Entanglement and Reference frames in Quantum Information Theory

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25th September 2009

Submitted in partial fulfilment of the requirements for the degree of Master of Science of the University of London

ABSTRACT. This review investigates and presents entanglement and reference frames as two rich subjects of Quantum Theory and how they can be treated in quantum information theory as useful physical resources. Use of entanglement to perform quantum informational tasks as such as quantum teleportation will be presented to illustrate entaglement as a resource and to highlight shared reference frames as a harnessable resource. The importance of considering reference frame in quantum information theory will be suggested and its connection to foundational problem of quantum theory as such as superselection rule will be explored. Then possible extension or further aspects of interest and perhaps further research will be discussed as such as problem of quantifying the resources and possible questions concerning both entanglement and reference frames as such as whether they are interconvertable will be raised.

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Introduction

In classical and quantum information theory, identifying possible resources and investigating treatment of resources in various consideration as such as efficiency, distillability, longevity and so on to name a few, are very important and interesting problems. In both classical and quantum computation, which falls under broad spectrum of information theory, physical resources as such as data storage space, computation time and energy have been important factors to consider for theory of computation, implementation of algorithms and realisation of computing devices. Physical space to store and implement data, number of gates required for algorithms and whether computation is reversible or not has been of much interest to classical information theory, for optimising these resources are crucial for application and therefore development of specific protocols. Same problems are equally applied to quantum information theory, for example, finding reversible gates and therefore reversible computation to account for unitary quantum gates studied in quantum information theory which are invertible by nature.

However in quantum information, new phenomenons as such as entanglement shows strange behaviour unique to quantum theory that is not present in classical information theory. As Einstein famously criticised entanglement and its non local property as 'spooky action at a distance', seemingly violating certain rules of physics, the non locality exhibited by a pair of entangled states is precisely one of the phenomenon we can harness as resources to use in quantum information theory. As we shall see in the quantum teleportation example, shared entanglement pair between two parties, Alice and Bob, and some classical communication channels, can transmit a qubit from Alice to Bob and transfer information, thereby depleting the entangled pair and using up the entanglement as a resource in exchange of using 'cheap' classical channel and certainty of the outcome. Also in this example, we identify the role of shared reference frame as a resource and begin our investigation of reference frame in quantum information.

Identifying entanglement and reference frame as resources of quantum information was a crucial step to study these fields and to apply these physical objects for better understanding the quantum theory. Therefore these two have suffered and still suffer bombardment of questions and investigation, bearing fruitful results, and we will attempt to raise some of these questions and glimpse still partial answers to some of the major problems.

Since entanglement is relatively widely known phenomenon, the definition was omitted. On the otherhand reference frames in quantum information theoretical sense needs some clear presentation and introduction. However before the definition of reference frame is given, the motivation for such definition will be revisited. Hence we look to the nature of the information we are dealing with for information process tasks.

As Landauer (1993) [**1**] claimed 'Information is physical' and from 'Quantum Information is physical too' (Rudolph, 1999) [**2**] emphasis on the importance of physical nature of the quantum information has been frequently restated to remind that high abstraction of information and treating information in forms of different physical medium as equivalent, somewhat obscures additional benefits quantum information theory can offer due to the quantum nature of the framework. Further to this, careful examination of nature of information is necessary. Fungible information [**3**] is information for which the physical nature of the information carrier is not important. The method of encoding the information is not important whether classic bit, cbit, is encoded via a laser being on or off, or an atom in excited state or in ground state. This type of information is typically dealt by classical information theory and such generalisation have allowed to focus the research on more abstract and inherent nature of the information theory, yielding great results as such as Shannon's theorem for coding information in noisy channel. However there are variety of information processing tasks that can not be implemented by using fungible information, as such as synchronising two atom clocks on different spaceships. This can not be done by just sending classical bits (which is fungible) but needs some kind of token physical system to share which has natural oscillation for the synchronisation. For the case of aligning a spatial direction in different Cartesian frames, a token physical system to share must have a direction to point to, for example spin $\frac{1}{2}$ particle with its axis of spin representing vertical-direction. These information is not describable with words or instructions or classical information without token physical system to compare to. These sort of information is called nonfungible information. Fungible and nonfungible information are also known as 'speakable' and 'unspeakable' information respectively (Peres and Scudo, 2002b) [**4**].

The physical nature of the quantum object which act as a information carrier, wether it be spin-1/2 particle (electron), or coherent superposition of ground and excited states of a two-level atom, needs some other system with respect to these can be relatively defined. Each observer does not define unspeakable information as such as directional information or phase information, against some absolute Newtonian space or absolute time, but can only describe them relative to another system at their disposal as such as gyroscopes, clocks, metre rule and so on in their laboratory for example. As with analogy to revelation of this non existence of abolute Newtonian space and time leading to theories of special relativity and general relativity, these systems with respect to which unspeakable information is defined are known as reference frames [**3**]. So the nonfungible or unspeakable informations exhibit obvious needs of a corresponding refernce frames in each communicating parties, Alice and Bob, in which to make any sense of the information. To the recipient, Bob, of a unspeakable information, since Bob would measure this respect to his own reference frame and Alice with respect to her own reference frame, the same unspeakable information could be described completely differently. Also agreeing on same measurements scheme would not be trivial as for example, their measurement of spin $\frac{1}{2}$ particles would not necessarily be aligned due to differing reference frames (Rudolph 1999 [**2**] identifies the problem of non-universal natural basis for example of spin measurements, unlike the energy eigenstates measurements, due to the Hamiltonian of the universe - lacking universal static magnetic field for example). This need of consideration of reference frames for unspeakable information is the crucial point for the case of importance of reference frames.

Now we have the definition of a reference frame. However it is not just unspeakable information that needs reference frames. Speakable information can be encoded into any degree of freedom of a system (for example, horizontal polarisation of a photon) but it still requires some reference frame to be chosen, corresponding to the degree of freedom that has been used. Therefore to lack the reference frame for a chosen degree of freedom for encoding information, for both speakable and unspeakable information, would entail some consequences for the information processing tasks that utilise them. It can be shown that this lack of reference frame can be viewed in quantum formalism as a form of decoherence or a quantum noise due to interaction of the system with the inaccessable reference frame. This is analogous to correlation of a system to an environment, which we have no access, to be treated as decoherence or noise and so both types of noises can possibly be treated via methods in quantum information theory which eradicates noise (such as use of decoherence-free subsystems) [**3**].

