TWO EXAMPLES OF NON STRICTLY CONVEX LARGE DEVIATIONS

STEFANO DE MARCO, ANTOINE JACQUIER, AND PATRICK ROOME

Abstract. We present here two examples of a large deviations principle where the rate function is not strictly convex. This is motivated by an example from mathematical finance, and adds a new item to the zoology of non strictly convex large deviations. For one of these examples, we also show that the rate function of the Cràmer-type of large deviations coincides with that of the Freidlin-Wentzell when contraction principles are applied.

1. Introduction

The Gärtner-Ellis theorem is a key result in the theory of (finite-dimensional) large deviations. Extending the results of Cràmer [9] for sequences of random variables not necessarily independent and identically distributed (iid), it provides a large deviations framework solely based on the knowledge of the cumulant generating function (cgf) of the sequence. The key assumptions are that the pointwise (rescaled) limit of these cgf satisfies some convexity property and becomes steep at the boundaries of its effective domain; this in turns implies that the rate function governing the large deviations, defined as the topological dual, is also convex. When convexity breaks down, no general result is known, and large deviations may or may not hold; the classical example is that of the iid sequence $(X_i)_{i\in\mathbb{N}}$ where $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Let $Y_n := \sum_{i=1}^n X_i$. Then $\Lambda(u) := \lim_{n\uparrow\infty} n^{-1}\log\mathbb{E}(\mathrm{e}^{nuY_n}) = |u|$ for all $u\in\mathbb{R}$. The dual is given by $\Lambda^*(x) := \sup_u(ux - \Lambda(u)) = 0$ if $|x|\leq 1$ and infinity otherwise. These certainly violate the assumptions of the Gärtner-Ellis theorem, which we recall in Appendix A, but it is immediate to check that the sequence $(Y_n)_{n\geq 1}$ actually satisfies a large deviations principle with rate function equal to zero on $\{-1,1\}$ and infinite otherwise, which does not correspond to Λ^*. More recently, O'Brien [25] and Comman [7] have strengthened this theorem, by partially relaxing the steepness and convexity assumptions. In the setting of topological vector spaces, Bryc's Theorem [6] (see also [10, Chapter 4.4]), or 'Inverse Varadhan's lemma', allows for large deviations of sequences of measures, with non strictly convex rate function, albeit with some exponential tightness requirement. However, several examples have been dug out which do not fall into this framework, such as in the setting of random walks with interface [14], occupation measures of Markov chains [21], the on/off Weibull sojourn process [13], or m-variate von Mises statistics [15].

Motivated by recent developments on large deviations in mathematical finance (see in particular [11], [18], [8] and the excellent review paper [26]), we study the small-time behaviour of the solution of the Feller stochastic differential equation (and an integral version of it) when the starting point is null. The absence of Lipschitz continuity of the diffusion coefficient and the degenerate starting condition make it not amenable to the classical Freidlin-Wentzell framework, and the absence of strict convexity of the limiting moment generating function

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violates the Gärtner-Ellis assumptions. It turns out that a large deviations principle however holds, and one can furthermore reconcile the pathwise large deviations to the marginal (Gärtner-Ellis one) by contraction. We believe this provides a nice example of non-strictly-convex large deviations principle in the context of continuous-time stochastic processes. It also sheds light on the importance of the starting point of the SDE being null, as opposed to the non-zero case where the Gärtner-Ellis theorem applies directly (see [16]). In Section 2, we present the model and state the large deviations results as time tends to zero; we also establish the connection with the Freidlin-Wentzell analysis via contraction principles. The proofs of the main results are gathered in Section 3.

Notations: For a set G in some topological vector space $\mathcal{T}$, we shall denote by G^o and $\overline{G}$ the respective interior and closure of G in $\mathcal{T}$.

2. Main results

We consider here the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = 0,\\ \mathrm{d}V_t &= (a+bV_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, && V_0 = 0,\\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{2.1}$$

where $a, \xi > 0$, $b < 0$, $|\rho| < 1$ and $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are two standard Brownian motions. We stress the importance of the parameter a to be strictly positive; otherwise, the process V, starting from zero, would just remain null, and the unique solution of (2.1) would simply be the two-dimensional zero process. We shall often make use of the notations $\bar{\rho} := \sqrt{1-\rho^2}$ and $\mu := 2a/\xi^2$. The Feller SDE for the variance process has a unique strong solution by the Yamada-Watanabe conditions [22, Proposition 2.13, page 291]). The Feller condition, $\mu \geq 1$, ensures that the origin is unattainable. Otherwise the origin is regular (hence attainable) and strongly reflecting [23, Chapter 15]. Define now the following functions:

$$\Lambda_X^*(x) = \left(u_-\mathbb{1}_{\{x<0\}} + u_+\mathbb{1}_{\{x\geq 0\}}\right)x, \tag{2.2}$$

$$\Lambda_V^*(x) = \begin{cases} 2x/\xi^2, & \text{if } x \geq 0,\\ +\infty, & \text{if } x < 0. \end{cases} \tag{2.3}$$

for all $x \in \mathbb{R}$, where the two real numbers u_- and u_+ read

$$\begin{aligned} u_- &:= \frac{2}{\xi\bar{\rho}}\arctan\left(\frac{\bar{\rho}}{\rho}\right)\mathbb{1}_{\{\rho<0\}} - \frac{\pi}{\xi}\mathbb{1}_{\{\rho=0\}} + \frac{2}{\xi\bar{\rho}}\left(\arctan\left(\frac{\bar{\rho}}{\rho}\right) - \pi\right)\mathbb{1}_{\{\rho>0\}},\\ u_+ &:= \frac{2}{\xi\bar{\rho}}\arctan\left(\frac{\bar{\rho}}{\rho}\right)\mathbb{1}_{\{\rho>0\}} + \frac{\pi}{\xi}\mathbb{1}_{\{\rho=0\}} + \frac{2}{\xi\bar{\rho}}\left(\arctan\left(\frac{\bar{\rho}}{\rho}\right) + \pi\right)\mathbb{1}_{\{\rho<0\}}. \end{aligned} \tag{2.4}$$

Note that u_- (resp. u_+) is a decreasing (resp. decreasing) function of ρ and maps the interval $(-1, 1)$ to $(-\infty, -2/\xi)$ (resp. $(2/\xi, +\infty)$). We shall use the subscript/superscript M to represent the quantities related to X or to V. For instance Λ_{M}^* represent Λ_X^* or Λ_V^*. We also denote $\mathcal{K}_X := \mathbb{R}\setminus\{0\}$ and $\mathcal{K}_V := (0, \infty)$.

