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New Ansatz for the C operator in PT-Symmetric Quantum Field Theory with Cubic Interaction

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Abstract

The reformulation of Quantum Mechanics in terms of $\mathcal{PT}$-symmetric Hamiltonians had its advent about a decade ago when it was realized that a Hamiltonian posessing unbroken $\mathcal{PT}$ symmetry exibits a real and positive spectrum which is bounded below and that the time evolution of such a theory is unitary (probability preserving). Thus the axiom in the Dirac-von Neumann formulation of Quantum Mechanics which states that a Hamiltonian should be Hermitian for the theory to be physical, although sufficient, is not necessary. This paper is started by giving a brief summary of the properties that a quantum mechanical theory described by such Hamiltonians must possess. If the Hamiltonian has an unbroken $\mathcal{PT}$ symmetry then there exists a hidden symmetry of the Hamiltonian which allows us to construct an inner product which is time-independet and has a positive norm. This symmetry is represented by a linear operator $\mathcal{C}$ which is believed to be non-unique. The construction of this operator requires the realization that $\mathcal{C}$ is a time-independent $\mathcal{PT}$-symmetric reflection operator with an operator representation given by $\mathcal{C} = e^{Q(\hat{x},\hat{p})}\mathcal{P}$, where $\mathcal{P}$ is the space reflection, or *parity*, operator. To perform the calculation for $\mathcal{C}$ we have to take an ansatz for Q based on the symmetry conditions that this operator has to satisfy. We will give a brief review of how the $\mathcal{C}$ operator has been determined for a Quantum mechanical theory described by a Hamiltonain of the form $H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2 + i\hat{x}^3$ and similarly for a field theory with cubic interaction. We then conjecture an alternative ansatz for the cubic field theory by adding two terms to the original one and proceed to try and determine the coefficient functions of the ansatz by imposing the condition that the $\mathcal{C}$ operator should transform as a Lorentz scalar. Although the conditions we impose on $\mathcal{C}$ are not enough to fully determine the coefficient functions, we find that requiring that $\mathcal{C}$ should transform as a Lorentz scalar leaves complete freedom on the coefficient functions of the terms dependent on the spatial derivatives, and thus the non-uniqueness of the $\mathcal{C}$ operator persists even after enforcing the Lorentz scalar conditon.

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Contents

Chapter 1

Introduction

1.1 Foundations of a Quantum Theory and conditions for the reality of the eigenvalues of a Hamiltonian

For a Quantum theory to be physically acceptable it must satisfy three main axioms. The energy spectrum must be real because all measurements of the energy of a system yield real results. The energy spectrum must be bounded below so that the system has a stable lowest-energy state, and the time evolution of a quantum system must be *unitary* (probability-conserving) because the expected result of a probability measurement of a state cannot grow or decay in time. A relativistic quantum theory has to satisfy in addition to these axioms the physical axioms of Lorentz covariance and causality.
In the Dirac-von Neumann formulation of quantum mechanics these axioms are satisfied be imposing that all physical observables be represented by self-adjoint or Hermitian operators on a Hilbert space, that is for any observable A we must have

$$A = A^{\dagger}. \tag{1.1}$$

This condition on the observables, which of course includes also the Hamiltonian of the theory, ensures that the theory satisfies the above axioms. Although the condition of Hermiticity ensures that the theory be physically acceptable it was noticed by Bender and colleagues that this restriction on the Hamiltonian is sufficient but not necessary. It was shown theoretically that a more general condition to be imposed on the Hamiltonian, which maintains the requirements for the theory to be physical, is to replace the condition of Hermiticity with the more physical condition that the Hamiltonian should posses space-time reflection symmetry, that is:

$$H = H^{\mathcal{PT}}. \tag{1.2}$$

Where the space-reflection operator, or *parity* operator, is represented by the symbol $\mathcal{P}$. This operator acts on the basic dynamical variables of a quantum theory, that is on the position operator $\hat{x}$ and the momentum operator $\hat{p}$, as:

$$\mathcal{P}\hat{x}\mathcal{P} = -\hat{x} \qquad and \qquad \mathcal{P}\hat{p}\mathcal{P} = -\hat{p}. \tag{1.3}$$

We remark that $\mathcal{P}$ is a linear operator and that it leaves invariant the fundamental commutation

relation (the Heisenberg algebra) of quantum mechanics,

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\mathbf{1}, \tag{1.4}$$

where $\mathbf{1}$ is the identity matrix. Similarly the time-reversal operator is represented by the symbol $\mathcal{T}$. This operator leaves $\hat{x}$ invariant but changes the sign of $\hat{p}$:

$$\mathcal{T}\hat{x}\mathcal{T} = \hat{x} \qquad and \qquad \mathcal{T}\hat{p}\mathcal{T} = -\hat{p}. \tag{1.5}$$

Like the parity operator $\mathcal{P}$, the time-reversal operator $\mathcal{T}$ leaves the commutation relation 1.4 invariant. This condition thus requires that $\mathcal{T}$ reverse the sign of the complex number i:

$$\mathcal{T}i\mathcal{T} = -i \tag{1.6}$$

This last property, equation 1.6, shows us that $\mathcal{T}$ is not a linear operator but rather $\mathcal{T}$ is said *antilinear*. The $\mathcal{P}$ and $\mathcal{T}$ are also reflection operators, which means that their squares are the unit operator:

$$\mathcal{P}^2 = \mathcal{T}^2 = \mathbf{1}. \tag{1.7}$$

Yet another property that these two operators satisfy is that they commute with each other:

$$[\mathcal{P}, \mathcal{T}] = 0. \tag{1.8}$$

We thus define the $\mathcal{PT}$-reflected Hamiltonian in 1.2 as $H^{\mathcal{PT}} \equiv (\mathcal{PT})H(\mathcal{PT})$.
It is then clear that if a Hamiltonian is $\mathcal{PT}$-symmetric, then it commutes with $\mathcal{PT}$:

$$[H, \mathcal{PT}] = 0. \tag{1.9}$$

The fact that a Hamiltonian need not necessarily be Hermitian but may be non-Hermitian and $\mathcal{PT}$ symmetric to describe a fully consistent quantum theory means that we are now in a position to reconsider many Hamiltonians which have been discarded in the past. One example of such a Hamiltonian is that used to describe the Lee model (see [3],[2]). This theory has been rejected in the past as the Hamiltonian is not Hermitian and was thus believed not to have real and positive eigenvalues. We now know that this theory describes a perfectly good quantum theory as the Hamiltonian has space-time reflection symmetry. Many strange looking Hamiltonians which are special cases deriving from the general parametric family of $\mathcal{PT}$-symmetric Hamiltonians

$$H = \hat{p}^2 + \hat{x}^2(i\hat{x})^\epsilon, \tag{1.10}$$

where the parameter ϵ is real can now also be considered because these Hamiltonians are all $\mathcal{PT}$-symmetric as they satisfy the condition in 1.2. It has been shown[1] that when $\epsilon \geq 0$ all of the eigenvalues of these Hamiltonians are entirely real and positive, but when $\epsilon < 0$ there are complex eigenvalues.The region with $\epsilon \geq 0$ is the region of *unbroken* $\mathcal{PT}$-symmetry while that of $\epsilon < 0$ is the parametric region of *broken* $\mathcal{PT}$-symmetry, see Figure 1.1.

Looking at Figure 1.1 we notice that as ϵ decreases past zero the real eigenvalues diminish giving way to complex conjugate pairs of eigenvalues. Note that the Hamiltonian for the Harmonic

[1] see [3],[2],[5] and [11]

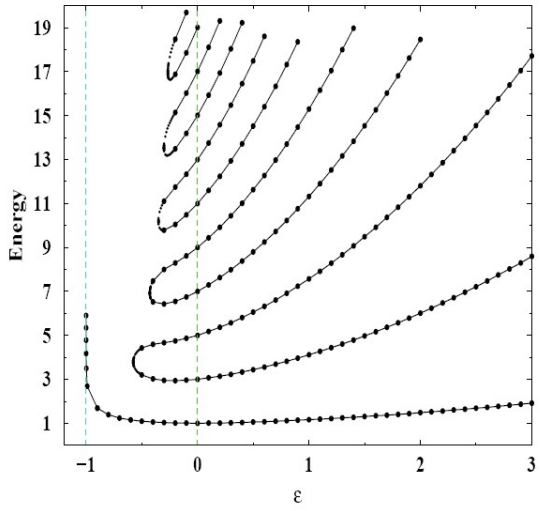

Figure 1.1: Taken from [3], these are the energy levels of the Hamiltonian $H = \hat{p}^2 + \hat{x}^2(i\hat{x})^\epsilon$ as a function of the real parameter ϵ. When $\epsilon \geq 0$, the spectrum is real and positive and the energy levels rise with increasing ϵ. $\epsilon = 0$ corresponds to the Harmonic oscillator, whose energy levels are $E_n = 2n + 1$. When $-1 < \epsilon < 0$, there are a finite number of real positive eigenvalues and an infinite number of complex conjugate pairs of eigenvalues. When $\epsilon \geq 0$ the $\mathcal{PT}$ symmetry is unbroken, but when $\epsilon < 0$ the $\mathcal{PT}$ symmetry is broken

oscillator is at the boundary of the broken and unbroken $\mathcal{PT}$ symmetry region, i.e. $\epsilon = 0$.

