Geometric Quantization of Hamiltonian Mechanics

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Abstract

We describe the symplectic geometry underlying the geometrical formulation of classical mechanics and how this is used to formulate the problem of quantization in a precise way. We describe prequantization and the use of real polarizations, and explain why a further construction is needed to give the correct Hilbert space.

Contents

1 Introduction

In the early part of the twentieth century Quantum Mechanics was discovered and it soon became clear that this new theory could provide explanaitions for many phenomena unnacounted for by classical physics, and also make extremlely accurate predictions of the outcomes of experiments testing the behaviour of matter at small scales. However, due to the abstract nature of the theory, quantum theories were often arrived at by starting with a classical theory and then "quantizing" it. Since different versions of quantum mechanics existed - those of Heisenberg and Schrodinger among others, it was not clear how well defined the idea of quantisation was.

Later, Dirac put quantum mechanics on a more rigourous, axiomatic footing and showed that the different quantum mechanics were just different representations of the same underlying theory. He also described the similarities between classical and quantum mechanics, in particular the similarity between poisson brackets and commutators. With these ideas, Dirac gave a set of rules that the quantisation of a classical theory should satisfy, which we give in section 2. This led to canonical quantisation, which has been used successfuly in many different situations from quantum field theory to condensed matter physics.

However, thare are several issues with canonical quantisation, mostly mathematical but also some philosophical - its not clear to what extent the quantisation of a particular classical system is unique. Geometric Quantization is an attempt at making quantization a rigorous mathematical process. The aim of this work is to describe the framework in which geometric quantisation works, and the basics of how it works as applied to non relativistic Hamiltonian mechanics.

The main sources used were [1, 2, 3, 4, 8, 11] in roughly equal amounts. However, since it is the classic reference on the subject I have mostly defined things as in [11] . I have tried throughout to include details of derivations that are not present in the texts, and also to give some original examples and proofs in order to demonstrate my understanding of the material.

2 From Classical to Quantum Mechanics

2.1 Dirac's rules

Any quantization approach has the same aim - to associate self adjoint (or symmetric) operators $\widehat{f}$ (which represent quantum observables) acting on some a Hilbert space $\mathcal{H}$ (which is the quantum phase space) to each classical observable f , that is to each function of the positions and momenta of the particles in the system. The rules given by Dirac are:[5]

1. The map $f \rightarrow \widehat{f}$ is linear over $\mathbb{R}$
2. The constant function $f(p,q) = 1$ is mapped to the multiplication operator 1
3. The operators $\widehat{f}$ are symmetric
4. If $\{f,g\} = h$ then $\left[\widehat{f},\widehat{g}\right] = \widehat{f}\widehat{g} - \widehat{g}\widehat{f} = -i\hbar\widehat{h}$

Where $\{f,g\}$ is the Poisson bracket of f with g .

2.2 Canonical Quantization

If, in generalized coordinates there are n momenta p_a and n positions q^a , canonical quantization works by assigning:

$$\begin{aligned} \widehat{q}^a &= q^a \\ \widehat{p}_a &= -i\hbar \partial/\partial q^a \end{aligned}$$

Other observables can then be expressed as polynomials in p_a and q^a and quantized using the rules above. This satisfies Dirac's rules but has several problems.

Firstly, canonical quantization does not determine the Hilbert space - on what space do the operators act? Experiments have shown that in the case of a single particle in 3 dimensions the “correct” Hilbert space is $L^2(\mathbb{R}^3)$ -

the space of square integrable functions complex functions on $\mathbb{R}^3$, with the inner product:

$$\langle \psi, \phi \rangle = \psi^* \phi d^3 x$$

In general the answer to this question is not clear.

Secondly, the choice of operator ordering is left undetermined since at the classical level the observables commute, e.g.

$$\begin{aligned} \widehat{(p_a q^b)} &= -i\hbar \left(q^b \partial / \partial q^a + \delta^a_b \right) \\ \widehat{(q^b p_a)} &= -i\hbar q^b \partial / \partial q^a \end{aligned}$$

This problem can be fixed by agreeing on a convention, e.g. put all momentum operators on the right, but this seems very unnatural.

Thirdly, why choose the canonical choices for $\widehat{p}_a$ and $\widehat{q}^a$? They satisfy the rules, but to what extent are they unique choice?

Finally, classical Hamiltonian mechanics is invariant under canonical transformations, but its clear that canonical quantization fixes a particular set of p's and q's and so breaks this.

2.3 Geometrical Quantization

In geometric quantization all the elements of the quantization - the Hilbert space, its inner product and the operators are constructed in a natural, coordinate independent way from the classical theory. We'll see that it successfully satisfies the Dirac's rules, but unfortunately without modification may give the wrong theory. It turns out that the quantization rules are not restrictive enough - they don't determine the quantum theory exactly.

3 Geometry and Mechanics

3.1 Symplectic Vector Spaces

3.1.1 Definition

A *Symplectic vector space* is a pair (V, ω) where V is a vector space over some field $\mathbb{F}$ (which here will always be $\mathbb{R}$ or $\mathbb{C}$) and ω is an antisymmetric, non degenerate bilinear form on V , i.e.

$$\begin{aligned} \omega : V \times V &\rightarrow \mathbb{F} \\ \omega(X, Y) &= -\omega(Y, X) \\ \omega(\alpha X + \beta Y, Z) &= \alpha\omega(X, Y) + \beta\omega(X, Z) \end{aligned}$$

$\forall X, Y, Z \in V$ and $\forall \alpha, \beta \in \mathbb{F}$. [11] If V is finite dimensional then the following two notions of non-degeneracy are equivalent:

$$\begin{aligned} \omega : V &\rightarrow V^* \\ X &\mapsto \omega(X, \bullet) \end{aligned}$$

is an isomorphism, or that

$$det(\omega_{ij}) \neq 0$$

where (ω_{ij}) is the matrix representation of ω with respect to some basis of V .

An antisymmetric matrix is always similar to a matrix of one of the forms:

$$\begin{pmatrix} 0 & -1_{n\times n} & 0 \\ 1_{n\times n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1_{n\times n} \\ 1_{n\times n} & 0 \end{pmatrix}$$

depending on whether it is $2n+1$ or $2n$ dimensional respectively ($-1_{n\times n}$ is the $n \times n$ identity matrix). Therefore, the non-degeneracy condition implies that any symplectic vector space is even dimensional - since the first matrix has zero determinant.

3.1.2 Examples

1. For example, let V be an 2n-dimensional real vector space with some basis $\{e_i \mid i = 1, 2, ..., 2n\}$ and ω some bilinear form on V . Then:

$$\begin{aligned} \omega(X, Y) &= X^i Y^j \omega(e_i, e_j) \\ &= X^i Y^j \omega_{ij} \end{aligned}$$

   and (V, ω) is a symplectic vector space if (ω_{ij}) is antisymmetric and non-degenerate.

2. The canonical example of a symplectic manifold will be a cotangent bundle, T^*Q , for some manifold Q . In this case, the tangent space T_mQ at some point $m \in Q$ has the following structure: Let W be some n-dimensional real vector space and let $V = W^* \oplus W$. We can make V into a symplectic vector space:

$$\begin{aligned} \omega(X, Y) &= \omega((p, q), (p\prime, q\prime)) \\ &\equiv \tfrac{1}{2}(p(q\prime) - p\prime(q)) \end{aligned} \tag{1}$$

where $X = (p, q)$ and $Y = (p\prime, q\prime)$ for some $p, p\prime \in W^*$ and $q, q\prime \in W$. [11]

3.1.3 Frames

It is very use full to use a particular kind of basis, $\{e^i, f_j \mid i, j = 1, ..., n\}$, to span some 2n dimensional (V, ω) , called a *symplectic frame*, if it satisfies:

$$\begin{aligned} \omega(e^i, e^j) &= 0 \\ \omega(f_i, f_j) &= 0 \\ \omega(e^i, f_j) &= \tfrac{1}{2}\delta^i_j \end{aligned}$$

This always exists, although it is not unique. [11] If $W \equiv span\{e^i\}$ and $S \equiv span\{f_j\}$ then $V \equiv S \oplus W$ and we can use ω to map S to W^* isomorphically:

$$\begin{aligned} \omega : S &\rightarrow W^* \\ X &\mapsto 2\omega(X, \bullet) \end{aligned}$$

We see that all symplectic vector spaces (V, ω) can be described in the canonical way as $V = W^* \oplus W$ with ω as in equation (1) .

3.1.4 Canonical Transformations and orientation

A linear map from a symplectic vector space to itself is called a *canonical transformation* if it preserves the symplectic structure, i.e. :

$$\begin{aligned} \rho : V &\rightarrow V \\ \omega(\rho X, \rho Y) &= \omega(X, Y) \end{aligned}$$

$\forall X, Y \in V$. [11] The maps ρ have unit determinant so V has a natural orientation.

3.2 Symplectic Manifolds

3.2.1 Definition

A *symplectic manifold* is a pair $(\mathcal{M}, \omega)$, where $\mathcal{M}$ is a smooth manifold and ω is a closed, non-degenerate 2-form globally defined on $\mathcal{M}$, [11] i.e.

$$\begin{array}{rcl} \omega & \in & \Omega^2(\mathcal{M}) \\ d\omega & = & 0 \end{array}$$

As for symplectic vector spaces, if $\mathcal{M}$ is finite dimensional there are two equivalent notions of non-degeneracy:

$$\begin{array}{rcl} \omega : T_m\mathcal{M} & \to & T_m^*\mathcal{M} \\ X & \mapsto & \omega_m(X, \ \bullet \ ) \\ & = & i_X\omega \end{array}$$

is an isomorphism $\forall m \in \mathcal{M}$, where ω_m is ω restricted to some point $m \in \mathcal{M}$ and $i_X\omega$ is the interior product of X with ω , or that

$$det\left(\omega_{m\ ij}\right) \neq 0$$

$\forall m \in \mathcal{M}$, where $(\omega_{m\ ij})$ are the components of ω_m .

The tangent spaces of $\mathcal{M}$ become symplectic vector spaces $(T_m\mathcal{M}, \omega_m)$ and so symplectic manifolds are always even dimensional.

For example, a real symplectic vector space (V, ω) can be made into a real symplectic manifold. Let V be 2n dimensional and $\{e^i, f_j \mid i, j = 1, ..., n\}$ be a symplectic frame. Then if $X \in V$, $X = \widetilde{X}^i f_i + X_i e^i$ for some scalars $X = \widetilde{X}^i, X_j \in \mathbb{R}$. Therefore, $\left(\widetilde{X}^i, X_j \mid -\infty < \widetilde{X}^i, X_j < +\infty\right)$ are global coordinates on V , and V as a manifold is diffeomorphic to $\mathbb{R}^{2n}$ as a manifold. The symplectic form on V is then $d\widetilde{X}^i \wedge dX_i$.