2.1. Large deviations results.

The main result of this paper is the following theorem, which provides an example of a sequence of random variables for which the limiting logarithmic cumulant generating function is zero (on its effective domain) but a large deviations principle still holds. This is to be compared to the Gärtner-Ellis theorem [10, Theorem 2.3.6] which requires this limiting function to be steep at the boundaries of its effective domain. As highlighted in the proof, understanding the pointwise limit of the (rescaled) cumulant generating function does not suffice any longer, and its higher-order behaviour is needed to prove large deviations.

Theorem 2.1. *For* $\mathrm{M} \in \{X, V\}$*, the family* $(\mathrm{M}_t)_{t\geq 0}$ *satisfies a LDP with speed* t *and rate function* Λ^*_{M} *as* $t \downarrow 0$*.*

Remark 2.2. A more in-depth analysis reveals a more precise behaviour of the small-time probabilities, which take the following form as t tends to zero:

$$\mathbb{P}(\mathrm{M}_t \geq x) = \begin{cases} 1 - C(x) t^{1-\mu} \exp\left(-\frac{\Lambda^*_{\mathrm{M}}(x)}{t}\right)(1+\mathcal{O}(t)), & \text{if } x < 0, \\ C(x) t^{1-\mu} \exp\left(-\frac{\Lambda^*_{\mathrm{M}}(x)}{t}\right)(1+\mathcal{O}(t)), & \text{if } x > 0, \end{cases}$$

for all $x \in \mathcal{K}_{\mathrm{M}}$, for some (smooth) strictly positive function C. This analysis is based on the so-called theory of sharp large deviations, developed in [4, 5], and used in [3, 18, 19] for diffusion processes and statistical estimators thereof. Note that the power $1-\mu$ here is pathological and not in line with the classical $1/2$ power common in the heat kernel literature. To our knowledge, there is no general result covering this special case.

2.2. **Intuitions from Freidlin-Wentzell analysis.** We now illustrate how piecewise linear rate functions such as (2.3) can arise from sample-path large deviations. In order to simplify the framework, consider the solution V of the equation obtained from (2.1) by setting $V_0 = v_0 > 0$ and $a = b = 0$. Setting $V^\varepsilon_t := V_{\varepsilon^2 t}$ for $\varepsilon > 0$, the process $(V^\varepsilon_t)_{t\geq 0}$ is the (weak) solution of the stochastic differential equation

$$\mathrm{d}V^\varepsilon_t = \varepsilon\xi\sqrt{V^\varepsilon_t}\mathrm{d}Z_t, \qquad V^\varepsilon_0 = v_0 > 0. \tag{2.5}$$

Pathwise large deviations (as ε tends to zero) for the solution of this SDE fall outside the scope of the classical Friedlin-Wentzell framework since the diffusion coefficient lacks the required global Lipschitz continuity property (see [10, Chapter 5.6]). Donati-Martin et al. in [12] proved that, for every $T > 0$, the process $(V^\varepsilon_t)_{t\in[0,T]}$ satisfies a large deviations principle on the path space $\mathcal{C}_T = C([0,T]; \mathbb{R}_+)$ of non-negative continuous functions, with speed ε^2 and rate function I_T given by

$$I_T(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \frac{\dot\varphi_t^2}{\xi\varphi_t}\mathbb{1}_{\{\varphi_t>0\}}\mathrm{d}t, & \text{if } \varphi \in \mathcal{C}_T \text{ is absolutely continuous and } \varphi_0 = v_0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.6}$$

where one sets $y^{-1}\mathbb{1}_{\{y>0\}} = 0$ when $y = 0$. More precisely, this means that the estimates

$$-\inf_{\varphi\in G^o} I_T(\varphi) \leq \liminf_{\varepsilon\downarrow 0} \varepsilon^2 \log \mathbb{P}(V^\varepsilon \in G) \leq \limsup_{\varepsilon\downarrow 0} \varepsilon^2 \log \mathbb{P}(V^\varepsilon \in G) \leq -\inf_{\varphi\in\overline{G}} I_T(\varphi) \tag{2.7}$$

hold for every set $G \subset \mathcal{C}_T$. By the contraction principle [10, Theorem 4.2.1], the path estimates (2.7) induce a LDP on $\mathbb{R}$ for the random variable $V^\varepsilon_1 \triangleq V_t|_{t=\varepsilon^2}$, where the rate function is now given by

$$\Lambda(x) := \inf\{I_1(\varphi) : \varphi \in \mathcal{C}_1, \varphi_0 = v_0, \varphi_1 = x\}. \tag{2.8}$$

This means that the sequence $(V_t)_{t\geq 0}$ satisfies a LDP with speed t as t tends to zero, namely for every $A \subset \mathbb{R}$,

$$-\inf_{x\in A^o} \Lambda(x) \leq \liminf_{t\downarrow 0} t \log \mathbb{P}(V_t \in A^o) \leq \limsup_{t\downarrow 0} t \log \mathbb{P}(V_t \in \overline{A}) \leq -\inf_{x\in\overline{A}} \Lambda(x).$$

Proposition 2.3. *The rate function* Λ *in* (2.8) *reads*

$$\Lambda(x) = \begin{cases} \frac{2}{\xi^2}\left(\sqrt{x} - \sqrt{v_0}\right)^2, & \textit{if } x \geq 0, \\ +\infty, & \textit{if } x < 0. \end{cases}$$

In particular, as v_0 *tends to zero, the function* Λ *converges pointwise to* Λ^*_V *given in* (2.3)*.*

