Dissertation

One-loop corrections : Introduction to precision calculations in the Standard Model and the MSSM

MSc Quantum Fields and Fundamental Forces

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Introduction

Over the last years, the overall importance of quantum corrections has been more and more emphasized. From a theoretical point of view, it is a key point for the survival of some models, as an example stands the MSSM which lightest Higgs would have been completely ruled out by experiments, were it not for the quantum corrections its mass receives. On the experimental side, the need for precision tools to analyse processes and discriminiate beetwen standard model contribution and new physics has been outlined by LEP and Tevatron, and will be of foremost importance at the forthcoming LHC. However we are currently far from having a single, efficient and polyvalent method to treat these corrections.

This issue can be viewed as a natural consequence of the intrinsic complexity of the calculations of nearly any observable in nearly any quantum field theory. Complexity about the renormalization procedure first, which is arbitrarily chosen and hence raises the amount of available methods, but also complexity of the numerical analysis of the Feynman expansion, which cannot be carried out but in an automatized way – hence the craving need for powerfull codes. The aim of this essay is to grasp the nature of the problems arising in calculations in a quantum field theory, together with the techniques to overcome these obstacles, by focusing on specific models, namely the Standard Model and its supersymmetric extension the MSSM, a given renormalization procedure and an automatized code – *Sloops* – to get a numerical outcome.

This dissertation is by no means a review of state-of-the-art precision calculations in MSSM like models, as I will mostly focus on standard processes, nor the latest predictions on standard observables since I limited my work to first order corrections. However the aim of presenting the full calculations of standard processes together with the description of a stable and realistic code for one-loop calculations is not a random choice, as we shall see later.

Chapter 1

Standard Model

The aim of this chapter is not to give a historical review of the Standard Model nor to give a self-sufficient description, but mainly to introduce the concept and notations that I will use for the rest of this thesis, together with their motivations.

The first section will be devoted to some aspects of quantum field theories in general, covering a wide range of models including the SM and the MSSM.

1.1 How to do calculations in a quantum field theory

As we said before, we would like to carry out some computations in our quantum field theory, find an interpretation of those calculations and finally match the whole thing to experiment. One natural way to achieve this goal is a three-point process :

1. Define the context of the quantum theory. I purposely use the general term “context” to cover different aspects such as : how is the space-time, what is a particle, what is an observable, how is defined a vacuum state, etc. I shall also refer to this as the “Situation” of the theory.
2. Define a spectrum and an action.
3. Any computation is then carried out by combining the proper definitions given in step 1, and the minimization of the action given in step 2.

To get a clearer insight in the first step which is so far a bit fuzzy, let me describe the one I used. This step is common either to the SM and the MSSM, whereas steps 2 and 3 are not, hence they will be explained in due time.

1.1.1 Situation

Space-time
The space-time in which physics will be happening will be a flat Minkowski $\mathbb{R}^4$. However we will also consider it as a common differentiable manifold endowed with a metric, which will allow us to represent each field as a n-form in a very convenient way.
As an example, fields in the vector representation of the Lorentz group will be one-forms.

Particles
To each particle of our theory is associated :

- an irreducible representation of the group $G = G_{Poincaré} \times G_{Gauge} = G_P \times G_G$
- a field, defined to be a smooth function from the space-time to the space of this representation

Without entering into the whole theory of representations of the Poincaré group, we may notice a few points. Each representation has a positive mass, which is the eigenvalue of P^2, and contains a representation of the Lorentz group $G_L = O_{\uparrow}^{+}(3,1)$. In most cases it is preferred to lift G_L to its universal covering group SPIN(4), which is $SL(2,\mathbb{C})$. Its irreducible representations are labelled by (i,j) where i and j are both half-integers. In the following we will refer to any representation of the Poincaré group by this couple and the mass.

One often oppose the Lorentz group and the gauge group by calling the latter “the internal group”. However they act very similarly, the Lorentz group being also “gauged”, the main difference being that while the gauge group is gauge fixed to a point, the Lorentz group is gauge fixed to a frame (that is to say a basis of the tangent space), hence the denomination “internal” of the former.

Quantization

The quantization, whether in its mathematical or interpretation side, will be the ’usual one’, meaning that we will use the standard notation and definitions (as can be found in [8] for instance) for Hilbert space, operators, observables, and that we will try to avoid as much as possible any trouble with the interpretation of Quantum Mechanics. The most important result being the formulae for the expectation value of an observable :

$$< \mathcal{O} >= \frac{\int \mathcal{D}\phi \mathcal{O}[\phi] e^{iS}}{\int \mathcal{D}\phi e^{iS}}$$

where ϕ runs over all particles in the spectrum, and S being the action, these two objects being defined in step 2.

Feynman expansion

We note that the formulae for finding expectation values require the computation of an exponential on all paths. We tend to prefer a Feynman expansion approach, which consists in the following

- Separating $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, where $\mathcal{L}_{int}$ stand for the interaction part.
- Computing propagators and vertex rules
- Add renormalization terms, with a well-defined renormalization procedure

Hence the context of the calculations we are willing to carry out has been made clear : we know how to add particles to a model and how to compute expectation values of observables. Let us have a look at the way it is done in the Standard Model.

1.2 Spectrum and Lagrangian of the Standard Model

1.2.1 Spectrum

The gauge group is $G_G = SU(3) \times SU(2) \times U(1)$, each being associated with the color, the weak representation, and the hypercharge respectively. We will associate a coupling constant g_a ($a = 1, 2, 3$) to each simple subgroup. The standard spectrum of particles is the following :

- Fermions ψ in a spinorial representation of G_P. We will encounter 2 cases:

  $(\frac{1}{2}, 0) \otimes$ any representation of G_G. Left-handed particle

  $(0, \frac{1}{2}) \otimes$ any representation of G_G. Right-handed particle

- Bosons A in a bosonic representation of G_P.

  The gauge bosons are by definition in the adjoint representation : $(\frac{1}{2}, \frac{1}{2}) \otimes$ ad

  The Higgs boson is in the representation $(0, 0) \otimes 1 \otimes 2 \otimes \frac{1}{2}$.

- We should also introduce ghosts, but as they are non-physical and they appear quite naturally in the gauge-fixing process, we will leave it aside for now.

To complete the description of the representations, we note that all fields are massless.

We recall that the vector representation of the Lorentz group (the spin 1 representation), will be the one-forms on space-time. Similarly the scalar representation will be the zero-forms. The spectrum is detailed in Table 1.1.

1.2.2 Lagrangian

The standard Lagrangian is the following

$$\mathcal{L} = \mathcal{L}_{fermions} + \mathcal{L}_{gauge} + \mathcal{L}_{Higgs} + \mathcal{L}_{GaugeFixing} + \mathcal{L}_{Ghost}$$

where :

- $\mathcal{L}_{fermions} = i\overline{\psi}\not{D}\psi$ sum on all fermions ψ.
- $\mathcal{L}_{gauge} = -\frac{1}{2}|F_A|^2$ sum on all gauge bosons A.
- $\mathcal{L}_{Higgs} = |DH|^2 + \mu^2|H|^2 - \lambda|H|^4$.
- $\mathcal{L}_{GF}$ and $\mathcal{L}_{Gh}$ to be defined later on.

We used the conventional notations for the covariant derivative and the field strength :

$$\begin{aligned} D &= d - i \sum_A g_A \rho(A) \\ F_A &= DA \end{aligned} \tag{1.1}$$

ρ being understood as the representation of the field on which D is acting.

		SU(3)	SU(2)	U(1)$_Y$	U(1)$_{\text{em}}$
quarks	$q_L^I = \begin{pmatrix} u_L^I \\ d_L^I \end{pmatrix}$	3	2	$\frac{1}{6}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
	u_R^I	$\bar{3}$	1	$-\frac{2}{3}$	$-\frac{2}{3}$
	d_R^I	$\bar{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$
leptons	$l_L^I = \begin{pmatrix} \nu_L^I \\ e_L^I \end{pmatrix}$	1	2	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	e_R^I	1	1	1	1
Higgs	$h = \begin{pmatrix} h^0 \\ h^- \end{pmatrix}$	1	2	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
gauge bosons	G	8	1	0	0
	W	1	3	0	$(0, \pm 1)$
	B	1	1	0	0

Table 1.1: The particle content of the Standard Model. Note that the ^{I} index stands for the generation.

We note that we have introduced a norm $|x|^2$ which must be understood as the combined norm of all representation spaces, the gauge and the Lorentz one. The first is easily defined by specifying an orthonormal basis of the representation space. The second is the norm on spinors (as defined in Appendix A.1) and the usual norm for r-forms $|u|^2 = u \cdot u = \int_{\mathcal{M}} u_\wedge * u$. With a slight abuse of notation I will use this norm either with the integral ($\int_{\mathcal{M}} u_\wedge * u$) when dealing with the action $\mathcal{S}$, and without ($u_\wedge * u$) when dealing with the Lagrangian $\mathcal{L}$.

For instance the orthonormal basis for the adjoint representation (ie an orthonormal basis of the gauge Lie algebra) will be $T_a, \frac{\sigma_i}{2}, Y$, where T_a are the Gell-Mann matrices of SU(3), σ_i the Pauli matrices, and Y the hypercharge factor, a 1.

In the following, I will use the notations $|x|^2$ and $x \cdot y$ for any representation. The point of using norms instead of indices is to allow us to treat each field as a whole thing, without entering into the details of its various representations. A more detailed justification is to be found in Appendix A.2.

1.2.3 Gauge Invariance

This Lagrangian is invariant under any transformation of the kind :

$$\psi, H \mapsto U\psi, UH \qquad A \mapsto A' \quad \text{such that} \quad D \mapsto UDU^{-1}$$

where U is a map from the space-time to the gauge group. This can be quickly seen from the Lagrangian by noticing that every such U is unitary (due to the particular gauge group) hence its action is norm-preserving, and it commutes with D.

The gauge invariance of the Lagrangian ensures the gauge invariance of any measurements, that is to say any expectation value of observables, hence we can talk of the gauge degrees of freedom as internal ones.

1.2.4 A little more on calculations in a quantum field theory

It may seem that we are done with the two first steps that lead us to actual calculations, and that the next part is only the techniques of those calculations. However, there is one of the settings of our theory that was not pointed out in step 1, because it was a very natural and apparently harmless concept, but due to the specific form of the action defined in step 2, it raises now an issue. This point is that all fields must have a zero vacuum expectation value. But, the vacuum defined by the minimization of the action of the standard model implies a non-zero vacuum expectation value of the Higgs field. Hence we need to redefine the Higgs fields, and this operation – called the symmetry breaking – will modify the physical spectrum, as we shall see now.

1.3 Symmetry Breaking

We said that the symmetry breaking was coming from the fact that, if we parametrize the Higgs boson by $H = \binom{h_1}{h_2}$ then the potential was not minimized for a 0 value of these fields, hence they were not physical. This can be easily seen by writting the Higgs potential

$$V_{Higgs} = -\mu^2|H|^2 + \lambda|H|^4 = \lambda\left(|H|^2 - \frac{\mu^2}{2\lambda}\right)^2 + \text{Cte}$$

which is indeed minimized – provided $\mu^2, \lambda > 0$ – for $|H|^2 = \frac{\mu^2}{2\lambda}$.

When I am speaking about minimizing the potential, I actually mean the true potential, hence the renormalized one evaluated at all orders. As the renormalization part will be explained a bit later, I do not focus on this point now, but the previous relation will turn out to be the zeroth order term of the true potential, hence it does hold. We will hence choose another parametrization :

$$H = \begin{pmatrix} \varphi^+ \\ \frac{1}{\sqrt{2}}(v + h + i\varphi_3) \end{pmatrix}$$

where φ^+ is a complex field, h, φ_3 reals, and v a constant called the vacuum expectation value.

Expanding $\mathcal{L}_{Higgs}$, we find out new mass terms for h and the gauge bosons such :

$$\mathcal{L}_{Higgs} = M_W^2 W^+ \cdot W^- + \frac{1}{2} M_Z^2 |Z|^2 + \frac{1}{2} M_h^2 h^2 + \text{ mixing and interaction terms}$$

$$\text{where} \qquad W^\pm = \frac{1}{\sqrt{2}} \left(W^1 \mp i W^2 \right) \qquad Z = c_w W^3 - s_W Y$$

$$M_W = g_1 \frac{v}{2} \qquad M_Z = \sqrt{g_1^2 + g_2^2} \frac{v}{2} \qquad M_h = \mu^2 - 3\lambda v^2$$

$$c_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}} \qquad s_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}$$

We have also ended up with a linear term $hv(\mu^2 - \lambda v^2)$ which vanishes for $v^2 = \frac{\mu^2}{\lambda}$. It is tempting to set v to this value, and hence obtain a seemingly more physical Lagrangian, but as previously said it would be a mistake, since renormalization has not been carried out yet. However those who understand the idea of renormalization would correctly guess that such an important relation is true at tree-level.

We see now that, if we want to use a Feynman type expansion, we will deal with the new fields $W^\pm, Z, B$ since their propagators do not mix in the free theory. We can then choose another basis of the gauge Lie algebra, ($\frac{\sigma_\pm}{2}, T_Z, Q$, see Appendix A.2) together with new coupling constants, so that :

$$g_1 W^i \frac{\sigma_i}{2} + g_2 BY = \frac{g}{\sqrt{2}} W^\pm \frac{\sigma_\pm}{2} + \frac{g}{c_W} Z T_Z + eAQ$$

By replacing this in the covariant derivative D, we get the Lagrangian as a function of the physical fields.

1.4 Propagators

After having chosen a basis of the fields in which we will carry our computations, the next step in the Feynman technique is to calculate the propagators of each particle. Although renormalization will introduce some new terms, we will count them as interactions, hence propagators will stay unchanged. The decomposition of the Lagrangian is elementary : constant terms are irrelevant, hence the quadratic terms will be the free Lagrangian, and linear terms together with the higher order terms the interaction. We recall that the only linear term is the Higgs tadpole.

By checking carefully the Lagrangian we obtain usual quadratic terms for fermions and bosons (that is to say mass terms and derivatives) but a bit more for bosons:

Higgs Lagrangian

$$\mathcal{L}_{Higgs}|_{\text{quadratic terms only}} = M_W^2 W^+ \cdot W^- + \frac{1}{2} M_Z^2 |Z|^2 - \frac{i}{2} M_W (d\varphi^+ \cdot W^- - d\varphi^- \cdot W^+) - M_Z d\varphi_3 \cdot Z$$

Hence we notice an unwanted mixing between the Goldstone modes and the weak bosons. However, as one may guess, we can cancel this mixing by fixing the gauge : indeed the straightforward unitary gauge eliminates the φ fields, hence the mixing.

1.5 Gauge Fixing

Nevertheless our approach will be a bit more subtle. As I will explain later on, it can be rewarding to keep some freedom in the gauge instead of having everything in the unitary gauge. The general gauge fixing that I will use, called generalised non-linear gauge fixing, is the following:

$$\mathcal{L}_{GF} = -\frac{1}{\xi_W} F^+ F^- - \frac{1}{2\xi_Z} |F^Z|^2 - \frac{1}{2\xi_A} |F^A|^2$$

where we used :

$$\begin{aligned} F^+ &= (d - ie\tilde{\alpha} A - ie\frac{c_W}{s_W}\tilde{\beta} Z) \cdot W^+ + i\xi_W \frac{e}{s_W}(v + \tilde{\delta} h - i\tilde{\kappa}\varphi_3)\varphi^+ \\ F^Z &= d \cdot Z + \xi_Z \frac{g}{2c_W}(v + \tilde{\epsilon} h)\varphi_3 \\ F^A &= d \cdot A \end{aligned}$$

We notice that the quadratic terms thus produced are either of the kind $-\frac{1}{\xi}(d \cdot A)^2$ and will be part of the propagator or are the same as those involved in the mixing occurring in the Higgs Lagrangian, but with an opposite sign, for any value of the ξ parameter. As an example we have

$$-\frac{1}{2\xi_Z}|F^Z|^2 \supset -\frac{1}{2\xi_Z} \times \frac{2d \cdot Z\xi_Z g v \varphi_3}{2c_w} \tag{1.2}$$

$$-\frac{gv}{2c_w} d \cdot Z\varphi_3 = -M_Z d \cdot Z\varphi_3 \tag{1.3}$$

$$\tag{1.4}$$

together with the $-M_Z d\varphi_3 \cdot Z$ from $\mathcal{L}_{Higgs}$ this yields a total derivative which hence does not contribute. Hence we managed to get rid of the mixing of the bosons.

