{0h})^{\frac{1}{2}} |h\rangle\rangle. \tag{4.3.2}$$

The Cardy conditions imply $n^h_{h'0} = \delta_{h'h}$. In this way we have constructed one boundary state $|h'\rangle$ for each highest weight h' from the original theory.

In order to check if this ansatz is really consistent with the Cardy conditions, we have to verify that

$$n^h_{h'h''} = \sum_k \frac{S_{hk} S_{h'k} S_{h''k}}{S_{0k}} \tag{4.3.3}$$

is a positive integer. But this is satisfied because of Verlinde's formula (c.f. eq. (2.8.2)) which states that the right hand side of the equation above equals the fusion rule coefficient $N^h_{h'h''}$. This reveals nice cross connections between boundary conformal field theory and the conformal field theory in the bulk and illustrates an application of concepts such as modular invariance or the fusion rules discussed in the first part of the thesis.

4.3.2 Application to the Ising model

We will demonstrate the construction of boundary states for the Ising model [3, 18] which was discussed in section 2.4.2. This will not only illustrate the general construction but will also give us an intuitive understanding of the boundary states.

The modular transformation matrix S is given by [3]

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{4.3.4}$$

if we choose (with the the notation from section 2.4.2) the ordering $(1, \epsilon, \sigma)$ for the primary fields corresponding to the highest weight states $(|0,0\rangle, |\frac{1}{2},\frac{1}{2}\rangle, |\frac{1}{16},\frac{1}{16}\rangle)$. A straightforward application of the procedure from above leads to the following three boundary states given in terms of the Ishibashi states $|0\rangle\rangle$,$|\frac{1}{2}\rangle\rangle$ and $|\frac{1}{16}\rangle\rangle$:

$$|0\rangle = \tfrac{1}{\sqrt{2}}|0\rangle\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\rangle + 2^{-\frac{1}{4}}|\tfrac{1}{16}\rangle\rangle \tag{4.3.5}$$

$$|\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}|0\rangle\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\rangle - 2^{-\frac{1}{4}}|\tfrac{1}{16}\rangle\rangle \tag{4.3.6}$$

$$|\tfrac{1}{16}\rangle = |0\rangle\rangle - |\tfrac{1}{2}\rangle\rangle \tag{4.3.7}$$

We expect to find the continuum analogues of the fixed $\sigma = 1$, $\sigma = -1$ or free boundary conditions known from the lattice realization of the Ising model. The expressions from above suggest to identify $|0\rangle$ and $|\frac{1}{2}\rangle$ with the fixed and $|\frac{1}{16}\rangle$ with the free boundary conditions.

5 Conclusions and Outlook

In this thesis we wanted to give an overview about general features of conformal field theories without focussing on a particular application for example in string theory or the in theory of critical phenomena.

In the first part the topics ranged from the conformal group over the Virasoro algebra and its representations to modular invariance and fusion rules. The chapter is intended to be complete in the sense that all important subtopics were at least touched. In addition to the references given at the appropriate places in text the standard reference [18] is in most cases a good starting point for more detailed discussions.

For pedagogical reasons we decided to present the pure Virasoro theory in the first part at the expense of a less compact presentation compared to [4] for example. In the second part we presented general consequences arising from an extended symmetry algebra and studied some general features of their representations. For Kac-Moody algebras we saw the Sugawara construction for the stress-energy tensor and the WZW- models as an explicit example for a conformal field theoretical model with a Kac-Moody symmetry. A more detailed discussion of these topics would require, in particular, a better understanding of the representation theory of (affine) Lie algebras in terms of roots and weights. Nice introductions particularly suited for the application to CFT are the corresponding chapters in [18, 4, 11, 17]. The other important example for extended symmetry algebras, superconformal algebras, was discussed very briefly. Especially the superspace formulation, theories with extended supersymmetry and their applications to string theory were not be covered due to space limitations.

In the third part of the thesis we considered CFTs on manifolds with a boundary, saw how to impose boundary conditions and found important differences to the theory in the bulk such as the occurrence of only one single Virasoro algebra. We derived the Cardy conditions as consistency conditions from the partition function on the annulus and found an explicit construction for the boundary states in a diagonal CFT. A more detailed treatment would have to include a discussion of Bulk-Boundary OPEs [3] and BCFTs with extended symmetry algebras.

We conclude this thesis with a few remarks on other important topics which could not be covered in this thesis. The first is the question of a rigorous formulation of conformal field theory for example in an axiomatic framework. There has been a lot of work on this subject and we refer the reader to references mentioned in the introduction.

Secondly we did not cover general construction methods for CFTs such as scalars with background charge or orbifolds [5, 18, 14]. The Coset construction was discussed only very briefly in the appendix because of its significance for minimal models.

The third point is a yet unsolved problem namely the classification of modular invariant partition functions both in ordinary CFT and in BCFT and the relation to graphical methods such as ADE Dynkin diagramms [27].

A Correlation functions in 2D CFT

A.1 Differential equations for n-point functions

In this short appendix we sketch the derivation of the results from section 2.1.4. First we derive three differential equations which every n-point function in conformally invariant field theory has to satisfy. The key idea is here to exploit the $SL(2,\mathbb{C})$ invariance of the vacuum.