Proof. Let $AC_+([0,1])$ denote the set of absolutely continuous functions on $[0,1]$. If $x<0$, then by definition of I_1, one has $I_1(\varphi)=+\infty$ for any φ such that $\varphi_1=x$. Then assume $x\geq 0$, and consider $\varphi\in\mathcal{C}_1$ such that $\varphi_1=x$ and $I(\varphi)<+\infty$. By the superposition principle (or the chain rule for absolutely continuous functions, see [24, Theorem 3.68]), the function $\psi\equiv\sqrt{\varphi}$ is absolutely continuous on every interval contained in the open set $\{\varphi>0\}$, with derivative almost surely equal to $\frac{\dot{\varphi}_t}{2\sqrt{\varphi_t}}\in L^2([0,1])$. On $\{\varphi=0\}$ one has $\psi\equiv 0$, therefore $\dot{\psi}_t=0$ for every accumulation point of $\{\varphi=0\}$ (the isolated points form a finite subset of $[0,1]$). In summary, it follows from [24, Corollary 3.26] that $\psi\in AC_+([0,1])$, and that

$$\int_0^1\frac{(\dot{\varphi}_t)^2}{\varphi_t}\mathbb{1}_{\{\varphi_t>0\}}\mathrm{d}t=4\int_{\{\varphi>0\}}(\dot{\psi}_t)^2\mathrm{d}t=4\int_0^1(\dot{\psi}_t)^2\mathrm{d}t.$$

Conversely, let $\psi\in AC_+([0,1])$ be such that $\dot{\psi}\in L^2([0,1])$, and set $\varphi\equiv\psi^2$; as the composition of a C^1 function and an absolutely continuous one, φ also belongs to $AC_+([0,1])$ and $\dot{\varphi}_t=2\psi_t\dot{\psi}_t=2\sqrt{\varphi_t}\dot{\psi}_t$ a.s. Therefore,

$$\begin{aligned}\Lambda(x)&=\inf\left\{\frac{1}{2\xi^2}\int_0^1\frac{\dot{\varphi}_t^2}{\varphi_t}\mathbb{1}_{\{\varphi_t>0\}}\mathrm{d}t:\varphi\in AC_+([0,1])\text{ and }\varphi_0=v_0,\varphi_1=x\right\}\\&=\inf\left\{\frac{2}{\xi^2}\int_0^1\dot{\psi}_t^2\mathrm{d}t:\psi\in AC_+([0,1])\text{ and }\psi_0=\sqrt{v_0},\psi_1=\sqrt{x}\right\}.\end{aligned}\tag{2.9}$$

It is well known that the last problem is solved by the straight line $\psi^*_t\equiv\sqrt{v_0}+t(\sqrt{x}-\sqrt{v_0})$. Substitution into (2.9) yields $\Lambda(x)=\frac{2}{\xi^2}\int_0^1(\dot{\psi}^*_t)^2=\frac{2}{\xi^2}(\sqrt{x}-\sqrt{v_0})^2$, and the proposition follows. □

We are not claiming here that Λ^*_V is the rate function for V_1^ε in (2.5) when $v_0=0$: in this case, the unique solution to (2.5) is the identically null process $V^\varepsilon\equiv 0$, so that the family V_1^ε satisfies a trivial LDP with rate function $\overline{I}(\varphi)=0$ if $\varphi\equiv 0$, and $\overline{I}(\varphi)=+\infty$ otherwise.

Returning to the small-time problem for the solution V_t to (2.1), set $V_t^\varepsilon:=V_{\varepsilon^2 t}$, which satisfies

$$\mathrm{d}V_t^\varepsilon=\varepsilon^2(a+bV_t^\varepsilon)\mathrm{d}t+\varepsilon\xi\sqrt{V_t^\varepsilon}\mathrm{d}Z_t,\qquad V_0^\varepsilon=0.\tag{2.10}$$

To our knowledge, large deviations for the solution to (2.10) are not covered by the existing literature (in [12], the authors considered the situation where the drift $a+bV$ is independent of ε). We leave it to future research to prove that a pathwise LDP holds for the solution to (2.10) with a rate function similar to (2.6).

Remark 2.4. It follows from [8, Theorem 1.1] that for any $v_0>0$, a pathwise LDP with rate function

$$\widetilde{I}_T(\varphi)=\begin{cases}\displaystyle\frac{1}{2}\int_0^T\frac{\dot{\varphi}_t^2}{\xi\varphi_t}\mathbb{1}_{\{\varphi_t>0\}}\mathrm{d}t, & \text{if }\varphi\in\mathcal{C}_T\text{ is absolutely continuous and }\varphi_0=0,\\ +\infty, & \text{otherwise,}\end{cases}$$

holds for the solution of the stochastic differential equation

$$\mathrm{d}\widetilde{V}_t^\varepsilon=\varepsilon^2a\mathrm{d}t+\varepsilon\xi\sqrt{\widetilde{V}_t^\varepsilon}\mathrm{d}Z_t,\qquad\widetilde{V}_0^\varepsilon=\varepsilon^2v_0.$$

Comparing with (2.10), note that the initial condition is strictly positive, but tends to zero as ε tends to zero. The rate function $\widetilde{I}_T$ is the same as the rate function in (2.6), except that the path φ is to be started at zero instead of $v_0>0$. Following analogous arguments to the proof of Proposition 2.3, the contraction principle applied to $\widetilde{I}_T$ yields the rate function (2.3), in line with Theorem 2.1.

3. Proof of Theorem 2.1

The standard method to prove large deviations [10] is to first prove an upper bound for the lim sup, and then prove a lower bound for the lim inf, for the logarithmic probability on all Borel subsets of the real line. We prove here directly that the limit holds for all open intervals of the form (x, ∞) for $x \in \mathbb{R}$, which is clearly sufficient. For any $t \geq 0$ and $\mathrm{M} \in \{X, V\}$, define the rescaled cumulant generating function (cgf) $\Lambda_{\mathrm{M}}(\cdot, t)$ of the random variable M_t and its effective domain $\mathcal{D}_t^{\mathrm{M}}$ by

$$\Lambda_{\mathrm{M}}(u,t) := t \log \mathbb{E}\left(\mathrm{e}^{u\mathrm{M}_t/t} | \mathcal{F}_0\right), \qquad \text{for all } u \in \mathcal{D}_t^{\mathrm{M}} := \{u \in \mathbb{R} : |\Lambda_{\mathrm{M}}(u,t)| < \infty\}, \tag{3.1}$$