I may point here that the gauge fixing procedure for Z and W bosons is quite different than this of the photon as it introduces new particles : ghosts. This is a general feature of non abelian fields, however as they are not physical but they will contribute to Feynman diagrams (through the gauge Lagrangian $\mathcal{L}_{Ghost}$) if we keep the gauge parameters free. Hence they will be taken into account in full automatized computations, however I will not compute their contribution explicitly in this essay.

We finally get the free Lagrangian, which will look like $\mathcal{L}_0 = -\frac{1}{2}\Phi^\dagger M \Phi$, where Φ is the vector containing all fields. The propagator is then M^{-1}, which in our case is diagonal as we discarded any mixing.

1.6 Conclusion

As a warning I would stress the fact that the parameters we wrote with conventional notations for couplings, masses, and so on, only have a physical significance at the tree level, as we will see later. From the renormalization global point of view, they are Lagrangian parameters, full stop. Moreover some of these parameters are redundant because of the relations between $M_W, M_Z, e, g_1 ...$. For the sake of renormalization, which the next step awaiting our freshly-made model, we will state the free set of parameters (that is to say independent parameters) that will be used :

$$M_Z, M_W, e, M_h, \mu^2$$

the original parameters $g_1, g_2, v, \mu^2.\lambda$ being recovered by the relations:

$$g_1^2 = \frac{e^2}{1 - \frac{M_W^2}{M_Z^2}} \quad g_2 = e\frac{M_Z}{M_W} \quad v^2 = 4\frac{M_W^2}{e^2}\left(1 - \frac{M_W^2}{M_Z^2}\right) \quad \lambda = \frac{\mu^2 - M_h^2}{2v^2}$$

We will include the Yukawa couplings by taking the masses of the fermions to be also parameters of the theory, hence we add m_u, m_e, etc to our set of parameters.

Notice that the relations we found are not relations among masses, couplings, and so on, these are relations between Lagrangian parameters. Once again they are relevant for tree-level computations but in a more general scope, one has to specify a renormalization scheme to exploit these relations for physical parameters, as we will see later.

By now, we have chosen the basis of field in which we want to do the Feynman expansion together with the associated coupling constant (g and e), we have obtained the propagator for these fields and all the vertices present in $\mathcal{L}_{int}$. Thus we have everything to compute observables such as cross-section, or decays, via the amplitude of Feynman graphs.

Chapter 2

MSSM

Once again, I am not presenting the full derivation of the MSSM along with its different motivation, but trying to apply the notations I used for the Standard Model, which will turn to be extremely useful when it comes to discriminate the extra supersymmetric part of the results from the standard one.

2.1 Situation

The situation is the same as in the Standard Model except that the Poincaré group is replaced by the super Poincaré group and fields are replaced by superfields (for the exact construction of the super Poincare group and superfields as one of its representation, see Appendix A.4).

2.2 Spectrum and Lagrangian

The superfields present in the MSSM are :

- Chiral Superfields Φ $(\lambda = 0 \oplus \lambda = -\frac{1}{2})$: matter,Higgs
- Vector Superfield V $(\lambda = \frac{1}{2} \oplus \lambda = -1)$: gauge multiplet

Note that all these superfields are massless.

We have exactly one chiral superfield for each fermionic field of the standard model, plus 2 Higgs chiral superfields : H_u in $1 \otimes 2 \otimes \frac{1}{2}$ and H_d in $1 \otimes 2 \otimes -\frac{1}{2}$. The vector superfield lies naturally in the gauge Lie Algebra.

The Lagrangian can be written either in terms of the superfields or in their fields-components, as we shall see now. I will not show the full derivation, which can be found either in [7] or in [4].

2.2.1 Lagrangian in superfields

Again, we start by decomposing the Lagrangian :

$$\mathcal{L} = \mathcal{L}_{chiralSF} + \mathcal{L}_{gauge} + \mathcal{L}_{gaugefixing} + \mathcal{L}_{ghost}$$

where :

- $\mathcal{L}_{chiralSF} = \int d^2\theta d^2\overline{\theta}\Phi^\dagger e^V \Phi + \int d^2\theta W(\Phi) + \int d^2\overline{\theta}(W(\Phi))^\dagger$ sum on all SF Φ.
- $\mathcal{L}_{gauge} = \frac{1}{8\pi}\int d^2\theta Tr\, W^2$
- $\mathcal{L}_{GF}$ and $\mathcal{L}_{Gh}$ to be defined later on.

The notation $\Phi^\dagger e^V \Phi$ stands for $\Phi^\dagger e^{\rho(V)}\Phi$ where ρ is the representation where the field lies. $W(\Phi)$ is the superpotential, which is an holomorphic function of the superfields and, for the sake of renormalizability, at most cubic in the chiral superfields. Last, $W_\alpha = -\frac{1}{4}\overline{D}^2(e^V D_\alpha e^V)$.

Note that the coupling constant do not appear here, until we rescale $V \mapsto gV$.

2.2.2 Lagrangian in fields

We decompose now the superfields :

$$V(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}v_\mu + i\theta^2\overline{\theta\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\overline{\theta}^2\mathcal{D}$$
$$\Phi(x,\theta,\overline{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 f(y) \qquad y = x + i\theta\sigma\overline{\theta}$$

The Lagrangian becomes :

- $\mathcal{L}_{chiralSF} = |D\phi|^2 - i\psi\sigma.D\overline{\psi} + |f|^2 + i\sqrt{2}g(\phi^\dagger\lambda\psi + \overline{\psi\lambda}\phi) + g\phi^\dagger\mathcal{D}\phi - \frac{\partial W}{\partial\Phi^i}f^i + h.c. - \frac{1}{2}\frac{\partial^2 W}{\partial\Phi^i\partial\Phi'^j}\psi^i\psi' j + h.c.$
- $\mathcal{L}_{gauge} = -\frac{1}{2}|F_v|^2 - i\lambda\sigma.D\overline{\lambda} + \frac{1}{2}|\mathcal{D}|^2)$

where we welcome back the covariant derivative $D = d - ig\rho(v)$ and the field strength $F = Dv$. Here again $\lambda\psi$ is to be understood as $\rho(\lambda)\psi$. Last, the derivatives of W (as $\frac{\partial W}{\partial\Phi^i}$) are evaluated in the scalar fields of all superfields, so this is just a shortcut for $\frac{\partial W}{\partial\Phi^i}(\phi)$ for instance.

We notice immediately that f and $\mathcal{D}$ are auxiliary fields, as there are no terms involving their derivatives. By solving the equations of motion one gets

$$f^\dagger = -\frac{\partial W}{\partial\Phi^i} \qquad \mathcal{D}^a = -g_a\sum_\phi \phi^\dagger\rho(T^a)\phi$$

where T^a is the orthonormal basis of the Lie algebra.

2.2.3 Analogy with the Standard Model

The expression of the supersymmetric Lagrangian in terms of fields is quite necessary in order to build Feynman rules, but actually, noticing that this Lagrangian is very much alike the standard model one, much of the work is already done.

The whole supersymmetric Lagrangian consists in the following parts :

Bosons

v_μ is the gauge boson (hence $v = G_a T_a + W_i\frac{\sigma}{2} + BY$) $\mathcal{L}_{gauge\ boson} = -\frac{1}{2}|F_v|^2$
ϕ is the scalar boson, for each chiral SF Φ $\mathcal{L}_{scalar} = |D\phi|^2$ sum on all SF

Fermions
λ and ψ are fermions of the gauge and chiral multiplets respectively

$$\mathcal{L}_{fermions} = -i\psi\sigma.D\overline{\psi} - i\lambda\sigma.D\overline{\lambda}$$

Scalar potential
Solving the equations of motion of the auxiliary fields lead us to a scalar potential.

$$V = \sum_{\Phi^i} |\frac{\partial W}{\partial \Phi^i}|^2 + \frac{1}{2}(g_a \sum_{\phi} \phi^\dagger \rho(T^a)\phi)(g_a \sum_{\phi} \phi^\dagger \rho(T^a)\phi)$$

running on all bosons of the chiral SF, that is to say the matter and the Higgses.

Additional Interactions Terms
There are also new interaction terms introduced in the MSSM:

$$\mathcal{L}_{int\ add} = i\sqrt{2}g(\phi^\dagger \lambda\psi + \overline{\psi\lambda}\phi) - \frac{1}{2}\frac{\partial^2 W}{\partial \Phi^i \partial \Phi'^j}\psi^i \psi' j + h.c.$$

The first term being an interaction between the bosons and fermions of a single chiral multiplets via the gauge bosons, and the second an interaction term involving two fermions from chiral multiplets (they can be different multiplets).

As we have seen the Lagrangian of the standard model is entirely included – except for the Higgs sector – in the MSSM one, and to that respect we can claim that it is actually an extension of the standard model. This also means that every Feynman graph of the MSSM will be built with standard vertices and propagators on one hand, and supersymmetric ones on the other.

2.3 Gauge Invariance

Provided that the superpotential W is gauge invariant, the whole Lagrangian is invariant under the transformation:

$$\text{for each } \Lambda \in g \qquad \Phi \mapsto e^{i\rho(\Lambda)}\Phi \qquad V \mapsto e^{i\Lambda^\dagger} V e^{i\Lambda}$$

Note that in the expression in components, V was expanded in the Wess-Zumino gauge, hence we do not have the full gauge invariance if we stick to the component notation.

2.4 Symmetry Breaking

The main part of specifying a supersymmetric model lies in the superpotential. In the MSSM it writes:

$$W = y_u u_R Q H_u + y_d d_R Q H_d + y_e e_R L H_d + \mu H_u H_d$$

where we recognise the Yukawa couplings. The standard Yukawa terms for fermions come from the additional $-\frac{1}{2}\frac{\partial^2 W}{\partial \Phi^i \partial \Phi'^j}\psi^i \psi' j + h.c.$ term in the MSSM which yields terms such as $y_u {h_u}^0 u_R u_L$, where the Higgs is such that $h_u = \binom{{h_u}^+}{{h_u}^0}$.

Following the same motivation than in the Standard Model, we write down the scalar potential:

$$V = |\mu|^2(|h_u|^2 + |h_d|^2) + \frac{g_1^2 + g_2^2}{8}(|h_u|^2 - |h_d|^2)^2 + \frac{g_1^2}{2}|h_u^\dagger h_d|^2$$

Here the small $h_{u/d}$ stand for the scalar field of $H_{u/d}$. We have here restrained ourselves to the Higgs part of the scalar potential, but it is easy to derive that the minimization of this potential requires a zero expectation value of the other fields, which also explains why we do not take the Yukawa coupling into account. See appendix for the derivation.

We notice immediately that there is no spontaneous breaking of symmetry, as the potential is minimized for a zero expectation value of all fields. This is why we have to introduce ourselves new supersymmetry breaking terms. We will precisely deal with the case of softly broken supersymmetry, which constrains the additional terms. We call this part $\mathcal{L}_{soft}$:

$$\mathcal{L}_{soft} = -b(h_u \cdot h_d + \overline{h_u} \cdot \overline{h_d}) - \phi^\dagger m_\phi^2 \phi - \frac{1}{2}\overline{\lambda}\tilde{m}\lambda + h.c.$$

Where ϕ runs over all the scalar fields of the model. And the dot $h_u \cdot h_d$ stands for $\epsilon_{ab} h_u^a h_d^b$. In the following we will also refer to b as m_{12}^2 as it is a cross mass term fro the two Higgs doublets.

Whereas nothing has changed for most of the scalars – they still have a zero vacuum value – the Higgs can now pick up a vev, for some values of b. The process is extremely similar to the one we saw in the standard model, hence we expand the Higgs as :

$$h_u = \begin{pmatrix} \varphi_u^+ \\ \frac{1}{\sqrt{2}}(v_u + \varphi_u^0 + i\varphi_u^3) \end{pmatrix} \qquad h_d = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_d + \varphi_d^0 + i\varphi_d^3) \\ \varphi_d^- \end{pmatrix}$$

Inserting this new parametrization in the Higgs scalar Lagrangian brings the very same electroweak symmetry breaking that we encountered in the standard Model, with a vacuum expectation value v given by

$$v^2 = v_u^2 + v_d^2$$

We notice that the gauge and Higgs sector, that have now to be considered as a whole because of the symmetry breaking, are a bit more complicated than in the standard model : we started with 8 Higgs real fields, 3 of them were eaten by the massive gauge bosons so we are left with 5 physical Higgses.

2.5 New Fields and propagators

The fields (W, Z, A) are defined as in the standard model, and it is also fairly easy to see that, due to the m_{12}^2 mass term of the Higgses, we will have to rotate the bases of H_u, H_d to get the mass eigenstates. This is done by the following rotations :

$$\begin{pmatrix} G^0 \\ A^0 \end{pmatrix} = R(\beta)\begin{pmatrix} \varphi_u^3 \\ \varphi_d^3 \end{pmatrix} \quad \begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = R(\beta)\begin{pmatrix} \varphi_u^\pm \\ \varphi_d^\pm \end{pmatrix} \quad \begin{pmatrix} H^0 \\ h^0 \end{pmatrix} = R(\alpha)\begin{pmatrix} \varphi_u^0 \\ \varphi_d^0 \end{pmatrix}$$

where β and α angles are defined by

$$\tan(\beta) = \frac{v_u}{v_d} \qquad \tan(2\alpha) = \frac{M_{A^0}^2 + M_Z^2}{M_{A^0}^2 - M_Z^2}\tan(2\beta)$$

$H^{\pm}$ are the charged Higgses, A^0 is the CP-odd scalar Higgs, and h^0, H^0 the CP-even Higgs, h^0 being the lightest one and hence plays the role of the h of the Standard Model.

The sfermions associated to the gauge and Higgs bosons also need to be rotated to form mass eigenstates, which is done by two separate matrices, one for charged fermions and another for neutral ones, as :

$$\begin{array}{ccll} (\tilde{W}^{\pm}, \tilde{h}^{\pm}_{u,d}) & \longrightarrow & \tilde{\chi}^{\pm}_{1,2} & \text{charginos} \\ (\tilde{B}, \tilde{W}^3, \tilde{h}^0_{u,d}) & \longrightarrow & \tilde{\chi}^0_{1..4} & \text{neutralinos} \end{array}$$

Having defined the new basis of fields (concerning usual matter, the sfermions do also mix), the next step towards the calculation of the propagators is the definition of the gauge fixing function.

2.6 Gauge fixing

The gauge fixing is ensured by $\mathcal{L}_{GF}$ as in the Standard Model, with the following modifications:

$$\begin{aligned} F^+ &= (\partial_\mu - ie\tilde{\alpha}\gamma_\mu - ie\frac{c_W}{s_W}\tilde{\beta}Z_\mu)W^{\mu+} + i\xi_W\frac{e}{2s_W}(v + \tilde{\delta}h^0 + \tilde{\omega}H^0 + i\tilde{\rho}A^0 + i\tilde{\kappa}G^0)G^+\,, \\ F^Z &= \partial_\mu Z^\mu + \xi_Z\frac{e}{s_{2W}}(v + \tilde{\epsilon}h^0 + \tilde{\gamma}H^0)G^0\,, \\ F^A &= \partial_\mu A^\mu\,. \end{aligned} \tag{2.1}$$

As in the standard model this specific form has been chosen for two reasons

- It discards the mixing between gauge and Higgs boson.
- It preserves some gauge dependence through the variables $\tilde{\alpha}, \tilde{\beta}$...

2.7 Conclusion

In this model too, we will have to carry out renormalization before talking about anything physical, and to that purpose we have to choose a free set of parameters. This is more arbitrary (meaning more discussed) than in the Standard Model, as it decides partly which renormalization scheme will be used. The one I will use is made of the following parameters :

$$e, M_W, M_Z, M_{A^0}, T_{\phi_1^0}, T_{\phi_2^0}, t_\beta$$

that are equivalent to the initial set

$$g, g', v_1, v_2, m_1^2, m_2^2, m_{12}^2$$

We also add the fermion masses that stand for the Yukawa coupling, and then renormalization can be safely operated.