We consider the correlation function $\langle \phi^{(-m)}(z,\bar{z}))\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle$ where ϕ and $\phi_1,\ldots\phi_n$ are primary fields and rewrite it using (2.3.17):

$$\begin{aligned}
&\langle(\hat{L}_{-m}\phi(z,\bar{z}))\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle \\
&\qquad = \mathcal{L}_{-m}\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle \\
&\qquad = \sum_{i=1}^{n}\left(\frac{(m-1)h_i}{(z_i-z)^m}-\frac{1}{(z_i-z)^{m-1}}\partial_{z_i}\right)\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle
\end{aligned} \tag{A.1.1}$$

By the $SL(2,\mathbb{C})$ invariance of the vacuum these correlation functions vanish for $m = 1,0,-1$. We set $\phi(z,\bar{z}) \equiv 1$ and $z=0$ and obtain the following three differential equations which every conformally invariant n-point function has to satisfy (in addition to the corresponding equations in $\bar{z}_i$):

$$0 = \sum_{i=1}^{n} \partial_{z_i} \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle \tag{A.1.2}$$

$$0 = \sum_{i=1}^{n} (h_i + z_i \partial_{z_i}) \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle \tag{A.1.3}$$

$$0 = \sum_{i=1}^{n} \left(2h_i z_i + z_i^2 \partial_{z_i}\right) \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle \tag{A.1.4}$$

They correspond to translational, dilational and special conformal invariance.

A.2 Form of the 2-point functions

For $n = 2$ (A.1.2) and its counterpart involving $\bar{z}_i$ imply that the 2-point function $G(z_1, \bar{z}_1, z_2, \bar{z}_2)$ depends only on $z_1 - z_2$ and $\bar{z}_1 - \bar{z}_2$. Then (A.1.3) constrains it to the form $C_{12}/(z_{12}^{h_1+h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2})$. Finally eq. (A.1.4) enforces $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$ unless the 2-point function vanishes identically.

By a similar analysis for the 3- and 4-point functions one can show that these are constrained to the forms given in section 2.1.4.

B Coset Conformal Field Theories

In this appendix we sketch the *Goddard-Kent-Olive (GKO) coset construction* [5, 18, 14, 11] which allows us to construct a CFT from two existing CFTs . This will give us an explicit realization of the minimal models from section 2.4 as Coset CFTs and will finally allows us to understand why these are unitary representations of the Virasoro algebra.

B.1 Coset Conformal Field Theories

Our starting point are two affine Lie algebras $\hat{g}$ and $\hat{h}$ with the property that the horizontal Lie subalgebra $\mathfrak{h}$ of $\hat{h}$ can be embedded in the horizontal Lie subalgebra $\mathfrak{g}$ of $\hat{g}$. For simplicity we assume that $\mathfrak{g}$ and $\mathfrak{h}$ are finite dimensional simple Lie algebras. Because of the embedding property the generators J^a of $\hat{h}$ are primary fields (with weight $h = 1$) both with respect to the Virasoro algebra L^g_m associated to $\hat{g}$ and with respect to L^h_m associated to $\hat{h}$. On the level of modes this is formulated as

$$[L^g_m, J^a_n] = -nJ^a_{m+n} \quad [L^h_m, J^a_n] = -nJ^a_{m+n}. \tag{B.1.1}$$

Now we define

$$L^{g/h}_m \equiv L^g_m - L^h_m \tag{B.1.2}$$

and get from the previous two equations $[L_m^{g/h}, J_n^a] = 0$. It follows that $[L_m^{g/h}, L_n^h] = 0$ because L_n^h is bilinear in the currents according to the Sugawara construction in eq. (3.2.7). Thus we have the decomposition

$$L_m^g = L_m^h + L_m^{g/h} \tag{B.1.3}$$

into the two commuting modes L_m^h and $L_m^{g/h}$. As both L_m^h and L_m^g satisfy the Virasoro algebra it is now easy to verify that the same is true for $L_m^{g/h}$. We find a central charge $c^{g/h} = c^g - c^h$ if we denote the central charges of the affine theories associated to $\hat{g}$ and $\hat{h}$ by c^g and c^h. We then refer to the quotient theory associated with the stress-energy tensor $T^{g/h} = T^g - T^h$ as *coset theory* $\hat{g}/\hat{h}$.

It is possible to generalize the construction by considering also semisimple Lie algebras and one can even consider arbitrary conformal field theories which contain an affine theory as a subtheory i.e. without assuming that the larger theory is an affine theory.

The representation of $T^{g/h}$ created from the vacuum is unitary by construction as it is realized on a subspace of a unitary representation of T^g.

B.2 Minimal conformal series realized as Coset CFTs

The proof that the minimal models from section are indeed unitary representation of the Virasoro algebra works by looking at the following coset theory (with the notation from the last section):

$$\hat{g} = \hat{su}(2)_K \otimes \hat{su}(2)_1 \quad \text{and} \quad \hat{h} = \hat{su}(2)_{K+1}, \tag{B.2.1}$$

where the subscripts denote the levels of the affine Lie algebras (c.f. (3.2.2)). From 3.2.14 with $g^\vee = 2$ for $su(2)$ we find for the central charge of the coset theory

$$c^{g/h} = \frac{K \cdot 3}{K+2} + 1 - \frac{(K+1) \cdot 3}{(K+1)+2} = 1 - \frac{6}{(K+2)(K+3)}. \tag{B.2.2}$$

These are precisely the central charges we found in (2.4.4) with $m = K + 2$. As these are the only possibly unitary representations of the Virasoro algebra for these central charges, the coset theories from above must provide an explicit realization of the minimal models. The precise argument leading to this identification is given in [19].

The coset theories $(\hat{su}(2)_K \otimes \hat{su}(2)_2)/\hat{su}(2)_{K+2}$ provide an explicit realization of the $\mathcal{N} = 1$ superconformal minimal models mentioned in section 3.3 [19, 14].

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