Define further $\mathcal{D}_{\mathrm{M}} := \cap_{t>0} \mathcal{D}_t^{\mathrm{M}}$. From [20], we know that

$$\begin{aligned} \Lambda_M(u,t) &= -\frac{\mu t}{2}\left[g_t^M(u) + 2\log f_t^M(u)\right], & f_t^X(u) &\equiv \cosh\left(\frac{d(\frac{u}{t})t}{2}\right) - \frac{g_t^X(u)}{td(\frac{u}{t})}\sinh\left(\frac{d(\frac{u}{t})t}{2}\right), \\ f_t^V(u) &\equiv 1 + \frac{u\xi^2}{2bt}\left(1 - \mathrm{e}^{bt}\right), & g_t^X(u) &\equiv bt + \rho\xi u, \qquad g_t^V(u) \equiv 0, \end{aligned} \tag{3.2}$$

where $d(u) \equiv [(b + \rho\xi u)^2 + u(1-u)\xi^2]^{1/2}$, so that the functions $\Lambda_X(\cdot, t)$ and $\Lambda_V(\cdot, t)$ are explicitly well defined on $\mathcal{D}_t^X$ and $\mathcal{D}_t^V$. The pointwise limit functions $\Lambda_{\mathrm{M}}(u) := \lim_{t \downarrow 0} \Lambda_{\mathrm{M}}(u,t)$, for $\mathrm{M} \in \{X, V\}$, read as follows:

Lemma 3.1. *The function Λ_{M} is null on $\mathcal{D}_{\mathrm{M}}$ and infinite outside, with $\mathcal{D}_X = (u_-, u_+)$ and $\mathcal{D}_V = (-\infty, 2/\xi^2)$.*

Proof. The lemma follows from a simple yet careful analysis of the functions $\Lambda_X(\cdot, t)$ and $\Lambda_V(\cdot, t)$ together with their effective domains. Clearly here $\mathcal{D}_t^V = (-\infty, u^V(t))$, where $u^V(t) \equiv 2bt/[\xi^2(\mathrm{e}^{bt} - 1)]$ converges from above to $2/\xi^2$ as t tends to zero. In [16], the authors showed that $u_+(t)$ (resp. $u_-(t)$) converges from above (resp. from below) to u_+ (resp. u_-) as t tends to zero, so that the limiting domain $\cap_{t>0}\mathcal{D}_t^X$ is equal to (u_-, u_+). The pointwise limits are then straightforward to prove. □

Define now the following functions on $\mathcal{D}_{\mathrm{M}}^o$:

$$\begin{cases} \mathrm{f}_0^X(u) := \cos(\bar{\rho}\xi u/2) - \dfrac{\rho}{\bar{\rho}}\sin(\bar{\rho}\xi u/2), \quad \mathrm{f}_0^V(u) := 1 - \dfrac{u\xi^2}{2}, \quad \mathrm{f}_1^V(u) := -\dfrac{bu\xi^2}{4}, \\ \mathrm{f}_1^X(u) := \begin{cases} \dfrac{\rho(\xi + 2b\rho)}{4\bar{\rho}^2}\cos(\bar{\rho}\xi u/2) + \left(\dfrac{\xi + 2b\rho}{4\bar{\rho}} - \dfrac{\xi\rho + 2b}{2u\xi\bar{\rho}^3}\right)\sin(\bar{\rho}\xi u/2), & \text{if } u \neq 0, \\ -b/2, & \text{if } u = 0, \end{cases} \\ \mathrm{g}_0^X(u) := \rho\xi u, \qquad \mathrm{g}_0^V(u) \equiv 0. \end{cases}, \tag{3.3}$$

Lemma 3.2. *For $\mathrm{M} \in \{X, V\}$, the expansions $f_t^M(u) = \mathrm{f}_0^M(u) + \mathrm{f}_1^M(u)t + \mathcal{O}(t^2)$ and $g_t^M(u) = \mathrm{g}_0^M(u) + \mathcal{O}(t)$ hold for all $u \in \mathcal{D}_{\mathrm{M}}^o$ as t tends to zero. They further hold uniformly on compacts.*

Proof. Let $\mathrm{M} = X$, and define the quantities $d_0 := \bar{\rho}\xi\mathrm{sgn}(u)$, $d_1 := \frac{\mathtt{i}(2\kappa\rho - \xi)\mathrm{sgn}(u)}{2\bar{\rho}}$, where $\mathrm{sgn}(u) = 1$ if $u \geq 0$, and -1 otherwise; then for any $u \in \mathcal{D}_X^o \backslash \{0\}$, $d(u/t) = \mathtt{i}ud_0/t + d_1 + \mathcal{O}(t)$, as t tends to zero, and hence

$$\begin{aligned} \frac{g_t^X(u)}{td(u/t)} &= \frac{\rho\xi u}{\mathtt{i}ud_0} + \frac{d_1\xi\rho - \mathtt{i}bd_0}{d_0^2 u}t + \mathcal{O}(t^2), \\ \cosh\left(\frac{d(u/t)t}{2}\right) &= \cos\left(\frac{d_0 u}{2}\right) + \frac{\mathtt{i}d_1}{2}\sin\left(\frac{d_0 u}{2}\right)t + \mathcal{O}(t^2), \\ \sinh\left(\frac{d(u/t)t}{2}\right) &= \mathtt{i}\sin\left(\frac{d_0 u}{2}\right) + \frac{d_1}{2}\cos\left(\frac{d_0 u}{2}\right)t + \mathcal{O}(t^2). \end{aligned} \tag{3.4}$$

The expansion for f_t^X in (3.2) for $u \in \mathcal{D}_X^o \backslash \{0\}$ follows after using the asymptotics in (3.4) and some simplification. When $u = 0$, straightforward computations reveal that $f_t^X(u) = 1 - bt/2 + \mathcal{O}(t^2)$, in agreement with (3.2). Note that f_1^X is continuous at the origin. The expansions for f_t^V and g_t^M follow analogous arguments and the lemma follows. Uniform convergence of the sequences $(g_t^{\mathrm{M}})_t$ and $(f_t^V)_t$ are trivial. Uniform convergence of compacts of the sequence $(f_t^X)_t$ holds as soon as $\sup_{u\in\mathcal{D}_X^o}\left|f_t^M(u)-f_0^M(u)-f_1^M(u)t\right|$ converges to zero when t tends to zero, which is tedious but straightforward to prove. $\square$