Chapter 3

Renormalization

3.1 Motivation of renormalization

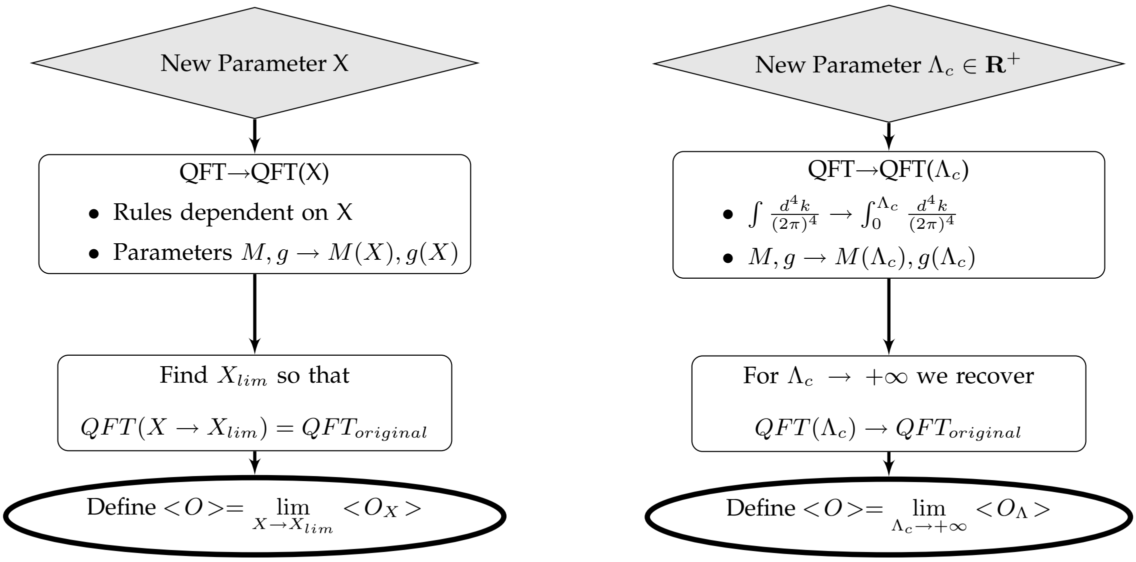
Having defined the numerous fields and the Feynman rules of our theory it seems that computing observables by the expansion on the couplings would be straightforward, however there is a last well-known subtlety : renormalization. Indeed, in such a calculation, most of the integrals we will encounter will be divergent, at least in the limit of high momentum. The very idea of renormalization is that these divergences do not impede us in finding physical results (though that is far from obvious) and that with a simple proceedure we can remove these divergences from all observables in a most realistic way.

3.2 Principle

The process is the following : first we introduce a new parameter – let us say X – in our theory, which usually requires a redefinition. For instance it modifies the rules of the theory, and the Lagrangian parameters which become functions of X. To make contact with our physical theory, we must then find a value of X for which we recover the original rules, let us call this value X_{lim}. Eventually, if all observables are well defined for X around X_{lim} (which has to be the case for any renormalization), the physical observable is set to be the limit of the observables $<O>_X$ for $X \to X_{lim}$. I recap this principle together with a classic example in fig 3.1.

The classic example, and quite the most intuitive one regarding what we said about the divergences is the momentum cut-off parametrization. The new parameter is Λ_c, which is a momentum, hence a real. The redefinition of the theory is the following : all parameters of the Lagrangian $(e, g, ...)$ get a specific dependence on Λ_c, and all integrals over momentum that would occur in the Feynman expansion are truncated at Λ_c. Hopefully, for any real value of the cut-off, observables will be finite as there will be no more high momentum divergences. Moreover the theory one recovers by taking the limit $\Lambda_c \mapsto +\infty$ is our original one, except for the values of the parameters (e,g,...). Indeed these parameters have a specific form to yield experiment-appliant result and which happens to diverge in the limit $\Lambda_c \mapsto +\infty$. But as they are not observables, this is not an issue.

At first glance, renormalization seems rather clumsy : we redefine drastically our theory and take our physical values to be the limit of unphysical values, obtained with unphysical parame-



Renormalization procedure — Mometum cutoff renormalization

Figure 3.1: Overall procedure

ters! Indeed, *a priori*, calculating a Feynman amplitude with integrals on momentum truncated is not physical, and Lagrangian parameters that diverge for the physical limit ($\Lambda_c \mapsto +\infty$) are not physical neither. But from the practical point of view you have very few parameters to set (for the Standard Model $M_Z, M_W, e, M_h, \mu^2, m_f$), and once they are set using a few observables (like bosons masses, cross-sections...) you can compute systematically all observables. As one may notice, I am not dealing with renormalization from the theoretical point of view, first because it is not the point of this essay, but above all because one barely need an explanation to be convinced of the predictive power of renormalization.

3.3 Systematics of renormalization

3.3.1 Expansion of the parameters

It is all right to claim that renormalization brings out finite (and correct!) results but there is a few shadow points. One may wonder how to write the Feynman expansion on a coupling constant that depends on a hidden parameter, and that may even diverges for some values of this parameter, the point is that the Feynman expansion is always on a finite (and usually small) coupling constant. More generally if we can express the general Lagrangian parameters – the one that depends on the renormalization parameter – as expansion on a finite coupling constant, we can still go through all the Feynman procedure. One may argue that all coefficients of the expansion will not be constants any more but variables of the renormalization parameter and that the mathematics involved by calculus on functional series is much more complicated[1]

[1]Indeed, if an observable has an expansion $O(\Lambda) = \sum_i f_i(\Lambda)\alpha^i$ where α is the finite coupling, it is not guaranteed that the coefficients of $O = \lim_{\Lambda\to\infty} O(\Lambda)$ correspond to $\lim_{\Lambda\to\infty} f_i(\Lambda)$, as is naively expected.

in this case (especially near the limit where these functions diverge), that is very true, but it is not my point. Renormalization still yields correct results.

In the electroweak model, the coupling constant of the expansion is e (remember that the coupling g_1, g_2 can be obtained from e with c_W, s_W). The point is then to find from the Λ-depending coupling constant, a Λ-independent constant. Here I use Λ to denote the renormalization parameter, which can be completely different from a momentum cut-off. For the electromagnetic coupling constant we have

$$e_\Lambda = \sum_{i>0} c_i(\Lambda) e^i$$

the zeroth-order term vanishing for the sake of transposing "correctly" an expansion in e_Λ in an expansion in e (see Appendix A.6.2). It is also straightforward to find the expansion of the other Lagrangian parameters in e. For instance

$$M_Z = \sum_{i\geq 0} z_i(\Lambda)(e,g)^i$$

where we kept the zeroth order.

These expressions can be slightly improved by noting that we want our observables to have a finite limit for $\Lambda \mapsto \Lambda_{lim}$, hence we need all orders in e to have a finite limit. The first non-zero order of each observable does not contain any loop (adding a loop always demand more vertices, hence more power of e), hence the only Λ contribution is from the lowest order of the Lagrangian parameters (that is to say $c_1, z_0, ...$). So the only way the lowest order of observables has a finite limit in Λ_{lim} is to impose that every lowest order of the Lagrangian parameters $c_1, z_0, ...$ has a finite limit $c_{1\ lim}, z_{0\ lim}, ...$. Inserting $c_1(\Lambda) = c_{1\ lim}$ (and similarly for the other parameter) does not modify the observables since they are defined in the very limit $\Lambda \mapsto \Lambda_{lim}$. Thus this allows us to rescale $e \mapsto \frac{e}{c_{1\ lim}}$ and re express every Lagrangian parameter in the satisfying form :

$$e_\Lambda = e + \delta e \qquad \text{where } \delta e = e^2 \sum_{i\geq 0} c'_{i+2}(\Lambda) e^i$$

$$M_{Z\ \Lambda} = M_Z + \delta M_Z \qquad \text{where } \delta M_Z = (e,g) \sum_{i\geq 0} z'_{i+1}(\Lambda)(e,g)^i$$

And so on for every parameter of the Lagrangian. It is quite usual to note the fields expansion $\varphi_\Lambda = \varphi + \delta\varphi$ by introducing a field renormalization constant :

$$\varphi_\Lambda = Z^{1/2}\varphi = \varphi + \delta Z^{1/2}\varphi$$

which is to be seen as a matrix acting on all fields (φ being the vector containing all fields). Indeed we can have terms $Z^{1/2}_{AB}$ where A and B are different particles. However, to have a consistent Lagrangian we must require that $Z^{1/2}_{AB} = 0$ if A and B are not in the same multiplet or representation (for instance they can be no such mixing between a boson and a fermion). We note $Z = Z^{1/2\ \dagger} Z^{1/2}$ (so that $\delta Z = \delta Z^{1/2\ \dagger} + \delta Z^{1/2}$) which is also used in calculations.

One very interesting consequence of this expression is that all Lagrangian parameters par_Λ are separated into a Λ-independent part *par* and a Λ-dependent part δpar such that δpar is at least one order above in the expansion. The δ are called counterterms.

Remark: when we rewrite the Lagrangian with this replacement, all δ fall in the interaction part. Hence the propagators are the same than those computed before the renormalization but with the lowest order value of each parameter instead the parameter itself.

Now that we have named our Lagrangian general parameters we need to specify them (that is, giving specific value to e, c_i, g_i, z_i...) so that observables are finite and compatible with experiments. The hard but however most amazing point is that by requiring just a limited number of observables to agree with the experiment we specify completely those parameters. We will call the set of observables and truncations that specify completely those parameters the renormalization scheme.

One may wonder why we attach so much importance to the parametrization of every parameter in a lowest-order term independent of Λ and higher order terms Λ-dependent, as the result we are interested in – the observable – is a full-order expression. The reason lies in the fact that we are often unable to give a general formula for every order, hence we will need to truncate the result. Hence we would rather appreciate to have coherent results as soon as possible in the expansion orders. This point being underlined by the fact that in most models the proliferation of Feynman diagrams for a given process is extremely quick : in the MSSM at one loop some processes already require more than ten thousands of diagrams. As the lowest orders of the Lagrangian parameters will yield the lowest orders of the observables, it is thus important to set them to a correct value. The appreciation of what should be a correct value being the whole story of assessing the quality of a renormalization scheme.

3.3.2 Obtaining the renormalized Lagrangian

The reason why we wanted so much the parametrization of the parameters is also that we can now express the renormalized Lagrangian in quite a simple way, compared to the large modifications we did.
Fortunately we saw that the terms in the Lagrangian of the Standard Model and the MSSM are very similar so the term we need to expand are:

- $|D\phi|^2$ for bosons
- $\overline{\psi}\not{D}\psi$ for fermions
- $\phi^\dagger M^2\phi$ for bosons
- $\overline{\psi}\tilde{M}\psi$ for fermions
- Interactions terms, gauge fixing ...

Analysing the case of the bosons mass terms gives :

$$\phi_\Lambda^\dagger M_\Lambda^2 \phi_\Lambda \to \phi^\dagger(1+\delta Z^{\frac{1}{2}\,\dagger})(M^2+\delta M^2)(1+\delta Z^{\frac{1}{2}})\phi \quad (3.1)$$

$$\to \phi^\dagger M^2\phi + \phi^\dagger \delta M^2 \phi + Re(\phi^\dagger M^2 \delta Z^{\frac{1}{2}}\phi) \quad (3.2)$$

where we have truncated the last line to the first order. We can expand all terms in a similar manner.

The expression for the renormalized Lagrangian depends on the basis of parameters that was chosen. It is commonplace to have to switch from a basis to another : for instance the W vertex involve the g coupling constant, which is not part of the basis we have chosen. However starting from the defining relation $g^2 = \frac{e^2}{1-\frac{M_W^2}{M_Z^2}}$ we can extract the δg^2 counterterm :

$$\frac{\delta g^2}{g^2} = \frac{\delta e^2}{e^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right)$$

where we truncated all δ at first order. This kind of manipulation occurs more often than not in the MSSM, where the total amount of masses and couplings is much larger than the set of parameters.

3.4 Renormalisation technique

Before introducing the renormalization schemes of the EW and MSSM models, I will present the technique of renormalization used, that is to say which renormalization parameter is created, and how it modifies the theory.

3.4.1 Dimensional regularisation and dimensional reduction

The new parameter is ϵ, a small real positive parameter. For any value of ϵ calculations are carried out in 4-ϵ dimensions, and all Lagrangian parameters depend on ϵ, in a way that will be defined later on. With a few computations one can check that the observables computed this way are finite for $\epsilon \neq 0$.
The physical observables are obtained by the limit $\epsilon \to 0$.

The point of this technique is that the ultraviolet divergences present in the bare theory will now show up as pole in ϵ, as $\frac{1}{\epsilon}, \frac{1}{\epsilon^2}$... It is then possible to insert those kind of poles in the Lagrangian parameters so that the renormalized result does not diverge any more. This will be seen in details in description of the renormalization of the electroweak model.

There is a necessary refinement to dimensional regularisation, called dimensional reduction. Its aim is to preserve 4 dimensional fields, while the integration is still carried in $4 - \epsilon$ dimensions. This is compulasory to maintain supersymmetry, hence it is required for computations in the MSSM. To maintain the dimensionless of the action we need to introduce another parameter μ which has the dimension of a mass and that appear by the replacement

$$\int \frac{d^4k}{(2\pi)^4} \to \mu^\epsilon \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}$$

We will refer to μ as the renormalization scale.

3.4.2 Bremsstrahlung

A more precise analysis of the observables show that they diverge because of some integrals on low momentum (whatever ϵ may be), which happens every time a massless boson is exchanged. As my work has been focused on electroweak models (standard or supersymmetric), I will deal only with the case of the photon. First these divergences can be regularised by creating a mass λ for the photon and computing observables in the cases $\lambda \neq 0$. However we can tackle the issue in a much simpler way – and intrinsically quite different – than the previous renormalization with ϵ. In the latter, divergences are cancelled by a singular behaviour of the Lagrangian parameters in the $\epsilon \to 0$ limit whereas the infrared divergences can also be cancelled in the physical limit where λ vanishes[2]. This is done by adding on top of all the diagrams contributing to the process those including the emission of an "unobservable" photon, also called "soft photon". What we understand by soft photon is an undetectable photon, that is to say either with such a small momentum that it cannot trigger a detector or collinear to another particle so that only the massive particle is visible.

Practically, computing the soft photon correction consists in integrating the diagrams with emission of a photon with a momentum from 0 to k_c, which is the threshold of the detector. The final amplitude of the process being now free from infrared divergences, but k_c dependent. However it is not that surprising to find that the result of a measurement depends on the measure apparatus, and if one wants to avoid this specific dependence, adding the hard photon part (ie the integration over k_c) is enough – though it redefines slightly the process.

One may point that the k_c may be brought as close as wished to 0, at least theoretically. That is true, but as long as it is different from zero, which has be the case in Quantum Mechanics where the quantum fluctuations of the vacuum cannot be avoided, the soft photon correction will still cancel the infrared divergences.

3.5 General renormalization scheme

Renormalization schemes are arbitrary. But the On-Shell (OS) scheme is by far the most used. This scheme is made of the three following point:

- The mass of each particle (as a parameter of the Lagrangian) is set to be the measured mass at all orders.
- On the mass shell of each particle, the residu of the pole of the propagator is set to 1.
- There is no mixing between particles on their mass shells.

Last, the electromagnetic coupling in the Thomson limit is set to be the measured coupling at all orders.

We recall that the mass of a particle is the pole of its propagator. Imposing it to be equal to the measured mass at any order induces that the zeroth order coefficient is exactly this measured mass and all other coefficients vanish. A quick analysis show that this zeroth order is precisely the zeroth order of $M_{X\ \Lambda}$ that is to say M_X, which hence ends up to be the physical mass of the particle X. The same goes for the electromagnetic coupling constant e.

