Imperial College London

Gauge-gravity Duality

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To my parents,

To Kiran and Claire for keeping me sane,

And to my coursemates for keeping me insane.

Contents

1 Introduction

String theory was originally mooted in the late 1960s, as a theory of the strong interaction. After QCD was discovered to be the correct theory for this phenomenon, string theory went out of favour. It then took ten years, and the remarkable successes of Schwarz, Greene, Neveu and Ramond to bring string theory back into the limelight as a remarkably elegant theory which could incorporate both gravity and the lower spin gauge theories of particle physics. In the 1990s, despite continuing successes (such as the discovery of D-branes and various 'dualities' which merged various string theories into one 'M-theory), string theory (by now also a theory of many other types of extended objects) was coming to be seen as an esoteric area of physics, far removed from experiment, with some critics to call it not physics at all, or worse still a religion (refs).

However, in the past ten years the new concept of 'gauge-gravity duality' has emerged [DF02]. In short, this is the statement that certain string theories on manifolds with a special asymptotic behaviour can be thought of as quantum gauge field theories without gravity, which are defined on the boundary space of those manifolds. This is an example of a 'holographic principle': something which has long been suspected to exist in quantum gravity, in which physics in a certain region can be entirely described by information on its surface. (ref Susskind)

Part of the reason people are now getting so excited about his topic is because it finally relates string theory to something experimentally observable and indeed ubiquitous: gauge theory. Furthermore, the regime in which the correspondence is mathematically simplest is that of 'strong gauge coupling'. Solving non-Abelian gauge theories (in particular QCD) at strong coupling has long been seen as the most important unsolved problem in quantum field theory. Systematic calculation of correlation functions in these theories has only been tractable at 'weak coupling' when the coupling constant g of the theory is small, enabling us to express them as a perturbation series in g.

This is the case at high energies, and the effective degrees of freedom are then quarks and gluons. At lower energies, g becomes large, perturbation theory is no longer possible, and we observe a phenomenon known as confinement. The effective degrees of freedom are now colour neutral hadrons. We still do not have a systematic method of obtaining the masses of these hadrons. So not only has string theory touched ground with experiment (at least in principle), but it has done so in the best possible place: where it could advance our understanding of the standard model, crowning bastion of the golden age of theoretical physics.

There are two main areas in which gravity theories are being used to study gauge theories. The first is in QCD, specifically in heavy ion collisions which take place at places such as RHIC in Brookhaven, NY. In these collisions, a strongly coupled quark-gluon plasma is created, which is a state of matter which exists just above the confinement energy of the theory. Interesting properties of theories which resemble QCD at these energies can be derived from the duality, leading to insights into the thermal properties of a large class of such gauged theories. The second is in condensed matter physics. Here effective field theories are commonplace, and have been used with great success over the past 50 years to study critical phenomena (phase transitions) in a wide variety of situations, such as ferromagnets, semi-conductors and super-conductors. But our understanding has been limited by the same lack of systematic understanding of non-perturbative phenomena. The duality provides a potential wealth of new models for effective field theories, and will hopefully lead to further understanding of new classes of condensed matter systems.

In this thesis we review the basic concepts and methods of the gauge-gravity correspondence.

In chapter two we review the geometry of Anti-di Sitter (AdS) space, in preparation.

In chapter three we present some basic arguments in favour of the duality, and outline the basic concepts.

In chapter four we give an explicit formalism for making calculations in gauge theory using gravity in AdS space. This naturally leads on to a discussion of the generic properties of the duality dynamics. We end the chapter doing some of the calulations.

In chapter five we describe a specific example of the duality, and show

how concepts introduced in chapter four can be used.

Finally in chapter six we give a hint of how to go on a create new dualities using the geometry of spacetime.

2 Anti de-Sitter Space

The whole geometrical structure of gauge theories which we will be describing is based on a spacetime manifold called anti-de Sitter space. Therefore the very first thing we need to do is define this manifold, and discuss its quirks, so that we will have collected all the necessary information before we start. The manifold comes in several different forms, and has different coordinate patches in which one can choose to describe it.

2.1 Basic anti-de Sitter space

Anti-de Sitter space in $d+1$ dimensions (AdS_{d+1}), in its most basic form, is defined as follows. Take the manifold $\mathbb{R}^{d+2}$, endowed with a metric:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^{d} dX_i^2 \tag{2.1}$$

Then basic AdS_{d+1} is the $d+1$-dimensional hypersurface defined by the embedding:

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{d} X_i^2 = -R^2. \tag{2.2}$$

To see what this looks like, it is useful to consider the simplest nontrivial case, namely $d = 1$. Here the the flat embedding space is $\mathbb{R}^3$, and $ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2$. Let us show the $X_{-1} - X_0$-plane as the vertical $y-z$ plane in the usual (x, y, z) construction, and let X_1 be the x coordinate. This is shown in 2.1. AdS_3 is the hyperboloid shown in 2.2.

We can satisfy the $d = 1$ embedding conditions with the following coordinates on AdS_3:

$$
\begin{aligned}
X_{-1} &= R \sin\tau \cosh\rho \\
X_0 &= R \cos\tau \cosh\rho \\
X_1 &= R \sinh\rho,
\end{aligned}
$$

where $-\inf < \rho < \inf$, and $0 \leq \tau < 2\pi$. These coordinates cover the entire manifold, and so are called the *global coordinates.* A position on the manifold is determined by a ρ coordinate (position in $\mathbb{R}$) and a τ coordinate (position in S^1); in fact there is a bijective mapping between the set of coordinates $\{\rho, \tau\}$ and points on the manifold, which maps open sets to open sets. Therefore AdS_3 has the topology $\mathbb{R} \times S^1$, which is a cylinder.

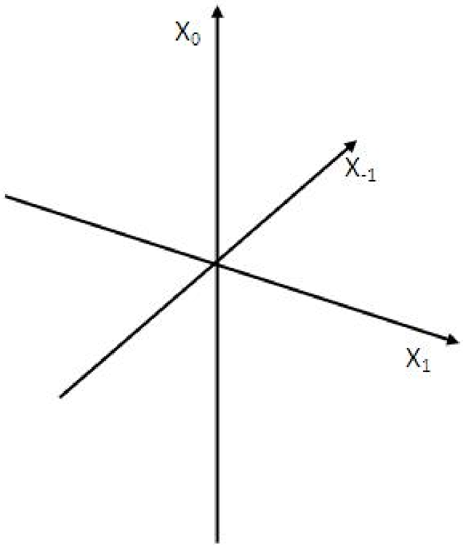

Figure 2.1: The setup of out axes for the embedding shown in 2.2.

This embedding induces the following metric on AdS_3:

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2). \tag{2.3}$$

In higher dimensions we have the following global coordinates and induced metric:

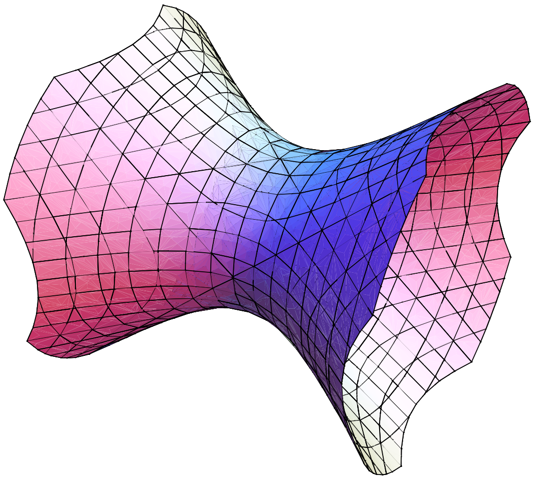

Figure 2.2: The two-dimensional hyperboloid of AdS_2.

$$\begin{aligned}
X_{-1} &= R \sin\tau \cosh\rho \\
X_0 &= R \cos\tau \cosh\rho \\
X_i &= R(\Omega_{d-1})_i \sinh\rho \\
ds^2 &= R^2 \left(\cosh^2\rho \, dt^2 + d\tau^2 + \sinh^2\rho \, d\Omega_{d-1}^2\right);
\end{aligned}$$

where $(\Omega_{d-1})_i, i = 1, 2 \cdots i)$ are the embedding coordinates for a unit (d-1)-sphere in $\mathbb{R}^d$, so that $\sum(\Omega_{d-1})_i = 1$. For example, $(\Omega_1)_1 = \cos\theta; (\Omega_1)_2 = \sin\theta$.

2.2 The universal cover

As can be seen from its topology, basic AdS space is not simply connected. In particular, it has a factor of S^1: a cylic time coordinate. This means there are severe problems for causality in this space. For example, to properly define a quantum field theory on a manifold, we need to concept of time-ordering. This is impossible to define if time is cyclic. For this reason, when physicists talk about AdS, they are always referring to its *universal cover*. This is produced if we decompactify the τ coordinate in the global coordinates, i.e. we take $-\infty < \tau < \infty$. This has an differential structure identical to basic AdS, but globally is now simply connected (hence its name, just as for example $SU(2)$ is the simply connected universal cover of $SO(3)$).

2.3 Coordinate patches

Apart from the global coordinates, there are a few other coordinate systems for AdS space. None of these cover the whole space, but only cover patches of it. The most useful of these to us will be the Poincaré coordinates $y, x^\mu, \mu = 0, \cdots d-1$. These are:

$$ds^2 = \frac{R^2}{y^2} dy^2 + \frac{y^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu \quad 0 \leq y < \infty \tag{2.4}$$

They only cover a wedge of the Penrose diagram, the ‘Poincaré patch’, as shown in 2.5. The boundary of the space in this patch consists of the hypersurface at $y = \infty$, and the point $y = 0$.

These coordinates have the advantage of making explicit some of the key features of AdS. Firstly, the metric has explicit Poincaré invariance in the transverse coordinates x^μ. In the the metric, the coefficient of the Minkowski metric is y^2. Therefore we can think of AdS as consisting of infinitely many copies of $\mathbb{M}$, one at each radial (y) coordinate, which get larger as we head out towards the boundary at large y.

Secondly we note another important isometry of the metric: scale invariance. By this we mean the following coordinate transformation:

$$x^\mu \mapsto a x^\mu \quad ; \quad y \mapsto \frac{y}{a} \quad , \quad a > 0 \quad . \tag{2.5}$$

If we invert the radial coordinate $z = \frac{R^2}{z}$, we find the ‘upper half space’

$$ds^2 = \frac{R^2}{z^2} \left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right) \quad 0 < z < \infty, \tag{2.6}$$

so called because they cover the region $z > 0$ in cartesians. In these coordinates the scale invariance is $x^\mu \mapsto a x^\mu; z \mapsto az$. Also, we can 'see' the boundary at $z = 0$. The rest of the boundary $z = \infty$ is a point (like $y = 0$), because the metric goes to zero there.

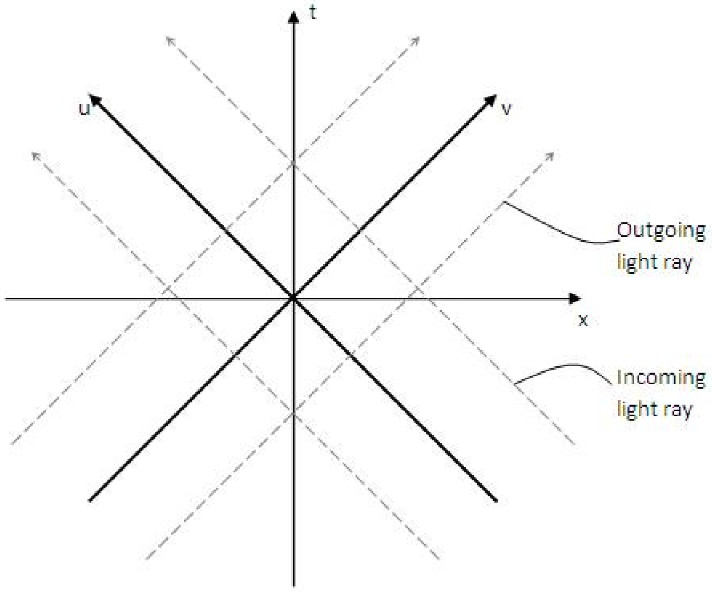

Figure 2.3: We rotate the axes to line up with null geodesics.

2.4 Causal structure

Anti-de Sitter space has a special causal structure, which enables the precise form of the correspondence to be formulated in (chapter). To see this, we need to study its *conformal compactification.* It will be informative to compare the causal structure of AdS with that of normal Minkowski space.

Here's the standard way of conformally compactifying $\mathbb{R}^{1,1}$. Take the metric

$$ds^2 = -dt^2 + dx^2 \qquad -\infty < t, x < \infty \tag{2.7}$$

and change coordinates to $u = t - x$; $v = t + x$. Lines of constant u are paths of outgoing light rays, lines of constant v are incoming light rays (2.3).

The metric is

$$ds^2 = -dudv \qquad -\infty < u, v < \infty. \tag{2.8}$$

Now *compactify* these coordinates, i.e. bring infinity to a finite value, by letting $u = \tan p$; $v = \tan q$. The metric is

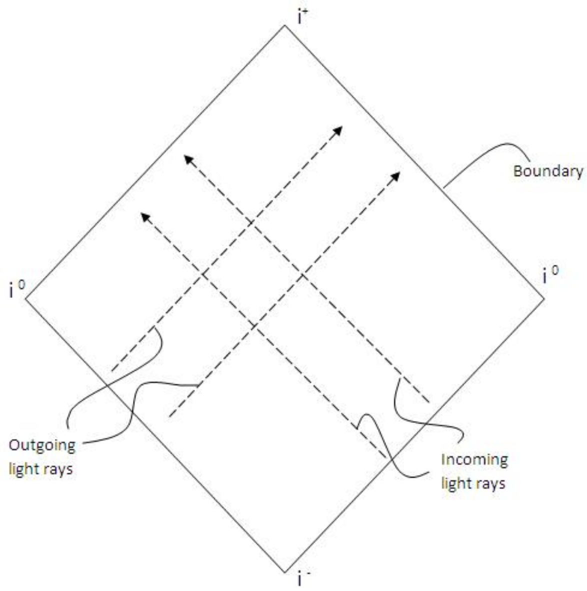

Figure 2.4: The Penrose diagram of $\mathbb{R}^{1,1}$. Timelike geodesics end on $i^{\pm}$, spacelike geodesic end on i^0.

$$ds^2 = -\sec^2 p \sec^2 q \, dpdq; \qquad -\frac{\pi}{2} < p, q < \frac{\pi}{2}. \tag{2.9}$$

Now rotate back to modified time and space coordinates $p = t' - x'$; $q = t' + x'$, to finally get

$$ds^2 = \sec^2 p \sec^2 q \left(-dt'^2 + dx'^2\right). \tag{2.10}$$

This is what we started with, but with the coordinates now having finite range, and with an extra factor out front. It takes the form of a diamond(fig. 2.4), and this factor goes to infinity at the edges of this diamond, reflecting the fact that there is an 'infinity' of space-time distance compressed into these edges. A *conformal transformation* is one in which we simply change the metric by a (in general space-time dependent) factor, $g_{\mu\nu} \to \Omega^2(x^\alpha)\, g_{\mu\nu}$, where $\Omega \in \mathbb{R}$. The key point is that this does not change the sign of norms of vectors on the manifold, since for a vector v:

$$v^2 \equiv g_{\mu\nu} v^\mu v^\nu \;\to\; \Omega(x^\mu) v^2, \tag{2.11}$$

and Ω is positive. So a conformal transformation maps timlike vectors to timelike vectors, null ones to null ones, and spacelike ones to spacelike ones. But this is the entire causal structure, which is therefore preserved under the transformation. So if we want to study the causal structure of a manifold, we might as well make a conformal transformation to bring the metric to the simplest possible form. In the present case, let $\Omega(x^\mu) = \frac{1}{\sec p \sec q}$, so that the metric becomes the $\mathbb{R}^{1,1}$ metric again. Lights rays follow $t' = \pm x'$, and we have the picture in fig. 2.4. Because we have divided the metric by a function which diverges asymptotically (p or $q = \frac{\pi}{2}$), the boundary of the new manifold is a finite proper distance away from any given point. We call this manifold the conformal compactification of $\mathbb{R}^{1,1}$. It encodes the entire causal structure of $\mathbb{R}^{1,1}$.