This generalisation of reference frame incorporated into quantum formalism as a form of decoherence will be further developed to show that it is equivalent to imposing superselection rule as additional restriction on the quantum theory. Superselection rule, mathematical restriction on possible states that can be prepared under certain conditions, as forbidding coherence between different eigenspaces of some observables, is additional restriction to the selection rule(conservation rule) in an attempt to adapt the theory to accomodate the experimental results. This restriction on coherence and therefore forcing decoherence is equivalent to lacking specific reference frame and this connection can be further developed to show that by choosing appropriate reference frame, superselection rule can be violated [**7**] and give some insight to the important question of whether superselection rule is fundamental axiomatic restriction of quantum theory. This point highlights that the study of reference frames is both motivated via application of quantum information theory and also via study of foundational problems of quantum theory and construction of quantum formalism.

From non locality of quantum theory, with aid of bipartite pure states between two parties, the bell states, notion of entanglement [**12**] emerged with powerful results and consequences for the quantum theory and in application into quantum information theory. This identified entanglement as useful resources in quantum information theory. From consideration of physical nature of information and absence of universal basis of some quantum systems [**2**], lead to discussion of reference frames in quantum theory with profound insight to its implications. These two aspects of quantum thoery as useful resources in quantum information theory and useful tools for investigation of quantum theory have not yet been fully understood. However they promise to answer or at least give interesting insights to many questions that they both share, as such as interconvertability of these two concepts as resources, how useful they are as resources in terms of longevity, distillablity and so on and what consequences they bring to formalism of quantum theory.

So now we introduce some notational conventions and basic results or tools assumed in this review before the discussion of concept of entanglement.

CHAPTER 1

Preliminaries

1. Notations

Before we turn to theory of entanglement and theory of reference frames, we introduce some notations and conventions that will be used in this review. If not stated specifically as a agreed notation in this chapter, the most general and common notations have been chosen where possible.

Classical bit is abbreviated to cbit; qubit for quantum bit; ebit for entanglement bit; refbit for reference frame bit and lbit or $|0_L\rangle$ for logical bit.

$|0\rangle$, $|1\rangle$ denote standard computational basis or number eigenstates depending on the context which will be clear.

The 4 Bell states: maximally entangled quantum states of two qubits are denoted as follows.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle \otimes |0_B\rangle - |1_A\rangle \otimes |1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle \otimes |1_B\rangle + |1_A\rangle \otimes |0_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle \otimes |1_B\rangle - |1_A\rangle \otimes |0_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Basic knowledge of group theory, linear algebra and representation theory will be assumed in this review. Specifically we will focus on finite groups, continuous (Lie) groups which are compact (ensures group-invariant Haar measure dg) and act on the Hilbert space via a unitary representation (completely reducible) [**3**], to ensure relatively simple treatment of our discussion.

Also for group identity notation, $\mathcal{I}_{\mathcal{N}}$ will be used for identity map over operators in the space $\mathcal{N}$ and 1_A will be used for identity map on system A.

Basic knowledge of quantum mechanics will also be assumed in this review as such as state space representation $|\psi\rangle$ of systems, use of density matrix representation ρ and their treatments and so on.

2. Schur's lemmas

Here we state without proof results from group representation theory, Schur's First Lemma and Schur's Second Lemma, which later on will be used to prove a result in considering reference frames. These form of Schur's Lemmas are from Bartlett (2007) [**3**], which follows Nielsen (2003) [**5**].

Lemma 2.1 (Schur's first). *If $T(g)$ is an irreducible representation of the group G on the Hilbert space $\mathcal{H}$, then any operator A satisfying $T(g)AT^{\dagger}(g) = A$ for all $g \in G$ is a multiple of the identity on $\mathcal{H}$.*

Lemma 2.2 (Schur's second). *If $T_1(g)$ and $T_2(g)$ are inequivalent representations of G, then $T_1(g)AT_2^{\dagger}(g) = A$ for all $g \in G$ implies $A = 0$.*

CHAPTER 2

Entanglement

In this review we mostly consider bipartite entanglement, i.e. entanglement between two parties, Alice and Bob. Some results of theory of bipartite entanglement extend to higher dimensions as such as Peres-Horodecki criterion, also known as positive partial transpose criterion (for example 2×2 and 2×3 dimensions i.e. two qubits case and one qubit and one qutrit case), however higher dimensions bipartite entanglement is generally difficult and much too complicated for scope of this review.

The Bell states as was shown in the section 1 in chapter 1, they are maximally entangled bipartite states. They are states that violate Bell's Inequalities maximally and with experimental evidence in favour of quantum mechanics, helped establish Bell's Theorem, that quantum theory is non local. Also these Bell states are the states that exhibit this non local behaviour maximally and therefore used widely in many quantum information processing tasks.

These Bell states have entanglement as their resources for doing information processing and this will be demonstrated in quantum teleportation example later on. Then if entanglement is viewed as a resource, we can ask several important and interesting questions about nature of this resource. Among several properties of entanglement as such as longevity, efficiency of extracting entanglement as resources out of systems, how hard it is to prepare entangled states or what other resources are needed to distill entanglement and many more interesing questions [**3**].

Entanglement is the property exhibited by bipartite states when they are not expressable as product states. For example, $\rho_A \otimes \sigma_B$, where ρ_A is a pure state in system A and σ_B is a pure state in system B, is a product state and therefore not a entangled state.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

One of the Bell states, $|\Phi^+\rangle$, is an example of entangled state where it is not a product state. Since for general states, $|\psi\rangle = a|0\rangle + b|1\rangle$, a product state would be,

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

Then for this to equal to the Bell state, $ac = bd = \frac{1}{\sqrt{2}}$ and $ad = bc = 0$. However there is no such combination of a, b, c, d since first equation gives $(ac)(bd) = abcd = \frac{1}{2}$ and second equation gives $(ad)(bc) = abcd = 0$, obtaining contradiction. This shows that Bell state $|\Phi^+\rangle$ is not a product state and therefore entangled state. The same is true for the other three Bell states and this can be easily checked as above example.

As will be shown later, this nonseparability into product state is what classifies entangled states. More precisely, it should be named non locally preparable states as this emphasis the physical nature of the states that they can not be prepared by just Local Operation and Classical Communications (LOCC) from a product state [**12**].