Setting the normalization of a field to 1 implies that its propagator writes

$$D = \frac{1}{k^2 - M^2 + O(k^4)}$$

[2]This means that λ is nothing but a *regularization* parameter, which is by far weaker than a *renormalization* parameter

that is to say the residu of its pole is 1.

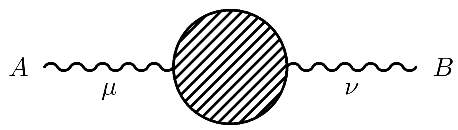
The on shell non mixing simply means that the propagator $D^{AB}(k^2)$ vanish at $k^2 = M_A^2$ and $k^2 = M_B^2$.

We may be a bit more precise now by specifying that a given renormalization scheme does not require all conditions on all particles, which would be a bad idea as there is often much less parameters than particles. Most of the time it requires the on-shell normalization and mixing condition on all, but set the masses for only some particles. Even the on-shell field-normalization can be avoided, if one is only interested in the S matrix.

3.5.1 Notations

To go a little bit inside the calculations of the counterterms at all order I will introduce some notations for Feynman diagrams. But I would like to remind the reader that for the purpose of calculating cross-sections and similar observables, we will have to resort to an automatized computation of these counterterms.

$i\Sigma^{AB}(k) =$



Where the shaded circle is to be understood as the sum of all one particle irreducible (1PI) diagrams.

For a gauge boson, the general form is

$$\Sigma_{\mu\nu}^{AB}(k) = k_\mu k_\nu L^{AB}(k^2) - g_{\mu\nu} G^{AB}(k^2) \tag{3.3}$$

$$\text{with} \quad G^{AB}(k^2) = G^{AB} + k^2 \Pi^{AB}(k^2) \tag{3.4}$$

A, B denote incoming and outgoing particles.
Contrary to what one could think, both of these equations are basic : the first is a consequence of the Lorentz representation and the second an expansion in k^2.

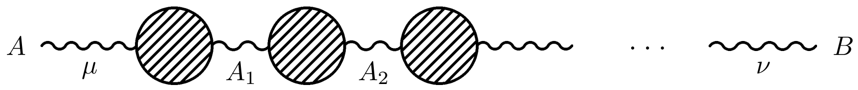
We will moreover note $\hat{\Sigma}$ the amplitude that also involves the interaction counterterms. This is the "true" amplitude.

We will also denote $D_0{}^{AB}$ ($D_0{}_{\mu\nu}^{AB}$ for gauge boson, $D_0{}_{\alpha\beta}^{AB}$ for fermions) the free propagator, as defined by the free Lagrangian, and D^{AB} for the full propagator, defined by the amplitude of the self energy diagrams. Last, note that we will often refer to those self-energy and propagators as matrices acting on all the vector containing all fields, hence omitting the A, B indices:

$$\Sigma_{\mu\nu}(k) = k_\mu k_\nu L(k^2) - g_{\mu\nu} G(k^2) \qquad \left(\hat{\Sigma}_{\mu\nu}(k) = k_\mu k_\nu \hat{L}(k^2) - g_{\mu\nu} \hat{G}(k^2)\right)$$

3.5.2 Full propagator

The full propagator for gauge bosons can be written as :



$$D_{\mu\nu} = D_{0\,\mu\lambda} \sum_{n\geq 0} (\hat{\Sigma} D_0)^{n\ \lambda}_{\ \ \nu} \tag{3.5}$$

The whole calculation of the propagator, though pretty easy, is not of much use now, since we are dealing with the renormalization procedure. Indeed we need the propagator to make contact with the OS conditions, but this condition are gauge independent – they are physical – whereas the propagator is not. More exactly its $k_\mu k_\nu$ part is not gauge independent. So we will focus on the $g_{\mu\nu}$ part, that is to says the transverse part, of the propagator, named D_T. The computation of the whole propagator is only useful when actually computing the amplitude of a process if one is willing to test the gauge independence of the result.

We see that in Eq. (3.5) the remaining $g_{\mu\nu}$ terms of the contractions are those implying only $g_{\lambda\rho}$ terms, so we can factorize out the $g_{\mu\nu}$ in the following way :

$$D_T(k^2) = (D_{0\,T}(k^2)) \sum_n (-G(k^2) D_{0\,T}(k^2))^n$$

We now use the formal identity $\sum_{n\geq 0} X^n = \frac{1}{1-X}$, where X is any matrix. The term formal means that in any cases we are dealing with integer series in e, so everytime we note $\frac{1}{1-x}$ we mean $\sum_{n\geq 0} x^n$ [3]. We can re express our full propagator:

$$D_T(k^2) = D_{0\,T}(k^2) \frac{1}{1 + G(k^2) D_{0\,T}(k^2)}$$

which eventually turns out to be

$$D_T(k^2) = \frac{1}{D_{0\,T}(k^2)^{-1} + G(k^2)} \tag{3.6}$$

A similar work can be carried on fermions and scalars.

3.5.3 Rewriting the OS conditions

Having found the simple relation

$$D_T^{-1}(k^2) = D_{0\,T}^{-1}(k^2) + \hat{G}(k^2)$$

we can re express the OS conditions in a more clever way. Remember that these conditions were

- There is no mixing between particles on theirs mass shells.

[3] The point in taking to the formal $\frac{1}{1-x}$ is that it is a bit more convenient to perform some calculations as it fortunately is compatible with a whole bunch of operations. Thus, we can write for instance $\frac{1}{1-x}\frac{1}{1-y} = \frac{1}{1-x-y+xy}$ which in reality means $\left(\sum_{n\geq 0} x^n\right)\left(\sum_{n\geq 0} y^n\right) = \sum_{n\geq 0}(x+y-xy)^n$ but proving the second result is much more complicated.

- The mass of a particle (as a parameter of the Lagrangian) is set to be the measured mass at all orders.
- On the mass shell of a particle, the residu of its propagator is set to 1.

They are efficiently re expressed in this way (all results of this section are quoted without proof, a formal derivation is to be found in Appendix A.5.2)

- $D_T^{AB}(M_A^2) = 0$ if $A \neq B$, for all particle A (non-mixing condition)
- $D_T^{-1\,AA}(M_A^2) = 0$ for all particle A (measured mass = M_A)
- $D_T^{-1\,AA\,\prime}(M_A^2) = 0$ for all particle A (field normalisation = 1)

We must add that these relations hold only on particles A concerned by renormalization. If a given particle has no requirements towards renormalization (mass, field normalization...) then these relations are not to be applied. Incidentally the $\prime$ in $D_T^{-1\,AA\,\prime}$ means the derivative along k^2.

Now we have to turn these conditions on D_L on conditions on $D_{0\,L}$ and $\hat{G}$, which leaves us with the following constraints :

$$D_{0\,L}^{-1\,AB}(M_A^2) + \hat{G}^{AB}(M_A^2) = 0 \text{ if } A \neq B \quad (3.7)$$

$$\hat{G}^{AA}(M_A^2) = 0 \quad (3.8)$$

$$\hat{G}^{AA\,\prime}(M_A^2) = 0 \quad (3.9)$$

These are the relations that we will use directly to enforce the OS renormalization. The reason of the last two sections is that by computing once and for all the propagators in terms of free propagators and renormalized self-energies, we fasten up the way to the computation of the counterterms. As we will see shortly, one just need to compute the amplitudes of the diagrams involved in the self energy and then plug the result in the equations (3.7-3.9).

3.6 Renormalization in the Standard Model

We may now apply this wise formalism to the Standard Model, for a start, and see if this renormalization can be carried out coherently. We will however focus on the electroweak model, since we are not dealing at all with QCD, which is required for a full renormalization of the Standard Model.

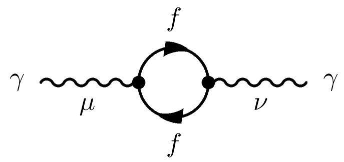
3.6.1 Computation of the self-energies of the bosons

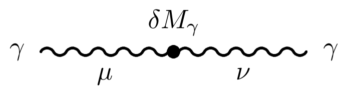
As we have just noticed, we need the $\hat{G}$ of the particles on which we want to enforce the renormalization scheme. Those particles are the photon, the Z and W bosons and the Higgs for the masses and normalization of the propagator (condition 2 and 3) and all bosons for the non-mixing condition. We will now compute this self-energy to the first order.

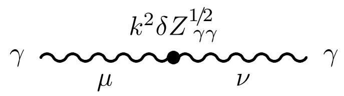
A first and very powerful operation is to choose a particular gauge to make our calculations as simple as possible. Going to the unitary gauge allows us to get rid of the Goldstone bosons, hence eases a lot the enforcement of the non-mixing constraint.

Calculating the non-diagonal element of $\hat{G}$ is not compulsory, as they do not appear often in the processes we are interested in, so I will directly go to computations of $\hat{G}_{\gamma\gamma}$, $\hat{G}_{ZZ}$, $\hat{G}_{WW}$, $\hat{G}_{hh}$. Again only results will be stated now, the full derivation being found in Appendix A.6.4

Photon
Diagrams accounting for this self-energy are

- 

being summed on all fermions f
- 

- 

We implicitly truncated at first order.

Computations of those diagrams (as done in Appendix A.6.4) leads us to the following expression

$$\Sigma^{\gamma\gamma}_{\mu\nu} = (k_\mu k_\nu - k^2 g_{\mu\nu})\frac{e^2}{12\pi^2}\sum_i N_{Q\,i} Q_i^2 \left(C_{UV} - 6B_2(k^2,m) + O(\epsilon)\right) \\ + \delta M_\gamma^2 g_{\mu\nu} - k^2 \delta Z_{\gamma\gamma} g_{\mu\nu} \quad (3.10)$$

where $B_2(k^2,m) = \int_0^1 x(1-x) ln\frac{m_i^2 - k^2x(1-x)}{\mu^2}dx$ and $C_{UV} = \frac{2}{\epsilon} + ln(4\pi) - \gamma$ as can be found in the Convention section.

Z boson

$$\hat{G}^{ZZ}(k^2) = -\frac{g^2}{c_W^2}\sum_\psi N_\psi \left(T_Z^2 G_{LL}(k^2,m_\psi) + 2s_W^2 T_{Z\,L} Q_R G_{LR}(k^2,m_\psi) + s_W^4 Q^2 G_{RR}(k^2,m_\psi)\right) \\ + \delta M_Z^2 - (k^2 - M_Z^2)\delta Z_{ZZ} \quad (3.11)$$

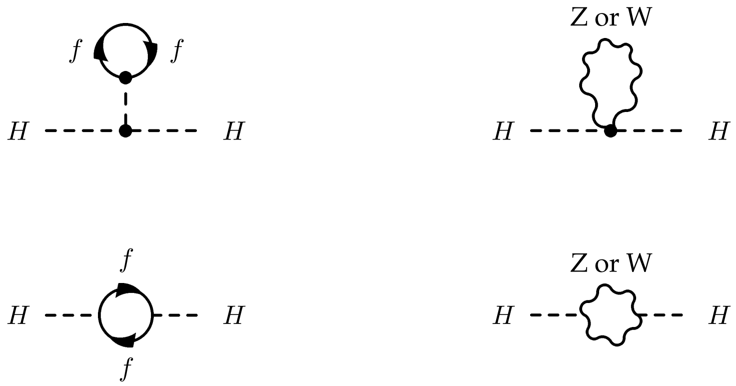
where G_{LL}.. are also defined in Appendinx A.6.4.

W boson

$$\hat{G}^{WW}(k^2) = -\frac{g^2}{2}\sum_\psi N_\psi G_{LL}(k^2, m_{\psi,2}, m_{\psi,1}) + \delta M_W^2 - (k^2 - M_W^2)\delta Z_{WW} \quad (3.12)$$

Higgs boson

The self energy of the Higgs is much intricate since it includes the following diagrams:



Hence the whole analytical expression is a bit lenghty so I do not include it here.

3.6.2 OS Renormalization

We saw in Chapter 1 that the propagator $D_{0\,T}$ was diagonal, hence the non-mixing condition writes

$$\hat{G}^{AB}(M_A^2) = 0 \qquad \text{if } A \neq B$$

However we said that we were not interested by these conditions, because the only counterterms $\delta Z^{1/2}_{AB}$ contributes and they usually do not contribute to most of the processes we want to compute. We will then leave those parameters aside, until we really need to compute some.

Enforcing the second OS renormalization condition (field normalization $\hat{G}^{AA}(M_A^2) = 0$) brings on

- $\delta M_\gamma = 0$
- $\delta M_Z^2 = \frac{g^2}{c_W^2} \sum_\psi N_\psi \big(T_Z^2 G_{LL}(M_Z^2, m_\psi) + 2 s_W^2 T_{Z\,L} Q_R G_{LR}(M_Z^2, m_\psi) + s_W^4 Q^2 G_{RR}(M_Z^2, m_\psi) \big)$
- $\delta M_W^2 = \frac{g^2}{2} \sum_\psi N_\psi G_{LL}(M_W^2, m_{\psi,2}, m_{\psi,1})$

The third condition yields the values of

$$\delta Z^{1/2}_{\gamma\gamma},\ \delta Z^{1/2}_{ZZ},\ \delta Z^{1/2}_{WW}$$

As the full expressions with C_{UV} and integrals are a bit heavy to carry in the calculations we will often writes the counterterms with the self-energies:

$$\begin{array}{lll} \delta M_\gamma = G^{\gamma\gamma}(0) & \delta M_Z^2 = G^{ZZ}(M_Z^2) & \delta M_W^2 = G^{WW}(M_W^2) \\ \delta Z^{\frac{1}{2}}_{\gamma\gamma} = -G^{\gamma\gamma\,\prime}(0) & \delta Z^{\frac{1}{2}}_{ZZ} = -G^{ZZ\,\prime}(M_Z^2) & \delta Z^{\frac{1}{2}}_{WW} = -G^{WW\,\prime}(M_W^2) \end{array}$$

The last line being also rewritten using Eq. (3.4)

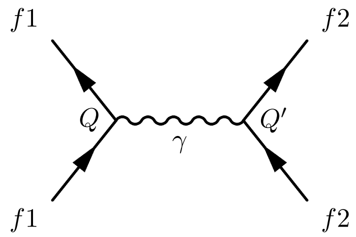
$$\delta Z^{\frac{1}{2}}_{\gamma\gamma} = -\Pi^{\gamma\gamma}(0) \qquad \delta Z^{\frac{1}{2}}_{ZZ} = -\Pi^{ZZ}(M_Z^2) \qquad \delta Z^{\frac{1}{2}}_{WW} = -\Pi^{WW}(M_W^2) \tag{3.13}$$

We may point out a weakness in our implementation of the OS scheme : indeed the $Z^{1/2}$ matrix does not cover all pairs of particles but only those in a same Poincaré multiplet. And yet, we can have some processes exchanging two particles of two distinct multiplets, but with the same gauge quantum numbers (as the $A_0 \to Z$ in the MSSM for instance) hence the δZ counterterms are not enough to ensure the overall non-mixing condition[4]. So we cannot *a priori* expect a general non-mixing.

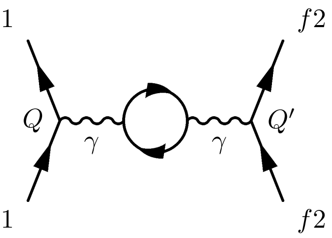
Last let us notice that the OS renormalization of the fermions will follow exactly the same pattern, except for the from of the propagator $D_{\alpha\beta}$ instead of $D_{\mu\nu}$, and will thus fix the value of the femions mass counterterms (or equivalently the Yukawa couplings) together with their field normalization.