The compactification of AdS_2 is a bit more straightforward. Take the the global coordinates

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 \tag{2.12}$$

and let $\sinh \rho = \tan \theta$. The metric becomes

$$ds^2 = \sec^2 \theta \left(-dt^2 + d\theta^2\right) \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \tag{2.13}$$

which diverges at $\theta = \pm\frac{\pi}{2}$. Following our general prescription, we conformally transform away the $\sec^2 \theta$ factor, leaving the Minkowski space metric with different coordinate ranges. The space is an infinitely long strip, as in fig. 2.14.

For higher dimensions (AdS_{d+1}), $0 \le \rho < \frac{\pi}{2}$, and each $\rho > 0$ position is to be thought of as an S^{d-1} of radius ρ, as in polar coordinates. For example AdS_3 can be thought of as the interior of an infinitely long cylinder (fig). The metric is:

$$ds^2 = \sec^2 \left(\theta(-dt^2 + d\theta^2 + \theta^2 d\Omega^2_{d-1}\right) \tag{2.14}$$

from which we can again remove the $\sec^2$ factor.

The first to notice about (fig. 2.13), is that the *boundary* of the space at $\theta = \frac{\pi}{2}$ is a *timelike* surface.

This has implications for dynamics on AdS. When we vary an action for a dynamcial field we in general obtain two sorts of terms:

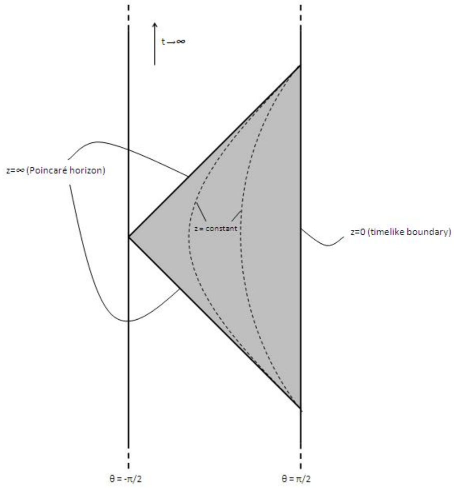

Figure 2.5: The Penrose diagram of AdS_2. It continues to infinity above and below. The Poincar is the shaded area.

- The field equations integrated over the bulk
- A boundary term.

In Minkowki space the standard procedure is to consistently set the fields to zero at infinity, so that the boundary term disappears. We do not need to know about the field far away. This works because because we can say we are considering a problem that is local in spacetime. Variations of the boundary conditions propagate along time curves, and so would take infinitely long to reach the interior, measured by a timelike observer (in (fig), all timelike curves end at timelike infinity i^+). Therefore interior solutions are determined independently of the boundary behaviour.

However, as can be seen from a comparison of (fig) and (fig. 2.14), the situation in AdS is qualitatively different: there is no timelike infinity, reflecting the fact that null curves propagate into the interior from the boundary in finite time, as measured by a timelike observer. So we can never ignore the boundary terms. We need to include a counter-term in the action to remove it. But to evaluate this counter-term, we need to know the behaviour of the fields at the boundary. They are needed to solve any wave equation on AdS. In fact there is a theorem (the Graham-Lee) theorem [**??**], which states that given boundary conditions for a field on AdS, there is a unique interior solution. This will be important for the formulation of the correspondence.

2.5 Euclidean vs. Lorentzian AdS

There is another form AdS space, in which the time component has been Wick rotated so that the metric signature is Euclidean. Let's formally define it.

Take the manifold $\mathbb{R}^{d+2}$, endowed with a metric:

$$ds^2 = -dX_{-1}^2 + dX_0^2 + \sum_{i=1}^{d} dX_i^2 \tag{2.15}$$

Then Euclidean AdS_{d+1} ($EAdS_{d+1}$) is the $(d+1)$-dimensional hypersurface defined by the embedding:

$$-X_{-1}^2 + X_0^2 + \sum_{i=1}^{d} X_i^2 = -R^2. \tag{2.16}$$

This is shown in fig. 2.6. AdS_2 is disconnected, but higher dimensional versions are not. To see the general topology, look at one piece of the the hyperboloid. $EAdS_{d+1}$ has the topology $\mathbb{R}^{d+1}$ for $d > 1$. It is therefore simply connected.

We can map *all* of $EAdS_{d+1}$ onto the $(d+1)$-dimensional upper half space:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \delta_{ij}dx^i dx^j\right) \tag{2.17}$$

. This is true even though the upper-half space only covers the Poincaré patch, i.e. some, of Lorentzian AdS.

A note on the uses of AdS and $EAdS$: many of the original calculations (e.g. in [[Wit98a]]) were done in $EAdS$ space. This has the following advantages:

- It has a postitive definite Euclidean metric.
- It can be completely covered by the very convenient upper-half space coordinates.

We have used the upper-half space coordinatization of the Poincaré patch of AdS in most of this review. As such there are several subtle points which

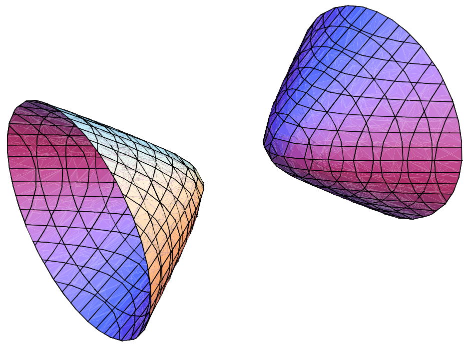

Figure 2.6: The hyperboloid of $EAdS_2$, showing the $\mathbb{R}^2$ topology. Higher dimensional versions are completely connected manifolds.

we have swept under the carpet. It should be noted that many of the calculations can be made more rigorous by Wick rotating to the Euclidean space.

3 Why, how gauge-gravity duality?

Why do we have any reason to think that gravity should be able to describe to describe a *non*-gravitational theory in one less dimension? And how does the relationship between the two theories work?

Evidence for a duality comes about from consideration of black holes. The work of Bekenstein and Hawking in the 1970s (ref) showed that the entropy of a black hole was proportional to its area. Actually, in natural units ($\hbar = c = 1$):

$$S_{\mathrm{BH}} = \frac{A}{4G_{\mathrm{N}}}.$$

this is actually the largest possible entropy that 'stuff' can have in a given region R of spacetime, of surface area A. There is a simple heuristic argument to show this: Let the entropy in R intially be larger than this entropy, $S_0 > S_{\mathrm{BH}}$. This means that there is not currently a black hole of area A in R covering all of R, though there might be smaller one. Now start to put more matter into R. Eventually, even if there wasn't a black hole in R to begin with, one will form. As we add matter, the mass of the black hole will increase, therefore so will its radius, until finally it fills the whole of R. Now we do have a black hole of area A, so the final entropy of the region is $S_{\mathrm{f}} = S_{\mathrm{BH}}$. Which means that $S_{\mathrm{f}} > S_0$. The entropy has decreased! So given the second law of thermodynamics, and given that there are gravitational interactions. S_{BH} is the maximum entropy in a region of surface area A.

Looking back at (eq), we see that the maximum entropy of R in proportional to its *area* A, *not* its volume V. If we are to treat gravity as a quantum field theory with even most basic of assumptions, we would deduce a maximum entropy proportional to V, so that fact that this is not true suggests that there is new physics at play. How might we hope to describe a physics in which $S_{\mathrm{max}} \propto A$?

One possible answer is a quantum field theory defined *on the boundary* of

R, this theory would have $S_{\max} \propto \text{volume}(\partial R) = A$. We call the interior R the *bulk*, and its boundary ∂R the *boundary.* The gravity and field theories are called the bulk and boundary theories, respectively.

Fine I hear you say, we are able to describe gravity in R somehow by stuff going on at its boundary. But surely there is whole *dimension*'s worth more information in the bulk theory. What has happened to all that information? The answer lies in the Wilsonian idea that when we consider a quantum theory theory, we must also consider the *scale* at which we are working. This is the smallest scale at which we take the theory to be valid. In doing caculations we do not consider excitations of the field with a smaller wavelength than this. This corresponds to some maximum energy, the *cutoff* of the theory, Λ. This parameter is as much a part of the theory as the Lagrangian, and studies of how QFTs depend on their scale has been of fundamental importance in understanding their interpretation. The basic idea is that the extra dimension should correspond to this energy scale.

3.1 AdS/CFT

There exist QFTs known as *conformal field theories.* These are theories which are invariant under conformal transformations of their background manifold, as described in (eq). The set of conformal transformations form a group known as the *conformal group.* An important subgroup of the conformal group is that of constant scaling of all the coordinates:

$$x^\mu \to a\, x^\mu \qquad a > 0. \tag{3.1}$$

Let us say we want our boundary QFT to be conformal. And let us say that there is certain limit of the bulk theory we can take, in which the gravity is dominates by a description involving a background metric g, over which there are perturbations. This is fair - it corresponds to *classical* Einstein gravity. If the QFT is independent of scaling the coordinates, which involves scaling the energy scale inversely as $\Lambda \to \frac{\Lambda}{a}$, then g, which provides the dual description, should also be invariant under this. Let us split up the bulk geometry into a radial coordinate u, which is the QFT energy scale, and the other, 'transverse' directions x^μ, on which we define the QFT. Then the transformation:

$$u \to \frac{u}{a} \tag{3.2}$$

$$x^\mu \to a\, x^\mu \tag{3.3}$$

should be an isometry of g.

Furthermore, we need a (non-dynamical) metric h on which to define the boundary theory. In order to have causality, there needs to be a time-like component to the boundary. If we want a standard QFT on Minkowski space-time, h should be Minkowski, which means it has the Poincaré group as its isometries.

All this implies we need a metric with scale invariance and Poincaré invariance in the transverse directions. There is only one candidate, as we saw in the last chapter: AdS space:

$$ds^2 = \frac{R^2}{u^2} du^2 + \frac{u^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu \quad . \tag{3.4}$$

Therefore a conformal field theory (CFT) on the boundary is dual to a gravity theory on AdS space, giving rise the popular term "AdS/CFT". This term is however misleading, because we emphasize that the boundary does not have to be a CFT in order to have a gravity dual. But the dual metric will not be pure AdS space - it will just have to be *asymptotically* AdS.

3.2 Large N

A theory of quantum gravity is hard to deal with. Be it string theory or loop quantum gravity, or something else, the quantum nature of spacetime changes things dramatically from the classical picture. If we have a gravity setup in the bulk which is overwhelmingly *classical*, so that we can use Einstein's theory, what does this mean for the boundary theory?

We can find this out by examining the number of degrees of freedom in the the field theory. We expect to go like the entropy of the field theory, which we have just said is $\frac{A}{4G_{\rm N}}$. Let's try to find these independently.

Will assume a pure AdS_{d+1} geometry, and work with Poincaré coordinates. Let's calculate the area of the boundary at some fixed t:

$$A = \int d^{d-1}x\,\sqrt{\gamma} \tag{3.5}$$

where γ is the induced metric on the spatial boundary, $\gamma = \left(\frac{R}{z}\right)^{d-1}$. (3.5) is infinite. This comes from two things: 1)the boundary is a copy of Minkowski space infinitely magnified by the $\frac{1}{z}$ factor in γ, and 2) the transverse space has an infinite volume. So we need to regulate the area by:

1. Taking the boundary at $z = \epsilon$
2. Putting the tranverse coordinates in a box, $0 < x^{\mu} < L$.

$$\Rightarrow \quad A = \int_0^L d^{d-1}x \left(\frac{R}{\epsilon}\right)^{d-1} = \left(\frac{LR}{\epsilon}\right)^{d-1} \quad . \tag{3.6}$$

Now let's calculate the number of degrees of freedom. Corresponding to the energy scale cutoff $z = \epsilon$ we used in (3.6), we will take the field theory on a lattice with lattice spacing ϵ. And we'll take the this space (the boundary) to be in a box (again as in **??**). We will say the that there are N^2 degrees of freedom per lattice site, so we have the picture in fig. 3.1. The total number of degrees of freedom on the boundary is N^2 times the number of lattice points:

$$\text{degrees of freedom} = \underbrace{\left(\frac{L}{\epsilon}\right)^{d-1}}_{\text{d.o.f./lattice site}} N^2 \tag{3.7}$$

Now we equate $\dfrac{A}{4G_{\rm N}}$ = degrees of freedom:

$$\begin{aligned} \Rightarrow \quad \left(\frac{LR}{\epsilon}\right)^{d-1} &= \left(\frac{L}{\epsilon}\right)^{d} N^2 \\ \Rightarrow \quad \frac{R^{d-1}}{G_{\rm N}} &\sim N^2 \end{aligned} \tag{3.8}$$

This is a very simple rough calculation, but it shows us something important. The LHS of (3.8) is the AdS radius (i.e. the typical radius of curvature of the bulk geometry), in units of the bulk Planck length ${G_{\rm N}}^{\frac{1}{d-1}}$. We rewrite (3.8):

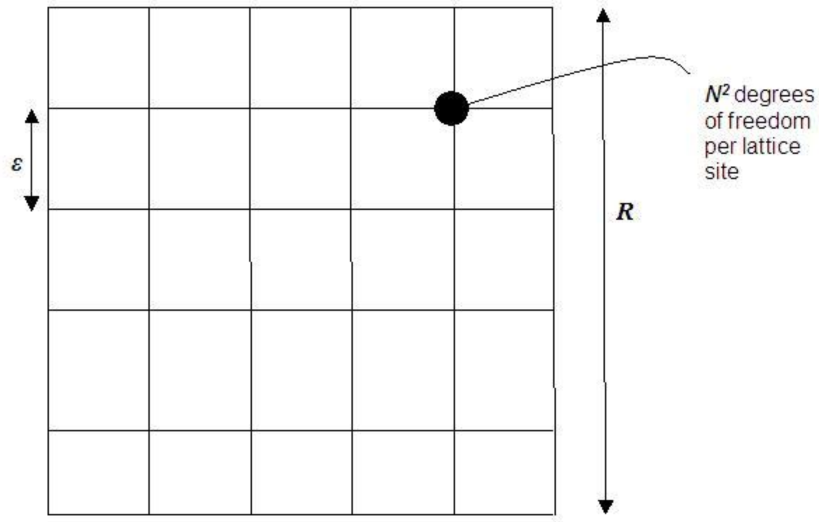

Figure 3.1: A two-dimensional planar cross-cross section of the boundary lattice.

$$\left(\frac{\text{curvature scale}}{\text{Planck length}}\right)^{d-1} = N^2 \quad . \tag{3.9}$$

So if there are a large number of degrees of freedom at each point in the field theory, the bulk curvatures will be small compared to the Planck scale and the energies will be low. We will have classical gravity.

large N^2	$\rightarrow$	Classical gravity dual

4 The Dynamical Statement

Up until now we have almost exclusively considered mappings between the respective symmetries of the gravity and gauge theories. (ordering?) However, if we are being honest with ourselves, this does not tell us that much. Consider the following: Given a theory with a certain symmetry group, G, we can always take objects which transform in a certain representation of G, and then take suitable products of them to give a composite object which transforms in any given representation we like, call it R. Then we take a second theory with the same symmetry group. It too will have objects transforming under representations of G. We again take products until we have an object transforming in R. Lastly, we define a map between the two composite objects. They transform in the same way under G, so things are looking good. Does this mean they are the same theory?? No! Why not?