First we focus on how to measure or qunatify the entanglement in order to quantify its resourcefulness.

1. Measure of Entanglement

We begin with the definition of separability which in turn defines entanglement.

DEFINITION 1.1. *A general mixed bipartite state is separable iff it can be written in the form*

$$\rho = \sum_{i,j} p_{ij}\, \sigma_i^A \otimes \sigma_j^B$$

if it is not separable then it is entangled.

However, with this definition of entanglement, it is often hard to determine if a bipartite state is entangled or not, since it is left to optimisation problem to check all the possible convex decompositions of ρ.

We introduce another definition for decomposing a general bipartite state.

DEFINITION 1.2. *If a general bipartite state between system A and system B, bipartite state can be written as,*

$$|\Psi_{AB}\rangle = \sum_{i,j} C_{ij} |a_i\rangle |b_j\rangle$$

then every such state can also be written as,

$$|\Psi_{AB}\rangle = \sum_{i} \sqrt{\lambda_i} |e_i\rangle_A |f_i\rangle_B$$

with $|e_i\rangle_A$ *as eigenbasis of* ρ_A *and* $|f_i\rangle_B$ *as eigenbasis of* ρ_B. *This is called Schmidt decomposition or biorthogonal decomposition.*

ρ_A is reduced density matrix of A, obtained by partial tracing out system B in the eigenbasis. Similarly for the ρ_B is reduced density matrix of B. $\rho_A = \sum_i \lambda_i |e_i\rangle\langle e_i|$ and so ρ_A and ρ_B share same eigenvalues.

Then for the bipartite states that has Schmidt decomposition, a measure is defined called von-Neumann entropy.

Definition 1.3. *von-Neumann entropy of Schmidt decomposable bipartite system is,*

$$S(\rho_A) = -\sum_i \lambda_i \log \lambda_i$$

This von-Neumann entropy measure is maximised when $\lambda_i = \frac{1}{d}$ uniformly where d is the dimension of ρ_A. and takes value 0 when only one such eigenvalue is 1 and others 0, i.e. when $\lambda_1 = 1$ and $\lambda_{i\neq 1} = 0$. This measure correlates with being maximum for maximally entangled states of $\sum_i \frac{1}{d}|e_i\rangle\langle e_i|$ and minimally entangled state when it is pure state of $|e_1\rangle\langle e_1|$ for example.

Then for bipartite state with density matrix ρ, which does not have Schmidt decomposition, Peres-Horodecki criterion or otherwise known as Positive Partial Transpose (PPT) criterion can be used to determine entangleness of a state.

Definition 1.4. *For general state ρ on bipartite system, $\mathcal{H}_A \otimes \mathcal{H}_B$,*

$$\rho = \sum_{ijkl} p^{ij}_{kl} \, |i\rangle_A \otimes |j\rangle_B \langle k|_A \otimes \langle l|_B$$

its partial transpose (with respect to the B) is defined as

$$\rho^{T_B} = \sum_{ijkl} p^{ij}_{kl} \, |i\rangle_A \otimes |l\rangle_B \langle k|_A \otimes \langle j|_B$$

If ρ^{T_B} has a negative eigenvalue, ρ is guaranteed to be entangled, if the dimenstion is not larger than 2×3 (i.e. not more than one qubit and one qutrit).

Thus these definitions and criterions give brief view of incomplete picture of how entanglement can be measured, quantified or validated. The inconclusivity of PPT criterion for higher dimensions and non Schmidt decomposable systems give huge area for further study of measure of entanglement and identifying entanglement. However this same question of finding a method of quantifying entanglement can be asked to another type of resource that is reference frame. Since study of quantum unspeakable information theory is very young in its stages, this area of interest is huge challenge for further investigation for understanding reference frame in quantum theory.

2. Further Properties of Entanglement

There are several other interesting properties of entanglement which will be briefly explored in this section. When considering entanglement of mixed states, as was with above example of identification of entangled states, the theory becomes much more rich and complex [**13**]. The new concepts which emerged as the outcome of research include bound entanglement, distillation, activation [**14**] and so on. These concepts give further insight to the role of entanglement as resource in information processing tasks.

Distillable states are states such that n copies of the state can be converted or 'distilled' into some number of maximally entangled pure states using LOCC with some fidelity [**12**]. Identifying which mixed states are distillable is not just hard unsolved problem, but it is not even known if it is possible at all [**12**]. This property of being able to use large supply of distillable states to obtain maximally entangled pure states between two parties illustrate how this resource of entanglement can be prepared and what possible resources can convert to yield entanglement in order to enable LOCC for example, thereby using cheap information processing methods for communication.

3. Quantum Teleportation

The Quantum Teleportation is perhaps one of the most exciting results of quantum information theory. As will be demonstrated in this example, entanglement is used up as a resources to transmit quantum information across two parties. By tapping into this resources, one can achieve 1 qubit of information transfer for every 1 ebit and 2 cbit transferred via classical communication channel. This is beneficial since using classical communication channel to transmit qubit is much cheaper and practical solution. This advantage in usage of resources is balanced by depleting the entanglement resources. This type of paradigm is known as Local Operation and Classical Communication (LOCC) where efficient and practical local operation and communication via classical channel is used for non local quantum information trasfer process.

This is one instant of what rich results in information theories can arise from restrictions on experimental operation or physical limitation [**6**]. Using entanglement as resources to enable cheap LOCC paradigm and acquiring restriction of being in this paradigm resulted in theory of entanglement. So using similar logic, one could hope that studying the restrictions of lacking a reference frame (and therefore gaining restrcition of superselection rule) [**3**] would open up new scope for the quantum information theory.

Now we give a simple illustration of quantum teleportation.

Say, Alice and Bob are in spacelike separated position or very far apart and they want to send a qubit of information to another. Luckily before they parted for the last time, they shared a bell state $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, with Alice taking system A and Bob taking system B on their long journey. Also they have Classical channel between them.

Alice has prepared an unknown state $|\psi\rangle = \alpha|0\rangle_S + \beta|1\rangle_S$ in system S. She wishes to teleport this state to Bob. Is it possible?