3.2.2 Cotangent bundles

Any Cotangent Bundle can be made into a symplectic manifold by choosing a certain 2 form constructed from the bundle structure. If Q is some real n-dimensional manifold then T^*Q , its cotangent bundle, is a 2n-dimensional manifold. It is a fibre bundle over Q where the fibre at $q \in Q$ is T_q^*Q , the cotangent space at Q , so T^*Q is the set:

$$T^*Q = \left\{(p,q) \mid q \in Q \ , \ p \in T_q^*Q\right\}$$

If (q^a) are some local coordinates on some open set in Q , then $\left(p_a, q^b\right)$ are local coordinates on T^*Q , where $p = p_a dq^a$ so that (p_a) are the components of p relative to the dual basis dq^a of the coordinate basis ${}^{\partial}\!/_{\partial q^a}$. [3, 2]

A 1-form, called the *canonical 1-form*, can be constructed in a coordinate independent way. Let π be the projection from the bundle T^*Q down to its base space Q :

$$\begin{aligned} \pi : T^*Q &\rightarrow Q \\ (p,q) &\mapsto q \end{aligned}$$

For each point $m = (p,q) \in T^*Q$ we can define a 1-form $\theta_m \in T_m^*(T^*Q)$ in the following way:

$$\theta_m(X) = p(\pi_* X) \tag{2}$$

where $X \in T_m(T^*Q)$ and π_* is the push forward of π :

$$\begin{aligned} \pi_* : T_m(T^*Q) &\rightarrow T_{\pi(m)}Q \\ X &\mapsto (\pi_* X) \end{aligned} \tag{3}$$

The coordinate expression of (3) is:

$$\begin{aligned} \widetilde{X}^a {}^{\partial}\!/_{\partial p^a} + X^a {}^{\partial}\!/_{\partial q^a} &\mapsto {}^{\partial q^a(\pi(m))}\!/_{\partial p_b} \widetilde{X}^b {}^{\partial}\!/_{\partial q^a} + {}^{\partial q^a(\pi(m))}\!/_{\partial q^b} X^b {}^{\partial}\!/_{\partial q^a} \\ &= 0 + \delta_b^a X^b {}^{\partial}\!/_{\partial q^a} \\ &= X^a {}^{\partial}\!/_{\partial q^a} \end{aligned}$$

The coordinate expression of (2) is then:

$$\begin{aligned} \left(\widetilde{\theta}_{ma} dp_a + \theta_{ma} dq^a\right)\left(\widetilde{X}^b {}^{\partial}\!/_{\partial p^b} + X^b {}^{\partial}\!/_{\partial q^b}\right) &= p_a dq^a \left(X^b {}^{\partial}\!/_{\partial q^b}\right) \\ \widetilde{\theta}_{ma} \widetilde{X}^b + \theta_{ma} X^a &= p_a X^a \end{aligned}$$

If this is true $\forall X \in T_m(T^*Q)$ then we find $\widetilde{\theta}_{ma} = 0$ and $\theta_{ma} = p_a$, so that:

$$\theta_m \;=\; p_a dq^a \tag{4}$$

We can then construct a 1-form $\theta \in \Omega^1(T^*Q)$ by setting $\theta(m) \equiv \theta_m$, and θ will be smooth as p_a is a smooth function on T^*Q .

The exterior derivative of the canonical one form is ω , the *canonical 2-form*:

$$\begin{aligned} \omega &\equiv d\theta \\ &= dp_a \wedge dq^a \end{aligned} \tag{5}$$

Since ω is exact it is closed - $d\omega = d^2\theta = 0$. If we make coordinates:

$$(y^\alpha \mid \alpha = 1, ..., 2n \text{ , } y^\alpha = p_a \text{ for } \alpha = a \text{ , } y^\alpha = q^a \text{ for } \alpha = a + n)$$

then relative to the basis dy^α , ω has components:

$$\omega_{\alpha\beta} \;=\; \begin{pmatrix} 0 & 1_{n\times n} \\ -1_{n\times n} & 0 \end{pmatrix}$$

so ω is non-degenerate and (T^*Q, ω) is a symplectic manifold. We will see in section 4 that cotangent bundles are a natural way to describe the phase space of a classical system.

3.2.3 Symplectic Potentials

Any symplectic manifold $(\mathcal{M}, \omega)$ has a symplectic 2-form ω that is closed, but it will not be exact in general. This will depend on the cohomology of $\mathcal{M}$. However, by Poincare's Lemma ω will be locally exact - there always exits a neighborhood U of any point $m \in \mathcal{M}$ and a 1-form θ_U defined on U such that:

$$\omega \mid_U = d\theta$$

and θ is then a *symplectic potential.*[11] But, θ is only defined up to the addition of a closed 1-form:

$$\begin{aligned} \omega &= d\theta \\ &= d(\theta + \sigma) \text{ if } d\sigma = 0 \end{aligned}$$

This will be important in pre-quantization, see section (5). An interesting result restricts the cohomology of compact symplectic manifolds without boundary. Let $\omega^n \equiv \omega \wedge \omega \wedge ... \wedge \omega$, let $(\mathcal{M}, \omega)$ be a compact symplectic manifold without boundary and *assume* that there exists a globally defined symplectic potential, so that $\omega = d\theta$, then:

$$\begin{aligned}
{}_{\mathcal{M}}\omega^n &= {}_{\mathcal{M}}d\theta \wedge \omega^{n-1} \\
&= {}_{\mathcal{M}}d\theta \wedge \omega^{n-1} + {}_{\mathcal{M}}\theta \wedge d\omega \wedge \omega^{n-2} + {}_{\mathcal{M}}\theta \wedge \omega \wedge d\omega \wedge \omega^{n-3} + ... \\
&= {}_{\mathcal{M}}d\left(\theta \wedge \omega^{n-1}\right) \\
&= {}_{\partial\mathcal{M}}\theta \wedge \omega^{n-1} \\
&= 0
\end{aligned}$$

since $\partial\mathcal{M} = 0$. But, ω^n is a positive volume form so the left hand side is non-vanishing and there is a contradiction. Therefore, compact symplectic manifolds without boundary must have non-trivial second de Rham cohomology groups. (The argument does not extend to non-compact $\mathcal{M}$ as ω is nowhere vanishing so has non-compact support and therefore Stokes's theorem doesn't apply). This result has consequences for the the pre quantization of symplectic manifolds representing internal degrees of freedom.

3.2.4 Canonical coordinates

The natural notion of "sameness" for symplectic manifolds of that of a *canonical diffeomorphism*. Let $(\mathcal{M}_1, \omega_1)$ and $(\mathcal{M}_2, \omega_2)$ be symplectic manifolds, then a diffeomorphism ρ is canonical if:

$$\begin{aligned}
\rho : \mathcal{M}_1 &\rightarrow \mathcal{M}_2 \\
\rho^*\left(\omega_2\right) &= \omega_1
\end{aligned}$$

where ρ^* is the pullback of ρ . [11]

A result known as *Darboux's theorem* shows that all symplectic manifolds are locally the same as a cotangent bundle of the same dimension. More precisely, if $(\mathcal{M}, \omega)$ is a symplectic manifold of dimension $2n$, and $m \in \mathcal{M}$, then there is a neighborhood U of m , and coordinates $\left(p_a, q^b\right)$ on U , where $a, b = 1, ..., n$, such that ω can be written as in (5) :

$$\omega = dp_a \wedge dq^a \text{ on U}$$

and (p_a, q^b) are then *canonical coordinates.* (So on a cotangent bundle T^*Q all coordinates (p_a, q^b) constructed from coordinates (q^a) on Q are canonical). Given any two points $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$ there always exists a local canonical diffeomorphism mapping a neighborhood of m_1 to one of m_2 . The significance of this is twofold. Firstly, in the context of Geometric Quantization this allows us to study the quantization of arbitrary symplectic manifolds in terms of a local structure which is well understood - that of a cotangent bundle. Secondly, from a purely mathematical point of view we see that symplectic manifolds have *no local geometric invariants*, other than their dimension. [3]

Note that this does not mean that locally defined symplectic potentials necessarily have the form of a canonical 1-form $p_a dq^a$, e.g. :

$$\begin{aligned} dp_a \wedge dq^a &= {}^1\!/\!_2 \left(dp_a \wedge dq^a - dq^a \wedge dp_a\right) \\ &= dp_a \wedge dq^a - {}^1\!/\!_2 \left(dp_a \wedge dq^a + dq^a \wedge dp_a\right) \\ &= d\left(p_a dq^a\right) - {}^1\!/\!_2 d\left(p_a dq^a + q^a dp_a\right) \end{aligned}$$

3.2.5 Integration

Taking the $2n$ form $\omega^n = \omega \wedge \omega \wedge ... \wedge \omega$ gives us a volume form on $(\mathcal{M}, \omega)$ since $\mathcal{M}$ has a natural orientation. [2] This will allow us to construct an inner product later.

3.3 Hamiltonian Vector Fields and Flows

The geometrical formulation of classical mechanics relies on using the natural symplectic structure on the phase space of the system to relate functions with vector fields in a coordinate independent way.

3.3.1 Hamiltonian Vector Fields

Let $f \in C^\infty_{\mathcal{M}}$ be a smooth function on a symplectic manifold $(\mathcal{M}, \omega)$. The *Hamiltonian Vector Field* of f is defined by the equation:[4]

$$i_{X_f}(\omega) = -df \tag{6}$$

This equation defines X_f uniquely, as can be seen from its local coordinate form. Let $\left(p_a, q^b\right)$ be local canonical coordinates, then (6) reads:

$$\begin{aligned} \omega\left(X_f, \bullet\right) &= -df \\ dp_a \wedge dq^a \left(\widetilde{X}_f^b \partial/\partial p^b + X_f^b \partial/\partial q^b\right) &= -\partial f/\partial p^a dp^a - \partial f/\partial q^a dq^a \\ \widetilde{X}_f^a dq^a - X_f^a dp^a &= -\partial f/\partial q^a dq^a - \partial f/\partial p^a dp^a \end{aligned}$$

and so X_f can be written locally as:

$$X_f = \partial f/\partial p^a \partial/\partial q^a - \partial f/\partial q^a \partial/\partial p^a \tag{7}$$

A general vector field X is called Hamiltonian if there exists a smooth function f such that $X = X_f$. We call the set of Hamiltonian vector fields $V_H\left(\mathcal{M}\right)$.

3.3.2 Hamiltonian Flows

The symplectic structure also allows us to associate a flow on $\mathcal{M}$ to each smooth function f - we define the *Hamiltonian Flow* of f , denoted ρ_f , to be the flow of X_f . Let γ be an integral curve of X_f :

$$\begin{aligned} \gamma : (a,b) \subset \mathbb{R} &\rightarrow \mathcal{M} \\ \gamma\left(s\right) &\mapsto m\left(s\right) \end{aligned}$$

$$\frac{d\gamma}{ds}\mid_m = X_f\mid_m \ \forall m \in \mathcal{M}$$

In local canonical coordinates this means:

$$\frac{dq^a(\gamma)}{ds}\partial/\partial q^a + \frac{dp_a(\gamma)}{ds}\partial/\partial p^a = \partial f/\partial p^a \partial/\partial q^a - \partial f/\partial q^a \partial/\partial p^a$$

and so, along the integral curves of X_f we have:

$$\begin{aligned} \frac{dq^a(\gamma)}{ds} &= \partial f/\partial p^a \\ \frac{dp_a(\gamma)}{ds} &= -\partial f/\partial q^a \end{aligned} \tag{8}$$

which we recognize as *Hamilton's equations*, see section (4). The integral curves of X_f are therefore solutions of Hamilton's equations for the function f . The flow of X_f is then:

$$\begin{array}{rcl} \rho_f : (s_1, s_2) \subset \mathbb{R} \times \mathcal{M} & \rightarrow & \mathcal{M} \\ (s, m) & \mapsto & \gamma_{f,m}(s) \end{array}$$

where $\gamma_{f,m}$ is the integral curve of X_f starting at m , i.e. the solution of (8) with initial condition $\gamma_{f,m}(0) = m$. Unless $\mathcal{M}$ is compact, the map ρ_f will in general only be defined for some subset (s_1, s_2) of $\mathbb{R}$. It therefore generates local diffeomorphisms of $\mathcal{M}$.