The answer is that a theory is not entirely defined by its symmetries. A classical theory is defined by its Lagrangian, and a quantum theory is defined by its path integral, or generating functional. To precisely define a duality between two physical theories, we must define a bijective map between the generating functionals of the two theories. This is precisely what Witten did in 1998 [[Wit98a]]. Witten's key idea was that *fields* in the gravity theory should be dual to *operators* in the gauge theory. Take a field φ in the bulk, with boundary condition $\varphi = \varphi_0$ at the boundary of the AdS space. Let φ be dual to an operator $\mathcal{O}$ in the boundary theory. Witten's ansatz is:

$$\boxed{Z_{QFT}[\varphi_0] = \int \mathcal{D}\,\varphi e^{iS[\varphi]|_{\varphi_0}} \quad ,} \tag{4.1}$$

where the LHS is the generating functional for $\mathcal{O}$ in the gauge theory as a function of the source φ_0, and the RHS is the path integral of the gravity theory over all field configurations of $\varphi(z, x^\mu)$ with $\varphi = \varphi_0$ at the boundary. This means the following: the field φ couples to an operator $\mathcal{O}$ on the boundary, and adds a term $\mathcal{L}_{int} = \varphi_0 \, \mathcal{O}$ to the Lagrangian of the boundary theory. Fluctuations of φ_0 then act as a source perturbing this theory with the operator $\mathcal{O}$. Then, in usual way, the generating functional for $\mathcal{O}$ is defined as the vacuum path integral in the presence of the source:

$$Z[\varphi_0] := \int \mathcal{D}\varphi \, e^{i \int d^4x (\mathcal{L} + \varphi_0 \mathcal{O})} \tag{4.2}$$

and the vacuum expectation value of products of this operator are found by finding the 'response' of the path integral to small fluctuations of the source about zero:

$$\langle \Omega | \, \mathcal{O}(x)\mathcal{O}(y) \cdots \mathcal{O}(z) \, | \Omega \rangle = \frac{\delta}{\delta\varphi_0(x)} \frac{\delta}{\delta\varphi_0(y)} \cdots \frac{\delta}{\delta\varphi_0(z)} Z[\varphi_0]|_{\varphi_0 = 0} \tag{4.3}$$

The correlation functions of a theory encode all of its information [**??**]. Therefore by knowing the correlators we are a long way to understanding the correspondence. It will be the goal of this chapter to calculate some correlation functions in the limit of classical gravity, and to outline the associated subtleties. In particular we will see that renormalization of these correlators corresponds to something geometrical in the gravity theory.

4.1 Preliminaries

So what will it take for us to calculate some juicy correlation functions? Firstly, we need to know what regime we are working in, so that we can make some all-important simplifying approximations and introduce some key concepts in the application of the precise duality statement. We will work in the limit of classical gravity in the AdS. In this limit, quantum (and stringy) fluctuations are small (see §5.1), and the path integral of the gravity theory is dominated by a classical solution of the coupled Einstein and matter field equations. In this case we have:

$$\int \mathcal{D}\varphi e^{iS} = e^{iS[\varphi_{\text{classical}}]}, \tag{4.4}$$

where $\varphi_{\text{classical}}$ is a classical solution to the equations of motion of some field φ.

The next issue we must address is the role of the boundary conditions φ_0

appearing in (4.1). To do this let us take the simplest field we can think of: a real scalar field, and try to solve its field equation out on the boundary of the *AdS* space. Asymtotically, as we mentioned, the space must be exactly *AdS*, so we know the metric we must take as our background.

Throughout this chapter, we are going to be working in the upper half space Poincaré coordinates (2.4), repeated here for your convenience:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \quad . \tag{4.5}$$

The action for our scalar field we will take to be:

$$-\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + m^2\varphi^2\right), \tag{4.6}$$

with equations of motion:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi\right) - m^2\varphi = 0. \tag{4.7}$$

It is an important assumption here the field configuration will not back-react on the metric, so that we can solve for the field on a gravity background without worrying about the fact that the two systems are dynamical coupled. We will come back to this point later.

For simplicity, let us consider the mode with zero momentum in the transverse directions, by letting $\varphi = \varphi(z)$. We will also take the AdS radius to be unity, $R = 1$, so we are measuring lengths in units of R. Now substituting in the metric (4.5), (4.7) becomes:

$$z^5\partial_z\left(\frac{1}{z^5}z^2\,\partial_z\varphi\right) - m^2\varphi = 0 \tag{4.8}$$

$$\Rightarrow \quad z^2\,\partial_z^2\varphi - 3z\,\partial_z\varphi - m^2\varphi = 0 \tag{4.9}$$

This is an ordinary differential equation of the Euler variety. Since in these coordinates $z \geq 0$, we can solve it using a power law ansatz: $\varphi(z) = z^\lambda$.

We find the indicial equation:

$$\lambda^2 - 4\lambda - m^2 = 0 \tag{4.10}$$

$$\Rightarrow \quad \lambda = 2 \pm \sqrt{4 + m^2} := \Delta_\pm \tag{4.11}$$

There are three separate cases:

- **$m^2 > 0$:** One root $\Delta_- < 0$, which causes the solution to blow up at the boundary ($z \to 0$). The other solution Δ_+ dies away towards the boundary. For a general solution with non-zero coefficient for Δ_-, Δ_- will be the dominant contribution at small z. What are we to make of this? In the end we are going to want to evaluate the classical action for a field configuration with a given boundary condition ϕ_0 at $z = 0$. But in general the solution is infinite there. Therfore we must regulate the boundary. That is to say, we take the boundary of AdS not at $z = 0$, but at $z = \epsilon$, so that we can impose finite boundary conditions.

- **$m^2 = 0$:** The roots are coincident, $\Delta_+ = \Delta_- = 0$. This means that all solutions to the scalar wave equation are radially constant near the boundary. Thus there seems no need to introduce a cutoff.

- **$m^2 < 0$:** Is this possible? The scalar potential for a negative mass is concave, and is classically unstable to the field configuration 'falling' off the top (see fig. 4.1). In Minkowski space, where the kinetic energy of the scalar in the vacuum is zero, this is indeed fatal, and it is impossible to define a sensible field theory [1] However, in AdS this is OK. Why? Well let us look at the solutions in this case: both roots $\Delta_+, \Delta_- > 0$. This means that any solution to the equations of motion will decrease towards the boundary. With this radial field variation will be associated a positive kinetic energy in the stress-energy tensor component T_{00}, which, as long the mass is not *too* negative, provides vacuum stabilization. Here we are faced with the same problem, namely: how are we supposed to impose finite boundary conditions on ϕ at the boundary, when all solutions go to zero there? So again, we must introduce a cut-off near the boundary at $z = \epsilon$.

[1] At least using conventional quantum mechanics.[[McH09]]

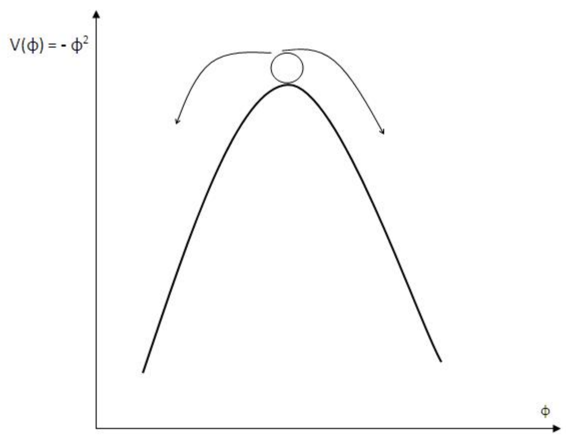

Figure 4.1: The upside-down potential. A static field configuration is unstable, but *AdS* forces momentum into the field, so it can some 'upside-down-ness'.

4.2 Conformal dimensions

If we are working at an energy/length scale at which the bulk space is *AdS*, i.e. asymptotically, the boundary theory behaves like a conformal field theory (CFT). If conformal tranformations of the boundary commute with the boundary Hamiltonian, the operator dual to some bulk field must transform in some irreducible representation of the boundary conformal group.

Of particular interest amongst the generators of the conformal group is the dilatation operator, which governs how a field transforms under scale transformations. If the coordinates are scaled as in (3.1) with $a = e^{\omega}$:

$$x^{\mu} \mapsto e^{\omega} x^{\mu} \tag{4.12}$$

then the field φ will transform as

$$\phi \mapsto e^{-\Delta\omega} \phi \tag{4.13}$$

and Δ is called the *conformal dimension* of ϕ. Notice the minus sign in the exponent, which means that a field of positive Δ scales up with the *energy*, and inversely with the coordinates. For a classical CFT, the conformal dimensions of the fields are just their classical energy dimensions, but quantum mechanics will in general renormalize the classical dimension. The change in dimension of an operator due to quantum loops in known as its *anomalous dimension*. From here on in, when we talk about the 'dimension' of a field, we will mean its full quantum conformal dimension.

The conformal dimensions add up for two fields multiplying each other: given two fields ϕ_1, ϕ_2 of conformal dimensions Δ_1, Δ_2, we have:

$$\Delta(\phi_1\phi_2) = \Delta_1 + \Delta_2 \tag{4.14}$$

Can we deduce what Δ will be for a field theory operator dual to some bulk field?

Let's look at the coupling term in the boundary action:

$$S_{\text{int}} = \int d^d x \mathcal{O}\varphi \tag{4.15}$$

The whole action must have $\Delta_S = 0$. The measure $d^d x$ naturally has dimension $\Delta = -d$. Now what is the dimension of φ? We have seen in §4.1 that a bulk field satisfying a second order differential equation of motion will have two different possible boundary behaviours near the boundary. E.g. for a massive scalar field we had:

$$\varphi \sim z^{\Delta\pm} \quad {}_0. \tag{4.16}$$

In general, as for the $m^2 > 0$ case, one will grow towards the boundary and one will die away. There is an elegant theory of linear response [[McG]], which describes how we should treat the growing solution z^{Δ}_{-} as the *source* for the operator, i.e. the φ leading to the φ_0 in (4.15). We should then interpret the other solution as the *response*, i.e. the vacuum expectation value of $\mathcal{O}$ in the boundary theory in the presence of the source term coupling in the Lagrangian.

We will see in §**??** that the φ we use in (4.15) is a renormalized field, which scales like $z^{-\Delta_-}$ with z, which in turns scales as the d^μ. That is, φ_0 has dimension $\Delta = \Delta_-$. Therefore to make the action conformally invariant $\mathcal{O}$ must have $\Delta = \Delta_+$.

So we have the following picture: a scalar field of mass m couples at the boundary to an operator of dimenion Δ_+, which is the large root of (4.11). From now on, we will refer to Δ_+ a simply Δ, and we will denote the dual operator with dimension Δ as $\mathcal{O}_\Delta$.

4.3 Renormalisation Group Flow

Where have we seen this idea of regulating something in order to calculate correlation functions? It appears in renormalization theory. Indeed we see that when we identify z with the distance scale of the theory, ϵ becomes like a short distance or high energy cutoff, exactly mirroring the procedure in any field theory. Concomitantly, the whole Wilsonian picture of varying the cutoff of a field theory to see how the couplings run with the scale is dual to a nice geometrical picture of changing how far towards the boundary the field theory 'lives'. As the 'home plane' of the field theory wanders off into the asymptotics of the AdS space, its couplings flow into the UV. As this plane comes in towards the interior of AdS, the field theory couplings run into the IR.

Let us see explicitly how this works. The dynamical statement of the duality says that each field in the bulk couples to an operator on the boundary:

$$\mathcal{L} \supset \varphi_0 \, \mathcal{O},$$

where φ_0 should be considered as the leading term in the solution, i.e. z^{Δ_-}. Again it is stressed that (last eq.) implies that each *field configuration* of the bulk corresponds to a different quantum field theory, with different values for the couplings or perhaps completely new terms. Given a field φ coupling to an operator $\mathcal{O}$ there two things we can do.

Firstly, we could 'turn on' (change the field configuration to) a solution of φ which is of finite value φ_0 at the boundary. This corresponds to turning on the coupling $\varphi_0 \, \mathcal{O}$ in the field theory, i.e. actually changing the theory.

Secondly, we could turn on a φ solution of infinitesimal value, small enough not to effect the dynamics of the field theory, and finding its differential effect on the path intergral. This corresponds to 'propagating' $\mathcal{O}$ in the field theory to find its correlation functions. This is what we do in the standard procedure for finding correlators described at the beginning of the theory, and we will do it explicitly in the next section.

Henceforth we will refer to turning on a finite solution as simply 'turning on' the solution, and refer to turning on an infinitesimal solution as 'perturbing'.

Since the correspondence tells us that the value of a bulk field solution at a particular z *is* the the coupling of the dual operator at a certain point in the RG flow, the variation of the solution is the beta function for the coupling.

Solutions like that for $m^2 < 0$ above are called normalizable modes (not to be confused with *re*normalizability, although the two are connected) [2]. They decrease towards the boundary, i.e. the coupling φ_0 dies away at high energies. Therefore the dual operator $\mathcal{O}$ is a *relevant* operator in the field theory.

Solutions like that for $m^2 > 0$, which become infinite at the boundary, are called non-normalizable modes [3]. They are couplings which increase as we flow into the UV, so their corresponding operators $\mathcal{O}$ are *irrelevant* operators. For these modes we need to reconsider our assumption that the metric and scalar field have decoupled from each other. If such a solution is turned on there will be a large energy density near the boundary, causing back-reaction on the metric, and the asymptotics of the space-time will be changed. One of two things could happen:

- The solution might blow up so dramatically as to disfigure the asymptotics beyond all recognition. There might not even be a boundary of the space-time on which to define boundary conditions for the fields. Or if we still have a boundary, it will not be time-like. This destroys the conditions for the Graham-Lee theorem, so that we couldn't uniquely define a solution in the interior given some boundary conditions. In short, our scheme for finding correlators in the theory will fail. However, given the large space-time curvatures involved, our classical gravity assumption will probably break down as well. The assumption is this that we will be saved by some UV completion of the bulk theory, such as string theory, which will tell us what to do at high energies. Without knowledge of the UV physics, we cannot compute anything. This corresponds to the existence of a Landau pole in the RG flow.

[2] In fact the definition of a non-normalisable mode is that it is a solution to the equations of motion for which the (non-regularized) classical action is infinite.
[3] The classical action is infinite.

- The solution might not change the metric asymptotics too drastically, in which case there will still be a time-like boundary. Thus, by definition, the space is asymptotically anti-de Sitter. We can still do all of our computations of correlators in the theory using the bulk field values at this boundary. We do not need to know what goes on past our chosen cutoff ϵ to do this. This corresponds to us being saved by a UV fixed point in the RG flow, so that we can still do renormalisation by ignoring higher energies.

Solutions like that for $m^2 = 0$ are radially constant near the boundary. They will in general be non-normalizable, since the volume of AdS is infinite. So, at least in the approximation we are working in, these couplings do not run. Therefore the operators $\mathcal{O}$ are *marginal*. Quantum corrections in the bulk may correct the operator dimension, making it relevant or irrelevant.

4.4 Remarks

The field theory on the boundary is determined by two things:

- The dual gravity theory Lagrangian. Amongst other things this determines the field content of the boundary theory. Excitations of operators in the field theory are dual to excitations of fields in the bulk.
- *The boundary conditions of the gravity theory.* This fixes the interaction terms in the boundary theory Lagrangian.