3.6.3 Charge Renormalization

The charge renormalization condition is that the measured electromagnetic coupling in the Thomson limit (that is when the momentum of the exchanged photon vanish) is precisely e.
The tree-level amplitude of the electromagnetic coupling between two charged particles is extracted from the following diagram :



The amplitude is proportionnal to $Q_1 Q_2 \frac{e^2}{k^2}$, Q_i being the charges of the fermions, hence the electromagnetic coupling is precisely e. So the renormalization condition at first order turns out to be that the sum of the first-order diagrams vanish. In this sum we find the photon self-energy



which vanishes in the Thomson limit because of the OS conditions, and counterterms, defined by :

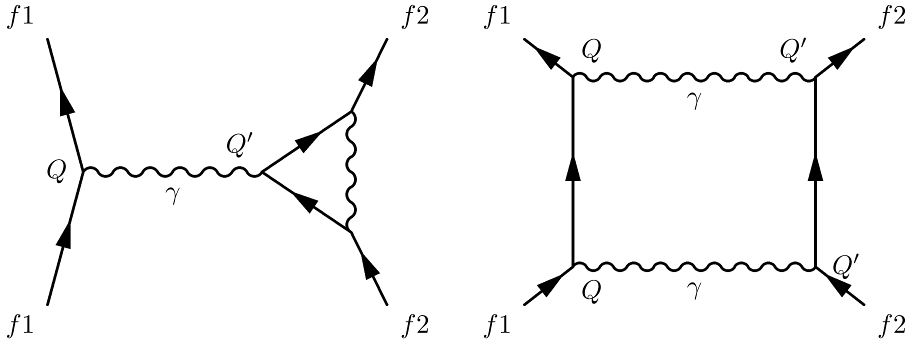
$$
\begin{array}{lcl}
e\overline{\psi}A\psi & \to & \delta e\overline{\psi}A\psi + \delta Z_{\psi\psi}^{1/2\,\dagger} e\overline{\psi}A\psi + \delta Z_{\psi\psi}^{\frac{1}{2}} e\overline{\psi}A\psi + \delta Z_{\gamma\gamma}^{1/2} e\overline{\psi}A\psi \\
 & \to & (\delta e + e\delta Z_{\psi\psi} + e\delta Z_{\gamma\gamma}^{1/2})\overline{\psi}A\psi \\
\frac{g}{c_W}\overline{\psi}Z\psi & \to & \frac{g}{c_W}\delta Z_{Z\gamma}^{1/2}\overline{\psi}A\psi + Z\ \textit{boson vertices}
\end{array}
$$

[4]As renormalization is a very flexible theory, one may propose to add terms such as $A \to Z_{AB}^{1/2} B + Z_{A\phi}^{1/2} d \cdot \phi$, where A, B are spin 1 and ϕ spin 0, for instance. Hence we extend the Z matrices to couple of particles in different Poincaré multiplets. However as long as it is not vital, we will avoid to introduce scores of renormalization terms and keep things simple.

Noticing that the $\delta Z_{\psi\psi}$ terms are cancelled by the renormalization of the external legs, we finally achieve the following condition :

$$\frac{\delta e}{e} = -(\delta Z^{1/2}_{\gamma\gamma} + \frac{s_W}{c_W}\delta Z^{1/2}_{Z\gamma})$$

Note that I omitted the following vertex and box corrections, which numerically turn out not to contribute much.



3.6.4 Electromagnetic coupling

Since we have defined the counterterm δe we can now express the electromagnetic coupling between two charged particle for any momentum k of the photon. Indeed, as the counterterms cancel each other, the amplitude is :

$$\mathcal{M} \sim e^2 D^{\gamma\gamma}_T(k^2)$$

and by inserting the value of the full photon propagator

$$\begin{aligned} D^{\gamma\gamma\,-1}_T &= D^{\gamma\gamma\,-1}_{0\,T} + \hat{G}^{\gamma\gamma} \\ &= k^2 + \delta Z^{\gamma\gamma}k^2 + G^{\gamma\gamma}(k^2) \\ &= k^2\left(1 + \delta Z^{\gamma\gamma} + \frac{G^{\gamma\gamma}(k^2)}{k^2}\right) \end{aligned}$$

Hence

$$D^{\gamma\gamma}_T = D^{\gamma\gamma}_{0\,T}\frac{1}{1 + \delta Z^{\gamma\gamma} + \frac{G^{\gamma\gamma}(k^2)}{k^2}}$$

We finally rewrites the amplitude as

$$\mathcal{M} \sim e^{*\,2}(k^2) D^{\gamma\gamma}_{0\,T}(k^2)$$

with

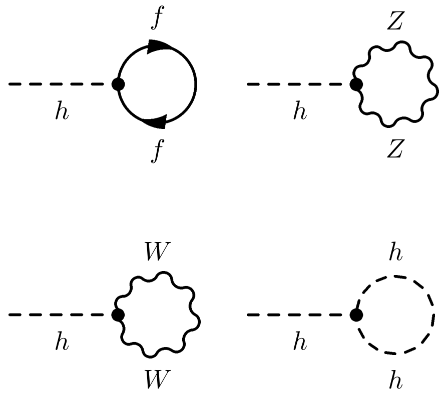
$$e^{*\,2}(k^2) = e^2\frac{1}{1 + \Pi^{\gamma\gamma}(k^2) - \Pi^{\gamma\gamma}(0)}$$

where we used Eq (3.13) to get $\delta Z^{\gamma\gamma}$ and the fact that $G^{\gamma\gamma}$ vanishes.

$e^*(k^2)$ is a running coupling constant, and also our first example of what we will call the effective couplings. But this I will explain in a few chapters.

3.6.5 Tadpole renormalization

The last renormalization condition of the Standard Model is the requirement that the Higgs tadpole vanishes at all order. The diagrams accounting for the tadpole at first order are the counterterms derived from $hv(\mu^2 - \lambda v^2)$ and some one-loop diagrams :



This operation fixes the value of our last parameter, μ^2.

As we have seen, renormalization in the Standard Model constrain all physical masses, namely $M_H, M_Z, M_W, M_{fermions}$, hence the most natural predictions will be decays, or simple interactions between particles, as we will see later. Note that in the case of the MSSM however, some masses are unconstrained, like the mass of neutralinos, thus leading to actual predictions of the quantum theory.

3.7 Renormalization of the MSSM

The renormalization scheme of the MSSM is mostly a very natural extension of the standard one : we impose field renormalization on all fields, masses conditions for W, Z, γ, A_0 and the fermions, general non-mixing and last the vanishing of the two Higgs tadpoles. However this leaves us with two issues, namely is it possible to avoid any mixing and how do we constrain our last parameter t_β.

Due to a lack of time I am unable to present a readable version of the full renormalization, so I may just point out a few comments about the issue of the t_β renormalization. An exact derivation of the renormalization can be found in [2].

First considering the way we can set the renormalization of t_β, we notice that the problem comes from the fact that it is not directly connected to a physical value. Indeed only $v^2 = v_u^2 + v_d^2$ is physical in the sense that it determines the masses of the Z, W bosons, hence there remains a rotational symmetry on (v_u,v_d). However this symmetry is broken by the Yukawa couplings to fermions, each Higgs yielding different masses. So we may extract the physical constrain on t_β by a coupling to fermion. This is done through the $A^0\tau\overline{\tau}$ coupling:

$$\mathcal{L} \supset i\frac{gm_\tau}{2M_W} t_\beta A^0 \tau\overline{\tau}$$

The actual value being extracted from the width $\Gamma_{A\tau\tau}$ determined by the decay of A^0.

Another scheme, called the DCPR scheme, is to impose the non mixing between the Z boson and the CP-odd Higgs

$$\hat{\Sigma}_{A^0 Z}(M_{A^0}^2) = 0$$

Last we can also choose to constrain another mass, for instance M_{H_0}, by imposing the OS condition

$$\hat{\Sigma}_{H_0 H_0}(M_{H_0}^2) = 0$$

Discussing the validity of these schemes requires an extensive discussion on the gauge invariance of the definitions of t_β as well as the size of the corrections they lead to. Indeed depending on the processes one is interested in some schemes can yield first loop corrections of the percent order, which is pointing to a very good result, whereas other may rise up to 30 percent, which indicates that the higher orders terms are likely to be quite important. An example of such a discussion can be found in [2].

Chapter 4

Automatisation

4.1 Principle

It is fairly easy to convince someone that automatisation of the Feynman procedure to compute observables is compulsory. For the standard model already it is a extremely comfortable tool, and much safer than hand calculation. In the MSSM, there is something like one thousand different diagrams for almost any calculation, and all hope vanishes.

To understand how we would implement this automatisation, we must analyse and generalise the process of calculating observables. Let us start with cross-sections. Decomposing into steps is a nice beginning :

- Define the in state and the out state : the particles and their momenta.
- Build all the Feynman diagrams whose external legs correspond to the states previously defined.
- For each diagram express the amplitude as tensorial integrals on the internal momenta.
- Express everything in term of standard scalar integrals (by tensorial reduction and other methods).
- Estimate the integral by numerical analysis.

In fact, this process presumes that we have all the rules to build the Feynman diagrams, which is rarely the case. Indeed this requires to handle the symmetry breaking and the resulting rotation of fields, splitting parameters into lowest order and counterterms and eventually computing these counterterms from the renormalization conditions. Hence the slight distinction between having a model – by which one usually means having a gauge group, a bunch of particles, a Lagrangian and the approximate sketch of a renormalization scheme – and having the tools to compute the Feynman amplitudes. Although I have shown explicitly (though not exhaustively) how to get these tools in the case of the SM and MSSM it would be nice to have an automatic procedure to carry out this step. Fortunately there exists such a program, and in fact there is one for each of the other steps, as detailed in Fig 4.1.

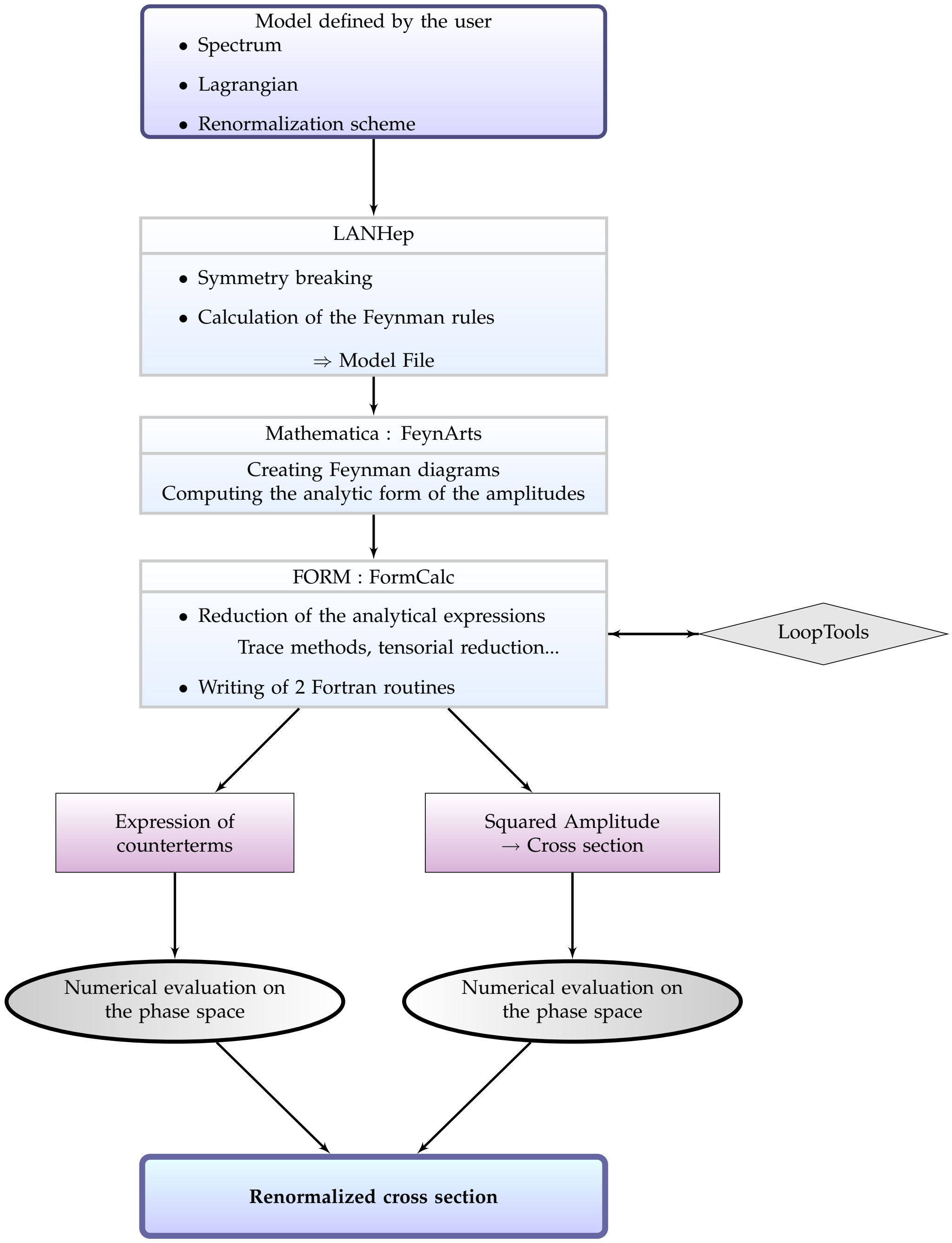


Figure 4.1: Automatisation Diagram

As a light correction to what I said before, we see that actual values of the counterterms are not part of the model as they are calculated only at the end of the computation just before evaluating the Feynman amplitude of the cross section. This is quite interesting regarding the fact that the program will not compute all counterterms (they can be extremely numerous), but only those which appear in the cross section.

4.2 Detailed Description

The very start of the procedure is the definition of the model, done through the model file.

4.2.1 Model file

In order to process any calculation, we need to specify which model we are using, which is the role of the model file. In this file one find :

- The list of the fields in which the Feynman expansion is carried, with their descriptions. The description includes the name of the particle, its representation, the mass term associated, etc.
- The Feynman rules for these fields.
- the renormalization conditions.

As noticed before, this file is generated by LanHEP. Note that the automatisation does not ask for a strong optimization as it is done only once for each model. Moreover it is often simpler to modify the model file by hand to create a variation of the model than to recreate it from LanHEP.

4.2.2 Computation of cross sections

As those computations actually demand a large amount of calculating power, care has been taken to carry each step in a well suited programming language. Those choices are the following:

- The analytic computation of Feyman amplitude is run under Mathematica by the FeynArts[6] package. Mathematica is a logical choice, as it handles formal expressions in a very powerful manner, and hence can express the amplitude in an analytic way, that is to say without carrying out any computation. An amplitude would hence look like

$$\int \frac{d^n k_1}{(2\pi)^2} \int \frac{d^n k_2}{(2\pi)^2} \int \frac{d^n k_3}{(2\pi)^2} ... \gamma_\mu \frac{i g^{\mu\lambda}}{k1^2 + i\epsilon} ...$$

- The next step toward the numerical estimation is to express the different traces in term of components of each particle, to collect all propagators denominators by the use of Feynman parameters, and to reduce all tensorial integrals down to scalar integrals. FormCalc[6], written in FORM, deals with this operation. Switching from Mathematica to FORM permits a much quicker computation. After this step we are done with all the formal calculations.
- This last step has produced a Fortran routine, which is specific to the process considered. We are left with the numerical estimation, that is replacing each variables by its actual value and computing the cross section on different points of the phase space; and this

does not require a formal language but a fast one, which justifies Fortran as a possible choice. Producing the actual Fortran files instead of just running it hidden from the user allow the latter to manipulate some parameters directly. For instance all fixed parameters (masses, couplings...) are available, hence one can compute the running cross section for different values. The same applies for gauge fixing parameters and renormalization parameters (including bremstrahlung cut-off). The main result is that the two first steps are just used once for each process, and inside the same process nearly all variations are done by modifying the Fortran routine.

4.2.3 Pros and Cons of an Automatic procedure

The main drawback in leaving the calculation task to the computer is that one may suspect whether there could be any mistake in the programme itself, as forgotten Feynman diagrams, wrong tensors reduction and so on, which are things not obvious to notice when confronted with a raw numerical value. But there are ways to test such a program. A good idea would be to compare results on a same process with exactly the same incoming and outgoing particles coming from different programs. However this requires that you already have a program that is reliable, so it is not an absolute test. A strong hint lies also in the variation of the non-physical parameter. Indeed, we can run the calculations over different value on the gauge-fixing parameters, along with the renormalisation parameter. Having a cross-section which changes its 12th decimal value when one vary the renormalization mass of 50 power of ten is not a proof for the program, but it is a strong relief for its consistency.