Using the identity $\mathcal{L}_X \alpha = d i_X \alpha + i_X d\alpha$ [8], which holds for all vector fields X and forms α, we see that:

$$\begin{aligned} \mathcal{L}_{X_f} \omega &= d i_{X_f} \omega + i_{X_f} d\omega \\ &= d(-df) + 0 \\ &= 0 \end{aligned}$$

From the definition of the Lie derivative;[8]

$$\mathcal{L}_{X_f} \omega (m) = \lim_{\epsilon \to 0} \tfrac{1}{\epsilon} \left(\rho^*_{X_f(\epsilon)} \left(\omega \mid_{\rho_{X_f(\epsilon)(m)}} \right) - \omega \mid_m \right)$$

we see that the vanishing of the Lie derivative of ω along X_f is equivalent to ω being invariant under the flow ρ_{X_f} ;

$$\rho^*_{X_f} : \omega \mapsto \omega$$

and so the Hamiltonian flow of f generates *local canonical diffeomorphisms.*

3.3.3 The Lie Algebra of Hamiltonian Vector Fields

Given two Hamiltonian vector fields X_f and X_g , any linear combination is also Hamiltonian:

$$\begin{aligned} i_{\alpha X_f + \beta X_g}(\omega) &= \alpha i_{X_f}(\omega) + \beta i_{X_g}(\omega) \\ &= -\alpha df - \beta dg \\ &= -d(\alpha f + \beta g) \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. Therefore, $V_H(\mathcal{M})$ is a subspace of $V(\mathcal{M})$. We'll show in section (3.4.2) that $V_H(\mathcal{M})$ is in fact a sub algebra.

3.4 Poisson Brackets

Poisson brackets are a key feature of the standard Hamiltonian description of classical dynamics. They allow us to express the change of an observable under time evolution, (or more generally under the evolution generated by any other observable) in a concise way. Poisson brackets also provide a simple way of relating symmetries and conserved quantities. As discussed in section 2 they are the key structure identified by Dirac in his attempt to axiomatise quantization.

The geometric formulation of mechanics gives a coordinate independent, entirely geometric definition of the Poisson bracket. This allows quantization to be formulated in a coordinate independent way.

3.4.1 Definition

We define the Poisson bracket as a map from pairs of smooth functions to smooth functions:[4]

$$\begin{array}{rcl} \{\ ,\ \} : C^\infty_{\mathcal{M}} \times C^\infty_{\mathcal{M}} & \rightarrow & C^\infty_{\mathcal{M}} \\ (f,g) & \mapsto & \{f,g\} \equiv \omega\left(X_f, X_g\right) \end{array} \tag{9}$$

The bracket can also be defined equivalently as $X_f\left(g\right)$:

$$\begin{array}{rcl} X_f\left(g\right) & = & dg\left(X_f\right) \\ & = & -i_{X_g}\omega\left(X_f\right) \\ & = & -\omega\left(X_g, X_f\right) \\ & = & \omega\left(X_f, X_g\right) \end{array}$$

and so:

$$\{f,g\} \ \equiv \ \omega\left(X_f, X_g\right) \ = \ X_f\left(g\right) \tag{10}$$

Using (10) we can derive the coordinate form of the Poisson bracket:

$$\begin{array}{rcl} X_f\left(g\right) & = & \left(\partial f/\partial p^a \,\partial/\partial q^a - \partial f/\partial q^a \,\partial/\partial p^a\right)\left(g\right) \\ & = & \partial f/\partial p^a \,\partial g/\partial q^a - \partial f/\partial q^a \,\partial g/\partial p^a \end{array}$$

which we recognize as the usual Poisson bracket of f with g in Hamiltonian mechanics.

3.4.2 The Lie Algebra of Smooth Functions

From the definition of the Poisson bracket (9) its obviously *antisymmetric* - due to the antisymmetry of ω . Using the following properties of the Lie derivative,[8]we can show that it also satisfies the *Jacobi identity*:

$$\begin{aligned} i_{[X,Y]}\alpha &= \mathcal{L}_X\left(i_Y\alpha\right)-\mathcal{L}_Y\left(i_X\alpha\right) \\ \mathcal{L}_{[X,Y]}T &= \mathcal{L}_X\mathcal{L}_YT-\mathcal{L}_Y\mathcal{L}_XT \end{aligned}$$

which hold for all $X,Y\in V\left(\mathcal{M}\right)$, $\alpha\in\Omega^p\left(\mathcal{M}\right)$ and tensor fields T . Let $X_f,X_g\in V_H\left(\mathcal{M}\right)$, then using the first identity identity:

$$\begin{aligned} i_{\left[X_f,X_g\right]}\omega &= \mathcal{L}_{X_f}\left(i_{X_g}\omega\right)-\mathcal{L}_{X_g}\left(i_{X_f}\omega\right) \\ &= \mathcal{L}_{X_f}\left(-dg\right)-\mathcal{L}_{X_g}\left(-df\right) \\ &= \left(i_{X_f}d+di_{X_f}\right)\left(-dg\right)-\left(i_{X_g}d+di_{X_g}\right)\left(-df\right) \\ &= -d\left(i_{X_f}dg\right)+d\left(i_{X_g}df\right) \\ &= -d\left\{f,g\right\} \end{aligned}$$

Comparing this with (6) we see that:

$$\left[X_f,X_g\right] = X_{\{f,g\}} \tag{11}$$

This shows that the set of Hamiltonian vector fields $V_H\left(\mathcal{M}\right)$ is closed under the Lie bracket, and so is a sub Lie algebra of the algebra of vector fields. We'll refer to it as $\left(V_H\left(\mathcal{M}\right),[\ ,\ ]\right)$.

Now, let $f,g,h\in C^\infty\left(\mathcal{M}\right)$ then using the second identity:

$$\begin{aligned} \mathcal{L}_{\left[X_f,X_g\right]}h &= \mathcal{L}_{X_f}\mathcal{L}_{X_g}h-\mathcal{L}_{X_g}\mathcal{L}_{X_f}h \\ \mathcal{L}_{X_{\{f,g\}}}h &= \mathcal{L}_{X_f}X_g\left(h\right)-\mathcal{L}_{X_g}X_f\left(h\right) \\ X_{\{f,g\}}\left(h\right) &= \mathcal{L}_{X_f}\left\{g,h\right\}-\mathcal{L}_{X_g}\left\{f,h\right\} \\ \left\{\left\{f,g\right\},h\right\} &= \left\{f,\left\{g,h\right\}\right\}-\left\{g\left\{f,h\right\}\right\} \end{aligned}$$

and so we have:

$$\left\{f,\left\{g,h\right\}\right\}+\left\{g,\left\{h,f\right\}\right\}+\left\{h,\left\{f,g\right\}\right\} = 0$$

which is the Jacobi identity. The set of smooth functions on $\mathcal{M}$ is therefore a Lie algebra under the Poisson bracket. We'll refer to this as $\left(C^\infty\left(\mathcal{M}\right),\{\ ,\ \}\right)$.

3.4.3 $C^{\infty}(\mathcal{M})$ and $V_H(\mathcal{M})$

The relationship between the two Lie algebras discussed is important in geometric quantization. The equation defining Hamiltonian vector fields, (6) gives us a map from one Lie algebra to the other:

$$\begin{array}{rcl} \left(C^{\infty}(\mathcal{M}), \{\ ,\ \}\right) & \rightarrow & \left(V_H(\mathcal{M}), [\ ,\ ]\right) \\ f & \mapsto & X_f \end{array}$$

As discussed above, the map is linear. The identity (11) then shows that its a homomorphism:

$$\{f, g\} \quad \mapsto \quad X_{\{f,g\}} = [X_f, X_g]$$

From (7), the coordinate expression of the Hamiltonian vector field of f , we see that X_f is the zero vector field iff all the derivatives of f vanish everywhere. The kernel of the homomorphism is therefore the set of constant functions on $\mathcal{M}$, which is isomorphic to $\mathbb{R}$, and so by the standard isomorphism theorem:[11]

$$\left(C^{\infty}(\mathcal{M}), \{\ ,\ \}\right)/\mathbb{R} \quad \cong \quad \left(V_H(\mathcal{M}), [\ ,\ ]\right)$$

It is this isomorphism, and the relation (11) that leads to it, that underlay geometric quantization. Notice that we are already seeing a correspondence between Poisson brackets and commutators, even at the classical level.

4 Geometric Formulation of Classical Mechanics

4.1 Constructing Phase Space

As mentioned in section (3), cotangent bundles are a natural way to describe the phase space of classical systems. Our starting point for quantization will always be a classical Hamiltonian theory, but we first try explain why cotangent bundles can be arrived at by considering Lagrangian mechanics and the Legendre transform.

4.1.1 Velocity Phase Space

In either the Lagrangian or Hamiltonian formulations of mechanics, we start with a *configuration space*, Q . This is the space of positions of the particles, so for a standard system in 3 dimensions with no constraints this will be $\mathbb{R}^{3n}$. Since classical mechanics can be formulated using *generalized coordinates* (as opposed to some fixed Euclidean coordinate system), it makes sense to think of Q as a manifold. $\mathbb{R}^{3n}$ has the obvious atlas consisting of the global chart $(\mathbb{R}^{3n}, i_d)$, where i_d is the identity map from $\mathbb{R}^{3n}$ to itself. Generalized coordinates systems then make up the complete atlas of all possible charts compatible with $(\mathbb{R}^{3n}, i_d)$.

In a more general situation, the particles' positions will be confined in some way, so Q will be some $3n$ dimensional manifold. For example, if the theory describes a single particle attached to one end of a light rigid rod of length r, the other end fixed at some point, then Q would be S^1 .

The Lagrangian is a function of the particles positions and velocities, so we need to construct the space of positions and velocities from Q . Take some point $q_0 \in Q$ and consider some curve $\gamma(t)$ through q_0 . Let $q_0 \in U \subset Q$ for some chart (U, ϕ) , so that q_0 has coordinates $\phi(q_0) = \left(q_0^1, ..., q_0^N\right)$, and parametrize γ such that $\gamma(t_o) = q_0$. The tangent vector to γ at q_0 is then:

$$\begin{aligned} \tfrac{d}{dt}\big|_{q_0} &\in T_{q_0}Q \\ \tfrac{d}{dt}\big|_{q_0} &= \tfrac{dq^a(\gamma(t))}{dt}\big|_{t_0} \, {}^{\partial}\!/\!_{\partial q^a} \end{aligned}$$

If we identify the parameter t with actual time, then $\gamma(t)$ describes the path of the particles, and the components of $\frac{d}{dt}\big|_{q_0}$ are their generalized velocities. The set of all possible generalized velocity vectors through q_0 is then $T_{q_0}Q$.

If we take TQ , the tangent bundle of Q , it has local coordinates $\left(q^a, v^b\right)$, where $v = v^a \partial/\partial q^a \big|_{q_0}$ is a tangent vector at some point $q_0 \in Q$, so that the coordinates v^b are the coordinates of tangent vectors. Therefore, TQ is the space of all position, velocity pairs. We'll call TQ the *velocity phase space*. In the case of some unconstrained 3 dimensional system, we recover the velocity phase space we'd expect:

$$\begin{aligned} T\left(\mathbb{R}^{3n}\right) &= \mathbb{R}^{3n} \times \mathbb{R}^{3n} \\ &= \mathbb{R}^{6n} \end{aligned}$$

as $\mathbb{R}^{3n}$ can be covered by a single chart, so the transition function between the fib res $T_qQ \cong \mathbb{R}^{3n}$ are trivial and so $T\left(\mathbb{R}^{3n}\right)$ is a trivial bundle.