The method goes like this. At Alice's disposal, she has system A and system S. She performs a measurement on her System of A and S which is given by projectors onto the Bell states, namely

$$|\Phi^+\rangle_{SA} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and so on with $|\Phi^-\rangle_{SA}$, $|\Psi^+\rangle_{SA}$, $|\Psi^-\rangle_{SA}$ in system S and A. These are orthogonal and measurement being performed is,

$$\left\{|\Phi^+\rangle_{SA}\langle\Phi^+| \otimes 1_B, \cdots, |\Psi^-\rangle_{SA}\langle\Psi^-| \otimes 1_B\right\}$$

Considering 3 systems, SAB,

$$(\alpha|0\rangle_S + \beta|1\rangle_S) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

If Alice gets the $|\Phi^+\rangle\langle\Phi^+|$ outcome from possible 4 outcomes,

$$\begin{aligned}\big(|\Phi^+\rangle\langle|\Phi^+|\otimes 1_B\big)\frac{1}{\sqrt{2}}(\alpha|00\rangle_{SA}|0\rangle_B+\alpha|01\rangle_{SA}|1\rangle_B\\ +\beta|10\rangle_{SA}|0\rangle_B+\beta|11\rangle_{SA}|1\rangle_B)\\ =\frac{1}{2}\left(\alpha|\Phi^+\rangle_{SA}|0\rangle_B+\beta|\Phi^+\rangle_{SA}|1\rangle_B\right)\\ =\frac{1}{2}|\Phi^+\rangle_{SA}\otimes(\alpha|0\rangle_B+\beta|1\rangle_B)\\ =\frac{1}{2}|\Phi^+\rangle_{SA}\otimes|\psi\rangle\end{aligned}$$

Thus state $|\psi\rangle$ has been teleported to system B, Bob, and entanglement between A and B is lost. The probability of obtaining this scenario is $\frac{1}{4}$ and for other three outcomes as such as $|\Phi^-\rangle$, Alice tells Bob via the 2 cbit channel that she obtained this outcome and Bob can apply corresponding unitary local transformation on system B, for this example Z gate, to yield quantum teleportation. Use of the entangle pair, taking measurement by projectors onto the 4 Bell states as orthogonal basis and using 2 cbit to transmit via classical communication channel to Bob which outcome Alice obtained and so which determinable unitary transformation Bob should apply to his system to obtain teleportation of 1 qubit. Important aspect of this simple scenario of quantum teleportation is that in order to utilise LOCC paradigm, use cheap resources and achieve quantum teleportation of 1 qubit, 1 ebit of entanglement was depleted as shown by entanglement no longer existing between system A and system B. As now system B is expressable tensor product of density matrix in system A and density matrix in system B. It is now separable.

Then the question that can be raised is what is the rate of resources of entanglement that is consumed in certain quantum information processing tasks? How quickly do you use up and how can you replenish stock of entangled states from the other resources at disposal? Although the general theory of entanglement is not yet complete enough to answer these questions in generality, some of answers to these questions can be found. Possibly extending the theory of entanglement to higher dimensions and sharing entanglement of more general dimenstional states between multiple parties can lead to much complex scenarios and rich results.

Now we leave the theory of entanglement briefly and venture into the theory of reference frames, in which we hope to discover new answers and insights to the questions we have been asking of entanglement theory and we begin by focusing on a detail in the quantum teleportation example already discussed.

CHAPTER 3

Reference Frames

In the quantum teleportation example that we looked at in terms of entanglement as resource for quantum information theory, there are many other details that a well equipped quantum information theorist might ask and might study further. Among them are precedures for doing the projective measurements in Bell states orthonormal basis which makes use of unitary transformation to change measurement basis. These are interesting but there is one detail which opens up new area of theory to explore and to experiment with.

The detail we are looking for is how both Alice and Bob knew what orientation their Cartesian frames were and how they agreed on the moment of time to do the measurements. Under certain circumstances as being in the same lab and sharing same clock, Alice and Bob would agree that they share same reference frame for Cartesian frame and clock synchronisation. If they had not shared a reference frame for the degrees of freedom for which the information was encoded in, even with the same protocol as already described quantum teleportation, they would not have been able to achieve it. The problem is that when Alice tells Bob which outcome out of 4 Bell states she recieved and therefore which unitary transformation Bob should implement, if they do not share the same Cartesian frames and therefore Bob's Z gate basis for example does not align with Alice's Z gate basis, Bob would not necessarily get back the qubit Alice was teleporting through using the wrongly oriented measurement basis. Orientation of particular Cartesian frame for example is an unspeakable information. Unspeakable information needs reference frame to be measured relative to and if they do not share reference frame, they would view the same state and operation differently in their own reference frames. Also if they did not know each other's orientation of reference frame or the relationship between their reference frames. They would only be able to do their best by averaging over all possible orientation of other's reference frame to estimate what states or operations others have used.

The assumed sharing of reference frame of directional orientation of which way is up and horizontal for example, between Alice and Bob has unknowingly played a crucial role in the quantum teleportation example as a resource.

Having a shared reference frame (SRF) enabled LOCC quantum teleportation useing entangled pair with certainty of success and lacking that reference frame of the degree of freedom of the system that the information was encoded into, would possibly jeopardise the success of the procedure. This share reference frame is acting as a resource to enable perhaps more efficient and effective quantum information process.

As the entanglement needed to be treated in quantum mechanics to apply to quantum information theory [**3**], reference frames have choice in being treated as a classical objects like measurement are treated as classical operation, or being incorporated into the quantum formalism. The analogous of this is the potential wall in the Hamiltonian of quantum object. If the potentail wall, modeled as infinite parallel potentail walls, is treated as external and classcial background to the quantum process, it breaks the translational symetry of the solution in the perpendicular direction to the direction of the potential wall. Possibly as with in quantum field theory, where symetry breaking is associated with appearnace of scalars and mass terms, this kind of symetry breaking by external system or specific reference frame choice might lead to explaination of some deep quantum theoretical issue. If the RFs that are parametrising the position space of the quantum states are taken as part of the system and described in quantum formalism, then the solution would remain translational invariant and would not show previous physical manifestation as such as superselection rule [**3**]. Also the treatment of RFs as not just a classical object but as quantised part of the quantum formalism induces 'back action' of the reference frame as a quantum system to other larger systems. These effects could manifest as deformed special relativity [**8**]. Here the implied relevence of these consideration of reference frames to deformed special relativity and possibly to quantum gravity could shed more light on nature of quantum information theory and perhaps beyond.