Consider now the (time-dependent) saddlepoint equation:

$$\partial_u\Lambda_M(u,t)=x,\qquad x\in\mathcal{K}_{\mathrm{M}},t>0. \tag{3.5}$$

The following lemma proves existence and uniqueness of the solution to this equation, as well as a small-time expansion. Let us first define the following functions on $\mathcal{K}_{\mathrm{M}}$:

$$\alpha_0^X(x):=u_-\mathbb{1}_{\{x<0\}}+u_+\mathbb{1}_{\{x>0\}},\qquad\qquad \alpha_0^V(x):=\frac{2}{\xi^2}\mathbb{1}_{\{x\geq 0\}}, \tag{3.6}$$

Lemma 3.3. *For any $x\in\mathcal{K}_{\mathrm{M}}$, $t>0$, Equation (3.5) admits a unique solution $u_{\mathrm{M}}^*(x,t)\in\mathcal{D}_t^{\mathrm{M}}$, and the expansion $u_{\mathrm{M}}^*(x,t)=\alpha_0^{\mathrm{M}}(x)+\mathcal{O}(t)$ holds uniformly on compacts. For $x=0$, Equation (3.5) also admits a unique solution $u_{\mathrm{M}}^*(0,t)$, which converges to zero as t approaches zero.*

Proof. We first prove existence and uniqueness of the solution of the saddlepoint equation (3.5). Consider first the case $\mathrm{M}=V$. Clearly, for any $t>0$, the map $\partial_u\Lambda_V(\cdot,t):\mathcal{D}_t^V\to\mathbb{R}$ is strictly increasing and the image of $\mathcal{D}_t^V$ by $\partial_u\Lambda_V(\cdot,t)$ is $\mathbb{R}_+^*$. Thus, for any $x>0$, (3.5) admits a unique solution $u_V^*(x,t)=\frac{2t}{\xi^2}\left(\frac{b}{\mathrm{e}^{bt}-1}-\frac{a}{x}\right)$, which converges to $\alpha_0^V(x)$. Consider now the case $\mathrm{M}=X$. We first start with the following claims, which can be proved using the convexity of the moment generating function and tedious computations.

(i) For any $t>0$, the function $\partial_u\Lambda_X(\cdot,t):\mathcal{D}_t^X\to\mathbb{R}$ is strictly increasing and maps $\mathcal{D}_t^X$ to $\mathbb{R}$;
(ii) For any $t>0$, $u_X^*(0,t)>0$ and $\lim_{t\downarrow 0}u_X^*(0,t)=0$, i.e. the unique minimum of $\Lambda_X(\cdot,t)$ converges to zero;
(iii) For each $u\in\mathcal{D}_X^o$, $\partial_u\Lambda_X(u,t)$ converges to zero as t tends to zero.

Now, choose $x>0$ (analogous arguments hold for $x<0$). It is clear from (i) that (3.5) admits a unique solution. Note further that (i) and (ii) imply $u_X^*(x,t)>0$. Next we introduce the following condition.

Condition A : There exists $t_1>0$ such that $u_X^*(x,t)\in\mathcal{D}_X^o$ for all $t<t_1$.

Suppose condition A is not true and further assume that the sequence $(u_X^*(x,t))_{t>0}$ does not converge to u_+^X as $t\downarrow 0$. Then there exists $t_1^*>0$ and $\varepsilon>0$ such that for all $t<t_1^*$ we have $u_X^*(x,t)\notin B(u_+^X,\varepsilon):=\{y\in\mathbb{R}:|y-u_+^X|<\varepsilon\}$. But since $\lim_{t\downarrow 0}\mathcal{D}_t^X=\mathcal{D}_X$, this implies that our sequence must then satisfy condition A, which is a contradiction. Therefore $u_X^*(x,t)$ converges to u_+^X. Next suppose that condition A is true. Again note that (i) and (ii) imply $u_X^*(x,t)>0$. From (iii) there exists $t_2>0$ such that the sequence $(u_X^*(x,t))_{t>0}$ is strictly increasing as t goes to zero for $t<t_2$. Now let $t^*=\min(t_1,t_2)$ and consider $t<t^*$. Then $u_X^*(x,t)$ is bounded above by u_+ (because $u_X^*(x,t)\in\mathcal{D}_X^o$) and therefore converges to a limit $L\in[0,u_+]$. Suppose that $L\neq u_+$. Since $s\mapsto u_X^*(x,s)$ is strictly increasing as s tends to zero (and $s<t^*$), and $\partial_u\Lambda_X(\cdot,t)$ is strictly increasing we have $\partial_u\Lambda_X(u_X^*(x,t),t)\leq\partial_u\Lambda_X(L,t)$; Combining this and (iii) yields $\lim_{t\downarrow 0}\partial_u\Lambda_X(u_X^*(x,t),t)\leq\lim_{t\downarrow 0}\partial_u\Lambda_X(L,t)=0\neq x$, which contradicts the assumption $x>0$. Therefore $L=u_+$ and the first part of the lemma follows.

Given existence and uniqueness of the solution to the saddlepoint equation, we now prove the expansion stated in the lemma. In light of (3.2), the saddlepoint equation (3.5) can be written explicitly as

$$-\frac{\mu t}{2}\left[\partial_u g_t^M(u_{\mathrm{M}}^*(x,t))f_t^M(u_{\mathrm{M}}^*(x,t))+2\partial_u f_t^M(u_{\mathrm{M}}^*(x,t))\right]=f_t^M(u_{\mathrm{M}}^*(x,t))x.$$

Using Lemma 3.2 in this equation and solving at each order yields the desired expansion. $\square$

For $\mathrm{M} \in \{X, V\}$ and $t > 0$, introduce now a time-dependent change of measure by

$$\frac{\mathrm{d}\mathbb{Q}^{\mathrm{M}}_{x,t}}{\mathrm{d}\mathbb{P}} := \exp\left(\frac{u^*_{\mathrm{M}}(x,t)\mathrm{M}_t}{t} - \frac{\Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t),t)}{t}\right). \tag{3.7}$$