Since the Lagrangian is a real function of the positions and velocities, we see that a Lagrangian system consists of a configuration space Q and some real function $L \in C^{\infty}\left(T\left(Q\right)\right)$. Its possible to then describe the Legendre transform in a coordinate independent way by mapping TQ to $T^{*}Q$. We can imagine momenta as co vectors since $p\left(v\right) = mv^{2} = 2KE$ and so momenta map velocity vectors to real numbers.

4.2 Hamiltonian Systems

4.2.1 Constituents

In the geometric formulation, a Hamiltonian system consist of a pair (Q, h) , where Q is some real n-dimensional manifold, and h is the *Hamiltonian*, a real function on the cotangent bundle of Q :[2]

$$h : T^{*}Q \rightarrow \mathbb{R}$$

As described in section 3.2.2 , the cotangent bundle $T^{*}Q$ can be made into a symplectic manifold in a natural way, by constructing the canonical one form θ and then taking its exterior derivative to give a symplectic structure ω . In local canonical coordinates:

$$\begin{aligned} \theta &= p_a dq^a \\ \omega &= dp_a \wedge dq^a \end{aligned}$$

From now on, cotangent bundles will be always considered as symplectic manifolds $(T^{*}Q, \omega)$.

4.2.2 Dynamics

Using the results derived in section 3 we can describe the dynamical evolution of the classical system. If $m \in T^{*}Q$ then it specifies some possible state of the system; $m = \left(p\left(m\right), q\left(m\right)\right)$ so it specifies the position and momenta of all the particles. The dynamics are then described by identifying the parametrization of curves in T^*Q with time. Using equations (7) and (8) The Hamiltonian vector field X_h of the Hamiltonian h is:

$$X_h = {}^{\partial h}\!/_{\partial p^a}{}^{\partial}\!/_{\partial q^a} - {}^{\partial h}\!/_{\partial q^a}{}^{\partial}\!/_{\partial p^a}$$

and along its integral curves:

$$\begin{aligned} \frac{dq^a}{dt} &= {}^{\partial h}\!/_{\partial p^a} \\ \frac{dp_a}{dt} &= -{}^{\partial h}\!/_{\partial q^a} \end{aligned}$$

which are Hamilton's equations. The integral curves of h are therefore the possible trajectories of the system under time evolution. Given some initial point m , the equations can be solved uniquely (at least in some neighborhood of m a solution always exists and is unique by the standard existence and uniqueness theorems for pde's [1]) In general, time evolution is thus described by the Hamiltonian flow of h :

$$\rho_h : (t, m) \mapsto \gamma_{h,m}(t) \tag{12}$$

Given a state of the system $m \in T * Q$, (12) then gives the state of the system at all future times t .

Its interesting to note that since (T^*Q, ω) is constructed canonically from Q , and the flow of h is then determined by X_f , Hamiltonian dynamics is essentially contained in the equation defining Hamiltonian vector fields:

$$i_{X_f}\omega = -df$$

4.2.3 Classical Observables

A *classical observable* is some real valued function of the particles positions and momenta, so in the geometric formulation they are functions $f \in C^\infty(T^*Q)$. We saw in section 3.4 that given some symplectic manifold $\mathcal{M}$, the set $C^\infty(\mathcal{M})$ is a Lie algebra under the Poisson bracket $\{\ ,\ \}$. Given some classical system with phase space T^*Q , the observables therefore form a Lie algebra, denoted $(C^\infty(\mathcal{M}), \{\ ,\ \})$, which we shall call the *algebra of classical observables.*

4.3 Quantization

We can now give a precise definition of the quantization of a symplectic manifold: It is a representation of the algebra of classical observables $(C^{\infty}(\mathcal{M}), \{\ ,\ \})$ consisting of symmetric operators acting on a Hilbert space.

4.3.1 General Symplectic Manifolds

Quantization can be applied to any symplectic manifold, not just cotangent bundles and Hamiltonian systems. We'll try to be as general as possible in our description, but the focus will be on cotangent bundles.

5 Prequantization

Prequantization is an attempt at quantizing symplectic manifolds. We shall see that it succeeds in satisfying the axioms outlined earlier. However, comparison with the canonical quantization of a simple example shows that it gives the wrong quantum theory in the physical sense.

5.1 The Hilbert Space of Functions $\mathcal{H}_{\mathbb{C}}$

5.1.1 Constructing $\mathcal{H}_{\mathbb{C}}$

As discussed in section 3.2.5 , the symplectic form ω on a symplectic manifold $(\mathcal{M}, \omega)$ gives an orientation and volume element, the Louisville measure:[11]

$$\begin{aligned} \epsilon &= \left(\tfrac{1}{2\pi n}\right)^{n} \omega^{n} \\ &= \left(\tfrac{1}{2\pi n}\right)^{n} dp_1 \wedge \ldots \wedge dp_n \wedge dq^1 \wedge \ldots \wedge dq^n \end{aligned}$$

This allows us to define an inner product on $C^{\infty}_{\mathbb{C}}(\mathcal{M})$, the set of smooth complex valued functions on $\mathcal{M}$:

$$\begin{aligned} \langle\ ,\ \rangle : C^{\infty}_{\mathbb{C}}(\mathcal{M}) \times C^{\infty}_{\mathbb{C}}(\mathcal{M}) &\rightarrow \mathbb{C} \\ (\psi, \phi) &\mapsto \langle \psi, \phi \rangle \end{aligned}$$

$$\langle \psi, \phi \rangle \equiv {}_{\mathcal{M}}\psi^{*}\phi\epsilon$$

We then take $L^2(\mathcal{M})$, the vector space of complex functions square integrable with respect to the Louisville measure:[11]

$$L^2(\mathcal{M}) \;\equiv\; \{\psi \in C^\infty_{\mathbb{C}}(\mathcal{M}) \mid \langle \psi, \psi \rangle < \infty\}$$

This is a Hilbert space, constructed entirely from the classical structure (as opposed to some arbitrary choice of Hilbert space) and so is our starting point for quantization. We will call $L^2(\mathcal{M})$ the *Hilbert space of functions*, denoted $\mathcal{H}_{\mathbb{C}}$:

$$\mathcal{H}_{\mathbb{C}} \;\equiv\; L^2(\mathcal{M})$$

5.1.2 The $(V_H(\mathcal{M}), [\ ,\ ])$ representation

The homomorphism from $(C^\infty(\mathcal{M}), \{\ ,\ \})$ to $(V_H(\mathcal{M}), [\ ,\ ])$ suggests that we try to quantize $\mathcal{M}$ by associating to each $f \in C^\infty(\mathcal{M})$ a vector field in $V_H(\mathcal{M})$, which then acts on the Hilbert space $\mathcal{H}_{\mathbb{C}}$ as a derivative operator:[11]

$$\begin{array}{rcl} \mathcal{P} : C^\infty(\mathcal{M}) & \to & V_H(\mathcal{M}) \subset \mathcal{O}(\mathcal{H}_{\mathbb{C}}) \\ f & \mapsto & \widehat{f} \\ & & \\ \widehat{f} & \equiv & -i\hbar X_f \end{array} \tag{13}$$

We will call $\widehat{f}$ the Prequantization of the classical observable f .

Since the map $f \mapsto X_f$ is linear over $\mathbb{R}$, then so is $f \mapsto \widehat{f}$:

$$\begin{array}{rcl} \mathcal{P}(\alpha f + \beta g) & = & -i\hbar X_{(\alpha f + \beta g)} \\ & = & -i\hbar(\alpha X_f + \beta X_g) \\ & = & \widehat{f} + \widehat{g} \end{array}$$

The operators $\widehat{f}$ are symmetric:

$$\begin{array}{rcl} \left\langle \psi, \widehat{f}\phi \right\rangle & = & -i\hbar_{\mathcal{M}}\psi^* X_f(\phi)\,\epsilon \\ & = & -i\hbar_{\mathcal{M}}\psi^* \mathcal{L}_{X_f}(\phi)\,\epsilon \\ & = & -i\hbar\left[{}_{\mathcal{M}}\mathcal{L}_{X_f}(\psi^*\phi\epsilon) - {}_{\mathcal{M}}\mathcal{L}_{X_f}(\psi^*)\,\phi\epsilon - {}_{\mathcal{M}}\psi^*\phi\mathcal{L}_{X_f}(\epsilon)\right] \\ & = & -i\hbar\left[{}_{\mathcal{M}}\mathcal{L}_{X_f}(\psi^*\phi\epsilon) - {}_{\mathcal{M}}\mathcal{L}_{X_f}(\psi^*)\,\phi\epsilon - \left(\frac{1}{2\pi n}\right)^n {}_{\mathcal{M}}\psi^*\phi\mathcal{L}_{X_f}(\omega^n)\right] \\ & = & {}_{\mathcal{M}} i\hbar X_f(\psi^*)\,\phi\epsilon \\ & = & {}_{\mathcal{M}}(-i\hbar X_f\psi)^*\,\phi\epsilon \\ & = & \left\langle \widehat{f}\psi, \phi \right\rangle \end{array}$$

where the first term in line four vanishes as it leads to boundary terms and $\partial\mathcal{M} = 0$, and the third term vanishes because $\mathcal{L}_{X_f}(\omega) = 0$. The identity (11) ensures that condition four is satisfied:

$$\begin{aligned} -i\hbar\mathcal{P}(\{f,g\}) &= (-i\hbar)^2 X_{\{f,g\}} \\ &= (-i\hbar)^2 [X_f, X_g] \\ &= [-i\hbar X_f, -i\hbar X_g] \\ &= \left[\widehat{f}, \widehat{g}\right] \end{aligned}$$

However, if f is the constant function $f(m) = 1$ on $\mathcal{M}$ then X_f is zero and so f is mapped to the zero operator. The map $\mathcal{P}$ therefore fails to satisfy all four quantization conditions, and we need to try something else.

5.1.3 The trick

Choose some symplectic potential $\theta \in \Omega^1(U)$, where $U \subset \mathcal{M}$. Then we can quantize observables restricted to U , by defining:

$$\widehat{f} = -i\hbar X_f - \theta(X_f) + f \tag{14}$$

which we can also write as:

$$\begin{aligned} \widehat{f}(\psi) &= -i\hbar\left(i_{X_f}(d\psi) - \tfrac{i}{\hbar} i_{X_f}\theta\psi\right) + f\psi \\ &= -i\hbar i_{X_f}\left((d\psi) - \tfrac{i}{\hbar}\theta\psi\right) + f\psi \\ &= -i\hbar i_{X_f}\left(d - \tfrac{i}{\hbar}\theta\right)\psi + f\psi \end{aligned}$$

This choice satisfies all the quantization conditions (we show this in section 5.3 in a more general way). But, this only quantizes observables on $U \subset \mathcal{M}$, and we could have chosen some other symplectic potential. Only if ω is exact, does this quantize all observables on $\mathcal{M}$. However, even in this case, θ is not determined exactly. If we are quanitizing some Hamiltonian system (as opposed to some arbitary symplectic manifold) then $\mathcal{M}$ is a cotangent bundle T^*Q and we can choose θ as the canonical one form. However, the above construction can be adapted to quantize a large class of symplectic manifolds, by formulating (14) in a global, coordinate independent way. Its worth doing this to see whether there are other quantization's of T^*Q .