The neatness of $\mathcal{PT}$-symmetric Quantum mechanics comes from the fact that the condition on the Hamiltonian that it should satisfy Dirac Hermiticity is replaced by the simple condition of $\mathcal{PT}$-symmetry, 1.2. This feature is referred to as more physical because the $\mathcal{P}$ and $\mathcal{T}$ are both elements of the homogeneous Lorentz group of spatial rotations and Lorentz boosts, [3]. Interestingly the first observation of $\mathcal{PT}$-symmetry breaking has been made recently via experiments using complex optical potentials, [12].

1.2 The Hamiltonian defines and determines the physical properties of a Quantum Theory

We here give a brief review of how the Hamiltonian defines and determines the physical properties of a quantum theory, how to calculate the eigenvalues of a $\mathcal{PT}$-symmetric Hamiltonian and what conditions must the eigenvalue equation satisfy for these eigenvalues to exist, how to determine whether non-Hermitian Hamiltonians such as that in 1.10 define physical theories of quantum mechanics and in what way the the Hilbert space for for such a Hamiltonian is constructed.

Quantum mechanics is a mathematical representation of experimentally measurable quantities on a Hilbert space where the norm of a vector in the Hilbert space represents a probability. Since a probability must be real and positive we require the norm on the Hilbert space to be real and positive. In addition the inner product between any two different vectors in the Hilbert space must be constant in time because probability is conserved. This last requirement is a fundamental property of any quantum theory, called unitarity, and it must not be violated. Of cental importance to any quantum theory is the Hamiltonian operator, which specifies three main properties of the

physical system being described[2] (see [3]):

(i) The Hamiltonian determines the energy eigenstates $|E_n\rangle$. These states are the eigenstates of the Hamiltonian operator and they solve the time-independent Schrodinger equation

$$H|E_n\rangle = E_n|E_n\rangle. \tag{1.11}$$

The energy eigenstates span the Hilbert space of physical state vectors. The eigenvalues E_n are the energy levels of the quantum theory. Because in principle one can measure these energy levels, the outcome of such a physical measurement is a real number, so it is essential that these energy eigenvalues be real.

(ii) The Hamiltonian H determines the time evolution of the theory. States $|t\rangle$ in the Schrödinger picture evolve in time according to the time-dependent Schrödinger equation $H|t\rangle = i\frac{d}{dt}|t\rangle$, whose formal solution is $|t\rangle = e^{iHt}|0\rangle$. Operators $A(t)$ in the Heisenberg picture evolve according to the time-dependent Schrödinger equation $\frac{d}{dt}A(t) = -i[A(t), H]$, whose formal solution is $A(t) = e^{iHt}A(0)e^{-iHt}$.

(iii) The Hamiltonian incorporates the symmetries of the theory. A quantum theory may have two kinds of symmetries: continuous symmetries, such as Lorentz invariance, and discrete symmetries, such as parity invariance and time reversal invariance. A quantum theory is symmetric under a transformation represented by an operator A if A commutes with the Hamiltonian that describes the quantum theory: $[A, H] = 0$. We also remark that if a symmetry transformation is represented by a linear operator A and if A commutes with the Hamiltonian, then the eigenstates of H are simoultaneous eigenstates of A. Two important Discrete symmetry operators are parity (space reflection) and time reversal $\mathcal{T}$, which were described in the previous section.

1.2.1 Properties of non-Hermitian Hamiltonians

As mentioned above the reality of the eigenvalues for a non-Hermitian Hamiltonian depend on the $\mathcal{PT}$-symmetry of the system. For the eigenvalues to be real we require the Hamiltonian to have unbroken $\mathcal{PT}$-symmetry. We say that the Hamiltonian has unbroken $\mathcal{PT}$-symmetry if all the eigenfunctions of H are simoultaneously eigenfunctions of $\mathcal{PT}$. Note that since the $\mathcal{PT}$ operator is not linear, i.e. antilinear, the condition that H commutes with $\mathcal{PT}$ does not imply that the eigenfunctions of H are necessarily eigenfunctions of $\mathcal{PT}$. In the case of the eigenvectors being simoultaneous eigenvectors of both H and $\mathcal{PT}$ it is then trivial to show that the $\mathcal{PT}$-symmetric Hamiltonian will have all of its eigenvalues which are real.
Assume that ϕ is an eigenstate of H with eigenvalue E and simoultaneously an eigenstate of $\mathcal{PT}$ with eigenvalue λ:

$$H\phi = E\phi \tag{1.12}$$

and

$$\mathcal{PT}\phi = \lambda\phi. \tag{1.13}$$

If we now multiply 1.13 on the left by $\mathcal{PT}$ and use properties 1.8 and 1.7 we conclude that $\phi = \lambda^*\lambda\phi$ and therefore that $\lambda = e^{i\alpha}$ for some real α. We thus see that the eigenvalue λ is a

[2] we here take $\hbar = 1$

pure phase. To simplify the future calculations we now introduce a convention, without loss of generality we replace the eigenfunction ϕ by $e^{-i\alpha/2}\phi$ so that its eigenvalues under the $\mathcal{PT}$ operator is unity:

$$\mathcal{PT}\phi = \phi. \tag{1.14}$$

Multiplying the eigenvalue equation 1.12 on the left by $\mathcal{PT}$ and use 1.9 we obtain $E\phi = E^*\phi$. Therefore $E = E^*$ and the eigenvalue E is real.
The eigenvalues satisfying the Schrödinger equation depend crucially on the bounadry conditions and the techniques used for calculating these, i.e. the energy levels, are those used for calculating the energy levels of conventional Hermitan Hamiltonians. The procedure is to first convert the formal eigenvalue problem $H\phi = E\phi$ to a Schrödinger differential equation whose solutions satisfy appropriate boundary conditions. The so obtained deifferential equation is then solved either numerically or by using approximate methods such as WKB[3]. As an example consider the Hamiltonian, [2];

$$H = \hat{p}^2 + (i\hat{x})^N. \tag{1.15}$$

The Schrödinger eigenvalue problem for this Hamiltonian is

$$-\phi_n''(x) - (ix)^N\phi_n(x) = E_n\phi_n(x) \tag{1.16}$$

where E_n is the nth eigenvalue. In the Dirac-von Neumann formulation of quantum mechanics, that is for a Hermitian Hamiltonian, the boundary conditions that give quantized energy levels, E_n, are that $\phi_n(x) \to 0$ as $|x| \to \infty$ on the real axis. This suffices for 1.16 when $1 < N < 4$, but when $N \geq 4$ we have to continue the eigenvalue problem into the complex-x plane. We therefore replace the real x-axis by a contour, C, in the complex plane along which our differential equation holds. The boundary conditions leading to quantization are then to be imposed at the end points of this contour. These end points lie in regions in the complex-x plane where $\phi_n(x) \to 0$ exponentially as $|x| \to \infty$. These regions are known as Stokes wedges, see Figure 1.2.

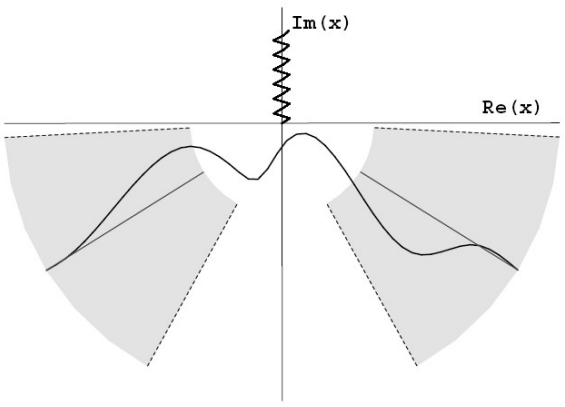

Figure 1.2: Taken from [2] this diagram shows the wedges in the complex-x plane containing the contour in which the eigenvalue problem for the differential equation 1.16 for $N = 4.2$. In this region $\phi(x)$ vanishes exponentially as $|x| \to \infty$

The location of the Stokes wedges are determined by the exponential contribution to the leading

[3]see [1]

behaviour of the solutions[4]. For example, say that the leading behaviours of solutions to a second-order differential equation are $e^{S_1(z)}$ and $e^{S_2(z)}$ as $z \to z_0$ (where z_0 is the point at which the solutions become asymptotic), then the Stokes lines are defined to be the asymptotes as $z \to z_0$ of the curves

$$Re[S_1(z) - S_2(z)] = 0. \tag{1.17}$$

Similarly the *anti*-Stokes lines[5] are the asymptotes as $z \to z_0$ of the curves

$$Im[S_1(z) - S_2(z)] = 0. \tag{1.18}$$

Now the Stokes and anti-Stokes lines have been defined as asymptotes because these asymptotes determine the opening angles of the wedges of validity of the asymptotic relations, [1]. It should be noted that these concepts are a local property of functions, and they are only meaningfull in the immediate vicinity of z_0.

The answer to the question of whether a non-Hermitian Hamiltonian defines a physically acceptable quantum theory was found in [[13],[7]]. The properties that a *physical* quantum theory must satisfy are:

(i) it must possess a Hilbert space of state vectors and the Hilbert space must have an inner product with a positive norm; (ii) the time evolution of the theory must be unitary, this means that the norm must be preserved in time.

