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Building an Inflationary Model of the Universe

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Abstract

The start of this dissertation reviews the Big Bang model and its associated problems. Inflation is then introduced as a model which contains solutions to these problems. It is developed as an additional aspect of the Big Bang model itself. The final section shows how one can link inflation to the Large Scale Structure in the universe, one of the most important pieces of evidence for inflation.

Dedicated to my Mum and Dad, who have always supported me in whatever I do, even quitting my job to do this MSc and what follows.

Thanks to Dan, Ali and Dr Contaldi for useful discussions and helping me understand the basics.

Thanks to Sinead, Ax, Benjo, Jerry, Mike, James, Paul, Valentina, Mike, Nick, Caroline and Matthias for not so useful discussions and endless card games.

Thanks to Lee for keeping me sane.

Contents

1 Introduction

1.1 The Big Bang Model

The central observation in developing the most basic models is the expansion of the universe. This was first observed by Edward Hubble in 1929 [1] as a result of observational data obtained from the red shift of recession speeds of distant galaxies. He noticed that as galaxies that where further away from us had signals such as visible light shifted towards the red end of the spectrum. The expansion of the universe has to be thought of as happening at all points, not originating from a centre and spreading out. So instead of an expanding sphere, it's better to think of raisins in a fruit cake, all moving away from each other as the cake rises in the oven.

This expansion implies that as time goes on, the universe is getting less dense and cooler. Conversely as we go further back in time, the universe gets hotter and denser. This led to the idea of a Big Bang, however the name gives a rather false pretence of what it actually is. The idea of a large explosion happening at a centre of the universe is not correct. Essentially it is a start to what is measured as time and space. At this point the universe was infinitely small, as it was at the start of expansion. If the conservation of energy is taken into account, then right after the big bang, all the energy in the universe was condensed to a small volume, resulting in very high temperatures at that time. The earlier on in the universe one goes, the more fundamental the matter of the universe becomes. Hence very large energy scales. As the universe expanded, it cooled [2]. As cooling occurs more complicated structure is formed. From the smallest of scales, a quantum (Planckian) scale, to the very theoretical particle physics, such as strings and grand unification, to the unification of the weak and electromagnetic forces, through the transitions of quarks to hadrons and to Big Bang Nucleosynthesis [3]. (Where the light elements H, He, Li were formed to abundances predicted quite accurately by Big Bang Theory.) Below is a table showing the time and energy scales of different events in the universe. Recombination will be discussed below. Note when inflation occurs.

	Time	Energy
Planck	$< 10^{-43}$s	10^{18}Gev
Grand Unification	$\geq 10^{-43}$s	$> 10^{18}$Gev
Inflation	$\geq 10^{-34}$s	$\geq 10^{15}$Gev
Electroweak Unification	10^{-10}s	1 Tev
Quark / Hadron Transition	10^{-4}s	100 Mev
Big Bang Nucleosynthesis	3 min	0.1 Mev
Recombination	10^5 years	0.1 ev
Galaxy Formation	$\sim 6 \times 10^8$ years	
Solar System	8×10^9 years	
Now	14×10^9 years	1 meV

Table 1: A Brief History of the Universe. (Adapted from [4]).

This introduces a limit to what we can test and build models for though. Experimental physics that can be tested here and now has an upper limit. Once we're past that limit, testing our very early universe models becomes difficult. This limit happens between inflation and the electroweak unification in the above table. Instead of directly testing a model, tests have to be done on consequences the model has to later times in the universe. One such consequence is the **Cosmic Microwave Background** (CMB), which was discovered in the 1960's [5] and measured in the 1990's by COBE [6] and 2000's by WMAP [7]. The CMB is a result of recombination. At this time the universe which was made up of a plasma of electrons, some light nuclei and photons. Photons had a short mean free path at this time, because of Thompson Scattering happening with the electrons at these energies. During this time temperatures dropped to a point where photons would no longer Thompson scatter and instead travel unhindered throughout the universe. These photons are the cause of the radiation which is now found in the microwave range. This spectrum has a consistent temperature of T=2.75 K. The discovery of the CMB and its measurements are further proof for the Big Bang. The CMB itself also gives a set of results for testing of theories. It is a touch stone for models which affect the universe before and after recombination. It is homogeneous which leads to postulations about the homogeneity of the universe on large scales. However there are variations of one part to every 10^5 [8].

1.2 Inflation

One model for the behaviour of the early universe is inflation. As seen in **Table 1** it is a theory which is placed in the "untestable" region of the history of the universe. Yet its effects are measurable in the CMB and beyond. This theory was introduced as way of explaining some problems with the Big Bang Model. As with most models, the ones for our universe require initial conditions. However a general model will work for any number of initial conditions, however the Big Bang requires the conditions of homogeneity and flatness for the model to work. The full details for these conditions are given found in Section 2.6.

Inflation is introduced as an additional component to the Big Bang model. It was first considered in the last 70's, early 80's by both American and Russian scientists in parallel [9, 10]. Essentially the model adds a period of rapid expansion is added to the universe. As shown in Table 1, this would happen very early on. This rapid expansion generalises the Big Bang models, such that homogeneity and flatness results from the Big Bang model and are no longer required external inputs.

Testing for inflation is difficult though. The conditions the universe was under throughout inflation aren't exactly testable in a lab. However inflation also has other effects on the universe, such as the varying behaviour of inhomogeneities of the universe on different scales. For inhomogeneities on the scale of the universe, inflation will smooth these out to result in the large-scale homogeneity seen later in the CMB and observed now. Details of how this happens are found in section 3.1. However if we look at inhomogeneities at a quantum scales, these variations in energy density, lead to a surprising result. These variations expand and get imprinted as the variations in the CMB. Hence we can use the CMB as a way of limiting the models of inflation, answering questions such as how long would inflation last for and how would it return to the Big Bang universe? [11, 12] showcases some early work done in this area, known as the **graceful exit**.

(The Big Bang universe is referred as the Friedmann universe as Friedmann introduced the metric which measures the expanding universe.)

The perturbations do continue past the age of recombination. One problem with the homogeneous universe is the small scale inhomogeneities, i.e. the clusters, galaxies. At some point in the history of the universe, perturbations of matter density would have caused gravitational instabilities which would have caused the formation of structure. This would have happened on the scales of clusters, then galaxies and finally solar systems. The surprising result though was the link between the gravitational instabilities and those initial quantum inhomogeneities at the early times. Due to inflation, these inhomogeneities should grow, then freeze at the point where they got larger than the universe. Once inflation is over, the universe would then expand (at a much slower rate) to the scale of these inhomogeneities. When this happened, the inhomogeneities caused a spectrum of curvature inhomogeneities, (that is curvature of the galaxy, this will be explained in detail later) which in turned caused the gravitational instabilities and structure to form. This was originally thought of in the late 80's [13].

On looking at all this evidence, one can argue that inflation is a useful addition to the Big Bang model. The following sections go into further detail on the inflation model. The conclusion will feature further discussion on the results of inflation and it's validity as a cosmological model.

2 Building a Model of the Universe

2.1 Starting Principles

How do we define the Universe? This is the first problem one has to tackle when creating a model. Is it infinite or finite? How do we know where it ends? Is there an edge? Some of the questions are indeed more philosophical in nature. For modelling purposes we introduce the concept of a horizon. One of the results of the Big Bang is that the universe has been around for a finite amount of time. Light rays have been decoupled from matter since recombination. In this said time, light rays have only been able to travel a certain finite distance. This distance is called the *Optical Horizon*. What lies beyond this horizon? It is impossible for us to be in contact with point beyond this horizon as the universe has not existed long enough for a signal to get there. All point within this optical horizon are said to be in *causal contact*.

The first assumptions that will go into a model are ones of **Homogeneity** and **Isotropy**. These together are known as the *Cosmological Principle*. Homogeneity implies translational invariance while isotropy implies rotational invariance. Isotropy implies homogeneity but the converse is not necessarily true. Why do we assume these? At scales of about 10 to 100 times the size of clusters the observed universe does indeed look the same. Yes, structures such as the clusters themselves and galaxies do exist, but they are not needed for the most basic models of the universe. Another implication of the Cosmological Principle is an extension of the Corpenican Principle, that is the Earth does not have a privileged position in the universe.

The next concept to add to a model it that of expansion. The universe is expanding. This can be summarised by the **Hubble law**: *Two observers will observe each other receding away from each other at a velocity proportional to the distance between them. H(t)* is called the Hubble rate, it has dimensions of inverse time and is what links the distance to this receding velocity. This rate depends on time.

$$v(t) = H(t)r \tag{2.1}$$

Following from the concept of expansion there are different possible means of measuring distance between two stationary observers. Measuring the physical distance between them would give us the r in the Hubble Law, but this is still a time dependent function due to the expansion of the universe. To make this measurement truly stationary, we introduce the concept of a comoving distance χ. The physical distance is then this comoving distance multiplied by a scale factor that will scale the distances. This comoving scale is dimensionless and depends on time. Figures (2.1) and (2.2) give an idea how these scales work.

$$r(t) = a(t)\chi \tag{2.2}$$

By differentiating (2.2) and then rearranging for χ, the Hubble rate is then linked to the scale factor by

$$v(t) = \dot{a}\chi = \frac{\dot{a}(t)}{a(t)}r = H(t)r \Rightarrow H(t) = \frac{\dot{a}}{a} \tag{2.3}$$

Having found a way of linking two length scales, one of which is constant and follows the Hubble expansion. It is now possible to use temperature, red shift, the scale factor and the Hubble constant to speak about specific times in the universe.

2.2 Geometry of the Universe

With the basic assumptions for the dynamics of the universe in place, a metric is needed. This metric essentially measures how the different space dimensions scale. Later on time will be included with the introduction of a relativistic picture. For now the geometry of the universe can be thought of as hypersurfaces of constant time. Imposing the conditions of homogeneity

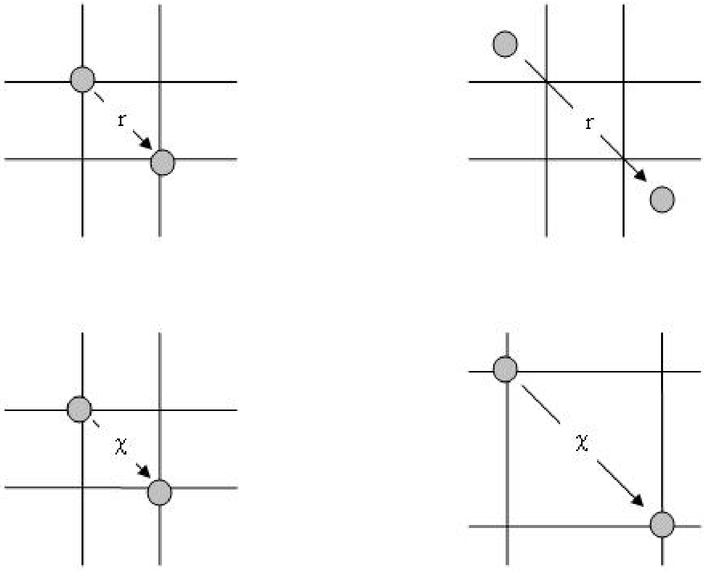

Figure 2.1: Comparing a physical scale, that does not take into account expansion *(top)* with a comoving scale *(bottom)*.

and isotropy results in 3 possible types of hypersurfaces;

- a 3-dimensional flat space,
- a 3-dimensional sphere with positive curvature,
- a 3-dimensional space with negative curvature.

In the case of the flat surface, drawing a triangle made by 3 geodesics would result in the angles adding up to π, for the positive hypersurfaces the angles would be greater than π and finally for the negative hypersurface it would be less than π. These hypersurfaces are embedded in four dimensional Euclidean space with co-ordinates (w,x,y,z). These spheres have a radius of α. α is real except for the negative curvature hypersurface where it is complex. As the hypersurfaces are spheres they satisfy:

$$w^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{2.4}$$

Differentiate, taking into account that alpha is constant, as the sphere is at a specific time and constant:

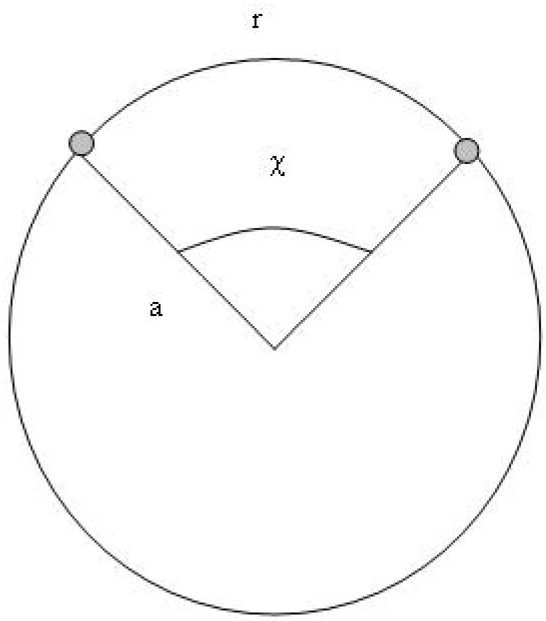

Figure 2.2: Another method of looking at different scales. It gives the basic idea, but encourages the *expansion from a point* which is incorrect.

$$2(wdw + xdx + ydy + zdz) = 2\alpha d\alpha = 0 \tag{2.5}$$

dw can be expressed in terms of the constant α and the other coordinates. Rearranging (2.4) for w, and using (2.5) gives:

$$dw^2 = \frac{(xdx + ydy + zdz)^2}{(\alpha^2 - x^2 - y^2 - z^2} \tag{2.6}$$

This gives us a change in the co-ordinate for w in 4-dimensional space on the sphere. This 4D Euclidean space has the following metric:

$$ds^2 = dw^2 + dx^2 + dy^2 + dz^2 = \frac{(xdx + ydy + zdz)^2}{(\alpha^2 - x^2 - y^2 - z^2} + dx^2 + dy^2 + dz^2 \tag{2.7}$$

This is the distance between two points on the 3-sphere embedded in the 4D Euclidean space. This equations are bounded by α. An easier way to express the above metric uses spherical polar co-ordinates (r,θ, ϕ). Here

$$\begin{aligned} x &= rsin(\theta)cos(\varphi) \\ y &= rsin(\theta)sin(\varphi) \\ z &= rcos(\theta) \end{aligned} \tag{2.8}$$

Giving the following:

$$\begin{gathered} r^2 = x^2 + y^2 + z^2 \Rightarrow rdr = xdx + ydy + zdz \\ dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta + r^2 sin(\theta) d\varphi = dr^2 + r^2 d\Omega \end{gathered} \tag{2.9}$$

It should be noted r is a co-ordinate and is bounded by 0 and α. The metric now becomes:

$$ds^2 = \frac{(rdr)^2}{\alpha^2 - r^2} + dr^2 + r^2 d\Omega^2 = \frac{dr^2}{1 - (\frac{r}{\alpha})^2} + r^2 d\Omega^2 \tag{2.10}$$

Using another change of co-ordinates the co-ordinate r can be related to the radius of the sphere α.

$$R = \frac{r}{\sqrt{|\alpha|^2}} dR = \frac{dr}{\sqrt{|\alpha|^2}} \tag{2.11}$$

It would also be useful to measure what type of hypersurface it is using another variable. So K is introduced such that:

$$K = \frac{|\alpha|^2}{\alpha^2} \tag{2.12}$$

K can either be positive (a closed sphere hypersurface), zero (flat hypersurface) or negative (so K is complex and it's an open hypersurface). K is defined as the curvature of spacetime. The metric now becomes

$$ds^2 = |\alpha|^2 (\frac{dR^2}{1 - KR^2} + R^2 d\Omega^2) \tag{2.13}$$

R is now linked to the comoving distance χ.

$$R = S_K(\chi) = \left\{ \begin{array}{cccc} sinh(\chi) & K = -1 & (open) & \chi : 0 \to \infty \\ \chi & K = 0 & (flat) & \\ sin(\chi) & K = +1 & (closed) & \chi : 0 \leqslant \chi \leqslant \pi \end{array} \right\} \tag{2.14}$$

In all these 3 cases (using trigonometric and hyperbolic trigonometric relationships):

$$d\chi^2 = \frac{dR^2}{1 - KR^2} \tag{2.15}$$

And the metric can be written as

$$ds^2 = |\alpha|^2(d\chi^2 + S_K^2(\chi)d\Omega^2) \tag{2.16}$$

This metric describes a line element on the hypersurface in a constant time. To introduce time as a variable, α is now the expansion constant *a(t)*. Also time has an opposite sign to the space components. This results in the Friedmann-Robertson-Walker (FRW) metric which is used to describe the spacetime of the universe.