If we consider the action of $\widehat{f}$ on some function $\psi \in \mathcal{H}_{\mathbb{C}}$:

$$\widehat{f}\left(\psi\right) \quad = \quad -i\hbar\left(X_f\left(\psi\right) - \tfrac{i}{\hbar}\theta\left(X_f\right)\psi\right) + f\psi \tag{15}$$

we see that $\widehat{f}\left(\psi\right)$ has $U\left(1\right)$ gauge covariance (similar to e.g. the covariant derivative in the abelian Higgs model) - it transforms covariantly under the following transformations:[11]

$$\begin{aligned} \theta &\mapsto \theta' \equiv \theta + du \\ \psi &\mapsto \psi' \equiv e^{iu/\hbar}\psi \end{aligned}$$

where $u \in C^{\infty}\left(\mathcal{M}\right)$.

$$\begin{aligned} \widehat{f'}\left(\psi'\right) &= \left[-i\hbar X_f - \theta\left(X_f\right) - du\left(X_f\right) + f\right] e^{iu/\hbar}\psi \\ &= X_f\left(u\right) e^{iu/\hbar}\psi - e^{iu/\hbar} i\hbar X_f\left(\psi\right) - e^{iu/\hbar}\theta\left(X_f\right)\psi - e^{iu/\hbar} du\left(X_f\right)\psi + e^{iu/\hbar} f\psi \\ &= e^{iu/\hbar}\left(-i\hbar X_f\left(\psi\right) - \theta\left(X_f\right)\psi + f\psi\right) \\ &= e^{iu/\hbar}\widehat{f}\left(\psi\right) \end{aligned}$$

This means that choosing a different symplectic potential θ' and mapping all the functions ψ to $e^{iu/\hbar}\psi$ is consistent. In general, gauge theories can be globally described by a principal G bundle, where G is the gauge group. The $U\left(1\right)$ covariance of $\widehat{f}\left(\psi\right)$ therefore suggests a $U\left(1\right)$ (S^1 with group multiplication) bundle over $\mathcal{M}$. Instead, we can use the associated vector bundle to formulate the gauge theory, which is a one dimensional vector bundle with typical fibre $\mathbb{C}$. [8]

5.2 Complex Line Bundles

We describe the necessary facts about complex line bundles. All definitions in this section are taken from [11] but we have filled in many missing details in the derivations.

5.2.1 Sections and One Forms

A *complex line bundle over* $\mathcal{M}$ is a fibre bundle $L \to \mathcal{M}$ where the typical fibre is the vector space $\mathbb{C}$ so that the fibre at a point $m \in \mathcal{M}$ is a vector space $V_m \cong \mathbb{C}$. If L is trivial then:

$$L = \mathcal{M} \times \mathbb{C}$$

otherwise it just has this structure locally. We denote the projection from L to $\mathcal{M}$ as, π , local trivializations by (U_i, τ_i) , the set of smooth sections of L as $C_L^\infty(\mathcal{M})$ and the set of L valued one-forms on $\mathcal{M}$ as $\Omega_L^1(\mathcal{M})$:

$$\pi : L \rightarrow \mathcal{M}$$

$$\tau_i : U_i \times \mathbb{C} \rightarrow \pi^{-1}(U_i) \subset L$$

$$\begin{aligned} C_L^\infty(\mathcal{M}) \ni s : \mathcal{M} &\rightarrow L \\ \pi_\circ s &= id_{\mathcal{M}} \end{aligned}$$

$$\Omega_L^1(\mathcal{M}) \ni \alpha : V(\mathcal{M}) \rightarrow C_L^\infty(\mathcal{M})$$

If L is trivial then $C_L^\infty(\mathcal{M})$ and $\Omega_L^1(\mathcal{M})$ are $C_\mathbb{C}^\infty(\mathcal{M})$ and $\Omega_\mathbb{C}^1(\mathcal{M})$, the sets of complex functions and one forms respectively.

5.2.2 Connections

A connection ∇ on L is a map:

$$\nabla : C_L^\infty(\mathcal{M}) \rightarrow \Omega_L^1(\mathcal{M})$$

$$\begin{aligned} \nabla(s_1 + s_2) &= \nabla s_1 + \nabla s_2 \\ \nabla(\psi s) &= (d\psi)\, s + \psi \nabla s \end{aligned}$$

where $\psi \in C_\mathbb{C}(\mathcal{M})$. Given some local trivialization (U_i, τ_i) , we can define its associated *unit section*, $s_i \in C_L^\infty(U)$ and its *potential 1-form*, $\Theta_i \in \Omega_\mathbb{C}^1(U)$:

$$s_i(m) \equiv \tau_i(m, 1)$$

$$\nabla s_i = -i \Theta_i s_i$$

The local trivialization restricts to a isomorphism when some point $m \in U$ is fixed:

$$\tau_i(m, \ ) : \mathbb{C} \ \rightarrow \ V_m$$

and since $\mathbb{C}$ and V_m are one dimensional it must be of the form:

$$\tau_i(m, \ )(\alpha) \ = \ \Gamma(m)\,\alpha$$

for some non-vanishing constant Γ (if Γ was zero then $\tau(m, \ )$ would not be 1-2-1), and so unit sections are non-vanishing. This allows us to describe the action of the connection on arbitrary sections in the following way: let $s \in C^{\infty}_{L}(U)$ be some section on U , then since s_i is non-vanishing, $s = \psi s_i$ for some function $\psi \in C^{\infty}_{\mathbb{C}}(\mathcal{M})$, which we call its *local representative*:

$$s(m) \ = \ \tau_i(m, \psi(m))$$

We use the notation $\nabla_X \equiv i_X \nabla$ for $X \in V(\mathcal{M})$ then:

$$\begin{aligned} \nabla_X s &= i_X\left((d\psi)\, s_i - i\psi\Theta_i s_i\right) \\ &= X(\psi)\, s_i - i\Theta_i(X)\,\psi s_i \\ &= \left(X(\psi) - i\Theta_i(X)\,\psi\right) s_i \end{aligned}$$

We can also write this as:

$$\nabla_X s \ = \ i_X\left(d - i\Theta_i\right)\psi s_i$$

and so the action of the connection on a section can be described entirely in terms of its local representative:

$$\nabla \psi s_i \ = \ \left[\left(d - i\Theta_i\right)\psi\right] s_i$$

5.2.3 Changing sections

The gauge covariance seen in $\widehat{f}(\psi)$ in equation (15) is described by the way that the potential one form changes under a change of local trivialization. If we have two local sections (U_i, τ_i) and (U_j, τ_j) , where $U_i \cap U_j \neq \emptyset$, then since their unit sections s_i and s_j are non-vanishing, we can define the *transition function* c_{ji}:

$$s_i \ = \ c_{ji} s_j$$

where c_{ji} is some complex function defined on $U_i \cap U_j$. We find that $s_i = c_{ji}s_j = c_{ji}c_{ij}s_i$, and so, $c_{ji}c_{ij} = 1$. We can use this and the properties of the connection to find the difference between the two potentials:

$$\begin{aligned}
\nabla s_i &= \nabla c_{ji}s_j \\
-i\Theta_i s_i &= (dc_{ji})\, s_j + c_{ji}\nabla s_j \\
-ic_{ji}\Theta_i s_j &= (dc_{ji})\, s_j - ic_{ji}\Theta_j s_j \\
-ic_{ji}(\Theta_i - \Theta_j)\, s_j &= (dc_{ji})\, s_j
\end{aligned}$$

and so:

$$\Theta_i = \Theta_j + i\tfrac{(dc_{ji})}{c_{ji}} \tag{16}$$

5.2.4 $U(1)$ Transition Functions

The functions $c_{ji} \in C^\infty_{\mathbb{C}}(U_i \cap U_j)$ are complex valued so can be written in the form:

$$c_{ji}(m) = r_{ji}(m)\, e^{i\phi_{ji}(m)}$$

where $r_{ji}, \phi_{ji} \in C^\infty(\mathcal{M})$. By introducing extra structure on the bundle L , we can normalize these. A *Hermitian Structure* $(\ ,\ )$ on L is an assignment of an inner product to each fibre in the bundle:

$$(\ ,\ )_m : V_m \times V_m \rightarrow \mathbb{C}$$

such that the inner products together form a smooth map from L to $\mathbb{C}$:

$$\begin{aligned}
(\ ,\ ) : L &\rightarrow \mathbb{C} \\
v &\mapsto (v,v)_m
\end{aligned}$$

where m is chosen for each $v \in L$ such that $v \in V_m$. The Hermitian structure allows us to take inner products between sections:

$$(\ ,\ ) : C^\infty_L(\mathcal{M}) \times C^\infty_L(\mathcal{M}) \rightarrow C^\infty_{\mathbb{C}}(\mathcal{M})$$

$$(s_1, s_2)(m) \equiv (s_1(m), s_2(m))_m$$

Now, let s_i be the unit section of some local trivialization (U_i, τ_i) , then define:

$$f_i \equiv (s_i, s_i)$$

Since (,) is an inner product so is non-degenerate and s_i is non-vanishing, f_i must be non-vanishing, and since for each m , $f_i\left(m\right)$ is the norm of some vector in V_m , f_i must be real. We can then construct a new local trivialization (U_i,τ_i') with a unit section s_i' :

$$\begin{aligned}
\tau_i' : U_i \times \mathbb{C} &\rightarrow \pi^{-1}\left(U_i\right) \\
\tau_i'\left(m,\alpha\right) &\equiv \left(f_i\right)^{-1/2}\tau_i\left(m,\alpha\right)
\end{aligned}$$

$$\begin{aligned}
s_i'\left(m\right) &= \tau_i'\left(m,1\right) \\
&= \left(f_i\right)^{-1/2}\tau_i\left(m,1\right) \\
&= \left(f_i\right)^{-1/2}s_i
\end{aligned}$$

The unit section s_i' is then normalized:

$$\begin{aligned}
\left(s_i',s_i'\right) &= \left(\left(f_i\right)^{-1/2}s_i,\left(f_i\right)^{-1/2}s_i\right) \\
&= f_i^{-1}\left(s_i,s_i\right) \\
&= 1
\end{aligned}$$

If we choose local trivializations covering $\mathcal{M}$ that have unit sections satisfying this, then the transition functions between them will be normalized:

$$\begin{aligned}
1 &= \left(s_i,s_i\right) \\
&= \left(c_{ji}s_j,c_{ji}s_j\right) \\
&= \left|c_{ji}\right|^2\left(s_j,s_j\right) \\
&= \left|c_{ji}\right|^2
\end{aligned}$$

The transition functions are then of the form:

$$c_{ji}\left(m\right) = e^{i\phi_{ji}(m)} \tag{17}$$

If the following identity holds:

$$i_X d\left(s_1,s_2\right) = \left(\nabla_X s_1,s_2\right)+\left(s_1,\nabla_X s_2\right)$$

for all $X\in V\left(\mathcal{M}\right)$ and $s_1,s_2\in C_L^\infty\left(\mathcal{M}\right)$ then (,) is *compatible* with ∇ . Then the choice of trivializations will also make the connection potential real:

$$\begin{aligned}
i_X d\left(s_i, s_i\right) &= \left(\nabla_X s_i, s_i\right) + \left(s_i, \nabla s_i\right) \\
i_X d\left(1\right) &= \left(-i\Theta_i\left(X\right) s_i, s_i\right) + \left(s_i, -i\Theta_i\left(X\right) s_i\right) \\
0 &= i\Theta_i^*\left(X\right)\left(s_i, s_i\right) - i\Theta_i\left(X\right)\left(s_i, s_i\right) \\
0 &= i\left(\Theta_i^*\left(X\right) - \Theta_i\left(X\right)\right)
\end{aligned}$$

which is true $\forall X \in V\left(\mathcal{M}\right)$ so that we must have $\Theta_i = \Theta_i^*$ and so:

$$\Theta_i \in \Omega^1\left(\mathcal{M}\right)$$

Using equations (16) and (17) we see that under a change of trivialization the potential will now change as:

$$\begin{aligned}
\Theta_i &= \Theta_j + i\frac{\left(dc_{ji}\right)}{c_{ji}} \\
&= \Theta_j + id\left(e^{i\phi_{ji}}\right)\left(e^{i\phi_{ji}}\right)^{-1} \\
&= \Theta_j - \left(d\phi_{ji}\right) e^{i\phi_{ji}}\left(e^{i\phi_{ji}}\right)^{-1} \\
&= \Theta_j - d\phi_{ji}
\end{aligned}$$

We know that $c_{ij}c_{ji} = 1$ so:

$$\begin{aligned}
c_{ij} &= c_{ji}^{-1} \\
e^{i\phi_{ij}} &= e^{-i\phi_{ji}}
\end{aligned}$$

and therefore the functions ϕ_{ij} are related as:

$$\phi_{ij} = -\phi_{ji}$$

If s is a section then its local representative ψ depends on local trivialization chosen:

$$\begin{aligned}
s &= \psi_i s_i \\
&= \psi_j s_j \\
&= \psi_j c_{ij} s_i
\end{aligned}$$

and so local representatives change as:

$$\psi_i = c_{ij}\psi_j$$

Combining these results, we find that under a change of local trivialization from (U_j, τ_j) to (U_i, τ_i) the potential one form and local representatives of sections change as:

$$\begin{aligned}\Theta_i &= \Theta_j + d\phi_{ij} \\ \psi_i &= e^{i\phi_{ij}}\psi_j\end{aligned}$$

which we recognize as $U(1)$ gauge transformations.