In constructing a theory described by a non-Hermitian Hamiltonian we find that we are not able to define *a priori* the inner product on the Hilbert space on which the Hamiltonian acts as is the case for ordinary Hermitian quantum mechanics[6]. In fact the inner product will have to be discovered throughout the course of the analysis. The core of the problem is that the Hamiltonian itself determines the the Hilbert space on which it acts and and a consequence it itself determines its own inner product. It was initially thought that a resonable inner product might be one with respect to $\mathcal{PT}$ conjugation, i.e.

$$(\psi, \phi) \equiv \int_C dx[\psi(x)]^{\mathcal{PT}} \phi(x) = \int_C dx[\psi(-x)]^* \phi(x), \tag{1.19}$$

where C is the contour in the Stokes wedges of Figure 1.2. This definition of the inner product satisfies the requirement that pairs of eigenfunctions of H associated with different eigenvalues are orthogonal, but it fails to satisfy the requirement that the norm of a state should always be positive. The solution to this problem was solved by discovering a hidden symmetry of the Hamiltonian itself. In order to construct an inner product which gives a positive norm on a Hilbert space defined by a complex non-Hermitian Hamiltonian having an *unbroken* $\mathcal{PT}$-symmetry, a new linear operator, $\mathcal{C}$, is constructed. This operator is required to satisfy $[\mathcal{C}, \mathcal{PT}] = 0$ and, being a symmetry of the Hamiltonian, it must also satisfy $[H, \mathcal{C}] = 0$. Having this new operator at our disposal we define a new inner product, an inner product with respect to $\mathcal{CPT}$ conjugation, i.e.

$$\langle \psi | \chi \rangle^{\mathcal{CPT}} = \int dx \psi^{\mathcal{CPT}}(x) \chi(x), \tag{1.20}$$

[4]See [1] and [2].

[5]These are the lines along which the leading behaviours are most unequal.

[6]In the Dirac-von Neumann formulation the inner product is an inner product with respect to hermitian conjugation.

where $\psi^{\mathcal{CPT}}(x) = \int dy \mathcal{C}(x,y)\psi^*(-y)$. This definition of inner product satifies the requirements for the quantum theory defined by H to have a positive norm and a unitary time evolution, [3].

1.2.2 Observables in $\mathcal{PT}$-symmetric quantum mechanics

In ordinary quantum mechanics the condition for a linear operator A to be an observable is that it should be Hermitian or *self-adjoint*, i.e. it must satify $A = A^\dagger$. This condition ensures that the expectation value of the operator in a state is real. Furthermore, we know that this Hermiticity condition is maintained in time since in the Heisenberg picture operators evolve in time according to $A(t) = e^{iHt}A(0)e^{-iHT}$. The equivalent condition to be satisfied by linear operators in $\mathcal{PT}$-symmetric quantum mechanics is that at time $t = 0$ the condition

$$A^T = \mathcal{CPT}A\mathcal{CPT}, \tag{1.21}$$

where A^T is the *transpose* of A has to hold. If this is the case then we know that this condition will continue to hold because we have assumed that H is $\mathcal{PT}$ symmetric. In addition this condition guarantees that the expectation value of A for all states is real. Because of 1.8 we see that the operator $\mathcal{C}$ itself satisfies 1.21 and is thus an observable. The Hamiltonian is also an observable. Contrary to ordinary quantum mechanics though the position and momentum operators are not observables, in fact the expectation value of $\hat{x}$ in the ground state is a negative imaginary number[7].

[7] for more details see [2] and [3]

Chapter 2

The $\mathcal{C}$ operator

2.1 Construction and properties of the $\mathcal{C}$ operator

The problem with 1.19 is that although it satisfies the requirement that pairs of eigenfunctions of H associated with different eigenvalues are orthogonal it fails to satisfy the requirement that the norm of a state should always be positive. This problem arises from the property that $\mathcal{PT}$-symmetric Hamiltonians which describe quantum theories have a space of quantum states which is spanned by energy eigenstates of which half have norm $+1$ and half have norm -1. This is expressed by the relation;

$$(\phi_m, \phi_n) = (-1)^n \delta_{mn}, \tag{2.1}$$

where ϕ_m and ϕ_n are eigenfunctions.
Since, see previous chapter, a quantum theory attributes a probabilistic interpretation to the norms of states, this indefiniteness of the metric, i.e. 2.1, is inconsistent. The resolution of this problem was achieved through the discovery that *any* theory having an unbroken $\mathcal{PT}$-symmetry has a hidden symmetry satisfied by the Hamiltonian. This symmetry is linked to the fact that there are equal numbers of positive and negative norm states, see 2.1. The way to describe this hidden symmetry is to construct a linear operator, the $\mathcal{C}$ operator, in position space as a sum over the eigenstates of the Hamiltonian[1]:

$$\mathcal{C}(x, y) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y). \tag{2.2}$$

A Hermitian Hamiltonian has a set of eigenstates which are complete, and there is good evidence that the eigenfunctions $\phi_n(x)$ for a $\mathcal{PT}$-symmetric Hamiltonian are also complete. The coordinate-space statement of completeness is

$$\sum_{n=0}^{\infty} (-1)^n \phi_n(x)\phi_n(y) = \delta(x - y), \tag{2.3}$$

where x and y are real. Using 2.3, 2.1 and 1.19 we are able to verify that the square of $\mathcal{C}$ is unity ([2], [3]), thus the eigenvalues of $\mathcal{C}$ are ± 1. Hence, as $\mathcal{C}$ is linear and as it is itself a symmetry of the Hamiltonian, the eigenstates of H have definite values of $\mathcal{C}$. In fact, if the eigenstates of H

[1] See [7].

satisfy 2.1, we have that

$$\mathcal{C}\phi_n(x) = \int dy \mathcal{C}(x,y)\phi_n(y) = \sum_{m=0}^{\infty} \phi_n(x) \int dy \phi_m(y)\phi_n(y) = (-1)^n \phi_n(x). \tag{2.4}$$

We thus understand that $\mathcal{C}$ represents the measurement of the sign of the $\mathcal{PT}$ norm of an eigenstate, 2.1.

2.2 Calculation of the $\mathcal{C}$ operator in Quantum mechanics

In $\mathcal{PT}$ Quantum mechanics the evaluation of the formal sum 2.2 is non-trivial, and thus makes it problematic to determine the $\mathcal{C}$ operator. The evaluation of the sum is non-trivial as it requires us to calculate all the eigenfunctions $\phi_n(x)$ of H. Furthermore such a procedure cannot be performed in quantum field theory as there exist no simple analogue of the schrödinger eigenvalue differential equation and its associated coordinate-space eigenfunctions, [2]. To avoid these difficulties we make use of a technique which relies on three properties of the $\mathcal{C}$ operator itself. This technique is readily generalized to quantum field theory. The three properties used for the calculation of $\mathcal{C}$ are:
i) $\mathcal{C}$ commutes with the space-time reflection operator $\mathcal{PT}$,

$$[\mathcal{C}, \mathcal{PT}] = 0, \tag{2.5}$$

it should be noticed that $\mathcal{C}$ does not commute with $\mathcal{P}$ or $\mathcal{T}$ separately.
ii) The square of $\mathcal{C}$ is the identity,

$$\mathcal{C}^2 = \mathbf{1}, \tag{2.6}$$

which means that we can interpret $\mathcal{C}$ as a reflection operator just as $\mathcal{P}$ and $\mathcal{T}$.
iii) $\mathcal{C}$ commutes with H,

$$[\mathcal{C}, H] = 0 \tag{2.7}$$

and is hence time independent. Therefore $\mathcal{C}$ is a time-independent $\mathcal{PT}$-symmetric reflection operator. To proceed with the calculation we introduce a general operator representation for $\mathcal{C}$ of the form, see [10];

$$\mathcal{C} = e^{Q(\hat{x},\hat{p})}\mathcal{P}, \tag{2.8}$$

where $Q(\hat{x},\hat{p})$ is a real function of the dynamical variables $\hat{x}$ and $\hat{p}$ and $\mathcal{P}$ is the parity operator. Using this representation we can now calculate $\mathcal{C}$ directly by using the three equations 2.5, 2.6 and 2.7. Substituting 2.8 in 2.5 we find that;

$$e^{Q(\hat{x},\hat{p})} = \mathcal{PT}e^{Q(\hat{x},\hat{p})}\mathcal{PT} = e^{Q(-\hat{x},\hat{p})}, \tag{2.9}$$

from this we can conclude that $Q(\hat{x},\hat{p})$ is an *even* function of $\hat{x}$. We then find a second condition on Q by substituting 2.8 in 2.6. This gives us;

$$e^{Q(\hat{x},\hat{p})}\mathcal{P}e^{Q(\hat{x},\hat{p})}\mathcal{P} = e^{Q(\hat{x},\hat{p})}e^{Q(-\hat{x},-\hat{p})} = 1 \tag{2.10}$$

which implies that $Q(\hat{x},\hat{p}) = -Q(-\hat{x},-\hat{p})$. Having determined in 2.9 that $Q(\hat{x},\hat{p})$ is an even

function of $\hat{x}$, we may conclude that it is also an *odd* function of $\hat{p}$. The condition we are thus left with is 2.7. Substituting 2.8 into 2.7 we get $e^{Q(\hat{x},\hat{p})}[\mathcal{P},H]+[e^{Q(\hat{x},\hat{p})},H]\mathcal{P}=0$. We will be showing how to perform the calculation explicitely for the $\mathcal{PT}$-symmetric Hamiltonian[2];