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega\right) \tag{2.17}$$

Using comoving scales the metric becomes;

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + S_K^2(\chi)d\Omega^2) \tag{2.18}$$

The metric is of the form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where μ and ν run from 0 to 3. 0 is time and 1 to 3 represent the spatial co-ordinates. For indices in general a greek index will run from 0 to 3, a latin index will run from 1 to 3. This form of a metric comes from differential geometry, a branch of mathematics which is used to model manifolds. Spacetime can be represented as a manifold with a metric, in this case the FRW metric. This manifold can contain different mathematical objects such as vectors, co variant vectors and tensors on it. Vectors are represented as v^α. A covariant vector is like an complementary vector, defined such that $\omega_\nu = g_{\nu\mu}v^\mu$. This is called lowering a index using the metric. Where an index appears twice, it is summed over 0 to 3. Tensor are products of vectors and covariant vectors such as $T^{\mu\nu}$. This tensor can be raised or lowered using $g_{\mu\nu}$ as follows $T_{\mu\nu} = g_{\mu\alpha}T^\nu_\alpha$ and $T^{\mu\nu} = g^{\mu\alpha}T^\alpha_\nu$. $g^{\mu\nu}$ is defined such that $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$, δ^μ_ν is 1 when $\mu = \nu$ and zero elsewhere. Using this set-up specific types of tensors can be derived from the metric. These are used with Einstein's equations to link spacetime to matter and gravitation.

2.3 Introducing matter into the universe

2.3.1 The Newtonian Picture

The next step to building a model of the universe is to model what matter is found in the universe. This is based on a perfect isotropic fluid with pressure p and energy density ρ. The ratio between them is defined as $\omega = \frac{p}{\rho}$ Different fluids are used to describe different types of matter, each with different values of ω. $\omega = 0$ represents non-relativistic, pressureless matter knows as **Dust**, allowing the use of Newtonian arguments to describe the equations of motion. Starting with a total mass of dust M in a sphere of radius $R(t) = a(t)\chi$. The density is expressed as:

$$\rho(t) = \frac{M}{\frac{4\pi}{3}R^3(t)} = \frac{M}{\frac{4\pi}{3}a^3(t)\chi^3} \tag{2.19}$$

Specifying the subscript 0 as representing the value of different time-dependent variables at the present time, the following ratio is derived:

$$R(t) = \frac{R_0 a(t)}{a_0} \tag{2.20}$$

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^3 \tag{2.21}$$

$\rho_0 = \frac{M}{\frac{4\pi}{3}a_0\chi^3}$ does not depends on time (remembering that the comoving distance χ is a constant). Differentiating (2.21) with respect to time gives:

$$\dot{\rho}(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^3 (-3)\dot{a}(t)\frac{1}{a(t)} \tag{2.22}$$

Using (2.19) and (2.3) the equation takes the form of the **Continuity Equation**.

$$\dot{\rho}(t) = -3H(t)\rho(t) \tag{2.23}$$

To further investigate the dynamics of the system gravitational forces have to be considered using Newtonian arguments. A test particle of mass m $\ll$ M is taken at radius R(t). The gravitational forces will act against the expansion forces and slow down the outward motion of the particle with respect to physical scales.

$$m\ddot{R}(t) = \frac{-GmM}{R(t)^2} = \frac{-4\pi}{3} Gm \frac{M}{\frac{4\pi}{3} R(t)^3} R(t) \tag{2.24}$$

Changing to comoving distance χ using $R(t) = a(t)\chi$

$$m\ddot{a}(t)\chi = \frac{-4\pi}{3} Gm \frac{M}{\frac{4\pi}{3} a(t)^3 \chi^3} a(t)\chi \tag{2.25}$$

Canceling m and χ and inserting ρ using (2.19) the equation is bought to the form known as the **acceleration equation**:

$$\ddot{a}(t) = \frac{-4\pi}{3} G\rho(t) a(t) \tag{2.26}$$

This equation is then used to determine how the system develops as time unfolds. From this equation it is possible to model open and closed universes, similar to those that arose from the geometry of the universe. Using the fact that $\rho(t) \propto \frac{1}{a(t)^3}$ to get the following form of the acceleration equation:

$$\ddot{a}(t) = \frac{-4\pi}{3} G\rho_0 \frac{a_0^3}{a^2(t)} \tag{2.27}$$

Multiplying both sides by $\dot{a}$ and integrating with respect to t together with the following relationships:

$$\tfrac{d}{dt}\dot{a}^2(t) = 2\ddot{a}\dot{a} \quad \tfrac{d}{dt}\tfrac{1}{a} = -\tfrac{\dot{a}}{a^2}$$

Gives:

$$\frac{1}{2}\dot{a}^2(t) - \frac{4\pi}{3} G\rho_0 \frac{a_0^3}{a(t)} = E \tag{2.28}$$

where E is a constant of integration. The equation is of the form of a projectile thrown from earth. Essentially *a(t)* becomes the distance and $\dot{a}(t)$ the velocity. Dependent on E, the "velocity" becomes an escape velocity, $(E > 0)$, an orbit $(E = 0)$ or a negative velocity and falls back $(E < 0)$. Another way of expressing (2.28) is using $\rho(t) \propto \frac{1}{a(t)^3}$.

$$H^2 = 2\frac{E}{a^2} + \frac{8\pi G\rho}{3} \tag{2.29}$$

Depending on the nature of E, *a(t)* will evolve differently. In the case where $E > 0$ *a(t)* grows at quite a fast rate, so the universe will expand rapidly. This is known as an open universe. The case $E = 0$ will also result in expansion, but at a much slower rate. This is known as the flat universe. With $E < 0$ a peaks and then gets smaller again until the scale factor becomes zero. This is known as a closed universe. Figure 2.3 shows the fate of a for different values of E.

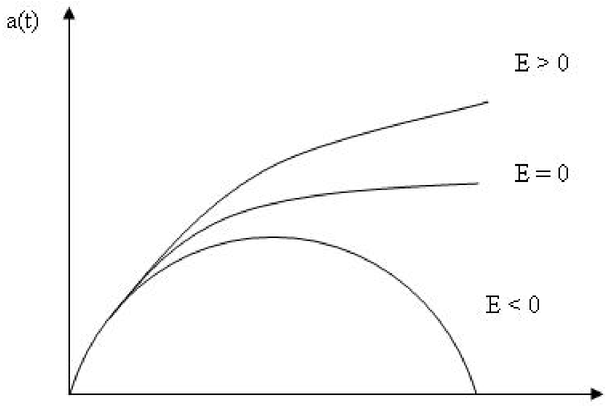

Figure 2.3: The behaviour of $\dot{a}$ with respect to different values of E.

Current data suggests that the universe is indeed flat. In this case we set E=0 and derive a **critical energy density**:

$$\rho_{crit}(t) = \frac{3H(t)^2}{8\pi G} \tag{2.30}$$

Note these are the same as names as we give the different universes possible due to the possible geometries of the universe. This is due to matter leading to a certain geometry. The matter density is measured as a ratio against the critical matter as follows:

$$\Omega(t) = \frac{\rho(t)}{\rho(t)_{crit}} \tag{2.31}$$

Ω is now used to describe the different possible energy densities, curvatures of the universe. A denser universe ($\Omega > 1$ $E < 0$) gives a closed universe. A less denser universe ($\Omega < 1$ $E > 0$) gives a closed universe. A universe with critical density gives a flat universe. It's important to note, Ω can change with respect to time. As the universe expands, the value for the critical density does get smaller.

2.3.2 Introducing Relativity

When relativistic matter is being considered, Einstein's equations are then used to link spacetime to matter. To get to Einstein's, equations a few different mathematical objects have to be derived. A starting point is to clarify the metric used. For the calculations presented here a flat isotropic universe has been assumed. The FRW metric (2.17) can be expressed as a matrix:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{2.32}$$

It's inverse is defined as $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}$, giving:

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)} \end{pmatrix} \tag{2.33}$$

A note on nation, a comma depicts a partial derivative with respect the index, i.e. $T(x^{\mu})_{,i} = \frac{\partial T}{\partial x^i}$. A semi-colon here represents a covariant derivative. Due to the non-simple space-time connection in general relativity, extra terms are required when looking at the derivatives which conserve energy and momentum:

$$A^{\mu}_{\nu;\gamma} = A^{\mu}_{\nu,\gamma} + \Gamma^{\mu}_{\alpha\gamma}A^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\gamma}A^{\mu}_{\alpha} = 0 \tag{2.34}$$

In general relativity, general motion of a particle with no external forces is described using the **geodesic equation**. This requires the use of an **affine parameter**, λ, being a parameter that divides a spacetime path into equal segments.

$$\frac{d^2x^{\mu}}{d\lambda^2} = -\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} \tag{2.35}$$

Γ described the connection between time and space co-ordinates and is based on derivatives of the metric. They are known as **Christoffel symbols** or **affine connections**:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}] \tag{2.36}$$

For the FRW metric, most are equal to zero. Here are some results and the calculation of non-zero connection. In general connections are symmetric in their lower indices.

$$\begin{aligned} \Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = \Gamma^i_{00} = \Gamma^i_{jk} = 0 \\ \Gamma^0_{ij} = \tfrac{g^{0\nu}}{2}[g_{i\nu,j} + g_{j\nu,i} - g_{ij,\nu}] = -\tfrac{1}{2}[\delta_{ij}a^2_{,t}] = -\tfrac{1}{2}[\delta_{ij}2\dot{a}a] = \delta_{ij}Ha^2 \\ \Gamma^i_{0j} = \tfrac{g^{i\nu}}{2}[g_{0\nu,j} + g_{j\nu,0} - g_{0j,\nu}] = \tfrac{1}{2}[\delta_{jk}a^2_{,t}] = \tfrac{1}{2}\delta^{ik}\tfrac{1}{a^2}\delta_{jk}2\dot{a}a] = \delta^i_j H \end{aligned} \tag{2.37}$$

The connections are combined to make a tensor which describes the curvature of the spacetime. This is the **Ricci tensor**:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{2.38}$$

For the flat FRW case, only the following two components of $R_{\mu\nu}$ are non-zero. Given is the derivation of the time-time component:

$$\begin{aligned} R_{00} &= -\Gamma^i_{0i,0} - \Gamma^i_{j0}\Gamma^j_{0i} \\ R_{00} &= -\delta^i_i H_{,t} - \delta^i_j H \delta^j_i H \\ R_{00} &= -3(\tfrac{\ddot{a}}{a} - \tfrac{\dot{a}^2}{a^2}) - 3H^2 \\ R_{00} &= -3\tfrac{\ddot{a}}{a} \\ R_{ij} &= \delta_{ij}[\ddot{a}a + 2\dot{a}^2] \end{aligned}$$

The tensor can be contracted to the **Ricci Scalar**. $\mathcal{R} = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$. In this case:

$$\begin{aligned} g^{00}R_{00} + g^{ij}R_{ij} \;\; \mathcal{R} &= 3\tfrac{\ddot{a}}{a} + \tfrac{1}{a^2(t)}3[\ddot{a}a + 2\dot{a}^2] \\ \mathcal{R} &= 6[\tfrac{\ddot{a}}{a} + (\tfrac{\dot{a}}{a})^2] \end{aligned} \tag{2.39}$$

These combine to form the **Einstein Tensor**:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} - g_{\mu\nu}\Lambda \tag{2.40}$$

Λ is the cosmological constant. In this context, Λ appears as an integration constant which appears in the derivation. The Einstein tensor is linked to the matter of the universe through the **Einstein Equations**:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.41}$$

$T_{\mu\nu}$ is the stress-energy tensor. This describes the energy and pressure for matter. In this case it takes the form for a perfect fluid:

$$T^{\mu\nu} = (\rho + p)U^{\mu}U^{\nu} - pg^{\mu\nu} \tag{2.42}$$

U^{μ} is the four velocity of the perfect fluid. In its comoving frame of reference the fluid is at rest with the expansion. The velocity is then $U^{\mu} = (1, 0, 0, 0)$. Using this in (2.42) gives the following stress-energy for the perfect fluid:

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.43}$$

Placing all these components together under the time-time components of the Einstein equation gives:

$$8\pi G T_{00} = R_{00} - \tfrac{1}{2}g_{00}\mathcal{R} - g_{00}\Lambda \; 8\pi G\rho = -3\tfrac{\ddot{a}}{a} + 3[\tfrac{\ddot{a}}{a} + H^2] + \Lambda$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} - \frac{K}{a^2} \tag{2.44}$$

This is known as the first Friedmann Equation. It describes how the expansion of the universe, through $H = \frac{\dot{a}}{a}$, is linked to the energy density of the universe. The last term appears when these derivations are done using a non-flat metric, therefore $K \neq 0$. Using the space-space components and inserting (2.44) gives a further equation of motion:

$$\begin{gathered} 8\pi G T_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}\mathcal{R} - g_{ij}\Lambda \\ \delta^i_j 8\pi G(a^2 p) = \delta_{ij}[\ddot{a}a + 2\dot{a}^2 - 3a^2(\tfrac{\ddot{a}}{a} + (\tfrac{\dot{a}}{a})^2) - a^2(t)\Lambda] \\ 8\pi G p + \tfrac{8\pi G\rho}{3} = -2\tfrac{\ddot{a}}{a} - \Lambda \end{gathered}$$

Which leads to the **second Friedmann Equation** or the **relativistic acceleration equation**:

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(3p + \rho) - \frac{\Lambda}{3} \tag{2.45}$$

This reduces to the Newtonian version (2.26) when p=0, which is when one is considering Dust. To further investigate the equation of motion for the FRW universe, one can look at the **Bianchi Identities** $G^{u}_{\nu;\mu}$. The Bianchi Identity in terms of the Stress-Energy tensor is as follows:

$$T^{\mu}_{\nu;\nu} = T^{\mu}_{\nu,\nu} + \Gamma^{\mu}_{\alpha\nu}T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\nu}T^{\mu}_{\alpha} = 0 \tag{2.46}$$

Taking the time-time component gives:

$$T^{0}_{0,0} + \Gamma^{i}_{j0}T^{j}_{i} - \Gamma^{0}_{ii}T^{0}_{0} = 0 \; -\tfrac{\partial\rho}{\partial t} + \delta^{i}_{j}H\delta^{j}_{i}p + \delta_{ii}\rho H a^{2} = 0$$

Which results in:

$$\frac{\partial\rho}{\partial t} + 3\frac{\dot{a}}{a}(p+\rho) = 0 \tag{2.47}$$

This can be written in terms of $\omega = \frac{p}{\rho}$. Using the chain rule, the variable for the partial derivative can be changed to a, $\frac{\partial\rho}{\partial a}\dot{a}$.

$$\frac{\partial\rho}{\partial a} = -3\frac{\rho}{a}(1+\omega) \tag{2.48}$$

This is the relativistic form of the continuity equation. Taking $\omega = 0$ or p=0 gives the non relativistic version (2.23). (2.48) is satisfied by power law solutions of type

$$\rho(a) = \rho_0(a/a_o)^n \tag{2.49}$$

Where $n = -3(1+\omega)$ and depends on the type of fluid.