5.2.5 The curvature of ∇ on L

The *curvature* of the connection ∇ is a 2-form $\Omega \in \Omega^2(\mathcal{M})$ defined locally as:

$$\Omega \equiv d\Theta_i$$

on U_i , the domain of some local trivialization (U_i, τ_i) with potential Θ_i . It is actually independent of the trivialization chosen, and satisfies the following identity:

$$\Omega(X,Y)\,s = i\left(\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s\right) \tag{18}$$

for all $X, Y \in V(\mathcal{M})$ and $s \in C_L^\infty(\mathcal{M})$. We will use this identity several times.

5.3 The Pre Quantum Bundle B

5.3.1 Summary of previous section

Collecting together the results of section 5.2 , if L is a Hermitian line bundle on a manifold $\mathcal{M}$, with a compatible connection ∇ , then in a local trivialization (U_i, τ_i) , ∇ acts on sections as:

$$\nabla\psi s_i = [(d - i\Theta_i)\,\psi]\,s_i$$

and if we change to a different trivialization (U_j, τ_j) , potential one form and local representatives change as:

$$\begin{aligned}\Theta_j &= \Theta_i + d\phi_{ji} \\ \psi_j &= e^{i\phi_{ji}}\psi_i\end{aligned}$$

Therefore, in order to describe:

$$\widehat{f}\left(\psi\right) = -i\hbar i_{X_f}\left(d - \tfrac{i}{\hbar}\theta\right)\psi + f\psi$$

in a global way, we need a Hermitian line bundle on $\mathcal{M}$ with a compatible connection such that:

$$\Theta_i = \tfrac{1}{\hbar}\theta$$

locally, which means the curvature must be:

$$\Omega = \tfrac{1}{\hbar}\omega$$

If such a bundle exists, we call it a *pre quantum bundle*, and label it B .

5.3.2 The pre quantum Hilbert space $\mathcal{H}_{\mathrm{B}}$

Given a pre quantum bundle B over a symplectic manifold $\mathcal{M}$ we can construct a Hilbert space called the *pre quantum Hilbert space*, $\mathcal{H}_{\mathrm{B}}$. As discussed in section 5.2 , the Hermitian structure on B gives a map on the space of smooth sections of B :

$$\begin{aligned} (\ ,\ ) : C^\infty_B\left(\mathcal{M}\right) \times C^\infty_B\left(\mathcal{M}\right) &\rightarrow C^\infty_{\mathbb{C}}\left(\mathcal{M}\right) \\ \left(s_1, s_2\right) &\mapsto \left(s_1\left(m\right), s_2\left(m\right)\right)_m \end{aligned}$$

We can use this to put an inner product on $C^\infty_B\left(\mathcal{M}\right)$:

$$\begin{aligned} \langle\ ,\ \rangle : C^\infty_B\left(\mathcal{M}\right) \times C^\infty_B\left(\mathcal{M}\right) &\rightarrow \mathbb{C} \\ \langle s_1, s_2\rangle &= {}_{\mathcal{M}}\left(s_1, s_2\right)\epsilon \end{aligned}$$

The pre quantum Hilbert space is then the space of square integrable sections of B :

$$\mathcal{H}_{\mathrm{B}} \equiv L^2\left(B\right) \equiv \left\{s \in C^\infty_B\left(\mathcal{M}\right) \mid \langle s, s\rangle < \infty\right\}$$

If B is trivial and so can be covered by a single local trivialization $\left(U_i, \tau_i\right)$, and ψ_1, ψ_2 are the local representatives of s_1, s_2 on U_i then the inner product is:

$$\begin{aligned} \langle s_1, s_2\rangle &= {}_{\mathcal{M}}\left(s_1, s_2\right)\epsilon \\ &= {}_{\mathcal{M}}\left(\psi_1 s_i, \psi_2 s_i\right)\epsilon \\ &= {}_{\mathcal{M}}\psi_1^*\psi_2\left(s_i, s_i\right)\epsilon \\ &= {}_{\mathcal{M}}\psi_1^*\psi_2\epsilon \end{aligned}$$

and $C_B^\infty(\mathcal{M}) = C_{\mathbb{C}}^\infty(\mathcal{M})$, so that in this case:

$$
\begin{aligned}
L^2(B) &= L^2(\mathcal{M}) \\
\mathcal{H}_{\mathrm{B}} &= \mathcal{H}_{\mathbb{C}}
\end{aligned}
$$

5.3.3 Prequantization of observables

We can now quantize $\mathcal{M}$ by mapping observables to operators on $\mathcal{H}_{\mathrm{B}}$:

$$
\begin{aligned}
C^\infty(\mathcal{M}) &\rightarrow \mathcal{O}(H_{\mathrm{B}}) \\
f &\mapsto \widehat{f}
\end{aligned}
$$

where $\widehat{f}$ is now globally defined as:

$$
\widehat{f} = -i\hbar\nabla_{X_f} + f
$$

Since the map $f \rightarrow X_f$ and the interior product is linear, $f \mapsto \widehat{f}$ is linear:

$$
\begin{aligned}
\nabla_{X_{\alpha f+\beta g}} &= i_{X_{\alpha f+\beta g}}\nabla \\
&= i_{\left(\alpha X_f+\beta X_g\right)}\nabla \\
&= \left(\alpha i_{X_f} + \beta i_{X_g}\right)\nabla \\
&= \alpha\nabla_{X_f} + \beta\nabla_{X_g}
\end{aligned}
$$

Secondly, $X_f = 0$ for constant functions, so if $f(m) = 1$ we have:

$$
\widehat{1} = 1
$$

Thirdly, $\widehat{f}$ is symmetric:

$$
\begin{aligned}
\left\langle s_1, \widehat{f}s_2\right\rangle &= {}_{\mathcal{M}}\left(s_1, -i\hbar\nabla_{X_f}s_2 + fs_2\right)\epsilon \\
&= {}_{\mathcal{M}}\left[-i\hbar\left(s_1, \nabla_{X_f}s_2\right) + (s_1, fs_2)\right]\epsilon \\
&= {}_{\mathcal{M}}\left[-i\hbar\left[i_{X_f}d(s_1, s_2) - \left(\nabla_{X_f}s_1, s_2\right)\right] + (fs_1, s_2)\right]\epsilon \\
&= {}_{\mathcal{M}}\left[\left(-i\hbar\nabla_{X_f}s_1 + fs_1, s_2\right)\right]\epsilon - i\hbar_{\mathcal{M}}i_{X_f}d(s_1, s_2)\epsilon \\
&= {}_{\mathcal{M}}\left[\left(\widehat{f}s_1, s_2\right)\right]\epsilon - i\hbar_{\mathcal{M}}i_{X_f}d(s_1, s_2)\epsilon
\end{aligned}
$$

where we have used the fact that ∇ is compatible with $(\, , \,)$. The second term then vanishes because $\mathcal{L}_{X_f}\omega = 0$:

$$\begin{aligned} {}_{\mathcal{M}}i_{X_f} d\left(s_1, s_2\right)\epsilon &= {}_{\mathcal{M}}\mathcal{L}_{X_f}\left[d\left(s_1, s_2\right)\right]\epsilon \\ &= {}_{\mathcal{M}}\mathcal{L}_{X_f}\left[d\left(s_1, s_2\right)\epsilon\right] \\ &= 0 \end{aligned}$$

Finally, by using the fact that $\frac{1}{\hbar}\omega$ satisfies the curvature identity (18) we see that this satisfies the last quantization condition:

$$\begin{aligned} \left[\widehat{f}, \widehat{g}\right] &= \left[-i\hbar\nabla_{X_f} + f, -i\hbar\nabla_{X_g} + g\right] \\ &= \left(-i\hbar\right)^2\left[\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f}\right] - i\hbar\left[dg\left(X_f\right) - df\left(X_g\right)\right] \\ &= \left(-i\hbar\right)^2\left[-\tfrac{i}{\hbar}\omega\left(X_f, X_g\right) + \nabla_{\left[X_f, X_g\right]}\right] - i\hbar\left[\{g, f\} - \{f, g\}\right] \\ &= \left(-i\hbar\right)^2\left[-\tfrac{i}{\hbar}\{f, g\} + \nabla_{X_{\{f,g\}}}\right] - 2i\hbar\{f, g\} \\ &= -i\hbar\left[-i\hbar\nabla_{X_{\{f,g\}}} + \{f, g\}\right] \end{aligned}$$

(Some terms cancel or are zero). Prequantization therefore satisfies all the quantization conditions set out earlier. However, we shall see that this fails to agree with canonical quantization for a theory where we know the "correct" quantization from actual physics.