By Lemma 3.3, $u^*_{\mathrm{M}}(x,t)$ belongs to the interior of $\mathcal{D}^{\mathrm{M}}_t$, and so $|\Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t))|$ is finite. Also $\mathrm{d}\mathbb{Q}^{\mathrm{M}}_{x,t}/\mathrm{d}\mathbb{P}$ is almost surely strictly positive and $\mathbb{E}[\mathrm{d}\mathbb{Q}^{\mathrm{M}}_{x,t}/\mathrm{d}\mathbb{P}] = 1$. Therefore (3.7) is a valid measure change for all $t > 0$ and $x \in \mathcal{K}_{\mathrm{M}}$. Define now the random variable $Z^{\mathrm{M}}_{x,t} := (\mathrm{M}_t - x)$, and denote its characteristic function in the $\mathbb{Q}^{\mathrm{M}}_{x,t}$-measure (3.7) by $\Phi^{\mathrm{M}}_{x,t}(u) := \mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}(\mathrm{e}^{\mathtt{i}uZ^{\mathrm{M}}_{x,t}})$. Its asympotic behaviour reads as follows:

Lemma 3.4. *For any $x \in \mathcal{K}_{\mathrm{M}}$, the expansion $\Phi^{\mathrm{M}}_{x,t}(u) = \mathrm{e}^{-\mathtt{i}ux}\left(1 - \frac{\mathtt{i}ux}{\mu}\right)^{-\mu}(1 + \mathcal{O}(t))$ holds for all $u \in \mathbb{R}$.*

Remark 3.5. Lévy's Convergence Theorem [28, Page 185, Theorem 18.1] implies that $Z^{\mathrm{M}}_{x,t}$ converges weakly to the zero mean random variable $Z_x := -x + \Xi$, where Ξ is a Gamma random variable with shape μ and scale $\nu := |x|/\mu$. When $x > 0$ the support of the Gamma density is $\mathbb{R}^*_+$, and for $\mathrm{M} = X$ and $x < 0$ the support is $\mathbb{R}^*_-$. For $x > 0$ the density of $Z^{\mathrm{M}}_{x,t}$ is $h(y) \equiv \mathbb{1}_{\{y+x>0\}}\mathrm{e}^{-(y+x)/\nu}(y+x)^{\mu-1}\nu^{-\mu}\Gamma(\mu)^{-1}$, so that $h \in L^2(\mathbb{R})$ when $\mu > 1/2$. A similar result holds for $\mathrm{M} = X$ and $x < 0$ where the density is given by $h(-y)$.

Proof. From the change of measure (3.7) and the re-scaled cgf given in (3.1) we can compute

$$\log \Phi^{\mathrm{M}}_{x,t}(u) = \log \mathbb{E}^{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}^{\mathrm{M}}_{x,t}}{\mathrm{d}\mathbb{P}}\mathrm{e}^{\mathtt{i}uZ^{\mathrm{M}}_{x,t}}\right) = -\mathtt{i}ux + \frac{1}{t}\Big[\Lambda_{\mathrm{M}}(\mathtt{i}ut + u^*_{\mathrm{M}}(x,t),t) - \Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t),t)\Big].$$

Using the definition of Λ_{M} in (3.2) then yields

$$\Phi^{\mathrm{M}}_{x,t}(u) = \left(\frac{f^M_t(\mathtt{i}ut + u^*_{\mathrm{M}}(x,t))}{f^M_t(u^*_{\mathrm{M}}(x,t))}\right)^{-\mu}\exp\left(-\mathtt{i}ux - \frac{2}{\mu}\left[g^M_t(u^*_{\mathrm{M}}(x,t) + \mathtt{i}ut) - g^M_t(u^*_{\mathrm{M}}(x,t))\right]\right), \tag{3.8}$$

and the lemma follows from careful manipulations of Lemma 3.2. $\square$

Lemma 3.6. *For any $x \in \mathcal{K}_{\mathrm{M}}$, there exists a constant $C > 0$ such that the expansion*

$$\exp\left[-\frac{xu^*_{\mathrm{M}}(x,t)}{t} + \frac{\Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t),t)}{t}\right] = C\exp\left(-\frac{\Lambda^*_{\mathrm{M}}(x)}{t}\right)t^{-\mu}(1 + \mathcal{O}(t))$$

holds as t tends to zero, with Λ^*_{M} in (2.2),(2.3).

Remark 3.7. The constant C above can be computed explicitly as

$$C = \left(-\frac{\mu\partial_u f^M_0(\alpha^M_0(x))}{x}\right)^{-\mu}\exp\left(-x\alpha^M_1(x) - \frac{\mu}{2}g^M_0(\alpha^M_0(x))\right).$$

Proof. From Lemma 3.3 and the characterisation of Λ^* in (2.2),(2.3), there exists $C > 0$, such that for small t,

$$\exp\left(-\frac{xu^*_{\mathrm{M}}(x,t)}{t}\right) = C\exp\left(-\frac{x\alpha^M_0(x)}{t}\right)(1 + \mathcal{O}(t)) = C\exp\left(-\frac{\Lambda^*_{\mathrm{M}}(x)}{t}\right)(1 + \mathcal{O}(t)). \tag{3.9}$$

The definition of Λ_{M} in (3.2) and Lemma 3.3 yield $\mathrm{e}^{\Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t),t)/t} = \mathcal{O}(t^{-\mu})$, and the lemma follows. $\square$

We now use the expansion of the characteristic function expansion in Lemma 3.4 and Fourier inversion methods to derive the tail probabilities of $Z^{\mathrm{M}}_{x,t}$ under the measure (3.7).