4.3 Test of the program

4.3.1 Comparison with GRACE-loop

The program GRACE-loop, designed ofr the Standard Model, had been intensively tested with experiment, so it is a natural place to start our verifications. I will now display results for standard processes, displayed in the following conventions :

- E_{beam} is the energy of incoming particles
- The tree-level cross section (tree) is expressed in pb.
- The one-loop corrections (δ) is expressed as percentage of the tree-level value.
- Calculations are done for unpolarized particles, meaning the average of the polarized cross sections.
- When IR divergences occur we introduce a Bremstrahlung with a energy cut-off $E_c = 0.1 E_{Beam}$

The set of measured values for renormalized parameters is given in Appendix 7.1.

Processes	GRACE-loop	Sloops
$e^+e^- \to W^+W^-$ $2E_{Beam} = 190$ GeV tree δ	 17.8623 -9.4923	 17.862 -9.923
$\gamma\gamma \to W^+W^-$ $2E_{Beam} = 500$ GeV tree δ	 77.552 -3.376	 77.5515 -3.380
$e\gamma \to W^-\nu_e$ $2E_{Beam} = 500$ GeV tree δ	 36.5873 -12.2803	 36.587 -12.281
$e\gamma \to eZ$ $2E_{Beam} = 2$ TeV tree δ	 0.046201 -39.529	 0.46201 -39.529

4.3.2 Gauge and renormalization Independence

Here we will vary the different gauge parameters ($\tilde{\alpha}$...) and also the renormalization parameter (ϵ, μ). Remember that all results had to be finite in the limit $\epsilon \to 0$, hence no $\frac{1}{\epsilon}$ should appears. Formally these terms are defined by the C_{UV} parameter, hence we carry the test on a running C_{UV}.

The process is $e\,\gamma \to e\,Z$, at an energy $2E_{Beam} = 500$ GeV.

Parameters	δ
$\varnothing$	-25.6900775195773
$\lambda = 10^{-30}$	-25.6900775195774
$C_{UV} = 10^7$	-25.6900775195773
$\tilde{\alpha} = 10^3$	-25.6900775195773
$\tilde{\beta} = 10^3$	-25.6900775195773
$\tilde{\delta} = 10^3$	-25.6900775195773
$\tilde{\kappa} = 10^3$	-25.6900775195773
$\tilde{\epsilon} = 10^3$	-25.6900775195773

4.3.3 IR cutoff

We said that introducing soft photons also leads to the appearance of a k_c cut-off in the result, that we could remove by adding the hard photon radiation. It is hence satisfying to check that the sum of these two contributions does not vary much with k_c, a good point being that the smaller k_c is, the more stable the quantum corrections are.

To make clear that such a behaviour is not a mere coincidence we also show the soft and hard contributions separately : it is then obvious that while they separately vary quite, their sum is more or less constant.

k_c	σ_{soft}	σ_{hard}	σ_{total}
$0.01 \times 2E_{Beam}$	$0.174373\,10^{-5}$	$0.284332\,10^{-5}$	$0.458706\,10^{-5}$
$0.001 \times 2E_{Beam}$	$-0.054858\,10^{-5}$	$0.511183\,10^{-5}$	$0.456325\,10^{-5}$
$0.0001 \times 2E_{Beam}$	$-0.284090\,10^{-5}$	$0.740235\,10^{-5}$	$0.456146\,10^{-5}$

4.3.4 Conclusion

These tables are but a pinch of the whole set of tests which the code *Sloops* has been through[1]. This successful step allows us to consider that first, the program does what was asked – meaning the Feynman procedure is carried through correctly (no missing diagrams for instance) – on one hand and that on the other the renormalization scheme is coherent with the latest measurements. Hence we may now use it for any computations.

[1]For a detailed description of one-loop automatized calculations, I refer the reader to [3]

Chapter 5

Probing New Physics

Here we are with a model extremely reliable, the SM, a most promising extension, the MSSM, and a fully automatized code to compute observables, *Sloops*. And fortunately, it turns out to be already enough to discuss some actual research issues. Indeed the quantum structure of the interactions between particles unveils an invaluable advantage in the fact that in any process, all particles can possibly contribute. Hence getting to know better what lies beyond the standard model, or even the not so precise parameters of the standard model does not require to go to specific processes or energies. Instead one just need to have a commonplace measurement, but accurate, and a good knowledge (from other experiments) of the standard parameters. Of course increasing the energy of the center of mass of particles (in the case of a collider for instance) would surely increase the quantum corrections and may create new particles, but this is not a reason to dismiss old experiments, as we shall see now.

5.1 Electromagnetic coupling

Starting with the case of the electromagnetic couplings, that can be easily computed analytically and can also be measured with an amazing precision, is kind of a must.

5.1.1 Predicting α*

As we quickly saw in the renormalization section, the photon exchange renormalization introduces a lot of contributions from matter particles that were not foreseen from the simple tree-level of the theory. Indeed for any measurement of the Coulomb repulsion between two charged particles, there is a contribution of each fermions to the amplitude, via the self-energy of the photon. So, measurement of electromagnetic interaction could possibly provides us some information about masses of other particles, except for measurement where the photon has a vanishing momentum, which was a renormalization condition.

The usual way to achieve this goal is to refer to $\alpha^*(k^2)$, which we will find in many different calculations. It is define by

$$\alpha^*(k^2) = \frac{e^{*\,2}(k^2)}{4\pi} \qquad\qquad \alpha = \alpha^*(0)$$

and we are mostly interested in $\Delta\alpha(k^2)$ defined by :

$$\alpha^*(k^2) = \frac{\alpha}{1 - \Delta\alpha(k^2)}$$

We saw that each fermions had a contribution to e^*, hence to $\Delta\alpha$ through the self energy of the photon. But if we want to be as realistic as possible, we must separate all the fermions in the following way : first the leptons whose contribution correspond precisely to the expression we gave in Eq(3.10), the quarks, which contribution is likely to be highly modified by QCD corrections, and the new particles, such as the Higgs, the sparticles and more generally all effects due to physics beyond the standard model. So it seems that knowing the QCD corrections with a satisfying precision allow us to probe new physics only by measurement of $\Delta\alpha$.

$$\Delta\alpha(k^2) = \Delta\alpha_{e\mu\tau}(k^2) + \Delta\alpha_{quarks}(k^2) + \Delta\alpha_{NewPhysics}(k^2)$$

We already know that for fermions – QCD corrections left aside – the contribution writes

$$\begin{aligned}\Delta\alpha(k^2) &= \Pi^{\gamma\gamma}(0) - \Pi^{\gamma\gamma}(k^2) \\ &= \frac{e^2}{2\pi^2} N Q^2 B_2(k^2, m)\end{aligned}$$

But the $B_2(m, m, k)$ term can be simplified in the limits of small and high mass (compared to the momentum) in such a way

$$\begin{array}{lcll} 6B_2(k^2, m) & \sim & -\frac{5}{3} + \ln\frac{-k^2}{m^2} & \text{for } k^2 >> m^2 \\ 6B_2(k^2, m) & \sim & -\frac{k^2}{5m^2} & \text{for } k^2 << m^2 \end{array}$$

We may notice here an issue with our first goal – which was to extract from the quantum corrections information about new particles – in the sense that contribution of massive particles is suppressed by a factor $\frac{1}{m^2}$. For instance, if we consider a measurement done at the Z boson scale ($k^2 \sim M_Z^2$), the contribution of a top quark with mass $M_{top} \sim 175$ GeV will yield a contribution $\Delta\alpha_{top}(M_Z^2) = -0.78 \times 10^{-4}$ whereas the contribution of lighter fermions e, μ, τ will be $\Delta\alpha_{e,\mu,\tau}(M_Z^2) = 314.98 \times 10^{-4}$.

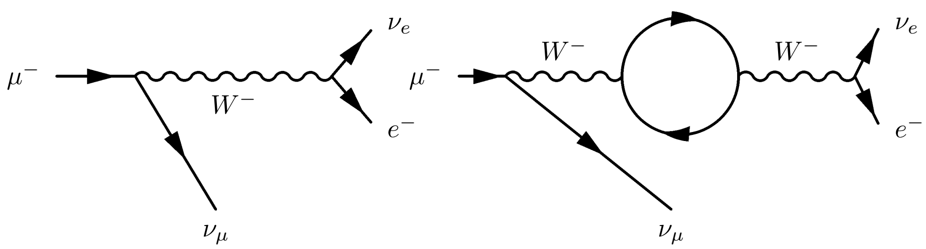
Using cutting methods – which I will not develop here – we are also able to compute the QCD contribution of partons which are $\Delta\alpha_{hadrons}(M_Z^2) = (276.3 \pm 1.6) \times 10^{-4}$. Hence we see clearly the bad consequences of the decoupling of massive particles : the corrections brought by the top quark is less than the imprecision on the hadronic corrections. It is hence hopeless to try to extract from measurements (at least at the electroweak scale) useful information about new physics, since even the corrections brought by a light Higgs would pass completely unnoticed. Hence we may want to think about other processes.

5.2 Predicting the muon decay

As we have seen, using $\Delta\alpha$ is not a so powerful tool to probe new physics because heavy particles decouple. We are then left to look for another process where the loop structure is such that massive particles corrections are not negligible any more. Fortunately there exists a well known candidate : the muon decay.

5.2.1 Amplitude of the muon decay

The main feature of the muon decay is that it includes a W internal propagator which is likely to be significantly modified by quantum corrections associated with new particles.



Tree-level One-loop corrections

With the self-energy contribution, we get :

$$\mathcal{M} \sim g^2 \frac{1}{k^2 - M_W^2 - \hat{G}_{WW}}$$

We then add the δg vertex correction with the $\delta Z_{WW} g$ vertex which exactly compensate the $\delta Z_{WW}(k^2 - M_W^2)$ of $\hat{G}_{WW}$. Incidentally, we note that the fact the field renormalization terms are useless is a general rule for all internal propagators.

$$\begin{aligned}\mathcal{M} &= \frac{g^2}{k^2 - M_W^2 - (G_{WW}(k^2) + \delta M_W^2)} + \frac{\delta g^2}{k^2 - M_W^2} \\ &= \frac{g^2}{k^2 - M_W^2}\left(1 + \frac{\delta g^2}{g^2} + \frac{G_{WW+\delta M_W^2}(k^2)}{k^2 - M_W^2}\right)\end{aligned}$$

Evaluating this for small momentum ($k^2 \to 0$) we comes to

$$\mathcal{M}(k^2 \to 0) = -\frac{g^2}{M_W^2}(1 + \Delta\hat{r})$$

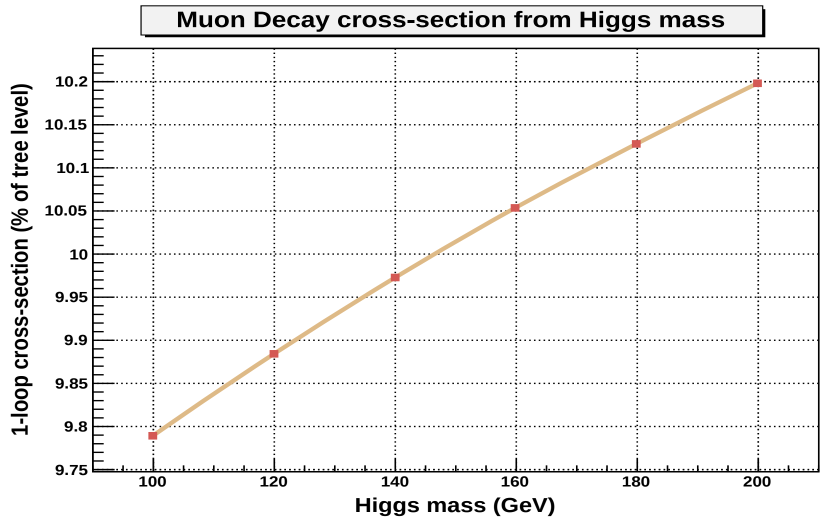
where

$$\Delta\hat{r} = \frac{\delta g^2}{g^2} - \frac{G_{WW}(0) - \delta M_W^2}{M_W^2}$$

Hence the quantum contribution to the decay is fully – up to the vertex and box corrections – encoded in $\Delta\hat{r}$.

5.2.2 Numerical approach

Without entering into the analytical form of $\Delta\hat{r}$, we can directly use the numerical results from the automatisation to get the variation of the predicted cross-section with parameters. Here I give the first order correction to the cross section for different masses of the Higgs boson (remember that the tree-level is independent from any new physics).



To give some sense to this graph which is not intrinsically relevant (of course we expected such a curve), we need to compute the uncertainties brought by uncertainties on the free parameters, hence the masses and the electromagnetic coupling. I enclose here a table of the cross-section obtained when those parameter vary in their uncertainty range.

Parameter	Range	δ_{min}	δ_{max}	Uncertainty, %
M_Z	91.1875 ± 0.0021	10.3526224	10.3554837	2.86×10^{-3}
M_W	80.426 ± 0.034	10.178442	10.2280581	49.6×10^{-3}
α^{-1}	137.03599235(73)	10.3542646	10.3543125	0.048×10^{-3}

Hence we remark that the corrections brought by the Higgs are well above the imprecisions on the other parameters, hence this observable suits our goal, which was extracting information on new physics from standard measurements. Note that this study should be completed by the estimation of the vertex and box corrections.

5.2.3 Analytic derivation

Now that we are convinced that this observable is indeed an efficient tool to probe new physics, it is enlightening to get the whole analytic expression. Moreover several well-known electroweak variables will show up, as we see here

$$\Delta\hat{r} = \Delta\alpha(M_Z^2) + \frac{c_W^2}{s_W^2}\epsilon_1 + \frac{c_W^2 - s_W^2}{s_W^2}\epsilon_2 + 2\epsilon_3$$

(The derivation being given in Appendix A.8).

On top of the $\Delta\alpha$, that we already encountered, we get three new variables which are all the more interesting since contributions from massive particle do not decouple. As an example the variable ϵ_1 has the following dependence on the top mass:

$$\epsilon_1 = \frac{3g^2}{64\pi^2 M_W^2} m_t^2 + \ ...$$

The question is then : why do we observe a decoupling for the electromagnetic interaction and not for the muon decay? By looking into the ϵ_1 variable we see that the contribution to the muon decay basically comes from:

$$2\,G_{LL}(k^2, m_2, m_1) - G_{LL}(k^2, m_1, m_1) - G_{LL}(k^2, m_2, m_2)$$

where the 1, 2 indices refer to the two components of the $SU(2)$ doublet. Hence it is quite obvious that the breaking of $SU(2)$ is responsible for the top contribution. However, were it not this symmetry breaking, there would still be a non vanishing contribution due to the chiral anomaly of the theory. Though it does not manifest as strongly (the symmetry breaking contribution being of order m_t^2), it is still a specific feature of the $\Delta\hat{r}$ correction compared to the electromagnetic coupling where no anomalies occur.

Thus mass uncertainties on the top quark will be detected, and in some extent the Higgs mass as other new physics corrections will be probed. As an example, the dependence on M_H that we observed on the full cross section is directly explained by these three variables.

Chapter 6

Effective theories

6.1 Effective couplings

Although computations of one-loop amplitude for a given process are always feasible, they do not give us a clear view on what is happening inside the process. However in some cases it is possible to reduce some one-loop corrections to an expression similar to the tree-level by introducing some new parameters.

We already had an insight of this technique when computing the charge renormalization. Indeed for the case of the electromagnetic interaction between two charged particles, we had the tree level amplitude with the simple form

$$\mathcal{M}_{tree} \sim Q_1 Q_2 \frac{e^2}{k^2}$$

but the first order expression was much more complicated for $k^2 \neq 0$. However, we defined a running coupling $e * (k^2)$ so that we could write the first order amplitude

$$\mathcal{M}_1 \sim Q_1 Q_2 \frac{e^{*\,2}(k^2)}{k^2}$$

So in this case e^* turns out to be an effective parameter, allowing us to express a first-order result in a tree-level shape. I recall that this is an approximation, because we do not take into account the vertex and box contribution.