The proposed effect of treating reference frames in quantum formalism and lacking the reference framce for some degrees of freedom of a system is manifestation of superselection rule. superselection rule is a seemingly axiomatised rule incorporated in to the foundation of quantum theory and is a mathematical rule that forbids preparation of quantum states showing coherence between different eigenstates of certain observables [**7**]. This superselection rule (SSR) gives further restriction to the quantum theory further than the selection rule (conservation rule). This mathematical rule was enforced to explain apparant decoherence between certain types of observable's eigenstates.

The early examples included not being able to prepare states with coherence between different eigenstates of electric charge [**3**]. However this treatment of RFs as resources, quantising the RFs and using relational encoding to do quantum information, has revealed many new insights. One of them being ability to violate certain superselection rules that were observed til rescently. Yet the examples of violation of superselection rule does not lift its necessity completely from the foundational explanation or axioms.

We show correlation between reference frames and superselection rule in the following section.

1. Reference Frames and Superselection Rule

Following proposal of correlation between reference frame theory and superselection rule and further discussions are heavily depedent on the review by Bartlett, Rudolph and Spekkens (2007) [**3**].

The superselection rule (SSR) was a notion introduced by Wick, Wightman and Wigner [**15**]. Mathematically, states $|\psi_1\rangle$ and $|\psi_2\rangle$ are separated by a superselection rule if $\langle\psi_1|A|\psi_2\rangle = 0$ for all physically realisable observables A. What is meant by separated by a selection rule is that coherence between the two states are not allowed if they are separated. The above equation implies that there are no possible measurement which can determine relative phase between the two states and verification or even preparation of such coherence is prohibited. This restriction of non mixing between states in different superselection sectors (partition the Hilbert space by the separation) results in full decoherence between such sectors and inhibit the superposition principle in preparation of the states.

In quantum theory, several background objects are still considered as classical objects. There include external potential, or position space of the particles which is parametrised relative to a spatial reference frame. The measuring devices are considered classic objects. So the quantum state of the system naturally relate to some reference frame in the way they are described.

Carrying on this idea, we consider a Hilbert space $\mathcal{H}$ and a quantum state $|\Psi_0\rangle$ relative to a reference frame. Then consider the active and passive transformation that changes the relation between the state in quantum system and the reference frame. This active and passive transformation can both be represented by unitary operator, $T(g)$, where g denotes the transformation so that initial state $|\Psi_0\rangle$ is transformed into $T(g)|\Psi_0\rangle$ in the new relationship between the state and the reference frame (either the state is actively transformed or the reference frame is passively transformed both leading to unitary transformation). This unitary operator shows properties of composition, associativity, existence of inverse (assumed unique here although non unique case can be considered with some complication) [**3**] and including identity, it is viewed as a group $g \in G$. g denotes an abstract transformation and T is the unitary represtation of the group G realised on this quantum system.

Then for example of simple case of group transformation as $U(1)$, we consider lack of phase reference and photon-number superselection rule [**3**]. Alice with her own phase reference, describes K optical modes as

Fock state basis for Hilbert space $\mathcal{H}^{(K)}$ as $|n_1, \cdots, n_K\rangle$ with n_i the number of photons in the mode i and $\hat{N}_i$ the number operator for the corresponding mode. Charlie, another party, then has a different phase reference that differ by angle ϕ. Then by applying Hamiltonian expressed in terms of the total number operator $\hat{N}_{tot} \equiv \sum_{i=1}^{K} \hat{N}_i$ and therefore undergoing active transformation of unitary $U(\phi)) = \exp(i\phi\hat{N}_{tot})$ actively advances her system by an angle ϕ. Then to Charlie, can achieve the same transformation with passive transformation of ϕ using $U \in U(1)$ on K modes given by the same unitary $U(\phi)$, he can represent Alice's state, $|\psi\rangle$ relative to Charlie's own phase referennce, as

$$U(\phi)|\psi\rangle = \exp(i\phi\hat{N}_{tot})|\psi\rangle.$$

For simple verification, let Alice prepare the single mode coherent state [**3**]

$$|\alpha\rangle \equiv \sum_{n=0}^{\infty} c_n|n\rangle, c_n \equiv \exp(\frac{-|\alpha|^2}{2})\frac{\alpha^n}{\sqrt{n!}},$$

with α complex. From the contribution of coefficients of $|n\rangle$, α^n, this state has phase $\arg(\alpha)$ relative to Alice's phase reference. Then in terms of Charlie's phase reference, the same state would be just shifted by ϕ in phase by passive transformation. So it would be $|\exp(i\phi)\alpha\rangle$. Since this is single mode state, the eigenvalue to the total number operator would be 1. Therefore $\exp(i\phi\hat{N})|\alpha\rangle = \exp(i\phi)|\alpha\rangle = |\exp(i\phi)\alpha\rangle$. This agrees with the result.

Another example of Alice preparing the Bell state, $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. This is eigenstate of $\hat{N}_{tot}$ since,

$$\hat{N}_{tot}\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\left(\hat{N}_{tot}|01\rangle - \hat{N}_{tot}|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(1|01\rangle - 1|10\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

with eigenvalue 1.

So when transformtion $U(\phi)$ is applied,

$$U(\phi)|\Psi^-\rangle = \exp(i\phi\hat{N}_{tot})|\Psi^-\rangle$$

$$= \exp(i\phi 1)|\Psi^-\rangle$$

$$= \exp(i\phi)|\Psi^-\rangle$$

So only overall phase has been changed, which doesn't affect physics of the state and therefore is not observed. Charlie then also represents this state with respect to his own phase reference as $|\Psi^-\rangle$ and this Bell state is unchanged. This is an example of an invariant state with respect to lacking phase reference and therefore defined as maximal entangled Bell state independently of phase refences. This is important point in this example, as we have identified, under lack of phase reference, that there is still some maximally entangled Bell state which can be used universally and therefore enable use of entanglement as resources in information processing tasks and also to exploit non-locality and use of LOCC by using $|\Psi^-\rangle$ and $|\Psi^+\rangle$, as this is also an eigenstate of $\hat{N}_{tot}$ with eigenvalue 1, as two orthogonal basis for measurement. So for example, quantum teleportation would still be possible without shared phase reference.

Also we decompose the Hilbert space into direct sum of subspaces in following way.