This gives an equation of motion for the different possible fluids. To model relativistic fluids, $\omega = 1/3$ is used. Think of thermodynamics and how a particle can move in 3 spatial dimensions.

A final type of matter is used one where $\omega = -1$. This represents what is called **Cosmological Constant** type matter. It's defined by the fact is has a constant energy density, ρ_Λ. This type of matter is quite counter-intuitive, essentially the matter has a negative pressure. This type of matter was originally introduced by Einstein in his calculations as a method of keeping the universe static, so it would counter the expansion. It originally appeared as an integration constant. However it has since been used to describe this specific type of matter. This matter plays an important role in inflation. The energy density is defined as $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. This allows the Friedmann equation to be written as

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{K}{a^2} \tag{2.50}$$

This also defines a ratio with the critical density:

$$\Omega_\Lambda = \frac{\rho_\Omega}{\rho_{crit}} = \frac{\Lambda}{3H^2} \tag{2.51}$$

Different fluids evolve with different powers of the scale factor. This can be seen by using the solution to (2.48).

- relativistic matter (ω= 1/3, n=-4) : $\rho_\gamma \propto a^{-4}$
- non-relativistic matter (ω= 0, n=-3) : $\rho_\gamma \propto a^{-3}$
- cosmological constant type matter (ω= -1, n=0) : $\rho_\gamma \propto$ constant

Using the fact that $\Omega_i(t) = \frac{\rho_i(t)}{\rho_{crit}}$ for different types of matter i, $\rho_{crit} = \frac{3H_0^2}{8\pi G}$, (2.49), the first Friedmann equation (2.44) can be rewritten:

$$H^2(a) = H_0^2[\Omega_\Lambda(a_0) + \Omega_m(a_0)a^{-3} + \Omega_\Gamma(a_0)a^{-4}] - \frac{K}{a^2} \tag{2.52}$$

Where a depends on a general time and a_0, Ω and ρ_{crit} depend on a specific time, i.e. now. To make the curvature term of the same form we introduce the term $\Omega_K = \frac{-K}{a_0^2 H_0^2}$. This leads to the following form for the first Friedmann equation:

$$H^2(a) = H_0^2[\Omega_\Lambda(a_0) + \Omega_m(a_0)a^{-3} + \Omega_\Gamma(a_0)a^{-4} - \Omega_K(a_0)a^{-2}] \tag{2.53}$$

The evolution of different types of matter can now be seen. Remember *a(t)* increases as time evolves. Initially the universe was radiation dominated but as expansion occurs, the density of relativistic matter fell. At this time the density of matter was also falling, but not as fast (a^{-3} compared with a^{-4}). So this lead to a matter dominated era. Eventually this also lead to a time where the curvature factor would play a larger factor in the calculation of H. However this is not strictly matter. Its inclusion in the above equation was to show how it compares in the evolution of H with matter. Finally the density of cosmological constant type matter remains constant as the universe expands. This is now leading to cosmological constant type matter domination.

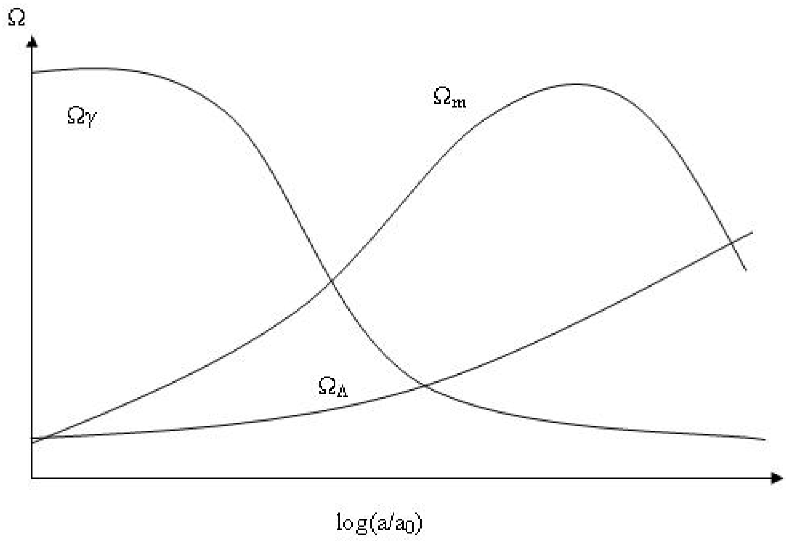

Figure 2.4: Different matter dominated eras of the universe

2.4 Horizons and Patches

To use the FRW metric (2.17) and its solutions for relativistic matter it is convenient to factorise the scale factor out. This requires the introduction of **conformal time**. This is defined using the following integral:

$$\eta = \int_0^t \frac{dt'}{a(t')} \tag{2.54}$$

As the scale factor increases, η gets *slower*. Using $dt = a(\eta)d\eta$ we derive the following relationship between conformal and physical time.:

$$\dot{a} = \frac{da(t)}{dt} = \frac{da(\eta)}{d\eta}\frac{\eta}{dt} = \frac{da}{d\eta}\frac{1}{a(\eta)} \tag{2.55}$$

Where the prime indicates a differential with conformal time, while the dot indicates a differential with actual time. The Hubble constant can also be redefined using conformal time:

$$\mathcal{H} = \frac{a'}{a} \tag{2.56}$$

The metric then becomes

$$ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + S_K^2(\chi)d\Omega^2) \tag{2.57}$$

It is now possible to derive trajectories for light particles using (2.57). Light and other forms of radiation are massless and follow null geodesics, that is $ds^2 = 0$. As the universe is considered isotropic, we can ignore the angular components of the metric. This gives us the following metric;

$$d\eta^2 = d\chi^2 \Rightarrow \chi(\eta) = \pm\eta + Constant \tag{2.58}$$

Here the advantage of using conformal time can be seem, if proper time would have been used, there would not have been a linear relationship. In this case light travels at $\frac{\pi}{2}$ in the η - χ plane. This trajectory defines a *light cone*. Two points are said to be *causally connected* if one point is inside or on the light cone of the other.

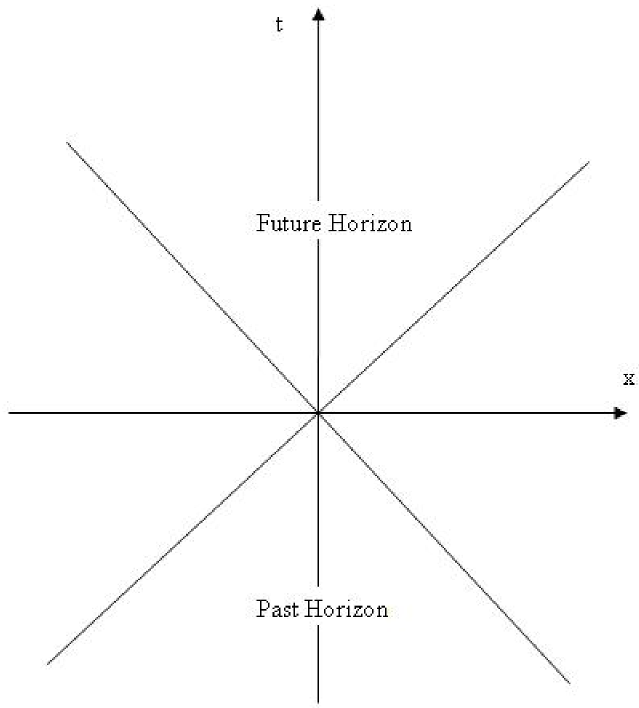

Figure 2.5: A light cone

These definitions can now be used to properly define the concept of the

particle horizon, introduced at the start of this section. (2.58) is used to define the maximum finite distance light can travel as

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{2.59}$$

In terms of physical coordinates

$$d_p(t) = \chi_p(\eta) = a(t) \int_{t_i}^{t} \frac{dt}{a(t)} \tag{2.60}$$

t_i is defined as the point where $a(t) = 0$ and is when the Big Bang occurred, that is when the universe started expanding. For now we will also assume that η_i also corresponds to t_i. However η_i does not have to be 0, for example in an inflationary the model. Physically this horizon represents the boundary in which it is possible to receive signals from. Taking the initial time as the time of recombination, we define the optical horizon.

Another quantity that needs defining for use later is the **Hubble Scale**. The Hubble scale is 1 / H(η) and describes a size of a local inertial frame. This is a scale which is characterised by the expansion. The particle horizon is defined using kinematic arguments. They are different concepts. The horizon has evolved from a start time to an end time, the scale is dependent on the specific time. Now in most situations these are actually in the same order of magnitude. (Specifically when the strong energy condition is preserved, more on this in the next section). Throughout inflation, the difference between these two quantities is a central idea.

2.5 Example: The de Sitter Universe

Different models of the universe can now be developed using the equations of motion derived in the previous sections. In particular there are different models where the universe is only full of a specific type of matter with a specific geometry. For example, a flat universe filled with dust is called the **Einstein-de Sitter Universe**. This can be extended to include just relativistic matter. There is also the **Milne universe**, one which models an empty open universe. Inflation models require a universe full of Λ type material, the **de Sitter solution** is a starting point.

The de Sitter universe is not empty, there is an energy / type of matter in it. This type of energy represents the vacuum. (This is the energy created by spontaneous creation of particle / anti-particle creation.) From a geometrical perspective this is just an integration constant that arises from the derivation of the Einstein equations.

Thinking of this as a type of matter, it's best to think of it as a fluid, just like dust or radiation matter. Specifically the energy density of the fluid is defined as $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. As $\omega = -1$, then $p_\Lambda = -\rho_\Lambda$. This is a type of matter that has negative pressure, this is the matter that is *slowing* down the expansion. (Note: *a(t)* is increasing, $\dot{a}$(t) is decreasing.) This pressure is also constant. As derived previously, the energy density also remains constant with expansion.

Applying the continuity equation to a constant ρ will result in ω =-1:

$$\frac{d\rho_\Lambda}{da} = 0 = -3\frac{\rho_\Lambda}{a}(1+\omega) \tag{2.61}$$

Taking the acceleration equation (2.45) gives the evolution of the scale factor in this universe.

$$\tfrac{\ddot{a}}{a} = -\tfrac{4\pi G}{3}\rho_\Lambda(1+3\omega)$$

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho_\Lambda \tag{2.62}$$

The scale factor has positive acceleration. The universe is then expanding and this scale factor gets larger as time evolves. This also looks like the equations of a harmonic oscillator. $\ddot{a} - a\frac{8\pi G}{3}\rho_\Lambda = 0$ Defining a constant Hubble rate (from 2.44):

$$H_\Lambda = \left(\frac{8\pi G}{3}\rho_\Lambda\right)^{\frac{1}{2}} \tag{2.63}$$

The acceleration equation now becomes $\ddot{a} = H_\Lambda^2 a$ which has solutions:

$$\begin{aligned} a(t) &= C_1 e^{(H_\Lambda t)} + C_2 e^{(-H_\Lambda t)} \\ \ddot{a}(t) &= H_\Lambda^2(C_1 e^{(H_\Lambda t)} + C_2^{(-H_\Lambda t)}) = H_\Lambda^2 a(t) \end{aligned} \tag{2.64}$$

The universe still has curvature though. And as seen from (2.44) this changes the dynamics of H(t).

$$(\tfrac{\dot{a}}{a})^2 = H_\Lambda^2 - \tfrac{K}{a^2} \Rightarrow \dot{a}^2 = H_\Lambda^2 a^2 - K$$

Substituting the solution for a gives:

$$\begin{aligned} K &= H_\Lambda^2 a^2 - \dot{a}2 \\ K &= 4H_\Lambda^2 C_1 C_2 \end{aligned} \tag{2.65}$$

The curvature of space then depends on the constants of integration. Making either C_1 or C_2 0 gives a flat universe. Assume $C_2 = 0$, the other has to take the value H_Λ^{-1} With an open universe, K=1, then so that $C_1 = C_2$, we take $C_1 = (2H_\Lambda)^{-1}$ at t=0. In the case K=-1 then $C_2 = -C_1$.

This results in:

$$a(t) = \frac{1}{H_\Lambda} \left\{ \begin{array}{cc} sinh^2(H_\Lambda t) & K = -1 \\ e^{2H_\Lambda t} & K = 0 \\ cosh^2(H_\Lambda t) & K = +1 \end{array} \right\} \tag{2.66}$$

The behaviour of *a(t)* become the same once $t > \frac{1}{H_\Lambda}$, that is $a(t) \sim e^{H_\Lambda t}$ Looking further into the spacetime structure of DeSitter space, one finds that it has translational invariance with respect to time. This results in all 3 cases above describing the same spacetime. Further details of this can be found in Mukhanov.

In this de Sitter space, one finds that the Hubble scale is:

$$\frac{1}{H(t)} = \frac{a}{\dot{a}} = \frac{e^{H_\Lambda t}}{H_\Lambda e^{H_\Lambda t}} = \frac{1}{H_\Lambda} \tag{2.67}$$

The Hubble scale remains constant in deSitter space.

a in terms of conformal time becomes $a(\eta) \propto \eta^{-1}$. This is shown in further detail below in Section 3.3.

2.6 The Big Bang Problems

As described in the introduction, the big bang model has many successes. However there are some fundamental issues with the model. These are problems associated with initial conditions. The Big Bang requires some specific initial conditions which should result from the model, not be entered as an assumption. This should not be the case, most physical models model a process and therefore a range of initial conditions. For example Newtonian Mechanics describes the movement of a classical particle. The theory works for any initial position and velocity. This following section describes how these specific conditions arise. The next section will deal with how these problems are solved with the introduction of inflation.

2.6.1 The Horizon Problem

The first problem is the horizon problem. As discussed earlier the universe is homogeneous on large scales. This homogeneity implies that some form of causal contact was required across most of the universe in its early formative stages. However the particle horizon can only be a length which is quite small at these formative stages. So this homogeneity should not be there, or more inhomogeneities should exist.

As defined earlier, the maximum distance a light ray can travel is the conformal time, η. This can be expressed as an integral of the comoving Hubble radius. H^{-1} was used as the Hubbles radius, so adding an a gives the comoving Hubble radius, aH^{-1}. This is the radius that describes the causal universe at a given instant.