$$H=\frac{1}{2}\hat{p}^2+\frac{1}{2}\hat{x}^2+i\epsilon\hat{x}^3. \tag{2.11}$$

Rewriting the Hamiltonian as $H=H_0+\epsilon H_1$, where H_0 is the Harmonic oscillator Hamiltonian $H_0=\frac{1}{2}\hat{p}^2+\frac{1}{2}\hat{x}^2$, which commutes with the parity operator $\mathcal{P}$, and $H_1=i\hat{x}^3$, which anticommutes with $\mathcal{P}$, the above condition becomes

$$2\epsilon e^{Q(\hat{x},\hat{p})}H_1=[e^{Q(\hat{x},\hat{p})},H]. \tag{2.12}$$

It is still not understood why, but only the odd powers of ϵ contribute to Q, therefore we may expand $Q(\hat{x},\hat{p})$ as a series in odd powers of ϵ:

$$Q(\hat{x},\hat{p})=\epsilon Q_1(\hat{x},\hat{p})+\epsilon^3 Q_3(\hat{x},\hat{p})+\epsilon^5 Q_5(\hat{x},\hat{p})+... \tag{2.13}$$

substituting this expansion into the exponential in 2.8 we obtain after some algebra a sequence of equations which have to be solved for the operator-valued functions $Q_n(\hat{x},\hat{p})$ $(n=1,3,5,..)$ subject to the symmetry constraints we have listed above. The first two of these equations are

$$\begin{aligned}[H_0,Q_1]&=-2H_1\\ [H_0,Q_3]&=-\frac{1}{6}[Q_1,[Q_1,H_1]].\end{aligned} \tag{2.14}$$

We will only be solving these equations to first order in ϵ. Higher order terms are solved analogously. The procedure is to substitute most general polynomial form for Q_n using arbitrary coefficient and then solve for these coefficients, [2]. So to solve the first equation in 2.14 we take as an asatz for Q_1 the most general Hermitian polynomial which is even in $\hat{x}$ and odd in $\hat{p}$:

$$Q_1(\hat{x},\hat{p})=M\hat{p}^3+N\hat{x}\hat{p}\hat{x}, \tag{2.15}$$

where M and N are undetermined coefficients. The operator equation for Q_1 is then satisfied if

$$M=-\frac{4}{3}\qquad and\qquad N=-2. \tag{2.16}$$

[2]See [15]

2.3 The $\mathcal{C}$ operator for an $i\varphi^3$ Quantum field theory

In this section we extend the operator techniques introduced in Chapter 2 to quantum field theory. We perform the calculation only up to the first order in ϵ. Consider the quantum field theory in (D+1)-dimensional Minkowski space-time described by the Hamiltonian;

$$H = \int d^D x \left\{ \frac{1}{2}\pi^2(\mathbf{x},t) + \frac{1}{2}[\nabla_{\mathbf{x}}\varphi(\mathbf{x},t)]^2 + \frac{1}{2}\mu^2\varphi^2(\mathbf{x},t) + i\epsilon\varphi^3(\mathbf{x},t) \right\}. \tag{2.17}$$

This theory is non-Hermitian but $\mathcal{PT}$-symmetric, it is thus a physically acceptable theory since the eigenvalues are real. We see that this Hamiltonian has the form $H = H_0 + \epsilon H_1$, where:

$$\begin{aligned} H_0 &= \int d^D x \left\{ \frac{1}{2}\pi^2(\mathbf{x},t) + \frac{1}{2}[\nabla_{\mathbf{x}}\varphi(\mathbf{x},t)]^2 + \frac{1}{2}\mu^2\varphi^2(\mathbf{x},t) \right\}, \\ H_I &= i\int d^D x \varphi^3(\mathbf{x},t). \end{aligned} \tag{2.18}$$

Note that the above integrals are performed in the spatial variable $\mathbf{x}$, which lies in R^D. From this point onwards we will use the notation $\int d\mathbf{x} = \int d^D x$ to represent integration on R^D. The field variables satisfy the usual equal-time canonical commutation relations[3]:

$$[\varphi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y}). \tag{2.19}$$

The formal expression for the parity operator is given by[4] $\mathcal{P} = exp(\frac{1}{2}i\pi\int d\mathbf{x}[\varphi^2(\mathbf{x},t)+\pi^2(\mathbf{x},t)-1])$. In analogy with quantum mechanics, where the operators $\hat{x}$ and $\hat{p}$ change sign under parity reflection, we take the fields to be *pseudoscalars* which means that they change sign under the action of $\mathcal{P}$:

$$\mathcal{P}\varphi(\mathbf{x},t)\mathcal{P} = -\varphi(-\mathbf{x},t), \qquad \mathcal{P}\pi(\mathbf{x},t) = -\pi(-\mathbf{x},t). \tag{2.20}$$

To proceed we use the product representation for $\mathcal{C}$, only that now the Q_{2n+1} $(n = 0, 1, 2)$ terms will be real *functionals* of the field variables $\varphi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$, and continue in analogy with the quantum mechanical case. Thus to find Q_1 we are to solve the first of the operator equations in 2.14

$$[\int d\mathbf{x}(\frac{1}{2}\pi^2(\mathbf{x},t) + \frac{1}{2}\mu^2\varphi^2(\mathbf{x},t) - \frac{1}{2}\varphi(\mathbf{x},t)\nabla^2_{\mathbf{x}}\varphi(\mathbf{x},t)), Q_1] = -2i\int d\mathbf{x}\varphi^3(\mathbf{x},t), \tag{2.21}$$

where we have changed the form of the third term in H_0 using integration by parts.
We know from the conditions on $\mathcal{C}$ that Q_1 is an even functional of $\varphi(\mathbf{x},t)$ and an odd functional of $\pi(\mathbf{x},t)$. Following these constraints Bender and collegues duduced an ansatz for Q_1 of the form:

$$Q_1 = \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}M_{\mathbf{xyz}}\pi_{\mathbf{x}}\pi_{\mathbf{y}}\pi_{\mathbf{z}} + \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}N_{\mathbf{x(yz)}}\varphi_{\mathbf{y}}\pi_{\mathbf{x}}\varphi_{\mathbf{z}} \tag{2.22}$$

where, in the fields, the time variable has been suppressed and the spatial dependences have been indicated with subscripts and the brakets denote the symmetry conditions on those variables in the braket. Therefore M is totally symmetric in all three indices while N is symmetric in the second and third indices.
Although the solution obtained for $\mathcal{C}$ with this ansatz for Q_1 is succesfull in producing a $\mathcal{C}$ operator

[3] From this point onwards we will take $\hbar = 1$
[4] [8]

which not only solves the operator equation in 2.14 but also satisfies the conditions for $\mathcal{C}$ to be a Lorentz scalar this ansatz is not the most general. In fact this paper was motivated by a consideration made by Shalaby in [14] where he proposed a different ansatz to the one proposed in [8]. His claim was that his ansatz leads to a less cumbersome procedure for the determination of the $\mathcal{C}$ operator and that it leads to a solution which is local. The innovation in his paper was to assume that the metric operator is not only a functional of the field operator φ and its conjugate field π but also a functional of the field gradient, $\nabla\varphi$. That is;

$$Q_1 = C_1 \int d\mathbf{x}\pi^3(x) + C_2 \int d\mathbf{x}\varphi(x)\pi(x)\varphi(x) + C_3 \int d\mathbf{x}\nabla\varphi(x)\pi(x)\nabla\varphi(x) \tag{2.23}$$

This is a reasonable assumption for two reasons:
i) It has been shown in [9] that the solution to the algebraic equations 2.14 is non unique and thus there exist alternative posibilities for the ansatz. ii) In field theory the Hamiltonian has an explicit dependence on $\nabla\varphi$.
Although the extra term is a reasonable addition to the ansatz, Shaloby used an ansatz which in addition to this extra term had the integral coefficients which were constant. Unfortunately the claim made by Shaloby that his ansatz would greatly simplify the calculation of the $\mathcal{C}$ operator performed by Bender, Jones and Brody in [8] is false as it was shown in [4] that his calculation is incorrect, and the corrected calculations lead to inconsistent equations for the coefficients.
Motivated by the previous work by Bender and collegues and the ansatz used by Shaloby we thus duduce an alternative ansatz of the form:

$$\begin{aligned} Q_1 = \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z} \big\{ & M_{(\mathbf{xyz})}\pi_\mathbf{x}\pi_\mathbf{y}\pi_\mathbf{z} + N_{x(yz)}\varphi_\mathbf{y}\pi_\mathbf{x}\varphi_\mathbf{z} \\ & + P_{\mathbf{y}(\mathbf{xz})}(\nabla_\mathbf{x}\varphi_\mathbf{x})\pi_\mathbf{y}(\nabla_\mathbf{z}\varphi_\mathbf{z}) + Q_{\mathbf{y}(\mathbf{xz})}\varphi_x(\nabla^2_\mathbf{y}\pi_\mathbf{y})\varphi_\mathbf{z}\big\} \end{aligned} \tag{2.24}$$