Let the Hilbert space $\mathcal{H}^{(K)}$ of K modes decompose into

$$\mathcal{H}^{(K)} = \oplus_{n=0}^{\infty} \mathcal{H}_n$$

with $\mathcal{H}_n$ subspace consisting of total n photons in K mode and n eigenvalue of $\hat{(N)}_{tot}$.

Then any state $|\psi_n\rangle \in \mathcal{H}_n$ transforms as

$$U(\phi)|\psi_n\rangle = \exp(i\phi \hat{N}_{tot})|\psi_n\rangle = \exp(in\phi)|\psi_n\rangle$$

with U representation of $U(1)$, group of transformation or reference frame corresponding to lack of phase reference.

Then any $|\psi\rangle \in \mathcal{H}^{(K)}$ transforms as

$$U(\phi)|\psi\rangle = \sum_n \exp(in\phi) \prod_n |\psi\rangle$$

with $\prod_n$ projector onto $\mathcal{H}_n$.

Until now Charlie had knowledge about relative reference difference. Now assume he has no idea about Alice's phase reference and tries to guess Alice's state as best as possible.

Then to Charlie, averaging over all possible values of ϕ angle for relating his reference to Alice's phase reference, he obtains mixed state, (Nielsen and Chuang 2000 for more information on operators, super-operator or quantum operator can be found) [**9**].

$$\mathcal{U}[|\psi\rangle\langle\psi|] \equiv \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi)|\psi\rangle\langle\psi|U(\phi)^\dagger$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{n,n'} \exp(in\phi) \prod_n |\psi\rangle\langle\psi| \prod_{n'} \exp(-in'\phi)$$

$$= \sum_{n,n'} \prod_{n} |\psi\rangle\langle\psi| \prod_{n'} \left(\int_0^{2\pi} \frac{d\phi}{2\pi} \exp(i(n-n')\phi) \right)$$

$$= \sum_{n,n'} \prod_{n} |\psi\rangle\langle\psi| \prod_{n'} \delta_{n,n'}$$

$$= \sum_{n} \prod_{n} |\psi\rangle\langle\psi| \prod_{n}$$

via substituting previous equation and using Kronecker delta for the integral.

Thus since this applies to any state, following also holds, applied to density matrix representation.

$$\mathcal{U}[\rho] = \sum_{n} \prod_{n} \rho \prod_{n}$$

So the map $\mathcal{U}$ removes all coherence between difference eigenstates of total photon number operator and also the map commutes with $U(\phi)$ for all ϕ as, for arbitary ϕ'

$$U(\phi')\mathcal{U}[\rho]U(\phi')^{-1} = U(\phi')\mathcal{U}[\rho]U(\phi')^{\dagger}$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi')U(\phi)|\psi\rangle\langle\psi|U(\phi)^{\dagger}U(\phi')^{\dagger}$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi'+\phi)|\psi\rangle\langle\psi|U(\phi'+\phi)^{\dagger}$$

$$= \int_0^{2\pi} \frac{d\phi''}{2\pi} U(\phi'')|\psi\rangle\langle\psi|U(\phi'')^{\dagger}$$

$$= \mathcal{U}[\rho]$$

Therefore $U(\phi')\mathcal{U}[\rho]U(\phi')^{-1} = \mathcal{U}[\rho]$ and so $U(\phi')\mathcal{U}[\rho] = \mathcal{U}[\rho]U(\phi')$ and then $[\mathcal{U}[\rho], U(\phi')] = 0$, for all ϕ.

Then to Charlie there is a restriction of states prepared by Alice, described relative to his reference frame. Charlie will always see the states prepared by Alice as block diagonal states in terms of toal photon number, according to the quantum operation $\mathcal{U}$. Also from the above commutation relation, it follows that $\mathcal{U}[\rho]$ is invariant under transformation in phase.

Also consideration of how the operations Alice performs on her states is described in Charlie's reference frame leads to following results. Assuming the angle ϕ is known which relates their phase references, σ is the description of the state in Charlie's reference frame and then unitary operation V is applied to the state by Alice. Then

$$U(\phi)VU(\phi)^{\dagger}\sigma U(\phi)V^{\dagger}U(\phi)^{\dagger}$$

is what the operation applied to the state would look like to Charlie. So to Charlie, this can be simplified to unitary $V_{\phi} = U(\phi)VU(\phi)^{\dagger}$. If the phase relationship is unknown and therefore ϕ is unknown, then Charlie would average out all the possibility of the relationship and get this map,

$$\tilde{\mathcal{V}}[\sigma] \equiv \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi)VU(\phi)^{\dagger}\sigma U(\phi)V^{\dagger}U(\phi)^{\dagger}$$

Then if the state was prepared by Alice and therefore $\sigma = \mathcal{U}[\rho]$ in terms of Charlie's reference frame and the map becomes,

$$\tilde{\mathcal{V}}[\sigma] = \int_0^{2\pi} \frac{d\phi}{2\pi} U(\phi)V\sigma V^{\dagger}U(\phi)^{\dagger}$$

using commutation relationship of $\mathcal{U}[\rho]$ and $U(\phi)$,

$$= \mathcal{U}[V\sigma V^{\dagger}]$$

This implies that $\tilde{\mathcal{V}}[\sigma]$ is also descirbed as block diagonal in total photon number as before and there is also restriction on what operations Alice can perform if viewed relative to Charlie's phase reference.

Now we can make the claim that this treatment of lack of reference frame is equivalent to superselection rule. As shown before this restriction of preparing states and possible operations to perform was what was described as superselection rule. Thus by lacking knowledge of reference frames and this leading to mixed state and mixture of operations in Charlie's view point, lead to forced decoherence on prepared states and possible operation Alice can perform.

This simplified example gave the glimpse of this insight of lack of reference frame showing up as superselection rule for the appropriate group of transformations. Later it will be discussed that it can be shown that in general this relationship holds true and perhaps is one of fundamental property of quantum theory.

The similar kind of treatment for the spatial reference also exhibits different kind of superselection rule emerging as lack of suitable reference frame. This can be found from [**3**] for which we have been following in this discussion of reference frames.

So if we are able to incorporate reference frame in to quantum system as quantised quantum object, then it may be possible to eradicate superselection rule phenomenon by careful choice of reference frame and question the necessity of the axiomtic restriction of superselection rule and perhaps view it as just another lack of information on the system as noise, with analogue to entanglement with environment with no access showing up as noise in quantum information communication.