In this case in co-moving co-ordinates (using $d(ln(a)) = \frac{da}{a}$):

$$d_p = \eta = \int_0^a \frac{da}{Ha^2} = \int_0^a d\,ln(a)\frac{1}{Ha} \tag{2.68}$$

Noting that $H^2 = H_0^2\Omega(\frac{a}{a_0})^{-3(1-\omega)}$ gives the following relationship between $(aH)^{-1}$ and a:

$$(aH)^{-1} \propto a^{\frac{1}{2}(1+3\omega)} \tag{2.69}$$

Here ω depends on what fluid is being modelled. Bringing the equations together gives a relationship between the particle Horizon and a:

$$\begin{array}{lll} \omega = \frac{1}{3} & \eta \propto a & \text{Radiation Dominated} \\ \omega = 0 & \eta \propto a^{\frac{1}{2}} & \text{Matter Dominated} \end{array} \tag{2.70}$$

The comoving horizon then grows without change throughout time. Scales that are now entering the particle horizon were therefore not in the horizon at an early time. This includes at the time of the CMB forming, our evidence for the large scale homogeneity of the universe.

2.6.2 The Flatness Problem

The second problem is known as the flatness problem, as a result of the curvature of the universe resulting being flat for the current universe to work. The flatness of the universe is actually linked to the initial Hubble velocities of constituents of the universe. The results of this problems are best viewed in terms of energies and velocities.

First consider a spherical universe filled with just matter. The total kinetic energy of the system due to Hubble expansion is E_k. The system also has a potential due to the gravitational self-interaction of the particle, E_p. This potential energy is negative and slows the expansion. Both these give the total energy in the system. This total energy remains constant throughout expansion. This can be shown in the following equation, here the subscript i represents the initial quantities and 0 represents the current quantities.

$$E^{tot} = E_i^k + E_i^p = E_0^k + E_0^p \tag{2.71}$$

The hubble velocity of the particle is proportional to the time derivative of the scale factor *a(t)*, which results in the following relationship between a and E^k.

$$E_i^k = E_0^k \left(\frac{\dot{a}_i}{\dot{a}_0}\right)^2 \tag{2.72}$$

The ratio between total energy and the kinetic energy of the system then becomes:

$$\frac{E_i^{tot}}{E_i^k} = \frac{E_i^k + E_i^p}{E_i^k} = \frac{E_0^k + E_0^p}{E_0^k} \left(\frac{\dot{a}_0}{\dot{a}_i}\right)^2 \tag{2.73}$$

Using the fact that the kinetic and potential energies today are in the same order of magnitude, and that $\frac{\dot{a}_0}{\dot{a}_i} \leq 10^{-28}$ results in the following ratio of energies:

$$\frac{E_i^{tot}}{E_i^k} = 10^{-56} \tag{2.74}$$

What does this mean for the early universe? There was a specific Hubble velocity distribution with very precise values such that the above relation can work. The Hubble velocities are very sensitive to the potential energy caused by the distribution of matter. If these velocities were a little too fast, then there would be too much kinetic energy and the universe empties out. Assuming the converse, too slow an initial velocity, and the universe recollapses.

Another way of looking at this particular problem is through the use of energy densities. Taking the first Friedman equation and re-writing it in the following form gives a better clue for a solution to this initial value problem.

$$\begin{aligned} H(t)^2 &= \tfrac{8\pi G\rho(t)}{3} - \tfrac{K}{a(t)^2} \\ (a(t)H(t))^2 &= \tfrac{8\pi G\rho(t)a^2(t)}{3} - K \\ \rho(t) &= \tfrac{3H^2(t)\Omega(t)}{8\pi G} \\ a(t)^2H(t)^2(1-\Omega^2(t)) &= -K \end{aligned}$$

$$\Omega(t) - 1 = \frac{K}{(H(t)a(t))^2} \tag{2.75}$$

This was also possible as $\Omega = E^p/E^k$ where Ω is time dependent. It is also dependent on the comoving Hubble radius. This radius gets bigger with time. The solution for this equation means we require K=0 and therefore $\Omega(a) = 1$. It is not a consequence of the equations, but a requirement to solve for a growing $(aH)^{-1}$.

3 Inflation - The Homogenous Model

3.1 A Simple Concept

The Big Bang problems all depended on the behaviour of the comoving Hubble radius. A solution to these problems would involve changing the behaviour of the radius. It is important to distinguish between the Hubble radius and the Particle Horizon at this point. The particle horizon is defined by the amount of distance a relativistic particle can travel in a given time. It is therefore a kinematical quantity. The Hubbles radius is a quantity defined as the inverse of the Hubble rate. This describes a causal horizon now, at this precise moment. It does not depends on the history of any particle. In most cases these two quantities are practically the same. For example, taking the particle horizon from the end of recombination, i.e. the optical horizon, this is close to the Hubble radius. However is we take these quantities from the big bang, the quantities could be very different. What would this imply? Essentially this could lead to different scenarios:

- Two particles are now separated by a distance larger than the particle horizon from the Big Bang, they were never in causal contact.
- Two particles are within the Hubble radius, then they are in causal contact now.
- The distance between the particles is greater than the Hubble radius, but less than the particle Horizon. This physically means at some point they were in causal contact in the past but not anymore.

How is this possible? This can be mathematically modelled, this is the central idea behind inflation. It is quite a counter-intuitive idea. Consider the 3rd case in more detail, two particles are to be in causal contact, so they are within each others Hubble radius'. This is followed by a short period of time when the horizon grows exponentially. As the physical particle horizon grows with the universe, the hubble radius remains practically the same, due to the short amount of time that has passed.

Thinking about it in comoving distances, the comoving particle horizon remains the same but the Hubble radius would get smaller. At the start, they would be identical. Then as inflation starts, what you're currently in causal contact with (the comoving radius) reduces to a small patch. Once normal Hubble expansion resumes, the Hubble radius starts to grow again. Concurrently the hubble radius also starts to grow again and then slowly casual contact is regained with some regions where it was lost earlier.

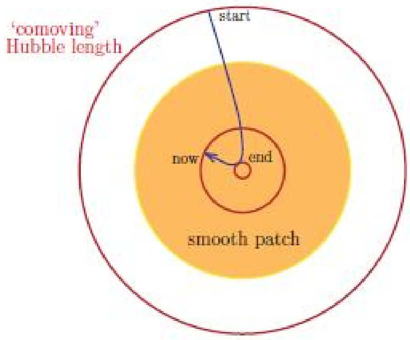

Figure 3.1: The behaviour of the comoving horizon. [4]

The following section will illustrate how this idea solves the Big Bang Problems and essentially generalise the Big Bang model to a range of initial conditions.

3.2 Solving the Big Bang Problems

Conceptually, the horizon problem is solved immediately. As the Hubble radius being initially large, i.e. larger than the Hubble radius now, it can set to be large enough such that the homogeneity required for the CMB is now possible. An interesting question to ask is if it's possible for a homogenous universe to emerge from inflation, starting with any amount of large inhomogeneity. The following argument is taken from Mukhanov's textbook [14]. First we introduce the concept of a perturbation on the energy density

ρ, $\delta\rho$. This varies depending on position and also evolves with time. Taking the perturbation at a specific time removes it's dependence on time.

$$(\frac{\delta\rho}{\rho})_{t_i} \sim \frac{1}{\rho}\frac{|\nabla\rho|}{a_i}H_i^{-1} = \frac{|\nabla\rho|}{\rho}\frac{1}{\dot{a}_i} \tag{3.1}$$

The ∇ is introduced as a spatial derivative in terms of comoving coridnates. ($\frac{\partial\rho}{\partial x^i} = \nabla\rho(a)^{-1}$), where the subscript i indicates the initial time of inflation. Here we assume the perturbation is of order of the Hubble radius, $(H)^{-1}$, making the order of the above equation to be about 1.

At later times t the perturbation grows. While they are still within the Hubble scale they are taken to be:

$$\left(\frac{\delta\rho}{\rho}\right)_t \sim \frac{1}{\rho}\frac{|\nabla\rho|}{a(t)}H(t)^{-1} \sim O(1)\frac{\dot{a}_i}{\dot{a}(t)} \tag{3.2}$$

due to $(\frac{\delta\rho}{\rho})_t \propto \frac{1}{a(t)}$.

The actual comoving derivative of ρ, $\frac{|\nabla\rho|}{a_i}$ is assumed not to change, due to it being a comoving scale. The perturbation are assumed not to evolve themselves within the comoving scales. (3.2) shows that as $\dot{a}$ grows, the scale of inhomogeneity get's smaller. So within the Hubble patch, the inhomogeneities are being ironed out. One can think of the inhomogeneity as a wave and the Hubble horizon is becoming concentrated on a tiny segment of it, and therefore at a later time, what is left of the perturbation with the Hubble radius is considered homogeneous. Figure 3.2 below shows a representation of this behaviour.

These large inhomogeneities are therefore removed during inflation. This argument considered inhomogeneities at the scale of the horizon. However inhomogeneities on quantum scales play an important role in structure formation and require further study. Chapter 4 summarises a simple treatment for studying these perturbations.

3.3 Defining Inflation

How exactly can Inflation be defined Mathematically? The first definition is essentially a shrinking comoving Hubble radius:

$$\frac{d}{dt}\frac{1}{aH} = \frac{d}{dt}\frac{a}{a\dot{a}} = \frac{\ddot{a}}{\dot{a}^2} < 0 \tag{3.3}$$

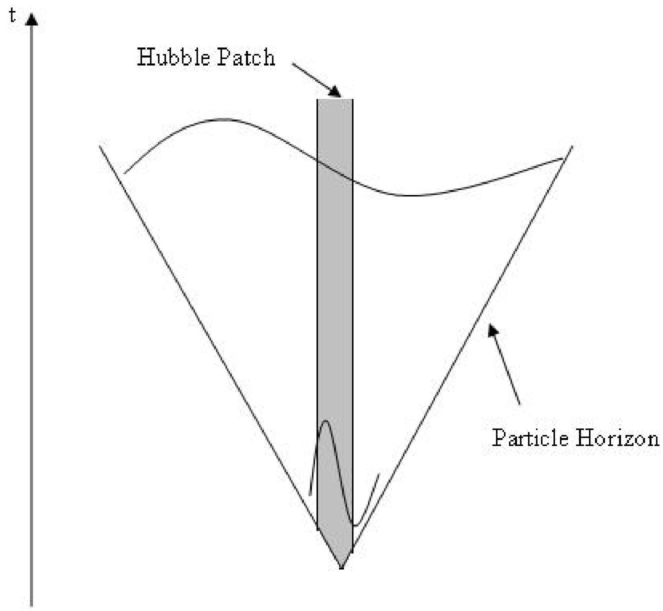

Figure 3.2: Diagram show evolution of a inhomogeneity as a wave.

$\dot{a}^2$ is positive and non-zero. This implies that $\ddot{a} > 0$, which is an accelerating universe,a property of inflation. Looking at the second Friedmann equation (2.45) gives $(\rho + 3p)$ as negative for an accelerating universe. This breaks the strong energy condition (a positive $(\rho + 3p)$).

A third expression for the definition of inflation uses this breaking of the strong energy condition: $p \leq \frac{-1}{3}$. This implies a negative pressure, like Λ type material. Could the de Sitter universe be used as a starting point for modelling inflation? One problem though, inflation can not last for ever. There has to be a point where inflation stops and the universe returns to an accelerated Friedmann state. This is known as the **graceful exit**. Too much inflation results in too much negative gravitational forces and the universe would fail to form. Figure (figadot) below shows the ideal behaviour of $\dot{a}$.

A method for visualising the effect inflation has on the Friedmann universe is to compare **conformal diagrams**. Taking the path of light rays and plotting them against an axis of time and space. Generally the upright vertical axis represents time moving forward. The horizontal axis represents the 3 space dimensions. A light ray is then represent by a line as 45^o,

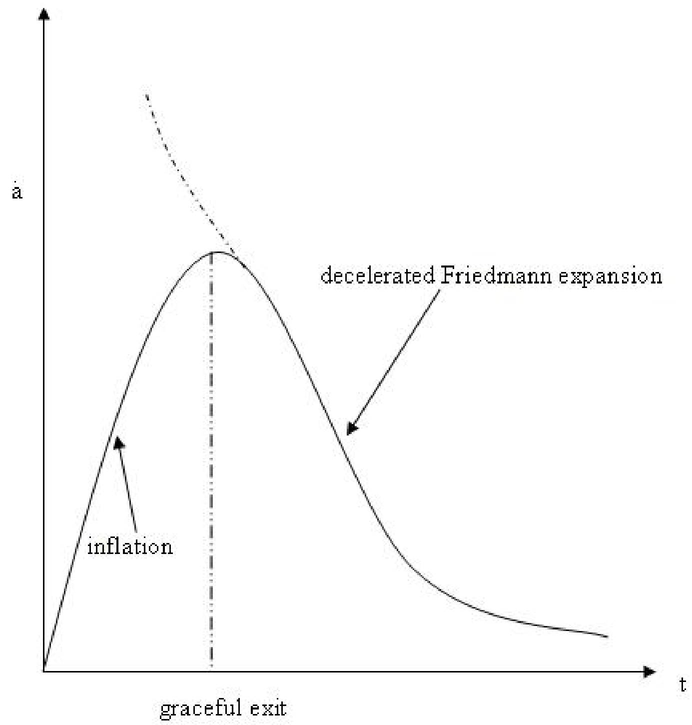

Figure 3.3: The ideal behaviour of $\dot{a}$. [14]

$d\eta = \sqrt{d\vec{x}}$. Taking a time η_0 as now we draw 45^o lines from there represents a light cone going back in time. For two point to have been in causal contact, their light cones would have to have intersected at some point in the past. Using the results for the development of η with respect to a 2.70 for the matter and radiation dominated universes, the relation of a with respect to η follows:

$$\begin{array}{c} \omega = \frac{1}{3} \quad a(\eta) \propto \eta \quad \text{Radiation Dominated} \\ \omega = 0 \quad a(\eta) \propto \eta^2 \quad \text{Matter Dominated} \end{array} \tag{3.4}$$

These relations imply that for $a = 0$, i.e. the Big Bang, $\eta = 0$, i.e. in a universe which has always been radiation and matter dominated, there is no possible negative conformal time. This adds a lower limit to the conformal diagram. Recombination happened relatively shortly after the Big Bang. Picking two points at recombination it's can be seen that they were never is causal contact. This is another way of viewing the horizon problem. The conformal diagram for this Friedmann universe is below Figure (3.4).