One may argue that we just hide the difficulty behind e^* but our result is much stronger. Indeed if we have any Feynman diagram at tree level including only electromagnetic vertices, we may try to turn it to the loop level just by replacing e by e^*. But to bring this technique to a useful level, we would need to find such effective variables for others parameters of the Lagrangian, for instance $M_W^*, s_W^*, Z^{1/2\,*}$... The problem arising is that the set of free parameters cannot be modified to fully represent the one-loop effects at tree-level : there is much more different contributions than parameters. Hence the aim is to find different effective variables, so that the effective amplitude is not true in general but is a good approximation in some kind of processes, or at some energies. The whole difficulty being to enter slightly into the analytic expression of an observable to get which diagrams contribute and which are negligible, without going too far in the large complexity of the first loop amplitudes; then building from these high-contribution diagrams an effective theory, and last assessing the quality of the approximation by carrying the full automatized computation on some chosen processes.

Conclusion

We finally achieved our initial goal, which was connecting a theoretical model to actual observables that can be measured in astrophysical or collider experiments. Though the procedure may seem extremely heavy overall, each step has a physical meaning and can be dealt with separately. Definig a model and a scheme for a symmetry breaking is the first theoretical issue. Defining a renormalization scheme is probably the most intricate point as the behaviour of the first order corrections (and to a certain extent the higher orders) will critically depend on it. Last, trying to simplify the analytic expressions encountered in calculations of a process by turning our original field theory to an effective field theory is an open question which answers could turn to be extremely usefull for analysing complex processes.

Thus this overall insight of the complete procedure left us with the satisfying conclusion that the whole computation could be easily automatised by using the right tools – here *Sloops* – and that those corrections could be measured with a great accuracy and, depending on the process, level the effect of new particles up to significant contributions. It hence seems that with those tools one can improve the predictions of physics in and beyond the standard model up to the experimental level, resulting in refining our knowledge of the known particles on the one hand, and discarding or restraining new models on the other.

One may argue that current precision calculations either deal with non standard processes or with corrections up to several orders, which seems a long way from what this essay was devoted to. Nevertheless, however specific were the steps I treated, the renormalization techniques clearly pave the way for processes including particles from the MSSM and similarly I have shown how the code could also handle those new processes. Hence the issue of the dark matter, specifically realized by sparticles of the MSSM, can be studied throught the computation of relic densities, which are actually not precisely known to the one-loop order. Those experimental issues of outstanding importance – as shown for instance by the PAMELA results – are thus closely connected to this work.

Appendix **A**

Appendix

A.1 Spinors, Fermions, and all that

Let me start by recalling that all Lorentz representations are subspaces of an $n = 4$ Grassmann algebra. Indeed taking the algebra being generated by the basis components of the $\left(\frac{1}{2}, 0\right)$ representation of $SL(2, \mathbb{C})$ together with those of $\left(0, \frac{1}{2}\right)$, plus the unit element, it then makes perfectly sense to take all other representations of $SL(2, \mathbb{C})$ in this algebra since these are the two fundamental representations of $SL(2, \mathbb{C})$.

Bosonic representations are subspaces of the even elements, and fermionic of the odd ones.

Thus the usual representations are $\psi_{L\alpha}$, α running over the first two Grassmann generators, for left-handed $\frac{1}{2}$ fermions, $\psi_{R\dot{\alpha}}$, $\dot{\alpha}$ running over the two last Grassmann generators, for right-handed ones, $v_{\alpha\dot{\alpha}}$ for the vector representation (spin 1 bosons), and so on.
It is quite instructive to make contact between the bispinor form $v_{\alpha\dot{\alpha}}$ and the Lorentz vector representation v_μ, which is done through the use of extended Pauli matrices σ_μ and $\overline{\sigma}_\mu$

$$v_\mu = \frac{1}{2} Tr(\sigma_{\mu\ \alpha\dot{\alpha}} v_{\alpha\dot{\alpha}})$$

where $\sigma_\mu = (1, \sigma_i)$, $\overline{\sigma}$ being in the dual representation.

In fact all $\sigma_\mu, \overline{\sigma}_\mu, \gamma_\mu$ are representation of the same object in different representations, namely

$$\sigma_\mu \leftrightarrow \frac{1}{2}_L \qquad \overline{\sigma}_\mu \leftrightarrow \frac{1}{2}_R \qquad \gamma_\mu \leftrightarrow \text{Dirac} \left(\frac{1}{2}_L \oplus \frac{1}{2}_R\right)$$

Hence the term $i\overline{\psi}\not{D}\psi$ must be seen as

- $i\psi_L^\dagger \sigma_\mu D^\mu \psi_L$ if in the $\frac{1}{2}_L$ representation
- $i\psi_R^\dagger \overline{\sigma}_\mu D^\mu \psi_R$ if in the $\frac{1}{2}_R$ representation
- $i\overline{\psi}\gamma_\mu D^\mu \psi$ if in the Dirac representation

Incidentally this also defines the norms : $|\psi|^2 = \psi^\dagger\psi$ for left and right representations and $|\psi|^2 = \overline{\psi}\psi$ for the Dirac representation.

A.1.1 Pros and cons of the Dirac notation

Propagators
As left and right fermions can be in different gauge representations, it is more natural to write the Lagrangian with fermions in left and right representations.

$$\mathcal{L}_{fermions} = -i\psi_L \not{D}\psi_L - i\psi_R \not{D}\psi_R + Y_e H\psi_L\psi_R + h.c.$$

However they do not really decouple, because of the Yukawa coupling[1]. Hence if we stick to the fundamental representations the propagators will be non diagonal. Indeed starting with the free Lagrangian :

$$\mathcal{L}_0 = \begin{pmatrix}\psi_L \\ \psi_R^\dagger\end{pmatrix}^\dagger \begin{pmatrix}\not{d} & m \\ m & \not{d}\end{pmatrix} \begin{pmatrix}\psi_L \\ \psi_R^\dagger\end{pmatrix}$$

we get the following propagator

$$\left\langle \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}^\dagger \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \right\rangle = \frac{i}{p^2 - m^2} \begin{pmatrix}\not{k} & -m \\ -m & \not{k}\end{pmatrix}$$

However we can ease most of the calculations by endowing ψ_L and ψ_R in a same Dirac representation

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

and rewrite the propagator as

$$\langle \psi^\dagger \psi \rangle = i\frac{\gamma_0(\not{k} + m)}{k^2 - m^2}$$

where the $\not{k}$ now means $k^\mu \gamma_\mu$. This result being rewritten at once as

$$\langle \overline{\psi}\psi \rangle = i\frac{\not{k} + m}{k^2 - m^2}$$

Couplings
In order to use this propagator in the Feynman expansion, one need to rewrite interaction terms with the Dirac notation. This is achieved by introducing the projectors $P_L = \frac{1-\gamma_5}{2}$ and $P_L = \frac{1+\gamma_5}{2}$. For instance the $\frac{g}{c_W} T_Z$ coupling writes

$$\frac{g}{c_W}\left(\frac{\sigma_3}{2} - s_W^2 Q\right) P_L - e\frac{s_W}{c_W} Q P_R$$

To conclude, if one want to use the Dirac representation instead of fundamental representations for $\frac{1}{2}$ fermions, one has less vertices and propagators, but the coupling will be more complicated : $gT \rightarrow g_L T_L\, P_L + g_R T_R\, P_R$, T being the charge matrix. It can be convenient to switch between notations : hence the photon self-energy is well-suited to the Dirac notation because in this case left and right coupling are equal so the overall Dirac coupling keeps the simple form gT. However the W self energy is clearly adapted to fundamental representations as there is no vertex with right-handed fermions.

[1]More generally the Lorentz structure imposes a left-right coupling for any fermion mass term

A.2 Why do we use norms instead of indices

First of all, let us notice that by replacing the norms with the components we get coherent relations, see for instance:

$$\begin{aligned} -\frac{1}{2}|DZ|^2 &= -\frac{1}{2}|D_\mu Z_\nu dx^\mu_\wedge dx^{nu}|^2 \\ &= -\frac{1}{2}(D_\mu Z_\nu - D_\mu Z_\nu)D^\mu Z^\nu \\ &= -\frac{1}{4}(D_\mu Z_\nu - D_\mu Z_\nu)(D^\mu Z^\nu - D^\mu Z^\nu) \end{aligned} \tag{A.1}$$

$$\begin{aligned} |DH|^2 &= |D_\mu H dx^\mu|^2 \\ &= |D_\mu H|^2 \\ &= (D_\mu H)^\dagger (D^\mu H) \end{aligned} \tag{A.2}$$

Hence we recover the usual expression for components of the Lorentz representation. Thus using the norm we get a much more compact notation, as it avoids enumerating all the components of each field in its Lorentz and gauge representation. Another advantage is that any calculation done with the norms is general for any field, whatever its representation should be. Eventually, when expressing a norm in a calculation, we can arbitrarily show some components and hide the others.

As an example, when looking at the symmetry breaking in the $-\frac{1}{2}|F_{total}|^2$ term we may write it with only the $su(2) \otimes u(1)$ components, ie

$$-\frac{1}{2}|F_{total}|^2 = -\frac{1}{2}|d\left(\overrightarrow{W}, B\right)|^2 + \text{Interactions terms}$$

And as we know that

$$\left(\overrightarrow{W}, B\right) = \mathcal{R}\left(W^\pm, Z, A\right)$$

where $\mathcal{R}$ is a rotation in the complex plane composed of the Hadamard matrix for W_1, W_2 and the θ_W rotation for W_3, B. The derivation keeping this relation, and as rotations leave norms unchanged, we get

$$-\frac{1}{2}|F_{total}|^2 = -\frac{1}{2}|d\left(W^\pm, Z, A\right)|^2 + \text{Interactions terms}$$

Remembering that $\left|\begin{array}{c} W^+ \\ W^- \end{array}\right|^2 = 2W^+ \cdot W^-$, we obtain the usual result.

$$-\frac{1}{2}|F_{total}|^2 = -dW^+ \cdot dW^- - \frac{1}{2}|dZ|^2 - \frac{1}{2}|dA|^2 + \text{Interactions terms}$$

A.3 Fields rotation in the EW breaking

The relations between the old and the new parameters are the following :

$$\begin{array}{ccc} \binom{Z}{A} = R(\theta_W)\binom{W^3}{B} & W^\pm = \frac{1}{\sqrt{2}}\left(W^1 \mp iW^2\right) & \\ \sigma_\pm = \sigma_1 \pm i\sigma_2 & Q = \frac{\sigma_3}{2} + Y & T_Z = \frac{\sigma_3}{2} - s_W^2 Q \\ g = g_1 & e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} & \end{array}$$

In order to simplify the computation of the action in the adjoint representation we compute

$$[\tfrac{\sigma_+}{2}, \tfrac{\sigma_-}{2}] = \sigma_3 \qquad [Q, \tfrac{\sigma_\pm}{2}] = \pm\tfrac{\sigma_\pm}{2}$$
$$\left[T_Z, \tfrac{\sigma_\pm}{2}\right] = \pm c_W^2 \tfrac{\sigma_\pm}{2} \qquad [T_Z, Q] = 0$$

This new basis of the Lie algebra, together with the fields rotated yields the satisfying result:

$$g_1 W^i \frac{\sigma_i}{2} + g_2 BY = \frac{g}{\sqrt{2}} W^\pm \frac{\sigma_\pm}{2} + \frac{g}{c_W} Z T_Z + eAQ$$

A.4 SUSY

It is fairly common to introduce different susy notations and conventions for every different paper, I will hence present, as quickly as possible, the one I used. They can also be found in [1].

Grassmann algebras and envelopes Λ_n stands for the Grassmann algebra with n generators on top of the unit element 1. Its graded structure is the following : the even elements are the one with a event number of generators and the odd are those with an odd number of generators.
The Grassmann envelope of a graded linear space K^{pq} is the set

$$K^{pq}(\Lambda) = \{c_i e_i + \gamma_j e'_j / c_i \in \Lambda^0, \gamma_j \in \Lambda^1, (e_i)\text{basis of } K^{0q}, (e'_j)\text{basis of } K^{p0}\}$$

Universal superalgebra and Poincare superalgebra We denote Σ the superalgebra generated by the even elements $P_{\alpha,\dot\alpha}, P$ and the odd elements $\partial_\alpha, \partial_{\dot\alpha}, \eta_\alpha, \eta_{\dot\alpha}$. where

$$P_{\alpha,\dot\alpha} = \sigma^\mu_{\alpha,\dot\alpha} P_\mu \quad P = \theta P \overline{\theta} \quad \eta_\alpha = P_{\alpha,\dot\alpha}\theta^{\dot\alpha} \quad \eta_{\dot\alpha} = \theta^\alpha P_{\alpha,\dot\alpha}$$

obeying the following anticommutations relations.

$$\{\theta^\alpha, \theta^\beta\} = \{\theta^\alpha, \overline{\theta}^{\dot\beta}\} = \{\overline{\theta}^{\dot\alpha}, \overline{\theta}^{\dot\beta}\} = 0 \tag{A.3}$$

$$\{\partial_\alpha, \partial_\beta\} = \{\partial_\alpha, \overline{\partial}_{\dot\beta}\} = \{\overline{\partial}_{\dot\alpha}, \overline{\partial}_{\dot\beta}\} = 0 \tag{A.4}$$

$$\{\partial_\alpha, \theta^\beta\} = \delta^\beta_\alpha \qquad \{\partial_\alpha, \overline{\theta}^{\dot\beta}\} = 0 \tag{A.5}$$

$$\{\overline{\partial}_{\dot\alpha}, \theta^\beta\} = 0 \qquad \{\overline{\partial}_{\dot\alpha}, \overline{\theta}^{\dot\beta}\} = \delta^{\dot\beta}_{\dot\alpha} \tag{A.6}$$

We can change our basis by defining $Q = \partial + \eta, \overline{Q} = \overline{\partial} + \overline{\eta}$, we notice then that the Q and P_μ operators obey the usual susy commutation relations:

$$[P, Q] = [P, \overline{Q}] = 0 \qquad \{Q, \overline{Q}\} = 2P^\mu \sigma_\mu \tag{A.7}$$

Then P, Q, $\overline{Q}$ generate the $\tilde{t}$ superalgebra that can be extended to the super Poincare algebra.

Superfields We will denote Λ_4 the Grassmann algebra generated by $\theta, \overline{\theta}$ and Λ the Grassmann algebra where the fields live.
$\Pi = \Lambda_4(\Lambda)$ is the set of the superfields. As the commutator of any element of Σ and a superfield is again a superfield, superfields do form a representation of Σ.

Note that we can construct different representation of the super Poincare group by specifying in which Poincare representations are the "fields-components" of the superfield. Moreover, as Q commutes with the gauge group, the fields must all be in the same representation of the gauge group. Hence, when we speak about a superfield, we specify the different Poincare representations of the fields and the general gauge representation. These representations fall into two cases that I will briefly describe (a more technical derivation to be found in [5]).

Massless superfields
Massless superfields are defined by the lowest helicity of the field components, that we name λ. The helicities of the other fields are raised by one half for each susy generator. In $N = 1$ supersymmetry, there will be two fields one with helicity λ and the other with helicity $\lambda + \frac{1}{2}$. However to have coherent representations for our particles we require a helicity symmetry, hence we add the states $-\lambda - \frac{1}{2}$ and $-\lambda$. For instance the chiral superfield looks like :

$$\left(\lambda = 0 \oplus \lambda = -\frac{1}{2}\right)$$

Massive superfields
The massive superfields are defined by a spin s and they have $2s+1$ field components, each with a spin going from $-s$ to s. As an example the scalar multilet is a spin 0 superfield. However, as most of the theories deal with massless multiplets, they will not be of much use in our scope.

A.5 Field Rotation in the MSSM

A.6 Renormalization

A.6.1 Transposing from e_Λ to e

For a given renormalized theory with a renormalization parameter Λ, the observables O_Λ write down as expansion in the coupling parameter e_Λ :

$$O_\Lambda = \sum_{i\geq 0} a_i e_\Lambda^i$$

Note that the a_i are Λ-dependent, this will be true for coefficients of all expressions, but for clarity I do not make it explicit.