In this ansatz there is now an extra two terms which differ from the one used in [8]. The third term has been inserted following Shalaby's paper, while the fourth term is a logical extension to this third term. The novelty introduced Shalaby was to introduce the extra term based on the fact that in field theory there is an extra term in the Hamiltonian which does not exist in quantum mechanics. It then only seems logical to introduce the fouth term as this is dimensionally equivalent to the third but completely independent to it[5]. To determine the coefficient functionals we now substitute this ansatz in the first operator equations in 2.14. After some algebra we are left with two operator identities which must be satisfied:

$$\begin{aligned} \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z} \big\{ & N_{\mathbf{x}(\mathbf{yz})}[\pi_\mathbf{x}\pi_\mathbf{y}\varphi_\mathbf{z} + \varphi_\mathbf{z}\pi_\mathbf{x}\pi_\mathbf{y}] + Q_{\mathbf{y}(\mathbf{xz})}[\pi_\mathbf{x}(\nabla^2_\mathbf{y}\pi_\mathbf{y})\varphi_\mathbf{z} + \varphi_\mathbf{x}(\nabla^2_\mathbf{y}\pi_\mathbf{y})\pi_\mathbf{z}] \\ & -[(\nabla_\mathbf{x}P_{\mathbf{y}(\mathbf{xz})})\pi_\mathbf{x}\pi_\mathbf{y}(\nabla_\mathbf{z}\varphi_\mathbf{z}) + (\nabla_\mathbf{z}P_{\mathbf{y}(\mathbf{xz})})(\nabla_\mathbf{x}\varphi_\mathbf{x})\pi_\mathbf{y}\pi_\mathbf{z}]\big\} \\ & = 3\iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}M_{(\mathbf{xyz})}\left\{\mu^2\pi_\mathbf{y}\varphi_\mathbf{x}\pi_\mathbf{z} - \pi_\mathbf{y}(\nabla^2_\mathbf{x}\varphi_\mathbf{x})\pi_\mathbf{z}\right\} \end{aligned} \tag{2.25}$$

[5]The third and fourth term are not related to each other by an integration by parts

$$\mu^2 \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}\left\{N_{\mathbf{x(yz)}}\varphi_{\mathbf{x}}\varphi_{\mathbf{y}}\varphi_{\mathbf{z}} + P_{\mathbf{y(xz)}}(\nabla_{\mathbf{x}}\varphi_{\mathbf{x}})\varphi_{\mathbf{y}}(\nabla_{\mathbf{z}}\varphi_{\mathbf{z}}) + (\nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}})\varphi_{\mathbf{x}}\varphi_{\mathbf{y}}\varphi_{\mathbf{z}}\right\}$$
$$- \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}\left\{N_{\mathbf{x(yz)}}\varphi_{\mathbf{y}}(\nabla^2_{\mathbf{x}}\varphi_{\mathbf{x}})\varphi_{\mathbf{z}} + P_{\mathbf{y(xz)}}(\nabla_{\mathbf{x}}\varphi_{\mathbf{x}})(\nabla^2_{\mathbf{y}}\varphi_{\mathbf{y}})(\nabla_{\mathbf{z}}\varphi_{\mathbf{z}}) + (\nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}})\varphi_{\mathbf{x}}(\nabla^2_{\mathbf{y}}\varphi_{\mathbf{y}})\varphi_{\mathbf{z}}\right\}$$
$$= -2\int d\mathbf{w}\varphi^3_{\mathbf{w}}. \tag{2.26}$$

By commuting 2.25 once with π and twice with φ, and 2.26 three times with π, we transform these two operator identities into two coupled partial differential equations for M, N, P and Q:

$$(\mu^2 - \nabla^2_{\mathbf{x}})[N_{\mathbf{x(yz)}} + \nabla_{\mathbf{y}}\nabla_{\mathbf{z}}P_{\mathbf{x(yz)}} + \nabla^2_{\mathbf{x}}Q_{\mathbf{x(yz)}}] + (\mu^2 - \nabla^2_{\mathbf{y}})[N_{\mathbf{y(xz)}} + \nabla_{\mathbf{x}}\nabla_{\mathbf{z}}P_{\mathbf{y(xz)}} + \nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}}]$$
$$+ (\mu^2 - \nabla^2_{\mathbf{z}})[N_{\mathbf{z(xy)}} + \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}P_{\mathbf{z(xy)}} + \nabla^2_{\mathbf{z}}Q_{\mathbf{z(xy)}}] = -6\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}-\mathbf{z}), \tag{2.27}$$

$$N_{\mathbf{y(xz)}} + N_{\mathbf{z(xy)}} + \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}P_{\mathbf{z(xy)}} + \nabla_{\mathbf{x}}\nabla_{\mathbf{z}}P_{\mathbf{y(xz)}} + \nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}} + \nabla^2_{\mathbf{z}}Q_{\mathbf{z(xy)}} = 3(\mu^2 - \nabla^2_{\mathbf{x}})M_{\mathbf{(xyz)}}. \tag{2.28}$$

To solve these eqution we Fourier transform to momentum space, such as to change these differential equations into algebraic equations. Denoting the D-dimensional Fourier transform of a function $f_{\mathbf{x}}$ by $\tilde{f}_{\mathbf{p}} \equiv \int d\mathbf{x} f_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}$ we obtain[6]:

$$\frac{1}{\tilde{G}_{\mathbf{p}}}[\tilde{N}_{\mathbf{p(qr)}} - \mathbf{qr}\tilde{P}_{\mathbf{p(qr)}} - \mathbf{p^2}\tilde{Q}_{\mathbf{p(qr)}}] + \frac{1}{\tilde{G}_{\mathbf{q}}}[\tilde{N}_{\mathbf{q(pr)}} - \mathbf{pr}\tilde{P}_{\mathbf{q(pr)}} - \mathbf{q^2}\tilde{Q}_{\mathbf{q(pr)}}]$$
$$+\frac{1}{\tilde{G}_{\mathbf{r}}}[\tilde{N}_{\mathbf{r(pq)}} - \mathbf{pq}\tilde{P}_{\mathbf{r(pq)}} - \mathbf{r^2}\tilde{Q}_{\mathbf{r(pq)}}] = -6(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}), \tag{2.29}$$

$$\tilde{N}_{\mathbf{q(pr)}} + \tilde{N}_{\mathbf{r(pq)}} - \mathbf{pq}\tilde{P}_{\mathbf{r(pq)}} - \mathbf{pr}\tilde{P}_{\mathbf{q(pr)}} - \mathbf{q}^2\tilde{Q}_{\mathbf{q(pr)}} - \mathbf{r}^2\tilde{Q}_{\mathbf{r(pq)}} = \frac{3}{\tilde{G}_{\mathbf{p}}}\tilde{M}_{\mathbf{(pqr)}}, \tag{2.30}$$

with $\tilde{G}_{\mathbf{p}} = (\mathbf{p}^2 + \mu^2)^{-1}$.

Note that just as in the calculation performed in [8] using 2.22 we have that the right hand side of 2.29 contains the factor $\delta(\mathbf{p}+\mathbf{q}+\mathbf{r})$, which implies that the four three-point functions M, N, P and Q conserve momentum. Continuing in analogy with [8] we introduce *reduced* representations of these vertex functions in which we factor off the delta function:

$$\tilde{M}_{\mathbf{(pqr)}} = (2\pi)^D\tilde{m}_{\mathbf{(pqr)}}\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}), \tag{2.31}$$

and similarly for $\tilde{N}$, $\tilde{P}$ and $\tilde{Q}$.

[6]From this point onwards, to avoid excessive symbols, we will be denoting the inner product between two vectors $\mathbf{x}\cdot\mathbf{y}$ by $\mathbf{xy}$

Thus the four three-point functions satisfy:

$$\frac{1}{\tilde{G}_{\mathbf{p}}}[\tilde{n}_{\mathbf{p(qr)}} - \mathbf{qr}\tilde{p}_{\mathbf{p(qr)}} - \mathbf{p^2}\tilde{q}_{\mathbf{p(qr)}}] + \frac{1}{\tilde{G}_{\mathbf{q}}}[\tilde{n}_{\mathbf{q(pr)}} - \mathbf{pr}\tilde{p}_{\mathbf{q(pr)}} - \mathbf{q^2}\tilde{q}_{\mathbf{q(pr)}}] + \frac{1}{\tilde{G}_{\mathbf{r}}}[\tilde{n}_{\mathbf{r(pq)}} - \mathbf{pq}\tilde{p}_{\mathbf{r(pq)}} - \mathbf{r^2}\tilde{q}_{\mathbf{r(pq)}}] = -6, \tag{2.32}$$

$$\tilde{n}_{\mathbf{q(pr)}} + \tilde{n}_{\mathbf{r(pq)}} - \mathbf{pq}\tilde{p}_{\mathbf{r(pq)}} - \mathbf{pr}\tilde{p}_{\mathbf{q(pr)}} - \mathbf{q}^2\tilde{q}_{\mathbf{q(pr)}} - \mathbf{r}^2\tilde{q}_{\mathbf{r(pq)}} = \frac{3}{\tilde{G}_{\mathbf{p}}}\tilde{m}_{\mathbf{(pqr)}}. \tag{2.33}$$