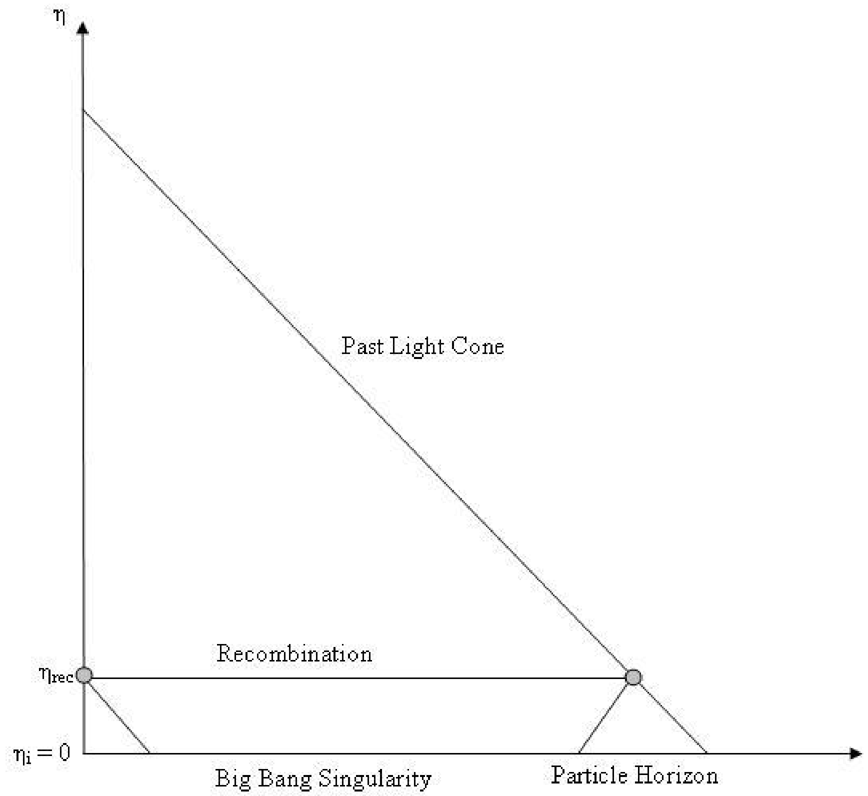

Figure 3.4: Conformal Diagram for the Friedmann universe. [4]

Including a very simple model of inflation into the diagram will solve this problem. At this point it will be assumed that the universe had a period of time where it was dominated by Λ type material. Essentially the universe was de Sitter space. For de Sitter space, as H is a constant, a in terms of conformal time was derived as using (2.70):

$$a(\eta) = -\frac{1}{H\eta} \propto \eta^{-1} \tag{3.5}$$

Taking the limit as a tends to 0 results in $\eta \to -\infty$. This period of inflation therefore takes place in negative conformal time. The Big Bang now takes places at $\eta \to -\infty$. While $\eta = 0$ represents a time when inflation ends. At this point a is tending towards infinity. This is why de Sitter space is not an exact model for inflation and a graceful exit is needed. But looking at the conformal diagram, the light cone for two points at the point of recombination (after inflation ends) will always intersect at some point throughout inflation and solve the horizon problem.

For the model to work a time evolving Hubble rate is required, but remaining similar to the de Sitter Hubble rate. Essentially $\dot{H}$ has to remain small until the end of inflation. From $H = (\frac{\dot{a}}{a})$ and differentiating both sides with time we get an alternate form for Friedmann's second equation (2.45).

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} \tag{3.6}$$

Looking at Figure(3.3) it can be seen that at the point of graceful exit $\ddot{a}$ is negative. Taking a derivative of the right side shows the terms changing at a rate of $2H\dot{H}$ and $\ddot{H}$ respectively. Assuming that $2H\dot{H} > |\ddot{H}|$ and that at the end of inflation, at t_f, $\ddot{a} \to 0$ then an estimate for t_f is $\frac{H_i}{|\dot{H}_i|}$. For the inhomogeneities to have been washed out by the CMB the condition $\dot{a}_i/\dot{a}_0 < 10^{-5}$ is needed. Using the chain rule and the definition of H in terms of a this is rewritten as:

$$\frac{\dot{a}_i}{\dot{a}_f}\frac{\dot{a}_f}{\dot{a}_0} = \frac{a_i}{a_f}\frac{H_i}{H_f}\frac{\dot{a}_f}{\dot{a}_0} < 10^{-5} \tag{3.7}$$

To get a further idea of scales, a numerical estimate for the ratio $\frac{\dot{a}_f}{\dot{a}_0}$ is needed. This is possible by looking at how the temperate of radiation has changed. The Planckian temperature was the primordial temperature, this was about 10^{32}K. The temperature today is of order 1. Temperature follows an inverse relation to the scale factor. Therefore:

$$\frac{\dot{a}_f}{\dot{a}_0} \sim \frac{a_f}{a_0}\frac{t_0}{t_f} \sim \frac{T_0}{T_{Pl}}\frac{t_0}{t_f} \tag{3.8}$$

t_f is estimated at about 10^{-43} seconds and t_0 is 10^{17} seconds. All this gives:

$$\frac{\dot{a}_f}{\dot{a}_0} \sim 10^{28} \tag{3.9}$$

(3.7) now becomes

$$\frac{a_f}{a_i} > 10^{33}\frac{H_i}{H_f} \tag{3.10}$$

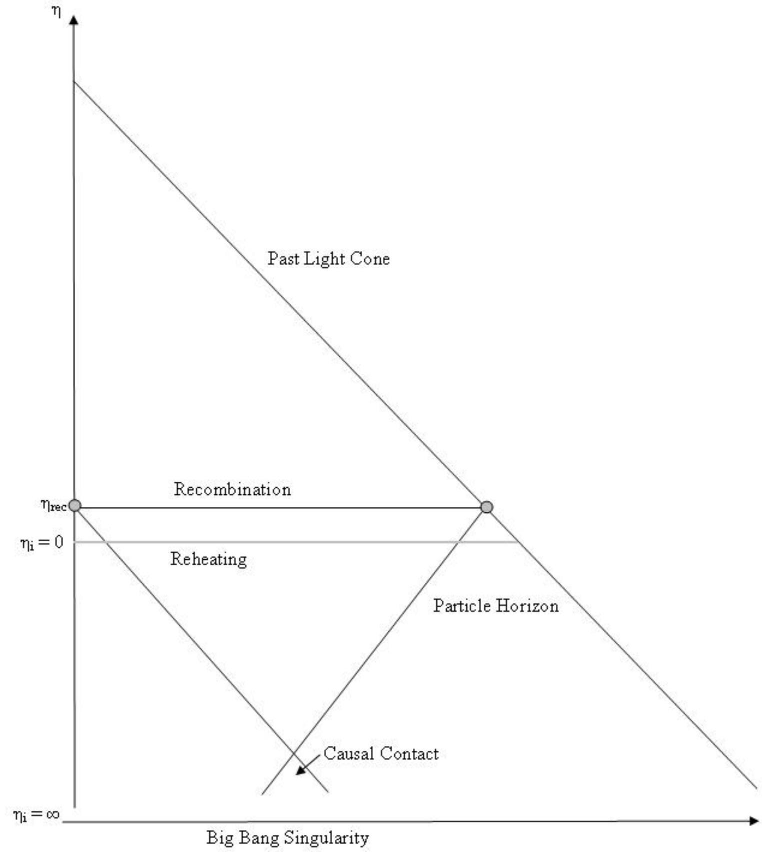

Figure 3.5: Conformal Diagram for an Inflationary universe. [4]

Simplifying (3.6) to just the H^2 term gives the following solution for a:

$$a_f = e^{H\Delta t} a_i \tag{3.11}$$

$\Delta t \sim t_f$ as $t_i \sim 0$ and using $t_f \sim \frac{H_i}{|\dot{H}_i|}$ gives:

$$\frac{a_f}{a_i} \sim e^{H\Delta t_f} \sim exp\left(\frac{H_i^2}{|\dot{H}_i|}\right) \tag{3.12}$$

Assuming the order of H does not change significantly, then (3.10) becomes:

$$exp\left(\frac{H_i^2}{|\dot{H}_i|}\right) > 10^{33} \tag{3.13}$$

Taking natural log on both sides gives $\frac{H_i^2}{|\dot{H}_i}| > 75$ or $t_f > 75(H_i)^{-1}$. A time unit of $(H_i)^{-1}$ is known as an e-folding or a Hubble time. Inflation then has to last 75 e-folds for both the inhomogeneities to iron out and a graceful exit to occur. This is normally seen in the following form:

$$\frac{|\dot{H}_i|}{H_i^2} < \frac{1}{75} \tag{3.14}$$

We can use this form of the equation and both Friedmann equations to estimate how accurate the de Sitter universe is for inflation. Taking K=0 and G=1 gives:

$$\frac{4\pi(\rho + p)}{8\pi\rho} < \frac{1}{75} \tag{3.15}$$

$$\frac{(\rho + p}{\rho} = (1 + \omega) < \frac{2}{225} \sim 10^{-2} \tag{3.16}$$

$\omega = -1$ makes (3.16) zero, which is or the order of 10^{-2}. Hence de Sitter is a good approximation. However the actual density of a more accurate model can't deviate from purely de Sitter by more than about 1

Inflation also solves the flatness problem. Looking at the first Friedmann equation in the following form shows the importance of a shrinking comoving Hubble radius.

$$\Omega(t) - 1 = \frac{K}{(H(t)a(t))^2} \tag{3.17}$$

With $(aH)^{-1}$ getting smaller, then $\frac{1}{(Ha)^2}$ is growing. The case where $K = 0$ and $\Omega(t) = 1$ is then an attractor solution for the equation. Previously at aH just grew, K could take any value. K=1 was a requirement for $\Omega(t) = 1$. Now this is a solution.

3.4 Introducing the scalar field

As a result of needing a graceful exit, it is not possible to model a period of time of inflation as a universe with just some cosmological constant type material. Λ type material would just cause this rapid expansion to continue and cause eternal inflation. Instead a more dynamical model is needed, which has similar properties to Λ but also decays after 75 e-folds. So far it has been useful to model matter using fluid mechanics, so developing something similar would allow easy comparison with other models. One such possibility is introducing a scalar field, called the **inflaton**, $\varphi(\vec{x}, t)$.

This is a classical scalar field, i.e. a value is assigned to each point in spacetime. (The next chapter will outline the quantum case.) This scalar field has a potential $V = (\varphi(\vec{x}, t))$. The energy-momentum tensor of a scalar field is defined as:

$$T^{\alpha}_{\beta} = \varphi^{,\alpha}\varphi_{,\beta} - (\frac{1}{2}\varphi^{,\gamma}\varphi_{,\gamma} - V(\varphi))\delta^{\alpha}_{\beta} \tag{3.18}$$

Defining the following as the energy density, pressure and normalised 4-velocity the energy-momentum becomes that of a perfect fluid (2.42).

$$\begin{aligned} \rho &= \tfrac{1}{2}\varphi^{,\gamma}\varphi_{,\gamma} + V(\varphi) \\ p &= \tfrac{1}{2}\varphi^{,\gamma}\varphi_{,\gamma} - V(\varphi) \\ u^{\alpha} &= \tfrac{\varphi^{,\alpha}}{\sqrt{\varphi^{,\gamma}\varphi_{,\gamma}}} \end{aligned} \tag{3.19}$$

Assuming homogeneity, $\partial\varphi/\partial x^i = 0$ the equations reduce to:

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \tag{3.20}$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \tag{3.21}$$

This is still not exactly what is needed. The condition required is $p < -\frac{1}{3}\rho$. Currently this is $p = -\rho + \dot{\varphi}^2$. Essentially the behaviour of the inflaton will determine whether the strong energy condition is broken or not. Therefore another condition needed is to make $\dot{\varphi} \ll V(\varphi)$, that is the kinetic energy of the inflation, defined as $1/2\dot{\varphi}^2$, is less than it's potential energy. When the potential energy is a lot larger, the strong energy condition breaks and inflation occurs. $V(\varphi)$ need to become smaller and then the kinetic energy can dominate again. This has to last the 75 e-folds, so that the Big Bang problems are solved. This is when inflation ends and the graceful exit occurs. Now this potential $V(\varphi)$ can change is a variety of ways. Some of the more prominent versions of $V(\varphi)$ are outlined in the next section. But when it comes to the kinematics during inflation they are all reduce to the same equations due to the large $V(\varphi)$.

Solving the equation for the inflaton in the Klein-Gordon equation ($\varphi_{;\alpha}^{;\alpha} + \frac{\partial V}{\partial \varphi}$) will give an equation of motion. This can also be achieved using the continuity equation (2.48). Using

$$\rho + p = \dot{\varphi}^2 \;\; \dot{\rho} = \dot{\varphi}\ddot{\varphi} + \dot{\varphi}V_{,\varphi}$$

Gives $\dot{\varphi}(\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi}) = 0$. Assuming $\dot{\varphi}$ is non-zero we then get the continuity equation for the inflaton as follows:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0 \tag{3.22}$$

The other equation of motion used to describe the inflation is simply Friedmann's equation for a flat universe (2.44):

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)) \tag{3.23}$$

Using these two equations, we can now model the behaviour of the inflaton throughout inflation. G is set to 1.

3.4.1 The Slow Roll Approximation

One of the most widely understood model in physics is that of the harmonic oscillator and its many variations. (3.22) takes the form of an oscillator with a friction term. This friction term is $\propto$ H. These oscillators have a certain type of behaviour when the friction term dominates over the acceleration term, this is known as the **Slow-Roll Regime**. In the case of the inflaton:

$$|\ddot{\varphi}| \ll 3H\dot{\varphi} \tag{3.24}$$

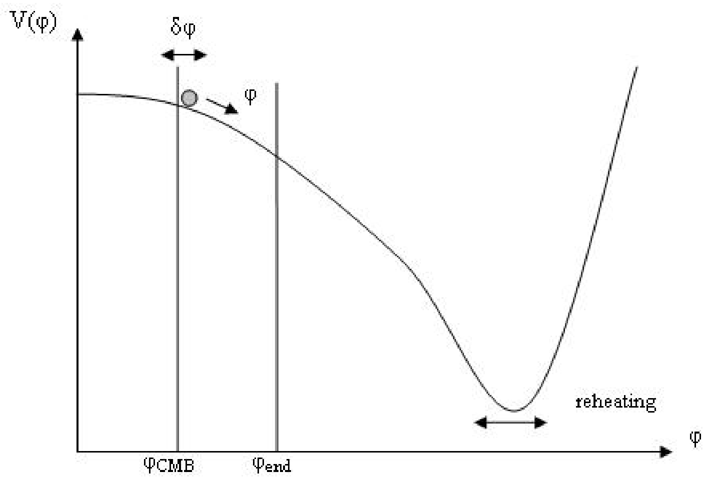

Figure 3.6: A typical potential. [4]

Making (3.22):

$$3H\dot{\varphi} \simeq -V_{,\varphi} \tag{3.25}$$

The other condition that need to be included was discussed in the previous section, i.e. the potential energy of the inflation is far greater than its kinetic energy. This gives an approximated form of the Hubble rate from (3.23).

$$H = \sqrt{\frac{8\pi}{3}}\sqrt{\rho} \simeq \sqrt{\frac{8\pi}{3}}\sqrt{V(\varphi)} \tag{3.26}$$

Overall we have a potential term which in turn is much larger than the velocity of the inflation, which in turn is much larger than it's acceleration. $V(\varphi) \ll |\dot{\varphi}| \ll |\ddot{\varphi}|$. H can also take a logarithmic form in terms of a, $\frac{dlna}{dt}$. A solution for the φ can now be found:

$$\begin{gathered} \frac{dlna}{dt} = \frac{d\varphi}{dt}\frac{dlna}{d\varphi} = -\frac{V_{,\varphi}}{3H}\frac{dlna}{d\varphi} \\ \sqrt{\frac{8\pi}{3}V} = -\frac{V_{,\varphi}}{3H}\frac{dlna}{d\varphi} \end{gathered} \tag{3.27}$$