5.4 Existence and Uniqueness of B

If $\mathcal{M}$ is a cotangent bundle T^*Q then the obvious choice for B always exists:

$$\begin{aligned} B &= \mathcal{M} \times \mathbb{C} \\ (\alpha, \beta) &= \alpha^*\beta \\ \nabla &= d - \tfrac{i}{\hbar}\theta \end{aligned}$$

where θ is the canonical one form. It is true in general that if the base space of a fibre bundle is contractible, then the bundle is trivial. This means that this choice of B is unique if T^*Q has trivial homotopy, i.e. :

$$\pi_1\left(T^*Q\right) = \{e\}$$

If T^*Q is also path connected then this condition is thatT^*Q is simply connected.[8] In general, there might not exist a pre quantum bundle on a symplectic manifold $(\mathcal{M}, \omega)$, see [11]

5.5 Examples

If we take the classical theory of a single particle in 3 dimensions canonical quantization gives a Hilbert space and operators representing momentum and position:

$$\begin{aligned}
\mathcal{H} &= L^2\left(\mathbb{R}^3\right) \\
\psi &= \psi\left(q\right) \\
\widehat{p}_a &= -i\hbar \partial / \partial q^a \\
\widehat{q}^a &= q^a \\
\left[\widehat{p}_a, \widehat{q}^b\right] &= -i\hbar\delta_a^b
\end{aligned}$$

The symplectic phase space is the cotangent bundle of $\mathbb{R}^3$:

$$T\mathbb{R}^3 = \mathbb{R}^6$$

It can be covered by a single canonical coordinate system, $\left(p_a, q^b\right)$. Since $\mathbb{R}^6$ is simply connected the trivial choice for B is unique:

$$B = \mathbb{R}^6 \times \mathbb{C}$$

As discussed earlier, since B is trivial $\mathcal{H}_P = \mathcal{H}_{\mathbb{C}}$ and so the pre quantum Hilbert space is:

$$\begin{aligned}
\mathcal{H}_P &= L^2\left(\mathbb{R}^6\right) \\
\psi &= \psi\left(p, q\right)
\end{aligned}$$

and the connection acts as:

$$\nabla_X \psi = X\left(\psi\right) - \tfrac{i}{\hbar}\theta\left(X\right)\psi$$

If we calculate the operators $\widehat{p}_a$ and $\widehat{q}^a$ we find:

$$\begin{aligned}
\widehat{p}_a &= -i\hbar X_{p_a} - \theta\left(X_{p_a}\right) + p_a \\
&= -i\hbar \partial/\partial q^a - p_b dq^b\left(\partial/\partial q^a\right) + p_a \\
&= -i\hbar \partial/\partial q^a - p_a + p_a \\
&= -i\hbar \partial/\partial q^a
\end{aligned}$$

$$\begin{aligned}
\widehat{q}^a &= -i\hbar X_{q^a} - \theta\left(X_{q^a}\right) + q^a \\
&= i\hbar \partial/\partial p_a - p_b dq^b\left(\partial/\partial p_a\right) + q^a \\
&= i\hbar \partial/\partial p_a + q^a
\end{aligned}$$

So Prequantization does not produce the correct quantum theory. The Hilbert space contains functions of position and momentum, and the operator corresponding to q^a is wrong. However, on restriction to functions $\psi(q)$ the derivative term in $\widehat{q}^a$ gives zero so can be dropped and we recover the correct form of the operator. What we have to do therefore, is construct a new Hilbert space containing functions of q only. In the general case, we therefore have to restrict to functions of half the coordinates of $\mathcal{M}$, and in order to do this in a coordinate independent and consistent way, we have to introduce a *polarization* on $\mathcal{M}$.

6 Polarizations

6.1 Identifying sections $s(q)$

If a section $s \in C_B^\infty(\mathcal{M})$ doesn't depend on the p coordinates then, in any local trivialization (U_i, τ_i) its local representative ψ_i should satisfy:

$$\partial/\partial p_a\left(\psi_i\right) = 0$$

Since $\theta\left(\partial/\partial p_a\right) = 0$ this is the same as:

$$\begin{aligned}
0 &= \partial/\partial p_a\left(\psi_i\right) \\
&= \partial/\partial p_a\left(\psi_i\right) s_i - i\theta\left(\partial/\partial p_a\right)\psi_i s_i \\
&= \nabla_{\partial/\partial p_a} s
\end{aligned}$$

We can therefore identify sections $s = s(q)$ in a global way by the condition:

$$\nabla_X s = 0$$

for all vector fields which can be locally expressed as $X^a \partial/\partial p_a$. These vector fields form a sub-bundle of the tangent bundle of $\mathcal{M}$, which "polarizes" the sections. We describe this more precisely in the next section.

6.2 Distributions

6.2.1 Distributions on $\mathcal{M}$

A *real polarization* starts with the idea of a *real distribution*, which is a sub-bundle of the tangent bundle $T\mathcal{M}$. If P is a real distribution then its a bundle:[11]

$$P \rightarrow \mathcal{M}$$

$$P_m \subset T_m\mathcal{M}$$

P picks out certain vector fields and functions on $\mathcal{M}$, those which are *tangent to* and *constant on* P , defined respectively as:[11]

$$V_P(\mathcal{M}) \equiv \{X \in V(\mathcal{M}) \mid X(m) \in P_m\}$$

$$C_P^\infty(\mathcal{M}) \equiv \{f \in C^\infty(\mathcal{M}) \mid df\mid_P = 0\}$$

The condition $df \mid_P = 0$ is the same as $df(X) = 0 \; \forall X \in V_P(\mathcal{M})$. The subspaces P_m have to be such that they vary smoothly as m varies so that $V_P(\mathcal{M})$ does contain actual vector fields.

We would like to be able to describe P in terms of coordinates on $\mathcal{M}$, which can be done if P is *integrable*. This means that each P_m is the tangent space of some sub manifold of $\mathcal{M}$. Let $\mathcal{M}$ be N dimensional, then since $P \subset T\mathcal{M}$ the fibres P_m must be $N-k$ dimensional for some $0 \le k \le N$. If P is integrable then for every point $m \in \mathcal{M}$ we can find local coordinates x^α:

$$x^\alpha = \left(x^1, x^2, ..., x^{k+1}, x^{k+2}, ..., x^N\right)$$

where fixing the values $x^{k+1}, x^{k+2}, ..., x^N$ gives a sub manifold of $\mathcal{M}$ with tangent spaces P_m .

For example, let $\mathcal{M} = \mathbb{R}^3$ with global coordinates (x, y, z) , and let $P \rightarrow \mathbb{R}^3$ be a distribution on $\mathcal{M}$ where P_m is defined by:

$$P_m = \{V \in T_m\mathbb{R}^3 \mid V = X{}^{\partial}\!/_{\partial x} \mid_m\}$$

If we choose two constants $c_1, c_2 \in \mathbb{R}$ then we can define a sub manifold Λ_{c_1,c_2} of $\mathbb{R}^3$ given by:

$$\Lambda_{c_1,c_2} = \{p \in \mathbb{R}^3 \mid p = (x, c_1, c_2)\}$$

Λ_{c_1,c_2} is the image of an embedding γ of $\mathbb{R}$ into $\mathbb{R}^3$:

$$\begin{array}{rcl} \gamma : \mathbb{R} & \rightarrow & \Lambda_{c_1,c_2} \subset \mathbb{R}^3 \\ x & \mapsto & (x, c_1, c_2) \end{array}$$

The tangent spaces of $\mathbb{R}$ are then pushed forward into subspaces of the tangent spaces of $\mathbb{R}^3$:

$$\begin{array}{rcl} \gamma_* : T_x\mathbb{R} & \rightarrow & T_{(x,c_1,c_2)}\mathbb{R}^3 \\ & & \\ X{}^{\partial}\!/_{\partial x} & \mapsto & X{}^{\partial x}\!/_{\partial x}{}^{\partial}\!/_{\partial x} + {}^{\partial c_1}\!/_{\partial x}{}^{\partial}\!/_{\partial y} + {}^{\partial c_2}\!/_{\partial x}{}^{\partial}\!/_{\partial z} \\ & = & X{}^{\partial}\!/_{\partial x} \end{array}$$

and so we see that:

$$P_{(x,c_1,c_2)} = \gamma_* (T_x\mathbb{R})$$

P is therefore an integral distribution, and the embedded sub manifolds Λ_{c_1,c_2} are then called the *leaves* of P . (In general the leaves only need be immersed sub manifolds - the map γ may be 2-1 but the push forwards of the tangent spaces are 1-1).

It turns out that a distribution is integrable iff the Lie bracket closes on the set of tangent vector fields, so that $V_P(\mathcal{M})$ is a sub algebra of $V(\mathcal{M})$:[1]

$$[X, Y] \in V_P(\mathcal{M}) \qquad \forall X, Y \in V_P(\mathcal{M})$$

Its possible to place restrictions on the subspaces P_m . If V is a symplectic vector space with symplectic form ω then a subspace $W \subset V$ is called *Lagrangian* if:[11]

$$\omega(X, Y) = 0 \qquad \forall X, Y \in W$$

A distribution P on $\mathcal{M}$ is called Lagrangian if for every $m \in \mathcal{M}$, P_m is a Lagrangian subspace of $T_m\mathcal{M}$ (with respect to the symplectic form ω restricted to $T_m\mathcal{M}$). Then we have:

$$\omega(X,Y) = 0 \qquad \forall X,Y \in V_P(\mathcal{M})$$

6.2.2 Polarized Sections

We saw that in the Prequantization of $\mathbb{R}^3$ we can identify sections $s = s(q)$ by the condition:

$$\nabla_X s = 0$$

for all vector fields $X = {}^{\partial}\!/_{\partial p^a}$. We can formulate this idea in a more general way using distributions.

Let B be a pre quantum bundle on a symplectic manifold $\mathcal{M}$ and consider some distribution P on $\mathcal{M}$ and the set of sections $P(B) \subset C_B^{\infty}(\mathcal{M})$ defined by:

$$P(B) \equiv \{s \in C_B^{\infty}(\mathcal{M}) \mid \nabla_X s = 0 \; \forall X \in V_P(\mathcal{M})\}$$

We call $P(B)$ the set of P polarized sections. Let $X,Y \in V_P(\mathcal{M})$ then using the curvature identity (18) satisfied by ω :

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X) s = \left(-\tfrac{i}{\hbar}\omega(X,Y) + \nabla_{[X,Y]}\right) s$$

If $s \in P(B)$ the left hand side is zero so we must have:

$$\nabla_{[X,Y]} s - \tfrac{i}{\hbar}\omega(X,Y) s = 0$$

This will always be satisfied if the two terms are individually zero, so that $\forall X,Y \in V_P(\mathcal{M})$:

$$\begin{aligned} [X,Y] &\in V_P(\mathcal{M}) \\ \omega(X,Y) &= 0 \end{aligned} \qquad (19)$$

Therefore, if there exists any polarized sections, so that $P(B) \neq \emptyset$, the conditions (19) must be satisfied which means that P is *integrable* and *Lagrangian.* Such a distribution is called a *Real Polarization.*

6.3 Real Polarizations

6.3.1 The Vertical Polarization

As described above, a real polarization P on a symplectic manifold $\mathcal{M}$ is an integrable Lagrangian distribution on $\mathcal{M}$. The canonical example is the *vertical polarization* of a cotangent bundle. Let T^*Q be a cotangent bundle, then we can construct a real polarization by taking the cotangent spaces T^*_qQ as the leaves. Each T^*_qQ is a surface of constant q which diffeomorphic to $\mathbb{R}^n$, where n is the dimension of Q . Taking canonical coordinates $\left(p_a, q^b\right)$ on T^*Q and coordinates (x_a) on $\mathbb{R}^n$, the cotangent spaces are embedded as:

$$\begin{array}{rcl} \gamma : \mathbb{R}^n & \rightarrow & T^*_qQ \subset T^*Q \\ x_a & \mapsto & \left(x_a, q^b\right) \end{array}$$

and so the pushed forward tangent spaces are:

$$\begin{array}{rcl} \gamma_* : T_x\mathbb{R}^n & \rightarrow & T_{(x,q)}\left(T^*Q\right) \\ X^a\partial/\partial x^a & \mapsto & X^a\left(\partial p_b/\partial x^a \partial/\partial p_b + \partial q^b/\partial x^a \partial/\partial q^b\right) \\ & = & X^a\partial/\partial p_a \end{array}$$

The vertical polarization therefore consists of "momentum vector fields" and:

$$V_P\left(\mathcal{M}\right) = \left\{X \in V\left(\mathcal{M}\right) \mid X_m = X^a\partial/\partial p_a\mid_m\right\}$$

It is integrable and Lagrangian since:

$$\begin{array}{rcl} \left[X^a\partial/\partial p_a, Y^b\partial/\partial p_b\right] & = & \left(X^a\frac{\partial Y^b}{\partial p_a} - Y^a\frac{\partial X^b}{\partial p_a}\right)\partial/\partial p_b \in V_P\left(\mathcal{M}\right) \\ \\ \omega\left(X, Y\right) & = & dp_a \wedge dq^a\left(X^a\partial/\partial p_a, Y^b\partial/\partial p_b\right) \\ & = & X^aY^b\left[dp_a \otimes dq^a\left(\partial/\partial p_a, \partial/\partial p_b\right) - dq^a \otimes dp_a\left(\partial/\partial p_a, \partial/\partial p_b\right)\right] \\ & = & 0 \end{array}$$

The canonical one form gives zero on restriction to P :

$$\begin{array}{rcl} \theta\left(X\right) & = & p_a dq^a\left(X^a\partial/\partial p_a\right) \\ & = & 0 \end{array}$$

The set of polarized sections $P\left(B\right)$ then contains sections which can be locally expressed as functions of q alone, so the vertical polarization gives a coordinate independent way of imposing what we need.