To solve these equations we first notice that the right hand side of 2.33 is totally symmetric in its indices, and can thus be used to obtain two new equations by permuting the indices:

$$\tilde{n}_{\mathbf{r(qp)}} + \tilde{n}_{\mathbf{p(qr)}} - \mathbf{qr}\tilde{p}_{\mathbf{p(qr)}} - \mathbf{pq}\tilde{p}_{\mathbf{r(pq)}} - \mathbf{r}^2\tilde{q}_{\mathbf{r(pq)}} - \mathbf{p}^2\tilde{q}_{\mathbf{p(qr)}} = \frac{3}{\tilde{G}_{\mathbf{q}}}\tilde{m}_{\mathbf{(pqr)}}, \tag{2.34}$$

$$\tilde{n}_{\mathbf{p(qr)}} + \tilde{n}_{\mathbf{q(rp)}} - \mathbf{pr}\tilde{p}_{\mathbf{q(rp)}} - \mathbf{qr}\tilde{p}_{\mathbf{p(qr)}} - \mathbf{p}^2\tilde{q}_{\mathbf{p(qr)}} - \mathbf{q}^2\tilde{q}_{\mathbf{q(pr)}} = \frac{3}{\tilde{G}_{\mathbf{r}}}\tilde{m}_{\mathbf{(pqr)}}. \tag{2.35}$$

To simplify the manipulations of these equations we define the four functions;

$$R_1(\tilde{p},\tilde{q}) = \frac{1}{\tilde{G}_{\mathbf{p}}}[\mathbf{qr}\tilde{p}_{\mathbf{p(qr)}} + \mathbf{p}^2\tilde{q}_{\mathbf{p(qr)}}] + \frac{1}{\tilde{G}_{\mathbf{q}}}[\mathbf{pr}\tilde{p}_{\mathbf{q(pr)}} + \mathbf{q}^2\tilde{q}_{\mathbf{q(pr)}}] + \frac{1}{\tilde{G}_{\mathbf{r}}}[\mathbf{pq}\tilde{p}_{\mathbf{r(pq)}} + \mathbf{r}^2\tilde{q}_{\mathbf{r(pq)}}] \tag{2.36}$$

$$R_2(\tilde{p},\tilde{q}) = \mathbf{pq}\tilde{p}_{\mathbf{r(pq)}} + \mathbf{pr}\tilde{p}_{\mathbf{q(pr)}} + \mathbf{q}^2\tilde{q}_{\mathbf{q(pr)}} + \mathbf{r}^2\tilde{q}_{\mathbf{r(pq)}} \tag{2.37}$$

$$R_3(\tilde{p},\tilde{q}) = \mathbf{qr}\tilde{p}_{\mathbf{p(qr)}} + \mathbf{pq}\tilde{p}_{\mathbf{r(pq)}} + \mathbf{r}^2\tilde{q}_{\mathbf{r(pq)}} + \mathbf{p}^2\tilde{q}_{\mathbf{p(qr)}} \tag{2.38}$$

$$R_4(\tilde{p},\tilde{q}) = \mathbf{pr}\tilde{p}_{\mathbf{q(rp)}} + \mathbf{qr}\tilde{p}_{\mathbf{p(qr)}} + \mathbf{p}^2\tilde{q}_{\mathbf{p(qr)}} + \mathbf{q}^2\tilde{q}_{\mathbf{q(pr)}}. \tag{2.39}$$

Therefore 2.32, 2.33, 2.34 and 2.35 now read:

$$\frac{1}{\tilde{G}_{\mathbf{p}}}\tilde{n}_{\mathbf{p(qr)}} + \frac{1}{\tilde{G}_{\mathbf{q}}}\tilde{n}_{\mathbf{q(pr)}} + \frac{1}{\tilde{G}_{\mathbf{r}}}\tilde{n}_{\mathbf{r(pq)}} = -6 + R_1(\tilde{p},\tilde{q}) \tag{2.40}$$

$$\tilde{n}_{\mathbf{q(pr)}} + \tilde{n}_{\mathbf{r(pq)}} = \frac{3}{\tilde{G}_{\mathbf{p}}}\tilde{m}_{\mathbf{(pqr)}} + R_2(\tilde{p},\tilde{q}) \tag{2.41}$$

$$\tilde{n}_{\mathbf{r(pq)}} + \tilde{n}_{\mathbf{p(qr)}} = \frac{3}{\tilde{G}_{\mathbf{q}}}\tilde{m}_{\mathbf{(pqr)}} + R_3(\tilde{p},\tilde{q}) \tag{2.42}$$

$$\tilde{n}_{\mathbf{p(qr)}} + \tilde{n}_{\mathbf{q(rp)}} = \frac{3}{\tilde{G}_{\mathbf{r}}}\tilde{m}_{\mathbf{(pqr)}} + R_4(\tilde{p},\tilde{q}) \tag{2.43}$$

respectively. Although we now have four equations in terms of four unknown functions we are not in a position to determine the solutions for all four these functions as we do not know the

symmetry conditions between all indices. What we are able to obtain though is an exact solution to $\tilde{m}$ and a solution to $\tilde{n}$ parametrised by $\tilde{p}$ and $\tilde{q}$. Solving for $\tilde{m}$ and $\tilde{n}$:

$$\tilde{m}_{(\mathbf{pqr})} = \frac{4\tilde{G}_{\mathbf{p}}^2\tilde{G}_{\mathbf{q}}^2\tilde{G}_{\mathbf{r}}^2}{\tilde{G}_{\mathbf{p}}^2\tilde{G}_{\mathbf{q}}^2 + \tilde{G}_{\mathbf{p}}^2\tilde{G}_{\mathbf{r}}^2 + \tilde{G}_{\mathbf{q}}^2\tilde{G}_{\mathbf{r}}^2 - 2\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}} + \tilde{G}_{\mathbf{q}} + \tilde{G}_{\mathbf{r}})} \tag{2.44}$$

$$\tilde{n}_{\mathbf{p}(\mathbf{qr})} = \frac{6\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{r}} + \tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}} - \tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}})}{\tilde{G}_{\mathbf{p}}^2\tilde{G}_{\mathbf{q}}^2 + \tilde{G}_{\mathbf{p}}^2\tilde{G}_{\mathbf{r}}^2 + \tilde{G}_{\mathbf{q}}^2\tilde{G}_{\mathbf{r}}^2 - 2\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}} + \tilde{G}_{\mathbf{q}} + \tilde{G}_{\mathbf{r}})} + \mathbf{qr}\tilde{p}_{\mathbf{p}(\mathbf{qr})} + \mathbf{p}^2\tilde{q}_{\mathbf{p}(\mathbf{qr})} \tag{2.45}$$

We were thus able, by substituting our ansatz for Q_1 in 2.21, to determine $\tilde{m}$ completely in terms of the momentum variables and to find an expression for $\tilde{n}$ in terms of the momenta and the reduced representations of the fourier transformed P and Q coefficients. Two observations worth making are the following: (1) We note that our expression for $\tilde{m}$ is the same as that obtained in [8] and can thus be inverted to obtain a closed form expression for M in terms of Bessel functions. (2) These expressions for the coefficient functions allow us to recover the results obtained for the case $D = 0$ (quantum mechanics). When $D = 0$ the operators $\nabla_{\mathbf{x}}\varphi_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}^2\pi_{\mathbf{x}}$ are not defined, and thus the P and Q functions are zero. In addition $\tilde{G}_{\mathbf{p}}$ reduces to $\tilde{G}_{\mathbf{p}} = \mu^{-2}$. Substituting this expression for $\tilde{G}$ into 2.44 and 2.45 and using $P = 0$ and $Q = 0$, we find that these equations once solved give precisely 2.16.

Although we have found expressions for the coefficient functions which are consistent with previous calculations, we are still not able to solve for $\tilde{n}$, $\tilde{p}$ and $\tilde{q}$ completely. The four equations we have solved have left us with a completely determined solution for M and an expression relating N, P and Q which we know from (2) is consistent with the quantum mechanical case.
In order to determine the explicit expressions for N, P and Q we have to look for further constraint on $\mathcal{C}$. One such constraint is that the $\mathcal{C}$ operator should transform as a Lorentz scalar and we will be exploiting this constraint in the next section.

Chapter 3

The $\mathcal{C}$ Operator Transforms as a Lorentz Scalar

In [6] it has been established that using ansatz 2.22 the $\mathcal{C}$ operator does indeed transform as a scalar by showing that the latter commutes with the generators of the Lorentz boost. We follow the same procedure here. Before doing this though we will briefly discuss an alternative operator representation for $\mathcal{C}$ which is preferred to 2.8 because, as is shown in [6], it transforms as a lorentz scalar.

3.1 The intrinsic parity operator

The operator representation 2.8 has been used untill now in both the quantum mechanical and field theoretic case. However, it was shown recently[1], that the correct field-theoretic form for the $\mathcal{C}$ operator is not $\mathcal{C} = e^{Q}\mathcal{P}$, but rather;

$$\mathcal{C} = e^{Q}\mathcal{P}_I, \tag{3.1}$$

where $\mathcal{P}_I$ id the *intrinsic* parity operator. $\mathcal{P}_I$ has the same effect as $\mathcal{P}$ on the field variables except that it does not change the sign of the spatial arguments of the fields. Thus it acts as;

$$\mathcal{P}_I\varphi(\mathbf{x},t)\mathcal{P}_I = \varphi(\mathbf{x},t) \tag{3.2}$$

for a scalar field, and as;

$$\mathcal{P}_I\varphi(\mathbf{x},t)\mathcal{P}_I = -\varphi(\mathbf{x},t) \tag{3.3}$$

for a pseudoscalar field. The fundamental difference betweent the two parity operators is seen in their Lorentz transformation properties, [6]. It is now believed that the correct representation for $\mathcal{C}$ is 3.1 rather than 2.8. Note that in the quantum mechanical case there is no difference between these two representations because for $D = 0$ we have that $\mathcal{P} = \mathcal{P}_I$. However in quantum field theory, where $D \neq 0$, we have that $\mathcal{P} \neq \mathcal{P}_I$ and the two representations are indeed different. It is via this new representation that it was understood that the $\mathcal{C}$ operator is the complex analytical continuation of the intrinsic parity operator[2].