This is rearranged as an integral with limits taken from an initial time, marked with a subscript i to a general time after. This is left as the variable without a subscript.

$$-8\pi\int_{\varphi_i}^{\varphi}\frac{V}{V_{,\varphi}}d\varphi=\int_{a_i}^{a}lna \tag{3.28}$$

Giving a general solution for a under slow roll conditions:

$$a(\varphi)=a_i exp(8\pi\int_{\varphi}^{\varphi_i}\frac{V}{V_{,\varphi}}d\varphi) \tag{3.29}$$

The slow roll conditions themselves ($|\dot{\varphi}^2|\ll V$, $|\ddot{\varphi}\ll 3H\dot{\varphi}\sim|V_{,\varphi}|$)can be expressed in terms of the potential, which is what changes for the different inflation models. By using (3.25, 3.26):

$$|V_{,\varphi}|\simeq 3H\dot{\varphi}|\Rightarrow {V_{,\varphi}}^2\simeq 9H^2\dot{\varphi}^2{V_{,\varphi}}^2\simeq 24\pi V\varphi\dot{\varphi}^2\sim V\varphi\dot{\varphi}^2 \tag{3.30}$$

$$|\dot{\varphi}^2|\ll V\Rightarrow(\frac{V_{,\varphi}}{V})^2 \tag{3.31}$$

For the second slow roll condition:

$$\begin{gathered}\dot{H}\simeq V_{,\varphi}\dot{\varphi}\sqrt{\tfrac{8\pi}{3V(\varphi)}}=\tfrac{V_{,\varphi}}{V}\dot{\varphi}H=-\tfrac{1}{3}\tfrac{{V_{,\varphi}}^2}{V}\\ \ddot{\varphi}=\tfrac{\dot{\varphi}}{H}(\tfrac{V_{\varphi\varphi}}{3}-\dot{H})\\ \ddot{\varphi}=\tfrac{\dot{\varphi}}{3H}(V_{\varphi\varphi}+\tfrac{{V_{,\varphi}}^2}{V})\\ \ddot{\varphi}=-\tfrac{\dot{\varphi}}{8\pi}(\tfrac{V_{\varphi\varphi}}{V}+\tfrac{{V_{,\varphi}}^2}{V^2})\\ |\ddot{\varphi}|\ll|V_{,\varphi}|\Rightarrow|\tfrac{\dot{\varphi}}{8\pi}(\tfrac{V_{\varphi\varphi}}{V}+\tfrac{{V_{,\varphi}}^2}{V^2})|\ll|V_{,\varphi}|\\ |\tfrac{V_{\varphi\varphi}}{V}|+(\tfrac{V_{,\varphi}}{V})^2|\ll 1\end{gathered}$$

From the previous condition, we know the second term is quite small, this then gives:

$$|\frac{V_{\varphi\varphi}}{V}|\ll 1 \tag{3.32}$$

Both these condition represent a regime in any inflation model, where the expansion will occur. Once these conditions are broken, inflation will end and a graceful exit to the expanding Friedmann universe can occur.

3.5 Reheating and Preheating

At the end of inflation, the universe is left with φ at a low potential. Reheating is the method by which the universe repopulates itself. The universe now is made up of fermions (ψ and it's spinor complement $\bar{\psi}$) and bosons (ξ). Having φ decay into such matter allows for a repopulation of the universe. Looking at most models of inflation, they have a potential well. The inflaton's potential is found in these wells at inflation's end. However the potential can oscillate, with these oscillations comes the decay into fermions and bosons. The coupling is done through the Lagrangian:

$$\Delta L = -g\varphi\chi^2 - h\varphi\bar{\psi}\psi \tag{3.33}$$

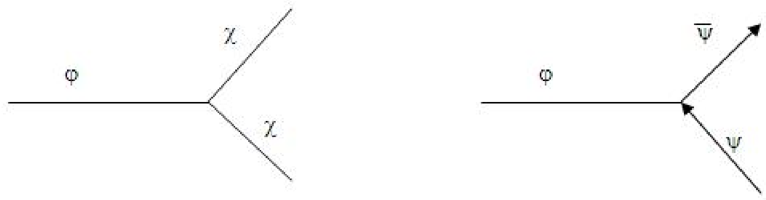

Figure 3.7: The decay of φ to bosons and fermions

The subject of rehating is vast. A good review of the subject can be found here [15]. When the coupling is non linear it is known as preheating. This causes the repopulation to occur faster.

3.6 Different models

Different inflation models have been developed over the years with different potentials. Shown below are some single scalar field models.

Other methods of getting repulsive gravity, higher derivative gravity and inflation with more than one scalar field. For further details on these alternative theories Mukhanov's [14] Chapter 5.6 has a quick review of these models. While Baumann's lectures, Lecture 1 Chapter 6.5, give a little more detail on other theories that aren't scalar field based [4].

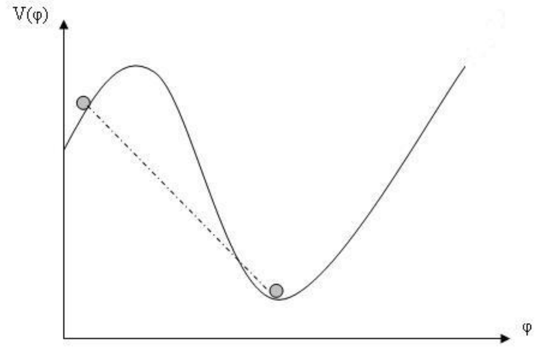

Figure 3.8: Old Inflation [14]

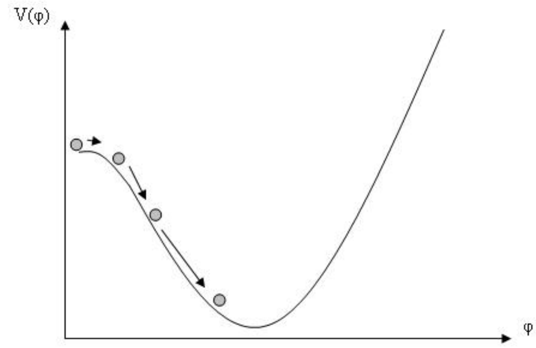

Figure 3.9: New Inflation [14]

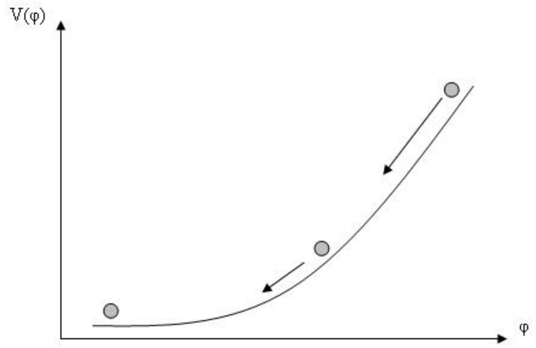

Figure 3.10: Chaotic Inflation [14]

4 Perturbations

4.1 What is a Perturbation?

The previous sections have dealt with the homogeneous Big Bang and Inflation models. In this section we will treat the quantities that are being measured as follows:

$$\delta X(\vec{X},\eta) = X(\vec{X},\eta) -^{(0)} X(\eta) \tag{4.1}$$

There is a background value, ${}^{(0)}X(\eta)$ that depends only on conformal time, and therefore are homogeneous. These represent the results from the previous chapter. $\delta X(\vec{X},\eta)$ are the perturbations on the homogeneous background, and in general $\delta X(\vec{X},\eta) \ll^{(0)} X(\eta)$. X could be any measured value, φ, T, G etc. The problem though comes when deciding how to split up X. There are many different possible way of achieving this, dependent on coordinate choice.

There is also a dependency on which *gauge* is chosen. To gain a deeper understanding of gauge choice, it's worth looking at the details of the geometry again. Thinking of X and ${}^{(0)}X$ as living on two different manifolds. To correspond between these two manifolds there is need of a function. A gauge choice is this function. δX is dependent on this gauge choice. To say δX is gauge-invariant implies that the correspondence between the manifolds is not dependent on which map is chosen between the manifold representing homogeneous spacetime and the one representing perturbed spacetime.

The dependency of δX on both the coordinate and gauge choices can create what are known as **fictitious perturbations**. Assume X is a homogenous quantity: $X(\vec{x},\eta) = X(\eta)$. Then the following coordinate transformation is possible in GR: $\tilde{\eta} = \eta + \delta\eta(\vec{x},\eta)$. The quantity now is dependent on $\tilde{\eta}$ and $\vec{x}$: $\tilde{X}(\vec{x},\tilde{\eta}) = (\eta(\vec{x},\tilde{\eta}))$.

$$X(t) = X((\tilde{\eta}) - \delta\eta(\vec{x}, t)) \simeq X(\tilde{\eta}) - \frac{\partial X}{\partial \eta}\delta\eta \equiv X\tilde{\eta} + \delta X(\vec{x}, \tilde{\eta}) \qquad (4.2)$$

On the left hand side, there is what can be interpreted as a background quantity. The term on the right hand side is that of a linear perturbation of the same quantity. This perturbation does not exist as the assumption was that X was indeed homogenous! Hence the production of a fictitious perturbation. The converse is also true, a real homogeneity can be removed through a coordinate choice. To avoid any ambiguity it is essential to consider both perturbations to the quantities that are being studied and those to the metric itself. Using a specific gauge transformation it is possible to go between these. It is important to use gauge-invariant quantities in order to avoid the generation of any further perturbations. (By definition a gauge-invariant quantities will be physical and can not be removed via a coordinate transformation.)

The calculations in this chapter follow the ones Mukhanov outlines in his textbook [14] and were used in his original papers [13]. Further details on perturbations and gauge choices can be found there or in a recent review on cosmic perturbations [16].

The aim is to show how the quantum perturbations of the inflaton, go on to cause a power spectrum of inhomogeneities of the curvature of spacetime. This power spectrum is then linked to the gravitational instabilities and caused the large scale structure formation of the universe.

4.2 Some Basic Quantities

4.2.1 Metric Perturbations

In this section the metric perturbations are calculated, the following will deal with the development of the perturbations of the inflaton. Assuming a flat (K=0) universe the metric can be expressed as:

$$ds^2 = [^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}(x^\gamma)]dx^\alpha dx^\beta \qquad (4.3)$$

As mentioned in the introduction the perturbed quantity is much smaller than the background quantity. Below is the metric using conformal time.

$$^{(0)}g_{\alpha\beta}dx^{\alpha}dx^{\beta} = a^2(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j) \quad (4.4)$$

$\delta g_{\alpha\beta}$ can be a scalar, vector or tensor quantity. The metric was defined to have a homogenous, isotropic background. At a given moment in time it is also invariant under spatial rotations and translations. Taking these limitations we have three possible sets of perturbations. One based on the time-time components, one on the time-space components and finally on the space-space components. Each of these in turn has scalar, vector or tensor terms in the perturbations.

$$\delta g_{00} = 2a^2\phi \quad (4.5)$$

Here ϕ is a 3-scalar.

$$\delta g_{0i} = a^2(B_{,i} + S_i) \quad (4.6)$$

Limitations on B are $B_{,i} = \frac{\partial B}{\partial x^i}$ where B is a scalar. S_i is a 3 vector with 2 independent co-ordinates as $S^i_(, i) = 0$.

$$\delta g_{ij} = a^2(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}) \quad (4.7)$$

Where ψ and E are scalars. F_i is a vector such that $F^i_{,i} = 0$. h_{ij} is a 3-tensor (i,j = 1,2,3) such that $h^i_i = 0$ (traceless) and $h^i_{j,i} = 0$ (transverse).

In total there are ten functions for the metric perturbations:

- Four functions for scalar perturbations (ϕ, ψ, B, E) These are the perturbations that arise from ρ and will lead to gravitational instabilities and structure formation.
- Four functions for vector perturbations (S, F) Each 3 vector has one conditions, so 2 functions each. These decay quickly, so do not play an important cosmological role.
- Two functions on tensor perturbations (One symmetric 3-tensor, so 6 individual components with a traceless and transverse constraints. This removes 4 components, leaving 2.) They go on to describe gravitational waves which are yet to be detected. For further analysis look at [14].

4.2.2 Gauge transformations

Having looked at the perturbations of the metric, it is now important to derive gauge-invariant quantities which can be used to express the metric perturbations. Consider a simple coordinate transformation:

$$x^{\alpha} \rightarrow \tilde{x}^{\alpha} + \xi^{\alpha} \tag{4.8}$$

Again ξ^{α} is very small and is inhomogenous: $\xi^{\alpha}(\vec{x}, t)$ Using this transformation we can use the following to link the metric measured in the two different sets of coordinates:

$$\tilde{g}_{\alpha\beta}(\tilde{x}^{\rho}) = \frac{\partial x^{\gamma}}{\partial \tilde{x}^{\alpha}} \frac{\partial x^{\delta}}{\partial \tilde{x}^{\beta}} g_{\gamma\delta} \tag{4.9}$$

Differentiating gives:

$$\frac{\partial x^{\gamma}}{\partial \tilde{x}^{\alpha}} = \delta^{\gamma}_{\alpha} - \xi^{\gamma}_{,\alpha}$$

(4.9) now becomes, to first order:

$$\tilde{g}_{\alpha\beta}(\tilde{x}^{\rho}) = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} g_{\gamma\delta}(x^{\rho}) - \delta^{\delta}_{\beta}\xi^{\gamma}_{,\alpha} g_{\gamma\delta} - \delta^{\gamma}_{\alpha}\xi^{\delta}_{,\beta} g_{\gamma\delta} \tag{4.10}$$

Splitting the metric values to the homogeneous and inhomogeneous gives:

$$g_{\alpha\beta}(x^{\rho}) =^{(0)} g_{\alpha\beta}(x^{\rho}) + \delta g_{\alpha\beta}$$
$$\tilde{g}_{\alpha\beta}(\tilde{x}^{\rho}) =^{(0)} g_{\alpha\beta}(\tilde{x}^{\rho}) + \delta \tilde{g}_{\alpha\beta}$$

Using the cooridnate transformation gives:

$$g_{\alpha\beta}(x^{\rho}) =^{(0)} g_{\alpha\beta}(\tilde{x}^{\rho} - \xi^{\rho})$$

As ξ^{ρ} is small, the perturbation can be thought of in terms of the differential:

$$g_{\alpha\beta}(x^{\rho}) \approx^{(0)} g_{\alpha\beta}(\tilde{x}^{\rho}) -^{(0)} g_{\alpha\beta,\gamma}\xi^{\gamma}$$

Substituting all the above in (4.10) gives the following relation between the metric perturbations in different coordinate systems, a **gauge transformation law**.

$$\delta \tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} -^{(0)} g_{\alpha\beta,\gamma}\xi^{\gamma} -^{(O)} g_{\gamma\beta}\eta^{\gamma}_{\alpha} -^{(0)} g_{\alpha\delta}\eta^{\delta}_{\beta} \tag{4.11}$$

To calculate the specific perturbations of the metric it is useful to introduce the following changes to the coordinate transformation: $\xi^\alpha = (\xi^0, \xi^i)$ where $\xi^i = \xi^i_\perp + \zeta^{,i}$ where $\xi^i_\perp = \xi_{\perp i}$ and has zero divergence ($\xi^i_{\perp,i} = 0$) and ζ is a scalar. The homogenous metric for a Friedmann universe has components ${}^{(0)}g_{00} = \alpha^2(\eta)$ and ${}^{(0)}g_{ij} = -\alpha^2(\eta)\delta_{ij}$.