We have discussed how the gravity theory needs to asymptote to a classical AdS geometry background. In order for this to happen, the gravity theory needs to have some classical solution of the coupled equations of motion for the metric and matter fields, in which the metric is AdS with a certain radius, and the fields have some definite radial behaviour. By *boundary conditions*, we mean that classical solution.

By turning on bulk field solutions, we are changing the boundary conditions of the gravity. We cannot turn on a mode dual to an irrelevant operator, because this would make the field theory non-renormalizable. We can however perturb this solution to find its correlation function in a theory were it hasn't been turned on.

So we see that given a gravity Lagrangian, we can construct lots of different dual field theories, which correspond to different boundary conditions for the bulk fields. We end this chapter with some explicit calculations of correlators. Thus fortified, we will be able to go on and put these tools to work in interesting and important situations.

4.5 Gravity calculation of correlators

If we can't do any explicit calculations, all of this is just a vague theoretical idea. They will be the subject of this section. In the field theory we want to calculate correlators of operators, each of which is dual to a bulk field. According to Witten, we must follow the prescription 4.1. There are two ways to do this. The first is in position space. This is easier to visualize and helps with intuition. The second is in momentum space. This is more abstract, but it is simpler to do and so more useful, especially in the case in two point functions, as we will see. We will present both methods for the two-point function of a scalar operator. Then we will calculate the three point function $\langle\Omega|\, \mathcal{O}_\Delta \mathcal{O}^*_\Delta \mathcal{J}^\mu \,|\Omega\rangle$ of two charged scalars and a conserved current.

4.5.1 Position space two-point function

This method was originally used in [[Wit98a]]. (4.1) says the first step is to evaluate the action for the classical solution of the bulk field with boundary condition $\varphi = \varphi_0$ at $z = \epsilon$, which we shall from now on refer to as the boundary. We do this using a Green's function method. That means, we find a solution to the field equation which is a delta function centred at a point (ϵ, y^μ) on the boundary. (Greek indices refer to Lorentz indices for the boundary metric, raised/lowered by $\eta_{\mu\nu}$, so y^μ, or just y for short, is a point on the boundary.) Then we multiply by $\varphi_0(y)$ and integrate over the boundary.

We can solve the equations of motion, as we saw in 4.1, simply by $\varphi(z, x^\mu) = c_\Delta z^{\Delta_+}$, where c_Δ is some constant. This solution becomes infinite at large z, and is singular at the point $z = \infty$ in the deep interior of the AdS.

Instead we need a function (a transverse space delta function) which is singular at a point y on the $z = 0$ surface (or rather $z = \epsilon$, but we'll come to this), and because of the transverse translational invariance of the metric we can take $y = 0$ without loss of generality. To proceed we note an important isometry of the metric which is used in almost all position space calculations in AdS. This is the *inversion* transformation:

$$
\begin{aligned}
z &\mapsto \frac{z}{z^2 + x^2} \\
x^\mu &\mapsto \frac{x^\mu}{z^2 + x^2} \quad ,
\end{aligned}
\tag{4.17}
$$

where $x^2 \equiv x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu$. This should not be confused with the inversion of just z, which we did in 2.3, and which is not an isometry. There is an important difference between them. Inverting just z changes the patch, so that the boundary is a different boundary after the transformation. Under (proper) inversion, which *is* an isometry, we stay with the same patch. All that has happened is:

- The point at $z = \infty$ in the interior has been mapped to the point $z = 0\,; x = 0$ on the boundary.
- The boundary is still at $z = 0$, and points on the boundary $z = 0\,; x \neq 0$ have been inverted, $x \mapsto \frac{x}{x^2}$.

We have found a solution that is a delta function at $z = \infty$. If we perform an inversion on it, it will still be a solution (because inversion is an isometry), and it will be singular instead at $z = 0\,; x = 0$. This is just what we need. So let us now perform the inversion on our solution (eq):

$$
c_\Delta z^{\Delta_+} \mapsto c_\Delta \frac{z^{\Delta_+}}{(z^2 + x^2)^{\Delta_+}} := c_\Delta K_\Delta(z, x; 0) \quad ,
\tag{4.18}
$$

So we have:

$$
K_\Delta(z, x; y) := \left(\frac{z}{z^2 + (x - y)^2} \right)^{\Delta_+}
\tag{4.19}
$$

(shows graphs of limit, in Mathematica? Contour/shaded plot?) where the reason for the K notation will become clear soon. As promised this is singular at $z = 0\,; x = 0$. In fact $z^{-\Delta_-} K_\Delta(z, x; 0)$ *is* proportional to a delta function on the boundary, centered around $x = 0$. We can see this by noting:

- As we take the limit

$$\lim_{x \to 0} z^{-\Delta_-} K_\Delta(z, x; 0) \tag{4.20}$$

  the function, which is a Lorentzian curve as a function of x for fixed z, becomes thinner and narrower, until at $z = 0$ it only has support at $x = 0$. This is shown in fig. 4.2.

- The integral

$$I = \int d^d x z^{-\Delta_-} K_\Delta(z, x; 0) \tag{4.21}$$

  stays constant as we take the limit. We see this by noting that I can only be $\propto z^\lambda$, for some $\lambda \in \mathbb{R}$. But by dimensional analysis $\lambda = 0$, so I is a constant. By normalizing K correctly, we can make $I = 0$.

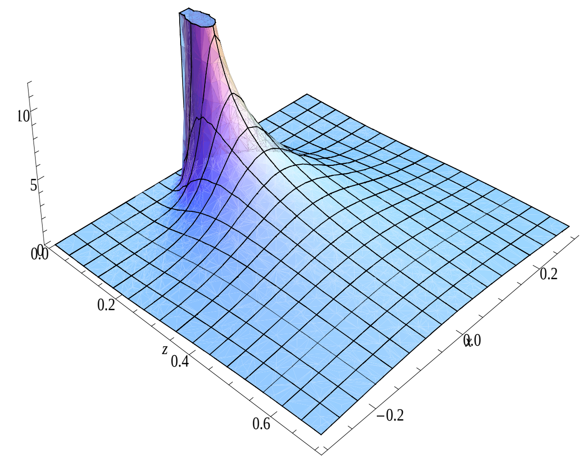

Figure 4.2: The bulk-to-boundary propagator (modulo a power of z), for a free scalar field in AdS. As $z \to 0$, its support falls away apart from at $x = 0$.

We can construct the solution with a given boundary behaviour by integrating over the boundary:

$$\overline{\varphi}(z,\, x) = \int d^d x' \varphi_0(x') K_\Delta(z, x;\, x') \quad . \tag{4.22}$$

For this reason, $K_\Delta(z, x;\, x')$ is known as the *bulk-to-boundary propagator*. It is the Green's function giving the effect on the bulk field of a delta function disturbance at the boundary. From the above argument, $\overline{\varphi}$ does not tend to $\varphi_0(x)$ at the boundary, but instead behaves like $z^{\Delta-}\varphi_0(x)$. Actually the φ_0 we have written here is actually a *renormalized* field. The bare field, $\varphi_0^{\rm B}(x)$ is what we need the field to be equal to at $z = 0$. We can therefore deduce:

$$\overline{\varphi}(x) = z^{\Delta_-}\, \varphi_0(x) = z^{\Delta_-}\, \epsilon^{-\Delta_-}\, \left(\epsilon^{\Delta_-}\, \varphi_0(x)\right) \tag{4.23}$$

$$\Rightarrow \varphi_0^{\rm B} = \epsilon^{\Delta_-} \varphi_0(x) \quad . \tag{4.24}$$

Notice that if we keep $\varphi_0(x)$ finite, then $\varphi_0^{\rm B}$ becomes infinite as $\epsilon \to 0$. We have literally scaled away this infinity, in exact parallel with the QFT renormaliztion scheme. For massive fields in bulk, we deal with renormalized fields and the bulk-to-boundary propagator, as above.

The next step is to calculate the action for this classical solution. Up to a constant factor we can express the action as, using integration by parts,

$$S[\varphi] = \int d^{d+1}x \sqrt{-g}\, g^{MN} \partial_M \varphi\, \partial_N \varphi \tag{4.25}$$

$$= \int d^{d+1}x \left\{ \partial_M \left(\sqrt{-g}\, g^{MN} \varphi\, \partial_N \varphi\right) \,-\, \varphi\, \partial_M (\sqrt{-g}\, g^{MN} \partial_N \varphi) \right\} \tag{4.26}$$

$$= \int_{AdS} \left\{ {\rm d}(\varphi * {\rm d}\varphi) \,-\, \varphi\, {\rm d}(*{\rm d}\varphi) \right\} \quad , \tag{4.27}$$

where capital letter indices denote are for the bulk spacetime. The equation of motion reads ${\rm d} * {\rm d}\overline{\varphi} = (*1)\, \Box \overline{\varphi} = 0$, so the second term vanishes on shell. Now, using Stokes' theorem, we can write the first term as a boundary term

$$S[\overline{\varphi}] = \int_{\partial AdS} \varphi * \mathrm{d}\varphi \tag{4.28}$$

$$= \int d^d x \sqrt{-\gamma_\epsilon}\, \varphi\, n^M \partial_M \varphi \quad , \tag{4.29}$$

where again greek indices are for the boundary spacetime, γ_ϵ is the metric induced by g on the boundary ($z = \epsilon$), and $n = n^M \frac{\partial}{\partial x^M}$ is the outward facing unit normal to this boundary. By symmetry, $n \propto \frac{\partial}{\partial z}$, so let $n = A\frac{\partial}{\partial z}$. Now if we require it to be a unit vector, we need

$$1 \overset{!}{=} g(z = \epsilon)_{MN} n^M n^N = \frac{R^2}{\epsilon^2} A^2 \quad \Rightarrow \quad A = \pm \frac{\epsilon}{R} \quad . \tag{4.30}$$

To make it point outwards, we choose the negative sign. We have $\sqrt{\gamma_\epsilon} = (\frac{R}{\epsilon})^d$. So up to constants:

$$S[\overline{\varphi}] = \int d^d x \left(\frac{R}{\epsilon}\right)^{d-1} \varphi\, \partial_z \varphi|_{z=\epsilon} \quad . \tag{4.31}$$

Now we have, using the notation $\alpha := z^2 + (x - x')^2$ for clarity:

$$\overline{\varphi}(z, x) = c_\Delta \int d^d x'\, \varphi_0(x) \frac{z^{\Delta_+}}{(z^2 + (x - x')^2)^{\Delta_+}} \tag{4.32}$$

$$\Rightarrow \quad \partial_z \overline{\varphi}(z,\, x) = c_\Delta \int d^d x'\, \varphi_0(x) \frac{\alpha^{\Delta_+} \Delta_+ z^{\Delta_+ - 1} - z^{\Delta_+} \Delta_+ 2z\alpha^{\Delta_+ - 1}}{\alpha^{2\Delta_+}} \tag{4.33}$$

$$= c_\Delta \int d^d x'\, \varphi_0(x) z^{\Delta_+ - 1} \alpha^{\Delta_+ - 1} \frac{-z^2 + (x - x')^2}{(z^2 + (x - x')^2)^{2\Delta_+}} \tag{4.34}$$

In the limit $z \to 0$ this becomes

$$c\, \Delta_+ z^{\Delta_+ - 1} \int d^d x' \varphi_0(x')\, \frac{1}{(x - x')^{2\Delta_+}} \tag{4.35}$$

We have already found that $\overline{\varphi}(x) \propto z^{\Delta_-} \varphi_0(x)$ at the boundary. So now, plugging both of these into (4.31), up to constants as always:

$$S[\overline{\varphi}(x)] = \epsilon^{1-d} \int d^d x \, d^d x' \, \epsilon^{\Delta_-} \varphi_0(x) \, \varphi_0(x') \frac{1}{(x-x')^{2\Delta_+}} \tag{4.36}$$

$$= \epsilon^{1-d+\Delta_+ + \Delta_- -1} \int d^d x d^d x' \, \varphi_0(x) \, \varphi_0(x') \frac{1}{(x-x')^{2\Delta_+}} \tag{4.37}$$

We have that $\Delta_+ + \Delta_- = d$, so we see that using our renormalized bulk fields the action becomes finite. So we have the result:

$$S[\overline{\varphi}(x)] = \int d^d x d^d x' \frac{\varphi_0(x) \, \varphi_0(x')}{(x-x')^{2\Delta_+}} \quad . \tag{4.38}$$

And finally:

$$\boxed{\langle \Omega | \, \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(x') \, | \Omega \rangle = \frac{\delta}{\delta \varphi_0(x)} \frac{\delta}{\delta \varphi_0(x')} \, S[\overline{\varphi}(x)] = \frac{1}{(x-x')^{2\Delta_+}}} \tag{4.39}$$

which is correct up to an uninteresting normalization. Note that this is the form the two-point function must take in a conformal field theory, for a field of conformal weight Δ_+. It is a triumph for Witten's postulate that this calculation works out correctly. We will now go on to show that this prescription seems to work for any type of bulk field, providing the correct form for a conformal field theory.

We have used a renormalization idea identical to that in QFT and found a correlation function. Except that instead of complicated Feynman integrals (over infinitely many fields, furthermore, since we are in general taking the large N field theory limit), we have only had to deal with much a simpler calculation in classical gravity. In this way the correspondence maps 'very quantum' physics (where the coupling is large and loop effects dominate) to 'very classical' physics in which quantum loops can be ignored. All we have to do is map between physics in completely different spacetimes! This is a revolutionary concept, and one which means we should definitely get excited about gauge-gravity duality.

4.5.2 Gauge Field Propagator

In the last section we considered the simplest possible action: that for a free scalar field. Of course, to be interesting the bulk theory will in general have fields in different representations of the Lorentz group, and interaction terms between them. For example we may have gauge fields A^a_μ, dual to boundary current operators $\mathcal{J}^{a\mu}$, where the index a is for the adjoint representation of a bulk G. Or there might be several scalars fields φ^a rotating amongst themselves in the adjoint of G, with G-invariant interactions $f^{abc}\varphi^a\varphi^b\varphi^c$. And there will have to be a renormalizable Einstein-Hilbert term to describe perturbative gravity fluctuations about the AdS background, i.e. $\frac{1}{16\pi G_\mathrm{N}}\mathcal{R}$. All of these fields will have duals in corresponding representations of the boundary symmetry group (which must be isomorphic to the bulk one). And the bulk field must couple to the boundary operators in a way which preserves this symmetry group.

For example let's work out the bulk-to boundary propagator for a $U(1)$ bulk gauge field.

Maxwell's equations read:

$$\mathrm{d} * \mathrm{d}A = 0 \tag{4.40}$$

Without loss of generality we can Lorentz transform in x^μ so to a frame in which the field has no momentum along the boundary: $A = A(z)$. We also work in a gauge in which the gauge field has a non-zero component only in one of the dx^μ, which we will take to be $x^0 = t$. So we have $A = A(z)\,dt$. Now we solve (4.40):

$$A = A(z)\,dt \tag{4.41}$$

$$\Rightarrow \quad \mathrm{d}A = A'(z) dz \wedge dt \tag{4.42}$$

$$\Rightarrow \quad *\mathrm{d}A = \frac{\sqrt{-g}}{(d-1)!}\, g^{zz} g^{tt} A'(z)\,(d-1)!\, dx^1 \wedge dx^2 \wedge dx^3 \wedge \cdots \wedge dx^{d-1} \tag{4.43}$$

$$= \frac{R^{d+1}}{z^{d+1}}\,\frac{z^2}{R^2}\,\frac{z^2}{R^2} g^{tt} A'(z)\,(d-1)!\, dx^1 \wedge dx^2 \wedge dx^3 \wedge \cdots \wedge dx^{d-1} \tag{4.44}$$

$$= \frac{R^{d-3}}{z^{d-3}}\, A'(z) dx^1 \wedge dx^2 \wedge dx^3 \wedge \cdots \wedge dx^{d-1} \tag{4.45}$$

$$\Rightarrow \quad \mathrm{d} * \mathrm{d}A = \left(\frac{L^{d-3}}{z^{d-3}} A'(z)\right)'\, dz \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \cdots \wedge dx^{d-1} \tag{4.46}$$

Substituting into (4.40) yields:

$$\left(\frac{L^{d-3}}{z^{d-3}}A'(z)\right)' = 0 \tag{4.47}$$

$$\Rightarrow \quad \frac{L^{d-3}}{z^{d-3}}A'(z) = \text{constant} \tag{4.48}$$

$$\Rightarrow \quad A'(z) = \text{const} \times \ z^{d-3} \tag{4.49}$$

$$\Rightarrow \quad A(z) \propto z^{d-2} \quad . \tag{4.50}$$

So just as for the massless scalar field the behaviour was $\phi \sim z^d$, here we have $A \sim z^{d-2}$. Now we can use exactly the same trick as we did there: we make the inversion (4.17).