[1] see [6]
[2] Again see [6].

3.2 The Lorentz scalar condition

For the $\mathcal{C}$ operator to transforms as a Lorentz scalar, we must show that $\mathcal{C}$ commutes with the Lorentz boost operator

$$J^{0i} = J_0^{0i} + \epsilon J_1^{0i}, \tag{3.4}$$

where;

$$J_0^{0i}(t) = t\int d\mathbf{x}\pi(\mathbf{x},t)\nabla^i_{\mathbf{x}}\varphi(\mathbf{x},t) - \int d\mathbf{x}x^i\mathcal{H}_0(\mathbf{x},t), \tag{3.5}$$

$$J_1^{0i}(t) = -\int d\mathbf{x}x^i\mathcal{H}_1(\mathbf{x},t). \tag{3.6}$$

So we expand the commutator $[\mathcal{C}, J^{0i}]$ in powers of ϵ. To order ϵ^2, we have

$$\begin{aligned}[\mathcal{C}, J^{0i}] =& [(1+\epsilon Q_1 + \frac{1}{2}\epsilon^2 Q_1^2)P_I, J_0^{0i} + \epsilon J_1^{0i}] + O(\epsilon^3)\\ =&[P_I, J_0^{0i}] + \epsilon([Q_1, J_0^{0i}]\mathcal{P}_I + Q_1[\mathcal{P}_I, J_0^{0i}] + [\mathcal{P}_I, J_1^{0i}])\\ &+\epsilon^2(\frac{1}{2}[Q_1^2, J_0^{0i}]\mathcal{P}_I + \frac{1}{2}Q_1^2[\mathcal{P}_I, J_0^{0i}] + [Q_1, J_1^{0i}]\mathcal{P}_I + Q_1[\mathcal{P}_I, J_I^{0i}]) + O(\epsilon^3).\end{aligned} \tag{3.7}$$

The leading order term vanishes because $[\mathcal{P}_I, J_0^{0i}] = 0$. Using the identity $[\mathcal{P}_I, J_1^{0i}] = -2J_1^{0i}\mathcal{P}_I$, we simplify 3.7 to

$$\begin{aligned}[\mathcal{C}, J^{0i}] =&\epsilon([Q_1, J_0^{0i}]\mathcal{P}_I - 2J_1^{0i}\mathcal{P}_I)\\ &+\epsilon^2(\frac{1}{2}Q_1[Q_1, J_0^{0i}]\mathcal{P}_I + \frac{1}{2}[Q_1, J_0^{0i}]Q_1\mathcal{P}_I - Q_1J_1^{0i}\mathcal{P}_I - J_1^{0i}Q_1\mathcal{P}_I) + O(\epsilon^3).\end{aligned} \tag{3.8}$$

We see that if the term of order ϵ vanishes, then the ϵ^2 term vanishes automatically.
The commutator 3.8 tells us that to show that $\mathcal{C}$ is a scalar we simply have to show that $[Q_1, J_0^{0i}] = 2J_1^{0i}$. 3.5 consists of two terms, and in [6] it was shown that the first part commutes with Q_1 by showing that for any functional of π and φ, say $f[\pi,\varphi]$, we have that $[f[\pi,\varphi], t\int d\mathbf{x}\pi(\mathbf{x},t)\nabla^i_{\mathbf{x}}\varphi(\mathbf{x},t)] = 0$. We now prove that this is the case also for any functional of π, φ, $\nabla\varphi$ and $\nabla^2\pi$, say $g[\pi,\varphi,\nabla\varphi,\nabla^2\pi]$. To show this we note that this commutator is explicitely an integral of a total derivative[3]. Start with

$$\begin{aligned}&[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi], t\int d\mathbf{y}\pi(\mathbf{y},t)\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)]\\ &\qquad = t\int d\mathbf{y}\left\{[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi],\pi(\mathbf{y},t)]\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t) + \pi(\mathbf{y},t)\nabla^i_{\mathbf{y}}[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi],\varphi(\mathbf{y},t)]\right\},\end{aligned} \tag{3.9}$$

then using the variational formulas

$$[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi],\varphi(\mathbf{x},t)] = -i\frac{\delta}{\delta\pi(\mathbf{x},t)}g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi] - i\nabla^2_{\mathbf{x}}\frac{\delta}{\delta(\nabla^2_{\mathbf{x}}\pi(\mathbf{x},t))}g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi] \tag{3.10}$$

$$[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi],\pi(\mathbf{x},t)] = i\frac{\delta}{\delta\varphi(\mathbf{x},t)}g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi] - i\nabla_{\mathbf{y}}\left(\frac{\delta}{\delta\left(\nabla_{\mathbf{x}}\varphi(\mathbf{x},t)\right)}g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi]\right) \tag{3.11}$$

[3] we assume here that the functional is expressible as a sum such as $g = A_n\pi^n + Bm\varphi^m + C_p(\nabla_{\mathbf{x}}\varphi)^p + D_q(\nabla^2_{\mathbf{x}}\pi)^q$.

.
we change 3.9 to:

$$[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi],t\int d\mathbf{y}\pi(\mathbf{y},t)\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)]$$
$$=it\int d\mathbf{y}\left\{\frac{\delta g}{\delta\varphi}\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)-\pi(\mathbf{y},t)\nabla^i_{\mathbf{y}}\left(\frac{\delta g}{\delta\pi}\right)-\nabla_{\mathbf{y}}\left(\frac{\delta g}{\delta\left(\nabla_{\mathbf{y}}\varphi\right)}\right)\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)\right.$$
$$\left.-\pi(\mathbf{y},t)\nabla^i_{\mathbf{y}}\left(\nabla^2_{\mathbf{y}}\left(\frac{\delta g}{\delta\left(\nabla^2_{\mathbf{y}}\pi\right)}\right)\right)\right\} \tag{3.12}$$

Integrating the second, third and fourth terms by parts, we get

$$[g[\pi,\varphi,\nabla\varphi,\nabla^2\varphi],t\int d\mathbf{y}\pi(\mathbf{y},t)\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)]$$
$$=it\int d\mathbf{y}[\frac{\delta g}{\delta\varphi}\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)+\nabla^i_{\mathbf{y}}\pi(\mathbf{y},t)\frac{\delta g}{\delta\pi}+\frac{\delta g}{\delta\left(\nabla_{\mathbf{y}}\varphi\right)}\nabla_{\mathbf{y}}\nabla^i_{\mathbf{y}}\varphi(\mathbf{y},t)+\nabla^2_{\mathbf{y}}\nabla^i_{\mathbf{y}}\pi(\mathbf{y},t)\frac{\delta g}{\delta\left(\nabla^2_{\mathbf{y}}\pi\right)}]=0 \tag{3.13}$$

Thus only the second term of J^{0i}_0 in 3.5 contributes to the commutator of Q_1 with J^{0i}_0,
i.e. $[Q_1,J^{0i}_0]=-[Q_1,\int d\mathbf{x}x^i\mathcal{H}_0(\mathbf{x},t)]$.
We have thus, in analogy with [6], reduced the problem of requiring that $\mathcal{C}$ is a scalar to solving the commutator identity

$$[Q_1,\int d\mathbf{x}x^i\left\{\frac{1}{2}\pi^2(\mathbf{x},t)+\frac{1}{2}\mu^2\varphi^2(\mathbf{x},t)+\frac{1}{2}[\nabla_{\mathbf{x}}\varphi(\mathbf{x},t)]^2\right\}]=2i\int d\mathbf{x}x^i\varphi^3(\mathbf{x},t). \tag{3.14}$$