2. Reference Frames and Decoherence free subsystem

We view briefly the implication of the general treatment of reference frame into quantum formalism. Using notions of superoperator and Positive Operator Valued Measure (POVM) further results can be derived.

As with the previous example, consider $g \in G$, a group element of transformations which relates Charlie's reference frame to Alice's reference frame. Assume finite or compact continuous Lie groups that has group invariant Haar measure and act on $\mathcal{H}$ as unitary representation T so that they are completely reducible [**3**].

With g completely unknown, Alice's prepared state ρ on $\mathcal{H}$ in her reference frame would take the following form in Charlie's frame

$$\tilde{\rho} = \int_G dgT(g)\rho T(g)^\dagger \equiv \mathcal{G}[\rho]$$

where $T(g)$ is a unitary representation of g on $\mathcal{H}$ and dg the group invariant Haar measure. So all states prepared by Alice in reference frame of Charlie are of the form $\tilde{\rho} = \mathcal{G}[\rho]$ and as was with the previous section, $\tilde{\rho}$ is G invariant as

$$[\tilde{\rho}, T(g)] = 0, \forall g \in G$$

Then define superoperator $\mathcal{T}(g)[\rho] = T(g)\rho T(g)^\dagger$, unitary representation of G on $\mathcal{B}(\mathcal{H})$. Then $\mathcal{G} = \int_G dg\mathcal{T}(g)$. Also transformations are represented by completely positivity preserving superoperator $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ [**3**] [**9**]. Then the superoperator in Charlie's frame is

$$\tilde{\mathcal{E}}[\rho] = \int_G dgT(g)\mathcal{E}[T(g)^\dagger \rho T(g)]T(g)^\dagger$$

and

$$\tilde{\mathcal{E}} = \int_G dg \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}(g^{-1})$$

Also considering POVM $\{\tilde{E}_k\}$, POVMs in Charlie's frame takes the form,

$$\tilde{E}_k = \mathcal{G}[E_k]$$

and this POVM is G invariant.

With these form of states, superoperator and the POVMs in Charlie's frame, it can now be shown that these restriction on Alice's prepared states correspond to the superselection rule.

The Hilbert space $\mathcal{H} = \oplus_q \mathcal{H}_q$ decomposes in to charge sectors $\mathcal{H}_q$ labeled by q. Then as each charge sector carries an inequivalent representation T_q of G,

$$\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q$$

decomposes further into tensor product of irreducible representation $\mathcal{M}_q$ of group of transformation and subsystem $\mathcal{N}_q$ carrying a trivial representation of the group. Then the irrep space is identified as decoherence full subsystem and the trivial rep space as decoherence free subsystems [**3**].

Then the following theorem about how the map $\mathcal{G}$ can be decomposed to simple form [**3**], establishes important step to the generalisation of lack of reference frame corresponding to superselection rule.

Theorem 2.1. *The action of $\mathcal{G}$ in terms of the decomposition*

$$\mathcal{H} = \oplus_q \mathcal{M}_q \otimes \mathcal{N}_q$$

is given by

$$\mathcal{G} = \sum_q \left(\mathcal{D}_{\mathcal{M}_q} \otimes \mathcal{I}_{\mathcal{N}_q}\right) \circ \mathcal{P}_q$$

where $\mathcal{P}_q$ *is the superoperator associated with projection into the charge sector q, that is,* $\mathcal{P}_q[\rho] = \prod_q \rho \prod_q$ *with* $\prod_q$ *the projection on to* $\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q$ *,* $\mathcal{D}_{\mathcal{M}}$ *denotes the trace-preserving operation that takes every operator on the Hilbert space* $\mathcal{M}$ *to a constant times the identity operator on that space, and* $\mathcal{I}_{\mathcal{N}}$ *denotes the identity map over operators in the space* $\mathcal{N}$.

This theorem is proved with aid of Schur's Lemmas presented in the beginning of the introduction. The representatoin $T(g) = \oplus_{q,\lambda} T_{q,\lambda}(g)$ for the group of transformation G is decomposed into a sum of irreducible representations.

$$\mathcal{G}[A] = \oplus_{q,q',\lambda,\lambda'} \int dg T_{q,\lambda}(g) A T^{\dagger}_{q',\lambda'}(g)$$

Then due to invariance of the measure dg,

$$T_{q,\lambda}(g) A_{q,q',\lambda,\lambda'} T^{\dagger}_{q',\lambda'}(g) = A_{q,q',\lambda,\lambda'}, \forall g \in G$$

with $A_{q,q',\lambda,\lambda'} = \int dg T_{q,\lambda}(g) A T^{\dagger}_{q',\lambda'}(g)$.

This with Schur's second lemma becomes

$$= \oplus_{q,\lambda,\lambda'} \int dg T_{q,\lambda}(g) A T^{\dagger}_{q,\lambda'}(g)$$

Then using projection of $\mathcal{H}$ onto the carrier space of $T_{q,\lambda}$, yields

$$\mathcal{G} = \sum_q \mathcal{G}_q \circ \mathcal{P}_q$$

Then considering decomposing charge sector of the Hilbert space into decoference full and decoherence free subsystems, projection operators into each subsystems yield

$$\mathcal{G}[A] = \sum_q \left(\mathcal{G}_{\mathcal{M}_q} \otimes \mathcal{I}_{\mathcal{N}_q}\right) \circ \mathcal{P}_q[A]$$

Then by Schurs first lemma, $\mathcal{G}_{\mathcal{M}_q}[B]$ act as multiple of identity on $\mathcal{M}_q$. Then from trace preserving property, $\mathcal{G}_{\mathcal{M}_q} = \mathcal{D}_{\mathcal{M}_q}$, the trace preserving map which takes every operator on decoherence full subsystem $\mathcal{M}_q$ to a constant times the identity on this subsystem[**3**].

Then the restriction for states of $\tilde{\rho}$ commuting with unitary representation of transformation $T(g)$, restriction for superoperators of $\tilde{\mathcal{E}}$ commuting with unitary representation of transformation $\mathcal{T}(g)$ of G on $\mathcal{B}(\mathcal{H})$, the set of all bounded operators on $\mathcal{H}$ and restriction for POVMs of $\tilde{E}_k$ commuting with $T(g)$, these three restriction of commutation relations of states, superoperators and POVMs with unitary representation of G are formally equivalent to the restrictions imposed by superselection rule associated with the group G in quantum information theory.