And we would like to express it as an expansion on a Λ-independent parameter e, as :

$$O_\Lambda = \sum_{i\geq 0} b_i e^i$$

by using an expansion of e_Λ :

$$e_\Lambda = \sum_{i\geq 0} c_i e^i$$

This is done by identifying the series with functions (f with a_i,g with b_i,h with c_i) so that

$O_\Lambda = f(h(e)) = g(e)$, and using the equalities :

$$g(0) = f(h(0)) \qquad \Longrightarrow b_0 = \sum_{i\geq 0} a_i (c_0)^i$$

$$g'(0) = h'(0) f'(h(0)) \qquad \Longrightarrow b_1 = c_1 \sum_{i\geq 0} (i+1)! a_{i+1} (c_0)^i$$

Clearly, the transposition makes more sense if $h(0) = 0$ or equivalently $c_0 = 0$ that is to say the e_Λ is at least in first order in e. In this case the i-th order coefficient in the expansion in e depends only on the coefficient of orders equal or less than i in the expansions of $e_\Lambda(e)$ and $O_\Lambda(e\Lambda)$.

The main reason of this choice is that now, when we truncate an observable to the ith order, we can do all computations with the expansions $O_\Lambda = \sum_{i\geq 0} a_i e_\Lambda^i$ and $e_\Lambda = \sum_{i>0} c_i e^i$ truncated at order i.

A.6.2 Obtaining the OS constraints

We show here a formal derivation of the OS constraints, as summarized by :

- $D_T^{AB}(M_A^2)$ if $A \neq B$ $\longrightarrow$ $\left(D_{0\,T}^{-1} + \hat{G}\right)^{AB} (M_A^2) = 0$ if $A \neq B$
- D_T^{AA} has a pole in $M^{2\,A}$ $\longrightarrow$ $\hat{G}^{AA}(M_A^2) = 0$
- the residue of D_T^{AA} in $M_{A\,2}$ is 1 $\longrightarrow$ $\hat{G}^{AA\,\prime}(M_A^2) = 0$

Concerning the first line we just write the full propagator in terms of the free one, and the self-energies (including counterterms), which we recall

$$D_T = (D_{L\,0}^{-1} + \hat{G})^{-1}$$

Then noticing that a non mixing condition holds equivalently on a propagator or its inverse (which is easily seen in $I_{AB} = (M^{-1}M)_{AB} = (MM^{-1})_{AB} = 0$ for $A \neq B$), we rewrite it as :

$$\boxed{\left(D_{0\,T}^{-1} + \hat{G}\right)^{AB} (M_A^2) = 0}$$

For the second line, we must first prove that D^{AA} having a pole in M_A^2 is equivalent to $(D^{-1})^{AA}$ having a zero in M_A^2, which can be rewritten as

$$(D^{AA})^{-1}(M_A^2) = 0 \Leftrightarrow (D^{-1})^{AA}(M_A^2) = 0$$

We use the relation for the inverse matrix

$$D^{-1} = \frac{1}{det(D)}\, {}^t com\, M$$

$$(D^{-1})^{AA} = \frac{1}{det(D)} \det Min_{A,A}(D)$$

$$\text{but} \qquad det\, D \overset{M_A^2}{\sim} D^{AA} \times Min_{A,A}(D) \quad \text{because } D_{AB} = 0 \text{ for } A \neq B$$

$$\text{hence} \qquad (D^{-1})^{AA} \overset{M_A^2}{\sim} (D^{AA})^{-1}$$

Hence D^{AA} having a pole in M_A^2 implies $(D^{-1})^{AA}(M_A^2) = 0$. But using the explicit writing of the full propagator, and a free propagator of the form $D_{0\,L}^{AA} = \frac{1}{k^2 - M_A^2}$, we get

$$\boxed{(D^{-1})^{AA}(M_A^2) = 0 \Rightarrow \hat{G}^{AA}(M_A^2) = 0}$$

Proving the last line is now straightforward, because having

$$(D^{-1})^{AA} \overset{M_A^2}{\sim} (D^{AA})^{-1} \quad \text{and} \quad (D^{-1})^{AA}(M_A^2) = 0$$

implies

$$(D^{-1})^{AA\,\prime} \overset{M_A^2}{\sim} (D^{AA})^{-1\,\prime}$$

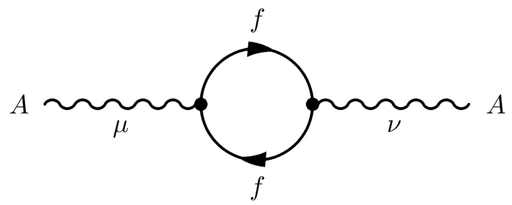
D^{AA} having a residue equal to one, means $(D^{-1})^{AA\,\prime}(M_A^2) = 1$, hence $(D^{AA})^{-1\,\prime}(M_A^2) = 1$, and with the explicit writing of the propagators

$$\boxed{\hat{G}^{AA\,\prime}(M_A^2) = 0}$$

So we trade the physical but not so practical conditions (stating that the propagator of a particle must be equivalent to $\frac{1}{k^2 - M_A^2}$ for $k^2 = M_A^2$) for precise relations directly related to self energies and counterterms. One of the point of this seemingly unnecessary exercise is to note that the well-used relations $\hat{G}^{AA}(M_A^2) = 0$ are valid uniquely if there is no mixing on the mass-shell.

A.6.3 Calculating self-energies

I will here detail calculations of the self energy of the gauge bosons, not including the counterterms. The one-loop diagram is the following :



This diagram comes from the following term of the exponential of the Lagrangian (for commodity reasons, I write the formulae for the photon, but it is general for any gauge boson).

$$(-ie\overline{\psi}\not{A}\psi)(-ie\overline{\psi}\not{A}\psi)$$

that we will re-express as :

$$\begin{aligned}(-ie\overline{\psi}\not{A}\psi)(-ie\overline{\psi}\not{A}\psi) &= (-ie)^2\overline{\psi}\gamma^\mu A_\mu Q\psi\overline{\psi}\gamma^\nu A_\nu Q\psi \\ &= (-ie)^2(-1)A_\mu A_\nu Tr(\gamma^\mu Q\psi\overline{\psi}\gamma^\nu Q\psi\overline{\psi})\end{aligned}$$

where the -1 comes from passing one $\overline{\psi}$ from first position to last – fermions anticommuting – and the trace is to be taken on Lorentz as well as gauge representation space. Using the Wick theorem the expectation value of this operator can be written as :

$$< A_\mu A_\nu Tr(\gamma^\mu Q\psi\overline{\psi}\gamma^\nu Q\psi\overline{\psi}) > =< A_\mu A_\nu > Tr(\gamma^\mu Q < \psi\overline{\psi} > \gamma^\nu Q < \psi\overline{\psi} >)$$
$$= D^{\gamma\gamma}_{\mu\nu}(k)Tr(\gamma^\mu Q D^{\psi\psi}(p)\gamma^\nu Q D^{\psi\psi}(p+k))$$

As the $i\Sigma_{\mu\nu}$ is the amputated diagram we get rid of $D^{\gamma\gamma}_{\mu\nu}(k)$.

The next step of the calculation is different following which gauge boson we have. Indeed for the photon, Q is diagonal and so is the propagator hence the trace on the gauge space turns out to be a sum on all elements of the representation, so we can write

$$i\Sigma^{\gamma\gamma}_{\mu\nu} = (-ie)^2(-1)\sum_i N_{Q\,i}Q_i^2 Tr\left(\gamma^\mu \frac{i}{\not{p} - m_i}\gamma^\nu \frac{i}{\not{p}+\not{k} - m_i}\right)$$

N_i stands for the number of fermions with the same mass and same charge in a same multiplet. As an example the quark up is a triplet of $SU(3)$ with equal masses and charges, so his contribution will have $N_i = 3$.
I will not go through the rest of the calculation (which can be found in , among others), and just state the result :

$$\Sigma^{\gamma\gamma}_{\mu\nu} = (k_\mu k_\nu - k^2 g_{\mu\nu})\frac{e^2}{12\pi^2}\sum_i N_{Q\,i}Q_i^2\left(C_{UV} - 6B_2(m_i, m_i, k^2) + O(\epsilon)\right)$$

where γ is the Euler-Mascheroni constant and μ a renormalization parameter.

Z Boson

For the Z boson we come back to the left-right representations.

$$(-i\frac{g}{c_W})^2(\psi_L^\dagger T_Z\psi_L + s_W^2\psi_R^\dagger Q\psi_R)(\psi_L^\dagger T_Z\psi_L + s_W^2\psi_R^\dagger Q\psi_R)$$

where we expand the product as

- $T_Z\psi_L\psi_L^\dagger T_Z\psi_L\psi_L^\dagger \longrightarrow T_Z^2\Sigma_{LL}(k^2)$
- $2s_W^2 T_Z\psi_L\psi_R^\dagger Q\psi_R\psi_L^\dagger \longrightarrow 2s_W^2 T_{Z\,L}Q_R\Sigma_{LR}(k^2)$
- $s_W^4 Q\psi_R\psi_R^\dagger Q\psi_R\psi_R^\dagger \longrightarrow Q^2\Sigma_{RR}(k^2)$

which ends up in

$$\Sigma^{ZZ}_{\mu\nu}(k^2) = -\frac{g^2}{c_W^2}\sum_\psi N_\psi\left(T_Z^2\Sigma_{\substack{LL\\ \mu\nu}}(k^2, m_\psi) + 2s_W^2 T_{Z\,L}Q_R\Sigma_{\substack{LR\\ \mu\nu}}(k^2, m_\psi) + Q^2\Sigma_{\substack{RR\\ \mu\nu}}(k^2, m_\psi)\right)$$

where $\Sigma_{LL}, \Sigma_{LR}, \Sigma_{RR}$ are defined using the right and left propagators. Their epressions can be found in the Convention section.

W boson

This case is not so straightforward since the $\sigma^{\pm}$ are not diagonal. We are faced with :

$$i\Sigma_{\mu\nu} = (-i\frac{g}{\sqrt{2}})^2(-1)Tr(\gamma^{\mu}\frac{\sigma^+}{2}D^{\psi\psi}(p)\gamma^{\nu}\frac{\sigma^-}{2}D^{\psi\psi}(p+k))$$

As fermions are usually in a doublet of $SU(2)$ (when not in the trivial representation) we focus on the action of $\sigma^{\pm}$ on a two by two propagator:

$$\frac{\sigma^+}{2}D\frac{\sigma^-}{2}D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} D_2 D_1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence by taking the trace we achieve the following result :

$$i\Sigma^{WW}_{\mu\nu} = (-i\frac{g}{\sqrt{2}})^2(-1)\sum_{\psi} N_{\sigma^{\pm}\,\psi} Tr\left(\sigma^{\mu}\frac{i\not{r}}{r^2 - m^2_{\psi,2}}\sigma^{\nu}\frac{i(\not{r}+\not{h})}{(r+h)^2 - m^2_{\psi,1}}\right)$$

Hence the result will be significantly different from the formulas for γ or Z since the two masses ni the loop are no more equal.

$$\Sigma^{WW}_{\mu\nu} = -\frac{g^2}{2}\sum_{\psi} N_{\psi}\Sigma_{\underset{\mu\nu}{LL}}(k^2, m_{\psi,2}, m_{\psi,1})$$

where $\Sigma_{\underset{\mu\nu}{LL}}(k^2, m_{\psi,2}, m_{\psi,1})$ is the same term as for the Z boson, but with two different masses for the propagators.

A.7 Automatization

A.7.1 Set of parameters

Here is the set of renormalized parameters used for the comparative tests.

Parameter	Value	Parameter	Value
α^{-1}	137.0359895		
M_Z	91.1875 GeV	m_μ	105.65839 MeV
M_W	80.385 GeV	m_τ	1.777 GeV
M_H	250 GeV	$m_u = m_d$	46 MeV
m_e	0.51099906 MeV	m_c	1.50 GeV
m_b	4.5 GeV	m_s	150 MeV
m_t	165 GeV		

A.8 Computation of $\Delta\hat{r}$

We start with

$$\frac{\delta g^2}{g^2} - \frac{G_{WW}(0) + \delta M_W^2(k^2)}{M_W^2}$$

Computing δg from the equation $g^2 = \frac{e^2}{1-\frac{M_W^2}{M_Z^2}}$ yields :

$$\frac{\delta g^2}{g^2} = \frac{\delta e^2}{e^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right)$$

where we can replace the counterterms by their values

$$\frac{\delta g^2}{g^2} = G^{\gamma\gamma'}(0) + \frac{G^{Z\gamma}(0)}{M_Z^2} + \frac{c_W^2}{s_W^2}\left(\frac{G^{ZZ}(M_Z^2)}{M_Z^2} - \frac{G^{WW}(M_W^2)}{M_W^2}\right)$$

By inserting the values of all the self energies, we get the analytical equation for $\Delta\hat{r}$, however the result in such a way is quite hard to handle, so we will rewrite it as :

$$\Delta\hat{r} = \Delta\alpha(M_Z^2) + \frac{c_W^2}{s_W^2}\epsilon_1 + \frac{c_W^2 - s_W^2}{s_W^2}\epsilon_2 + 2\epsilon_3$$

where

- $\epsilon_1 = g^2 \frac{G_{33}(0) - G_{11}(0)}{M_W^2}$
- $\epsilon_2 = g^2(\Pi_{11}(M_W^2) - \Pi_{33}(M_Z^2))$
- $\epsilon_3 = g^2(\Pi_{3Q}(M_Z^2) - \Pi_{33}(M_Z^2))$

where we defined

$$\begin{aligned} G_{11}(k^2) &= \sum_\psi \tfrac{N_\psi}{2} G_{LL}(k^2, m_{\psi,2}, m_{\psi,1}) \\ G_{33}(k^2) &= \sum_\psi N_\psi T_3^2 G_{LL}(k^2, m_\psi) \\ G_{3Q}(k^2) &= \sum_\psi N_\psi T_3 Q(G_{LL}(k^2, m_\psi) - G_{LR}(k^2, m_\psi)) \end{aligned}$$

and we implicitly assume the usual decomposition

$$G(k^2) = G + k^2\Pi(k^2)$$

Chapter 2

Conventions

$$\Sigma_{\mu\nu}(k) = k_\mu k_\nu L(k^2) - g_{\mu\nu} G(k^2)$$
$$\text{with} \quad G(k^2) = G + k^2 \Pi(k^2)$$

$$C_{UV} = \frac{2}{\epsilon} + ln(4\pi) - \gamma$$
$$\Delta(m_1, m_2, k^2) = m_1^2 + x(m_2^2 - m_1^2) - x(1-x)k^2)^2$$

$$B_0(m_1, m_2, k^2) = \int_0^1 dx\, ln \frac{\Delta(m_1, m_2, k^2)}{\mu^2}$$
$$B_1(m_1, m_2, k^2) = \int_0^1 dx\, xln \frac{\Delta(m_1, m_2, k^2)}{\mu^2}$$
$$B_0(m_1, m_2, k^2) = \int_0^1 dx\, x(1-x)ln \frac{\Delta(m_1, m_2, k^2)}{\mu^2}$$

$$\Sigma_{\substack{LL\\ \mu\nu}}(k^2, m_{\psi,2}, m_{\psi,1}) = \int \frac{d^4 r}{(2\pi)^4} Tr \left(\sigma_\mu \frac{\not r}{r^2 - m_1^2} \sigma_\nu \frac{\not r + \not k}{(r+k)^2 - m_2^2} \right)$$
$$\Sigma_{\substack{LR\\ \mu\nu}}(k^2, m_{\psi,2}, m_{\psi,1}) = \int \frac{d^4 r}{(2\pi)^4} Tr \left(\sigma_\mu \frac{m_1}{r^2 - m_1^2} \sigma_\nu \frac{\not r + \not k}{(r+k)^2 - m_2^2} \right)$$
$$\Sigma_{\substack{RR\\ \mu\nu}}(k^2, m_{\psi,2}, m_{\psi,1}) = \int \frac{d^4 r}{(2\pi)^4} Tr \left(\sigma_\mu \frac{m_1}{r^2 - m_1^2} \sigma_\nu \frac{m_2}{(r+k)^2 - m_2^2} \right)$$

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