Note how this equation, apart from an integration by parts is structurally similar to 2.21. The difference is that in 3.14 there are extra factors of x^i in the integrand.
Following the analysis of Chapter 2, we introduce the same ansatz for Q_1 to solve 3.14. Performing the commutator in 3.14 we obtain two functional equations:

$$\iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}\varphi_{\mathbf{x}}\varphi_{\mathbf{y}}\varphi_{\mathbf{z}}\left\{[x^i(\mu^2-\nabla^2_{\mathbf{x}})-\nabla^i_{\mathbf{x}}]N_{\mathbf{x(yz)}}+[y^i(\mu^2-\nabla^2_{\mathbf{y}})-\nabla^i_{\mathbf{y}}]\nabla_{\mathbf{x}}\nabla_{\mathbf{z}}P_{\mathbf{y(xz)}}\right.$$
$$\left.+[y^i\nabla^2_{\mathbf{y}}(\mu^2-\nabla^2_{\mathbf{y}})-\nabla^{i3}_{\mathbf{y}}]Q_{\mathbf{y(xz)}}\right\}=-2\int d\mathbf{w}w^i\varphi^3_{\mathbf{w}}, \tag{3.15}$$

$$\iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}\left\{(y^i\pi_{\mathbf{y}}\pi_{\mathbf{x}}\varphi_{\mathbf{z}}+z^i\varphi_{\mathbf{y}}\pi_{\mathbf{x}}\pi_{\mathbf{z}})N_{\mathbf{x(yz)}}+\left(x^i\pi_{\mathbf{x}}\pi_{\mathbf{y}}\varphi_{\mathbf{z}}+z^i\varphi_{\mathbf{x}}\pi_{\mathbf{y}}\pi_{\mathbf{z}}\right)\nabla_{\mathbf{x}}\nabla_{\mathbf{z}}P_{\mathbf{y(xz)}}\right.$$
$$\left.+\left(x^i\pi_{\mathbf{x}}\pi_{\mathbf{y}}\varphi_{\mathbf{z}}+z^i\varphi_{\mathbf{x}}\pi_{\mathbf{y}}\pi_{\mathbf{z}}\right)\nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}}\right\}=\iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}\left\{\varphi_{\mathbf{x}}\pi_{\mathbf{y}}\pi_{\mathbf{z}}(x^i(\mu^2-\nabla^2_{\mathbf{x}})-\nabla^i_{\mathbf{x}})M_{\mathbf{(xyz)}}\right.$$
$$\left.+\pi_{\mathbf{x}}\varphi_{\mathbf{y}}\pi_{\mathbf{z}}(y^i(\mu^2-\nabla^2_{\mathbf{y}})-\nabla^i_{\mathbf{y}})M_{\mathbf{(xyz)}}+\pi_{\mathbf{x}}\pi_{\mathbf{y}}\varphi_{\mathbf{z}}(z^i(\mu^2-\nabla^2_{\mathbf{z}})-\nabla^i_{\mathbf{z}})M_{\mathbf{(xyz)}}\right\}. \tag{3.16}$$

Next, commute 3.15 three times with π and commute 3.16 once with π and twice with φ to transform these two operator identities into coupled differential equations for M, N, P and Q:

$$[x^i(\mu^2-\nabla^2_{\mathbf{x}})-\nabla^i_{\mathbf{x}}][N_{\mathbf{x(yz)}}+\nabla_{\mathbf{y}}\nabla_{\mathbf{z}}P_{\mathbf{x(yz)}}+\nabla^2_{\mathbf{x}}Q_{\mathbf{x(yz)}}]+[y^i(\mu^2-\nabla^2_{\mathbf{y}})-\nabla^i_{\mathbf{y}}][N_{\mathbf{y(xz)}}+\nabla_{\mathbf{x}}\nabla_{\mathbf{z}}P_{\mathbf{y(xz)}}+\nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}}]$$
$$+[z^i(\mu^2-\nabla^2_{\mathbf{z}})-\nabla^i_{\mathbf{z}}][N_{\mathbf{z(xy)}}+\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}P_{\mathbf{z(xy)}}+\nabla^2_{\mathbf{z}}Q_{\mathbf{z(xy)}}]=-6x^i\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}-\mathbf{z}), \tag{3.17}$$

$$y^i[N_{\mathbf{z(xy)}} + \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}P_{\mathbf{z(xy)}} + \nabla^2_{\mathbf{z}}Q_{\mathbf{z(xy)}}] + z^i[N_{\mathbf{y(xz)}}+\nabla_{\mathbf{x}}\nabla_{\mathbf{z}}P_{\mathbf{y(xz)}} + \nabla^2_{\mathbf{y}}Q_{\mathbf{y(xz)}}]$$
$$= 3[x^i(\mu^2 - \nabla^2_{\mathbf{x}}) - \nabla^i_{\mathbf{x}}]M_{\mathbf{(xyz)}}. \quad (3.18)$$

Introducing the condition that $\mathcal{C}$ should be a Lorentz scalar has thus given us two coupled differential equations, independent from the ones obtained from the conditions 2.5, 2.6 and 2.7 , which we may use to try and determine the undetermined functions N, P and Q. We also note that the right hand side of 3.18 is totally symmetric in the three indices just as 2.33, we are thus able to obtain another two equations by eploiting this symmetry condition. Although permuting the indices in 3.18 will give us two extra equation we can use to solve for the undetermined functions, we will see that these are not necessary. The expression we have obtained for the undetermined functions is in terms of their Fourier transform, see 2.45, thus to make use of 3.17, 3.18 we will either require inverse Fourier transforming 2.45 back to position space, or to Fourier transform 3.17 and 3.18 to momentum space. We here perform the latter.
Fourier transforming to momentum space we have:

$$[\nabla^i_{\mathbf{p}}\tilde{G}^{-1}_{\mathbf{p}} - p^i][\tilde{N}_{\mathbf{p(qr)}} - \mathbf{qr}\tilde{P}_{\mathbf{p(qr)}} - \mathbf{p}^2\tilde{Q}_{\mathbf{p(qr)}}] + [\nabla^i_{\mathbf{q}}\tilde{G}^{-1}_{\mathbf{q}} - q^i][\tilde{N}_{\mathbf{q(pr)}} - \mathbf{pr}\tilde{P}_{\mathbf{q(pr)}} - \mathbf{q}^2\tilde{Q}_{\mathbf{q(pr)}}]$$
$$+[\nabla^i_{\mathbf{r}}\tilde{G}^{-1}_{\mathbf{r}} - r^i][\tilde{N}_{\mathbf{r(pq)}} - \mathbf{pq}\tilde{P}_{\mathbf{r(pq)}} - \mathbf{r}^2\tilde{Q}_{\mathbf{r(pq)}}] = -6\nabla^i_{\mathbf{p}}(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}), \quad (3.19)$$

$$\nabla^i_{\mathbf{q}}[\tilde{N}_{\mathbf{r(pq)}} - \mathbf{pq}\tilde{P}_{\mathbf{r(pq)}} - \mathbf{r}^2\tilde{Q}_{\mathbf{r(pq)}}] + \nabla^i_{\mathbf{r}}[\tilde{N}_{\mathbf{q(pr)}} - \mathbf{pr}\tilde{P}_{\mathbf{q(pr)}} - \mathbf{q}^2\tilde{Q}_{\mathbf{q(pr)}}] = 3[\nabla^i_{\mathbf{p}}\tilde{G}^{-1}_{\mathbf{p}} - p^i]\tilde{M}_{\mathbf{(pqr)}}. \quad (3.20)$$

To make use of these algebraic equations we first change the reduced expression for $\tilde{N}$, i.e. $\tilde{n}$, to the full expression which is simply;

$$\tilde{N}_{\mathbf{p(qr)}} = \frac{6\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{r}} + \tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}} - \tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}})}{\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}} + \tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{r}} + \tilde{G}^2_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}} - 2\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}} + \tilde{G}_{\mathbf{q}} + \tilde{G}_{\mathbf{r}})}\cdot(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r})$$
$$+ \mathbf{qr}\tilde{P}_{\mathbf{p(qr)}} + \mathbf{p}^2\tilde{Q}_{\mathbf{p(qr)}}. \quad (3.21)$$

We now permute the indice in 3.20 such as to get the algebraic equation satisfied by $\tilde{N}_{\mathbf{p(qr)}}$ and then substitute 3.21 into this. We find that no matter which of the equations we use to determine the conditions on $\tilde{P}$ and $\tilde{Q}$, these terms cancel. Therefore enforcing the condition that the $\mathcal{C}$ operator should be a Lorentz scalar does not allow us to determine the coefficient functionals $\tilde{N}$, $\tilde{P}$ and $\tilde{Q}$ completely. We discuss the implications of this result in the conclusion section.

Chapter 4

Conclusion

In [9] it was concluded that the three conditions 2.5, 2.6 and 2.7 allow for a whole family of solutions. The question was then posed of whether there existed a fourth constraint that could be used to supplement the above three conditions such as to give rise to the "best form" for the $\mathcal{C}$ operator. In this paper we have found that with the generalized ansatz the three constraints are not sufficient to fully determine the coefficient functionals. We thus looked for a fourth constraint on $\mathcal{C}$. Using this extra constraint did not allow us to determine all the coefficients completely, but it allowed us to conclude that even in the context of quantum field theory there exists a whole family of solutions. These solutions are given in terms of a fully determined coefficient, M, which involve Bessel functions[1] and a coefficient N parametrized by P and Q, that is the terms dependent on the spatial derivatives of the fields. Hence we can conclude that the non-uniqueness of $\mathcal{C}$ is inherent in quantum theories described by non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians. Using 2.22 as an ansatz Bender, Jones and Brody showed that the two coefficient functionals, i.e. N and M, both contained Bessel functions. This meant that Q_1 would represent a *nonlocal* interaction of the three fields[2]. However, since associated Bessel functions decrease exponentially rapidly for large arguments, the degree of nonlocality is small[3]. The huge parametric freedom in $\mathcal{C}$ that we have found using a more general ansatz could possibly be used to impose the condition of locality. In fact it could be that the dependences of Q on the spatial derivatives allow us to replace the Bessel function by a spatial delta function so that $\mathcal{C}$ becomes local, [6]. An additional constraint could be to impose that $\mathcal{C}$ should transform as a rotational scalar.

[1] See [8]
[2] Recall that we are workin with an $i\varphi^3$ field theory
[3] [8]

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