$$\begin{aligned}
\delta g_{00} \to \delta\tilde{g}_{00} = \delta g_{00} -^{(0)} g_{00,0}\xi^0 - 2^{(0)}g_{00}\xi^0_{,0} \\
\delta\tilde{g}_{00} = \delta g_{00} - \tfrac{\partial a^2(\eta)}{\partial\eta}\xi^0 - 2a^2(\eta)\tfrac{\partial\xi}{\partial\eta} \\
\delta\tilde{g}_{00} = \delta g_{00} - 2a(a'(\eta)\xi^0 + a(\eta)\xi^{0\,\prime}) \\
\delta\tilde{g}_{00} = \delta g_{00} - 2a(a\xi^0)'
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\delta g_{0i} \to \delta\tilde{g}_{0i} = \delta g_{01} -^{(0)} g_{0i,\gamma}\xi^\gamma - (0)g_{\gamma i}\xi^\gamma_{,0} - (0)g_{0\gamma}\xi^\gamma_{,i} \\
\delta\tilde{g}_{0i} = \delta g_{01} + a^2(\eta)[\delta_j i\xi^j_{,0} - \xi^0_{,i}] \\
\delta\tilde{g}_{0i} = \delta g_{01} + a^2(\eta)[[\xi^i_\perp + \zeta^{,i}]_{,0} - \xi^0_{,i}] \\
\delta\tilde{g}_{0i} = \delta g_{01} + a^2(\eta)[[\xi^{i\,\prime}{}_\perp + (\zeta' - \xi^0)_{,i}]
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
\delta g_{ij} \to \delta\tilde{g}_{ij} = \delta g_{ij} - {}^{(0)}g_{ij,0}\xi^0 - (0)g_{\gamma j}\xi^\gamma_{,i} - (0)g_{i\gamma}\xi^\gamma_{,j} \\
\delta\tilde{g}_{ij} = \delta g_{ij} + a^2(\eta)_{,0}\delta_{ij}\zeta^0 + a^2(\eta)[\delta_{kj}\zeta^k_{,i} + \delta_{ik}\zeta^k_{,j}] \\
\delta\tilde{g}_{ij} = \delta g_{ij} + a^2(\eta)[2\tfrac{a'}{a}\delta_{ij}\zeta^0 + 2\zeta_{,ij} + (\zeta^j_{,i} + \zeta^i_{,j})]
\end{aligned} \tag{4.14}$$

These are the different metric perturbations. Only taking the metric scalar perturbations gives the following metric:

$$ds^2 = a^2[(1+2\phi)d\eta^2 + 2B_{,i}dx^i d\eta - ((1-2\psi)\delta_{ij} - 2E_{,ij})dx^i dx^j] \tag{4.15}$$

Using (4.12,4.13, 4.14) and taking the values from (4.15) gives the following coordinate transformations for the scalar perturbations of the metric. Here $\xi_\perp$ has been ignored due to its vector nature.

$$\begin{aligned}
2a^2\phi \to 2a^2\phi - 2a(a\xi^0)' \\
\phi \to \phi - \tfrac{1}{a}(a\xi^0)'
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
a^2B_{,i} \to a^2(B + \zeta' - \xi^0)_{,i} \\
B \to B + \zeta' - \xi^0
\end{aligned} \tag{4.17}$$

$$
\begin{aligned}
a^2(2\psi\delta_{ij} - 2E_{,ij}) \rightarrow a^2(2\psi\delta_{ij} + 2\tfrac{a'}{a}\delta_{ij}\xi^0 - 2E_{,ij} + 2\zeta, ij) \\
\psi \rightarrow \psi + \tfrac{a'}{a}\xi^0 \\
E \rightarrow E + \zeta
\end{aligned}
\tag{4.18}
$$

Therefore the only elements of the coordinate transformations ($\tilde{x}^\alpha \rightarrow x^{\alpha+\xi^\alpha}$) that affect the scalar perturbations of the metric are ξ^0 and ζ. The four scalar functions (ϕ,ψ, B, E) can be used to create two new scalar functions. The simplest one of these, which is in fact gauge-invariant, is the Longitudinal (Newtonian) Gauge.

$$
\begin{aligned}
\Phi = \phi - \tfrac{1}{a}[a(B - E')]' \\
\Psi = \psi + \tfrac{a'}{a}(B - E')
\end{aligned}
\tag{4.19}
$$

The following shows the above Scalar Perturbations of the metric are not affected by coordinate transformations:

$$
\begin{aligned}
\Phi \rightarrow \tilde{\Phi} = \tilde{\phi} - \tfrac{1}{a}[a(\tilde{B} - \tilde{E}')]' \\
\tilde{\Phi} = \phi - \tfrac{1}{a}(a\xi^0)' - \tfrac{1}{a}[a(B + \zeta' - \xi^0 - E' - \zeta')]' \\
\tilde{\Phi} = \phi - \tfrac{a'}{a}\xi^0 - \xi^{0\,\prime} - \tfrac{a'}{a}[B - \xi^0 - E'] - B' + \xi^0 + E'' \\
\tilde{\Phi} = \phi - \tfrac{1}{a}[a(B - E')]' \\
\tilde{\Phi} = \Phi
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
\Psi \rightarrow \tilde{\Psi} = \tilde{\psi} + \tfrac{a'}{a}(\tilde{B} - \tilde{E}') \\
\tilde{\Psi} = \psi + \tfrac{a'}{a}(\xi^0 + b + \zeta' - \xi^0 - E' - \zeta') \\
\tilde{\Psi} = \psi + \tfrac{a'}{a}(\tilde{B} - \tilde{E}') \\
\tilde{\Psi} = \Psi
\end{aligned}
\tag{4.21}
$$

Therefore if they are zero in one co-ordinate, they are zero in all. So setting $B_l = E_l = 0$ removes all fictitious perturbations and the metric becomes:

$$
ds^2 = a^2[(1 + 2\phi_l)d\eta^2 - (1 - 2\psi_l)\delta_{ij}dx^i dx^j] \tag{4.22}
$$

4.2.3 Expressing Perturbations of the Einstein Equation in terms of Metric Perturbations

A coordinate transformation for the perturbations for energy density can be expressed using the above metric perturbations. $\rho(\vec{x},t) = {}^{(0)}\rho(t) + \delta\rho(\vec{x},t)$transform as follows under a co-ordinate transformation:

$$\begin{gathered}
\tilde{\rho}(\tilde{x}^\alpha) = \rho(x^\alpha) \\
{}^{(0)}\rho(\tilde{x}^\alpha) + \delta\tilde{\rho} = {}^{(0)}\rho(x^\alpha) + \delta\rho \\
{}^{(0)}\rho(x^\alpha) = {}^{(0)}\rho(\tilde{x}^\alpha - \xi^\alpha) = {}^{(0)}\rho(\tilde{x}^\alpha) - \rho_{,\gamma}\xi^\gamma \\
\delta\tilde{\rho} = \delta\rho - {}^{(0)}\rho_{,\alpha}\xi^\alpha
\end{gathered} \tag{4.23}$$

As ${}^{(0)}\rho(t)$ is only dependent on time, the only component of ξ that remains is the time component. In the Newtonian Gauge, using the coordinate transformations of the scalar perturbations, gives $\xi^0 = (B - E')$. Resulting in the following:

$$\delta\tilde{\rho} = \delta\rho - {}^{(0)}\rho'(B - E') \tag{4.24}$$

The perturbations of the Einstein equation can also be expressed using the Newtonian gauge. As the Einstein equation is linear, it is easily split between the homogenous background and the perturbation.

$${}^{(0)}G^\alpha_\beta + \delta G^\alpha_\beta G = 8\pi G({}^{(0)}T^\alpha_\beta + \delta T^\alpha_\beta) \tag{4.25}$$

$$\delta G^\alpha_\beta G = 8\pi G \delta T^\alpha_\beta \tag{4.26}$$

Perturbations of G and T are not gauge-invariant. However combining them with a the metric perturbation gauge-invariant quantities are derived. For a general (1,1) tensors (such as G^α_β and T^α_β) a coordinate transforms as follows:

$$\delta\tilde{T}^\alpha_\beta = \delta T^\alpha_\beta + \xi^\alpha_{,\gamma}\,{}^{(0)}T^\gamma_\beta - \xi^\gamma_{,\beta}\,{}^{(0)}T^\alpha_\gamma - {}^{(0)}T^\alpha_{\beta,\gamma}\xi^\gamma \tag{4.27}$$

The different components can then be derived, remembering that only scalar perturbations are of current interest.

$$\delta\tilde{T}^0_0 = \delta T^0_0 - {}^{(0)}T^0_{0,0}\xi^0 - {}^{(0)}T^0_{0,i}\xi^i - {}^{(0)}T^0_{0,i}\xi^i_{\perp}$$
$$\delta\tilde{T}^0_0 = \delta T^0_0 - {}^{(0)}T^0_0{}'(B - E') \tag{4.28}$$

$$\delta\tilde{T}^0_i = \delta T^0_i + \xi^0_{,\gamma}\,{}^{(0)}T^{\gamma}_i - \xi^0_{,i}\,{}^{(0)}T^0_0 - \xi^k_{,i}\,{}^{(0)}T^0_k - {}^{(0)}T^0_{i,\gamma}\xi^{\gamma} \tag{4.29}$$

$$\delta\tilde{T}^i_j = \delta T^i_j + \alpha(\xi^i_{,k}\delta^k_j - \xi^k_{,j}\delta^i_k) - {}^{(0)}T^i_{j,0}\xi^0 - {}^{(0)}T^i_{j,k}\xi^k$$
$$\delta\tilde{T}^i_j = \delta T^i_j + {}^{(0)}T^i_j{}'(B - E') \tag{4.30}$$

As ${}^{(0)}T^i_j \propto \delta^i_j$, then each diagonal elements of ${}^{(0)}T^i_j$ is the trace divided by 3, i.e. $\frac{{}^{(0)}T^k_k}{3}$. G^i_j has similar perturbations. This gives the following perturbations of the Einstein equation:

$$\delta\tilde{G}^{\alpha}_{\beta} = 8\pi G\delta\tilde{T}^{\alpha}_{\beta} \tag{4.31}$$

Using the following components of the Einstein tensor in terms of conformal time:

$$(0)G^0_0 = \tfrac{3\mathcal{H}^2}{a^2}$$
$${}^{(0)}G^0_i = 0$$
$${}^{(0)}G^i_j = \tfrac{1}{a^2}(2\mathcal{H}' + \mathcal{H}^2)\delta^i_j \tag{4.32}$$

Gives the following perturbations of the Einstein equation:

$$\nabla\psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi Ga^2\delta\tilde{T}^0_0 \tag{4.33}$$

$$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi Ga^2\delta\tilde{T}^0_i \tag{4.34}$$

$$[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{2}\nabla(\Phi - \Psi)]\delta^i_j - \frac{1}{2}(\Phi - \Psi)_{,ij} = -4\pi Ga^2\delta\tilde{T}^i_j \tag{4.35}$$

These perturbations of the Einstein equation have been expressed using the chosen Newtonian gauge. This can be reversed using the definitions of the gauge and then set to another gauge.

4.3 Perturbing the Inflaton

As done in the previous section the variable is separated into a homogenous component and its perturbation:

$$\varphi(\vec{x},\eta) = \varphi_0(\eta) + \delta\varphi(\vec{x},\eta) \tag{4.36}$$

Just as in the homogeneous case this inflation has the following action in curved spacetime:

$$S = \int (\frac{1}{2}g_{\gamma}\delta)\varphi_{,\gamma}\varphi_{,\delta} - v)\sqrt{-g}d^4x \tag{4.37}$$

Where $g = detg^{\alpha\beta}$. Following a similar method to the homogenous case, the perturbed inflaton scalar field is placed into the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\sqrt{-g}g^{\alpha\beta}\frac{\partial\varphi}{\partial x^{\beta}} + \frac{\partial V}{\partial\varphi} = 0 \tag{4.38}$$

However in this case the metric we use must also include the perturbations. (4.15) is then used in the above equation. The solution to the homogeneous components in this case is similar to just the homogenous case (3.22):

$$\varphi_0'' + 2\mathcal{H}\varphi_0' + a^2V_{,\varphi} = 0 \tag{4.39}$$

The inhomogenous component of the inflation approximately solves to:

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla(\delta\varphi - \varphi_0'(B - E')) + a^2V_{,\varphi\varphi}\delta\varphi - \varphi_0'(3\psi + \phi)' + 2a^2V_{,\varphi}\phi = 0 \tag{4.40}$$

∇ represents the spatial derivative in terms of comoving coordinates. The metric perturbations have only been taken to a linear order. This is valid for any coordinate system. To reduce the components the equation can be expressed in terms of the gauge-invariant variable derived in the previous section, Ψ and Φ. In this coordinate system the following gauge invariant scalar field perturbation can be used $\delta\tilde{\varphi} = \delta\varphi - \varphi_0'(B - E')$. This is derived using the same method as ρ in the previous section.

$$\delta\tilde{\varphi}'' + 2\mathcal{H}\delta\tilde{\varphi}' - \nabla(\delta\tilde{\varphi}) + a^2 V_{,\varphi\varphi}\delta\tilde{\varphi} - \varphi_0'(3\Psi + \Phi)' + 2a^2 V_{.\varphi}\phi = 0 \quad (4.41)$$

The Einstein equations can also be used to get further equations for the behaviour of the inflaton perturbation. For scalar fields the energy-momentum tensor is (3.18). Together with (4.29) gives the 0,i tensor gauge-invariant perturbation as:

$$\delta\tilde{T}_i^0 = \frac{1}{a^2}\varphi_0'\delta\varphi_{,i} - \frac{1}{a^2}\varphi_0'^2(B - E')_{,i} = \frac{1}{a^2}(\varphi_0'\delta\tilde{\varphi})_{,i} \quad (4.42)$$

Which ois then used in (4.34) to give the Einstein equation in terms of gauge-invariant metric and inflaton perturbations. G has been set to 1.

$$\Psi' + \mathcal{H}\Phi = 4\pi\varphi_0'\delta\tilde{\varphi} \quad (4.43)$$

(4.43) and (4.41) can now be used to solve the behaviour of the inflaton in two separate case. $\Psi = \Phi$ due to the fact that non-diagonal components of the perturbations to the tensor are 0.