In the scalar case (4.18) we only needed to transform the coordinates, but this is a tensor field so we need to worry about its non-trivial Lorentz transformation:

$$\bar{A}_A = \frac{\partial x^B}{\partial \bar{x}^A} A_B \tag{4.51}$$

where $A,\ B = 0, 1, ...d$ are bulk spacetime indices.

We need to calculate the inverse Jacobian factor in (4.51) for the inversion transformation:

$$\bar{x}^A = \frac{x^A}{z^2 + x_\mu x^\mu} = \frac{x^A}{x_B x^B} \tag{4.52}$$

$$\Rightarrow \quad x^A = \frac{\bar{x}^A}{\bar{x}_B \bar{x}^B} \tag{4.53}$$

$$\Rightarrow \quad \frac{\partial x^A}{\partial x^B} = \frac{\partial}{\partial \bar{x}^B}\left(\frac{1}{\bar{x}_C \bar{x}^C}\right)\bar{x}^A + \frac{1}{\bar{x}_C \bar{x}^C}\frac{\partial \bar{x}^A}{\partial \bar{x}^B} \tag{4.54}$$

$$= -\frac{1}{(\bar{x}_C \bar{x}^C)^2} \cdot 2\bar{x}_B \bar{x}^A + \frac{1}{\bar{x}_C \bar{x}^C}\,\delta^A_B \tag{4.55}$$

$$= \frac{-2}{(\bar{x}_C \bar{x}^C)^2}\left\{\bar{x}^A \bar{x}_B - \frac{1}{2}\delta^A_B\,(\bar{x}_C \bar{x}^C)\right\} \tag{4.56}$$

$$= \frac{1}{\bar{x}_C \bar{x}^C}\left\{\delta^A_B - 2\frac{\bar{x}^A \bar{x}_B}{\bar{x}_C \bar{x}^C}\right\} \tag{4.57}$$

This gives us the bulk-to-boundary propagator for a gauge field of polarization in the x^μ direction at boundary point y^μ to propagate in to point

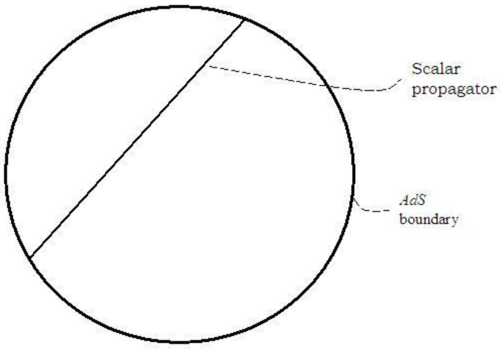

Figure 4.3: The correlation function $\langle\Omega|\, \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(x')\,|\Omega\rangle$ expressed as a Witten Diagram.

(z, x^μ) with polarization along x^A:

$$K^i_A(z,x;y) = \frac{z^{d-2}}{(z^2+(x-y)^2)^{d-1}}\left\{\delta^i_A - 2\frac{x^i x_A}{z^2+(x-y)^2}\right\} \quad . \qquad (4.58)$$

4.5.3 Witten diagrams

Notice that the scalar two-point function is just the limit of the bulk-to-boundary propagator, as $z \to 0$, up to a (divergent) factor:

$$\lim_{z\to 0} K_\Delta(z,x;x') = \frac{z^{\Delta_+}}{(x-x')^{2\Delta}} \propto \langle\Omega|\, \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(x')\,|\Omega\rangle \quad . \qquad (4.59)$$

We could choose to represent it as in figure fig. 4.3.

That is, we 'propagate' the scalar field from one point x on the boundary to another, x'. This may seem superfluous and useless, but read on and we will see why these little diagrams are so conceptually and computationally useful.

Using a free field action, we can calculate the two point functions of the dual boundary operators for different types of bulk field, as we did for the scalar case. For example we could have calculated the two-point function of two boundary currents, $\langle\Omega|\, \mathcal{J}^\mu(x_1)\mathcal{J}^\nu(x_2)\,|\Omega\rangle$. Just as in the scalar case, the two-point functions are the boundary limit of the corresponding bulk-to-boundary propagators.

Now what happens when we introduce interactions? These will cause the classical action to be hard to evaluate exactly, but we can solve as a perturbation series in the coupling constant, call it λ. The result we obtain is identical to the tree level approximation of Feynman diagram representing the different possible bulk processes for the correlation function. To calculate boundary correlation functions, we must scatter the dual bulk fields and find the amplitudes! This is our classical dual of the horrible strongly coupled quantum mess which occurs in the bulk.

As an example let's take a complex scalar field, minimally coupled to a $U(1)$ gauge field, all in the bulk. The action is:

$$S = -\int d^{d+1}x \left\{ \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \ + (\partial_\mu \phi + ieA_\mu \phi)(\partial^\mu \phi^* - ieA^\mu \phi^*) + m^2 \phi^* \phi \right\} \tag{4.60}$$

The cubic interaction term in the Lagrangian is:

$$ieA^\mu \left\{ \phi \partial_\mu \phi^* \ - \ \phi^* \partial_\mu \phi \right\} \tag{4.61}$$

In the dual theory we want to find the three-point correlation function

$$\langle \Omega | \, \mathcal{J}^\mu(x_3) \mathcal{O} *_\Delta (x_1) \mathcal{O}_\Delta(x_2) \, |\Omega\rangle \quad . \tag{4.62}$$

We use our new technique. Instead of laboriously finding the classical solution, evaluating the action and differentiating, we just write down the corresponding Witten diagram for the $O(e)$ vertex (fig. 4.4). We evaluate it using the familiar position space Feynman rules.

Holographically, we can treat each of ϕ, ϕ^* as a real scalar field of dimension Δ given by (4.11), with dual operators $\mathcal{O}_\Delta, \mathcal{O}^*_\Delta$, as above. Each has the propagator (4.19). So the diagram becomes, up to constants:

$$e \int d^{d+1}x \sqrt{-g}\, K^i_A(z,x;x_3)\, \partial^A K_\Delta(z,x;x_1) K_\Delta(z,x;x_3) \tag{4.63}$$

If we substitute the respective propagators into this expression, we get a bit of mess. To evaluate, it the trick is go into the most convenient coordinate system. First, we set $x_3 = 0$ without loss of generality, because of translational invariance. Then we redo the inversion calculation leading to (4.58), so that K^i_A becomes $\bar{x}^{d-2}$ again.

We must also work out how the scalar propagators in (**??**) transform under

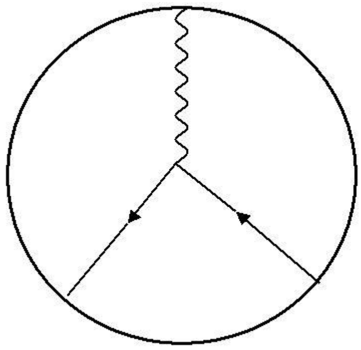

Figure 4.4: The three-point interaction vertex for the scalar QED theory, as a Witten diagram. From it we can immediately write down the expression for a boundary correlation function.

the inversion. In general for the inversion we have for two $(d+1)$-vectors x^A and y^A:

$$(x-y)^2 \equiv (x-y)_A(x-y)^A = \left(\frac{\bar{x}}{\bar{x}^2} - \frac{\bar{y}}{\bar{y}^2}\right)^2 \tag{4.64}$$

$$= \left(\frac{\bar{x}^2}{\bar{x}^4} + \frac{\bar{y}^2}{\bar{y}^4}\right)^2 - 2\frac{\bar{x}\bar{y}}{\bar{x}^2\bar{y}^2} \tag{4.65}$$

$$= \frac{1}{\bar{x}^2} + \frac{1}{\bar{y}^2} - 2\frac{\bar{x}\bar{y}}{\bar{x}^2\bar{y}^2} \tag{4.66}$$

$$= \frac{\bar{x}^2 + \bar{y}^2}{\bar{x}^2\bar{y}^2} - 2\frac{\bar{x}\bar{y}}{\bar{x}^2\bar{y}^2} \tag{4.67}$$

$$= \frac{(\bar{x} - \bar{y})^2}{\bar{x}^2\bar{y}^2} \quad . \tag{4.68}$$

Then considering the $(d+1)$-vectors $X \equiv (z, x)$ and $X_1 \equiv (0, x_1)$ we have the tranformation:

$$K_\Delta(z,x;x_1) = \left(\frac{z}{z^2+(x-x_1)^2}\right)^\Delta \tag{4.69}$$

$$= \left(\frac{\bar{z}/\bar{X}^2}{(\bar{X}-\bar{X}_1)^2/\bar{X}^2\bar{X}_1^2}\right)^\Delta \tag{4.70}$$

$$= \left(\frac{\bar{z}}{(\bar{X}-\bar{X}_1)^2}\right)^\Delta \cdot \bar{X}^2 \tag{4.71}$$

$$= \left(\frac{\bar{z}}{\bar{z}^2+(\bar{x}-\bar{x}_1)^2}\right)^\Delta \cdot \bar{X}^2 \quad . \tag{4.72}$$

Now we plug the transformed propagators back into (4.63). Things to notice in transforming the intergral:

- Since inversion is an isometry, $\sqrt{-g}$ does not need to be transformed.
- The index on the ∂_A dervative does not transform because it is contracted with the gauge field propagator. The correlator becomes

$$g\bar{x}_1^2\bar{x}_2^2 \int d^{d+1}\bar{x}\frac{1}{\bar{z}^{d+1}}\bar{z}^{\Delta-2}\delta_A^i \overbrace{\bar{z}^2}^{=g^{\mu\mu}} \partial_A\left(\frac{\bar{z}}{z^2+(\bar{x}-\bar{x}_1)^2}\right)^\Delta \tag{4.73}$$

$$= g\bar{x}_1^2\bar{x}_2^2 \int d^{d+1}\bar{x}\frac{1}{\bar{z}}\partial_i\left(\frac{\bar{z}}{z^2+(\bar{x}-\bar{x}_1)^2}\right)^\Delta\left(\frac{\bar{z}}{\bar{z}^2+(\bar{x}-\bar{x}_2)^2}\right)^\Delta - (\bar{x}_1 \leftrightarrow \bar{x}_2) \tag{4.74}$$

This still looks quite difficult. To proceed we use a nifty trick of the type Feynman liked, by noting following:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\bar{z}}{z^2+(\bar{x}-\bar{x}_1)^2}\right)^\Delta = \frac{\partial}{\partial x_1^\mu}\left(\frac{\bar{z}}{z^2+(\bar{x}-\bar{x}_1)^2}\right)^\Delta \quad . \tag{4.75}$$

This allows us to change the ∂_μ in (4.74) with $\partial_{x_1^\mu}$, which we can then take outside the integration sign. (4.74) becomes:

$$e\bar{x}_1^2\bar{x}_2^2\,\partial_{x_1^\mu} \int d^{d+1}\bar{x}\frac{1}{\bar{z}}\left(\frac{\bar{z}}{z^2+(\bar{x}-\bar{x}_1)^2}\right)^\Delta\left(\frac{\bar{z}}{\bar{z}^2+(\bar{x}-\bar{x}_2)^2}\right)^\Delta - (\bar{x}_1 \leftrightarrow \bar{x}_2) \quad . \tag{4.76}$$

Now let's look at this integral:

- It is translationally invariant on the boundary: if we shift $\bar{x}_1$ and $\bar{x}_2$ by the same distance each, we can just shift the integration variable to bring the integral back to the same form.

- Given the above, it can be evaluated up to a numerical consant by dimensional analysis.

The integral has length dimension $= d - 2\Delta$, therefore it is proportional to

$$|\bar{x}_1 - \bar{x}_2|^{d-2\Delta} \quad . \tag{4.77}$$

Putting this into (4.76), we have

$$\langle\Omega| \mathcal{J}^{\mu}(x_3)\mathcal{O}*_{\Delta}(x_1)\mathcal{O}_{\Delta}(x_2) |\Omega\rangle = \text{constant} \times e(\partial^{\bar{x}_1}|\bar{x}_1 - \bar{x}_2|^{d-2\Delta} - (\bar{x}_1 \leftrightarrow \bar{x}_2) \tag{4.78}$$

Doing the derivative:

$$\text{constant} \times e(\bar{x}_1^2\bar{x}_2^2)^{\Delta}\, 2\bar{x}_1^{\mu}(\frac{d}{2} - \Delta - 2)|\bar{x}_1 - \bar{x}_2|^{d-2\Delta} \; - \; (\bar{x}_1 \leftrightarrow \bar{x}_2) \tag{4.79}$$

Finally, changing back to the original coordinates:

$$\text{constant} \times e\,\frac{x_1^{2\Delta}x_1^{2\Delta}}{x_1^{4\Delta}x_1^{4\Delta}}\,\frac{x_1^{\mu}}{x_1^2}\left(\frac{|x_1 - x_2|}{|x_1||x_2|}\right)^{d-2\Delta-2} - (x_1 \leftrightarrow x_2) \tag{4.80}$$

$$= \text{constant} \times e\left(\frac{x_1^{\mu}}{x_1^2} - \frac{x_2^{\mu}}{x_2^2}\right)\frac{|x_1 - x_2|^{d-2\Delta-2}}{(|x_1||x_2|)^{d-2}} \tag{4.81}$$

This is exactly the form we expect from conformal covariance. This may seem like a tautology, since surely this was bound to happen? But remember that although there is a shared symmetry group we have no *a priori* reason to expect that the *AdS* geometry describes the conformal field theory using a dynamical correspondence of the form (4.1). Every different correlator we calculate gives further evidence of the gauge-gravity duality.

We have in fact done something more than this. If we had really done the integral in (4.76), which is possible using Feynman parameters, we would have calculated the coefficient of the correlator, which is non-trivial since it depends on the detailed dynamics of the CFT. Furthermore, as will be made clear in §5.3, the fact that we have used the Einstein gravity approximation to the bulk theory means we would have done this at strong field theory coupling. So we have done an inherently non-perturbative problem, using perturbative bulk dynamics. This shows that the strongly coupled physics is like the weakly coupled version, only seen with another dimension, and gravity!

5 Examples ; $\mathcal{N} = 4$ Super Yang-Mills

Up until now, we have studied the basic ideas of gauge-gravity duality. But, perhaps infuriatingly, we have provided no concrete, 'real life' examples. Which are these gauge theories? Do they include the sort of gauge theories that have been so successful in describing particle physics since the Second World War, namely $SU(3) \times SU(2) \times U(1)$?

Now, excitingly, with all the tools at our disposal, we are able to give examples of explicit gauge theories, and the quantum gravities that they are conjectured to be dual to. Currently, nobody knows the gravity dual for the standard model. In particular, as mentioned in the introduction, it is a long term goal of physics to find a dual to Quantum Chromodynamics, which is of particular interest because no precise analytical structure has yet been developed to deal with its low-energy, strong coupling regime consisting of baryons and mesons.