6.3.2 Other Real Polarizations

It turns out that the conditions a real polarization must satisfy are very restrictive, so that there are not many possibilities, and locally they all have the structure of vertical polarizations. More precisely, given a real polarization P there exists a neighborhood of each point $m \in \mathcal{M}$ with canonical coordinates $\left(p_a, q^b\right)$ and a symplectic potential θ such that around m , P_n consists of the "vertical vectors" spanned by $\{{}^{\partial}/_{\partial p_a}\}$ and $\theta\mid_P = 0$. These coordinates and the potential θ are then "adapted" to P . [11]

We can use this to show that for any polarization:

$$f \in C_P^{\infty}\left(\mathcal{M}\right) \quad iff \quad X_f \in V_P\left(\mathcal{M}\right)$$

Let $f \in C_P^{\infty}\left(\mathcal{M}\right)$ then for any $X \in V_P\left(\mathcal{M}\right)$:

$$\begin{aligned} 0 &= X\left(f\right) \\ &= X^a {}^{\partial f}/_{\partial p_a} \end{aligned}$$

and so f must be of the form:

$$\begin{aligned} {}^{\partial f}/_{\partial p_a} &= 0 \\ f &= f\left(q\right) \end{aligned}$$

and the Hamiltonian vector field of f is:

$$X_f = -{}^{\partial f}/_{\partial q^a}{}^{\partial}/_{\partial p_a} \in V_P\left(\mathcal{M}\right)$$

Conversely, let $X_f \in V_P\left(\mathcal{M}\right)$ then simply looking at the form of X_f :

$$X_f = {}^{\partial f}/_{\partial p_a}{}^{\partial}/_{\partial q^a} - {}^{\partial f}/_{\partial q^a}{}^{\partial}/_{\partial p_a}$$

we see that ${}^{\partial f}/_{\partial p_a} = 0$ and so $f \in C_P^{\infty}\left(\mathcal{M}\right)$.

6.3.3 Polynomial Observables

We shall see later that only certain observables map polarized sections to other polarized sections, but this actually has another geometric origin.

Given an observable $f \in C^{\infty}(\mathcal{M})$ it generates a flow through its Hamiltonian vector field X_f , which acts as local differemorphisms on $\mathcal{M}$. This maps $T\mathcal{M}$ to itself in the following way:

$$\begin{aligned} T\mathcal{M} &\rightarrow T\mathcal{M} \\ (m, T_m\mathcal{M}) &\mapsto \left(\rho_{X_f}(m), \rho_{X_f*}T_m\mathcal{M}\right) \end{aligned}$$

We can ask the question, which observables "preserve" a polarization P , i.e. for all $m \in \mathcal{M}$:

$$\rho_{X_f*}(P_m) \subset P_{\rho_{X_f}(m)}$$

which is the same as:

$$\rho_{X_f*}(X) \in V_P(\mathcal{M}) \quad \forall X \in V_P(\mathcal{M})$$

Choose some point $m \in \mathcal{M}$ and say that for some choice of parameter t :

$$\begin{aligned} \rho_{X_f}(t, m) &= n \\ \rho_{X_f,t*} : T_m\mathcal{M} &\rightarrow T_n\mathcal{M} \end{aligned}$$

and also, for convenience let t be small enough that m and n can be covered by some canonical coordinates $\left(p_a, q^b\right)$ adapted to P . We can then write the push forward of $Y \in P_m$ as:

$$Y^a \partial/\partial p_a\,|_m \mapsto Y^a \left(\partial p(n)_b/\partial p^a \partial/\partial p_b\,|_n + \partial q(n)^b/\partial p^a \partial/\partial q^b\,|_n\right)$$

If the result is a vector in $V_P(\mathcal{M})$ then the second term must be zero, so:

$$\partial q(n)^a/\partial p_b = 0$$

The coordinates $q(n)^a$ of n are determined by Hamilton's equations which describe the flow of f :

$$\frac{dq^a}{dt} = \frac{\partial f}{\partial p_a}$$

Therefore, we must have:

$$\frac{\partial^2 f}{\partial p_a \partial p_b} = 0$$

So if an observable f preserves a polarization P , then in any adapted coordinates it must be linear in the momentum coordinates and so of the form:

$$f = v^a (q) \, p_a + u (q)$$

[11]We denote the set of observables preserving P by $S_P (\mathcal{M})$:

$$S_P (\mathcal{M}) \equiv \{ f \in C^\infty (\mathcal{M}) \mid f = v^a (q) \, p_a + u (q) \}$$

Note that the functions constant along P are of the form $f (q)$ so that:

$$C_P^\infty (\mathcal{M}) \subset S_P (\mathcal{M})$$

7 Full Quantization

We have seen that if a symplectic manifold $\mathcal{M}$ is is pre quantized with pre quantum bundle B its possible to single out sections of the pre quantum bundle depending only on the position coordinates by introducing a polarization, P , which determines a set of polarized sections $P (B) \subset C_B^\infty (\mathcal{M})$. Its natural to then try and construct the quantum Hilbert space as:

$$\mathcal{H} = \mathcal{H}_B \cap P (B)$$

But, we shall see that this doesn't quite work and must be modified.

7.1 Operators

If we are going to construct the quantum Hilbert space in some way from polarized sections $P (B)$ then its clear that any operators we let act on $P (B)$ must map it to itself. We will call the set of these operators $\mathcal{O} (P (B))$. If $\widehat{f}$

is such an operator, then for all $X \in V_P(\mathcal{M})$ and $s \in P(B)$ we must have:

$$\begin{aligned}
0 &= \nabla_X \left(\widehat{f} s\right) \\
&= \nabla_X \left(-i\hbar \nabla_{X_f} s + f s\right) \\
&= \left(-i\hbar \nabla_X \nabla_{X_f} + X(f) + f \nabla_X\right) s \\
&= \left[-i\hbar \left(-\tfrac{i}{\hbar}\omega(X, X_f) + \nabla_{X_f}\nabla_X + \nabla_{[X,X_f]}\right) + X(f) + f\nabla_X\right] s \\
&= \left[-i\hbar \left(-\tfrac{i}{\hbar}X(f) + \nabla_{X_f}\nabla_X + \nabla_{[X,X_f]}\right) + X(f) + f\nabla_X\right] s \\
&= \left[\left(-i\hbar \nabla_{X_f}\nabla_X + f\nabla_X\right) - i\hbar \nabla_{[X,X_f]}\right] s \\
&= \widehat{f}\nabla_X s - i\hbar \nabla_{[X,X_f]} s \\
&= -i\hbar \nabla_{[X,X_f]} s
\end{aligned}$$

Therefore:

$$[X, X_f] \in V_P(\mathcal{M})$$

If we choose $X = \partial/\partial p_a$ then:

$$\begin{aligned}
\left[\partial/\partial p_a, X_f\right] &= \left[\partial/\partial p_a, \partial f/\partial p_b \partial/\partial q^b - \partial f/\partial q^b \partial/\partial p_b\right] \\
&= \partial^2 f/\partial p_a \partial p_b \partial/\partial q^b - \partial^2 f/\partial p_a \partial q^b \partial/\partial p_b
\end{aligned}$$

and so f satisfies:

$$\partial^2 f/\partial p_a \partial p_b = 0$$

which we recognize from section 6.3.3 as the condition for f to preserve P . The set of operators that map $P(B)$ to itself is therefore the set of operators corresponding to observables that preserve the polarization:

$$\mathcal{O}(P(B)) = \left\{\widehat{f} = -i\hbar \nabla_{X_f} + f \;\middle|\; f \in S_P(\mathcal{M})\right\}$$

This means we are reduced to only constructing observables of the form:

$$f = v^a(q)\, p_a + u(q)$$

The Hamiltonian vector field of such a function is:

$$X_f = v^a \partial/\partial q^a - \left(p_a \partial v^a/\partial q^b \partial/\partial q^b + \partial u/\partial q^b\right) \partial/\partial p_b$$

The operator $\widehat{f}$ then acts on $s \in P(B)$ as:

$$\begin{aligned} -i\hbar\nabla_{X_f} s + fs &= \left(-i\hbar v^a \partial/\partial q^a - p_a v^a\right) s + \left(v^a p_a + u\right) s \\ &= \left[-i\hbar v^a (q)\, \partial/\partial q^a \psi_i + u(q)\, \psi_i\right] s_i \end{aligned}$$

in some local trivialization.

7.2 The Hilbert Space $\mathcal{H}$

7.2.1 Problems with Polarized Sections

The problem with the definition:

$$\mathcal{H} \ = \ \mathcal{H}_B \cap P(B)$$

is that $\mathcal{H} = \emptyset$! There are no polarized square integrable sections because the inner product will diverge as we're integrating freely over the momentum coordinates (p_a) which span all of $\mathbb{R}^4$. In the case of $\mathcal{M} = T^*\mathbb{R}^n$ this is easily seen as:

$$\begin{aligned} \langle s, s\rangle &= {}_{\mathbb{R}^{2n}} (s, s)\, \epsilon \\ &= {}_{\mathbb{R}^{2n}} \left|\psi(q)\right|^2 \epsilon \\ &= \left({}_{\mathbb{R}^n} \left|\psi(q)\right|^2 d^n q\right) \left({}_{\mathbb{R}^n} d^n p\right) \end{aligned}$$

The solution to this problem is to construct an inner product defined by integration over Q . In order to do this, we have to consider objects which are "square roots" of n-forms on Q , so that when squared they give a volume form which can be integrated. This is called *half-density quantization.*

8 Concluding Remarks

We have seen that Prequantization satisfies Dirac's rules given at the beginning of this work, but the construction has to be altered on physical grounds by introducing a polarization. The final stage is the use of half densities, or one of the other methods that can be used e.g. metaplectic structures and kahler polarizations, see [11]. If I had more time I would have liked to have described some of these techniques.

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References

[1] R. Abraham, J. E. Marsden, *Foundations of Mechanics, Second Edition,* Benjamin/Cummings

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics,* Springer-Verlag

[3] J. Baez, *Lectures on Classical Mechanics*, lecture notes available at http://math.ucr.edu/home/baez/classical/

[4] M. Blau, *Symplectic Geometry and Geometric Quantization*, lecture notes available at http://www.blau.itp.unibe.ch/Lecturenotes.html

[5] P. A. M. Dirac, *The Principles of Quantum Mechanics, Fourth Edition,* Oxford

[6] B. Hall, *Lie Groups, Lie Algebras and Representations*, Springer

[7] I. Madsen, J. Tornehave, *From Calculus to Cohomology*, Cambridge

[8] M. Nakahara, *Geometry, Topology and Physics, Second Edition,* Taylor & Francis

[9] W. G. Ritter, *Geometric Quantization*, math-ph/0208008

[10] W. Rudin, *Functional Analysis, TMH Edition,* Tata McGraw-Hill Publishing

[11] N. M. J. Woodhouse, *Geometric Quantization, Second Edition,* Oxford