4.3.1 The Subhorizon Limit

The first of the two cases involves looking at scales where $\lambda_{ph} \ll (H)^{-1}$. Or in terms of the wavevector $\vec{k}$, $\frac{a}{k} \ll (H)^{-1} \Rightarrow k = |\vec{k}| \gg aH \sim |\eta|^{-}1$. It is convenient at this point to look at the functions of the scalar perturbations in terms of the fourier transform. This is defined as:

$$f(\vec{x}) = \int f_{\vec{k}} e^{i\vec{k}x} \frac{d^3k}{(2\pi)^{3/2}} \quad (4.44)$$

In fourier space spatial derivatives essentially becomes k^2 terms. For large $k|\eta|$ as $\delta\varphi = exp(ik\eta)$ then the spatial derivative in (4.41) will dominate ($\delta\varphi'' = -k^2 exp(ik\eta)$).Taking into account of the slow roll conditions during inflation $V_{,\varphi\varphi} \ll V \sim H^2$ reduces (4.41) to just the first 3 terms:

$$\delta\tilde{\varphi}_k'' + 2\mathcal{H}\delta\tilde{\varphi}_k' + k^2\delta\tilde{\varphi}_k \approx 0 \quad (4.45)$$

The Mukhanov Variable is now introduced toreduce the equation further:

$$\begin{aligned} u_k &\equiv a\delta\tilde{\varphi}_k \\ \delta\tilde{\varphi}_k &= \tfrac{1}{a}u_k \\ \delta\tilde{\varphi}'_k &= \tfrac{a'}{a^2}u_k + \tfrac{1}{a}u'_k \\ \delta\tilde{\varphi}''_k &= \tfrac{1}{a}u + k'' - \tfrac{2a'}{a^2}u'_k + (\tfrac{a''}{a^2} + \tfrac{2a'^2}{a^3})u_k \end{aligned} \tag{4.46}$$

Using this variable reduces (4.45) to:

$$\begin{aligned} \tfrac{1}{a}u''_k - \tfrac{a''}{a^2}u_k + k^2\tfrac{1}{a}u_k = 0 \\ u''_k + (k^2 - \tfrac{a''}{a})u_k = 0 \end{aligned} \tag{4.47}$$

As $k \gg \frac{1}{|\eta|} \to k^2 \gg \frac{1}{\eta^2}$ and $a \sim -\frac{1}{\eta} \to \frac{a''}{a} \sim \frac{1}{\eta^2}$ gives a simpler equation:

$$u''_k + k^2u_k = 0 \tag{4.48}$$

Which solves as a simple harmonic oscillator:

$$\delta\tilde{\varphi}_k \approx \frac{C_k}{a}exp(\pm ik\eta) \tag{4.49}$$

C_k is a constant of integration and has to be fixed. As the inflation perturbations are quantum fluctuations, vacuum quantum fluctuations on a scale of L can be considered to set the amplitude of the solution. Consider the quantum fluctuation in a volume which is L^3. The action becomes of the form $S \simeq \frac{1}{2}\int(\dot{X}^2 + ...)dt$ where $X = \delta\varphi_L$, the perturbation of the inflaton at the scale of the length of the box. The dot is a derivative of time. $P = \dot{X} \sim X/L$ can be thought of as a conjugate to X. As they are each others conjugate they satisfy the uncertainty relation: $\Delta X\Delta P \sim 1$. $P = \delta\varphi_l L^{1/2}$. For the minimum X, $X_{min} \sim L^{1/2}$ gives $\delta\varphi_L \sim L^{-1} \sim \frac{k}{a} \sim |\delta\varphi_k|k^{3/2}$. Which results in the following: $|\delta\varphi_k| \sim \frac{k}{k^{3/2}a} = \frac{k^{-1/2}}{a}$. In this limiting case $\eta \to 0 \Rightarrow \delta\varphi_k \to \frac{C_k}{a}|$ and hence $C_k \sim K^{-1/2}$.

To calculate the power spectrum we need to define a few statistical quantities first. For Gaussian statistics, which is what is being considered, the dimensionless variance is:

$$\delta_f^2(k) = \frac{\sigma_k^2k^3}{2\pi^2} \tag{4.50}$$

σ_k is the variance and characterises the gaussian process. In fourier modes, as only quantum perturbations are being considered, each mode evolves independently. Each k-mode evolves individually and follows a gaussian process so the variance σ_k^2 is simply the function $|f_k|^2$. $\delta_f^2(k)$ is known as the power spectrum.

In this case, at the moment where the inhomogeneity crosses the horizon of the Hubble scale, $k \sim Ha_k$:

$$\delta_\varphi^2(k) = \frac{|\delta\varphi|^2 k^3}{2\pi^2} \sim \left(\frac{k}{a_k}\right)^2 \sim H^2_{k\sim aH} \tag{4.51}$$

Throughout the rest inflation, the inflation perturbations remain frozen out and expanding. The study how the perturbations behave after crossing one needs to study the large wavelength limit.

4.3.2 The Large Wavelength Limit

The first assumption that will be used in this limit is the slow-roll approximations (3.31,3.32). As $\ddot{\varphi}$ is small this gives the following relation for the homogeneous background:

$$3H \simeq \frac{-V_{,\varphi}}{\dot{\varphi}_0} \tag{4.52}$$

These conditions are given in proper time, so first (4.43) and (4.41) have to be written in terms of proper time, and rewriting $\delta\tilde{\varphi} = \delta\varphi$:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \nabla\delta\varphi + V_{,\varphi\varphi}\delta\varphi - 4\dot{\varphi}_0\dot{\Phi} + 2V_{,\varphi}\Phi = 0 \tag{4.53}$$

$$\dot{\Phi} + H\Phi = 4\pi\dot{\varphi}_0\delta\varphi \tag{4.54}$$

As $\lambda_{ph} \gg (H)^{-1}$ the spatial derivative $\nabla\delta\varphi \sim k \sim (\lambda)^{-1}$ is therefore considered quite small. $\delta\ddot{\varphi}$ and $\dot{\Phi}$ terms are also ignored as in this large wavelength limit the perturbations are at a later time, so it is assumed they have not decayed. This simplifies the equations further to:

$$3H\delta\dot{\varphi} + V_{,\varphi\varphi}\delta\varphi + 2V_{,\varphi}\Phi \simeq 0 \tag{4.55}$$

$$H\varphi \simeq 4\pi\dot{\varphi}_0\delta\varphi \tag{4.56}$$

Similarly to the subhorizon limit, a new variable is introduced to simplify the equations to a more familiar form:

$$y = \frac{\partial\varphi}{V_{,\varphi}} \tag{4.57}$$

$$\dot{y} = \frac{\partial\dot{\varphi}}{V_{,\varphi}} - \frac{\partial\varphi}{\frac{\partial V}{\partial\varphi}^2}\dot{\varphi}V_{\varphi\varphi} \tag{4.58}$$

Substituting (4.57, 4.58) into (4.55, 4.56) and using (4.52) gives the following:

$$3H\dot{y} + 2\Phi = 0 \tag{4.59}$$

$$H\phi = 4\pi\dot{V}y \tag{4.60}$$

Using $3H^2 = 8\pi V$ (from Friedmann's second equation (2.45) during inflation $1/2\dot{\varphi}^2 \to 0$)

$$\begin{gathered} 3H\dot{y} + 2\Phi = 0 \times 8\pi/H \\ \dot{y}V + \tfrac{H\Phi}{4\pi} = 0 \\ \dot{y}V + y\dot{V} = 0 \\ \tfrac{d(yV)}{dt} = 0 \end{gathered} \tag{4.61}$$

This integrates to a simple equation:

$$y = A/V \tag{4.62}$$

A is an integration constant that requires fixing. From the definition of y (4.57) the following final equation is reached for the long wavelength approximation of the inflaton perturbations:

$$\delta\varphi_k = A_k\frac{V_{,\varphi}}{V} \tag{4.63}$$

V represents the potential of inflation which is dependent on ρ and p. Another important assumption was that these perturbations were non-decaying and also developed throughout the slow roll regime after the perturbation crossed the horizon. A_K is a constant we will fix in the next section.

One can also determine the perturbations to the curvature of spacetime, i.e. perturbations to the metric, Φ using (4.60).

$$\Phi_k = 4\pi A_k \frac{\dot{\varphi} V_{,\varphi}}{HV} \tag{4.64}$$

Using the fact that $H^2 = -\frac{8\pi}{3}V$ and the slow roll condition for $\dot{\varphi}_0$ gives:

$$\Phi_k = -\frac{1}{2} A_k \left(\frac{V_{,\varphi}}{V}\right)^2 \tag{4.65}$$

4.3.3 Bringing them together

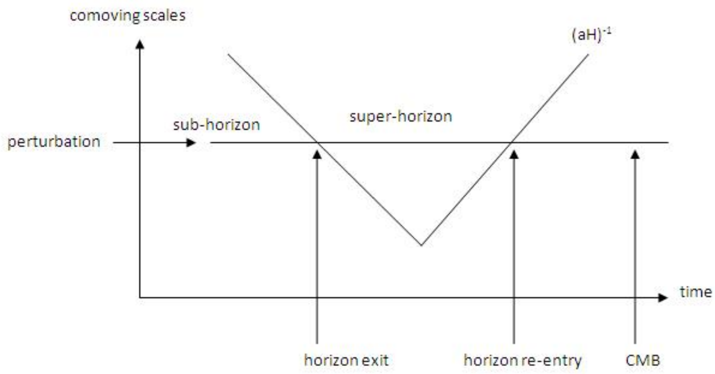

Figure 4.1: A perturbation mode in comoving scales

This figure shows the evolution of a perturbation in the inflaton from before horizon to now. Its start in a sub-horizon, this was explored in Section 4.3.1. It then crossed the comoving horizon and froze out. As the comoving horizon continued to get smaller throughout inflation the perturbation remained steady. Once the comoving horizon started to grow again, it reentered the horizon on super-horizon scales, Secion 4.3.2. This perturbation is then linked to the perturbation in curvature (4.65) which causes gravitational instability. This in turn causes the variations in the CMB and later causes the large scale structure of the universe to form.

So the following step is to determine the power spectrum of these curvature perturbations. Equating the perturbation from the sub-horizon to the super-horizon is done by equating (4.49) and (4.63) at the time of horizon crossing, $k \sim aH$. The constant A_k can then be set.

$$
\begin{aligned}
|\delta\varphi_k| \sim \tfrac{k^{-1/2}}{a_k} = A_k \left(\tfrac{V_{,\varphi}}{V}\right)_{k\sim aH} \\
A_k \sim \tfrac{k^{-1/2}}{a} \left(\tfrac{V}{V_{,\varphi}}\right)_{k\sim aH}
\end{aligned}
\tag{4.66}
$$

Taking (4.64), the curvature perturbation to a time where slow-roll conditions are violated and inflation end, $\left(\frac{V_{,\varphi}}{V}\right) \sim 1$, gives the following:

$$
\Phi_k = -\frac{1}{2}\frac{k^{-1/2}}{a}\left(\frac{V_{,\varphi}}{V}\right)^2 \tag{4.67}
$$

The power spectrum is then calculated as:

$$
\delta_\Phi^2(k) = \tfrac{|\delta\Phi|^2 k^3}{2\pi^2} \sim \tfrac{k^{-1}k^3}{a_k^2}\left(\tfrac{V}{V_{,\varphi}}\right)^2_{k\sim aH} \sim \tfrac{k^2}{a_k^2}\left(\tfrac{V}{V_{,\varphi}}\right)^2_{k\sim aH} \sim \tfrac{a_k^2 H^2}{a_k^2}\left(\tfrac{V}{V_{,\varphi}}\right)^2_{k\sim aH}
$$

Then as $H^2 \propto V$:

$$
\delta_\Phi^2(k) \sim \left(\frac{V^3}{V_{,\varphi}^2}\right)_{k\sim aH} \tag{4.68}
$$

This can then be linked to power spectrum of the CMB fluctuations and large scale structure. This treatment is a basic one meant to give a physical intuition into the larger picture. A more complex and thorough treatment is given in Chapter 8 of Mukhanov's textbook [14]. A simpler treatment of the link between experiment and the theory is found in Baumann's lectures [4].

5 Conclusions

This dissertation has given a very strong case for the theory of inflation.

- The solution of the Big Bang problems. This was one of the reason why inflation was postulated.
- Explains the variations found in the CMB.
- Generates a power spectrum which is related to the spectrum of Large Scale Structure in the universe.
- Uses the same physics, quantum perturbations, to solve the two problems listed above.

But somehow there is still a bit of doubt. The theory itself also seems to have grown in a bit of a segmented manner. From the initial idea to the introduction of reheating and then different scalar potentials. There is not one method that works for all. Using the conditions required for the model to work, some general conditions are derived for all scalar field models of inflation.

In the introduction, Table 1, it was shown the inflation occurred at such a time in the universe's history which is impossible for us to replicate. What exactly happened in these early moments? The physics itself would be completely foreign to us. However the model does reduce this unknown phsyics to a working mathematical model which fits in with observed data from later in the universe.

But what exactly is a scalar field? Mathematically we know it's an object, living on a manifold that assigns a value to every point on the manifold. But what does it physically mean in terms of a field that encompasses the entire universe? What is the inflaton? Using the model it must have a potential of some work, which slopes to a minimum. And there are still many possible different potential fields being modelled.

As to inflation models themselves, further development can be done on the starting conditions of inflation. There are extensive models and ideas for the period of expansion ending, but the beginning is still a little nebulous.

What other theories could there be? Ones that would solve the Big Bang models are ones that take off from $\rho = -p$. These could involve having two scalar fields or modifying gravity. This later theory changes the Einstein action at high energies. This is then transformed into a scalar field with potential. A recent review on approaches to understanding inflation as a period of acceleration includes a section on modified gravities [17]. However, how do these solve the CMB variations and large scale structure?

Our solutions to the Big Bang problems involved changing the behaviour of the comoving hubble radius, what if instead of altering this, the speed of light is changed? This could solve the horizon problem if light travelled faster in the early universe. The reference shows how having a variable speed of light solves the horizon problem [18].

What other alternative could there be? A larger variety of possibilities open up if the Big Bang Models are questioned. The dissertation discussed the view of the Big Bang Model as problem of initial conditions. Without them the flatness and homogeneity are required as a condition. There is a possibility that indeed our universe does require these conditions. Unlike other physical models there is only one "test run" and one set of data, it's what is possible to measure. So questions such as would the universe form with another set of conditions?

If the Big Bang problems did not require solving, then models developed that would explain the CMB variations and large scale structure would suffice. Other questions, what is beyond our current Hubble patch? One probably expect more homogeneity. But is there anything beyond that? If inflation is correct, would the initial Hubble patch contain all the possible surrounding homogenous regions close to our visible universe, or was it in causal contact with areas of large inhomogeneities are large scales. Would our Hubble patch ever reach these? These questions are very fundamental, but quite difficult to answer due to the limiting factor, our current Hubble patch.

Looking more closely within our own region though there is great scope for fine tuning our models of the universe, with or without inflation. The Planck satellite has just started to send a more detailed view of the CMB. This will refine the data available to set the models to and get a greater degree of accuracy. Planck will also provide more detailed data for looking at Non-Gaussianity in the primordial modes of the CMB. Will the quantum fluctuations of the inflaton hold up to more detailed data?

What would cast aside doubts about inflation would be concrete evidence for inflation that could not be explained by any other model. The leading candidate are gravitational waves. In inflation there are caused by the tensor perturbations of the metric and scalar field. Detecting this would be further proof of inflation as a successful model.

Inflation overall is a good start. It's achievements are many and the idea is deceptively simple. I do think there is scope for growth and change. A final definitive model is not quite set in stone yet. The theory has to stand many challenges from competing theories. Inflation has to finalise it's smaller components, the potential, it's own initial conditions and reheating. It's very much a live theory which is added onto as more time goes by. However much the theory has to go, one can not forget it's achievements. The threading of the CMB fluctuations and the Large Scale Structure together as the quantum fluctuations of a hypothetical scalar field is quite impressive. Especially when the limitations are based on what we need for the universe to work, i.e. these explanations are a result of conditions set by the solution of the Big Bang Problems. It's a powerful predictive theory that still needs further work and refinement.

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