Proposition 3.8. *For any $x \in \mathcal{K}_{\mathrm{M}}$, the following holds as t tends to zero:*

$$\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[\exp\left(-\frac{u^*_{\mathrm{M}}(x,t)Z^{\mathrm{M}}_{x,t}}{t}\right)\mathbb{1}_{\{Z^{\mathrm{M}}_{x,t}\geq 0\}}\right]\mathbb{1}_{\{x>0\}} + \mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[\exp\left(-\frac{u^*_{\mathrm{M}}(x,t)Z^{\mathrm{M}}_{x,t}}{t}\right)\mathbb{1}_{\{Z^{\mathrm{M}}_{x,t}\leq 0\}}\right]\mathbb{1}_{\{x<0\}} = \mathcal{O}(t).$$

Proof. We only prove the lemma in the case $x > 0$, the other side being completely analogous. Lemma B.2 implies that for small enough t,

$$\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[\exp\left(-\frac{u^*_{\mathrm{M}}(x,t)Z^{\mathrm{M}}_{x,t}}{t}\right)1\!\!1_{\{Z^{\mathrm{M}}_{x,t}\geq 0\}}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Phi^{\mathrm{M}}_{x,t}(u)\mathrm{d}u}{u^*_{\mathrm{M}}(x,t)/t+\mathtt{i}u}. \tag{3.10}$$

Using the asymptotics of $u^*_{\mathrm{M}}(x,t)$ in Lemma 3.3 yields $\left(u^*_{\mathrm{M}}(x,t)/t+\mathtt{i}u\right)^{-1} = t/\alpha_0^M(x)+\mathcal{O}(t^2)$ for small enough t. Finally combining this with Lemma 3.4 and (3.10) we obtain

$$\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[\exp\left(-\frac{u^*_{\mathrm{M}}(x,t)Z^{\mathrm{M}}_{x,t}}{t}\right)1\!\!1_{\{Z^{\mathrm{M}}_{x,t}\geq 0\}}\right] = \frac{t}{2\pi\alpha_0^M(x)}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathtt{i}ux}\left(1-\frac{\mathtt{i}ux}{\mu}\right)^{-\mu}(1+\mathcal{O}(t))\,\mathrm{d}u.$$

Clearly $\left|\int_{-\infty}^{\infty}\mathrm{e}^{-\mathtt{i}uz}\left(1-\frac{\mathtt{i}uz}{\mu}\right)^{-\mu}\mathrm{d}u\right| < \infty$ for all $z\neq 0, \mu>0$, and the proposition thus follows. □

We now put all the pieces together. Using the time-dependent change of measure (3.7), we have for $x>0$

$$\begin{aligned}\mathbb{P}(\mathrm{M}_t\geq x) = \mathbb{E}\left[1\!\!1_{\{\mathrm{M}_t\geq x\}}\right] &= \exp\left(\frac{\Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t)}{t}\right)\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[\exp\left(-\frac{u^*_{\mathrm{M}}(x,t)\mathrm{M}_t}{t}\right)1\!\!1_{\{\mathrm{M}_t\geq x\}}\right]\\ &= \exp\left(-\frac{xu^*_{\mathrm{M}}(x,t)-\Lambda_{\mathrm{M}}(u^*_{\mathrm{M}}(x,t),t)}{t}\right)\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[\exp\left(-\frac{u^*_{\mathrm{M}}(x,t)Z^{\mathrm{M}}_{x,t}}{t}\right)1\!\!1_{\{Z^{\mathrm{M}}_{x,t}\geq 0\}}\right],\end{aligned}$$

with $Z^{\mathrm{M}}_{x,t}$ defined on page 7. The theorem then follows from Lemma 3.6 and Proposition 3.8. An analogous argument holds for probabilities $\mathbb{P}(\mathrm{M}_t\leq x)$ when $x<0$, and Theorem 2.1 follows.

Appendix A. The Gärtner Ellis Theorem

We provide here a brief review of large deviations and the Gärtner-Ellis theorem. For a detailed account of these, the interested reader should consult [10]. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathbb{R}$, with law μ_n and cumulant generating function $\Lambda_n(u)\equiv\log\mathbb{E}(\mathrm{e}^{uX_n})$.

Definition A.1. The sequence X_n is said to satisfy a large deviations principle with speed n and rate function I if for each Borel mesurable set $E\subset\mathbb{R}$,

$$-\inf_{x\in E^o}I(x)\leq\liminf_{n\uparrow\infty}\frac{1}{n}\log\mathbb{P}\left(X_n\in E\right)\leq\limsup_{n\uparrow\infty}\frac{1}{n}\log\mathbb{P}\left(X_n\in E\right)\leq-\inf_{x\in\bar{E}}I(x).$$

Before stating the main theorem, we need one more concept:

Definition A.2. Let $\Lambda:\mathbb{R}\to(-\infty,+\infty]$ be a convex function, and $\mathcal{D}_\Lambda:=\{u\in\mathbb{R}:\Lambda(u)<\infty\}$ its effective domain. It is said to be essentially smooth if

- The interior $\mathcal{D}^o_\Lambda$ is non-empty;
- Λ is differentiable throughout $\mathcal{D}^o_\Lambda$;
- Λ is steep: $\lim_{n\uparrow\infty}|\Lambda'(u_n)|=\infty$ whenever (u_n) is a sequence in $\mathcal{D}^o_\Lambda$ converging to a boundary point of $\mathcal{D}^o_\Lambda$.

Assume now that the limiting cumulant generating function $\Lambda(u):=\lim_{n\uparrow\infty}n^{-1}\Lambda_n(nu)$, exists as an extended real number for all $u\in\mathbb{R}$, and let $\mathcal{D}_\Lambda$ denote its effective domain. Let $\Lambda^*:\mathbb{R}\to\mathbb{R}_+$ denote its (dual) Fenchel-Legendre transform, via the variational formula $\Lambda^*(x)\equiv\sup_{\lambda\in D_\Lambda}\{\lambda x-\Lambda(\lambda)\}$. Then the following holds:

Theorem A.3 (Gärtner-Ellis theorem)**.** *If the origin lies in the interior of $\mathcal{D}_\Lambda$ and if Λ is lower semicontinuous and essentially smooth, then the sequence $(X_n)_n$ satisfies a large deviations principle with rate function Λ^*.*

Appendix B. Inverse Fourier Transform Representation

Let $g(z) := \exp\left(-u^*_{\mathrm{M}}(x,t)z/t\right)1\!\!1_{\{z\geq 0\}}$. The main result of this appendix is the following representaion:

Lemma B.1. *There exists $t_1^* > 0$ such that for all $t < t_1^*$ and all $x > 0$:*

$$\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}\left[g(Z^{\mathrm{M}}_{x,t})\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Phi^{\mathrm{M}}_{x,t}(u)\mathrm{d}u}{u^*_{\mathrm{M}}(x,t)/t + \mathtt{i}u}. \tag{B.1}$$

The proof of Lemma B.1 proceeds in two steps: We first prove that the integrand in the right-hand side of Equality (B.1) belongs to $L^1(\mathbb{R})$ (and hence the integral is well-defined), and we then prove that this very equality holds. The first step is contained in the following lemma.