The standard model theories are hard to dualize essentially because they have too little symmetry. But if we impose more symmetry on our gauge theories, i.e. we make them supersymmetric[1]. Indeed a large part of the work in gauge-gravity duality in the 12 years since its inception has consisted of non-trivial checks on the duality between superstring theory and SUSY gauge theories.

[1] The Coleman-Mandula theorem limits us to this type of extended symmetry. ([CM67]), there arise natural candidates for their duals. These are the supersymmetric string theories and M-theory, in 10 and 11 space-time dimensions respectively. The first example of such a duality was conjectured by Maldecena in his seminal paper of 1997 [Mal98], and caused an explosion in the study of all related topics[2]

5.1 The Original - Discussion

This is the original example of the correspondence, was proposed by Maldacena in [Mal98], and in many ways is the simplest to understand. The most precise way to state it is this:

$D = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory is dual to type IIb superstring theory with $AdS_5 \times S^5$ boundary conditions.

We have discussed what we mean by the boundary conditions in the last chapter, i.e. there is some low energy effective theory for IIb string theory. This is known to be 10-dimensional type IIb supergravity. There is a solution to the supergravity equations of motion, in which the metric is that of $AdS_5 \times S^5$. *Asymptotically*, this is the boundary condition we impose on the quantum gravity theory.

So now we know what must happen near the boundary of spacetime, in order for the dual field theory to be $\mathcal{N} = 4$ Super Yang-Mills. What can happen in the interior? The answer is that the full string theory happens in the interior. This includes the high energy string excitations, and highly quantum effects that occur at strong coupling. These bulk dynamics are described by the path integral on the RHS of (4.1), and the integral is taken over all possible string/field states in the string theory.

The landscape of string theory has not yet been fully charted; in a full, non-perturbative description of string/M-theory, we should expect that the fields themselves are effective degrees of freedom formed by states of strings, branes, or even other, as yet unknown exotic objects.

But there are certain limits of string theory which are well known. In type IIb superstring theory, this is limit in which the string coupling g_s and string length $\sqrt{\alpha'}$ are both small. Then we can consider:

- A background metric/field configuration, which is an exact solution to the classical equations of supergravity.
- Quantum excitations excitations of the fields about this classical backgroud. These are governed by a perturbation series in g_s.
- Stringy excitations about the classical background: 1) closed strings

moving freely on the background, which are like gravitons and 2) open strings tied at each end the p-branes appearing in the supergravity backgroud. These are governed by a perturbation series in α'.

When the coupling is small enough, we can make a tree-level approximation to the quantum excitations, i.e. ignore loops. This the approximation we have been using so far in this review, for example when we calculated correlation functions at the end of the last chapter.

We have seen how to use this approximation. The one question remaining is: what should we use as the classical background? We know from this discussion that it should a) be a solution of classical IIb supergravity, and b) be asymptotically the $AdS_5 \times S^5$ solution.

The answer is this:

Different interior gravity solutions correspond to different equilibrium *conditions* for the field theory.

What do we mean by different conditions for the field theory? We mean things like different temperatures, chemical potentials, or sources for field (e.g. static heavy quarks sitting at a point in spacetime, which is like considering quantum electrodynamics between charged plates or next to a current-carrying wire). The thing that all these conditions have in common is this: at high enough energies, the effects of the conditions go away, and we are left with the 'pure' field theory, which is at zero temperature with no sources or thermodynamic potentials present.

This has a dual geometrical interpretation: low energies correspond to the interior of the bulk, where there is some choice of background. As we increase the energy, we approach the boundary, where we have specified the boundary conditions. Indeed, as we discussed in the remarks at the end of the last chapter, there is a one-to-one correspondence between boundary conditions and field theories. Now we can add to this statement, by saying that there is also a one-to-one correspondence between equilibrium conditions for a field theory and interior backgrounds for the bulk. Being on shell in the bulk theory corresponds to being in equilibrium in the field theory, and fluctuations about equilibrium are the quantum perturbations about the classical bulk background.

5.2 Ingredients

If we take particular boundary conditions, then the dual field theory is at zero temperature, with no sources or interesting thermodynamics. These are the field theories used to describe high-energy physics at particle colliders, for example at CERN, where experiments may eventually begin at some point. They are also the starting point for most studies of field theory, since they are the simplest. In these theories, the equilibrium condition is the ground state $|\Omega\rangle$ of the theory, and we consider quantum excitations above this state.

For the $\mathcal{N} = 4$ duality, the dual geometry the zero temperature, no source theory is the the same as the asymptotic geometry, i.e. $AdS_5 \times S^5$, with the corresponding field configurations. In this section we will describe $\mathcal{N} = 4$ Super Yang-Mills, then describe type IIb supergravity and its $AdS_5 \times S^5$ solution. Then in the next section we will get down comparing to two.

5.2.1 $\mathcal{N} = 4$ Super Yang-Mills

In $D = 4$, $\mathcal{N} = 4$ is the most supersymmetry possible without gravity. Then the unique type of supermultiplet is the vector multiplet, consisting of: 1 scalar A_μ; 4 Weyl fermions λ^b_α; and 6 scalars ϕ^i. This can be constructed by taking a Clifford vacuum and acting with lowering operators, as shown in fig. 5.1. There is an $SU(\mathcal{N}) = SU(4)$ global R-symmetry, with 15 generators T^{bc}, which rotates fields of the same spin: the $F_{\mu\nu}$ in the $\mathbf{1}$, the ϕ^b's in the $\bar{\mathbf{4}}$, the ϕ^i's in the $\bar{\mathbf{6}}$.

To get the Yang-Mills theory we take N vector multiplets and let them transform in the adjoint of the gauge group, which we can choose but take to be $SU(N)$. The unique action is:

$$S = \frac{1}{g^2_{YM}} \mathrm{Tr} \int \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_b i \bar{\lambda}_{\dot{\alpha}\, b} \bar{\sigma}^{\mu\, \dot{\alpha}\alpha} D_\mu \lambda^b_\alpha - \sum_i D_\mu \phi^i D^\mu \phi^i \right. \tag{5.1}$$
$$\left. + \sum_{b,c,i} \left(C_{ibc} \lambda^{b\alpha} [\phi^i, \lambda^c_\alpha] + \bar{C}^{bc}_i \bar{\lambda}_{b\dot{\alpha}} [\phi^i, \bar{\lambda}_{c\dot{\alpha}}] \right) + \frac{1}{2} \sum_{i,j} [\phi^i, \phi^j]^2 \right\}$$

α's and $\dot{\alpha}$'s, and other letters from the beginning of the Greek alphabet are Lorentz spinor indices; μ's and ν's are Einstein indices. Roman letters are $SU(4)_R$ indices; gauge indices have been suppressed.

There is also a possible instanton term, $\frac{\theta_I}{8\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F^{\mu\nu}})$, which is topological in nature, and will not appear in our analysis.

$$|0\rangle \qquad \mathbf{1}$$

$$Q_1|0\rangle \quad Q_2|0\rangle \quad Q_3|0\rangle \quad Q_4|0\rangle \qquad \mathbf{4}$$

$$Q_1Q_2|0\rangle \quad Q_1Q_3|0\rangle \quad Q_1Q_4|0\rangle \quad Q_2Q_3|0\rangle \quad Q_2Q_4|0\rangle \quad Q_3Q_4|0\rangle \qquad \mathbf{6}$$

$$Q_1Q_2Q_3|0\rangle \quad Q_1Q_2Q_4|0\rangle \quad Q_1Q_3Q_4|0\rangle \quad Q_2Q_3Q_4|0\rangle \qquad \bar{\mathbf{4}}$$

$$Q_1Q_2Q_3Q_4|0\rangle \qquad \mathbf{1}$$

Figure 5.1: Clifford state diagram of the $D = 4$, $\mathcal{N} = 4$ massless state diagram. (The states are massless because explicit mass terms would break gauge invariance.) The $SU(4)_R$ symmetry mixes the states in each 'layer', in the labelled representation. The fields are obtained by gathering together these states into representations of $SL(2, \mathbb{C})$

In general the Lagrangian of any theory will change as we flow in the renormalisation group. Each coupling flows according to its dimension, and additional terms appear. The 4D Yang-Mills coupling is dimensionless, so *classically* the coupling is scale invariant, and BRST symmetry means that no terms appear in the flow. But quantum corrections cause the coupling to flow, so that we still have to renormalize the theory. These are generic features for any 4D Yang-Mills theory, and still apply if we couple scalars and fermions to the gauge field.

Supersymmetry helps with this situation, by cancelling some of the quantum corrections, e.g. in $\mathcal{N} = 1$ SYM the remaining corrections are all logarithimic ($\propto \log \Lambda$). The more supersymmetry the better, as it were, and for $\mathcal{N} = 4$, miraculously they completely disappear. This is very unusual for a quantum field theory.

This fact means that $\mathcal{N} = 4$ SYM is fully scale invariant, and in fact it is invariant under the full conformal group. It is therefore a CFT. The super-Poincaré algebra (P_μ; $M_{\mu\nu}$; Q^b_α; T^{bc}) and conformal algebra (K_μ; K) combine into the *superconformal* superalgebra. This contains the conformal and super-Poincaré as subalgebras, but also contains *conformal supersymmetries* S^b_α. These arise as the commutator between the conformal and supersymmetry generators. In summary, an element of the superconformal algebra looks like this:

$$\left(\begin{array}{c|c} P_\mu\, M_{\mu\nu}\, K_\mu\, D & Q^b_\alpha\, \bar{S}^b_{\dot\alpha} \\ \hline \bar{Q}^b_{\dot\alpha}\, S^b_\alpha & T^{bc} \end{array}\right) \tag{5.2}$$

This superalgebra has the structure $SU(2,2|4)$.

5.2.2 Type IIb Supergravity

Supergravity [[vN82]] is a supersymmetric effective field theory of gravity. It is non-renormalizable, but is a well-defined classical theory. It can be constructed by requiring an action which is invariant under:

- Local Poincaré transformations. This is the veilbein formulation of the diffeomorphism invariance of Einstein gravity.
- Local supersymmetry transformations.

Said another way, we need to gauge the Super-Poincaré group. All of these gauge symmetries must have gauge fields associated with them:

- Translations: the vielbein one-form $e^a = e^a_\mu dx^\mu$.
- Lorentz transformations: the connection one-form $\omega^{ab} = \omega^{ab}_\mu dx^\mu$.
- Supersymmetry transformations: the gravitino one-form $\psi = \psi_\mu$.

Here a, b, c etc. are local Lorentz indices, and μ, ν are Einstein indices.

To build a gauge invariant action, we need a covariant derivative. For a vector v^μ we should first express it in the vielbien basis: $v^a = e^a_\mu v^\mu$, then the covariant derivative is

$$D_\mu v^a = \partial_\mu v^a + \omega_{\mu\,b}^{\,a} v^b \quad . \tag{5.3}$$

Acting on a spinor, it is

$$D_\mu \lambda = \partial_\mu \lambda + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\lambda \quad , \tag{5.4}$$

where the γ^a's are the Dirac matrices in the relevant dimension, so that $\frac{1}{2}\gamma^{ab} \equiv \frac{1}{4}[\gamma^a,\, \gamma^b]$ are the generators of Lorentz transformations in the spinor representation.

From these fields we can form the Riemann curvature tensor:

$$R_{ab}(e,\omega) \equiv D\omega^{ab} = \mathrm{d}\omega^{ab} + \omega^a_{\ c} \wedge \omega^{cb} \tag{5.5}$$

Now we can define the one-form object $R^a_\mu(e,\omega) \equiv e^\nu_b R^{ab}_{\mu\nu}$ (like the Ricci tensor) and the Ricci scalar:

$$\mathcal{R} \equiv e^a_\mu e^b_\nu R(\omega)^{ab}_{\mu\nu} \quad . \tag{5.6}$$

With these we can form the simplest possible supergravity: $\mathcal{N} = 1$, with no cosmological constant:

$$S_{\mathrm{SUGRA}} = -\frac{1}{2}\int d^d x e\Big(R(e,\omega) + \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho\Big) \quad . \tag{5.7}$$

These terms are common to all supergravity actions. But in general more fields will have to be included, to complete the supermultiplets and make the action (5.7) invariant under superymmetry. These are:

$(p+1)$-form fields:

$$A_{p+1} = \frac{1}{(p+1)!} A_{\mu_1\mu_2\mu_3\cdots\mu_{p+1}}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge \cdots \wedge dx^{\mu_{p+1}} \quad . \tag{5.8}$$

Left and right-handed spinors: λ_α, $\bar{\lambda}_{\dot{\alpha}}$. We will not be as concerned with these.

Let us lot at the form fields in more detail. These are the generalized gauge fields of local $U(1)$ symmetries. We can define the corresponding $(n+1)$-form field strengths $F_{p+1} \equiv \mathrm{d}A_{p+1}$.

(p+1)-forms naturally couple to objects defined by a (p+1) dimensional submanifold ω of spacetime:

$$S_{\mathrm{int}} = \int_\omega A_{p+1} \quad . \tag{5.9}$$

We call these objects *p-branes*, and they are intimately connected with the full UV-completed string theory picture. Any field solution will be called a p-brane solution if there is a non-zero $(p+1)$-form field.

Forms have kinetic terms which look like $\frac{1}{2}F \wedge *F$; their equations of motion take the generic form:

$$\mathrm{d} * F = \mathcal{J} \quad , \tag{5.10}$$

where $\mathcal{J}$ is a $(d-p-1)$-form current which sources A_{p+1}.

Let's integrate the equations of motion over some $(d-p-1)$ dimensional submanifold Ω:

$$\int_\Omega \mathrm{d} * F = \int_{\partial\Omega} *F = \int_\Omega \mathcal{J} \tag{5.11}$$

The second equality follows from Stokes' theorem. This is Gauss' law for a generalized gauge field: the flux of $*F$ through the boundary of Ω equals the 'charge' inside Ω. Dirac presented an argument (ref) that this should be quantized, and indeed we will see an example of this later on.

Type IIb supergravity is a theory in $9+1$ dimensions. It contains just massless fields, so that all the fields fall into irreducible representations of the little group, $SO(8)$. They can all be labelled with their highest weights in the Cartan basis. The $SO(8)$ algebra is has rank 4 and is nice because it possesses an symmetry called *triality*. The Dynkin diagram is is show in fig. 5.2.

Looking at the rotational symmetry of the diagram, we can see that in a highest weight labelling $(r_1\, r_2\, r_3\, r_4)$, $r_{1,3,4}$ may be interchanged without affecting the dimension of the representation. Thus there is an infinite sequence of groups of three representations of the same dimension, e.g. there are three $\mathbf{8}$'s: $\mathbf{8}_V$, $\mathbf{8}_S$, $\mathbf{8}_C$ corresponding to (1000), (0010) and (0001) respectively. The dimensions of a representation equal its number of on-shell degrees of freedom.