3. Further on Reference Frames

From the previous section, we have demonstrated that lacking a reference frame completely puts restrictions on the quantum theory, which is formally equivalent to the axiomatic superselection rule. So one can aks the question of whether the theory of reference frame puts forward any evidence of rejection of superselection rule as a axiomatic rule. This is an interesting area of study with far reaching implications in quantum theory and formalism of quantum theory. There is an indication that although this treatment of reference frame gives some insight to manifestation of superselection rule, there is not enough evidence for rejecting superselection as a axiomatic rule yet [**3**].

Also as seen from the theorem proved in previous section, the decomposition of Hilbert space and the action of the transformation on the Hilbert space gives some insight to possible usage of the subsystems for doing quantum information processing tasks. Using standard tool of using decoherence free subsystem to encode information and to do communication, the decoherence free subsytem of the decomposition we saw earlier can be used to encode and do secure quantum communications [**3**]. It is interesting that with this scheme of doing quantum information with lack of reference frames, one transmitted qubit can not send any information as, to Bob, every state prepared by Alice will look like $\mathcal{E}_1[\rho] = \frac{1}{2}\mathcal{I}$ as the completely mixed state [**3**].

Then by consideration of $SU(2)$, two transmitted qubits and a classical channel yields communication of one classical bit, via appropriate choice of orthogonal measurement basis to utilise the decoherence free subsystem to encode information. For the case of three trasmitted qubits and a quantum channel, one logical qubit can be communicated via choosing appropriate protocol [**3**].

These example illustrate the viewpoint of reference frames as resources in quantum information theory. They do point out that information processing communication is still possible without shared reference frame. Although, as we have seen, lacking a shared reference frame forces the information processing task to use up more expensive resources as such as quantum channels and multiple transmitted qubits for achieving less satisfactory rate of information exchange. However another suprising result is that asymptotically, the number of logical qubits being able to be encoded in number of physcal transmitted qubits approach one to one ratio and therefore recover the efficiency as if there was shared reference present. Obviously, increasing the dimensionality of the number of qubits to transfer ensures that decoherence free subsystem gets larger in dimension and this in turn increases the effectiveness of the information transfer [**3**].

The intersting questions of whether entanglement is distillable without shared reference frame is one of the quetions that consider both the entanglement and reference frames as the resources of information theory. The answer to this question is fortunately yes [**3**] and perhaps this is suggestion of whether they are interconvertable as resources and this direction of research will be very fruitful in the future.

CHAPTER 4

Conclusion

From results seen so far, it is clear that there is much work to be done in first of all general quantum unspeakable information theory. These include find measure for quantifying entanglement and reference frames. We have seen few examples of such a measure for entanglement in special cases, but for reference frames it is wide open problem. Perhaps relative entropy of frameness as a measure for the reference frame might shed some insight on the subject [**10**].

Theory of investigating quantum communication without a shared reference frames yielded suprising results that you can perform quantum teleportation and send classical and quantum information and also communicate unspeakable information to certain degree of confidence [**3**]. On one qubit of transmitted information, it is strange that reciever can not extract any information out of it when there is lack of reference frame as all the states are indistinguishable from the maximally mixed state, but of course now we have seen shared reference frame is resource and therefore lacking a shared referece frame require additional other resources for information process. If there are two transmitted qubits and a classical channel, one classcial bit can be transfered. With three transmitted qubits and a quantum channel, it can be shown that [**3**] by decomposing into irreducible representation and using orthogonal basis of the system, decoherence free subsystem can be used as resources to transfer one logical qubit [**3**]. The realisation of lack of reference frame as a noise of the quantum system lead to utilise known methods of combating these noises as such as decoherence free subsystems.

There are so many questions left to ask in terms of reference frame in quantum theory. As suggested, this may give some insight to some problem of quantum gravity [**8**] via considering quantum reference frames and its 'back-action' to produce effects of deformed special relativity. The problem of distillation of entanglement through shared or privately shared reference frame [**3**] and using private shared reference frame to utilise in quantum key distribution [**3**] are few of many questions to the quantum information theory.

The connection of entanglement and reference frames as resources has not been in perfect correlation in terms of their properties and how they are implemented quantum theoretically. They do not seem interconvertible in terms of resources [**3**] in the way they are deplete, distilled and implemented. However they both share an important quality that the questions either one presents and answers seem to have huge impact on the other in terms of asking the same questions or finding if there is analogous answer to each other.

Perhaps considering different reference frames for multiple party correspondence will give easier and more profound insight on the quantum theory and combined with the resources of reference frames harnessed to create entanglement and vice versa, therefore giving two different aspect or view to the variety of questions of quantum information theory can offer.

CHAPTER 5

Appendix

1. Schur's Lemmas

The two important results from the representation theory, namely, Schur's first lemma and Schur's second lemma, are concerned with properties of matrix representation which commutes with irreducible representations, where first lemma deals with commutation with all matrices of a given irreducible representation and second lemma dealing with communtation with distinct irreducible representations. These results are important building blocks in formulating the Great orthogonality theorem which is central in representation theory in terms of identifying irreducible representations and in constructing character tables which is useful in application to physics. It is interesting to note that these two lemmas from representation theory were evoked to give proof of theorem 2.1, decomposition of the action of $\mathcal{G}$, transformation of state ρ between two different uncorrelated reference frames, into simple forms acting on subsystems of charge sectors of Hilbert space $\mathcal{H}$, thereby showing lack of knowledge of reference frames corresponds to emergence of superselection rule associated with the group G, the group of transformation of reference frames. By formulating the treatment of reference frames in group theoretic and representation theoretic sense, with additional restrictions and assumptions, relatively basic properties of such theories were needed to highlight this correlation of reference frame and superselection rule. This might suggest that considering reference frames in this manner readily gives profound insight to the foundational problems of quantum theory as such as superselection rule. Also it might be suggested that consideration of reference frames in quantum theory as fundamental and elementary property is essential for effective and powerful theory.

The proof of the two lemmas are relatively elementary and only basic knowledge of representation theory is needed. Therefore the proof is apologetically omitted and can be found in any representation theory textbook.

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