Lemma B.2. *There exists $t_0^* > 0$ such that $\int_{-\infty}^{\infty}\left|\frac{\Phi^{\mathrm{M}}_{x,t}(u)}{u^*_{\mathrm{M}}(x,t)/t+\mathtt{i}u}\right|\mathrm{d}u < \infty$ for all $t < t_0^*$ and $x > 0$.*

Proof. Let $\mathrm{M} = V$. Using (3.8) we see that $\Phi^V_{x,t}(u) = \mathrm{e}^{-\mathtt{i}ux}\left(1 + \mathtt{i}ud_t\right)^{-\mu}$ where $d_t := \frac{\xi^2 t(1-\mathrm{e}^{bt})}{2bt+u^*_V(x,t)\xi^2(1-\mathrm{e}^{bt})}$ (recall also from Lemma 3.4 that $\lim_{t\downarrow 0} d_t = -x/\mu$). The modulus is then given by $|\Phi^V_{x,t}(u)| = (1+u^2d_t^2)^{-\mu/2}$ and hence for small enough t we have that $|\Phi^V_{x,t}(u)| \leq D|u|^{-\mu}$ for some $D > 0$. Furthermore, we easily see that $|(u^*_{\mathrm{M}}(x,t)/t + \mathtt{i}u)^{-1}| \leq \min(1/u, t/u^*_{\mathrm{M}}(x,t))$ and hence we compute

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{\Phi^V_{x,t}(u)}{u^*_V(x,t)/t+\mathtt{i}u}\right|\mathrm{d}u &= \int_{|u|\leq 1}\left|\frac{\Phi^V_{x,t}(u)}{u^*_V(x,t)/t+\mathtt{i}u}\right|\mathrm{d}u + \int_{|u|>1}\left|\frac{\Phi^V_{x,t}(u)}{u^*_V(x,t)/t+\mathtt{i}u}\right|\mathrm{d}u\\ &\leq 2t/u^*_V(x,t) + D\int_{|u|>1}|u|^{-\mu-1}\mathrm{d}u.\end{aligned}$$

The last inequality is finite for sufficently small t since $u^*_V(x,t)$ converges to $2/\xi^2$ as t tends to zero and $\mu > 0$. The case $\mathrm{M} = X$ follows from anlogous yet tedious computations. □

We now move on to the proof of Lemma B.1. We only look at the case $\mathrm{M} = V$, the other cases being completely analogous. We denote the convolution of two functions $f, h \in L^1(\mathbb{R})$ by $(f * h)(x) := \int_{\mathbb{R}} f(x-y)h(y)\mathrm{d}y$, and recall that $(f*h) \in L^1(\mathbb{R})$. For such functions, we denote the Fourier transform by $(\mathcal{F}f)(u) := \int_{-\infty}^{\infty}\mathrm{e}^{\mathtt{i}ux}f(x)\mathrm{d}x$ and the inverse Fourier transform by $(\mathcal{F}^{-1}h)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathtt{i}ux}h(u)\mathrm{d}u$. We have that

$$\mathcal{F}\left(g(z)\right)(u) := \int_{-\infty}^{\infty}\exp\left(-\frac{u^*_V(x,t)z}{t} + \mathtt{i}uz\right)1\!\!1_{\{z\geq 0\}}\mathrm{d}z = \frac{1}{u^*_V(x,t)/t - \mathtt{i}u}, \tag{B.2}$$

if $u^*_V(x,t) > 0$, which holds for t small enough since by Lemma 3.3 $u^*_V(x,t)$ converges to u^V_+ and $u^V_+ > 0$. We write

$$\mathbb{E}^{\mathbb{Q}^V_{x,t}}\left[g(Z^V_{x,t})\right] = \int_{\mathbb{R}} q(x-y)p(y)\mathrm{d}y = (q*p)(x),$$

with $q(z) \equiv g(-z)$ and p denoting the density of V_t. On the strips of regularity ($x > 0$) we know there exists $t_0 > 0$ such that $q \in L^1(\mathbb{R})$ for $t < t_0$. Since p is a density, $p \in L^1(\mathbb{R})$, and therefore

$$\mathcal{F}(q*p)(u) = \mathcal{F}q(u)\mathcal{F}p(u). \tag{B.3}$$

We note that $\mathcal{F}q(u) \equiv \mathcal{F}g(-u) \equiv \overline{\mathcal{F}g(u)}$ and hence using (B.2)

$$\mathcal{F}q(u)\mathcal{F}p(u) \equiv \mathrm{e}^{\mathtt{i}ux}\frac{\Phi^V_{x,t}(u)}{u^*_V(x,t)/t+\mathtt{i}u}, \tag{B.4}$$

since the complex conjugate of w^{-1} is equal to $(\Re(w) - \mathtt{i}\Im(w))^{-1}$, for $w \in \mathbb{C}$. Thus by Lemma B.2 there exists an $t_1 > 0$ such that $\mathcal{F}q\mathcal{F}p \in L^1(\mathbb{R})$ for $t < t_1$. By the inversion theorem [27, Theorem 9.11] this then implies from (B.3) and (B.4) that for $t < \min(t_0, t_1)$:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^V_{x,t}}\left[g(Z^V_{x,t})\right] &= (q * p)(x) = \mathcal{F}^{-1}\left(\mathcal{F}q(u)\mathcal{F}p(u)\right)(x) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathtt{i}ux}\mathcal{F}q(u)\mathcal{F}p(u)\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\Phi^V_{x,t}(u)}{u^*_V(x,t)/t + \mathtt{i}u}\mathrm{d}u.\end{aligned}$$

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CMAP, Ecole Polytechnique Paris
E-mail address: demarco@cmap.polytechnique.fr

Department of Mathematics, Imperial College London
E-mail address: a.jacquier@imperial.ac.uk

Department of Mathematics, Imperial College London
E-mail address: p.roome11@imperial.ac.uk