The bosonic fields of IIb supergravity are as follows. They are divided into two sectors: the Ramond-Ramond (R-R) sector and the Neveu/Schwarz-Neveu/Schwarz (NS) sector:

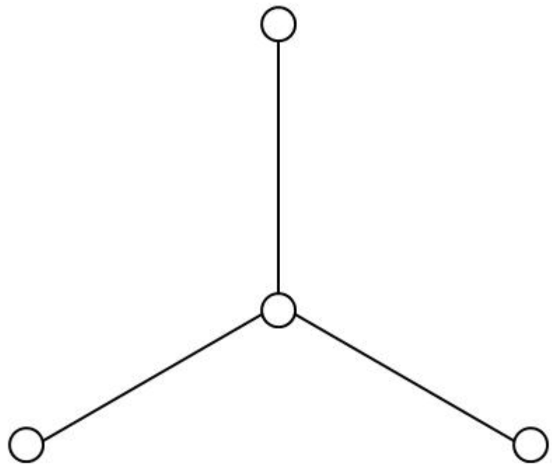

Figure 5.2: The dynkin diagram for the $SO(8)$ dynkin diagram. Each circle is a simple root. $SO(8)$ has an automorphism group S_3, corresponding to the symmetries of this diagram when rotated by 120^o or 240^o.

	field	$SO(8)$ rep.	highest weight	name
	e^a_μ	$\mathbf{8}_V$	(2000)	vielbein/graviton (rank 2 tracless symmetric tensor)
R-R	$B_{\mu\nu}$	**28**	(0100)	'B field' (2-form)
	ϕ	**1**	(0000)	dilaton (scalar)
	C_4^+	$\mathbf{35}_3$	(0020)	self-dual 4-form
NS	C_2	**28**	(0100)	R-R 2-form
	C_0	**1**	(0000)	axion (scalar)

The fermionic fields are:

field	$SO(8)$ rep.	highest weight	name
$2 \times \psi^\mu_\alpha$	$2 \times \mathbf{56}_1$	$2 \times (1010)$	two left-handed gravitini
$2 \times \lambda_\alpha$	$2 \times \mathbf{8}_S$	$2 \times (0010)$	two left-handed spinors

By counting up the bosonic then the fermionic representation dimensions we can that the number of on-shell degrees of freedom are the same (128) for each, which we must have for supersymmetry. Also notice that all of the fermions are left-handed, therefore IIb is a chiral theory.

The form fields all have corresponding field strengths. We say:

$$F_{n+1} \equiv \mathrm{d}C_n \qquad H_3 \equiv \mathrm{d}B_2 \quad . \tag{5.12}$$

We will be most interested in the dilaton ϕ, and the self-dual four-form C_4^+. Represented on $\mathbb{R}^8$, C_4^+ is self-dual itself, but when represented on the full $9+1$ dimensional spacetime its five-form field strength $F_5 \equiv \mathrm{d}C_4^+$ is self-dual instead. Precisely, this is the statement:

$$\tilde{F_5} = *\tilde{F_5} \tag{5.13}$$

where

$$\tilde{F_5} \equiv F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{5.14}$$

We can write down an action for the bosonic part of the theory [[BBS07]]:

$$\begin{aligned} S_{\mathrm{IIb}} = \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\Big\{R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{-\phi}|H_3|^2\Big\} \\ - \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\Big\{e^{2\phi} + e^{\phi}|\tilde{F_3}|^2 + \frac{1}{2}|\tilde{F_5}|^2\Big\} \\ + -\frac{1}{4\kappa^2}\int C_4^+ \wedge H_3 \wedge F_3 \quad . \end{aligned} \tag{5.15}$$

Note that this does not describe the theory completely, because we need to supplement it with the self-duality condition on $\tilde{F_5}$. It has always been a problem with IIb supergravity, but does not matter much because we always solve for the equations of motion rather than quantize the theory, and at this level it is easy to impose the constraint (5.13).

It turns out that we can decouple $g_{\mu\nu}$, F_5, ϕ and C_0 from the rest of the theory and consider a solution where these are the only non-zero fields. Then we have $\tilde{F_5} = F_5$. Since we have a five-form, we are going to try to construct a 3-brane solution, by imposing Poincaré symmetry on 4+1 of the dimensions ($\mathbb{R}^4 \times SO(3,1)$), and we work in a gauge in which C_4^+ has non-zero components in these directions only. In the rest the of the coordinates we impose rotational symmetry $SO(6)$.

It can be found that the solution is characterized by a single radial function:[3]

$$\begin{aligned} ds^2 &= H(r)^{-\frac{1}{2}} dx_\mu dx^\mu + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \\ F_5 &= \epsilon_{abcdef}\partial^f H \\ \phi &= \text{constant} \\ C &= \text{constant}' \end{aligned} \tag{5.16}$$

Here x^μ are the Poincaré directions and (r, Ω_i) are spherical polars in the $SO(6)$ ('transverse') directions. The a, b indices are for the cartesian version of the transverse coordinates. All of the other fields are zero. The symmetries are evident.

(5.16) solves the IIb equations of motion iff $H(r)$ is a harmonic function [[HS91]]. By this we mean:

$$\eta^{ab}\partial_a\partial_b H(r) = 0 \quad . \tag{5.17}$$

It has the general solution:

$$H(r) = 1 + \frac{L^4}{r^4} \tag{5.18}$$

with L some length scale which is a free parameter.

We have indeed found a 3-brane solution. This will finally provide us with the asymptotic $AdS_5 \times S^5$ solution we must use in the duality. Because let us take the limit of $r << R$ (the 'near-horizon' limit):

$$\begin{aligned} r << R \quad &\Rightarrow \quad H(r) = 1 + \frac{R^4}{R^2} \simeq \frac{R^4}{R^2} \\ &\Rightarrow \quad H(r)^{\frac{1}{2}} \simeq \frac{R^2}{r^2} \\ &\Rightarrow \quad ds^2 \simeq \frac{r^2}{R^2} dx_\mu dx^\nu + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \end{aligned} \tag{5.19}$$

[3]The calculation is best done in the 'string frame', i.e. by working with the metric $(g_{\mathrm{Str}})_{\mu\nu} \equiv e^{\frac{\phi}{2}} g_{\mu\nu}$. The equations of motion should be used instead of the action when there is a non-zero F_5, for the reason discussed in the text.

We see that in the metric the two sets of coordinates are now separated out, corresponding to a product geometry. Furthermore, we can recognize the first two terms in the metric as the Poincaré coordinates for AdS_5. The last term is the metric for a 5-sphere. So we have found the near horizon geometry $AdS_5 \times S^5$.

Regularity of this solution however requires that

$$R^4 = 4\pi g_s N(\alpha')^2 \quad . \tag{5.20}$$

This in turn implies that the flux of F_5 over through the S^5 is quantized:

$$\int_{S^5} F_5 = N \tag{5.21}$$

agreeing with the Dirac charge quantization, because (5.11) tells us the flux is equal to the charge enclosed. We interpret this in string theory as the presence of N D3 branes.

The group of supersymmetry transformations which preserve the solution corresponding to the metric (5.19) is $SU(2,2|4)$.

5.3 Maldacena's Correspondence

The theories ($\mathcal{N} = 4$ SYM and the IIb supergravity/superstring theory solution) have the following parameters:

$\mathcal{N} = 4$ SYM	g_{YM} N
IIb SUGRA/strings	$g_{\mathrm{S}} = e^{\phi} \sim \frac{G_{\mathrm{N}}}{\kappa_{10}}$ $\alpha' = l_{\mathrm{s}}^2$ R

. (5.22)

Maldacena considered the large N limit of the gauge theory. There is an elegant theory, due to 't Hooft [[tH74]][[Col85]], for Feynman diagrams in this limit, in which only a certain type of diagram survives. This can be compared with string perturbation theory [[McG]], and the natural outcome of the analysis is to conclude that $g_{\mathrm{S}} = g_{\mathrm{YM}}^2$. Once this is done, we can use it in (5.20) to show that $R^4 = 4\pi g_{\mathrm{YM}}^2 N(\alpha')^2$.

So we have the following identifications between the parameters of the two theories:

$$g_\mathrm{S} = g^2_\mathrm{YM} \qquad \frac{R^4}{(\alpha')^2} = 4\pi g^2_\mathrm{YM} N \quad . \tag{5.23}$$

There are two limits we can take:

- Large N. This was the starting point for the suggestion of the correspondence. We take $N \to \infty$ while keeping $g^2_\mathrm{YM}N$ fixed. This means that $g_\mathrm{S} \to 0$. Looking at table 5.22 we see that this makes G_N small, and so gives an explicit example of the large N argument in §3.2: many degrees of freedom imply a classical gravity dual. This limit corresponds to *classical string theory.*
- Large $g^2_\mathrm{YM}N$. If we now let g^2_YM take a finite value (it does not have to be large), then $g^2_\mathrm{YM}N$ becomes infinite. Looking at (5.23), we can see that this corresponds to α' becoming much smaller than R^2. Again from (5.22),this means that the string length is much smaller than the bulk curvature scale. That means we can ignore the stringy corrections, and use supergravity. This limit corresponds to *classical supergravity.*

The various supergravity limits are illustrated in (5.24) below.

	small α'	large α'
small g_s	Classical Supergravity	Classical String Theory
large g_s	Quantum Supergravity	Quantum String Theory

(5.24)

5.4 Symmetries

The theories have the same symmetry group, $SU(2,2|4)$, with 16 real supercharges. The maximal bosonic subgroup is $SU(2,2) \times SO(4,2) \simeq SO(4,2) \times SO(6)$.

$SO(4,2)$ arises as the isometry group of AdS_5, or the conformal symmetry group of the SYM. This is related to the fact that the boundary is defined using a conformal compactification of AdS.

$SO(6)$ arises as the isometry group of S^5, or the R-symmetry group of the SYM.

We build representation of $SU(2,2|4)$ as follows. Start with bosonic subgroup. Label representations of the subgroup by choosing a Cartan subalgebra. We'll choose it such that a representation is:

$$\underbrace{(s_1, s_2)}_{SO(3,1)} \times \underbrace{(\Delta)}_{SO(1,1)} \times \underbrace{(r_1, r_2, r_3)}_{SO(6)} \tag{5.25}$$

Each of these representations corresponds to a bulk field, and to a boundary operator which is its dual. $s_{1,2}$ are the spins of the boundary operator, Δ is the boundary dilation eigenvalue, i.e. the scaling dimension of the operator $\mathcal{O}_\Delta$.

Now to represent the full supersymmetry we state that the Q's and $\bar{Q}$'s rotate between these different bosonic representations. One first finds an operator known as the *superconformal primary operator* for a given supermultiplet($SU(2,2|4)$ representation). For a given multiplet this is defined operator of lowest dimension in that multiplet. The test of whether this is true is if the operator is annihilated by all of the superconformal supercharges $S_\alpha, \bar{S}_{\dot{\alpha}}$, since these are the only generators in the algebra with negative dimension. If they can't lower the dimension nothing else can, so that we have found the lowest dimension operator in the multiplet. Furthermore, the other operators are built from the Q's and $\bar{Q}$'s, which all have dimension $+\frac{1}{2}$. Therefore all the other operators in the multiplet have higher dimension, so the superconformal primary is unique.

What explicit form do the primary operators take? We know it cannot be formed by acting with Q's on another operator. Let us look at the effect of the supercharges on the SYM fields:

$$\begin{array}{ll} \{Q, \phi\} \sim \lambda & \{Q, \lambda\} \sim F + [\phi, \phi] \\ \{Q, \bar{\lambda}\} \sim D\phi & [Q, F] \sim D\lambda \end{array} . \tag{5.26}$$

So we see that to be a primary the operator cannot involve F or the λ's. Furthermore it can't involve a commutator of the ϕ's. So it must be a completely symmetrized product of ϕ's:

$$\mathcal{O}_{\text{primary}} = \text{Tr}(\phi^{i_1} \phi^{i_2} \cdots \phi^{i_n}) \tag{5.27}$$

Since each ϕ is a $\mathbf{6}$ under $SO(6)_R$, (5.27) will transform as a symmetric $SO(6)$ tensor. For the orthogonal groups these are reducible because we can take out the traces.

5.4.1 Mapping BPS states

Supersymmetry is very useful for testing hypotheses because it removes quantum fluctuations, and the fact that we have supersymmetry in this example of the duality makes it easy verify in the following case.

There is a certain type of primary operator which is annihilated by *half* of the supercharges. Their multiplets are called *1/2-BPS* multiplets. These are the analogue of chiral superfields in $\mathcal{N} = 1$ superspace which satisfy $\bar{D}_{\dot{\alpha}}\Phi = 0$. Thus some of the states built from applying supercharges to the primary are zero, and the multiplets are correspondingly shorter. The primaries of these representations look like:

$$(0, 0) \times (\Delta) \times (\Delta, 0, 0) \tag{5.28}$$

Treating the three sets of quantum numbers in turn, (5.27) means that the 1/2 BPS primaries

- are scalars (all primaries are)
- have scaling dimension Δ
- are traceless symmetric $SO(6)$ tensors of rank Δ.

i.e. their tensor rank is the same as their scaling dimension. The important thing to realize is that for BPS multiplets these numbers *are protected against quantum corrections by supersymmetry.* So even in the full quantum theory, there is a discrete series of representations parametrized by an integer $\Delta \geq 2$.

Do these representations appear in either theory? If they appear in one, they'd better appear in the other, otherwise we would have shown the correspondence to be false. Happily, this is not the case:

- In SYM these are the operators (5.28) with all possible traces removed.
- We should have a field per representation. To find the supergravity fields visible in AdS_5 we must do a decomposition of the fields into spherical harmonics on S^5:

$$\phi(x,\Omega) = \sum_{\Delta=0}^{\infty} \tilde{\phi}(x,\Omega)\, Y^{\Delta}(\Omega) \quad . \tag{5.29}$$

Here x are the AdS coordinates, Ω are the S^5 coordinates. When we do this we find a number of series of fields, whose masses are quantised because of the cylic boundary conditions on the sphere [[AGM$^+$00]][4]. The ϕ^{Δ}'s are traceless symmetric $SO(6)$ representations. Amongst these we find a series of massive scalar fields with masses:

$$m^2 R^2 = \Delta(\Delta - \frac{d^2}{4}) \tag{5.30}$$

Setting $R = 0$ and $d = 4$, we see that this is just what we expect from the dual of a dimension Δ scalar operator, according the analysis after (4.11). So we have found the dual bulk fields corresponding to the 1/2 BPS primaries, with a dimension equal to the $SO(6)$ tensor rank.

There is much more we could do here. The $\mathcal{N} = 4$ SYM duality has been extensively studied since 1997, and there are many branches of its study. One concerns finding the anomalous dimensions of non-BPS gauge operators, with a mind to matching these to the supergravity. This is more non-trivial than for BPS states, since the quantum corrections depend on the exact dynamics of the gauge theory and not just its symmetries. The systems is found to be integrable [[MZ03]], and the theory surrounding them is elegant.

Another branch concerns quantizing strings on an $AdS_5 \times S^5$ background [**??**], which is a hard thing to do. Matching of string states beyond the supergravity approximation is important for both gauge and string theory.

There is still much work to do in establishing a full mapping between the two theories, ensuring work for string theorists for years to come.

[4]for a complete Kaluza-Klein decomposition, see [[GM85]].

6 Finite Temperature

There are, broadly speaking, two different approaches to gauge-gravity duality. One, which we have been outlining in the last chapter, is based on string theory, and aims to discover new examples of the duality by studying different brane configurations. The other approach, which we have emphasized in this review, is the 'purer' approach, in which a new field theory is built by specifying the bulk dynamics and studying what this implies for the boundary. If the bulk theory is classical, the field theory is necessarily strongly coupled. In this way we hope to be able to find new tools for dealing with strongly coupled gauge theories, which are traditionally intractable.

There has recently been some excitement surrounding the possibility of constructing models of strongly coupled condensed matter systems using a gravity dual. There are two mains observations to be made about such possible field theories: firstly, there are not supersymmetric. Although the subject was discovered by the study of SUSY gauge theories, we have emphasized in this review that the duality is a completely general principle, which does not rely on supersymmetry for its construction. Secondly, these field theories will be at *finite temperature*. That is, they are based on field excitations around a thermally mixed state, as opposed to theories at zero temperature (such as we deal with in high energy particle physics), in which excitations are around the ground state $|\Omega\rangle$ of the theory.

How do we construct a gravity dual of a finite temperature (thermal) field theory? Thermal field theory at equilibrium is obtained by Wick rotating to a Euclidean time ($t \mapsto i\tau$), and making τ periodic, by identifying $\tau \sim \tau + \beta$). The structure of the manifold $\mathbb{R}^{d-1,1}$ changes to $\mathbb{R}^{d-1} \times S^1$. The period (circumference of the 'thermal circle') is related to the temperature by $\beta = \frac{1}{T}$, so that field theory at high temperature is like the theory on a small circle, and at low temperature the circle unwinds and we recover the original theory on Minkowski space.

We have already noted that the conditions of the boundary theory are determined by the interior of the gravity solution. We need a classical solution for the metric to describe field theory at a temperature. It follows that to describe thermal field theory on the boundary, the bulk manifold must have boundary metric which is $\mathbb{R}^{d-1} \times S^1$ asymptotically. We choose the simplest possible action for the bulk that will give us anti-de Sitter type geometries, namely

$$-\frac{1}{16\pi G_{\mathrm{N}}} \int d^{\mathrm{d}}x\sqrt{-g}\left(\mathcal{R} - 2\Lambda\right) \quad . \tag{6.1}$$

In order to have the specified boundary conditions, this manifold must asymptote to Euclidean anti-de Sitter space with periodic time. For the action (6.1), there are two known possibilities, and they are believed to be the only ones:

1. **Thermal AdS**. By this we mean Poincaré AdS space, with cyclic Euclidean time:

$$ds^2 = \frac{R^2}{z^2}\left(dt^2 + dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu + dx^i dx^i\right) \quad , \tag{6.2}$$

   where we identify $t \sim t + \beta$. This seems the simplest option.

2. **AdS black brane**. This is Poincaré AdS with a 'black brane' sitting inside it:

$$ds^2 = \frac{R^2}{z^2}\left(f(z)dt^2 + \frac{dz^2)}{f(z)} + \eta_{\mu\nu}dx^\mu dx^\nu + dx^i dx^i\right) \tag{6.3}$$

$$f(z) \equiv 1 - \frac{z^d}{z_0^d} \quad . \tag{6.4}$$

   The Euclidean time is automatically compactified in this metric, if we require its regularity at z_0. To do this we look at it near $z = z_0$ Taylor expanding $f(z) \approxeq f'(z_0)(z - z_0) = 2\kappa(z - z_0) = \kappa^2 z_0^2 \rho^2$, where we have defined the surface gravity $\kappa \equiv \frac{|f'(z_0)|}{2}$ and the new coordinate $\rho^2 \equiv \frac{2}{\kappa z_0^2}(z - z_0)$. Now

$$dz = kz_0^2\rho d\rho \quad \Rightarrow \quad dz^2 = \kappa^2 z_0^4 \rho^2 d\rho^2 \tag{6.5}$$

$$\Rightarrow \quad \frac{dz^2}{f(z)} \approxeq \frac{dz^2}{f'(z_0)(z - z_0)} = \frac{\kappa^2 z_0^4 \rho^2 d\rho^2}{\kappa^2 z_0^2 \rho^2} = z_0^2 d\rho^2 \quad , \tag{6.6}$$

so that we have $ds^2 = L^2(\kappa^2\rho^2 dt^2 + d\rho^2)$. This looks like polar coordinates $ds^2 = r^2 d\theta^2 + dr^2$, which only describe a space smooth at $r = 0$ if $\theta \sim \theta + 2\pi$. Otherwise there is a conical singularity. So in our black brane metric we conclude that for smoothness we must have $\kappa t \sim \kappa t + 2\pi \Rightarrow \beta = \frac{2\pi}{\kappa}$. Much like a Schwarzschild black hole, the metric has a temperature $T = \frac{1}{\beta} = \frac{\kappa}{2}$ associated with it.

An interesting feature of the black brane solutions, which only occurs for the Euclidean version, is that the spacetime stops at $z = z_0$, so that $0 \leq z \leq z_0$.

Which one of these metrics, 1 or 2, should we choose? In the full quantum gravity we expect a path integral over metrics, and the metric which dominates will be the one which has the smaller action. Let'd do this calculation explicitly. We will construct the difference in actions $S_1 - S_2$. If this is positive, the black brane solution wins out, if negative it is the thermal AdS.

Both metrics solve the field equations with $\Lambda = -\frac{d(d-1)}{2R^2}$. This also means that $\mathcal{R} = -\frac{d(d+1)}{R^2}$. Which these two facts, the gravity action (6.1) becomes

$$S = \frac{d}{8\pi G_{\mathrm{N}}} \int d^{\mathrm{d}}x\, \sqrt{g} \tag{6.7}$$

The change $\sqrt{-g} \to \sqrt{g}$ comes from the Euclideanization. So we see the the action for either solution is directly proportional to its volume, therefore it suffices to work with the volumes. To keep things finite, we introduce a radial cutoff, as usual, at $z = \epsilon$. We have:

$$S_1 \propto \int_\epsilon^\infty dz(\frac{R}{z})^{d+1} \int_0^{\beta_1} dt \int d^{d-1}x = \mathrm{Vol}(\mathbb{R}^{d-1})\beta_1 \frac{R^{d+1}}{d}\frac{1}{\epsilon^d} \tag{6.8}$$

$$S_2 \propto \int_\epsilon^{z_0} dz(\frac{R}{z})^{d+1} \int_0^{\beta_2} dt \int d^{d-1}x = \mathrm{Vol}(\mathbb{R}^{d-1})\beta_2 \frac{R^{d+1}}{d}\left(\frac{1}{\epsilon^d} - \frac{1}{z_0^d}\right) \quad . \tag{6.9}$$

The commmon factor $\mathrm{Vol}(\mathbb{R}^{d-1})$ (which must be suitably regularized) is just some positive constant, so we will drop $\mathrm{Vol}(\mathbb{R}^{d-1})\frac{R^{d+1}}{d}$.

The next step is to fix the relation between β_1 and β_2. This is done by equating the radii of the thermal circles at the boundary, so that we are comparing two metrics which give field theories at the same temperature, which is the point of the calculation. To do this we measure circumference of the timelike circle at $z = \epsilon$, and equate this between the metrics. Using

$$\text{circumference} = \int_{S^1} dt \sqrt{g_{tt}|_{z=\epsilon}} \tag{6.10}$$

we find that we need

$$\beta_1 (\frac{R}{\epsilon})^{d+1} = \beta_2 (\frac{R}{\epsilon})^{d+1} \sqrt{f(\epsilon)} \tag{6.11}$$

$$\Rightarrow \quad \frac{\beta_1}{\beta_2} = \sqrt{f(\epsilon)} \simeq 1 - \frac{1}{2}\frac{\epsilon^d}{z_0^d} \quad . \tag{6.12}$$

The last equation is valid for small ϵ. Now finally we take the action difference

$$S_1 - S_2 \propto \frac{1}{\epsilon^d}(\beta_1 - \beta_2) + \frac{\beta_2}{z_0^d} = \beta_2 \left((\frac{\beta_1}{\beta_2} - 1) + \frac{1}{z_0^d} \right) = \frac{1}{2}\frac{1}{z_0^d} \tag{6.13}$$

So we see that the black brane spacetime always dominates, and that therefore this is the correct dual to describe finite temperature conformal field theory on $\mathbb{R}^{d-1} \times S^1$.

We make two comments on this calculation:

The Einstein-Hilbert action for AdS type spacetimes has additional boundary terms. These cancel in the preceding because 1 and 2 are asymptotically the same.

Secondly, we have used for our manifolds the Poincaré type AdS. These are separate manifolds in their own right. They are to be compared with the corresponding metrics in the *global* AdS. These are

1. Thermalized global AdS:

$$ds^2 = (1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + dx^i dx^i \tag{6.14}$$

2. and the Schwarzschild AdS ($SAdS$) solution:

$$ds^2 = (1 + \frac{r^2}{R^2} + \frac{a}{r^{d-2}})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2} + \frac{a}{r^{d-2}}} + dx^i dx^i \tag{6.15}$$

.

The same calculation can be done with these two metrics [Wit98b], and in contrast we find that the thermal AdS dominates at low temperature, the black hole at high temperature. This is called the Hawking-Page phase transition [HP83], and implies a thermal phase transition in the field theory. This difference in results comes from the fact that the boundary of Poincaré AdS is Minkowski space, whereas the boundary of full AdS has a spacial sphere. So in the global case we are studying finite temperature field theory on a sphere, where we therefore predict the phase transition. This was to be expected - conformal field theory on flat space can have no phase transition since there is no scale to compare T to, whereas on a spatial sphere the radius provides a transition scale.

6.1 Entropy

We now have an exact prescription for dealing with the equilibrium thermodynamics of a large class of strongly coupled gauged theories: the above black hole metric is the dual geometry to the boundary gauge theory at finite temperature with no sources or potentials.

The duality tells us the partition function of the thermal field theory, as a function of temperature:

$$Z(T) = e^{-S(g_{\mathrm{BH}})} \quad . \tag{6.16}$$

All equilibrium thermodynamic properties can be derived from this partition function:

$$F = -T \log Z(T) \qquad E = -T^2 \frac{\partial}{\partial T} Z(T) \tag{6.17}$$

$$S = -\frac{\partial F}{\partial T} \qquad \text{etc.} \tag{6.18}$$

So we need to caculate the action for the the black hole metric. We did somthing like this in the last section. However now we must take account of the boundary terms which cancelled in that calculation:

$$S = S_{\rm EH} + \int_{\partial AdS} d^d x \sqrt{\gamma_\epsilon} \Big\{ - n^z \gamma^{\mu\nu}\, \partial_z \gamma_{\mu\nu} \; + \frac{2(d-1)}{R} \Big\} \tag{6.19}$$

The first term is the so-called 'Gibbons-Hawking' term [**??**]. It arises because varying the Einstein-Hilbert action yields not only the equations of motion but also a boundary term, which we need to cancel in order for the bulk solution to be a solution of the Euler-Lagrange equations. See the discussion in §2.4.

The second term is a renormalization counter-term designed to cancel the divergence in the action coming from integration near the boundary. This simply corresponds to the fact that the metric is always a non-normalizable mode - it determines the metric on which the field is defined, so its effects are present at all energies.

This integral can be done, but it would not be particularly instructive to do it explicitly here. We merely state the result:

$$S(g_{\rm BH}) = -\frac{R^{d-1}}{16\pi G_{\rm N}} \frac{V}{z_0^d T} \tag{6.20}$$

$$\Rightarrow \quad F = -T \log Z(T) = -T S(g_{\rm BH}) = \frac{R^{d-1}}{16\pi G_{\rm N}} \frac{V}{z_0^d} \quad . \tag{6.21}$$

Using $T = \frac{\kappa}{2\pi} = \frac{|f'(z_0)|}{4\pi} = \frac{d}{4\pi}\frac{1}{z_0}$, we have

$$\frac{F}{V} = -\frac{(4\pi)^d}{16\pi d^d} \frac{R^{d-1}}{G_{\rm N}} T^d \tag{6.22}$$

$$\frac{S}{V} = \frac{(4\pi)^d}{16\pi d^d} \frac{R^{d-1}}{G_{\rm N}} T^{d-1} \tag{6.23}$$

$$. \tag{6.24}$$

To find this entropy (6.22) in terms of field theory parameters, we need to know the precise mapping between the theories. This differs from example to example. For $\mathcal{N} = 4$ SYM we have

$$S = \frac{\pi^2}{2} N^2 V T^3 \tag{6.25}$$

This was to be expected: since the entropy is a extensive quantity, it is proportional to V. It should also scale with number of degrees of freedom per field, which is $N^2 - 1 \simeq N^2$(for large N) in this case. The only other scale is T, since we are considering a CFT, so that dependence follows from dimensional analysis. The only unknown thing is the numerical factor. The correspondence predicts this at strong coupling $g >> 1$. It can also be worked out at small coupling [[GKP96]], and we find:

$$S_{\text{strong}} = \frac{3}{4} S_{\text{weak}} \tag{6.26}$$

This demonstrates a general rule: the physics (at least thermodynamics) at strong coupling doesn't seem to be too different than at weak coupling. AdS/CFT may be trying to give us a lesson: when we leave one coupling regime, we seem to enter another which can just be mapped back into the first.

7 Conclusion

Gauge-gravity duality is a very powerful tool. It enables us to probe the strongly coupled dynamics of gauge theory, and perhaps more excitingly in principle it allows us to study the non-perturbative aspects of string theory. In fact, it is the best way we have yet actually to *define* string theory, since it says that string theory is dual to something we can define, i.e. supersymmetric gauge theories.

In this review, we gave one example of the correspondence in string theory, i.e. IIb superstrings asymptotic to $AdS_5 \times S^5$. This describes string theory with a boundary condition that looks like a D3 brane. But in the ten years since its inception, gauge-gravity duality has been used to give many holographic duals to string and M theory asymptoting to different brane solutions. Topical examples are the $\mathcal{N} = 8$ superconformal Chern-Simons theories living on M2-branes, as holographic probes of M theory.

In other developments, work has been done on extending the correspondence to *de Sitter* spacetimes. Is it reasonable to assume that all of string and M theory be defined by gauge theories living on all the possible boundaries conditions of the theory?

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Bibliography

[AGM+00] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rep. **323**, 183–386 (2000).

[BBS07] K. Becker, M. Becker and J. Schwarz, *String Theory and M-theory*, Cambridge University Press, 2007.

[CM67] S. Coleman and J. Mandula, *All possible symmetries of the S-matrix*, Phys. Rev. **158**, 1251–1256 (1967).

[Col85] S. Coleman, 1/N, in *Aspects of Symmetry*, page 351, Cambridge University Press, 1985.

[DF02] E. D'Hoker and D. Z. Freedman, Supersymmetric Gauge Theories and the AdS/CFT Correspondence, 2002.

[GKP96] S. S. Gubser, I. R. Klebanov and A. Peet, *Entropy and Temperature of Black 3-Branes*, Phys. Rev. D **54**, 3915 (1996), arXiv:hep-th/9602135.

[GM85] M. Günaydin and N. Marcus, *The Spectrum of the S^5 compactification of the chiral $\mathcal{N} = 2$, $D = 10$ supergravity and the unitary supermultiplets of $U(2, 2|4)$*, Class. Quant. Grav. **2**, L11–L17 (1985).

[HP83] S. W. Hawking and D. N. Page, *Thermodynamics Of Black Holes In Anti-De Sitter Space*, Commun. Math. Phys. **87**, 577 (1983).

[HS91] G. T. Horowitz and A. Strominger, *Black Strings and P-branes*, Nucl. Phys. B **360**, 197 (1991).

[Mal98] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2**, 231–252 (1998).

[McG] J. McGreevy, Applied AdS/CFT, Lecture notes from the KITP Mini-Program *Quantum Criticality and the AdS/CFT Correspondence*", Santa Barbara, July 2009. Available online at http://adscmt-m09.wikispaces.com/.

[McH09] J. McHugh, September 2009.

[MZ03] J. A. Minahan and K. Zarembo, *The Bethe-Ansatz for N=4 Super Yang-Mills*, JHEP **0303**, 013 (2003), arXiv:hep-th/0212208.

[str] http://online.itp.ucsb.edu/online/strings98/procon2.html.

[tH74] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nucl. Phys. B **75**, 461 (1974).

[vN82] P. van Nieuwenhuizen, *Supergravity*, Phys. Rep. **68** (1982).

[Wit98a] E. Witten, *Anti de Sitter Space and Holography*, Adv. Math. Theor. Phys. **2**, 253–291 (1998).

[Wit98b] E. Witten, *Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories*, Adv. Math. Theor. Phys. **2**, 505–532 (1998).